Entanglement and symmetries in one-dimensional quantum field theories

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La mia vita è nella fretta
La mia strada assai ristretta
La mia casa è una cantina
La mia vita è in officina

Il lavoro a me mi stende
E per giunta non mi rende
Per comprarmi una maglietta
Ho venduto la mia Lambretta

Quando prendo lo stipendio
In gelati me lo spendo
I gelati sono buoni
Ma costano milioni

—Skiantos, *Gelati*

LIST OF PUBLICATIONS

During my PhD, I co-authored the following works, which have either been published or are currently under peer review:

- [1] Capizzi, Luca and Castro-Alvaredo, Olalla A. and De Fazio, Cecilia and **Mazzoni, Michele** and Santamaria-Sanz, Lucia "Symmetry resolved entanglement of excited states in quantum field theory. Part I. Free theories, twist fields and qubits" *JHEP*, 2022, 127(2022). https://doi.org/10.1007/JHEP12(2022)127

- [2] Capizzi, Luca and De Fazio, Cecilia and **Mazzoni, Michele** and Santamaria-Sanz, Lucia and Castro-Alvaredo, Olalla A. "Symmetry resolved entanglement of excited states in quantum field theory. Part II. Numerics, interacting theories and higher dimensions" *JHEP*, 2022, 128(2022). https://doi.org/10.1007/JHEP12(2022)128

- [3] Capizzi, Luca and **Mazzoni, Michele** and Castro-Alvaredo, Olalla A. "Symmetry resolved entanglement of excited states in quantum field theory. Part III. Bosonic and fermionic negativity" *JHEP*, 2023, 74(2023). https://doi.org/10.1007/JHEP06(2023)074

- [4] Castro-Alvaredo, Olalla A. and **Mazzoni, Michele** "Two-point functions of composite twist fields in the Ising field theory" *J. Phys. A: Math. Theor.*, 2023, 56, 124001. https://doi.org/10.1088/1751-8121/acbe82

- [5] Capizzi, Luca and **Mazzoni, Michele** "Entanglement asymmetry in the ordered phase of many-body systems: the Ising field theory", *JHEP*, 2023, 144(2023). https://doi.org/10.1007/JHEP12(2023)144

- [6] **Mazzoni, Michele** and Travaglino, Riccardo and Castro-Alvaredo, Olalla A. "Expectation Values of Conserved Charges in Integrable Quantum Field Theories out of Thermal Equilibrium", *arXiv preprint arXiv:2402.14788*, 2024. https://doi.org/10.48550/arXiv.2402.14788

- [7] Travaglino, Riccardo and **Mazzoni, Michele** and Castro-Alvaredo, Olalla A. "Generalised

Hydrodynamics of $T\bar{T}$-Deformed Integrable Quantum Field Theories", *JHEP*, 2024, 90(2024). https://doi.org/10.1007/JHEP08(2024)090

- [8] Capizzi, Luca and **Mazzoni, Michele** "Entanglement content of kink excitations", *arXiv preprint arXiv:2409.03048*, 2024. https://doi.org/10.48550/arXiv.2409.03048

This thesis is based on the works [1–5].

Contents

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DECLARATION OF AUTHENTICITY

This report contains genuine work conducted originally by the author. The work presented herein has not been submitted and/or accepted for the award of any other degree or diploma in any university. To the best of my knowledge and belief, this thesis contains no materials previously published or written by other person, except where due references has been made.

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ABSTRACT

In this thesis, we investigate the interplay between entanglement and internal symmetries in 1+1D quantum field theories (QFTs). Although the notion of symmetry-resolved entanglement has been widely studied in various systems, its behavior in excited states of massive QFTs remains largely unexplored. In the first part of the thesis, we focus on theories with global $U(1)$ symmetry and compute the symmetry-resolved entanglement entropy and logarithmic negativity of zero-density excited states of the massive free complex boson and Dirac fermion. We find that the excess of symmetry-resolved entropy (and negativity) of these states with respect to the ground state is largely independent on the details of the excited state and it has a very simple dependence on the $U(1)$ charge. We test our results numerically on a one-dimensional Fermi gas and on a one-dimensional harmonic chain, and we propose an interpretation of our formulae in terms of simple multi-qubit states. Next, we generalise the field-theoretic computation of symmetry-resolved entanglement measures to higher-dimensional, non-integrable field theories using semi-local twist operators, which are defined through their commutation relations with an algebra of local observables in a QFT. In the second part of the thesis, we turn our attention to the one-dimensional massive Ising QFT, which possesses a $\mathbb{Z}_2$ symmetry. The ground state in the paramagnetic (disordered) phase of the theory is symmetric, and its $\mathbb{Z}_2$-resolved entanglement entropy can be obtained from a two-point function of composite twist fields. We provide an exact expression for the cumulant expansion of this two-point function. In contrast, the ferromagnetic (ordered) phase features two $\mathbb{Z}_2$-breaking vacua. The extent to which the symmetry is broken can be quantified by a relative entropy measure known as entanglement asymmetry. By making use of twist operators, we develop a method to compute entanglement asymmetry in massive 1+1D QFTs with discrete internal symmetry and apply this approach to the Ising model.

CHAPTER

ONE

INTRODUCTION

Over the past decades, advances in experimental technologies have led to extensive exploration of the role of entanglement in quantum many-body systems [9]. In particular, the study of entanglement measures in quantum field theory (QFT), which started in the 1990s with the seminal works [10, 11] and was then crucially extended by Calabrese and Cardy [12], brought results to the attention of a much wider scientific community. These theoretical results, in conjunction with numerical and analytical work in integrable spin chain models [13–16], revealed how certain entanglement measures, such as entanglement entropy [17], display universal scaling at conformal critical points [18]. This observation has many implications, a very important one being that computing the entanglement entropy of a pure state is one of the most numerically effective ways of determining if the theory is critical.

A recent development in this field is the growing interest in a type of entanglement termed symmetry-resolved entanglement. In the context of conformal field theory (CFT), a definition of this quantity was given in [19], where it was related to correlation functions of generalised (or composite) branch-point twist fields (CBPTF). In the context of entanglement, such fields were first introduced in [12] as associated with conical singularities in conformal maps and in [18, 20, 21] as symmetry fields associated to cyclic permutation symmetry in 1+1D QFT (both critical and gapped). The basic idea is that in theories that posses an underlying symmetry (say $U(1)$ symmetry in a complex free boson theory or in sine-Gordon theory) entanglement can be expressed as a sum over contributions from different symmetry sectors. A strong motivation to study the symmetry-resolved entanglement measures is that the different contributions to the total entropy are experimentally measurable [22–24] and can be related to the operationally accessible entanglement in quantum computing [25–27]. Prior to the works presented in this thesis, several results existed for the

symmetry-resolved entanglement of symmetric ground states of CFT, massive QFT and spin chains [28], as well as for low-lying excited states of critical theory, but not for excited states of massive QFTs[1].

Alongside with progress in the study of symmetry-resolved entanglement measures, in the past two years growing attention has been devoted to the notion of entanglement asymmetry. Introduced in [29], entanglement asymmetry provides the first information-based probe of symmetry breaking: it quantifies how much a certain state breaks an underlying internal symmetry of the system, spontaneously or explicitly. Initially, such quantity was used to analyse the restoration (or lack thereof) of a $U(1)$ symmetry in the quench dynamics of quantum spin chains [29–31]. Shortly after these first works, entanglement asymmetry studies were conducted on CFT [32], critical spin chains with larger symmetries [33] and black-hole radiation [34]. The largest part of the research regarding entanglement asymmetry has thus far been devoted to the dynamics of the entanglement asymmetry, whereas a characterisation of this measure in a field-theoretic setting and for more complex symmetry groups is mostly still lacking.

In part I and part II of this thesis, we contribute to filling the two large gaps in the literature that we mentioned above. Namely, in part I we characterise the symmetry-resolved entanglement content of excited states of massive QFTs, and in part II we introduce a field-theoretic framework for the study of entanglement asymmetry in ordered phases of massive field theories. The two parts of the thesis can also be distinguished according to the type of symmetry we treat. In part I, we deal with theories that possess an internal $U(1)$ symmetry, namely the complex boson and the Dirac fermion. In part two of the thesis, on the other hand, we consider the massive Ising field theory, the most paradigmatic quantum field theory featuring discrete $\mathbb{Z}_2$ symmetry. Chapter 4, in which we derive exact formulae for a correlation function in the disordered phase of the Ising QFT, is included in part II of thesis according to this criterion. Among our main contributions in the two parts of this thesis is the introduction of a class of operators, that we call *twist operators* and that generalise objects known in $1+1$D dimensions as branch-point twist fields. We will define twist operators in Chapter 2 and make a consistent use of them throughout the rest of the thesis.

The remaining part of this Introduction contains an essential review of the theoretical foundations necessary for the following chapters: the definition and main properties of entanglement measures, the replica model in $1+1$D quantum field theories, the definition of symmetry-resolved entanglement and entanglement asymmetry. A more detailed outline of the thesis structure concludes this chapter.

[1] Throughout this thesis, the expression "massive QFT", without further specification, refers to either a free or interacting local QFT with a mass gap.

1.1 Entanglement in many-body quantum systems

Entanglement is arguably *the* quantum phenomenon that irremediably marks the departure from deterministic, classical physics. This is, at least, the opinion of someone who might have had a say in the matter [35]. Entanglement is essentially present when there are correlations of an intrinsic quantum nature between two parts of a system prepared in a certain state. Thus, entanglement depends on the state and on the way the system is partitioned. Quantum correlations can be revealed by a measurement on one of the two parts of the system, which causes the *collapse of the wavefunction* according to the Copenhagen interpretation of quantum mechanics [36]. The non-locality of the collapse was at the heart of a debate about the *completeness* of quantum mechanics as a fundamental theory of reality, a debate that culminated in the famous 1935 article by Einstein, Podolski and Rosen [37]. In this paper, the authors proposed that quantum correlations between distant particles should be explained by means of some local hidden variables: our ignorance about the probability distribution associated with these variable would then explain the most counter-intuitive features of quantum mechanics. This position on the fundamental properties of quantum mechanics is sometimes referred to as *local realism*. However, if there are local hidden variables of a purely classical nature, then the possible results of a measurement should satisfy certain inequalities, the first of which were derived by Bell in 1964 [38]. The violation of Bell inequalities was crucially proved by Alain Aspect and collaborators in a series of experiments that led to the 2022 Nobel Prize in Physics [39–41]. The experimental proof that there can be no local hidden variables marks the end of local realism: quantum mechanics is complete and quantum entanglement cannot be explained in terms of classical correlations.

With the developments in quantum technology achieved in the 1980s and 1990s, entanglement started to be considered a resource in the field of information and communication protocols. Just to mention two examples, it is possible to encode in a single *qubit* (quantum bit) the information carried by two classical bits, as long as the qubit belongs to an entangled pair: this is known as *dense coding* [42]. Moreover, by exploiting entanglement it is possible to achieve *quantum teleportation* [43], i.e. transferring a state between two observers (without neither of them knowing what the state actually is).

There are different ways of measuring entanglement, depending on the different points of view that may be adopted. For instance, one may be interested in measuring the entanglement between two spins at different sites in a quantum chain, ignoring the rest of the system: this is an example of pairwise entanglement, which can be measured for instance by the entanglement of formation [9, 44]. Conversely, one may be interested in measuring how entangled are two large subsystems of a larger system: in this case, one talks of entanglement of regions. This is the point of view we will adopt in

this thesis.

We give the following definition of entanglement. Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the Hilbert space corresponding to a total system partitioned into subsystems A and B. Let $\{|\phi_i\rangle_A\}$, $\{|\chi_i\rangle_B\}$ be basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Then a state:

$$|\psi\rangle = \sum_{i,j} \alpha_{i,j} |\phi_i\rangle_A \otimes |\chi_j\rangle_B \in \mathcal{H}, \quad \alpha_{i,j} \in \mathbb{C}, \quad \sum_{i,j} |\alpha_{i,j}|^2 = 1, \tag{1.1}$$

is said to be separable if there exist some $|\phi\rangle_A \in \mathcal{H}_A$, $|\chi\rangle_B \in \mathcal{H}_B$ such that

$$|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B. \tag{1.2}$$

A state is entangled with respect to a given bipartition if it is not separable.

In quantum many-body systems, it is often convenient to adopt the viewpoint of density operators. A density operator, or density matrix, is a map $\rho : \mathcal{H} \to \mathcal{H}$ that encodes all the characteristics of a state. For a state $|\psi\rangle \in \mathcal{H}$, the density matrix is defined as the projector:

$$\rho = |\psi\rangle\langle\psi|. \tag{1.3}$$

A state for which the density matrix can be written as a projector, i.e. every state defined by a vector as in (1.1), is called a *pure state*. However, the true advantage of the density matrix formalism is that density matrices incorporate a notion of classical probability, which is present if the state of the system is a statistical mixture. Indeed, suppose that one wants to describe an ensemble of states $\{|\psi_i\rangle \in \mathcal{H}\}$ in which each state is associated to a certain probability p_i. The correct way to represent this ensemble is via the density operator:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i p_i = 1. \tag{1.4}$$

This is called a *mixed state*. The paradigmatic example of a mixed state is given by the usual Gibbs ensemble (or thermal state), which in quantum statistical mechanics is described by a density matrix:

$$\rho_\beta = \frac{e^{-\beta \hat{H}}}{Z} = \sum_n \underbrace{Z^{-1} e^{-\beta E_n}}_{p_n} |n\rangle\langle n|, \quad \sum_n p_n = 1, \tag{1.5}$$

where $\{|n\rangle\}$ are the energy eigenstates. The Gibbs ensemble displays all the natural requirements of a density operator, namely, it is a positive semi-definite, Hermitian operator of unit trace:

$$\rho = \rho^\dagger, \quad \langle\psi|\rho|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}, \quad \mathrm{Tr}_{\mathcal{H}} \rho = 1. \tag{1.6}$$

Every operator with these properties can be written in the form (1.4). Additionally, density matrices of pure states (and of pure states only) are idempotent, $\rho^2 = \rho$: for this to hold, all the probabilities in (1.4) must be zero except for one $p_i = 1$, and the state reduces to (1.3) for a certain $|\psi\rangle$.

In the state ρ, the expectation value of a local observable represented by an operator $\mathcal{O} : \mathcal{H} \to \mathcal{H}$ is given by $\mathrm{Tr}_{\mathcal{H}}(\rho\mathcal{O})$. However, in systems characterised by a large number of degrees of freedom, it often occurs that an observer can access only a portion of the total system. Suppose the observer is interested in the expectation value of a local operator $\mathcal{O}_{\mathcal{A}} : \mathcal{H}_A \to \mathcal{H}_A$. This is easily embedded in the algebra of operators acting on the total space by defining $\mathcal{O} := \mathcal{O}_{\mathcal{A}} \otimes \mathbb{1}_B$. Then, by picking an orthonormal basis $\{|e_i, e_j\rangle := |e_i\rangle_A \otimes |e_j\rangle_B\}$ for the total Hilbert space, the expectation value of $\mathcal{O}$ reads:

$$\begin{aligned} &\langle\mathcal{O}\rangle = \mathrm{Tr}_{\mathcal{H}}(\rho(\mathcal{O}_{\mathcal{A}} \otimes \mathbb{1}_B)) \\ &= \sum_{i,j,i',j'} \langle e_i, e_j|\rho|e_{i'}, e_{j'}\rangle\langle e_{i'}, e_{j'}|(\mathcal{O}_{\mathcal{A}} \otimes \mathbb{1}_B)|e_i, e_j\rangle = \mathrm{Tr}_{\mathcal{A}}(\rho_A\mathcal{O}_A). \end{aligned} \tag{1.7}$$

The quantity

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho, \tag{1.8}$$

is the *reduced density matrix* (RDM) of the subsystem A. It encodes the description of the system according to an observer who can only access a portion of the total space: the degrees of freedom associated to the complementary region B describe then the environment, and are traced out. When taking the partial trace of a pure state $\rho = |\psi\rangle\langle\psi|$, one may end up with a state ρ_A which is mixed. Namely, this happens when (and only when) $|\psi\rangle$ is not separable. Therefore, the mixedness of the reduced density matrix is a criterion for the presence of entanglement in a pure state of the total system: the state $|\psi\rangle \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_B$ is separable if and only if its reduced density matrix has only one non-vanishing eigenvalue. We do not specify which of the two density matrices: indeed, as a consequence of the *Schmidt decomposition*, the two operators ρ_A and ρ_B have the same spectrum, and can only differ in the number of their zero eigenvalues.

The inspection of the reduced density matrix is sufficient to reveal whether a state is separable, however, it does not provide information about the amount of entanglement in that state. To do so, one needs to define a *measure of entanglement*. In particular, for most of this thesis we will be interested in quantifying the bipartite entanglement of pure states. The reason is that these states play a prominent role in many-body systems: they describe the (non-degenerate) ground state of spin chains and of zero-temperature QFTs. Also the particular excitations we will consider in Chapter 2, i.e. zero-density excited states of quantum field theories, are pure states. There are several ways to define entanglement measures, but all good measures should be *entanglement monotones*. An entanglement monotone is a functional E from the set of quantum states to the real numbers, which

satisfies the fundamental properties [44]:

- non-negativity: $E(\rho) \geq 0$ for every state ρ, and $E(\rho) = 0$ if ρ is separable.
- monotonicity under LOCC: if the state σ is obtained from ρ via local operations and classical communications (LOCC) then $E(\rho) \geq E(\sigma)$.

The notion of LOCC can be given a precise mathematical formulation by means of Kraus operators [45]. Here, we only remark that requiring E to be non-increasing under LOCC is fundamental for at least two reasons. First, from a technological point of view, local quantum operations and classical communications between separate subsystems are necessary for the implementation of protocols such as quantum teleportation: asking that the state which is teleported does not become more entangled during the process seems quite obvious. Second, LOCC operations can be employed to distinguish between classical and quantum correlations: classical correlations are the ones that can be generated by acting on a certain quantum system via LOCC.

The prototypical entanglement monotone for pure states in a bipartite quantum system is the entanglement entropy. To define the entanglement entropy, we start from the von Neumann entropy, i.e. the quantum analog of the Shannon entropy of classical information theory [46]. For a state $\rho : \mathcal{H} \to \mathcal{H}$ (pure or mixed), the von Neumann entropy:

$$S[\rho] := -\mathrm{Tr}(\rho \log \rho), \tag{1.9}$$

has (among others) the following properties:

- $S[\rho] \geq 0$ for every state ρ and $S[\rho] = 0$ if and only if ρ is pure.
- If $\dim(\mathcal{H}) = N$, $\max_{\rho} S[\rho] = \log N$, obtained for the maximally mixed state $\rho = \frac{1}{N}\mathbb{1}_{\mathcal{H}}$.
- It is concave: if $\sum_i \lambda_i = 1$, $S\left[\sum_i \lambda_i \rho_i\right] \geq \sum_i \lambda_i S[\rho_i]$.
- It is sub-additive: if $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and ρ_A, ρ_B are the two RDMs, then $S[\rho] \leq S[\rho_A] + S[\rho_B]$. The equality holds only if $\rho = \rho_A \otimes \rho_B$.

The entanglement entropy is the von Neumann entropy of the reduced density matrix:

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}(\rho_A \log \rho_A). \tag{1.10}$$

The von Neumann entropy vanishes on pure states, but the reduced density matrix ρ_A is pure if and only if the full state ρ is separable. This means that the entanglement entropy of a bipartition is zero if and only if ρ is not entangled. Moreover, because the two reduced density matrices have the same

non-vanishing eigenvalues, $S_A = S_B$. In the following chapters, we will equivalently refer to the von Neumann or entanglement entropy of a bipartition.

The entanglement entropy is the most ubiquitous measure of entanglement, being widely used in quantum many-body physics [9], quantum information theory [47], quantum and conformal field theory [12, 18, 20, 48], black-hole physics and holography [10, 11, 49]. There are several reasons behind the popularity of entanglement entropy in condensed matter and low-dimensional QFT. First of all, it recently became experimentally accessible through measurements of a strictly related quantity, the Rényi entropy. The Rényi entropy of order α, or αth Rényi entropy, for $\alpha \in \mathbb{R}^+$ and $\alpha \neq 1$, is defined as [50]:

$$S_\alpha[\rho_A] := \frac{1}{1-\alpha} \log \mathrm{Tr}(\rho_A^\alpha), \tag{1.11}$$

and it reproduces the entanglement entropy in the limit $\alpha \to 1$:

$$S_A = \lim_{\alpha \to 1} S_\alpha[\rho_A]. \tag{1.12}$$

The authors of [22] were able to measure the second Rényi entropy in a trapped gas of ultra-cold bosonic atoms by letting two many-body wavefunctions interfere[2]. In the next section, we will show how the Rényi entropies can be computed in a *replica version* of a quantum field theory in $1+1$D: the "replica trick"will then be extensively used throughout the entire thesis.

Other than the experimental and computational advantages, the entanglement entropy is a prime measure of entanglement because it encodes universal features and scaling properties of a theory. The universality stems from the fact that the ground state entanglement entropy (as well as the thermal entropy) of a critical system, described for instance by a two-dimensional conformal field theory, contains information about the central charge of such theory, and thus identifies a certain universality class: we will provide some more details about this in the next section. The scaling properties, on the other hand, are related to the celebrated *area law* [51]. This is the statement that the entanglement entropy S_A of a bipartite system generally scales with the (generalised) area of the surface ∂A that separates the two subsystems A and B. In D dimensions, this means:

$$S_A \sim |\partial A| \sim \ell^{D-1}, \tag{1.13}$$

where ℓ is a characteristic length scale associated with either of the two subsystems. The physical intuition behind this law is that the entanglement entropy is due to those pairs of entangled particles for which one particle is in A and one is in B. For systems with short-range correlations, the number of such pairs is roughly proportional to the area of the separating surface. The validity of the area law

[2]For the sake of precision, the quantity that was experimentally measured is the *quantum purity* $\mathrm{Tr}(\rho_A^2)$, the logarithm of which is proportional to $S_2[\rho_A]$.

for the ground state of one-dimensional gapped spin chains (i.e. with finite correlation length $\xi < \infty$) was proved by Hastings in [52], whereas gapless spin chains, which are described by $1+1$D CFTs in the continuum limit, violate the area law. A good review about the area law of entanglement entropy can be found in [53].

We conclude this section by introducing a different measure of entanglement. We mentioned that the entanglement entropy is a good measure for bipartite systems in a pure state. When the state is mixed, the entropy is no longer an adequate measure of entanglement because it fails to discern true quantum correlations from the statistical correlations present in the initial ensemble. A similar scenario occurs in many-body systems in two paradigmatic cases: when the initial state is thermal, as in equation (1.5), or when the state itself is pure, but the system is not bipartite, i.e. the subsystems A and B are not complementary. In this thesis, we focus on the second situation. Let us consider a tripartite Hilbert space:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C, \tag{1.14}$$

where the subsystem C plays the role of the environment, and suppose we want to measure the entanglement between subsystems A and B in a pure state ρ of the total Hilbert space. The reduced density matrix $\rho_{A\cup B} = \mathrm{Tr}_{\mathcal{H}_C}\rho$ is mixed. A necessary condition for this state to be separable is that its partial transpose $\rho_{A\cup B}^{T_B}$ has no negative eigenvalues (this is known as the Peres-Horodecki criterion [54, 55]). The quantity $\rho_{A\cup B}^{T_B}$ is defined by transposing the matrix only with respect to the subsystem B: if $\{|e_i, e_j\rangle := |e_i\rangle_A \otimes |e_j\rangle_B\}$ is an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_B$, then

$$\langle e_i, e_j|\rho_{A\cup B}^{T_B}|e_k, e_l\rangle := \langle e_i, e_l|\rho_{A\cup B}|e_k, e_j\rangle. \tag{1.15}$$

According to the Peres-Horodecki criterion, the *logarithmic negativity* [56, 57]:

$$\mathcal{E}[\rho_{A\cup B}] := \log \mathrm{Tr}(|\rho_{A\cup B}^{T_B}|), \tag{1.16}$$

where we introduced the trace norm of a matrix, $\mathrm{Tr}|A| := \mathrm{Tr}\sqrt{A^\dagger A}$, is a good indicator of entanglement of the state $\rho_{A\cup B}$ (and an entanglement monotone). Indeed, it is possible to show that $\rho_{A\cup B}^{T_B}$ has real spectrum, but in general it is not positive semi-definite, and the trace norm of the partial transpose can be equivalently expressed as the sum of all its singular values. Therefore, a logarithmic negativity $\mathcal{E} > 0$ necessarily indicates that $\rho_{A\cup B}^{T_B}$ has at least one negative eigenvalue. This means that the subsystems A and B are entangled. We postpone a more detailed discussion of the logarithmic negativity to Chapter 3.

1.2 Bipartite entanglement in $1+1$D quantum field theories

In this section, we review the replica method for the computation of bipartite entanglement entropy in $(1+1)$-dimensional QFTs. As we show below, in a replica theory a central role is played by some fields which implement cyclic permutations of replicas and display semi-locality properties with respect to the fundamental fields of the theory: these are the *branch-point twist fields* (BPTF). In general, twist fields are associated to internal symmetries of a theory, and they first appeared in the context of orbifold CFT, where they describe the propagation of bosonic strings in a background described by a $\mathbb{Z}_N$ orbifold [58, 59]. In statistical field theory, a prominent example of twist fields is given by the order and the disorder fields of the Ising field theory [60–62], which are associated to the $\mathbb{Z}_2$ symmetry of the model and are semi-local with respect to the fermion field. The replica method for the computation of bipartite entanglement in CFT was employed for the first time in the pioneering works [10, 11]. These results were then extended in [12], where the entanglement of an interval in a replica theory of a CFT (at finite temperature and in the ground state) was computed by means of correlation functions of some primary operators of the CFT. The full picture was finally unveiled in [20], where the case of massive integrable QFT was treated: here, the branch-point twist fields were first defined in relation to the symmetry of the replica theory under permutation of copies. These fields are the massive counterpart of the CFT primary operators introduced in [12], i.e. the fields that flow to the CFT branch-point twist fields in the UV limit. The form-factor program for the BPTF was also developed in [20], and extended to theories with a boundary in [63]. We refer to [64] for a definition of twist fields associated to an internal continuous symmetry. In the following review, we mostly follow [12], [20] and [18].

Let us consider the ground state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ of a one-dimensional QFT, where $A = [x_1, x_2]$, $B = (-\infty, x_1) \cup (x_2, \infty)$. We aim to compute the bipartite entanglement entropy of this state. This is done by first obtaining an expression for the Rényi entropy (1.11) for $\alpha = n \in \mathbb{N}$, $n > 1$, and then performing an analytic continuation over real values of n. The density matrix $\rho = |\psi\rangle\langle\psi|$ is obtained by taking the limit $\beta \to \infty$ of the thermal state (1.5), whose matrix elements can be expressed by an Euclidean path integral as:

$$\langle\varphi_1|\rho_\beta|\varphi_2\rangle = \frac{1}{Z(\beta)} \int_{\varphi(x,0)=\varphi_1(x)}^{\varphi(x,\beta)=\varphi_2(x)} \mathcal{D}\varphi\, e^{-S_E[\varphi]}, \quad Z(\beta) = \int_{\varphi(x,0)=\varphi(x,\beta)} \mathcal{D}\varphi\, e^{-S_E[\varphi]}, \tag{1.17}$$

where we assume that $\varphi(x,\tau)$ is the fundamental scalar field in the theory and the Euclidean action is $S_E[\varphi] = \int_0^\beta \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}x \mathcal{L}[\varphi](x,\tau)$. The reduced density matrix is obtained by splitting the field φ in the components φ_A and φ_B defined on the regions A, B respectively and tracing out the degrees of freedom φ_B:

$$\langle\varphi_{1,A}|\rho_{\beta,A}|\varphi_{2,A}\rangle = \frac{1}{Z(\beta)} \int \mathcal{D}\varphi_B \int \mathcal{D}\varphi\, e^{-S_E[\varphi]}, \tag{1.18}$$

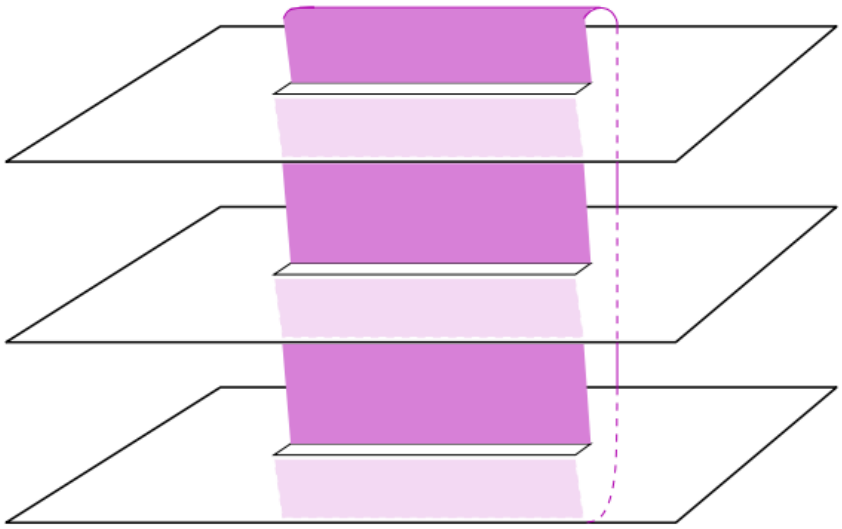


Figure 1.1 Replica manifold $\mathcal{R}_n$ for $n = 3$. The branch cuts in each sheet are along the interval $A = [x_1, x_2]$. Picture taken from [20].

where the boundary conditions for the integration in $\mathcal{D}\varphi$ are $\varphi(x,0) = \varphi_{1,A}(x)$ and $\varphi(x,\beta) = \varphi_{2,A}(x)$ for $x \in A$, $\varphi(x,0) = \varphi(x,\beta) = \varphi_B(x)$ for $x \in B$. From the expression above, after taking the zero-temperature limit, it is straightforward to compute:

$$\mathrm{Tr}\rho_A^n = \int_{\mathrm{BC}} \left[\prod_{i=1}^{n} \mathcal{D}\varphi\right] \prod_{i=1} \langle\varphi_{i,A}|\rho_{\beta,A}|\varphi_{i+1,A}\rangle = \frac{Z_n}{(Z_1)^n}, \tag{1.19}$$

where $\varphi_{n+1} \equiv \varphi_1$ and the presence of the factor $(Z_1)^n$ ensures that $\mathrm{Tr}\rho_A = 1$. In the above expression we introduced the partition function

$$Z_n = \int \prod_i [\mathcal{D}\varphi_i]_{\mathcal{R}_n} \, e^{-S_{E,\mathcal{R}_n}[\{\varphi_i\}]}, \tag{1.20}$$

over the n-sheeted Riemann surface $\mathcal{R}_n$. This surface is the *replica manifold*, and its structure is dictated by the boundary conditions (BC) imposed on the path integral, which we specify below. $\mathcal{R}_n$ consists in n sheets which are flat everywhere except at the branch points $(x_1, 0)$ and $(x_2, 0)$. The branch cuts extending between these points (that is, on the interval A) cyclically connect the sheets, as depicted in Figure 1.1.

The replica action $S_{E,\mathcal{R}_n}[\{\varphi_i\}]$ in equation (1.20) is given by:

$$S_{E,\mathcal{R}_n}[\{\varphi_i\}] = \int_{\mathcal{R}_n} \mathrm{d}\tau \mathrm{d}x \, \mathcal{L}^{(n)}[\{\varphi_i\}](x,\tau), \quad \mathcal{L}^{(n)}[\{\varphi_i\}] = \sum_{i=1}^{n} \mathcal{L}[\varphi_i]. \tag{1.21}$$

Thus, the total energy of the theory is the sum of the single-replica energies, which allows interpreting the sheets as independent copies of the theory. The fields φ_i, $i = 1, \ldots, n$, each of which is defined on

a different copy, are not interacting, being connected only through the boundary conditions:

$$\mathcal{C}(x_1,x_2): \quad \begin{cases} \varphi_i(x,0^+) = \varphi_{i+1}(x,0^-), & \forall x \in [x_1,x_2] \\ \varphi_i(x,0^+) = \varphi_i(x,0^-), & \forall x \notin [x_1,x_2] \end{cases}, \quad i=1,\dots,n, \quad n+1 \equiv 1 \tag{1.22}$$

By explicitly employing these conditions, together with the fact that the Lagrangian density $\mathcal{L}^{(n)}[\{\varphi_i\}]$ is insensitive to the structure of the manifold, we can rewrite Z_n as a partition function on $\mathbb{R}^2$:

$$Z_n = \int_{\mathcal{C}(x_1,x_2)} \prod_i [\mathcal{D}\varphi_i]_{\mathbb{R}^2} \, e^{-S_{E,\mathcal{R}^2}[\{\varphi_i\}]}. \tag{1.23}$$

The equality between the two expressions (1.20) and (1.23) reflects the presence of a symmetry in the replica version of the theory. In fact, the replica partition function is invariant under the full group S_n of replica permutations. As evident from the boundary conditions (1.22), among the permutations a special role is played by the subgroup $\mathbb{Z}_n$ of cyclic permutations. The branch-point twist field $\mathcal{T}$ and its Hermitian conjugate $\tilde{\mathcal{T}}$ are the symmetry fields associated to the generator σ of cyclic permutations of replicas and to its inverse σ^{-1}, respectively:

$$\mathcal{T} = \mathcal{T}_\sigma, \quad \sigma: \quad i \mapsto i+1 \bmod n, \quad \tilde{\mathcal{T}} = \mathcal{T}_{\sigma^{-1}}, \quad \sigma^{-1}: \quad i \mapsto i-1 \bmod n. \tag{1.24}$$

The fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ are *defined* by the property of implementing the boundary conditions (1.22). Namely, if $\mathcal{O}_i(x,\tau)$ is a local observable defined on the ith sheet, its replica correlation function in the theory described by the Lagrangian density $\mathcal{L}$ is:

$$\langle \mathcal{O}_i(x,\tau)\dots\rangle_{\mathcal{L};\mathcal{R}_n} = \frac{\langle \mathcal{T}(x_1,0)\tilde{\mathcal{T}}(x_2,0)\mathcal{O}_i(x,\tau)\dots\rangle_{\mathcal{L}^{(n)};\mathbb{R}^2}}{\langle \mathcal{T}(x_1,0)\tilde{\mathcal{T}}(x_2,0)\dots\rangle_{\mathcal{L}^{(n)};\mathbb{R}^2}}. \tag{1.25}$$

Pictorially, one can think of $\mathcal{T}$, $\tilde{\mathcal{T}}$ as generating semi-infinite branch cuts at their insertion points, along which the permutations (1.24) are realised. These fields, being associated to a global symmetry of the theory, are local with respect to the replica action. However, they do not have vanishing equal-time commutators with the fields φ_i. Indeed, it follows from (1.25):

$$\varphi_i(x)\mathcal{T}(y) = \begin{cases} \mathcal{T}(y)\varphi_{i+1}(x) & y > x \\ \mathcal{T}(y)\varphi_i(x) & y < x \end{cases} \quad \text{and} \quad \varphi_i(x)\tilde{\mathcal{T}}(y) = \begin{cases} \tilde{\mathcal{T}}(y)\varphi_{i-1}(x) & y > x \\ \tilde{\mathcal{T}}(y)\varphi_i(x) & y < x \end{cases}. \tag{1.26}$$

Semi-local fields with the properties outlined above are not unique, but they can be fixed by imposing that $\mathcal{T}$, $\tilde{\mathcal{T}}$ correspond to the lightest, spinless primary fields of the underlying CFT (of

central charge c), with scaling dimension[12, 20, 58, 59]:

$$d_{\mathcal{T}} = 2\Delta_{\mathcal{T}} = 2\Delta_{\tilde{\mathcal{T}}}, \quad \Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right). \tag{1.27}$$

Thus, by employing the CFT normalisation of these fields and equation (1.25) one can finally express:

$$\mathrm{Tr}\rho_A^n = \frac{Z_n}{(Z_1)^n} = \zeta_n \epsilon^{4\Delta_{\mathcal{T}}} \langle \mathcal{T}(x_1,0)\tilde{\mathcal{T}}(x_2,0)\rangle_{\mathcal{L}^{(n)};\mathbb{R}^2}, \tag{1.28}$$

where the correlator is on the replicated ground state of the theory[3]. In the expression above, ϵ is a short-distance cutoff, which can be identified for instance with a lattice spacing $\epsilon \sim a$, while ζ_n is a non-universal normalisation factor such that $\zeta_1 = 1$, consistently with the fact that for $n = 1$ $\mathcal{T}$ and $\tilde{\mathcal{T}}$ reduce to the identity operator, and $\frac{d\zeta_n}{dn}\Big|_{n=1} = 0$. The nth Rényi entropy directly follows from the previous equation, while the entanglement entropy is obtained as:

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\left[\epsilon^{4\Delta_{\mathcal{T}}}\langle \mathcal{T}(x_1,0)\tilde{\mathcal{T}}(x_2,0)\rangle\right]. \tag{1.29}$$

We mention that analytically continuing the replica index n to real values usually requires some care. We shall come back to this at the end of Chapter 4.

A replica picture for the computation of logarithmic negativity in QFT was devised in [65, 66]. In these works, the authors were able to obtain the *Rényi negativities*:

$$\mathcal{E}_n[\rho_{A\cup B}^{T_B}] = \log \mathrm{Tr}\left(\rho_{A\cup B}^{T_B}\right)^n, \quad \text{for} \quad n \in \mathbb{N}, \quad n > 1, \tag{1.30}$$

for a subsystem $A \cup B$ consisting of two disjoint intervals $A = [x_1, x_2]$, $B = [x_3, x_4]$, by means of a four-point function of BPTFs:

$$\mathcal{E}_n[\rho_{A\cup B}^{T_B}] = \log\left[\xi_n \epsilon^{8\Delta_{\mathcal{T}}}\langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\tilde{\mathcal{T}}(x_3)\mathcal{T}(x_4)\rangle\right], \tag{1.31}$$

where the correlator is again taken on the replicated ground state of the theory. The replica manifold associated to the computation of the above correlation function is more complicated than the one depicted in Figure 1.1, as there are in this case two branch cuts in each copy which cyclically connect the sheets (see for instance [65] for a pictorial representation). We remark that the logarithmic negativity (1.16) is obtained from the expression (1.30) by analytically continuing n to the reals and taking the limit $n \to 1$ when n is even. Indeed, the nth Rényi entropy has a different expression in terms of the singular values $|\lambda_i|$ of the partial transpose $\rho_{A\cup B}^{T_B}$ according to the parity of n, as can be seen from direct inspection: if n_e is an even positive integer and $n_o > 1$ is an odd positive integer,

[3] However, as we will discuss in Chapter 2, this relation is valid even in zero-density excited state

then

$$\mathcal{E}_{n_e}[\rho_{A\cup B}^{T_B}] = \log\left[\sum_{\lambda_i>0}|\lambda_i|^{n_e} + \sum_{\lambda_i<0}|\lambda_i|^{n_e}\right], \quad \mathcal{E}_{n_o}[\rho_{A\cup B}^{T_B}] = \log\left[\sum_{\lambda_i>0}|\lambda_i|^{n_o} - \sum_{\lambda_i<0}|\lambda_i|^{n_o}\right], \tag{1.32}$$

and the trace norm in (1.16) is correctly reproduced only by taking the limit $\lim_{n_e\to 1}\mathcal{E}_{n_e}[\rho_{A\cup B}^{T_B}]$.

We now come back to the bipartite entanglement entropy and briefly discuss the main findings for the ground state of a $1+1$D QFT. In a critical theory, for an interval A of length ℓ, S_A is proportional to the logarithm of ℓ [11–14]:

$$S_A \sim \frac{c}{3}\log\frac{\ell}{\epsilon}, \quad \text{for} \quad \epsilon \ll \ell \ll \xi, \tag{1.33}$$

where c is the central charge of the CFT, ϵ is a short-distance cutoff and ξ is the correlation length of the system. The above expression violates the area law (1.13), which predicts that for one-dimensional systems the bipartite entropy is independent of the size[4]. The situation is more complex in $D>1$, where some systems characterised by a divergent correlation length satisfy the area law [67], whereas the violation persists in other cases, see for instance [68].

On the other hand, one-dimensional off-critical systems satisfy the area law. These systems are described by gapped quantum spin chains or, in the continuum limit limit, by massive QFTs. In this case, the entropy S_A of an interval saturates to a constant value in the limit of large ℓ:

$$S_A \sim \frac{c}{3}\log\frac{\xi}{\epsilon}, \quad \text{for} \quad \epsilon \ll \xi \ll \ell, \tag{1.34}$$

where the correlation length is typically given by the inverse of a characteristic mass scale in the theory, $\xi = m^{-1}$. The off-critical behaviour of the bipartite entropy was derived for $1+1$D massive quantum field theories in [12, 20], and observed in several gapped spin chains, see for instance [69–73].

We conclude this section by mentioning how the subleading corrections to the entanglement saturation (1.34) can be obtained in (relativistic) massive $1+1$D QFTs. This is done by a spectral expansion of the branch-point twist field two-point function in (1.29). In one-dimensional quantum field theories, a basis for the Hilbert space of the theory is provided by the asymptotic multi-particle states, labelled by the rapidities θ_j of the particles in the theory together with their quantum numbers μ_j:

$$|\theta_1,\dots,\theta_n\rangle_{\mu_1,\dots,\mu_n}, \tag{1.35}$$

where the rapidity $\theta\in\mathbb{R}$ of an on-shell particle of mass m parametrises its energy and momentum as $E(\theta)=m\cosh\theta$, $P(\theta)=m\sinh\theta$. In a replica version of a QFT, the quantum numbers $\mu_j=(i_j,a_j)$

[4] However, consistently with the area law, when A is a disjoint union of intervals (see e.g. [12, 18]), S_A is proportional to the number of boundary points, which is precisely the value of $|\partial A|$ in $D=1$

account for internal degrees of freedom a_j as well as the replica indices $i_j \in \{1, \dots, n\}$. Moreover, the rapidities in the above state are usually ordered such that $\theta_1 > \cdots > \theta_n$ if the state is incoming, i.e. far in the past before any interaction, and $\theta_1 < \cdots < \theta_n$ if the state is far in the future, after any interaction. Using the expansion in asymptotic states, one can write [20]:

$$ {}^n\langle 0|\mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)|0\rangle^n = \sum_{k=1}^{\infty} \sum_{\mu_1,\dots,\mu_k} \int \frac{\mathrm{d}\theta_1 \dots \mathrm{d}\theta_k}{k!(2\pi)^k} e^{-\ell \sum_{j=1}^k m_{a_j} \cosh\theta_j} |{}^n\langle 0|\mathcal{T}(0)|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k}|^2. \tag{1.36} $$

In the above equation, $\ell = |x_2 - x_1|$, $|0\rangle^n$ is the replica ground state and the quantity:

$$ F^{\mathcal{T}|_{\mu_1,\dots,\mu_K}}(\theta_1, \dots, \theta_k) :=^n \langle 0|\mathcal{T}(0)|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots,\mu_k}, \tag{1.37} $$

is the k-particle form factor of the branch-point twist fields $\mathcal{T}$. In general, form factors are matrix elements of local or semi-local operators of a field theory, and they are used to characterise its operator content. Form factors can be defined in every $1+1$D QFT, as long as the asymptotic states (1.35) are provided together with a description of the interaction in terms of scattering amplitudes between those states. However, it is only in *integrable* quantum field theories (IQFT) that form factors can be exactly obtained in a systematic way[5]. The reason behind the prominent role of integrability is that the presence of infinitely many conserved quantities severely constrains the dynamics of a model: in particular, once the scattering amplitudes are known, the form factors of a given operator can be obtained by imposing that some natural requirements on their structure are satisfied: this is the so-called form factor *bootstrap program* [77–79]. The form factor bootstrap program has been the object of intense research in the past forty years, leading to the classification of the operator content in several integrable QFTs, featuring diagonal and non-diagonal scattering, and even the presence of unstable bound states: see for instance [62, 80, 81] for the Ising model, [82] for the Federbush model, [83] for the sinh-Gordon model, [84] for the scaling Lee-Yang model, [85] for the sine-Gordon model, [86, 87] for the homogeneous sine-Gordon model, and [88] for the Bullough-Dodd model. In the case of operators which are semi-local with respect to the fundamental fields of the theory, the form-factor bootstrap equations need to be slightly modified. The first treatment of form factors of semi-local operators can be found in [62], for the order and disorder fields of the Ising IQFT, and in [89, 90] for the $U(1)$ vertex operator of the sine-Gordon model. The form factor equations for the branch-point twist field were first derived in [20], and employed to obtain a large-ℓ expansion of the two-point function (1.36) in the Ising and sinh-Gordon model. The result of [20] was then generalised in [21] to the case of massive QFT, not necessarily integrable: for an interval A of length ℓ, the ground state

[5] In this thesis, we will only be concerned with form factors of free IQFT, so we refer to [74–76] for good reviews on one-dimensional IQFTs.

bipartite entropy is given by

$$S_A = -\frac{c}{3}\log(m_1\epsilon) + U - \frac{1}{8}\sum_{\alpha=1}^{N} K_0(2\ell m_\alpha) + \mathcal{O}(e^{-3\ell m_1}), \tag{1.38}$$

where U is a constant, m_α are the masses of the N species of particles in the theory ($m_1 = \xi^{-1}$ being the lightest), and K_0 is a modified Bessel function [91]. In general, the inclusion of up to N-particle scattering processes in the spectral expansion of the BPTF two-point function yields a correction to the saturation law (1.34) which at leading order is $\mathcal{O}(e^{-Nm_1\ell})$.

In this thesis, we will perform expansions of the form (1.36) for the free massive Dirac fermion and complex boson (Chapter 2) and for the Ising field theory (Chapters (4) and (5)), both at finite and infinite size. We will provide the necessary details as needed throughout the work.

1.3 The role of internal symmetries

Investigating the relationship between the internal symmetries of a system and its entanglement properties has emerged as one of the most productive research directions in the field of many-body physics over the past five years. The first work on this subject was [92], in which the notion of spin-resolved entanglement was introduced to better characterise the spectrum of entanglement of the critical XXZ spin-$\frac{1}{2}$ chain. A more general notion of symmetry-resolved entanglement entropy (SREE) was then put forward in [93] and [19]. In [93], the SREE was employed to explain the mechanism behind many-body localisation in disordered systems whereas, crucially, [19] contains a field-theoretic characterisation of this quantity for $U(1)$ and discrete symmetries within a replica picture. The equipartition of the symmetry-resolved entropy, i.e. the fact that different symmetry sectors yield the same leading-order contribution to the total entanglement, was proved in [94]. After these first works, the SREE for a generic non-abelian Lie group in a $1+1$D CFT was obtained in [95], where the author focused on the ground state of Wess-Zumino-Witten models. Among the several other papers on this topic, we mention the generalisations to other entanglement measures, namely the charge-imbalance resolution of negativity [96] and symmetry-resolved quantum distances [97]. Following [19], we present the notion of symmetry-resolved entanglement entropy for a $U(1)$ symmetry.

The key idea is that if a system possesses an internal symmetry, then the reduced density matrix has a block-diagonal structure. Let us consider a pure state ρ in a bipartite geometry, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and an internal $U(1)$ symmetry of the system generated by the charge $\hat{Q}$:

$$\hat{Q} = \hat{Q}_A \oplus \hat{Q}_B := \hat{Q}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{Q}_B. \tag{1.39}$$

Then the Hilbert space is decomposed into irreducible representations of the symmetry, $\mathcal{H} = \oplus_q \mathcal{H}_q$,

$q \in \mathbb{Z}$. We further assume that the state ρ is symmetric, and so is the reduced density matrix[6] ρ_A, so that

$$[\rho, \hat{Q}] = 0, \quad [\rho_A, \hat{Q}_A]. \tag{1.40}$$

The symmetry allows us to write ρ_A in a block-diagonal form:

$$\rho_A = \oplus_q p(q)\rho_A(q), \quad p(q) := \mathrm{Tr}\left(\Pi_q \rho_A\right), \quad \rho_A(q) = \frac{\Pi_q \rho_A}{\mathrm{Tr}\left(\Pi_q \rho_A\right)}, \tag{1.41}$$

where $p(q)$ is the probability of measuring a charge eigenvalue q in the state ρ_A and we introduced the projector $\Pi_q : \mathcal{H}_A \to \mathcal{H}_{A,q}$ onto the charge eigenspace of $\mathcal{H}_A$ labeled by q. In the case of $U(1)$, Π_q can be written as

$$\Pi_q = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha \, e^{2\pi i \alpha(\hat{Q}_A - q)}, \quad q \in \mathbb{Z}. \tag{1.42}$$

It is straightforward to check from the above expression that Π_q is idempotent, Hermitian and it commutes with ρ_A.

We then define the charged replica partition function:

$$\mathcal{Z}_n(q) = \mathrm{Tr}\left(\Pi_q \rho_A^n\right), \quad n \in \mathbb{N}, \quad n > 1, \tag{1.43}$$

together with the nth symmetry-resolved Rényi entropy (SRRE):

$$S_n(q) = \frac{1}{1-n} \log \frac{\mathcal{Z}_n(q)}{[\mathcal{Z}_1(q)]^n} = \frac{1}{1-n} \log \mathrm{Tr}\left[\rho_A(q)\right]^n, \tag{1.44}$$

and the symmetry-resolved entanglement entropy (SREE):

$$S_A(q) = \lim_{n \to 1} S_n(q) = -\mathrm{Tr}\left[\rho_A(q) \log \rho_A(q)\right]. \tag{1.45}$$

Although the SREE, and particularly the charged partition function, has an interesting interpretation in terms of a modified replica manifold, which we describe below, in many situations one is ultimately interested in the total bipartite entropy S_A. This is related to the symmetry-resolved entropy as follows:

$$\begin{aligned} S_A &= -\sum_q p(q) \mathrm{Tr}\left[\Pi_q \rho_A \log \rho_A\right] \\ &= \underbrace{-\sum_q p(q) \log p(q)}_{S_A^f} + \underbrace{\sum_q p(q) S_A(q)}_{S_A^c}. \end{aligned} \tag{1.46}$$

[6] If the group acts unitarily on both $\mathcal{H}_A$ and $\mathcal{H}_B$ then it is easy to show that the symmetry of ρ implies that of ρ_A [95].

In the expression above, in which the second line follows from the definition (1.41) after some straightforward algebraic manipulation, we highlighted two contributions to the total entropy: the *configurational entropy* S_A^c and the *fluctuation entropy* S_A^f (also referred to as *number entropy* [22]). The quantity S_A^f measures the entropy related to fluctuations of the total charge in the subsystem A, which arise because of particle motion through the boundary ∂A. On the other hand, S_A^c is the averaged symmetry-resolved entanglement entropy of the two subsystems, where each contribution at a fixed charge q is weighted by the corresponding probability $p(q)$. Although both quantities can be experimentally probed [22], it is only S_A^c that can be used as a resource in quantum information. Namely, S_A^c provides an upper bound to the *operationally accessible entanglement*, i.e. the entanglement that can be transferred to a quantum register by means of LOCC [25–27].

The charged partition function (1.43) is usually computed by means of its charged moments. For a fixed value of n, the charged moment $Z_n(\alpha)$ is the Fourier transform of $\mathcal{Z}_n(q)$ (or its Fourier series in the case of a discrete group). For a $U(1)$, this is defined as

$$Z_n(\alpha) := \mathrm{Tr}\left(\rho_A^n e^{2\pi i\alpha}\right), \tag{1.47}$$

so that, using expression (1.42) one obtains:

$$\mathcal{Z}_n(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha Z_n(\alpha) e^{-2\pi i\alpha q}. \tag{1.48}$$

Expression (1.47) appeared in [19], although similar quantities were previously employed to describe Rényi entropies charged by a magnetic potential in holographic settings [98–100], theories with symmetry-protected topological phases [101], 2D CFTs and free-fermionic field theories in higher dimensions [102]. Physically, the computation of (1.47) amounts to evaluating the partition function over a replica manifold with the insertion of an Aharonov-Bohm flux $2\pi\alpha$ along the branch cut [103]. This is represented in Figure 1.2. Thus, a charged particle that encircled the branch point x_1 for n times, crossing all the replicas and returning to the initial position, would acquire a phase $e^{2\pi i\alpha}$. By fractionalising the total flux, so that the flux between consecutive sheets is $\frac{2\pi\alpha}{n}$, the boundary conditions (1.22) imposed on the (now charged) fields φ_i are modified to:

$$\mathcal{C}_\alpha(x_1,x_2): \quad \begin{cases} \varphi_i(x,0^+) = e^{\frac{2\pi i\alpha}{n}}\varphi_{i+1}(x,0^-), & \forall x \in [x_1,x_2] \\ \varphi_i(x,0^+) = \varphi_i(x,0^-), & \forall x \notin [x_1,x_2] \end{cases}, \quad i=1,\dots,n, \tag{1.49}$$

for a bipartition $A=[x_1,x_2]$, $B=\bar{A}$. These boundary conditions are implemented in the path integral by *charged* or *composite* branch-point twist fields (CBPTF). If $\mathcal{T}$ is the the branch-point twist field defined in equations (1.24), (1.25), and if $\mathcal{V}_\alpha$ is the operator that generates the $U(1)$ flux, then the

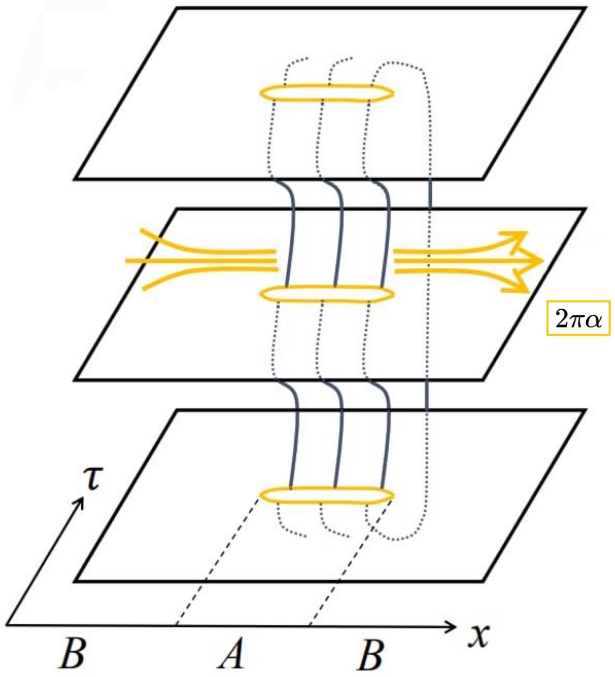


Figure 1.2 Replica manifold $\mathcal{R}_n$ for $n = 3$ with the insertion of a flux $2\pi\alpha$ along the branch cut in the region $A = [x_1, x_2]$. Picture taken from [19] and re-adapted.

CBPTF $\mathcal{T}^\alpha$ is obtained by "fusing" $\mathcal{T}$ and $\mathcal{V}_\alpha$. In particular, in a conformal field theory $\mathcal{T}^\alpha$ is defined as the lightest operator[7] appearing in the operator product expansion (OPE) of those two fields, and the conjugate field $\tilde{\mathcal{T}}^\alpha$ is analogously defined starting from $\tilde{\mathcal{T}}$ and $\mathcal{V}_{-\alpha}$. The off-critical deformation of the CFT fields $\mathcal{T}^\alpha$, $\tilde{\mathcal{T}}^\alpha$ define the composite branch-point twist fields in massive $1+1$D QFTs with internal symmetries, and in the case of integrable theories they can be fully characterised by means of their form factors. We provide a more detailed characterisation of the composite fields associated to continuous and discrete (abelian) internal symmetries in Chapters 2 and 4 respectively. We remark that since the charge operator $\hat{Q}_A$ commutes with the reduced density matrix ρ_A, different choices of the flux fractionalisation will lead to the same expectation values of physical observables. However, a homogeneous fractionalisation of the flux among copies keeps the replica symmetry manifest and it is essential to diagonalise the action of twist fields in free theories. In the following Chapters, we will specify every time which choice of flux fractionalisation we are adopting.

In the bipartite geometry considered above, the expression of the charged moment $Z_n(\alpha)$ in the ground state of a $1+1$D QFT in terms of CBPTFs is the natural generalisation of equation (1.28):

$$Z_n(\alpha) = \zeta_{n,\alpha} \epsilon^{4\Delta_{\mathcal{T}^\alpha}} \langle \mathcal{T}^\alpha(x_1, 0) \tilde{\mathcal{T}}^\alpha(x_2, 0) \rangle_{\mathcal{L}^{(n)};\mathbb{R}^2}, \tag{1.50}$$

where $\zeta_{n,\alpha}$ is a normalisation constant and $\Delta_{\mathcal{T}^\alpha}$ is the conformal dimension of the CBPTFs (or of their critical counterpart). By means of charged moments, the bipartite SREE was computed in

[7] That is, the operator in the OPE with the lowest scaling dimension Δ above the identity operator (for which $\Delta = 0$).

the ground state of $1+1$D CFTs [19, 94, 95], in the lowest-lying CFT excited states [104], in the ground state of massive $1+1$D IQFTs with continuous [105, 106] and discrete [107, 108] symmetries. In Chapter 2, we provide a natural completion to this picture by computing charged moments and (excess of) SREEs of excited states in massive QFT, obtaining results which extend to non-integrable and higher-dimensional theories as well.

Before proceeding, we briefly comment on the notion of symmetry decomposition of logarithmic negativity, introduced in [96]. In a tripartite space $A \cup B \cup C$, the partial transpose $\rho_{A\cup B}^{T_B}$ admits a decomposition according to the irreducible representations of some internal charge of the theory. In this case, if $\hat{Q} = \hat{Q}_A \oplus \hat{Q}_B \oplus \hat{Q}_C$ is the charge operator on the total Hilbert space, then the operator that generates the symmetry of $\rho_{A\cup B}^{T_B}$ is the *charge imbalance*:

$$\hat{Q}_A - \hat{Q}_B^T := \hat{Q}_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \hat{Q}_B^T. \tag{1.51}$$

In the replica theory, the charged moments of the partial transpose are naturally interpreted as partition functions on the Riemann surface, with the insertion of an Aharonov-Bohm flux only on region A. We defer a more detail discussion and a literature review on the subject to Chapter 3, where we obtain the symmetry-resolved logarithmic negativity and Rényi negativities of excited states in a QFT.

The idea of using entanglement measures to probe the *breaking* of an internal symmetry appeared about five years after the first works on symmetry-resolved entanglement in many-body systems. The notion of *entanglement asymmetry* was defined in [29] as a quantifier of the amount of symmetry breaking in a non-symmetric state ρ_A. A similar notion had already appeared in [109], where it was used as a measure of inseparability of a global state ρ with a conserved charge. In [29], the authors quenched an initial $U(1)$ symmetry-breaking state to the XX Hamiltonian, which preserves the symmetry, and observed a dynamical restoration of the latter. Surprisingly, the symmetry was restored more quickly when it was initially more broken, a phenomenon that was dubbed *quantum Mpemba effect*. The reference is to the "classical" Mpemba effect, i.e. the counterintuitive phenomen for which, in a certain range of initial temperature and pressure, hot water freezes faster than cooler water [110]. A further investigation of the $U(1)$ entanglement asymmetry in ground states of spin chains, together with a quasiparticle interpretation of the dynamical restoration of symmetry (or of its absence) was the object of [30, 31, 111]. The breaking and restoration of $U(1)$ symmetry in random qubit states was also used to model the loss of information in black-hole radiation [34]. The case of the discrete group $\mathbb{Z}_n$ was first addressed in [112], where the specific example of $\mathbb{Z}_2$ symmetry-breaking in the XY ground state was discussed. In [32], the authors computed the $U(1)$ entanglement asymmetry in the ground state of a CFT by relating it to the presence of topological defects in the replica surface.

Finally, the symmetry breaking pattern $SU(2) \to U(1)$ in the XXZ critical chain was first investigated in [33].

To define the $U(1)$ entanglement asymmetry, following [29, 30], we once again consider a pure state ρ of a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, together with a global symmetry of the system generated by $\hat{Q} = \hat{Q}_A \oplus \hat{Q}_B$. We further assume that the RDM $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho$ is not symmetric:

$$[\rho_A, \hat{Q}_A] \neq 0. \tag{1.52}$$

The symmetrised state $\rho_{A,Q}$ is built by retaining only the part of ρ_A which is block-diagonal in the eigenbasis of $\hat{Q}_A$, that is:

$$\rho_{A,Q} = \sum_q \Pi_q \rho_A \Pi_q, \tag{1.53}$$

where Π_q, for $q \in \mathbb{Z}$, are the projectors onto the $U(1)$ charge eigenspaces. The state $\rho_{A,Q}$ satisfies $[\rho_{A,Q}, \hat{Q}_A] = 0$ by construction and it is immediate to check that $\rho_A = \rho_{A,Q}$ if and only if the state is symmetric. The entanglement asymmetry ΔS_A is defined as the difference of the von Neumann entropies of the states $\rho_{A,Q}$ and ρ_A:

$$\Delta S_A := S[\rho_{A,Q}] - S[\rho_A] = -\mathrm{Tr}\left(\rho_{A,Q} \log \rho_{A,Q}\right) + \mathrm{Tr}\left(\rho_A \log \rho_A\right), \tag{1.54}$$

and it can be conveniently obtained by taking the limit $n \to 1$ of the *Rényi asymmetry*

$$\Delta S_A^{(n)} := \frac{1}{1-n} \left[\log \mathrm{Tr}\left(\rho_{A,Q}^n\right) - \log \mathrm{Tr}\left(\rho_A^n\right)\right]. \tag{1.55}$$

Moreover, it is possible to express ΔS_A as the relative entropy between the states ρ_A and $\rho_{A,Q}$ [113]:

$$\Delta S_A = \mathrm{Tr}\left[\rho_A \left(\log \rho_A - \log \rho_{A,Q}\right)\right] =: S(\rho_A || \rho_{A,Q}). \tag{1.56}$$

The above equality follows from the definition (1.54) noting that

$$\begin{aligned} \mathrm{Tr}\left(\rho_{A,Q} \log \rho_{A,Q}\right) &= \sum_q \mathrm{Tr}\left(\rho_A \Pi_q \log \rho_{A,Q} \Pi_q\right) \\ &= \sum_q \mathrm{Tr}\left(\rho_A \log \rho_{A,Q} \Pi_q\right) = \mathrm{Tr}\left(\rho_A \log \rho_{A,Q}\right), \end{aligned} \tag{1.57}$$

where we used $[\Pi_q, \log \rho_{A,Q}] = 0$, together with the idempotence of the projector and completeness of the expansion in charge eigenstates. The fact that ΔS_A can be written as a relative entropy implies the following important properties:

- $\Delta S_A \geq 0$ for every state ρ_A,

- $\Delta S_A = 0$ if and only if $\rho_A = \rho_{A,Q}$.

Like the SREE, the entanglement asymmetry can be computed by means of charged moments. Using the explicit expression (1.42), we can write

$$\rho_{A,Q} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha \, e^{-2\pi i \alpha \hat{Q}_A} \rho_A e^{2\pi i \alpha \hat{Q}_A}, \tag{1.58}$$

from which it follows that

$$\mathrm{Tr}\left(\rho_{A,Q}^n\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n \, Z_n(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \tag{1.59}$$

where:

$$Z_n(\boldsymbol{\alpha}) = \mathrm{Tr}\left(\rho_A e^{2\pi i(\alpha_1 - \alpha_2)\hat{Q}_A} \rho_A \dots \rho_A e^{2\pi i(\alpha_n - \alpha_1)\hat{Q}_A}\right). \tag{1.60}$$

We stress that, because ρ_A and $\hat{Q}_A$ do not commute, the order of the operators in the above expression is relevant.

In a replica picture, $Z_n(\boldsymbol{\alpha})$ is the partition function on a charged Riemann manifold similar to that depicted in Figure 1.2, with the important difference that the total Aharonov-Bohm flux, i.e. the sum of all the fluxes inserted between consecutive replicas, is zero. In Chapter 5, we propose a way to compute entanglement asymmetry in terms of generalisations of branch-point twist fields and we show that twist fields associated to a vanishing total flux are indeed unitarily equivalent to the standard (uncharged) BPTFs.

1.4 Structure of the thesis

This thesis is organised as follows. In Chapter 2, which is based on the works [1, 2], we compute the ratio of $U(1)$ charged moments between zero-density excited states and the ground state in free $1+1$D massive fermionic and bosonic QFTs by means of a form factor expansion of the twist field two-point function. The ratio of charged moments is then used to obtain the excess of SREE between the excited states and the ground state. This quantity displays the same universal features already observed for the total excess of entropy in [114, 115] and a very simple dependence on the $U(1)$ charge. We then show how the QFT results can be derived within a much simpler framework, in which the excitations are represented by qubit states. Moreover, we prove that the validity of our formulae extends beyond the domain of free two-dimensional QFTs: this is done by explicitly computing the SREE of one- and two-magnon states of a spin chain and by developing the formalism of twist operators for generic algebras of observables in D-dimensional QFTs. In particular, the algebraic twist operator approach allows us to bypass the computation of form factors and establish the validity of the free-theory results

for local interacting theories, not necessarily integrable. We conclude the chapter by numerically checking our results on two one-dimensional discrete models with $U(1)$ symmetry: a chain of spinless fermions and a complex harmonic chain.

We complete our investigation of symmetry resolution of entanglement for theories with a global $U(1)$ symmetry in Chapter 3, based on [3], where we focus on symmetry-resolved negativity. Employing the qubit picture and the algebraic twist operator formalism introduced in Chapter 2, we obtain universal results for the difference of symmetry-resolved Rényi and logarithmic negativity between zero-density excited states and the ground state. Our results extend those found in [116] for free massive $1+1$D QFTs with no internal symmetry, and at leading order in the large-volume limit they also apply to interacting and higher-dimensional theories. We provide distinct treatments of bosonic and fermionic excitations, as the correct definition of entanglement negativity for fermions requires a slightly different construction of the partial transpose of the RDM. We test our results numerically on a 1D chain of spinless fermions.

The first two chapters form part I of the thesis. Part II starts with Chapter 4, based on the work [4]. In this chapter we turn our attention to the massive Ising QFT in $1+1$D, which features a global $\mathbb{Z}_2$ symmetry, and obtain an exact formula for the cumulant expansion of $\log\langle\mathcal{T}_\mu(x_1,0)\tilde{\mathcal{T}}_\mu(x_2,0)\rangle$ in the paramagnetic ground state, where $\mathcal{T}_\mu$ is the composite twist field obtained by fusing the BPTF $\mathcal{T}$ and the disorder field μ. The cumulant expansion for the correlator of standard BPTFs is known for free $1+1$D theories [63, 117], but in the case of composite twist fields there are convergence issues which we show how to solve. We conclude the chapter by performing the analytic continuation of the cumulant expansion to real values of the replica index.

In Chapter 5, based on [5], we propose a field-theoretic framework for the computation of entanglement asymmetry in the ordered phase of $1+1$D, which display symmetry-breaking vacua. Our approach, valid for any discrete global symmetry G, is based on the algebraic construction of generalised twist operators and is applied to the Ising QFT in the ferromagnetic phase. We characterise the generalised twist operators by means of form factor bootstrap and obtain a formula for the $\mathbb{Z}_2$ entanglement asymmetry in the Ising QFT which is valid up to two-particle contributions. Moreover, we show as a byproduct that the total bipartite entropy in the ferromagnetic phase differs from the one in the dual point of the paramagnetic phase. We close the chapter by presenting a conjecture on the general form of the entanglement asymmetry in the case of partial symmetry breaking $G \to H$.

We conclude with a final summary of what we accomplished in this thesis and outline what we believe are the most promising research directions opened up by this work.

PART I : SYMMETRY-RESOLVED ENTANGLEMENT OF LOCALISED EXCITATIONS IN MASSIVE COMPLEX FREE THEORIES

CHAPTER

TWO

SYMMETRY-RESOLVED ENTROPY OF EXCITED STATES

The excess entanglement resulting from exciting a finite number of quasiparticles above the ground state of a free 1+1D QFT has been investigated quite extensively in the literature. It has been found that it takes a very simple form, depending only on the number of excitations and their statistics. There is now mounting evidence that such formulae also apply to interacting and even higher-dimensional quantum theories. In this chapter, based on [1, 2], we extend the known results by studying the symmetry-resolved entanglement entropy of such zero-density excited states in 1+1D QFTs that possess an internal symmetry. The ratio of charged moments between the excited and ground states, from which the symmetry-resolved entanglement entropy can be obtained, takes a very simple and universal form, which in addition to the number and statistics of the excitations, now depends also on the symmetry charge. Using form factor techniques, we obtain both the ratio of moments and the symmetry-resolved entanglement entropies in complex free theories which possess $U(1)$ symmetry. The same formulae are found for simple multi-qubit states. We then generalise our results in two directions: by showing that they apply also to some excited states of quantum spin chains (one- and two-magnon states) and by developing a higher-dimensional generalisation of the branch-point twist field picture, leading to results in (interacting) higher-dimensional models. Finally, we provide numerical evidence for our formulae by computing functions of the charged moments in two free lattice theories: a 1D Fermi gas and a complex harmonic chain.

2.1 Introduction and summary of results

Symmetry-resolved entanglement and zero-density states. Starting from the basic ideas that we presented in Section 1.3, SREEs have been computed and discussed for many classes of models,

ranging from 1+1D CFTs [19, 92, 94, 95, 97, 104, 118–122], to free [105, 123, 124] and interacting integrable QFT [106–108], holographic settings [98, 102, 125–127], lattice models [26, 27, 92, 94, 118, 119, 123, 128–133], out of equilibrium systems [23, 118, 131, 134–137] and for systems with more exotic types of dynamics [138–143]. In this chapter, we compute the SREEs of zero-density excited states in 1+1D gapped systems in the scaling limit, focusing of excitations of free QFTs, the massive Dirac fermion and the complex boson. We then extend these results to generic zero-density excited states of interacting QFTs and spin chains.

The systems we consider are at zero temperature and finite volume with periodic boundary conditions. The two complementary regions A and $\bar{A}$ have lengths ℓ and $L-\ell$. Eventually, we will consider the infinite-volume limit in which the relative size of the two subsystems is fixed:

$$\ell, L \to \infty, \quad \text{with } r := \frac{\ell}{L} \text{ fixed}, \quad r \in [0,1]. \tag{2.1}$$

In this limit, a zero-density excited state is a state describing a finite number of excitations above the QFT vacuum whose momenta are fixed as the volume increases. In the series of papers [114–116, 144], the excess of entanglement entropy and logarithmic negativity[1] of the zero-density states with respect to their ground state values was computed and found to take a remarkably universal and simple form: it depends only on r, on the number of excitations and on their statistics. The results were originally derived by employing the branch-point twist field approach in free fermion and free boson theories. However, it was argued in [114] (and illustrated with the example of one- and two-magnon states) that the formulae should hold much more generally, for interacting and even higher-dimensional theories[2], as long as a notion of localised excitations exists. These claims have been substantiated through additional recent results. In particular, a series of works by Rajabpour and collaborators [145–150] has expanded previous work in various directions: by obtaining finite-volume corrections, new formulae for systems where quasiparticles are not localised, and finally by establishing that the formulae indeed hold for generic magnon states, thus also in interacting theories. Similar formulae have also been found for interacting higher-dimensional theories in [151] and even in the presence of an external potential, arising from a semiclassical limit [152]. Indeed, the formulae found in [114] were not entirely unexpected as they can be derived for semiclassical systems [153], however their wide range of applicability, well beyond the semiclassical regime, as well as their derivation in the context of QFT were new.

In CFT, the excess of entanglement entropy of some excited states with respect to the ground state was studied in [154, 155]. The states considered therein are the lowest-lying excited states of the

[1] In some of these works, more complex partitions were also considered, e.g. multiple disconnected regions.

[2] In [144] the same formulae were shown to hold for free bosons in any dimension if r is replaced by the ratio of generalised volumes.

theory in the same geometric setting described above. Namely, for states of the form:

$$|\Upsilon\rangle := \lim_{z,\bar{z}\to 0} \Upsilon(z,\bar{z})|0\rangle, \tag{2.2}$$

where $\Upsilon(z,\bar{z})$ is a primary field[3] of conformal weights $(\Delta,\bar{\Delta})$, the excess of entropy with respect to the vacuum state in the limit of small subsystem size $r \ll 1$ is given by:

$$S_\Upsilon - S_0 \simeq \frac{2\pi^2}{3}(\Delta+\bar{\Delta})r^2 + \mathcal{O}(r^{2d_\Psi}), \tag{2.3}$$

with $d_\Psi = \Delta_\Psi + \bar{\Delta}_\Psi$ the scaling dimension of the field defined by the operator product expansion $\Upsilon \times \Upsilon^\dagger = 1 + \Psi + \dots$. This expression was then checked for the lowest-lying excited states of the complex compactified boson ($c = 2$), which has $(\Delta,\bar{\Delta}) = (1,0)$. The computation of the excess of $U(1)$ symmetry-resolved entropy for these low-lying states was performed in [104]. On the other hand, for zero-density excited states of the massive free boson and massive free fermion, the excess entropy is [114, 115]:

$$S_{\text{exc}} - S_0 = -r\log r - (1-r)\log(1-r). \tag{2.4}$$

By comparing these two expressions, it is clear how the entanglement content of zero-density excitations in massive QFT and that of low-lying states of critical systems are captured by different physical pictures.

In this chapter we combine these two topics, symmetry-resolved entropies and excited states, to investigate how the entropy of excited states may be seen as a sum over symmetry sectors in the presence of an internal symmetry. We will initially focus our attention on the complex free fermion and boson theories. The total excited state entanglement of (real) free fermions and bosons was obtained in [115, 116], while the SREE in the ground state of the $1+1$D massive Dirac fermion and complex boson was studied in [105]. This chapter can be seen as a generalisation of these works. Our motivation to study these types of states from this viewpoint is, first and foremost, to provide exact formulae for the SREEs of at least a class of excited states in 1+1D QFT. This is interesting because the SREE of the ground state of 1+1D QFTs has generally a very complicated form, only accessible perturbatively in some parameter, as discussed in many papers [19, 92, 94, 95, 97, 104–108, 118–124]. On the other hand, it is possible to show that the SREEs of zero-density excited states are as complex as that of the ground state, i.e. knowing the symmetry-resolved entanglement content of the ground state is sufficient to reconstruct that of the excited states. Moreover, for special cases when the ground state is trivial, the SREEs can be obtained exactly. Further motivation is provided by the fact that, unlike the total entropy, the SREEs are entanglement measures that allow us to distinguish between

[3] The complex coordinates $(z,\bar{z})$ are introduced as customary in the radial quantisation scheme [74], in which $|z| \to 0$ corresponds to the infinite past in Euclidean time.

charged and neutral excitations, although in the present chapter we only focus on charged particles. An example where both charged and neutral excitations are present is the sine-Gordon model in the interacting regime, which was studied in [106].

Main results and outline. The main results can be summarised as follows. Let $Z_n^{\Psi}(L,\ell,\alpha)$ be the charged moments of the nth symmetry-resolved Rényi entropy (SRRE) of a connected region of length ℓ, in a pure state $|\Psi\rangle_L^n$ of an n-replica theory in finite volume L. Then, the ratio of moments

$$M_n^{\Psi}(r;\alpha) := \lim_{L\to\infty} \frac{Z_n^{\Psi}(L,rL;\alpha)}{Z_n^{0}(L,rL;\alpha)}, \tag{2.5}$$

between the state $|\Psi\rangle_L^n$ and the ground state $|0\rangle_L^n$, in the infinite-volume limit with r fixed, is given by a universal formula, which depends very simply on r and α. We call this quantity a charged ratio. There are two particularly useful cases from which more general formulae can be constructed. When $|\Psi\rangle_L^n = |1^\epsilon\rangle_L^n$ is a state of a single particle excitation with $U(1)$ charge $\epsilon = \pm 1$ we have that

$$M_n^{1^\epsilon}(r;\alpha) = e^{2\pi i\epsilon\alpha} r^n + (1-r)^n, \tag{2.6}$$

whereas for a state of k identical excitations, that is excitations with the same momenta and charges ϵ, we have

$$M_n^{k^\epsilon}(r;\alpha) = \sum_{j=0}^{k} [f_j^k(r)]^n e^{2\pi i j\epsilon\alpha}, \quad f_j^k(r) := \binom{k}{j} r^j (1-r)^{k-j}. \tag{2.7}$$

Formula (2.7) is the building block for all other results (formula (2.6) is the $k=1$ case of (2.7)). A generic state comprising s groups of $k_i^{\epsilon_i}$ identical particles of charge ϵ_i will have

$$M_n^{k_1^{\epsilon_1}\dots k_s^{\epsilon_s}}(r;\alpha) = \prod_{i=1}^{s} M_n^{k_i^{\epsilon_i}}(r;\alpha). \tag{2.8}$$

Note that the possibility of having identical excitations is excluded for fermionic theories. For $\alpha = 0$ these formulae reduce to those found in [114, 115], later generalised to entanglement measures of multiple disconnected regions [116] and to higher dimensions for free bosons in [144]. These results in turn have been extended in a series of works [145–150] to deal with finite-volume corrections and non-localised excitations. More recently, some of the $\alpha = 0$ results were recovered as a semiclassical limit in the presence of an interaction potential [152]. This semiclassical picture had already been invoked much earlier, see for instance [153]. However, it is worth emphasizing that our formulae are not merely semiclassical limits but hold for genuine quantum theories. The quantum nature of the model is encoded in the symmetry-resolved entanglement entropy of the ground state (and its associated moments), which is indeed highly non-trivial. In other words, it is only the ratios (2.5) that are simple, not the individual charged moments.

In order to obtain the SREE it is necessary to isolate the charged moments of the excited state. This can be easily done analytically, and it allows us to write the SREEs in terms of the ground state entropies. For instance, for the state $|\Psi\rangle_L^n = |1^\epsilon\rangle_L^n$ considered above, the SREEs (Rényi and von Neumann) are given by

$$S_n^{1^\epsilon}(r;q) = \frac{1}{1-n}\log\frac{\mathcal{Z}_n^{1^\epsilon}(r,q)}{(\mathcal{Z}_1^{1^\epsilon}(r,q))^n} = \frac{1}{1-n}\log\frac{\mathcal{Z}_n^0(q-\epsilon)r^n + \mathcal{Z}_n^0(q)(1-r)^n}{(\mathcal{Z}_1^0(q-\epsilon)r + \mathcal{Z}_1^0(q)(1-r))^n}, \tag{2.9}$$

and

$$\begin{aligned} S_1^{1^\epsilon}(r;q) \;=\; & -\frac{\mathcal{Z}_1^0(q-\epsilon)r\log r + \mathcal{Z}_1^0(q)(1-r)\log(1-r) + [r\partial_n\mathcal{Z}_n^0(q-\epsilon) + (1-r)\partial_n\mathcal{Z}_n^0(q)]_{n=1}}{r\mathcal{Z}_1^0(q-\epsilon) + (1-r)\mathcal{Z}_1^0(q)} \\ & + \log(\mathcal{Z}_1^0(q-\epsilon)r + \mathcal{Z}_1^0(q)(1-r)), \end{aligned} \tag{2.10}$$

in terms of the ground state partition functions and their derivatives, which can be related back to the ground state entropies. Here $\mathcal{Z}_n^\Psi(r,q)$ are the symmetry-resolved partition functions in the state $|\Psi\rangle$ and $\mathcal{Z}_n^0(q)$ are those of the ground state, which are independent of r in the scaling limit considered here. The formulae for the SREEs of other states are rather cumbersome and we discuss more general cases in Section 2.2.

This chapter is organised as follows: In Section 2.2 we employ the $U(1)$ composite branch-point twist fields to express the charged ratio of moments between zero-density states and the ground state. We show how the problem of computing this quantity can be simplified by diagonalising the fields in the replica space and present explicit form factor computations for zero-density excited states of the complex free boson and fermion. We then discuss how the symmetry-resolved entropies can be obtained from the ratios of charged moments. Some details of the calculations are left to Appendix 2.A and Appendix 2.B, while Appendix 2.C contains a form factor expansion of the two-point function of composite twist fields in the ground state. In Section 2.3 we show how the results obtained via the twist field approach can be derived in a simpler framework of qubit states. Multi-qubit states have coefficients that represent the probabilities of finding a certain number of excitations in a certain space region. In this case the symmetry-resolved entanglement of the excited states can be obtained explicitly. In Section 2.4 we generalise the results to interacting and higher dimensional theories. In the first part of the section we show that the formulae derived for free theories also hold in some magnonic states of spin chains in the presence of interactions. In the second part of the section we introduce the notion of twist operator, which extends that of twist field to theories (either free or interacting) in higher space-time dimensions. Twist operators will be used extensively in other chapters of this thesis. In Section 2.5 we present numerical results for two (free) discrete systems: a 1D lattice Fermi gas and a 1D complex harmonic chain. We find that in both cases the formulae presented above are reproduced with great precision, even if the scaling limit of the Fermi gas is a massless free fermion.

For the complex free boson, we show how the SREE can be computed by employing a wave-functional method, some details of which are left in Appendix 2.D. Conclusions and outlook are in Section 2.6.

2.2 Charged moments and SREE of free theories: form factor approach

In this section we express the charged moments of excited states of $U(1)$ free theories as two-point functions of composite branch-point twist fields (CBPTF) in finite volume, and we show how their computation can be much simplified by diagonalising the action of the twist fields in the replica space. Once we have expressed the twist fields and the excited states in the diagonal basis, we compute the two-point functions via a form factor expansion, obtaining the ratio of charged moments for the complex boson and the Dirac fermion. Finally, from the ratio of charged moment and the knowledge of the charged moments in the ground state, we reconstruct the SREE of the excited states by performing simple Fourier transforms.

2.2.1 Composite branch-point twist field factorisation

Let us consider, as above, a system of total length L with periodic boundary conditions, and a connected subsystem A extending from $x = 0$ to $x = \ell$. The moments of the RDM of a pure state in the replica theory, that is, a tensor product $|\Psi\rangle_L^n := |\Psi\rangle_L \otimes \cdots \otimes |\Psi\rangle_L$ of n identical states, can be obtained from the equal-time correlator of two branch-point twist fields, thanks to the identification[4]:

$$\mathrm{Tr}\rho_A^n = \varepsilon^{4\Delta_{\mathcal{T}}} {}_L^n\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\Psi\rangle_L^n, \tag{2.11}$$

where ε is a short-distance cut-off and $\Delta_{\mathcal{T}}$ is the conformal dimension of the branch-point twist field, defined in (1.27). Because of the definitions (1.9), (1.11), differences of Rényi or von Neumann entropies are independent of ε. They depend only on the ratio

$$R_n^{\Psi}(\ell, L) := \frac{{}_L^n\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\Psi\rangle_L^n}{{}_L^n\langle 0|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|0\rangle_L^n}, \tag{2.12}$$

where $|0\rangle_L^n := \otimes_{j=1}^n |0\rangle_{j,L}^n$ is the finite-volume replica ground state. In the limit (2.1) this becomes a function $R_n^{\Psi}(r)$ of $|\Psi\rangle^n$ and r only and, for the states considered in the previous section, it is given by the same equations (2.6), (2.7) and (2.8) if we set $\alpha = 0$. In theories possessing a $U(1)$ symmetry the formulation is a very natural generalisation of the previous case, in which the branch-point twist field

[4]Note that here we do not include a non-universal normalisation factor to account for the norm of the state. The reason is that we are computing the correlator in finite volume and, as we will show, working in finite volume provides a natural way to regularise the divergences arising from the norm of the state when the large-volume limit is taken.

$\mathcal{T}$ is replaced by the CBPTF $\mathcal{T}^\alpha$ and the charged moment is given by the two-point function

$$\mathrm{Tr}_A(\rho_A^n e^{2\pi i\alpha\hat{Q}_A}) = Z_n^\Psi(L,\ell;\alpha) = \varepsilon^{4\Delta_{\mathcal{T}^\alpha}} {}^n_L\langle\Psi|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|\Psi\rangle^n_L. \tag{2.13}$$

The field $\mathcal{T}^\alpha$ and its conjugate $\tilde{\mathcal{T}}^\alpha$ can be understood as massive counterparts of the corresponding CFT field. This means that the field $\mathcal{T}^\alpha$ in the massive theory obtained as a certain relevant perturbation of a CFT flows to the field:

$$\mathcal{T}^\alpha(y) =: \mathcal{T}\mathcal{V}_\alpha : (y) = n^{2\Delta_\alpha-1}\lim_{x\to y}|x-y|^{2\Delta_\alpha(1-\frac{1}{n})}\sum_{j=1}^n \mathcal{T}(y)\mathcal{V}_\alpha^j(x), \tag{2.14}$$

when the conformal limit is performed. Above, $\mathcal{V}_\alpha$ is the vertex operator associated with the $U(1)$ symmetry of the theory, corresponding to the insertion of an Aharonov-Bohm phase $e^{2\pi i\alpha}$ on the Riemann surface, Δ_α is the conformal dimension of this field, and $\mathcal{V}_\alpha^j$ is a copy of this field living in copy j of the replica theory. That is, $\mathcal{T}^\alpha$ is the lightest field appearing in the operator product expansion (OPE) of $\mathcal{T}$ and $\mathcal{V}_\alpha$. As shown in [105, 124, 156–158], the conformal dimension of the fields $\mathcal{T}^\alpha$, $\tilde{\mathcal{T}}^\alpha$ is

$$\Delta_{\mathcal{T}^\alpha} := \Delta_{\mathcal{T}} + \frac{\Delta_\alpha}{n}, \tag{2.15}$$

where, for $\alpha \in [-\frac{1}{2},\frac{1}{2}]$:

$$\Delta_\alpha = \frac{\alpha^2}{2} \qquad \text{for Dirac fermions,} \tag{2.16}$$

$$\Delta_\alpha = \frac{|\alpha|-\alpha^2}{2} \qquad \text{for complex bosons.} \tag{2.17}$$

The main result of this section is the finding that, similar to the quantity (2.12), also the ratio of the moments (2.13) between an excited state and the ground state:

$$M_n^\Psi(r;\alpha) = \lim_{L\to\infty}\frac{{}^n_L\langle\Psi|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(rL)|\Psi\rangle^n_L}{{}^n_L\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(r\,L)|0\rangle^n_L}, \tag{2.18}$$

takes a simple, universal form which is a function of the ratio r, the charge α and the number and statistics of excitations in the state $|\Psi\rangle^n$.

Let us now discuss how these ratios may be computed in practice, employing a form factor approach. A key technical problem that was solved in [115] is the question of how to evaluate finite-volume matrix elements of the branch-point twist field. The same question arises for the composite field. Although a finite-volume form factor program for generic local fields exists [159, 160] this cannot be directly employed for twist fields (its extension to this case is still an open problem). In the absence of such a program, an alternative approach can be used for complex free theories, where the internal $U(1)$ symmetry on each replica can be exploited to diagonalise the action of the CBPTF [20, 161, 162].

In fact, this diagonalisation procedure can also be employed in infinite volume to compute the form factors of $\mathcal{T}^\alpha$, as done in [105]. The idea is the following. Let us denote by $\Phi = (\Phi_1, \dots, \Phi_n)^T$ the multiplet of free complex fields the replica manifold, i.e. Φ_j is the local field in the jth replica (for now, we use the same symbol Φ to denote a multiplet of free complex bosons or free Dirac fermions). Since there is a $U(1)$ symmetry in each copy, and the free replica action is quadratic, the theory is enhanced with a $SU(n)$ symmetry. Thus, there is a global $SU(n)$ transformation $\Phi \mapsto \tilde{\Phi} = (\tilde{\Phi}_1, \dots, \tilde{\Phi}_n)^T$ such that the action of $\mathcal{T}^\alpha$ and $\tilde{\mathcal{T}}^\alpha$ is diagonal in the new basis of local fields. Assuming a homogeneous fractionalisation of the phase $e^{2\pi i\alpha}$ among all replicas, the fields that diagonalise the action of $\mathcal{T}^\alpha$ and $\tilde{\mathcal{T}}^\alpha$ are:

$$\tilde{\Phi}_p = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{-\frac{2\pi i j p}{n}} \Phi_j, \quad \begin{cases} p = 1, \dots, n & \text{for complex bosons} \\ p = -\frac{n-1}{2}, \dots, \frac{n-1}{2} & \text{for Dirac fermions} \end{cases}, \tag{2.19}$$

and the eigenvalues of the transformation are given by

$$\lambda_p = e^{\frac{2\pi i(p+\alpha)}{n}}. \tag{2.20}$$

In free theories, correlation functions of the composite branch-point twist fields factorise in free theories, and therefore we can write $\mathcal{T}^\alpha(x,t)$, $\tilde{\mathcal{T}}^\alpha(x,t)$ as products of fields acting non-trivially only on one copy. Namely, the factorisation is:

$$\mathcal{T}^\alpha(x,t) = \prod_{p=1}^{n} \mathcal{T}_{p+\alpha}(x,t), \qquad \tilde{\mathcal{T}}^\alpha(x,t) = \prod_{p=1}^{n} \mathcal{T}_{-p-\alpha}(x,t), \tag{2.21}$$

for complex free bosons and

$$\mathcal{T}^\alpha(x,t) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \mathcal{T}_{p+\alpha}(x,t), \qquad \tilde{\mathcal{T}}^\alpha(x,t) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \mathcal{T}_{-p-\alpha}(x,t), \tag{2.22}$$

for complex free fermion. The factors $\mathcal{T}_{p+\alpha}$ are all $U(1)$ fields with charge $p+\alpha$, resulting from the fusion of two $U(1)$ fields of charges p and α: namely, the fields $\mathcal{T}_p$ employed in [115], in terms of which one can decompose the (standard) branch-point twist field (1.26) in a free theory, and the $U(1)$ vertex operators $\mathcal{V}_\alpha$. This factorisation was already employed in [105, 124], albeit with a different normalisation of the parameter α. There, it was shown that the conformal dimension of these fields is $\Delta_{\frac{\alpha+p}{n}}$ for the free fermion and $\Delta_{\frac{|\alpha|+p}{n}}$ for the free boson, with Δ_α given by (2.17), as indeed by summing these quantities over the allowed values of p one obtains (2.15) in the two cases. The fields $\mathcal{T}_{\pm(p+\alpha)}$ satisfy the usual equal-time exchange relations for $U(1)$ fields, which involve what is termed a factor of local commutativity $\gamma^{\pm}_{p+\alpha} = \exp(\pm 2\pi i(p+\alpha)/n)$, that is, the phase that a field $\tilde{\Phi}_p(x)$ of

charge $+1$ accrues when taking a trip around the $U(1)$ field:

$$\mathcal{T}_{\pm(p+\alpha)}(x,t)\tilde{\Phi}_q(y,t) = \begin{cases} (\gamma^{\pm}_{p+\alpha})^{\delta_{p,q}}\tilde{\Phi}_q(y,t)\mathcal{T}_{\pm(p+\alpha)}(x,t) & y > x \\ \tilde{\Phi}_q(y,t)\mathcal{T}_{\pm(p+\alpha)}(x,t) & y < x \end{cases}. \tag{2.23}$$

As we will review shortly, the factor of local commutativity is the key ingredient in determining the form factors of these fields.

The computation presented in [115] for the total entanglement entropy may be easily extended to the case of the ratio $M_n^{\Psi}(r;\alpha)$ in excited states. First, a word is due regarding the excited state $|\Psi\rangle_L^n$. In general, any state in the replica QFT can be characterised in terms of the rapidities and quantum numbers of the excitations above the ground state. Considering a free complex theory, we may define creation operators $(a_j^{\epsilon})^{\dagger}(\theta)$ where $\epsilon = \pm 1$ is the $U(1)$ charge of the particle, θ is its rapidity and $j = 1, \dots, n$ is the copy number. Unlike the works [114–116, 144], where complex theories were considered only in order to access results for real ones, here we are interested in obtaining results for complex models. The type of k-particle excited state that we are interested in consists of n identical copies of a standard k-particle state:

$$|\Psi\rangle_L^n = \prod_{j=1}^{n}\left(\prod_{i=1}^{k}(a_j^{\epsilon_i})^{\dagger}(\theta_i)\right)|0\rangle_L^n. \tag{2.24}$$

We will start by considering the complex boson and then move to the complex fermion.

2.2.2 Complex free boson

In order to represent the state, it is convenient to move to the basis (2.19) in which the action of the CBPTF is factorised and diagonal. In this basis, the state can be expressed in terms of creation operators $\mathfrak{a}_j^{\dagger}(\theta)$ and $\mathfrak{b}_j^{\dagger}(\theta)$ associated with bosons of charge $+1$ and -1 respectively. They are related to the creation operators in the standard basis as [115]

$$\mathfrak{a}_p^{\dagger}(\theta) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} e^{\frac{2\pi i j p}{n}}(a_j^{+})^{\dagger}(\theta) \qquad \text{and} \qquad \mathfrak{b}_p^{\dagger}(\theta) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} e^{-\frac{2\pi i j p}{n}}(a_j^{-})^{\dagger}(\theta), \tag{2.25}$$

where $p = 1, \dots n$. That is, the sets of creation operators in the two basis are Fourier modes of each other. The annihilation operators $\mathfrak{a}_p(\theta)$, $\mathfrak{b}_p(\theta)$ are defined analogously and the only non-vanishing commutation relations are:

$$[\mathfrak{a}_p(\theta_1), \mathfrak{a}_p^{\dagger}(\theta_2)] = [\mathfrak{b}_p(\theta_1), \mathfrak{b}_p^{\dagger}(\theta_2)] = \delta(\theta_1 - \theta_2). \tag{2.26}$$

As an example, let us consider the case of one single excitation of charge ϵ, and rapidity θ, which

we write[5] as $|1^\epsilon\rangle_L^n$. In the original basis, this would be the state $\prod_{j=1}^n (a_j^\epsilon)^\dagger(\theta)|0\rangle_L^n$, that is a state where a single complex boson of rapidity θ and charge ϵ is present in each replica. In the diagonal basis, by inverting the relations (2.25), such a state takes the form

$$|1^+\rangle_L^n = \sum_{\{N^+\}} A_n(\{N^+\}) \prod_{p=1}^n [\mathfrak{a}_p^\dagger(\theta)]^{N_p^+} |0\rangle_{p,L}^n, \quad |1^-\rangle_L^n = \sum_{\{N^-\}} A_n(\{N^-\}) \prod_{p=1}^n [\mathfrak{b}_p^\dagger(\theta)]^{N_p^-} |0\rangle_{p,L}^n, \tag{2.27}$$

where the indices $\{N^\pm\} := \{N_1^\pm, \dots, N_n^\pm\}$ are boson occupation numbers in each sector and they are constrained by the condition that they must add up to n

$$\sum_{p=1}^n N_p^\pm = n. \tag{2.28}$$

The coefficients $A(\{N^\pm\})$ can be obtained systematically from equations (2.25) and their inverses. Combining the factorisation of the CBPFT and equation (2.27), we can expand the two-point function in the excited state $|1^\pm\rangle_L^n$ as follows:

$$\begin{aligned} {}_L^n\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle_L^n &= \sum_{\{N^+\}} \sum_{\{M^+\}} A_n^*(\{N^+\})\, A_n(\{M^+\}) \\ &\times \prod_{p=1}^n {}_{p,L}^n\langle 0|[\mathfrak{a}_p\,(\theta)]^{N_p^+}\, \mathcal{T}_{p+\alpha}(0)\, \mathcal{T}_{-p-\alpha}(\ell)\, [\mathfrak{a}_p^\dagger(\theta)]^{M_p^+}\, |0\rangle_{p,L}^n\,, \end{aligned} \tag{2.29}$$

$$\begin{aligned} {}_L^n\langle 1^-|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^-\rangle_L^n &= \sum_{\{N^-\}} \sum_{\{M^-\}} A_n^*(\{N^-\})\, A_n(\{M^-\}) \\ &\times \prod_{p=1}^n {}_{p,L}^n\langle 0|[\mathfrak{b}_p\,(\theta)]^{N_p^-}\, \mathcal{T}_{p+\alpha}(0)\, \mathcal{T}_{-p-\alpha}(\ell)\, [\mathfrak{b}_p^\dagger(\theta)]^{M_p^-}\, |0\rangle_{p,L}^n\,. \end{aligned} \tag{2.30}$$

This can be computed in the standard way by inserting a sum over a complete set of states between the two fields as detailed in Appendix 2.A. A particular subtlety of this kind of computation is that, because of finite volume, the momenta of the excitations are quantised and non-zero matrix elements correspond to particular quantisation conditions that take the monodromy of the fields into account. In particular we have:

$$P(\theta_i^\pm) = m \sinh\theta_i^\pm = 2\pi J_i^\pm \pm \frac{2\pi(p+\alpha)}{n}, \qquad J_i^\pm \in \mathbb{Z}, \tag{2.31}$$

where θ_i^+ denotes rapidities of particles created by $\mathfrak{a}_j^\dagger$ and θ_i^- denotes rapidities of particles created by $\mathfrak{b}_j^\dagger$. The reason is that these states, being inserted between the two composite twist fields, belong to a twisted sector of the Hilbert space (see [115] for more details), and thus the monodromy relation

[5] The fact that we do not label the state using the rapidity θ is justified *a posteriori*, as our results do not depend on the energy and momentum of the zero-density excitations.

is $\tilde{\Phi}_p(x+L) = e^{\pm\frac{2\pi i(p+\alpha)}{n}}\tilde{\Phi}_p(x)$, from which (2.31) follows. On the other hand, the rapidity θ of the external "untwisted"states is quantised through $P(\theta) = 2\pi I$ for $I \in \mathbb{Z}$, i.e. the usual Bethe-Yang quantisation condition for a free theory [163–165]. Note that the quantity $\frac{p+\alpha}{n}$ is never an integer for $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$ and $p \neq n$ ($p = n$ corresponds to the identity field). This guarantees that only non-diagonal form factors (that is matrix elements involving only distinct right and left states) will be involved in the computation of the leading large-volume contribution to (2.29).

Once a sum over a complete set of states is inserted in (2.29), the problem reduces to the computation of matrix elements of the $U(1)$ fields $\mathcal{T}_{p+\alpha}$. Such matrix elements have been known for a long time but they were re-derived in [105, 115]. Because of the free nature of the theory, all matrix elements are given in terms of the two-particle form factor

$$F_n^{p+\alpha|+-}(\theta_{12}) := {}_p\langle 0|\mathcal{T}_{p+\alpha}(0)\,\mathfrak{a}_p^\dagger(\theta_1)\mathfrak{b}_p^\dagger(\theta_2)|0\rangle_p = -\tau_{p+\alpha}\sin\frac{\pi(p+\alpha)}{n}\frac{e^{\left(\frac{p+\alpha}{n}-\frac{1}{2}\right)\theta_{12}}}{\cosh\frac{\theta_{12}}{2}}, \tag{2.32}$$

where $\tau_{p+\alpha}$ is the vacuum expectation value of $\mathcal{T}_{p+\alpha}$:

$$\tau_{p+\alpha} = {}_p\langle 0|\mathcal{T}_{p+\alpha}|0\rangle_p, \tag{2.33}$$

and $\theta_{12} = \theta_1 - \theta_2$. Since the composite twist field preserves the total charge of the state, it follows that

$$F_n^{p+\alpha|++}(\theta_{12}) := {}_p\langle 0|\mathcal{T}_{p+\alpha}(0)\,\mathfrak{a}_p^\dagger(\theta_1)\mathfrak{a}_p^\dagger(\theta_2)|0\rangle_p = 0, \tag{2.34}$$

$$F_n^{p+\alpha|--}(\theta_{12}) := {}_p\langle 0|\mathcal{T}_{p+\alpha}(0)\,\mathfrak{b}_p^\dagger(\theta_1)\mathfrak{b}_p^\dagger(\theta_2)|0\rangle_p = 0. \tag{2.35}$$

The two-point function (2.32) is the only solution of the Watson equations:

$$F_n^{p+\alpha|\pm\mp}(\theta) = F_n^{p+\alpha|\mp\pm}(-\theta), \quad F_n^{p+\alpha|\pm\mp}(\theta+2\pi i) = \gamma_{p+\alpha}^{\pm}F_n^{p+\alpha|\pm\mp}(\theta), \tag{2.36}$$

and of the kinematic residue equation:

$$\mathrm{Res}_{\theta=0}F_n^{p+\alpha|\pm\mp}(\theta+i\pi) = i(1-\gamma_{p+\alpha}^{\pm})\tau_{p+\alpha}. \tag{2.37}$$

Higher-particle form factors simply follow from Wick's Theorem, which for the free boson reads:

$$\begin{aligned} F_2^{p+\alpha,n}m(\theta_1,\dots,\theta_m,\beta_1,\dots,\beta_m) &:= {}_p\langle 0|\mathcal{T}_{p+\alpha}(0)\,\mathfrak{a}_p^\dagger(\theta_1)\dots\mathfrak{a}_p^\dagger(\theta_m)\mathfrak{b}_p^\dagger(\beta_1)\dots\mathfrak{b}_p^\dagger(\theta_m)|0\rangle_p \\ &= \tau_{p+\alpha}\sum_{\sigma\in S_m} f_{p+\alpha}^n(\theta_{\sigma(1)}-\beta_1)\dots f_{p+\alpha}^n(\theta_{\sigma(m)}-\beta_m), \end{aligned} \tag{2.38}$$

where S_m is the set of permutations of the first m positive integers. For free fermions, one has to

take into account also the parity of the permutations σ. In the formula above, we introduced the normalised two-particle form factor:

$$f^n_{p+\alpha}(\theta) := \frac{F^{p+\alpha|+-}_n(\theta)}{\tau_{p+\alpha}}. \tag{2.39}$$

The finite-volume matrix elements can then be obtained thanks to the results of [159, 160] (see also Appendix 2.A).

In summary, all results obtained in [115] follow through for the CBPTF with the replacement $p \to p+\alpha$ and the choice of an appropriate state. In particular, the ratio of charged moments for a state describing one excitation is given by

$$M^{1^\pm}_n(r;\alpha) = \sum_{\{N^\pm\}} |A_n(\{N^\pm\})|^2 \prod_{p=1}^n (N^\pm_p!)[g^n_{\pm(p+\alpha)}(r)]^{N^\pm_p} = e^{\pm 2\pi i\alpha} r^n + (1-r)^n, \tag{2.40}$$

which is, as anticipated, the formula (2.6) and where

$$g^n_p(r) := 1 - (1 - e^{\frac{2\pi i p}{n}})r. \tag{2.41}$$

For free bosons, this can be generalised to states containing k identical excitations to find (2.7). For states containing k different excitations (with different rapidities and any combination of charges ϵ_i) the result is

$$\begin{aligned} M^{1^{\epsilon_1}\dots 1^{\epsilon_k}}_n(r;\alpha) &= \prod_{s=1}^k \sum_{\{N^\pm\}} |C_n(\{N^\pm\})|^2 \prod_{p=1}^n N^+_{p,s}!\, N^-_{p,s}! \left(g^n_{p+\alpha}(r)\right)^{N^+_{p,s}} \left(g^n_{-p-\alpha}(r)\right)^{N^-_{p,s}} \\ &= \prod_{j=1}^k \left[e^{2\pi i\epsilon_j\alpha} r^n + (1-r)^n\right]. \end{aligned} \tag{2.42}$$

In these formulae, $C_n(\{N^\pm\}$ and $A_n(\{N^\pm\}$ are coefficients which are determined by the form of the state in the diagonal basis. Both results are special cases of (2.8).

2.2.3 Complex free fermion

For complex free fermions the computation is very similar, although states involving identical excitations are forbidden. The Fourier modes of the creation operators $(a^\pm_j)^\dagger(\theta)$ are:

$$\mathfrak{a}^\dagger_p(\theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^n e^{\frac{2\pi i j p}{n}} (a^+_j)^\dagger(\theta) \qquad \text{and} \qquad \mathfrak{b}^\dagger_p(\theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^n e^{-\frac{2\pi i j p}{n}} (a^-_j)^\dagger(\theta), \tag{2.43}$$

where now $p = -\frac{n-1}{2}, \dots \frac{n-1}{2}$, and the only non-vanishing anticommutators are:

$$\{\mathfrak{a}_p(\theta_1), \mathfrak{a}_p^\dagger(\theta_2)\} = \{\mathfrak{b}_p(\theta_1), \mathfrak{b}_p^\dagger(\theta_2)\} = \delta(\theta_1 - \theta_2). \tag{2.44}$$

For a free fermion, the two-particle form factor of the factorised CBPTF is modified to [82, 89]

$$F_n^{p+\alpha|+-}(\theta_{12}) := {}_p\langle 0|\mathcal{T}_{p+\alpha}(0)\,\mathfrak{a}_p^\dagger(\theta_1)\mathfrak{b}_p^\dagger(\theta_2)|0\rangle_p = i\tau_{p+\alpha}\sin\frac{\pi(p+\alpha)}{n}\frac{e^{\left(\frac{p+\alpha}{n}\right)\theta_{12}}}{\cosh\frac{\theta_{12}}{2}}. \tag{2.45}$$

The structure of a state consisting of a single particle excitation is as for the free boson, namely

$$|1^+\rangle_L^n = \prod_{j=1}^n (a_j^+)^\dagger(\theta)|0\rangle_L^n = \prod_{j=1}^n \frac{1}{\sqrt{n}} \sum_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \omega^{jp}\mathfrak{a}_p^\dagger(\theta)|0\rangle_L^n, \tag{2.46}$$

$$|1^-\rangle_L^n = \prod_{j=1}^n (a_j^-)^\dagger(\theta)|0\rangle_L^n = \prod_{j=1}^n \frac{1}{\sqrt{n}} \sum_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \omega^{-jp}\mathfrak{b}_p^\dagger(\theta)|0\rangle_L^n, \tag{2.47}$$

with $\omega = e^{-\frac{2\pi i}{n}}$. For instance, for $n = 2$ we have:

$$|1^+\rangle_L^2 = \frac{1}{2}(i\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta) - i\mathfrak{a}^\dagger_{\frac{1}{2}}(\theta))(-\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta) - \mathfrak{a}^\dagger_{\frac{1}{2}}(\theta))|0\rangle_L^2 = -i\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta)\mathfrak{a}^\dagger_{\frac{1}{2}}(\theta)|0\rangle_L^2, \tag{2.48}$$

and

$$|1^-\rangle_L^2 = \frac{1}{2}(-i\mathfrak{b}^\dagger_{-\frac{1}{2}}(\theta) + i\mathfrak{b}^\dagger_{\frac{1}{2}}(\theta))(-\mathfrak{b}^\dagger_{-\frac{1}{2}}(\theta) - \mathfrak{b}^\dagger_{\frac{1}{2}}(\theta))|0\rangle_L^2 = i\mathfrak{b}^\dagger_{-\frac{1}{2}}(\theta)\mathfrak{b}^\dagger_{\frac{1}{2}}(\theta)|0\rangle_L^2. \tag{2.49}$$

Similarly, for $n = 3$:

$$|1^+\rangle_L^3 = i\mathfrak{a}^\dagger_{-1}(\theta)\mathfrak{a}^\dagger_0(\theta)\mathfrak{a}^\dagger_1(\theta)|0\rangle_L^3 \quad , \quad |1^-\rangle_L^3 = -i\mathfrak{b}^\dagger_{-1}(\theta)\mathfrak{b}^\dagger_0(\theta)\mathfrak{b}^\dagger_1(\theta)|0\rangle_L^3. \tag{2.50}$$

Due to the anticommutation relations, many contributions now cancel off and the states take extremely simple forms in the diagonal basis. It is easy to show that the general structure of the states (2.46) and (2.47) is:

$$|1^+\rangle_L^n = e^{i\kappa}\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\mathfrak{a}_p^\dagger(\theta)|0\rangle_L^n, \quad |1^-\rangle_L^n = e^{-i\kappa}\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\mathfrak{b}_p^\dagger(\theta)|0\rangle_L^n, \tag{2.51}$$

with κ a real parameter that can be computed for each specific state and will not play a role in our computation. Making use of the factorisation (2.22) we can expand the fermionic two-point function in terms of a sum over the form factors (2.45). The details are presented in Appendix 2.B. For a state

consisting of a single excitation the result is

$$M_n^{1^\pm}(r;\alpha) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} g^n_{\pm p\pm\alpha}(r) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left[1-(1-e^{\pm\frac{2\pi i(p+\alpha)}{n}})r\right].$$

Since the quantities $e^{\pm\frac{2\pi ip}{n}}$ are the nth roots of $+1$ for n odd, and the nth roots of -1 for n even, it follows that

$$\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} (x - e^{\pm\frac{2\pi ip}{n}}y) = x^n + (-y)^n, \tag{2.52}$$

which, after setting $x = 1-r$, $y = -re^{\pm\frac{2\pi i\alpha}{n}}$ gives:

$$\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} g^n_{\pm p\pm\alpha}(r) = e^{\pm 2\pi i\alpha}r^n + (1-r)^n, \tag{2.53}$$

that is, the same result as (2.40) for free bosons, albeit resulting from a rather different product of g-functions. Similarly, all free boson formulae presented in the previous subsection are recovered for free fermions, as long as we consider only distinct excitations.

2.2.4 Symmetry-resolved entanglement entropies

Having obtained the ratios of charged moments we now proceed to computing the SREE of excited states. To this aim, we need to isolate the charged moments of the excited state and then compute their Fourier transform to obtain the charged partition functions (1.48). In other words, we need to multiply our results of the previous section by the ground state correlator in the infinite-volume limit considered here. Note that this ground state correlator will be different for different theories, even if formulae (2.6)-(2.8) are always satisfied.

For (local) 1+1D QFTs, such as complex free theories, the ground state correlator in our scaling limit reduces to its disconnected part, that is the square of the vacuum expectation value (VEV) of the field $\mathcal{T}^\alpha$. This result follows simply from clustering of correlators in local QFT, but can also be demonstrated explicitly from the finite-volume expansion of the ground state two-point function. This expansion is presented in Appendix 2.C for complex free fermions[6]. In particular, looking at equation (2.218) we can see how, despite the complexity of the expansion, in infinite volume the only surviving term in the sum corresponds to the product of VEVs $|\tau_{p+\alpha}|^2$. The same statement holds for complex free bosons. As mentioned earlier, it is common to normalise the correlators by the inclusion of a UV

[6] A large-distance expansion of the two-point function of $\mathcal{T}^\alpha$ in free complex theories was performed in [105], although in that case the authors considered QFTs in infinite volume from the very beginning.

cut-off, so that the natural quantity to compute is

$$Z_n^{\Psi}(r;\alpha) = Z_n^0(\alpha) M_n^{\Psi}(r;\alpha) \qquad \text{with} \quad Z_n^0(\alpha) = \varepsilon^{4\Delta_\alpha} \langle \mathcal{T}^\alpha \rangle^2, \tag{2.54}$$

where $Z_n^{\Psi}(r;\alpha)$ are the charged moments of the excited state in our particular scaling limit, $Z_n^0(\alpha)$ are the moments of the ground state and $\langle \mathcal{T}^\alpha \rangle$ is the VEV of the composite twist field. From general dimensionality arguments, as can be found for instance in [166], the VEV has a very particular dependence on the mass scale and the conformal dimension of the CBPTF. In fact, we have

$$\langle \mathcal{T}^\alpha \rangle = v_n^\alpha m^{2\Delta_{\mathcal{T}^\alpha}}, \tag{2.55}$$

where v_n^α is a function that depends on the model and can be determined by requiring CFT normalisation of the composite twist field (that is, that the CFT two-point function has numerical coefficient of 1) and $\Delta_{\mathcal{T}^\alpha}$ is given by (2.15). The Fourier transform of the ground state moments has been studied in detail for free QFTs in [105, 124], thus we will not revisit its computation here. Instead, we show that, assuming $Z_n^0(\alpha)$ to be known , it is possible to express the symmetry-resolved partition functions and entropies of excited states fully in terms of those of the ground state. The reason for this is that the functions $M_n^{\Psi}(r;\alpha)$ depend on α in an extremely simple manner, namely through factors of the form $e^{\pm 2\pi i j \alpha}$ only. Thus, in order to compute the SREE of an excited state, the only non-trivial integrals that we need to consider are of the form

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha \, Z_n^0(\alpha) e^{-2\pi i \alpha (q \pm j)} = \mathcal{Z}_n^0(q \pm j). \tag{2.56}$$

For instance, using (2.6), the simple example of a single excitation of charge ϵ gives the following relationship amongst partition functions

$$\mathcal{Z}_n^{1^\epsilon}(r;q) = \mathcal{Z}_n^0(q-\epsilon) r^n + \mathcal{Z}_n^0(q)(1-r)^n. \tag{2.57}$$

Therefore, the symmetry-resolved Rényi and von Neumann entropies of such a state are given by formulae (2.9) and (2.10), respectively. They can in turn be written in terms of the SREE and partition function of the ground state (i.e. eliminating derivative terms) by recalling that

$$\partial_n \mathcal{Z}_n^0(q)\big|_{n=1} = -\mathcal{Z}_1^0(q)[S_1^0(q) - \log Z_1^0(q)]. \tag{2.58}$$

Similar relations are found for more complicated cases, such as (2.7), that is an excited state of k

identical excitations of charge ϵ. In this case we find instead

$$S_n^{k^\epsilon}(r;q) = \frac{1}{1-n}\log\frac{\sum_{j=0}^k \left[f_j^k(r)\right]^n \mathcal{Z}_n^0(q-\epsilon j)}{\left[\sum_{j=0}^k f_j^k(r)\mathcal{Z}_1^0(q-\epsilon j)\right]^n}, \tag{2.59}$$

and the symmetry-resolved von Neumann entropy

$$\begin{aligned} S_1^{k^\epsilon}(r;q) &= -\frac{\sum_{j=0}^k \left[\mathcal{Z}_1^0(q-\epsilon j) f_j^k(r)\log f_j^k(r) + f_j^k(r)\,\partial_n \mathcal{Z}_n^0(q-\epsilon j)\big|_{n=1}\right]}{\sum_{j=0}^k f_j^k(r)\mathcal{Z}_1^0(q-\epsilon j)} \\ &\quad + \log\sum_{j=0}^k f_j^k(r)\mathcal{Z}_1^0(q-\epsilon j). \end{aligned} \tag{2.60}$$

Unlike for the charged moments, the entropies of other states are not expressed by particularly simple formulae. However, the kind of integrals involved are of the same type so that the computation can be performed in a similar manner for any excited state. As a last example, let us consider the ratio of charged moments for an excited state of two particles of opposite charges. We have that

$$\begin{aligned} \mathcal{Z}_n^{1^+1^-}(r;q) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha\, Z_n^0(\alpha)(r^n + e^{2\pi i\alpha}(1-r)^n)(r^n + e^{-2\pi i\alpha}(1-r)^n)e^{-2\pi i\alpha q} \\ &= \mathcal{Z}_n^0(q)(r^{2n} + (1-r)^{2n}) + (\mathcal{Z}_n^0(q-1) + \mathcal{Z}_n^0(q+1))r^n(1-r)^n, \end{aligned} \tag{2.61}$$

so that the Rényi entropy is

$$S_n^{1^+1^-}(r;q) = \frac{1}{1-n}\log\frac{\mathcal{Z}_n^0(q)(r^{2n} + (1-r)^{2n}) + (\mathcal{Z}_n^0(q-1) + \mathcal{Z}_n^0(q+1))r^n(1-r)^n}{[\mathcal{Z}_1^0(q)(r^2 + (1-r)^2) + (\mathcal{Z}_1^0(q-1) + \mathcal{Z}_1^0(q+1))r(1-r)]^n}, \tag{2.62}$$

from which the von Neumann entropy follows as above.

In conclusion, the SREE of the kind of excited states considered here can be expressed in terms of the SREE and partition function of the ground state. This statement holds for any system where formulae (2.6)-(2.8) apply and where the ground state contribution is well-defined. As we shall see below, this includes a wide range of models, well beyond free QFTs. As a final remark, we recall that a key property of the SREEs of the ground state both in QFT [19] and interacting quantum spin chains [94] is the property of equipartition at leading order. That is, within a certain range of parameters[7] the SREEs of all charge sectors are charge independent. It is clear from the formulae above that this property also holds for the SREEs of excited states, as their charge dependence is solely encoded in the symmetry-resolved partition function and entropies of the ground state. Thus, if the entropy is equipartite in the ground state it will also be so in excited states.

[7] For massive QFT this range typically corresponds to the double limit of large subsystem size $\ell \gg 1$ and $|\log(m\epsilon)| \ll 1$ where m is a typical mass scale and ϵ a UV cut-off (see e.g. [106, 108]).

2.3 The qubit picture

Besides the QFT approach based on twist fields that we have presented so far, there are alternative ways in which the entanglement of excited states may be studied. In the works [114–116] several models and approaches were considered, including the study of the entanglement of certain qubit states. In the present context, such states are also useful as they provide a simpler way of obtaining our formulae for the ratios of charged moments, even if their associated SREEs will be different, in fact much simpler than those of QFT states.

2.3.1 Charged moments of multi-qubit states

The main idea behind the qubit picture is that of representing a localised excitation in $A \cup \bar{A}$ via a superposition of two two-qubit states:

$$|\Psi_{\rm qb}^{(1)}\rangle = \sqrt{r}|1\rangle_A \otimes |0\rangle_{\bar{A}} + \sqrt{1-r}|0\rangle_A \otimes |1\rangle_{\bar{A}}, \tag{2.63}$$

where the qubit state 1(0) represents the presence (absence) of the excitation in the corresponding spacial region. The coefficients are chosen in such a way to reproduce a uniform probability of finding the excitation in $A \cup \bar{A}$, a choice dictated by the fact that the SREEs computed via the form factor approach are independent on the energy of the excitation. A state consisting of k indistinguishable excitations (same rapidity and same charge) is described by defining a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, with:

$$\mathcal{H}_A = \text{span}\{|q\rangle_A, q = 0, \dots, k\} \simeq \mathcal{H}_{\bar{A}} \simeq \mathbb{C}^{k+1}, \tag{2.64}$$

and a subspace

$$\mathcal{H}^{(k)} = \text{span}\{|q\rangle_A \otimes |k-q\rangle_{\bar{A}}, q = 0, \dots, k\} \subset \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}. \tag{2.65}$$

Then:

$$|\Psi_{\rm qb}^{(k)}\rangle := \sum_{q=0}^{k} \sqrt{f_q^k(r)}|q\rangle_A \otimes |k-q\rangle_{\bar{A}} \in \mathcal{H}^{(k)}, \tag{2.66}$$

where the function $f_q^k(r)$ (see the definition in (2.7)) represents the probability of finding q out of k indistinguishable particles in the region A. The RDM can be computed in the orthonormal multi-qubit basis of $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, yielding:

$$\begin{aligned}
\rho_A^{(k)} &= \text{Tr}_{\bar{A}}|\Psi_{\rm qb}^{(k)}\rangle\langle\Psi_{\rm qb}^{(k)}| \\
&= \text{Tr}_{\bar{A}} \sum_{q,q'} \sqrt{f_q^k(r)\, f_{q'}^k(r)}|q\rangle_{AA}\langle q'| \otimes |k-q\rangle_{\bar{A}\bar{A}}\langle k-q'| \\
&= \sum_{q=0}^{k} f_q^k(r)|q\rangle_{AA}\langle q|,
\end{aligned} \tag{2.67}$$

that is, the reduced density matrix is diagonal. Assuming that the charge operator associated with the internal symmetry is $\hat{Q} = \hat{Q}_A \oplus \hat{Q}_{\bar{A}}$ and that, without loss of generality, all the particles in the state above have charge $\epsilon = +1$, then $e^{2\pi i\alpha \hat{Q}_A}|q\rangle_A = e^{2\pi i\alpha q}|q\rangle_A$ and the charged moments can be easily computed as follows:

$$\begin{aligned}
&\mathrm{Tr}\left[\left(\rho_A^{(k)}\right)^n e^{2\pi i\alpha\hat{Q}_A}\right] \\
&= \sum_{q_1,\dots,q_n=0,\dots,k} {}_A\langle q_1|\rho_A^{(k)} e^{\frac{2\pi i\alpha\hat{Q}_A}{n}}|q_2\rangle_A\, {}_A\langle q_2|\rho_A^{(k)} e^{\frac{2\pi i\alpha\hat{Q}_A}{n}}|q_3\rangle_A \cdots {}_A\langle q_n|\rho_A^{(k)} e^{\frac{2\pi i\alpha\hat{Q}_A}{n}}|q_1\rangle_A \\
&= \sum_{\{q_i\}}\prod_{i=1}^{n} f_{q_i}^k(r) e^{\frac{2\pi i\alpha q_i}{n}}\delta_{q_i,q_{i+1}} = \sum_{q=0}^{k}[f_q^k(r)]^n e^{2\pi i\alpha q}.
\end{aligned} \tag{2.68}$$

Thus, the result (2.7) is reproduced. The more general situation one can consider is that of a state containing N sets of indistinguishable excitations, with the set j formed by k_j particles of the same rapidity and charge ϵ_j, for $j = 1, \dots, N$. By keeping fixed the total number of particles $k = \sum_j k_j$, the Hilbert space of the multi-qubit states is now $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, with:

$$\mathcal{H}_A = \mathrm{span}\{|q_1^{\epsilon_1}, \dots, q_N^{\epsilon_N}\rangle_A, q_j = 0, \dots, k_j, \epsilon_j = \pm 1, j = 1, \dots, N\} \simeq \mathcal{H}_{\bar{A}} \simeq \bigotimes_{j=1}^{N} \mathbb{C}^{k_j+1}. \tag{2.69}$$

A multi-qubit state with k particles uniformly distributed over $A \cup \bar{A}$ is:

$$|\Psi_{\mathrm{qb}}^{(k_1,\epsilon_1;\dots;k_N,\epsilon_N)}\rangle := \sum_{q_1,\dots,q_N} \sqrt{\prod_{j=1}^{N} f_{q_j}^{k_j}(r)}\,|q_1^{\epsilon_1}, \dots, q_N^{\epsilon_N}\rangle_A \otimes |\bar{q}_1^{\epsilon_1}, \dots, \bar{q}_N^{\epsilon_N}\rangle_{\bar{A}}, \quad \bar{q}_j := k_j - q_j. \tag{2.70}$$

The state is normalised:

$$\langle\Psi_{\mathrm{qb}}^{(k_1',\epsilon_1';\dots;k_N',\epsilon_N')}|\Psi_{\mathrm{qb}}^{(k_1,\epsilon_1;\dots;k_N,\epsilon_N)}\rangle = \prod_{j=1}^{N}\delta_{k_j,k_j'}\delta_{\epsilon_j,\epsilon_j'}, \tag{2.71}$$

and factorises into a product of multi-qubit states (2.66):

$$|\Psi_{\mathrm{qb}}^{(k_1,\epsilon_1;\dots;k_N,\epsilon_N)}\rangle = \bigotimes_{j=1}^{N}|\Psi_{\mathrm{qb}}^{(k_j,\epsilon_j)}\rangle = \bigotimes_{j=1}^{N}\left(\sum_{q_j=0}^{k_j}\sqrt{f_{q_j}^{k_j}(r)}|q_j^{\epsilon_j}\rangle_A \otimes |\bar{q}_j^{\epsilon_j}\rangle_{\bar{A}}\right). \tag{2.72}$$

Because of this factorisation, the charged moments are obtained in a straightforward way from those of the state (2.66). Indeed, the RDM matrix is:

$$\rho_A^{(k_1,\epsilon_1;\dots;k_N,\epsilon_N)} = \bigotimes_{j=1}^{N}\rho_A^{(k_j,\epsilon_j)} = \bigotimes_{j=1}^{N}\sum_{q_j=0}^{k_j} f_{q_j}^{k_j}(r)|q_j^{\epsilon_j}\rangle_{A\,\bar{A}}\langle q_j^{\epsilon_j}|, \tag{2.73}$$

and, decomposing $\hat{Q}_A = \bigoplus_{j=1}^{N} \hat{Q}_A^{(j)}$, we obtain the expression:

$$\mathrm{Tr}\left[\left(\rho_A^{(k_1,\epsilon_1;\ldots;k_N,\epsilon_N)}\right)^n e^{2\pi i\alpha\hat{Q}_A}\right] = \prod_{j=1}^{N}\mathrm{Tr}\left[\left(\rho_A^{(k_j,\epsilon_j)}\right)^n e^{2\pi i\alpha\hat{Q}_A^{(j)}}\right] = \prod_{j=1}^{N}\sum_{q_j=0}^{k_j}[f_{q_j}^{k_j}(r)]^n e^{2\pi i\alpha q_j\epsilon_j}, \tag{2.74}$$

which indeed reproduces the most general result (2.8) (with $s = N$).

2.3.2 SREE of multi-qubit states

We close this section by noting that for multi-qubit states the results obtained are directly the charged moments of the state rather than ratios of moments. This is so because the qubit ground state $|0\rangle_A \otimes |0\rangle_{\bar{A}}$ is trivial from the point of view of entanglement. This means that the formulae (2.6)-(2.8) are the quantities we need to Fourier-transform in order to obtain the SREEs. The simplicity of the formulae allows us to obtain the SREEs exactly, something that is typically beyond reach for QFT. By using

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha\, e^{-2\pi i\alpha x} = \delta_{x,0}, \qquad \text{for} \quad x \in \mathbb{Z}, \tag{2.75}$$

it immediately follows from (2.68) that

$$S_n^{k^\epsilon}(r;q) = \frac{1}{1-n}\log\frac{\sum_{j=0}^{k}\left[f_j^k(r)\right]^n\delta_{q,\epsilon j}}{\left[\sum_{j=0}^{k}f_j^k(r)\delta_{q,\epsilon j}\right]^n}, \tag{2.76}$$

and in particular

$$S_n^{1^\epsilon}(r;q) = \frac{1}{1-n}\log\left[\frac{\delta_{q,\epsilon}r^n + \delta_{q,0}(1-r)^n}{(\delta_{q,\epsilon}r + \delta_{q,0}(1-r))^n}\right]. \tag{2.77}$$

The von Neumann entropies easily follow from these expressions. Due to the simplicity of the states, however, we see that all the entropies above are identically zero whenever any of the delta-functions is 1. This can be interpreted as the statement that the symmetry-resolution of the entropy does not give any additional information about these states. Another way to put this is to say that the only property that matters in establishing formulae (2.77)-(2.76) is whether particles are distinguishable or not and in both formulae particles are identical by construction, so that specifying the charge does not add any relevant information.

The situation is different if we consider states containing distinct excitations. For instance, for a state of k distinct excitations of the same charge ϵ the charged moments are given by

$$(r^n + e^{2\pi i\epsilon\alpha}(1-r)^n)^k = \sum_{j=0}^{k}\binom{k}{j}(1-r)^{nj}e^{2\pi i\epsilon\alpha j}r^{n(k-j)}, \tag{2.78}$$

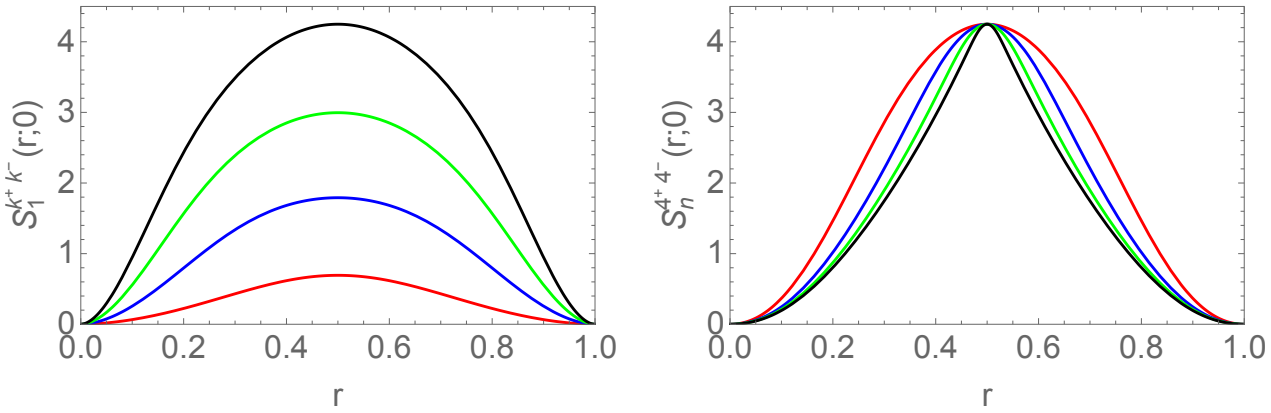


Figure 2.1 Symmetry-resolved entropies of various qubit states in the zero charge sector as functions of r. Left: The symmetry-resolved von Neumann entropy of the charge zero sector for states of equal numbers k of identical positively and negatively charged particles. In the figure $k = 1, 2, 3, 4$ giving larger entropy for higher k. The maxima at $r = 1/2$ are $\log 2, \log 6, \log 20$ and $\log 70$, that is $\log(2k)! - 2\log(k!)$ which counts the number of distinct arrangements of two groups of k identical particles. Right: The symmetry-resolved Rényi entropy of the charge zero sector of a state consisting of four identical positively and four identical negatively charged excitations for $n = 2, 4, 8, 20$. The larger n is, the more sharply peaked at $r = 1/2$ the functions become. The value at $r = 1/2$ is $\log 70$, independent of n.

so performing the Fourier transform we get

$$S_n^{1^\epsilon 1^\epsilon \dots 1^\epsilon}(r;q) = \frac{1}{1-n}\log\frac{\sum_{j=0}^k \binom{k}{j}(1-r)^{nj}\delta_{q,\epsilon j} r^{n(k-j)}}{\left[\sum_{j=0}^k \binom{k}{j}(1-r)^{j}\delta_{q,\epsilon j} r^{(k-j)}\right]^n}, \tag{2.79}$$

thus for $q = \epsilon j$, $j = 0, \dots, k$, we have

$$S_n^{1^\epsilon 1^\epsilon \dots 1^\epsilon}(r;\epsilon j) = \frac{1}{1-n}\log\frac{\binom{k}{j}(1-r)^{nj} r^{n(k-j)}}{\left[\binom{k}{j}(1-r)^{j} r^{(k-j)}\right]^n} = \log\binom{k}{j}. \tag{2.80}$$

In this case the symmetry-resolved Rényi entropy tells us about the number of equally likely configurations which produce a charge ϵj in region A, and it is independent of n. Many other configurations can be considered, all of which produce different results, with similar interpretations. For instance, for a state with one positively and one negatively charged particle, the Fourier transform of the function

$$(r^n + e^{2\pi i\alpha}(1-r)^n)(r^n + e^{-2\pi i\alpha}(1-r)^n), \tag{2.81}$$

yields the simple formula

$$S_n^{1^+1^-}(r;q) = \frac{1}{1-n}\log\frac{(r^{2n} + (1-r)^{2n})\delta_{q,0} + r^n(1-r)^n(\delta_{q,1} + \delta_{q,-1})}{[(r^2 + (1-r)^2)\delta_{q,0} + r(1-r)(\delta_{q,1} + \delta_{q,-1})]^n}, \tag{2.82}$$

and

$$S_n^{1^+1^-}(r;0) = \frac{1}{1-n}\log\frac{r^{2n}+(1-r)^{2n}}{(r^2+(1-r)^2)^n}, \quad S_n^{1^+1^-}(r;\pm 1) = 0. \tag{2.83}$$

In this case the $q=0$ result is n-dependent and gives a non-trivial symmetry-resolved von Neumann entropy:

$$S_1^{1^+1^-}(r;0) = \log(r^2+(1-r)^2) - \frac{r^2\log r^2 + (1-r)^2\log(1-r)^2}{r^2+(1-r)^2}, \quad S_1^{1^+1^-}(r;\pm 1) = 0. \tag{2.84}$$

In this example the SREE of the $q=0$ sector is non-trivial as there are now two possible configurations that we can associate with such a charge, namely both particles being in region A and no particle being in region A. Thus there is a difference in the SREEs of states involving two particles with the same or distinct charges, even for the simple states considered here. Additional examples are presented in Fig. 2.1.

It is worth noting that all formulae in this section are in agreement with those in Section 2.2.4 if one identifies the ground state partition function $\mathcal{Z}_n^0(q-\epsilon j)$ with the quantity $\delta_{q,\epsilon j}$. Therefore, the study of multi-qubit states provides a neat application of the general results of the previous section to the case of a trivial, unentangled, ground state.

Because of the simplicity and explicit nature of the formulae for the multi-qubit states, it is possible to compute precisely the two contributions to the total von Neumann entropy, that is the configurational entropy and the number entropy [19, 93, 94]. Calling $S_1^\Psi(r)$ the total von Neumann entropy of the state $|\Psi\rangle$, we can write

$$S_1^\Psi(r) = \sum_q (p(q)S_1^\Psi(q;r) - p(q)\log p(q)), \tag{2.85}$$

where $p(q) := \mathcal{Z}_1^\Psi(r;q)$, that is the symmetry-resolved partition function of the state for $n=1$, and the term $\sum_q p(q)\log p(q)$ is the number entropy. The quantity $p(q)$ represents the probability of obtaining the value q when measuring the charge. It is easy to work out an explicit example and see the features of these two contributions. For the state consisting of two distinct excitations with different charges the SREEs are given by (2.84). Thus, from

$$p(0) = r^2+(1-r)^2, \qquad p(\pm 1) = r(1-r), \tag{2.86}$$

it follows that the number entropy is simply

$$(r^2+(1-r)^2)\log(r^2+(1-r)^2) + 2r(1-r)\log(r(1-r)), \tag{2.87}$$

while the configuration entropy is

$$(r^2 + (1-r)^2) S_1^{1^+1^-}(r;0), \tag{2.88}$$

By summing these contributions we recover the known formula for the total von Neumann entropy of a state of two distinct excitations $-2r\log r - 2(1-r)\log(1-r)$ as found in [114, 115]. We note also that the number entropy takes its maximum (absolute) value $3/2\log 2$ at $r = 1/2$, and that it can itself be considered a measure of entanglement, as discussed for other examples in [93, 109].

We close this section by noting that the entropy formulae for qubit states considered here do not have the property of equipartition, that is, they depend explicitly on the charge sector as we see for instance from Eq. (2.83). This is no contradiction, since the property of equipartition [94] is typically a leading order property (for instance in [94] it holds for small magnetisation), whereas in the case of multi-qubit states we have exact formulae rather than leading order expressions. Indeed, these states provide probably the simplest example where such a dependence on the charge sector can be fully computed.

2.4 Generalisations to interacting theories and higher dimensions

2.4.1 Magnon states

The agreement of the results obtained via form factors of CBPTFs in Section 2.2 and via multi-qubit states in Section 2.3 provide evidence that the formulae we obtained are correct. This is further substantiated by the numerical results we will present in the next Section. We now consider how our results might be applicable in a broader context. A natural starting point are magnon states. Such states describe the eigenstates of a variety of spin chain Hamiltonians, with or without interactions. They admit a simple explicit form in the spin basis so that entanglement computations are easy to perform. We also know from [115, 150] that the total entanglement entropy of magnon states is described by our formulae with $\alpha = 0$. As we see below, even in the presence of non-trivial scattering, the agreement extends to $\alpha \neq 0$.

The main idea behind this construction is somewhat similar in spirit to the qubit picture [1, 115], namely, that the entanglement content of quasiparticles can be easily understood if one factors out the zero-point fluctuations. In other words, instead of considering the full quantum theory where the quasiparticles are constructed on top of a nontrivial ground state, which in general has its own entanglement content, we consider a simpler theory in which particles are constructed above a trivial ground state. It turns out that the entanglement of this simpler model keeps track of the exact entanglement of the quasiparticle and explicitly discards explicitly the entanglement of the true ground

state. Our magnon states belong to a Fock space generated by multi-particle configurations, endowed with an internal symmetry that acts as a phase on each multi-particle state. This phase is directly related to the quantum numbers of each particle.

2.4.1.1 One-magnon states

We firstly focus on a single magnon state on the lattice, belonging to the one-particle sector of a quantum spin-$\frac{1}{2}$ chain of length L. A one-magnon state can be written as a superposition of localised excitations with momentum p:

$$|\Psi_1\rangle = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ipj} |j\rangle, \quad |j\rangle = |\uparrow\rangle_1 \otimes \ldots |\downarrow\rangle_j \otimes \ldots |\uparrow\rangle_L. \tag{2.89}$$

If one imposes boundary conditions on the chain, the momentum p is quantised as follows

$$p \in \frac{2\pi}{L}\mathbb{Z}. \tag{2.90}$$

We introduce the action of the symmetry operator $e^{2\pi i\alpha\hat{Q}}$, where $\hat{Q}$ is associated with an internal symmetry. For our purposes we just need to specify its action on the vacuum state $|0\rangle = \otimes_{j=1}^{L}|\uparrow\rangle_j$ and on the one-particle sector. In addition we assume that the magnon is charged with respect to $\hat{Q}$, and it has charge $+1$.

We are interested in the entanglement of the one-magnon state with respect to the partition of spin sites into $A = \{1, 2, \cdots, \ell\}$ and $\bar{A} = \{\ell + 1, \ldots, L\}$. We associate to the region A a restricted symmetry generator $e^{2\pi i\alpha\hat{Q}_A}$, which acts as

$$e^{i2\pi\alpha\hat{Q}_A}|0\rangle = |0\rangle \qquad e^{2\pi i\alpha\hat{Q}_A}|j\rangle = e^{2\pi i\alpha\delta_{j\in A}}|j\rangle \tag{2.91}$$

where $\delta_{j\in A} = 1$ if $j \in A$ and $\delta_{j\in A} = 0$ otherwise. The reduced density matrix of the region A is

$$\rho_A := \mathrm{Tr}_{\bar{A}}\left(|\Psi_1\rangle\langle\Psi_1|\right) = \frac{1}{L}\sum_{j,j'\in A} e^{ip(j-j')}|j\rangle_A\,{}_A\langle j'| + (1-r)|0\rangle_A\,{}_A\langle 0|, \tag{2.92}$$

where the states $|0\rangle_A$, $|j\rangle_A$ are defined by restricting the tensor products to the sites in A and $r = \ell/L$. The two terms appearing in the formula above are interpreted as the contributions associated to the presence/absence of the magnon in subsystem A. It is easy to show that

$$e^{2\pi i\alpha\hat{Q}_A}\rho_A^n = \left(\frac{1}{L}\sum_{j,j'\in A} e^{ip(j-j')}|j\rangle_A\,{}_A\langle j'|\right)^n e^{2\pi i\alpha} + (1-r)^n|0\rangle_A\,{}_A\langle 0|, \tag{2.93}$$

and that $[\rho_A, e^{2\pi i\alpha\hat{Q}_A}] = 0$. After a straightforward calculation one gets

$$\mathrm{Tr}_A \left(\frac{1}{L} \sum_{j,j' \in A} e^{ip(j-j')} |j\rangle_A \, {}_A\langle j'| \right)^n = r^n, \quad \mathrm{Tr}_A \left((1-r)^n |0\rangle_A \, {}_A\langle 0| \right) = (1-r)^n. \tag{2.94}$$

Putting the pieces together, we arrive at the expected final result

$$\mathrm{Tr}\left(\rho_A^n e^{2\pi i\alpha\hat{Q}_A} \right) = r^n e^{2\pi i\alpha} + (1-r)^n, \tag{2.95}$$

which provides the exact charged moments of a single magnon state.

2.4.1.2 Two-magnon states

In the following we consider a state of two magnons with the same symmetry charge. This example is more interesting because it allows us to test whether the presence of non-trivial interaction changes our results. Given a pair of momenta p and p', we parametrise the two-magnon state in the following way

$$|\Psi_2\rangle = \frac{1}{\sqrt{L}} \sum_{j,j'}^{L} S_{j,j'} e^{ipj+ip'j'} |jj'\rangle, \tag{2.96}$$

where S is a scattering matrix and $|jj'\rangle$ is the state with two localised magnons in sites j and j'. The choice of the S-matrix is not really relevant for our purpose, but for the sake of concreteness we set

$$S_{jj'} = \begin{cases} e^{i\varphi} \text{ for } j > j', \\ 1 \text{ for } j < j', \\ 0 \text{ for } j = j', \end{cases} \tag{2.97}$$

using the same conventions as in [114]. The action of the restricted symmetry operator $e^{2\pi i\alpha\hat{Q}_A}$ on the two-particle sector of the Hilbert space is

$$e^{2\pi i\alpha\hat{Q}_A} |jj'\rangle = e^{2\pi i\alpha(\delta_{j\in A}+\delta_{j'\in A})} |jj'\rangle. \tag{2.98}$$

It is possible to decompose $\rho_A = \mathrm{Tr}_{\bar{A}} \left(|\Psi_2\rangle\langle\Psi_2| \right)$ in a block-diagonal way as follows

$$\rho_A = \frac{1}{L} \left(\rho_A^{(1)} + \rho_A^{(2)} + \rho_A^{(3)} \right), \tag{2.99}$$

where $\rho_A^{(1)}$ is the two-particle contribution (both particles in A), $\rho_A^{(2)}$ is the vacuum contribution (no particles in A) and $\rho_A^{(3)}$ is the one-particle contribution (one particle in A and one in $\bar{A}$). The

introduction of the flux gives rise to the following relation

$$\rho_A^n e^{2\pi i\alpha\hat{Q}_A} = \frac{1}{L^n}\left((\rho_A^{(1)})^n e^{4\pi i\alpha} + \left(\rho_A^{(2)}\right)^n + \left(\rho_A^{(3)}\right)^n e^{2\pi i\alpha}\right). \tag{2.100}$$

No approximation was made up to this point, but the explicit expressions of $\rho_A^{(j)}$, given in [114], are cumbersome and not particularly enlightening for our purpose. However, one can show that in the limit (2.1) and with $p \neq p'$ kept fixed, $\mathrm{Tr}_A\left((\rho_A^{(j)})^n\right)$ simplifies drastically, and the leading contributions are:

$$\mathrm{Tr}_A\left((\rho_A^{(1)})^n\right) \simeq L^n r^{2n}, \quad \mathrm{Tr}_A\left((\rho_A^{(2)})^n\right) \simeq L^n(1-r)^{2n}, \quad \mathrm{Tr}_A\left((\rho_A^{(3)})^n\right) \simeq 2L^n r^n(1-r)^n. \tag{2.101}$$

Putting all the pieces together, one finally gets

$$\mathrm{Tr}_A\left(\rho_A^n e^{2\pi i\alpha\hat{Q}_A}\right) \simeq r^{2n}e^{4\pi i\alpha} + 2r^n(1-r)^n e^{2\pi i\alpha} + (1-r)^{2n} = (r^n e^{2\pi i\alpha} + (1-r)^n)^2. \tag{2.102}$$

This computation shows that in this particular scaling limit the interaction between particles has no effect on the final result, and the total charged moment is just a product of two single-particle charged moments. A different result is obtained if $p = p'$ and fixed. In that case, the magnons are indistinguishable and one can prove that

$$\mathrm{Tr}_A\left((\rho_A^{(1)})^n\right) \simeq L^n r^{2n}, \quad \mathrm{Tr}_A\left((\rho_A^{(2)})^n\right) \simeq L^n(1-r)^{2n}, \quad \mathrm{Tr}_A\left((\rho_A^{(3)})^n\right) \simeq 2^n L^n r^n(1-r)^n, \tag{2.103}$$

so that

$$\mathrm{Tr}_A\left(\rho_A^n e^{2\pi i\alpha\hat{Q}_A}\right) \simeq r^{2n}e^{4\pi i\alpha} + 2^n r^n(1-r)^n e^{2\pi i\alpha} + (1-r)^{2n}, \tag{2.104}$$

which no longer factorises into one-magnon contributions. Both results are special cases of (2.6) and (2.7).

The results of this section generalise previous work for the excess entanglement entropy of excited states [115] and are also related to the results of [150] where the entanglement of magnon states was considered more generally. In particular, it was shown that for states consisting of several magnons, entanglement will factorise into the contributions of groups of magnons which are well-separated from each other in momentum space (that is, their momentum difference is of order $\mathcal{O}(1)$ rather than $\mathcal{O}(1/L)$, as the length of the system grows). Such results also apply to the present case up to the introduction of the appropriate phases.

2.4.2 Twist operator approach

So far we have derived the behaviour of the charged moments of the SREE of quasiparticle excited states making use of two different types of formalism: the form factor expansion in 1+1D (free) IQFTs [1] and the analysis of qubit/magnon states on the lattice. Unfortunately, these techniques are suited for a limited range of situations. Indeed, on the one hand, the description of the Rényi entropy as a correlation function of branch-point twist fields inserted at different points is special of 1+1D QFT [12, 20]. On the other hand, while the description of excited states as magnons or qubit states can be generalised to higher dimensions, it has the disadvantage of not taking into account the zero-point fluctuations. In this Section, we want to consider instead a generic QFT in higher dimensions. To this aim we introduce a slightly different approach.

Despite the technical limitations outlined above we expect that, in the particular scaling limit we are considering, the universal entanglement content of the symmetry-resolved Rényi entropy should not depend on dimensionality, on the presence of interactions and even on the integrability of the theory. The computation performed on interacting two-magnon states in the previous Subsection partially supports this claim. Concerning theories in higher dimensions, at least one precedent for this generalisation already exists. In a previous work [144], the excitations of the free massive boson in $D := d + 1$ dimensions were analysed and their Rényi entropy was computed in terms of graph partition functions. The results obtained therein agree with the formulae in [115], with r replaced by the ratio of generalised volumes.

In this section we slightly generalise the formalism of [144] to take into account possible interactions and provide, as a proof of concept, a simple calculation of symmetry-resolved entanglement of a single-particle excited state. The key ingredients we need are the description of the excited states as local operators acting on a vacuum state and a semi-local twist operator, which generalises the composite branch-point twist field to higher dimensional settings. The only strong assumption we make in our derivation is the presence of a finite mass gap m, with correlation length $\xi = m^{-1}$ much smaller than the typical lengths of the system.[8]

We anticipate here that our formulae (2.6)-(2.8) are unchanged in higher dimensional theories, up to the identification

$$r = \frac{V_A}{V}, \tag{2.105}$$

which is the ratio between the (generalised) volumes of subsystem A and of the total system. It may seem surprising that the results should only depend on r and not on other features of the entanglement region, such as the connectivity and smoothness of its boundary. Indeed, the charged moments and

[8] The emergence of the universal entanglement content is also expected for some high-energy states in massless theories (see [148, 150] for the analysis of the gapless XY chain). However, here we keep the assumption of a finite mass gap to avoid technical complications.

symmetry-resolved entropies of both the ground state and excited states depend on such properties, as would finite-volume corrections to our results. However, our computations deliver results for the ratio of charged moments in the infinite-volume limit, and it is this ratio in this limit which is universal and independent of boundary features. This independence of boundary features has been analytically shown for the ratio of (uncharged) moments in the case of one-dimensional disconnected regions, where the same formulae as for one connected region were found to apply, with r the sum of the lengths of all disconnected parts [116].

2.4.2.1 Excited states and operator algebra

Let us consider the vacuum state $|0\rangle$ of a Hilbert space $\mathcal{H}$, together with an algebra $\mathcal{A}$ of observables[9] acting on $\mathcal{H}$ which has $|0\rangle$ as a cyclic vector (see [167] for a modern review of this algebraic viewpoint in QFT). This allows us to represent any state $|\Psi\rangle$ of the Hilbert space as

$$|\Psi\rangle = \mathcal{O}|0\rangle \quad \text{with} \quad \mathcal{O} \in \mathcal{A}. \tag{2.106}$$

We would like to assume that the vacuum state is translationally invariant, namely that it is invariant under a faithful representation of the translation group in d dimensions. However, since we consider a finite-size system, we modify this requirement by defining the system on a d-dimensional torus $\mathcal{M}$ of volume V and requiring that $|0\rangle$ is invariant under the isometries of the torus. Other boundary conditions can be considered too, but they do not change the picture in the scaling limit we are interested in. We also require locality of the observables, asking that at any point $\mathbf{x} \in \mathcal{M}$, $\mathcal{A}$ is generated by a set of fields $\{\mathcal{O}(\mathbf{x})\}$.

We define the Fourier transform of the field $\mathcal{O}(\mathbf{x})$ as

$$\mathcal{O}(\mathbf{p}) = \int_{\mathcal{M}} \mathrm{d}^d x \, e^{-i\mathbf{p}\mathbf{x}} \mathcal{O}(\mathbf{x}), \tag{2.107}$$

where we adopt the same symbol $\mathcal{O}$ in real and momentum space for notational convenience. The Fourier-transformed fields are building blocks for the following set of translationally invariant states

$$\mathcal{O}_1(\mathbf{p}_1) \dots \mathcal{O}_k(\mathbf{p}_k)|0\rangle, \tag{2.108}$$

which correspond to k particles distributed over $\mathcal{M}$ with momenta $\mathbf{p}_1, \dots, \mathbf{p}_k$, and the choice of the fields $\{\mathcal{O}_j\}$ depend on the particle species and quantum numbers. This construction is similar to the usual way of generating particle states in free theories acting with creation operators on the vacuum of a Fock space. However, the advantage of our formulation is that it is directly related to local

[9] In the case of a single real boson, $\mathcal{A}$ is just the algebra of operators generated by the field $\Phi(x)$ and its conjugated momentum $\Pi(x)$.

observables, a property which is fundamental to correctly define entanglement measures.

Let us take a set of orthogonal fields $\{\mathcal{O}_j\}$, so that the correlation function $\langle 0|\mathcal{O}_i^\dagger(\mathbf{x})\mathcal{O}_j(\mathbf{x}')|0\rangle$ vanishes for $i \neq j$. In other words, the fusion rule

$$[\mathcal{O}_i^\dagger] \times [\mathcal{O}_j] \to [1], \tag{2.109}$$

is present only if $i = j$ and the Operator Product Expansion (OPE) can be expressed formally as

$$\mathcal{O}_i^\dagger(\mathbf{x})\mathcal{O}_j(\mathbf{x}') \simeq \delta_{ij}\langle 0|\mathcal{O}_i^\dagger(\mathbf{x}-\mathbf{x}')\mathcal{O}_i(0)|0\rangle + \dots, \tag{2.110}$$

where in the right-hand side we neglected contributions coming from operators less relevant than the identity. The exact evaluation of the correlation function above can be hard, but the assumption of a finite gap m ensures that the latter is exponentially damped for $|\mathbf{x}-\mathbf{x}'| \gg m^{-1}$. This is the only property we really need in the following discussion.

We now consider a restriction of the modes $\mathcal{O}(\mathbf{p})$ with support in a subsystem only, that is a spacial region $A \subseteq \mathcal{M}$:

$$\mathcal{O}_A(\mathbf{p}) = \int_A \mathrm{d}^d x\, e^{-i\mathbf{p}\mathbf{x}}\mathcal{O}(\mathbf{x}). \tag{2.111}$$

Given any two regions A, $A' \subseteq \mathcal{M}$, we can compute[10] $\mathcal{O}_A^\dagger(-\mathbf{p})\mathcal{O}_{A'}(\mathbf{p}')$ by making use of some approximations. First, we consider only the most relevant term in the fields OPE

$$\begin{aligned}\mathcal{O}_A^\dagger(-\mathbf{p})\mathcal{O}_{A'}(\mathbf{p}') &= \int_A \mathrm{d}^d x \int_{A'} \mathrm{d}^d x' e^{i\mathbf{p}\mathbf{x}-i\mathbf{p}'\mathbf{x}'}\mathcal{O}^\dagger(\mathbf{x})\mathcal{O}(\mathbf{x}') \\ &\simeq \int_A \mathrm{d}^d x \int_{A'} \mathrm{d}^d x' e^{i\mathbf{p}\mathbf{x}-i\mathbf{p}'\mathbf{x}'}\langle 0|\mathcal{O}^\dagger(\mathbf{x})\mathcal{O}(\mathbf{x}')|0\rangle.\end{aligned} \tag{2.112}$$

Second, since we are working in the limit of small correlation length (compared to the geometry), the leading contribution comes from the insertion of the fields at small distances, which is present if $\mathbf{x}, \mathbf{x}' \in A \cap A'$; this observation motivates the change of variable $\mathbf{x}'' = \mathbf{x}' - \mathbf{x}$, and the subsequent approximation[11]

$$\mathcal{O}_A^\dagger(-\mathbf{p})\mathcal{O}_{A'}(\mathbf{p}') \simeq \int_{A\cap A'} \mathrm{d}x\, e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}} \int_{\mathcal{M}} \mathrm{d}x''\, e^{-i\mathbf{p}'\mathbf{x}''}\langle 0|\mathcal{O}^\dagger(0)\mathcal{O}(\mathbf{x}'')|0\rangle. \tag{2.113}$$

The second integral may be difficult to compute and in principle it could require a UV regularisation

[10] One should note that Hermitian conjugation and Fourier transform do not commute. Indeed, with our notation $\mathcal{O}_A^\dagger(-\mathbf{p}) = (\mathcal{O}_A(\mathbf{p}))^\dagger$.

[11] To be more rigorous: in approximating the integration domain $\mathbf{x} \in A$, $\mathbf{x}' \in A'$ with $\mathbf{x}$, $\mathbf{x}' \in A \cap A'$ we are neglecting terms where at least one of the two variables, say $\mathbf{x}$, lies outside of $A \cap A'$. If the distance between $\mathbf{x}$ and $A \cap A'$ is smaller than the correlation length there is no exponential suppression, however, the integration over $\mathbf{x}$ produces a term which grows with the area of the boundary $\partial(A \cap A')$ and hence is subleading in the large-volume limit. On the other hand, the integration domain of the variable $\mathbf{x}''$ depends in principle on the set $A \cap A'$, and its approximation with $\mathcal{M}$ is valid up to terms exponentially damped in the correlation length.

for $|\mathbf{x}''| < \epsilon \ll m^{-1}$. However, because the integration is over $\mathcal{M}$, the result does not depend on the regions A, A' and in our computation this integral appears only as a multiplicative constant. In conclusion, we end up with

$$\mathcal{O}_A^\dagger(-\mathbf{p})\mathcal{O}_{A'}(\mathbf{p}') \propto V_{A\cap A'}\delta_{\mathbf{p},\mathbf{p}'}, \tag{2.114}$$

where $V_{A\cap A'}$ is the volume of $A\cap A'$. Equation (2.114) is the main result of this Subsection. Since the volume in (2.114) emerges from the integrals (2.113), which involve a Fourier transform, we require that subsystem $A\cap A'$ consists of a finite number of disconnected regions, whose boundaries are piecewise smooth.

It is natural to ask how the discussion above would be modified for a vanishing gap $m = 0$. The main change is in the scaling of correlations functions: exponential localisation of the correlation function in a region of typical length m^{-1} does not hold any longer, due to the long algebraic tails of the correlation functions. We conjecture that, as long as the momenta are fixed in the infinite-volume limit, the main conclusion (2.114) is unchanged. A qualitative argument is that in this case the inverse momentum, say the De Broglie length, plays the role of typical length scale. In order to make this consideration more precise, let us analyse Eq. (2.113) for a 1+1D CFT, where $\mathcal{O}$ is a field of conformal dimension $\Delta_\mathcal{O}$. We focus on the following integral

$$\int_\mathcal{M} \mathrm{d}x\, e^{-i\mathbf{p}\mathbf{x}}\langle 0|\mathcal{O}^\dagger(0)\mathcal{O}(\mathbf{x})|0\rangle, \tag{2.115}$$

which we regulate both in the UV, with a cutoff ϵ, and in the IR, with a cutoff L, as follows

$$\int_\epsilon^L \mathrm{d}x\, e^{-ipx}\frac{1}{x^{4\Delta_\mathcal{O}}} + (\text{c.c.}). \tag{2.116}$$

This integral can be explicitly computed. However, the important feature is that for $\Delta_\mathcal{O} > 0$, $p > 0$ and $\epsilon > 0$ all fixed, the integral converges to a finite value when $L \to +\infty$. This is no longer the case if $p \sim 1/L$ in the infinite-volume limit. In practice, this means that for small momentum and scaling dimension $0 < \Delta_\mathcal{O} \leq 1$ the considerations we made so far regarding the scaling at large sizes cannot be applied. As a matter of fact, for free CFTs the scaling dimensions of the fundamental fields are smaller than 1: the fermionic field Ψ has dimension $1/2$ while the derivative of a compact boson $\partial_x\Phi$ has dimension 1. While these considerations are not mathematically rigorous in establishing convergence of the OPE in the large-volume limit, they are sufficient to explain why low-energy states of gapless theories, or multi-particle states with small momenta difference, are not well captured by our predictions. Indeed, for such states the excess entanglement was computed in [154, 155], and the results obtained therein are different from those of [114, 115].

2.4.2.2 Replica construction for symmetry-resolved entanglement

Consider now a replica version of the theory, consisting of n copies of the latter. For any state $|\Psi\rangle$ we denote its replicated version by $|\Psi\rangle^n$. Our goal is to define a $U(1)$ composite twist operator which generalises to higher dimensions the composite branch-point twist field defined in [19]. We construct this operator in such a way that its expectation value over $|\Psi\rangle^n$ gives exactly the charged moments $Z_n(\alpha)$, allowing for the computation of symmetry-resolved Rényi entropies in higher dimensional QFTs. This type of operator was already considered in the literature, mostly without the flux insertion (see for example [144, 168–170]). However, the novelty of our approach consists in establishing the relation between the twist operator and the algebra of local operators.

The first point we have to clarify regards the symmetry and its action on the space of fields. Starting from $e^{2\pi i\alpha Q}$, the global generator of $U(1)$ symmetry in the non-replicated theory, we say that $\mathcal{O}(\mathbf{x})$ has charge $\kappa_{\mathcal{O}}$ if

$$e^{2\pi i\alpha\hat{Q}}\mathcal{O}(\mathbf{x})e^{-2\pi i\alpha\hat{Q}} = e^{2\pi i\alpha\kappa_{\mathcal{O}}}\mathcal{O}(\mathbf{x}). \tag{2.117}$$

Since one can decompose the space of fields in irreducible representations of $U(1)$, we restrict our analysis to charged fields. Going back to the replicated theory, we define the algebra of replicated observables $\mathcal{A}^n$ as the algebra generated by the tensor product of n observables in $\mathcal{A}$. Thus, to any field $\mathcal{O}(\mathbf{x}) \in \mathcal{A}$, we associate $\mathcal{O}^j(\mathbf{x}) \in \mathcal{A}^n$ defined as

$$\mathcal{O}^j(\mathbf{x}) = 1 \otimes \cdots \otimes 1 \otimes \mathcal{O}(\mathbf{x}) \otimes 1 \cdots 1, \tag{2.118}$$

where $\mathcal{O}(\mathbf{x})$ is inserted in the jth replica. By requiring orthogonality of local fields in different copies, the momentum space OPE (2.114) generalises to:

$$\mathcal{O}_A^{i,\dagger}(-\mathbf{p})\mathcal{O}_{A'}^j(\mathbf{p}') \propto V_{A\cap A'}\delta_{\mathbf{p},\mathbf{p}'}\delta_{i,j}. \tag{2.119}$$

Consider now a spacial region $A \subset \mathcal{M}$ and its complement $\bar{A}$. We define a composite twist operator T_A^α which implements the structure of the n-sheeted, cyclically connected, Riemann surface where the replica theory is defined. That is, T_A^α implements the "gluing"of the replicas along A with an additional flux insertion due to the action of the $U(1)$ symmetry. The commutation relations of T_A^α with any charged field $\mathcal{O}^j(\mathbf{x})$ generalise in an obvious way those of a standard CBPTF:

$$T_A^\alpha\mathcal{O}^j(\mathbf{x}) = \begin{cases} e^{2\pi i\kappa_{\mathcal{O}}\alpha\delta_{j,n}}\mathcal{O}^{j+1}(\mathbf{x})T_A^\alpha & \mathbf{x}\in A, \\ \mathcal{O}^j(\mathbf{x})T_A^\alpha & \mathbf{x}\in\bar{A}. \end{cases} \tag{2.120}$$

Here, the flux is inserted only between the nth and the first replica, but other choices are possible. We emphasize that a similar definition has already appeared in the context of 1+1D integrable QFTs

(see [105–108]). In particular, for $A = [0, \ell]$ one can identify

$$T_A^\alpha = \mathcal{T}_n^\alpha(0)\tilde{\mathcal{T}}_n^\alpha(\ell), \tag{2.121}$$

and the commutation relations for T_A^α can be obtained from those of the CBPTF and of its Hermitian conjugate. However, such a representation in terms of the usual CBPTFs of one-dimensional theories is in general not possible in higher dimensions. Regarding the definition of the twist operator T_A^α, it is worth mentioning also that in QFT it is known[171, 172] that the twist operators are not local observables of the algebra $\mathcal{A}^n$. Rather, they are observables of the orbifolded algebra $\mathcal{A}^n/\mathbb{Z}_n$, which is obtained by taking the coset of $\mathcal{A}^n$ over the cyclic symmetry of replicas.

We are now ready to relate the twist operator to the symmetry-resolved entanglement. The charged moments of $|\Psi\rangle$ are given by

$$Z_n^\Psi(\alpha) = \frac{{}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n}{{}^n\langle\Psi|\Psi\rangle^n}. \tag{2.122}$$

The above definition, together with the commutation relations (2.120) and the momentum space OPE (2.119), is enough to obtain the explicit expression for the ratio of charged moments between the state $|\Psi\rangle$ and the ground state, as defined in (2.5). We now show how these ideas come together with a simple example.

2.4.2.3 Single-particle state

In this Section, we consider an excited state $|\Psi\rangle$ consisting of a single quasiparticle with momentum $\mathbf{p}$ generated by a charged field $\mathcal{O}$. Its explicit expression is given by

$$|\Psi\rangle = \mathcal{O}(\mathbf{p})|0\rangle, \tag{2.123}$$

and the replicated version is just

$$|\Psi\rangle^n = \mathcal{O}^1(\mathbf{p})\dots\mathcal{O}^n(\mathbf{p})|0\rangle^n. \tag{2.124}$$

For the sake of convenience, we split the observable

$$\mathcal{O}^j(\mathbf{p}) = \mathcal{O}_A^j(\mathbf{p}) + \mathcal{O}_{\bar{A}}^j(\mathbf{p}), \tag{2.125}$$

so that its commutation relations with T_A^α become more transparent. Indeed, using (2.120) one can write

$$\begin{aligned} T_A^\alpha|\Psi\rangle^n = T_A^\alpha(\mathcal{O}_A^1(\mathbf{p}) + \mathcal{O}_{\bar{A}}^1(\mathbf{p}))\dots(\mathcal{O}_A^n(\mathbf{p}) + \mathcal{O}_{\bar{A}}^n(\mathbf{p}))|0\rangle^n = \\ (\mathcal{O}_A^2(\mathbf{p}) + \mathcal{O}_{\bar{A}}^1(\mathbf{p}))\dots(\mathcal{O}_A^1(\mathbf{p})e^{2\pi i\alpha\kappa_\mathcal{O}} + \mathcal{O}_{\bar{A}}^n(\mathbf{p}))T_A^\alpha|0\rangle^n. \end{aligned} \tag{2.126}$$

Up to now, everything is exact. However, to compute the expectation value ${}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n$ we make use of the OPE contraction (2.119). Among all the terms which are generated, all but two are vanishing and they give

$$\begin{aligned} {}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n \simeq e^{2\pi i\alpha\kappa_{\mathcal{O}}}\ {}^n\langle 0|(\mathcal{O}^\dagger)_A^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)_A^1(-\mathbf{p})\mathcal{O}_A^2(\mathbf{p})\dots\mathcal{O}_A^n(\mathbf{p})\mathcal{O}_A^1(\mathbf{p})T_A^\alpha|0\rangle^n \\ +{}^n\langle 0|(\mathcal{O}^\dagger)_{\bar{A}}^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)_{\bar{A}}^1(-\mathbf{p})\mathcal{O}_{\bar{A}}^1(\mathbf{p})\mathcal{O}_{\bar{A}}^2(\mathbf{p})\dots\mathcal{O}_{\bar{A}}^n(\mathbf{p})T_A^\alpha|0\rangle^n \\ \propto \left(e^{2\pi i\alpha\kappa_{\mathcal{O}}}V_A^n+(V-V_A)^n\right)\frac{{}^n\langle 0|T_A^\alpha|0\rangle^n}{{}^n\langle 0|0\rangle^n}. \end{aligned} \tag{2.127}$$

To evaluate the first of the two terms we needed to commute the local observable before applying the OPE contraction, but the commutators always produce terms which are subleading in the volume, as it is obvious e.g. from the inspection of the free theory. Similarly, we can evaluate the norm ${}^n\langle\Psi|\Psi\rangle^n$ which does not require the splitting of $\mathcal{O}^j(\mathbf{p})$

$${}^n\langle\Psi|\Psi\rangle^n = {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})(\mathcal{O}^\dagger)^1(-\mathbf{p})\mathcal{O}^1(\mathbf{p})\dots\mathcal{O}^n(\mathbf{p})|0\rangle^n \propto V^n. \tag{2.128}$$

In the evaluation of the ratio

$$\frac{{}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n}{{}^n\langle\Psi|\Psi\rangle^n}, \tag{2.129}$$

the proportionality constant (which is non-universal and could be absorbed in a redefinition of the field) cancels out, and one can write

$$\frac{{}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n}{{}^n\langle\Psi|\Psi\rangle^n} \simeq \left(e^{2\pi i\alpha\kappa_{\mathcal{O}}}r^n+(1-r)^n\right)\frac{{}^n\langle 0|T_A^\alpha|0\rangle^n}{{}^n\langle 0|0\rangle^n} \tag{2.130}$$

with $r=\frac{V_A}{V}$. In the expression above the term in brackets is universal, while the term factored out is not universal and it is the nth charged moment of the ground state. Therefore, as anticipated, the ratio of charged moments is given by:

$$M_n^\Psi(r,\alpha) = \frac{{}^n\langle\Psi|T_A^\alpha|\Psi\rangle^n}{{}^n\langle\Psi|\Psi\rangle^n}\frac{{}^n\langle 0|0\rangle^n}{{}^n\langle 0|T_A^\alpha|0\rangle^n} \simeq e^{2\pi i\alpha\kappa_{\mathcal{O}}}r^n+(1-r)^n. \tag{2.131}$$

Results for multi-particle states can be obtained in a similar fashion.

In conclusion, the striking simplicity of these results relies on the truncation of the OPE in (2.110), which is expected to become exact in the limit $m^dV\gg 1$. We expect that for finite m^dV further contributions in the OPE can be recast as a (possibly non-integer) power series in $(m^dV)^{-1}$, which generalises the explicit $(mL)^{-1}$ power expansion that is obtained for 1+1D free theories using form factor techniques (see Appendix 2.C). In massless theories, we instead expect corrections as a power series in $(|\mathbf{p}|^dV)^{-1}$. The explicit evaluation of these non-universal corrections, as well as the treatment of possible divergences in the power series, are all beyond the purpose of this Thesis.

2.5 Numerical results

In this section we present numerical results for two very different discrete models. First we consider a 1D lattice Fermi gas, which has a gapless phase but it also possesses highly excited states whose entanglement is well described by our formulae, and then we look at the harmonic chain, which becomes a $U(1)$ massive free boson in the scaling limit. While for the first model we can only consider distinct excitations, for the second we consider also identical excitations. The treatment of the complex harmonic chain is based on a generalisation of the wave-functional method as presented in [115] which we derive in detail. The good agreement with the theoretical predictions confirms the picture already put forward for the total entropy in [114], i.e. that our formulae hold under the broad assumption of localised excitations.

2.5.1 1D lattice Fermi gas

In this subsection we analyse a particle-hole excited state of a 1D lattice Fermi gas, comparing our analytical predictions with the numerical data. Even though the model is critical, it was realised in [147] that certain highly energetic quasiparticle excitations still have a universal entanglement content. More precisely, if one considers a set of quasiparticles with small enough De Broglie wavelengths (compared to the typical geometric lengths) and sufficiently separated momenta, then these quasiparticles will be essentially uncorrelated with each other and with respect to zero-point fluctuations. We refer to [145–150] for further details about the universal entanglement content of quasiparticles in critical systems.

Our goal here is to briefly review the numerical techniques involved in the characterisation of fermionic Gaussian states [173] and their application to the computation of symmetry-resolved measures. We start by considering the Hamiltonian of free spinless fermions on a circle of length L

$$H = -\frac{1}{2}\sum_j f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1} + \mu \sum_j f_j^\dagger f_j, \tag{2.132}$$

where μ is the chemical potential and $\{f_j\}_{j=1,\dots,L}$, $\{f_j^\dagger\}_{j=1,\dots,L}$ are the ladder operators obeying the standard anticommutation relations

$$\{f_j, f_{j'}\} = \{f_j^\dagger, f_{j'}^\dagger\} = 0, \qquad \{f_j, f_{j'}^\dagger\} = \delta_{jj'}. \tag{2.133}$$

When $|\mu| < 1$ the theory is gapless, as can be seen by moving to momentum space, where the Hamiltonian is diagonal:

$$H = \sum_{q=0}^{L-1} \left[\mu - \cos\left(\frac{2\pi q}{L}\right)\right] \tilde{f}_q^\dagger \tilde{f}_q, \tag{2.134}$$

and the ground state is a Fermi sea with Fermi momentum $k_F = \arccos(\mu)$. The two-point function evaluated in the ground state at Fermi momentum k_F takes the following form [174, 175]:

$$C_0(j,j') := \langle f_j^\dagger f_{j'} \rangle_0 = \frac{\sin k_F(j-j')}{L \sin \frac{\pi(j-j')}{L}}. \tag{2.135}$$

Here, we analyse the quasiparticle excited state described by the two-point function

$$C(j,j') = C_0(j,j') + \frac{1}{L} e^{-i(k_F+\frac{\pi}{4}-\frac{\pi}{L})(j-j')} - \frac{1}{L} e^{-i(k_F-\frac{\pi}{4}+\frac{\pi}{L})(j-j')}, \tag{2.136}$$

which corresponds to the insertion of a fermion of momentum $k = k_F + \frac{\pi}{4} - \frac{\pi}{L}$ above the ground state and the removal of another fermion (or equivalently, the insertion of a hole) at $k = k_F - \frac{\pi}{4} + \frac{\pi}{L}$. The choice of the momentum shift is not important in the continuum limit, where the only necessary condition is that $|k - k_F|$ remains finite when $L \to \infty$ [12]. We now have to specify the symmetry of the model. The Hamiltonian (2.132) is invariant under an internal $U(1)$ symmetry associated to the number of fermions, generated by the operator

$$\hat{Q} = \sum_j f_j^\dagger f_j, \tag{2.137}$$

which satisfies the locality condition $\hat{Q} = \hat{Q}_A + \hat{Q}_{\bar{A}}$, with

$$\hat{Q}_A = \sum_{j \in A} f_j^\dagger f_j, \quad \hat{Q}_{\bar{A}} = \sum_{j \in \bar{A}} f_j^\dagger f_j. \tag{2.138}$$

As done in Section 2.4.1, the subsystems A and $\bar{A}$ are defined by the sets of fermion sites $A = \{1, \dots, \ell\}$, $\bar{A} = \{\ell+1, \dots, L\}$. We denote by C_0^A and C^A the $\ell \times \ell$ matrices resulting from projection of the matrices C_0 and C (defined by equations (2.135) and (2.136) respectively) onto subsystem A, so that the indices j, $j' \in A$. Following [104] we express the charged moments of the particle-hole state and the ground state by means of the determinants

$$\mathrm{Tr}_A(\rho_A^n e^{2\pi i \alpha \hat{Q}_A}) = \det\left((C^A)^n e^{2\pi i\alpha} + (1-C^A)^n\right), \tag{2.139}$$

$$\mathrm{Tr}_A(\rho_{A,0}^n e^{2\pi i \alpha \hat{Q}_A}) = \det\left((C_0^A)^n e^{2\pi i\alpha} + (1-C_0^A)^n\right), \tag{2.140}$$

with ρ_A and $\rho_{A,0}$ the RDMs of the two states. According to our analytical predictions, because the fermion and the hole are distinguishable quasiparticles with opposite charges ± 1, we expect the ratio

[12] In the work [104] a particle-hole state satisfying $|k - k_F| \sim 1/L$ was considered. Unlike the present case, the entanglement measures of such a low-lying state is captured by CFT predictions, due to the strong correlation effects between the particle/hole and the zero-point fluctuations.

of charged moment to take the universal form

$$\frac{\mathrm{Tr}_A(\rho_A^n e^{2\pi i\alpha\hat{Q}_A})}{\mathrm{Tr}_A(\rho_{A,0}^n e^{2\pi i\alpha\hat{Q}_A})} \simeq (r^n e^{2\pi i\alpha} + (1-r)^n)(r^n e^{-2\pi i\alpha} + (1-r)^n), \tag{2.141}$$

where the equality only holds in the scaling limit of the lattice model, with $r = \ell/L$ fixed.

To test the validity of Eq. (2.141) we consider two entanglement measures, namely the excess of (total) Rényi entropy and the so-called (following the terminology of [104]) "excess of variance". The excess of entropy is recovered from our formulae for $\alpha = 0$, and for two distinct excitations it takes the simple form

$$\Delta S_n = \frac{1}{1-n}\log\frac{\mathrm{Tr}_A(\rho_A^n)}{\mathrm{Tr}_A(\rho_{A,0}^n)} \simeq \frac{\log\left(r^n + (1-r)^n\right)^2}{1-n}. \tag{2.142}$$

We define the variance[13] associated to ρ_A as

$$\langle\Delta\hat{Q}_A^2\rangle_n := \frac{\mathrm{Tr}_A(\rho_A^n\hat{Q}_A^2)}{\mathrm{Tr}_A(\rho_A^n)} - \left(\frac{\mathrm{Tr}_A(\rho_A^n\hat{Q}_A)}{\mathrm{Tr}_A(\rho_A^n)}\right)^2 = \frac{1}{(2\pi i)^2}\frac{d^2}{d\alpha^2}\log\frac{\mathrm{Tr}_A(\rho_A^n e^{2\pi i\alpha\hat{Q}_A})}{\mathrm{Tr}_A(\rho_A^n)}\bigg|_{\alpha=0}. \tag{2.143}$$

Similarly, we denote by $\langle\Delta\hat{Q}_A^2\rangle_{n,0}$ the variance of the ground state $\rho_{A,0}$. From (2.141) it follows that the excess of variance is given by

$$\langle\Delta\hat{Q}_A^2\rangle_n - \langle\Delta\hat{Q}_A^2\rangle_{n,0} \simeq \frac{2r^n(1-r)^n}{(r^n+(1-r)^n)^2}. \tag{2.144}$$

A way to physically interpret the result (2.144) is to regard this excess of variance as twice the contribution associated to a single quasiparticle, since particles and antiparticles contribute in the same way. The latter is just the variance of a Bernoulli random variable with success probability given by

$$p = \frac{r^n}{r^n + (1-r)^n}, \tag{2.145}$$

namely the probability one associates to the presence of a quasiparticle in A computed with the density matrix ρ_A^n. Since the variance of a Bernoulli variable with probability p is just $p(1-p)$, one gets Eq. (2.144).

In Fig. 2.2 we report our analytical predictions and the numerical values of ΔS_n and $\langle\Delta\hat{Q}_A^2\rangle_n - \langle\Delta\hat{Q}_A^2\rangle_{n,0}$, computed from (2.139) and (2.140) using exact diagonalisation of the correlation matrices $C_A, C_{A,0}$. We keep L fixed, analysing different values of $r = \ell/L$. Our choice is motivated by the expectation that these plots should be "universal" at large L, meaning that data obtained with different values L should collapse to the same universal prediction (independent of lattice details such as the value of k_F) when $L \to \infty$. As we see from the plots of both measures, the match between numerics

[13] The choice of this terminology comes from the fact that for $n = 1$ this measure reproduces the physical variance of the charge in the state ρ_A.

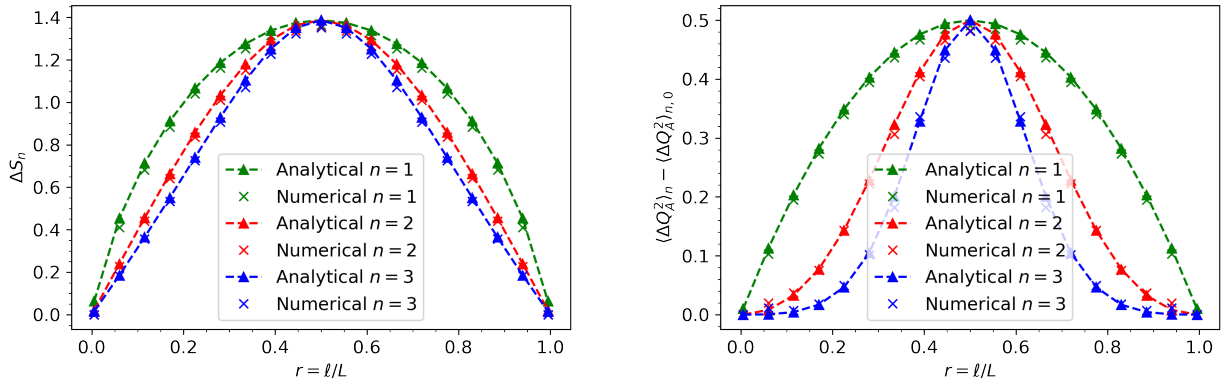


Figure 2.2 Numerical data versus analytical prediction for the particle-hole excited state described by the correlation function (2.136). Here, $k_F = \pi/2$, $L = 200$ and we considered the first values of n for $r = \ell/L \in [0, 1]$. Left: Excess Rényi entropy checked against Eq. (2.142). Right: Excess variance, checked against Eq. (2.144). The numerical results are in very good agreement with the analytical formulae.

and analytics is really good.

2.5.2 Complex free boson and 1D harmonic chain

In this section we consider a complex massive free boson. Unlike the 1D Fermi gas, this model and its lattice version allow us to test formulae for states containing two or more identical excitations. Our numerical computation is based on the wave-functional method introduced in [115] (see Appendix A of that paper). Here we need to extend the technique to a complex theory and to the symmetry-resolved moments. These extensions are not entirely trivial and for that reason we review the wave-functional method in detail.

Let us consider a 1D complex massive boson on the line $[0, L]$ with Hamiltonian:

$$H = \int_0^L \mathrm{d}x \left(\Pi^\dagger \Pi + (\partial_x \Phi)^\dagger (\partial_x \Phi) + m^2 \Phi^\dagger \Phi \right), \tag{2.146}$$

where

$$\Pi(x) = \dot{\Phi}^\dagger(x), \quad \Pi^\dagger(x) = \dot{\Phi}(x). \tag{2.147}$$

Alternatively, one can introduce a pair of real bosons Φ_1, Φ_2, and express Φ and Π as

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}, \quad \Pi = \frac{\Pi_1 + i\Pi_2}{\sqrt{2}}, \tag{2.148}$$

so that the Hamiltonian becomes that of two real bosons. The only non-vanishing equal-time commutators are:

$$[\Phi(x), \Pi(y)] = [\Phi^\dagger(x), \Pi^\dagger(y)] = i\delta(x - y). \tag{2.149}$$

Since space is compact, there are discrete energy levels with dispersion relation:

$$E_p = \sqrt{m^2 + p^2}, \quad p \in \frac{2\pi}{L}\mathbb{Z}, \tag{2.150}$$

and the Hamiltonian is diagonalised via the introduction of two sets of creation/annihilation operators. The annihilation operators are:

$$A_p = \frac{1}{\sqrt{2LE_p}} \int_0^L \mathrm{d}x e^{-ipx}(E_p\Phi(x) + i\Pi^\dagger(x)), \tag{2.151}$$

$$B_p = \frac{1}{\sqrt{2LE_p}} \int_0^L \mathrm{d}x e^{-ipx}(E_p\Phi^\dagger(x) + i\Pi(x)). \tag{2.152}$$

and as usual $A_p^\dagger$ ($B_p^\dagger$ resp.) creates from the a positively (negatively) $U(1)$-charged particle with momentum p from the vacuum. These operators satisfy:

$$[A_p, A_{p'}^\dagger] = [B_p, B_{p'}^\dagger] = \delta_{p,p'}. \tag{2.153}$$

$\Phi(x)$, $\Pi(x)$ then admit the usual Fourier decomposition

$$\Phi(x) = \sum_p \frac{1}{\sqrt{2LE_p}} \left(A_p e^{ipx} + B_p^\dagger e^{-ipx}\right), \tag{2.154}$$

$$\Pi(x) = -i\sum_p \sqrt{\frac{E_p}{2L}} \left(B_p e^{ipx} - A_p^\dagger e^{-ipx}\right). \tag{2.155}$$

Finally, the charge operator corresponding to the $U(1)$ symmetry of the theory is:

$$Q = i\int_0^L \mathrm{d}x \, : \left(\Phi^\dagger(x)\Pi^\dagger(x) - \Phi(x)\Pi(x)\right) := \sum_p (A_p^\dagger A_p - B_p^\dagger B_p), \tag{2.156}$$

where $:\mathcal{O}:$ denotes the usual normal ordering of $\mathcal{O}$.

2.5.2.1 The wave-functional method

A very useful way to represent the states in the theory is provided by the wave-functional formalism. In this approach, we associate to every state a functional $\mathbf{\Psi}$ acting on the space of classical field configurations $\phi, \phi^\dagger : [0, L] \to \mathbb{C}$ and formally defined by:

$$\mathbf{\Psi}[\phi, \phi^\dagger] = \langle \phi, \phi^\dagger | \mathbf{\Psi} \rangle. \tag{2.157}$$

This definition resembles that of the wave function $\psi(x) = \langle x|\psi\rangle$ in non-relativistic quantum mechanics, and the action of the operators Φ, Π on the state $|\phi, \phi^\dagger\rangle$ mimics that of the position and

momentum operators on the state $|x\rangle$:

$$\Phi(x)\mathbf{\Psi}[\phi,\phi^\dagger]=\phi(x)\mathbf{\Psi}[\phi,\phi^\dagger],\quad i\Pi(x)\mathbf{\Psi}[\phi,\phi^\dagger]=\frac{\delta\mathbf{\Psi}[\phi,\phi^\dagger]}{\delta\phi(x)},\tag{2.158}$$

and analogously

$$\Phi^\dagger(x)\mathbf{\Psi}[\phi,\phi^\dagger]=\phi^\dagger(x)\mathbf{\Psi}[\phi,\phi^\dagger],\quad i\Pi^\dagger(x)\mathbf{\Psi}[\phi,\phi^\dagger]=\frac{\delta\mathbf{\Psi}[\phi,\phi^\dagger]}{\delta\phi^\dagger(x)}.\tag{2.159}$$

Notice that, for consistency, when $i\Pi$, $i\Pi^\dagger$ act on a ket $|\phi,\phi^\dagger\rangle$ there is a minus sign in front of the functional derivative. The vacuum state functional is defined by:

$$A_p\mathbf{\Psi}_{\text{vac}}=B_p\mathbf{\Psi}_{\text{vac}}=0\quad\forall\, p\in\frac{2\pi}{L}\mathbb{Z},\tag{2.160}$$

and the only solution to these functional differential equations up to normalisation is the Gaussian functional:

$$\mathbf{\Psi}_{\text{vac}}[\phi,\phi^\dagger]=\exp\left[-\int_0^L\mathrm{d}x\int_0^L\mathrm{d}y\,\phi^\dagger(x)K(x-y)\phi(y)\right],\quad K(x-y)=\frac{1}{L}\sum_p E_p e^{ip(x-y)}.\tag{2.161}$$

Notice that $K(x)$ is a real and even function of x. The functionals of the positively and negatively charged one-particle states are obtained through the action of $A_p^\dagger$ and $B_p^\dagger$:

$$A_p^\dagger\mathbf{\Psi}_{\text{vac}}=\alpha_p[\phi^\dagger]\mathbf{\Psi}_{\text{vac}},\quad \alpha_p[\phi^\dagger]=\sqrt{\frac{2E_p}{L}}\int_0^L\mathrm{d}x\,e^{ipx}\phi^\dagger(x),\tag{2.162}$$

$$B_p^\dagger\mathbf{\Psi}_{\text{vac}}=\beta_p[\phi]\mathbf{\Psi}_{\text{vac}},\quad \beta_p[\phi]=\sqrt{\frac{2E_p}{L}}\int_0^L\mathrm{d}x\,e^{ipx}\phi(x),\tag{2.163}$$

and the functional of a state with k^+ positive excitations and k^- negative excitations (all with different momenta) is:

$$\mathbf{\Psi}^{k^+,k^-}_{\{p_i,q_j\}}[\phi,\phi^\dagger]=\prod_{i=1}^{k^+}A_{p_i}^\dagger\prod_{j=1}^{k^-}B_{q_j}^\dagger\mathbf{\Psi}_{\text{vac}}[\phi,\phi^\dagger]=\prod_{i=1}^{k^+}\alpha_{p_i}[\phi^\dagger]\prod_{j=1}^{k^-}\beta_{q_j}[\phi]\mathbf{\Psi}_{\text{vac}}[\phi,\phi^\dagger].\tag{2.164}$$

A correct choice of the normalisation in (2.161) ensures that the functional above has unit norm with respect to the bra-ket product. It is then immediate to construct the functionals of multi-particle states with identical excitations. If there are k_i^+ (k_i^-) positively (negatively) charged particles with momentum p_i (q_i), for $i=1,\dots,m^+$ (m^-) we define:

$$\mathbf{\Psi}^{\{k_i^+,k_j^-\}}_{\{p_i,q_j\}}[\phi,\phi^\dagger]=\prod_{i=1}^{m^+}\frac{(A_{p_i}^\dagger)^{k_i^+}}{\sqrt{k_i^+!}}\prod_{j=1}^{m^-}\frac{(B_{q_j}^\dagger)^{k_j^-}}{\sqrt{k_j^-!}}\mathbf{\Psi}_{\text{vac}}[\phi,\phi^\dagger].\tag{2.165}$$

The action of the charge operator on a wave-functional immediately follows from that of the fields:

$$Q\mathbf{\Psi}[\phi,\phi^\dagger] = \int_0^L \mathrm{d}x \left(\phi^\dagger(x)\frac{\delta}{\delta\phi^\dagger(x)} - \phi(x)\frac{\delta}{\delta\phi(x)}\right)\mathbf{\Psi}[\phi,\phi^\dagger], \tag{2.166}$$

and in particular one finds that the vacuum functional and the functionals of multi-particle states are charge eigenstates:

$$Q\mathbf{\Psi}_{\mathrm{vac}} = 0 \quad , \quad Q\mathbf{\Psi}^{k^+,k^-}_{\{p_i,q_j\}} = (k^+ - k^-)\mathbf{\Psi}^{k^+,k^-}_{\{p_i,q_j\}}. \tag{2.167}$$

However, because $|\phi,\phi^\dagger\rangle$ is not associated to any charged state in the Fock space, it is not a charge eigenstate. The exponential of the charge operator acts on $|\phi,\phi^\dagger\rangle$ by introducing phases (notice the minus sign in front of the integral):

$$e^{2\pi i\alpha Q}|\phi,\phi^\dagger\rangle = \exp\left[-2\pi i\alpha\int_0^L \mathrm{d}x \left(\phi^\dagger(x)\frac{\delta}{\delta\phi^\dagger(x)} - \phi(x)\frac{\delta}{\delta\phi(x)}\right)\right]|\phi,\phi^\dagger\rangle = |e^{2\pi i\alpha}\phi, e^{-2\pi i\alpha}\phi^\dagger\rangle. \tag{2.168}$$

Employing these results it is possible to show (see Appendix 2.D for the derivation) that

$$\mathrm{Tr}_A(\rho_{0,A}^n e^{2\pi i\alpha Q_A}) = \int \mathcal{D}\phi_1\mathcal{D}\phi_1^\dagger\ldots\mathcal{D}\phi_n\mathcal{D}\phi_n^\dagger \exp\left[-\mathcal{G}_\alpha\right], \tag{2.169}$$

where $\mathcal{G}_\alpha$ is a known Gaussian functional of the fields $\phi(x),\phi^\dagger(x)$ given in (2.248). Results for the harmonic chain can then be obtained by discretisation, as we see in the next Subsection.

2.5.2.2 The harmonic chain

Since the Hamiltonian (2.146) reduces to the sum of two Hamiltonians for the real bosons Φ_1, Φ_2 with prefactors $\frac{1}{2}$, the discretisation proceeds exactly as for the real boson [115]. We divide the interval $[0,L]$ in N parts by introducing a spacing:

$$\Delta x = \frac{L}{N}, \tag{2.170}$$

and we define $x = \frac{L}{N}\bar{x}$, $\bar{x} \in \{0,1,\ldots,N-1\}$, so that we can replace every integral with a sum:

$$\int_{A\cup\bar{A}}\mathrm{d}x \to \frac{L}{N}\sum_{x=0}^{L-\Delta x}, \quad \int_A \mathrm{d}x \to \frac{L}{N}\sum_{x=0}^{\ell-\Delta x}, \quad \int_{\bar{A}}\mathrm{d}x \to \frac{L}{N}\sum_{x=\ell}^{L-\Delta x}. \tag{2.171}$$

If we discretise the Laplace operator as:

$$\partial_x^2\Phi(x) \to \frac{\Phi(x+\Delta x)+\Phi(x-\Delta x)-2\Phi(x)}{(\Delta x)^2}, \tag{2.172}$$

and impose periodic boundary conditions $\Phi(0) = \Phi(L)$, $\Phi^\dagger(0) = \Phi^\dagger(L)$, the Hamiltonian (2.146) reduces (after integration by parts) to two independent harmonic chains for real fields. The set of

momenta is now restricted to the first Brillouin zone, $p = \frac{2\pi}{L}\bar{p}, \bar{p} \in \{0, 1, \ldots, N-1\}$, and the dispersion relation becomes:

$$E_p = \sqrt{m^2 + \left(\frac{2N}{L}\sin\frac{pL}{2N}\right)^2}, \tag{2.173}$$

from which the relativistic relation $E_p^2 = m^2 + p^2$ is obtained when $\frac{pL}{2N} \ll 1$. Notice that since we restrict the set of momenta, the function $K(x)$ defined in (2.161) becomes a finite sum:

$$K(x) = \frac{1}{L}\sum_{p=0}^{2\pi(N-1)/L} E_p e^{ipx}, \tag{2.174}$$

thus it is no longer an even function of x, though the property $K^*(x) = K(-x)$ still holds.

For the sake of simplicity we will take Φ and $\Phi^\dagger$ to be the fundamental degrees of freedom in the following, while keeping in mind that the real degrees of freedom are recovered using (2.148). In the formula (2.251) the functions $U_i^\pm$, $V_i^\pm$ are modified by simply replacing the integrals with sums following the prescription (2.171). On the other hand, discretisation of the measure $\mathcal{G}_\alpha$ leads to a finite-dimensional $(nN) \times (nN)$ matrix G which couples the fields $\phi_i^\dagger(x)$ and $\phi_j(y)$:

$$\begin{aligned}\mathcal{G}_\alpha = \left(\frac{L}{N}\right)^2 \sum_{i=1}^{n} \Bigg[& 2\left(\sum_{x\in A, y\in A} + \sum_{x\in \bar{A}, y\in \bar{A}}\right)\phi_i^\dagger(x)K(x-y)\phi_i(y) \\ & + \sum_{x\in A, y\in \bar{A}} \left(\phi_i^\dagger(x) + \phi_{i+1}^\dagger(x)e^{-2\pi i\alpha\delta_{i,n}}\right)K(x-y)\phi_i(y) \\ & + \sum_{x\in \bar{A}, y\in A} \phi_i^\dagger(x)K(x-y)\left(\phi_i(y) + \phi_{i+1}(y)e^{2\pi i\alpha\delta_{i,n}}\right)\Bigg] =: \sum_{i,j=1}^{n}\sum_{x,y=0}^{L}\phi_i^\dagger(x)G_{ix,jy}\phi_j(y),\end{aligned} \tag{2.175}$$

where we explicitly wrote the complex conjugate in (2.248) before discretising the integrals. Wick's theorem ensures that the Gaussian average in (2.251) can be computed from the contractions of pairs of fields, which are in turn obtained via the inversion of the matrix G:

$$\overbrace{\phi_i(x)^\dagger\phi_j}(y) = (G^{-1})_{ix,jy}. \tag{2.176}$$

The matrix G has a block structure, consisting of n^2 blocks $G_{i\cdot,j\cdot}$, each of which is an $N \times N$ matrix. From the above expression we see that the only non-vanishing blocks are either in the diagonal $G_{i\cdot,i\cdot}$ or just off the diagonal, $G_{i\cdot,(i\pm1)\cdot}$. Each block G admits a sub-block structure in terms of the matrices $K_{Q_1Q_2}$, $Q_1, Q_2 \in \{A, \bar{A}\}$, whose elements are:

$$(K_{Q_1Q_2})_{xy} = \left(\frac{L}{N}\right)^2 K(x-y), \quad x \in Q_1, y \in Q_2. \tag{2.177}$$

Notice that K_{AA} is a square matrix with dimensions $\frac{\ell}{L}N \times \frac{\ell}{L}N$, $K_{A\bar{A}}$ has dimensions $\frac{\ell}{L}N \times \frac{L-\ell}{L}N$

and so on. From (2.175) we obtain the following basic structures:

- Diagonal blocks $G_{i\cdot,i\cdot}$

$$\left(\sum_{x\in A,y\in A}+\sum_{x\in\bar{A},y\in\bar{A}}+\sum_{x\in A,y\in\bar{A}}+\sum_{x\in\bar{A},y\in A}\right)\phi_i^\dagger(x)G_{ix,iy}\phi_i(y)$$
$$=\left(\frac{L}{N}\right)^2\Bigg[2\sum_{\substack{x\in A\\y\in A}}\phi_i^\dagger(x)K(x-y)\phi_i(y)+2\sum_{\substack{x\in\bar{A}\\y\in\bar{A}}}\phi_i^\dagger(x)K(x-y)\phi_i(y)$$
$$+\sum_{x\in A,y\in\bar{A}}\phi_i^\dagger(x)K(x-y)\phi_i(y)+\sum_{x\in\bar{A},y\in A}\phi_i^\dagger(x)K(x-y)\phi_i(y)\Bigg]$$
$$\Rightarrow G_{i\cdot,i\cdot}=\begin{pmatrix}2K_{AA} & K_{A\bar{A}}\\ K_{\bar{A}A} & 2K_{\bar{A}\bar{A}}\end{pmatrix}.$$

- Off-diagonal blocks $G_{i\cdot,(i+1)\cdot}$

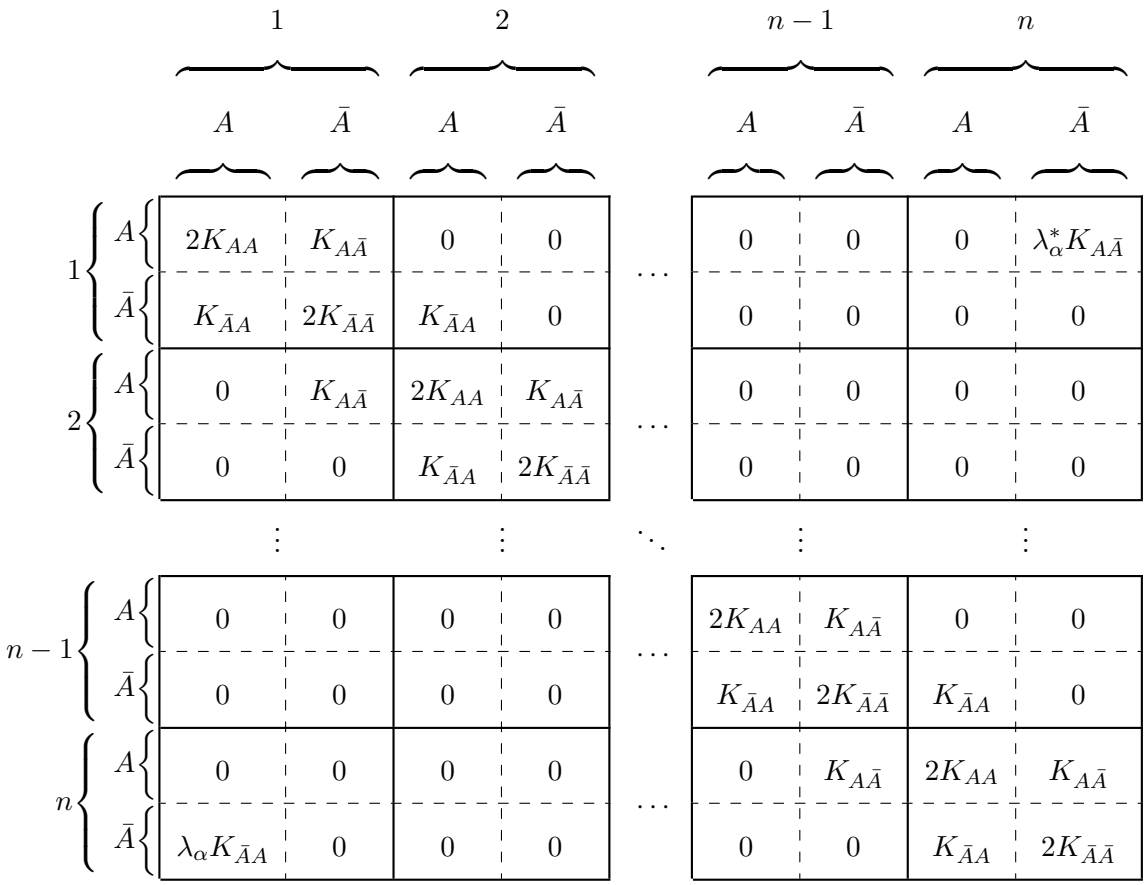
$$\sum_{x\in\bar{A},y\in A}\phi_i^\dagger(x)G_{ix,(i+1)y}\phi_{i+1}(y)=\left(\frac{L}{N}\right)^2\sum_{x\in\bar{A},y\in A}\phi_i^\dagger(x)K(x-y)\phi_{i+1}(y)e^{2\pi i\alpha\delta_{i,n}}$$
$$\Rightarrow G_{i\cdot,(i+1)\cdot}=\begin{pmatrix}0 & 0\\ K_{\bar{A}A}e^{2\pi i\alpha\delta_{i,n}} & 0\end{pmatrix}.$$

- Off-diagonal blocks $G_{(i+1)\cdot,i\cdot}$

$$\sum_{x\in A,y\in\bar{A}}\phi_{i+1}^\dagger(x)G_{(i+1)x,iy}\phi_i(y)=\left(\frac{L}{N}\right)^2\sum_{x\in A,y\in\bar{A}}\phi_{i+1}^\dagger(x)K(x-y)\phi_i(y)e^{-2\pi i\alpha\delta_{i,n}}$$
$$\Rightarrow G_{(i+1)\cdot,i\cdot}=\begin{pmatrix}0 & K_{A\bar{A}}e^{-2\pi i\alpha\delta_{i,n}}\\ 0 & 0\end{pmatrix}.$$

Note that the block structure is different from that in [115] because the roles of regions A and $\bar{A}$ are now exchanged. Although this exchange has no effect on the form of the entanglement entropy, for the SREE it makes a difference as the symmetry between $A,\bar{A}$ is broken when we choose to place the charge in subsystem A.

In terms of the $N\times N$ blocks above, we can schematically write the matrix G as follows:



where we introduced $\lambda_\alpha = e^{2\pi i\alpha}$.

2.5.2.3 Numerical results

In Appendix 2.D we have explicitly derived the ratio of charged moments for excited states. We thus have all the ingredients needed to obtain numerical results. Let us take a bi-partition where A is a segment made of $N_A \leq N$ consecutive sites, with $N_A/N = r$ and analyse the behaviour of $M_n^\Psi(r;\alpha)$.

In Fig. 2.3 we compare results for two kinds of two-particle excited states: those of particles with identical charges and either distinct or equal momenta p_1 and p_2. Our analytical predictions for $M_n(r;\alpha)$ are

$$\begin{aligned} M_n^{1^+1^+}(r;\alpha) &= (r^n e^{2i\pi\alpha} + (1-r)^n)^2, \quad p_1 \neq p_2, \\ M_n^{2^+}(r;\alpha) &= r^{2n} e^{4\pi i\alpha} + 2^n (1-r)^n r^n e^{2\pi i\alpha} + (1-r)^{2n}, \quad p_1 = p_2. \end{aligned} \tag{2.178}$$

In our numerics we have chosen $L = N = 30$, so that the lattice spacing $L/N = 1$. We also fix the mass scale to $m = 0.1$, which corresponds to a typical correlation length of $\xi = m^{-1} = 10$ sites. Finally we choose either $p_1 = p_2 = \pi$ or $p_1 = \pi$ and $p_2 = \frac{2\pi}{5}$, both in units of the lattice spacing.

Similarly, Fig. 2.4, we consider the following three-particle excited states: a state of three equal momenta, that is $p_1 = p_2 = p_3$, a state of two equal momenta among the three, that is $p_1 = p_2 \neq$

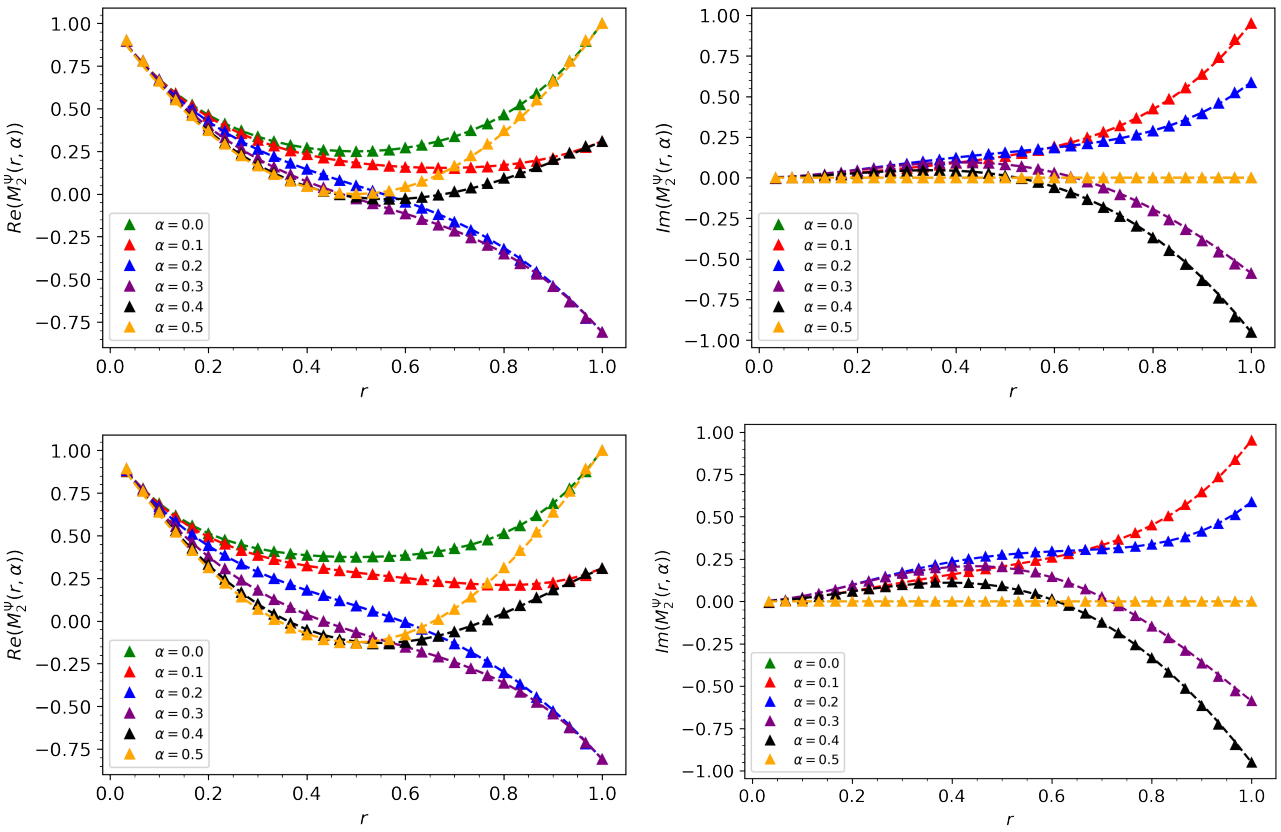


Figure 2.3 Charged moments of two-particle excited states in the 1D harmonic chain
Numerical data (triangles) versus analytical predictions (dashed lines) for $M_2^{1^+1^+}(r;\alpha)$ (top row) and $M_2^{2^+}(r;\alpha)$ (bottom row). We consider $n = 2$, system size $L = 30$ with $m = 0.1$. The left/right panels in each row show the real/imaginary part of the function. In both rows we take values of the flux $\alpha = 0, 0.1, \ldots 0.5$. For the numerics of the top row figures we use momenta $p_1 = \pi, p_2 = 2\pi/5$ whereas for the bottom row we take equal momenta $p_1 = p_2 = \pi$.

p_3, and a state with three distinct momenta, that is p_1, p_2, p_3 distinct. In this case the analytical predictions are

$$M_n^{3^+}(r;\alpha) = r^{3n}e^{6i\pi\alpha} + 3^n r^{2n}(1-r)^n e^{4i\pi\alpha} + 3^n r^n (1-r)^{2n} e^{2i\pi\alpha} + (1-r)^{3n}, \quad p_1 = p_2 = p_3, \tag{2.179}$$

and

$$M_n^{2^+1^+}(r;\alpha) = M_n^{2^+}(r;\alpha)(r^n e^{2i\pi\alpha} + (1-r)^n), \quad p_1 = p_2 \neq p_3, \tag{2.180}$$

$$M_n^{1^+1^+1^+}(r;\alpha) = (r^n e^{2i\pi\alpha} + (1-r)^n)^3, \quad p_1 \neq p_2 \neq p_3. \tag{2.181}$$

The set of momenta is $p_1 = p_2 = p_3 = \pi$ for the first excited state, $p_1 = p_2 = \pi, p_3 = \pi/3$ for the second state, and $p_1 = \pi, p_2 = \pi/3, p_3 = \pi/5$ for the third one.

In all our figures we chose non-negative values of α. Given the formulae above, taking $\alpha < 0$ is equivalent to complex conjugation with α positive, so the figures for negative α are identical except for a change of sign in the imaginary part of all functions. We have also considered the value $\alpha = 0$ (in

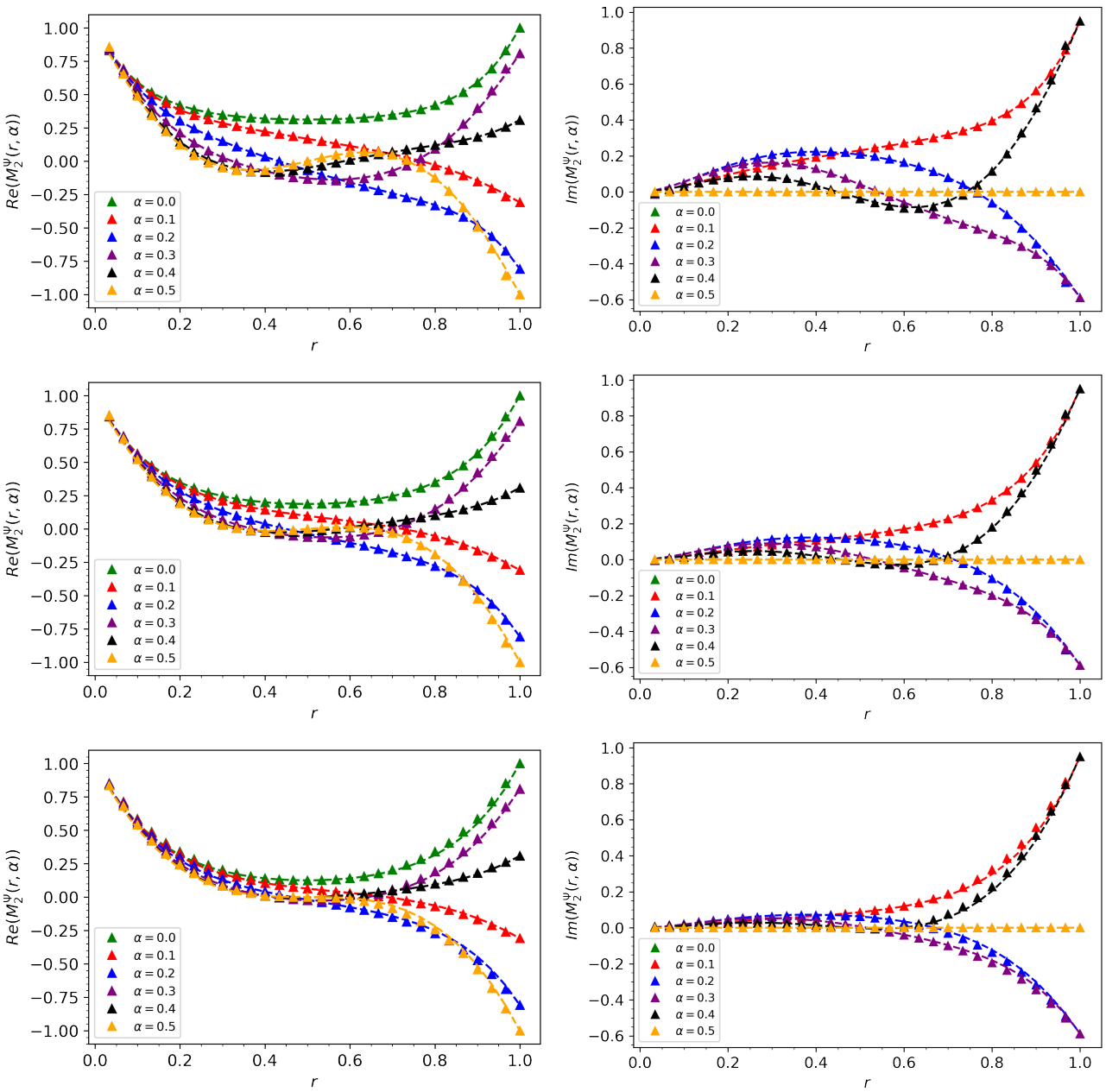


Figure 2.4 Numerical data (triangles) versus analytical predictions (dashed lines) for $M_2^{3^+}(r;\alpha)$ (top row), $M_2^{2^+1^+}(r;\alpha)$ (central row), and $M_2^{1^+1^+1^+}(r;\alpha)$ (bottom row). We consider $n = 2$, system size $L = 30$ with $m = 0.1$. The left/right panels in each row show the real/imaginary part of the function. In each rows we take values of the flux $\alpha = 0, 0.1, \dots 0.5$. For the numerics of the top row figures we use momenta $p_1 = p_2 = p_3 = \pi$, for the central row $p_1 = p_2 = \pi, p_3 = \pi/3$, whereas for the bottom row we take $p_1 = \pi, p_2 = \pi/3, p_3 = \pi/5$.

green) which is the limit where there is no flux. As expected, in this case our formulae recover those for the excess Rényi Entropies in [114, 115], which are symmetric in r and have vanishing imaginary part. Despite the fact that the correlation length is not particularly small with respect to the system size L ($\xi = m^{-1} = \frac{L}{3}$), we took highly energetic states (momenta being fixed in the large-volume limit) and we thus expect our predictions to remain valid.

In both Fig. 2.3 and 2.4, we plot the numerical data (triangles) against analytical predictions

((2.178) and (2.181)) as functions of r fixing $n = 2$ for several values of α between 0 and $\frac{1}{2}$, which correspond to flux ± 1, respectively. At these two points, the ratio becomes purely real. The figures show excellent agreement between numerical data and analytical predictions.

2.6 Concluding remarks

In this chapter we have computed the symmetry-resolved entanglement entropy and its moments for zero-density excited states. These are excited states consisting of a finite number of excitations above the ground state in a scaling limit where both the volume of the system and the volume of each subsystem are taken to infinity, keeping their ratio constant.

It is known from previous work [114–116, 144–150] that the difference between the entanglement entropy of the excited state and that of the ground state, also known as excess entropy, takes an extremely simple and universal form for non-interacting 1+1D QFTs and also for certain highly excited states of CFT [104]. Since this excess of entanglement represents the extra contribution to entanglement of an excited state above a non-trivially entangled ground state, the same extra contribution is obtained when the ground state is trivial. For this reason both a free QFT and a qubit picture lead to the same results, even if the underlying theories are extremely different. Finally, it has also been shown that the results extend to free bosons in any dimension [116] and more generally they are expected to hold in any situations where excitations are localised. Here, we extended the work summarised above to the computation of excess symmetry-resolved Rényi entropies and entanglement entropies in excited states of theories possessing a $U(1)$ symmetry. The ratio of charged moments between the excited and ground states takes the same universal form in the 1+1D massive Dirac fermion and 1+1D complex free boson. By employing the form factor program for composite branch-point twist fields, we computed the ratio of charged moments and from the latter we obtained exact expressions for the SREE of the excited states. Interestingly, these expressions can be written solely in terms of the SREE and symmetry-resolved partition function of the ground state. As we expected, the results obtained in free 1+1D QFTs are recovered using the qubit picture also in the symmetry-resolved case.

The greatest novelty of the work presented in this chapter consists in showing that our results, i.e. the formulae for the excess of SREE, apply much more broadly than the form factor computation would suggest: namely, they apply to any localised excitations of interacting and higher dimensional theories with $U(1)$ symmetry as long as the correlation length is much smaller than the typical subsystem size. As a first, simple example we considered one- and two-magnon states of quantum spin chains, with and without interaction. Their entanglement entropies can be computed analytically via free fermion techniques and we showed that, irrespective of interaction, the formulae presented in Section 2.1 still

apply, in line with observations made in [115, 150]. Furthermore, we proposed a general method to compute the charged moments of zero-density excited states of QFTs in higher dimensions with or without interactions. Our method is based on the notion of algebra of local observables of a theory, and on the commutation relations between composite twist operators -generalisations of CBPTFs to higher dimensions- and the local observables. The formulae already obtained for free 1+1D theories can then be derived in full generality by making few natural assumptions on the OPE of observables in the theory and on the correlation length of the latter.

Finally, we performed numerical tests of our results. For this, we considered a 1D Fermi gas and a complex harmonic chain. Although both the models are discrete and amenable to numerical computations, their microscopic features are quite different. Whereas the 1D Fermi gas is a fermionic theory which possesses a gapless phase, the complex harmonic chain is a gapped theory which becomes the complex massive free boson in continuous limit. It is therefore quite remarkable that the same set of formulae for the ratios of charged moments apply for both theories. This is nonetheless the case. Whereas for the 1D Fermi gas our formulae hold for highly excited states containing excitations of large momenta, thus small De Broglie wavelengths, for the complex harmonic charge the formulae hold as long as the correlation length is small compared to subsystem size.

In the following chapter, we will extend our investigation of zero-density excited states to another symmetry-resolved measure of entanglement, the logarithmic negativity [23, 116, 136]. In the conclusion of that chapter, we will also discuss some research directions that naturally emerge from our work.

2.A Complex free boson computation

In this appendix we present the form factor computation of the ratio of charged moments in detail, focusing on the complex free boson theory.

2.A.1 Single-particle excited states

The two-point function in (2.29) can be computed by inserting a sum over a complete set of states between the $U(1)$ fields as follows:

$$
\begin{aligned}
{}^n_L\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle^n_L = \sum_{\{N^+\}}\sum_{\{M^+\}} A^*_n(\{N^+\})\, A_n(\{M^+\}) \prod_{p=1}^{n}\sum_{m^\pm=0}^{\infty}\sum_{\{J^\pm\}}\prod_{j=1}^{m^+}\prod_{r=1}^{m^-}\frac{1}{m^+!\,m^-!} \\
\times {}^n_{p,L}\langle 0|[\mathfrak{a}_p\,(\theta)]^{N_p^+}\,\mathcal{T}_{p+\alpha}(0)\mathfrak{a}^\dagger_p(\theta^+_j)\mathfrak{b}^\dagger_p(\theta^-_r)|0\rangle^n_{p,L} \;\times\; {}^n_{p,L}\langle 0|\mathfrak{a}_p(\theta^+_j)\mathfrak{b}_p(\theta^-_r)\mathcal{T}_{-p-\alpha}(\ell)\,[\mathfrak{a}^\dagger_p(\theta)]^{M_p^+}\,|0\rangle^n_{p,L}\,,
\end{aligned}
\tag{2.182}
$$

and similarly for the $|1^-\rangle_L^n$ case. As shown in [115] following [159, 160], the finite-volume matrix elements involved in the expression above differ from the corresponding infinite-volume form factors for a term $\mathcal{O}(e^{-\mu L})$, with μ a characteristic mass scale, in this case $\mu = m$. Thus, we can rewrite the previous expression up to exponentially decaying corrections as

$$
\begin{aligned}
{}_L^n\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle_L^n = \sum_{\{N^+\}}\sum_{\{M^+\}} A_n^*(\{N^+\})\, A_n(\{M^+\}) \prod_{p=1}^{n}\sum_{m^\pm=0}^{\infty}\sum_{\{J^\pm\}} \frac{1}{m^+!\, m^-!} \qquad (2.183)\\
\times \frac{e^{i\ell\left(\sum_{j=1}^{m^+} P(\theta_j^+)+\sum_{r=1}^{m^-} P(\theta_r^-)-M_p^+ P(\theta)\right)}}{\sqrt{LE(\theta)}^{N_p^+ + M_p^+}\prod_{j=1}^{m^+} LE(\theta_j^+)\prod_{r=1}^{m^-} LE(\theta_r^-)} F^{p+\alpha,n}_{N_p^+ + m^+ + m^-}(\theta_1^+ \dots \theta_{m^+}^+, \hat\theta, \dots \hat\theta, \theta_1^- \dots \theta_{m^-}^-)\\
\times F^{n-p-\alpha,n}_{M_p^+ + m^+ + m^-}(\theta \dots \theta, \hat\theta_1^- \dots \hat\theta_{m^-}^-, \hat\theta_1^+ \dots \hat\theta_{m^+}^+),
\end{aligned}
$$

being $\hat\theta_j^\pm = \theta_j^\pm + i\pi$, and $E(\theta)$, $P(\theta)$ are the one-particle energy and momentum. The momenta are quantised according to the usual Bethe-Yang condition or to equation (2.31) for the intermediate rapidities $\theta_j^\pm$. The complete formula for the form factors above was given in [115] and they can be fully expressed as sums of products of two-particle form factors. They are non-vanishing for $N_p^+ = M_p^+ = m^+ - m^-$ and zero otherwise.

If the same intermediate rapidity θ_j^+ is paired up (in the Wick-contraction sense) with the rapidity of the excited state θ from the in- an out-states, the dominant contribution in the form factor product will come from kinematic poles. In other words, if $\theta_j^+ \simeq \theta$ two-particle form factors will appear as follows:

$$
\begin{aligned}
&F^{p+\alpha,n}_{N_p^+ + m^+ + m^-}(\theta_1^+ \dots \theta_{m^+}^+, \hat\theta \dots \hat\theta, \dots) \simeq N_p^+ f^n_{p+\alpha}(\theta_j^+ - \hat\theta)\\
&\quad \times F^{p+\alpha,n}_{N_p^+ + m^+ + m^- - 2}(\theta_1^+ \dots \theta_{j-1}^+ \theta_{j+1}^+ \dots \theta_{m^+}^+, \hat\theta \dots \hat\theta \dots)\\
&F^{n-p-\alpha,n}_{M_p^+ + m^+ + m^-}(\theta \dots \theta, \dots \hat\theta_1^+ \dots \hat\theta_{m^+}^+) \simeq M_p^+ f^n_{n-(p+\alpha)}(\hat\theta - \theta_j^+)\\
&\quad \times F^{n-p-\alpha,n}_{M_p^+ + m^+ + m^- - 2}(\theta \dots \theta, \dots \hat\theta_1^+ \dots \hat\theta_{j-1}^+ \hat\theta_{j+1}^+ \dots \hat\theta_{m^+}^+),
\end{aligned}
$$

where the number of rapidities $\hat\theta$ (θ) in the arguments of the form factors in the right-hand side are now $N_p^+ - 1$ ($M_p^+ - 1$). The main property of the matrix elements in (2.182) in determining the final formula for (2.29) is the infinite-volume limit of the terms such as

$$
\begin{aligned}
&\sum_{J^+\in\mathbb{Z}} \frac{f^n_{p+\alpha}(\theta^+ - \hat\theta) f^n_{n-p-\alpha}(\theta - \hat\theta^+) e^{i\ell(P(\theta^+) - P(\theta))}}{\cosh\theta \cosh\theta^+} \simeq\\
&(mL)^2 \sum_{J_i^+\in\mathbb{Z}} \frac{\sin^2\frac{\pi(p+\alpha)}{n}}{\pi^2} \frac{e^{2\pi i r(J^+ - I + \frac{p+\alpha}{n})}}{(J^+ - I + \frac{p+\alpha}{n})^2} = (mL)^2 g^n_{p+\alpha}(r), \qquad (2.184)
\end{aligned}
$$

with $g^n_{p+\alpha}(r)$ the functions defined in (2.41) and the indices J^+, I are integers resulting from the quantisation conditions of the rapidities of intermediate states (2.31) and of the rapidity of the

physical one-particle state respectively. We can proceed in an analogous way for (2.30) obtaining $(mL)^2 g^n_{-(p+\alpha)}(r)$ as the leading contribution.

Once all possible contractions with a rapidity of the excited in- and out- state have been carried out, the leading large-volume contribution from the summation over the quantum number J^+ is of order $\mathcal{O}(L^0)$ and comes from terms with $N = M$, as shown in Appendix B of [115]. This contribution reads

$$
{}^n_L\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle^n_L = \sum_{\{N^+\}} |A_n(\{N^+\})|^2 \prod_{p=1}^n N_p^+! \left[g^n_{p+\alpha}(r)\right]^{N_p^+} \prod_{q^+=0}^\infty \prod_{m^-=0}^\infty \frac{1}{q^+!\, m^-!} \sum_{\{J^\pm\}\in\mathbb{Z}}
$$
$$
\times \frac{e^{i\ell\left(\sum_{j=1}^{q^+} P(\theta_j^+)+\sum_{r=1}^{m^-} P(\theta_r^-)\right)}}{\prod_{j=1}^{q^+} L^2E(\theta_j^+)\prod_{r=1}^{m^-} L^2E(\theta_r^-)} F^{p,n}_{q^++m^-}(\theta_1^+\dots\theta_{q^+}^+,\theta_1^-\dots\theta_{m^-}^-)F^{n-p,n}_{q^++m^-}(\hat\theta_1^+\dots\hat\theta_{q^+}^+,\hat\theta_1^-\dots\hat\theta_{m^-}^-)
$$

with $q^+ = m^+ - N_p^+$. Dividing by the finite-volume vacuum two-point function:

$$
\prod_{p=1}^n {}^n_{p,L}\langle 0|\mathcal{T}_{p+\alpha}(0)\mathcal{T}_{-p-\alpha}(\ell)|0\rangle^n_{p,L},
$$

we obtain the formula (2.40) for the ratio of moments of the SREE for a one excitation state. We stress that the approximations made in considering the leading contributions become exact if the large-volume limit (2.1) is taken while keeping the mass m fixed.

2.A.2 Free boson ($k = 1$,$n = 2$)

In this Section we work out an example in detail. Consider a single particle excited state consisting of a complex boson excitation above the ground state. The relevant state is

$$
(a_1^+)^\dagger(\theta)(a_2^+)^\dagger(\theta)|0\rangle^2_L = \frac{1}{2}(-\mathfrak{a}_1^\dagger(\theta)+\mathfrak{a}_2^\dagger(\theta))(\mathfrak{a}_1^\dagger(\theta)+\mathfrak{a}_2^\dagger(\theta))|0\rangle^2_L, \tag{2.185}
$$

and comparing this expression to (2.27) we obtain $A_2(2,0) = -A_2(0,2) = -\frac{1}{2}$. Thus

$$
\begin{aligned}
M_n^{1^+}(r;\alpha) &= \frac{2!}{4}(g^2_{1+\alpha}(r)^2 + g^2_{2+\alpha}(r)^2) \\
&= \frac{1}{2}\left(1-r+re^{\pi i(1+\alpha)}\right)^2 + \frac{1}{2}\left(1-r+re^{\pi i(2+\alpha)}\right)^2 \\
&= \frac{1}{2}((1-r)^2 + r^2e^{2\pi i\alpha} - 2r(1-r)e^{i\pi\alpha} + (1-r)^2 + r^2e^{2\pi i\alpha} + 2r(1-r)e^{i\pi\alpha}) \\
&= (1-r)^2 + e^{2\pi i\alpha}r^2.
\end{aligned} \tag{2.186}
$$

2.A.3 Multi-particle excited states

Below, we describe in detail the computation of the ratio of moments of the SREE for a state consisting of k particle excitations with equal rapidities and charge signs. These states have the form:

$$|k^{\pm}\rangle_L^n = \prod_{j=1}^{n}(a_j^{\pm})^\dagger(\theta)|0\rangle_L^n = \frac{1}{(\sqrt{k!})^n}\sum_{\{N^\pm\}} D_n(\{N^\pm\})\prod_{p=1}^{n}\left[\left(\tilde{a}_p^{\pm}\right)^\dagger(\theta)\right]^{N_p^\pm}|0\rangle_L^n, \tag{2.187}$$

where in the second equality we used the expression of the creation operators in the diagonal basis (2.25), with the convenient notation $(\tilde{a}_p^+)^\dagger = \mathfrak{a}_p^\dagger$, $(\tilde{a}_p^-)^\dagger = \mathfrak{b}_p^\dagger$. After inserting a the resolution of the identity between the twist fields, the two-point function reads:

$$\begin{aligned}{}_L^n\langle k^{\pm}|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|k^{\pm}\rangle_L^n = \frac{1}{(k!)^n}\sum_{\{N^\pm\}}\sum_{\{M^\pm\}} D_n(\{N^+\})\, D_n^*(\{M^+\})\prod_{p=1}^{n}\sum_{m^\pm=0}^{\infty}\sum_{\{J^\pm\}}\prod_{j=1}^{m^+}\prod_{r=1}^{m^-}\frac{1}{m^+!\,m^-!}\\ \times{}_{p,L}^{n}\langle 0|[\tilde{a}_p^{\pm}(\theta)]^{N_p^\pm}\,\mathcal{T}_{p+\alpha}(0)\mathfrak{a}_p^\dagger(\theta_j^+)\mathfrak{b}_p^\dagger(\theta_r^-)|0\rangle_{p,L}^{n}\,{}_{p,L}^{n}\langle 0|\mathfrak{a}_p(\theta_j^+)\mathfrak{b}_p(\theta_r^-)\mathcal{T}_{-p-\alpha}(\ell)\,[(\tilde{a}_p^{\pm})^\dagger(\theta)]^{M_p^\pm}|0\rangle_{p,L}^{n},\end{aligned} \tag{2.188}$$

Employing the relation between these matrix elements and the finite-volume form factors, together with the action of the translation operator, we get:

$$\begin{aligned}{}_L^n\langle k^{\pm}|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|k^{\pm}\rangle_L^n = \frac{1}{(k!)^n}\sum_{\{N^+\}}\sum_{\{M^+\}} D_n(\{N^+\})\, D_n^*(\{M^+\})\prod_{p=1}^{n}\sum_{m^\pm=0}^{\infty}\sum_{\{J^\pm\}}\frac{1}{m^+!\,m^-!}\\ \times\frac{e^{i\ell\left(\sum_{j=1}^{m^+}P(\theta_j^+)+\sum_{r=1}^{m^-}P(\theta_r^-)-M_p^\pm P(\theta)\right)}}{\sqrt{LE(\theta)}^{N_p^\pm+M_p^\pm}\prod_{j=1}^{m^+}LE(\theta_j^+)\prod_{r=1}^{m^-}LE(\theta_r^-)}F^{n,p}_{N_p^\pm+m^++m^-}(\theta_1^+\dots\theta_{m^+}^+,\hat{\theta},\dots\hat{\theta},\theta_1^-\dots\theta_{m^-}^-)\\ \times F^{n-p,n}_{M_p^++m^++m^-}(\theta\dots\theta,\hat{\theta}_1^-\dots\hat{\theta}_{m^-}^-,\hat{\theta}_1^+\dots\hat{\theta}_{m^+}^+).\end{aligned}$$

Once all possible intermediate rapidities have been paired up with the same rapidity of the excited state in both form factors and the contribution of the ground state has been factored, the leading large-volume contribution of the ratio of moments can be written as:

$$\begin{aligned}M_n^{k^\pm}(r;\alpha) &= \frac{1}{(k!)^n}\sum_{\{N^\pm\}}|D_n(\{N^\pm\})|^2\prod_{p=1}^{n}N_p^\pm!\,\left(g^n_{\pm(p+\alpha)}(r)\right)^{N_p^\pm}\\ &= \sum_{p=0}^{k}\left[\binom{k}{p}r^p(1-r)^{k-p}\right]^n e^{\pm 2\pi i\alpha p}.\end{aligned} \tag{2.189}$$

On the other hand, if the k-particle excitations have distinct rapidities there can be two different cases:

- k-particle excitations with distinct rapidities but equal charge sign
- k-particle excitations with distinct rapidities and charge sign

We can summarize the computations for both cases if we consider the following ansatz for the excited state:

$$|1^{\epsilon_1}\,1^{\epsilon_2}\dots 1^{\epsilon_k}\rangle_L^n = \prod_{i=1}^{k}\prod_{j=1}^{n}(a_j^{\epsilon_i})^\dagger(\theta_i)|0\rangle_L^n = \sum_{\{N^\pm\}} C_n(\{N^\pm\})\prod_{s=1}^{k}\prod_{p=1}^{n}[\mathfrak{a}_p^\dagger(\theta_s)]^{N_{p,s}^+}[\mathfrak{b}_p^\dagger(\theta_s)]^{N_{p,s}^-}|0\rangle_L^n, \quad (2.190)$$

where each ϵ_i with $i = 1,\dots,k$ can be either $+$ or $-$. If we consider k-particle excitations with distinct rapidities but equal charge sign $+$ $(-)$ then all the ϵ_i are the same sign and $N_{p,s}^-$ $(N_{p,s}^+)$ vanish. When pairing up the intermediate rapidities with the same rapidity of the excited state in both form factors, the conditions for the matrix elements to be non vanishing are:

$$\begin{aligned} N_{p,s}^- + m^+ &= N_{p,s}^+ + m^-, \\ m^- + M_{p,s}^+ &= m^+ + M_{p,s}^-. \end{aligned}$$

Subtracting the contribution of the ground state, the leading large-volume contribution to the ratio of moments can be written as:

$$M_n^{1^{\epsilon_1}\,1^{\epsilon_2}\dots 1^{\epsilon_k}}(r;\alpha) = \prod_{s=1}^{k}\left[\sum_{\{N^\pm\}}|C_n(\{N^\pm\})|^2\prod_{p=1}^{n}N_{p,s}^+!\;N_{p,s}^-!\;[g_{p+\alpha}^n(r)]^{N_{p,s}^+}\,[g_{-p-\alpha}^n(r)]^{N_{p,s}^-}\right]. \quad (2.191)$$

Notice that if we study k-particle excitations with distinct rapidities but equal charge sign, the above expression reduces to:

$$M_n^{1^\pm\,1^\pm\dots 1^\pm}(r;\alpha) = \prod_{s=1}^{k}\left[\sum_{\{N^\pm\}}|C_n(\{N^\pm\})|^2\prod_{p=1}^{n}N_{p,s}^\pm!\;[g_{\pm(p+\alpha)}^n(r)]^{N_{p,s}^\pm}\right], \quad (2.192)$$

with the conditions $m^\mp + M_{p,s}^\pm = m^\pm = N_{p,s}^\pm + m^\mp$ for the $\pm$ sign state.

2.A.4 Free boson ($k = 2$,$n = 2$)

As an example, consider the following two-particle excited states with distinct rapidities:

$$\begin{aligned} |1^+1^+\rangle_L^2 &= (a_1^+)^\dagger(\theta_1)(a_1^+)^\dagger(\theta_2)(a_2^+)^\dagger(\theta_1)(a_2^+)^\dagger(\theta_2)|0\rangle_L^2 = \frac{1}{4}\Big([\mathfrak{a}_1^\dagger(\theta_1)]^2\,[\mathfrak{a}_1^\dagger(\theta_2)]^2 \\ &+ [\mathfrak{a}_2^\dagger(\theta_1)]^2\,[\mathfrak{a}_2^\dagger(\theta_2)]^2 - [\mathfrak{a}_1^\dagger(\theta_2)]^2\,[\mathfrak{a}_2^\dagger(\theta_1)]^2 - [\mathfrak{a}_1^\dagger(\theta_1)]^2\,[\mathfrak{a}_2^\dagger(\theta_2)]^2\Big)\;|0\rangle_L^2, \quad (2.193) \\ |1^+1^-\rangle_L^2 &= (a_1^+)^\dagger(\theta_1)(a_1^-)^\dagger(\theta_2)(a_2^+)^\dagger(\theta_1)(a_2^-)^\dagger(\theta_2)|0\rangle_L^2 = \frac{1}{4}\Big([\mathfrak{a}_1^\dagger(\theta_1)]^2\,[\mathfrak{b}_1^\dagger(\theta_2)]^2 \\ &+ [\mathfrak{a}_2^\dagger(\theta_1)]^2\,[\mathfrak{b}_2^\dagger(\theta_2)]^2 - [\mathfrak{a}_2^\dagger(\theta_1)]^2\,[\mathfrak{b}_1^\dagger(\theta_2)]^2 - [\mathfrak{a}_1^\dagger(\theta_1)]^2\,[\mathfrak{b}_2^\dagger(\theta_2)]^2\Big)\;|0\rangle_L^2. \quad (2.194) \end{aligned}$$

The charged ratios for these excited states are given by

$$\begin{aligned} M_2^{1^+1^+}(r;\alpha) &= \frac{1}{4}\left([g^2_{1+\alpha}(r)]^4+[g^2_{2+\alpha}(r)]^4+2\,[g^2_{1+\alpha}(r)]^2\,[g^2_{2+\alpha}(r)]^2)\right) \\ &= \left((1-r)^2+r^2e^{2\pi i\alpha}\right)^2, \qquad (2.195) \\ M_2^{1^+1^-}(r;\alpha) &= \frac{1}{4}\left([g^2_{1+\alpha}(r)]^2\,[g^2_{-1-\alpha}(r)]^2+[g^2_{2+\alpha}(r)]^2\,[g^2_{-2-\alpha}(r)]^2+[g^2_{1+\alpha}(r)]^2\,[g^2_{-2-\alpha}(r)]^2\right. \\ &+ \left.[g^2_{-1-\alpha}(r)]^2\,[g^2_{2+\alpha}(r)]^2\right) = \left((1-r)^2+r^2e^{2\pi i\alpha}\right)\left((1-r)^2+r^2e^{-2\pi i\alpha}\right). \qquad (2.196) \end{aligned}$$

On the other hand, if we consider ta two-particle excited state with coinciding rapidities:

$$\begin{aligned} |2^-\rangle^2_L &= [(a_1^-)^\dagger(\theta)]^2\,[(a_2^-)^\dagger(\theta)]^2|0\rangle^2_L \\ &= \frac{1}{8}\left([\mathfrak{b}_1^\dagger(\theta)]^4+[\mathfrak{b}_2^\dagger(\theta)]^4-2[\mathfrak{b}_1^\dagger(\theta)]^2\,[\mathfrak{b}_2^\dagger(\theta)]^2\right)|0\rangle^2_L, \qquad (2.197) \end{aligned}$$

the charged moment is

$$\begin{aligned} M_2^{2^-}(r;\alpha) &= \frac{1}{8^2}(4!\,[g^2_{-1-\alpha}(r)]^4+4!\,[g^2_{-2-\alpha}(r)]^4+4^2\,[g^2_{-1-\alpha}(r)]^2\,[g^2_{-2-\alpha}(r)]^2) \\ &= (1-r)^4+4\,r^2(1-r)^2e^{-2\pi i\alpha}+e^{-4\pi i\alpha}r^4. \qquad (2.198) \end{aligned}$$

2.B Complex free fermion computation

In this appendix we present the form factor computation of the ratio of charged moments in detail, focusing on the complex free fermion theory.

2.B.1 Single-particle excited states

Below, we present the explicit computation of the fermionic two-point function in an excited state consisting of a single positively-charged particle. Thanks to the factorisation (2.22), the latter can be cast as:

$$\begin{aligned} &{}^n_L\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle^n_L \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} {}^n_{p,L}\langle 0|\mathfrak{a}_p\,(\theta)\;\mathcal{T}_{p+\alpha}(0)\;\mathcal{T}_{-p-\alpha}(\ell)\;\mathfrak{a}_p^\dagger(\theta)\;|0\rangle^n_{p,L} \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{s=0}^{\infty}\sum_{\{J_i^\pm\}}\frac{1}{s!(s+1)!}{}^n_{p,L}\langle 0|\mathfrak{a}_p\,(\theta)\;\mathcal{T}_{p+\alpha}(0)\;\mathfrak{a}_p^\dagger(\theta_1)\dots\mathfrak{a}_p^\dagger(\theta_{s+1})\mathfrak{b}_p^\dagger(\theta_{s+2})\dots\mathfrak{b}_p^\dagger(\theta_{2s+1})|0\rangle^n_{p,L} \\ &\times {}^n_{p,L}\langle 0|\mathfrak{a}_p\,(\theta_1)\;\dots\mathfrak{a}_p(\theta_{s+1})\mathfrak{b}_p(\theta_{s+2})\dots\mathfrak{b}_p(\theta_{2s+1})\mathcal{T}_{-p-\alpha}(0)\;\mathfrak{a}_p^\dagger(\theta)|0\rangle^n_{p,L}e^{i\ell(\sum_{i=1}^{2s+1}P(\theta_i)-P(\theta))} \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{s=0}^{\infty}\sum_{\{J_i^\pm\}}\frac{|F^{p+\alpha,n}_{2s+2}(\theta_1,\dots,\theta_{s+1};\theta+i\pi,\theta_{s+2},\dots,\theta_{2s+1})|^2}{s!(s+1)!LE(\theta)\prod_{i=1}^{2s+1}(\theta_i)LE(\theta_i)}e^{i\ell(\sum_{i=1}^{2s+1}P(\theta_i)-P(\theta))}, \qquad (2.199) \end{aligned}$$

where the resolution of the identity is inserted in such a way as to preserve the total charge of the one-particle state and the Bethe quantum numbers $\{J_i^{\pm}\}$ are defined as in (2.31). Notice that since the excitations are fermionic, one could either have $J_i^{\pm} \in \mathbb{Z}$ or $J_i^{\pm} \in \mathbb{Z} + \frac{1}{2}$: for the sake of simplicity, we consider the case where these numbers are integer, as this assumption does not make a difference in the final result.

The non-vanishing contributions in the large-volume limit come from the terms in the previous expression in which the rapidity of the excited state is contracted with θ_i, $i = 1, \dots, s+1$ in both form factors. The $s+1$ possible contractions in $F_{2s+2}^{p+\alpha,n}$ give rise to:

$$\begin{aligned} &F_{2s+2}^{p+\alpha,n}(\theta_1, \dots, \theta_{s+1}; \hat{\theta}, \theta_{s+2}, \dots, \theta_{2s+1}) \\ &\simeq f_{p+\alpha}^n(\theta_i - \hat{\theta})\, F_{2s}^{p+\alpha,n}(\theta_1, \dots, \check{\theta}_i, \dots, \theta_{s+1}; \theta_{s+2}, \dots, \theta_{2s+1}), \end{aligned} \tag{2.200}$$

where around the pole:

$$f_{p+\alpha}^n(\theta_i - \hat{\theta}) \underset{\theta \simeq \theta_i}{\simeq} \frac{mL \sin\frac{\pi(p+\alpha)}{n} \cosh\theta\, e^{\frac{i\pi(p+\alpha)}{n}}}{\pi(J_i^+ - I + \frac{p+\alpha}{n})}. \tag{2.201}$$

Thus, considering also the contraction coming from the form factor of $\mathcal{T}_{-p-\alpha}$, we can separately perform the $s+1$ summations over the quantum numbers J_i^+ as

$$\begin{aligned} &\sum_{J_i^+ \in \mathbb{Z}} \frac{|f_{p+\alpha}^n(\theta_i - \hat{\theta})|^2 e^{i\ell(P(\theta_i) - P(\theta))}}{Lm\cosh\theta\, Lm\cosh\theta_i} \\ &\underset{\theta \simeq \theta_i}{\simeq} \sum_{J_i^+ \in \mathbb{Z}} \frac{\sin^2\frac{\pi(p+\alpha)}{n} e^{2\pi i r\left(J_i^+ - I + \frac{p+\alpha}{n}\right)}}{\pi^2(J_i^+ - I + \frac{p+\alpha}{n})^2} = g_{p+\alpha}^n(r). \end{aligned} \tag{2.202}$$

We therefore obtain, in the limit (2.1) and for fixed m:

$$\begin{aligned} &{}_L^n\langle 1^+|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^+\rangle_L^n \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} g_{p+\alpha}^n(r) \sum_{s=0}^{\infty} \frac{1}{(s!)^2} \sum_{\{J_i^\pm\}} |F_{2s}^{p+\alpha,n}(\theta_1, \dots, \theta_s; \beta_1, \dots, \beta_s)|^2 \frac{e^{i\ell\sum_{i=1}^s (P(\theta_i) + P(\beta_i))}}{\prod_{i=1}^s LE(\theta_i) LE(\beta_i)} \\ &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} g_{p+\alpha}^n(r) \times {}_{p,L}^n\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|0\rangle_{p,L}^n, \end{aligned} \tag{2.203}$$

where we re-labeled the rapidities of the negatively charged intermediate states as $\beta_i = \theta_{s+i+1}$ for $i = 1, \dots, s$. We can now make use of (2.53) in the evaluation of the ratio, so that we finally obtain

for the free fermionic one-particle states:

$$M_n^{1^+}(r;\alpha) = \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} g_{p+\alpha}^n(r) = (1-r)^n + e^{2\pi i\alpha} r^n. \tag{2.204}$$

An analogous result can be obtained for a negatively charged particle, where the phase above picks up an extra minus sign.

2.B.2 Multi-particle excited states

The anticommuting nature of the creation/annihilation operators allows us to obtain an exact expression for the ratio of the charged moments in the fermionic case, which (unlike for the free boson) does not require a case-by-case calculation. This is because in the free fermion case, the structure of the states in the transformed base is extremely simple, as we shall see. We have non-vanishing two-point functions only with two kind of states:

- k-particle excitations with distinct rapidities, irrespective of the charge signs;
- 2-particle excitations with equal rapidities and different charge signs: $|1^+1^-\rangle_L^n$.

Below, we consider in detail the case of k-particle states with distinct rapidities. Such states are written exactly as in the bosonic case:

$$|1^{\epsilon_1}\, 1^{\epsilon_2} \dots 1^{\epsilon_k}\rangle_L^n = \prod_{i=1}^{k} \prod_{j=1}^{n} (a_j^{\epsilon_i})^\dagger(\theta_i)|0\rangle_L^n, \tag{2.205}$$

where $\epsilon_i = \pm 1$, $\theta_i \neq \theta_{i'}$ if $i \neq i'$. Unlike the bosonic case, however, all the operators anticommute, so that we can make the ansatz:

$$|1^{\epsilon_1}\, 1^{\epsilon_2} \dots 1^{\epsilon_k}\rangle_L^n = e^{i\kappa} \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \prod_{i=1}^{k} (\mathfrak{a}_p^{\epsilon_i})^\dagger(\theta_i)|0\rangle_L^n, \tag{2.206}$$

with the identification $(\mathfrak{a}_p^+)^\dagger(\theta_i) = \mathfrak{a}_p^\dagger(\theta_i)$, $(\mathfrak{a}_p^-)^\dagger(\theta_i) = \mathfrak{b}_p^\dagger(\theta_i)$ and the only unspecified parameter is the phase $\kappa = \kappa(k, n; \{\epsilon_i\})$. Notice that the order of the operators in the double product can be arbitrarily altered, resulting only in a change in the phase. Without giving a full proof of the validity of this formula, let us consider a few simple cases and introduce the notations $k^\pm$ to indicate the number of positively/negatively charged excitations in the state, with $k = k^+ + k^-$:

- $n = 2$, $k^+ = 2$:

$$
\begin{aligned}
|1^+1^+\rangle_L^2 &= \prod_{i=1}^{2}\prod_{j=1}^{2}\frac{1}{\sqrt{2}}\sum_{p=-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{2\pi i j p}{2}}\mathfrak{a}_p^\dagger(\theta_i)|0\rangle_L^2 \\
&= \frac{1}{4}\prod_{i=1}^{2}(i\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta_i) - i\mathfrak{a}^\dagger_{\frac{1}{2}}(\theta_i))(-\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta_i) - \mathfrak{a}^\dagger_{\frac{1}{2}}(\theta_i))|0\rangle_L^2 \\
&= \frac{1}{4}\prod_{i=1}^{2}(-2i\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta_i)\mathfrak{a}^\dagger_{\frac{1}{2}}(\theta_i))|0\rangle_L^2 = -\prod_{p=-\frac{1}{2}}^{\frac{1}{2}}\prod_{i=1}^{2}\mathfrak{a}_p^\dagger(\theta_i)|0\rangle_L^2
\end{aligned}
$$

- $n = 2$, $k^+ = 2$, $k^- = 1$:

$$
\begin{aligned}
|1^+1^+1^-\rangle_L^2 &= \left(\prod_{i=1}^{2}\prod_{j=1}^{2}\frac{1}{\sqrt{2}}\sum_{p=-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{2\pi i j p}{2}}\mathfrak{a}_p^\dagger(\theta_i)\right)\prod_{j=1}^{2}\frac{1}{\sqrt{2}}\sum_{p=-\frac{1}{2}}^{\frac{1}{2}} e^{\frac{2\pi i j p}{2}}\mathfrak{b}_p^\dagger(\theta_3)|0\rangle_L^2 \\
&= \left(\frac{1}{4}\prod_{i=1}^{2}(-2i\mathfrak{a}^\dagger_{-\frac{1}{2}}(\theta_i)\mathfrak{a}^\dagger_{\frac{1}{2}}(\theta_i))\right)\frac{1}{2}(2i\mathfrak{b}^\dagger_{-\frac{1}{2}}(\theta_3)\mathfrak{b}^\dagger_{\frac{1}{2}}(\theta_3))|0\rangle_L^2 \\
&= -i\prod_{p=-\frac{1}{2}}^{\frac{1}{2}}\left(\prod_{i=1}^{2}\mathfrak{a}_p^\dagger(\theta_i)\right)\mathfrak{b}_p^\dagger(\theta_3)|0\rangle_L^2
\end{aligned}
$$

- $n = 3$, $k^+ = 2$:

$$
\begin{aligned}
|1^+1^+\rangle_L^3 &= \prod_{i=1}^{2}\prod_{j=1}^{3}\frac{1}{\sqrt{3}}\sum_{p=-1}^{1} e^{-\frac{2\pi i j p}{3}}\mathfrak{a}_p^\dagger(\theta_i)|0\rangle_L^3 \\
&= \prod_{i=1}^{2}\frac{1}{3^{\frac{3}{2}}}\left(e^{\frac{2\pi i}{3}}\mathfrak{a}^\dagger_{-1}(\theta_i) + \mathfrak{a}^\dagger_0(\theta_i) + e^{-\frac{2\pi i}{3}}\mathfrak{a}^\dagger_1(\theta_i)\right) \\
&\times\left(e^{\frac{4\pi i}{3}}\mathfrak{a}^\dagger_{-1}(\theta_i) + \mathfrak{a}^\dagger_0(\theta_i) + e^{-\frac{4\pi i}{3}}\mathfrak{a}^\dagger_1(\theta_i)\right)(\mathfrak{a}^\dagger_{-1}(\theta_i) + \mathfrak{a}^\dagger_0(\theta_i) + \mathfrak{a}^\dagger_1(\theta_i))|0\rangle_L^3 \\
&= -\prod_{p=-1}^{1}\prod_{i=1}^{2}\mathfrak{a}_p^\dagger(\theta_i)|0\rangle_L^3.
\end{aligned}
$$

For a fixed value of n, the structure of more complicated states can be easily worked out following these simple examples. Using equation (2.206) and the twist field factorisation (2.22), the two-point

function reads:

$$
\begin{aligned}
&{}_L^n\langle 1^{\epsilon_1}\,1^{\epsilon_2}\dots 1^{\epsilon_k}|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^{\epsilon_1}\,1^{\epsilon_2}\dots 1^{\epsilon_k}\rangle_L^n \\
&= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} {}_{p,L}^n\langle 0|\mathfrak{a}_p(\theta_1)\dots\mathfrak{a}_p(\theta_{k^+})\mathfrak{b}_p(\beta_1)\dots\mathfrak{b}_p(\beta_{k^-})\mathcal{T}_{p+\alpha}(0)\mathcal{T}_{-p-\alpha}(\ell)\mathfrak{a}_p^\dagger(\theta_1)\dots\mathfrak{a}_p^\dagger(\theta_{k^+})\mathfrak{b}_p^\dagger(\beta_1)\dots\mathfrak{b}_p^\dagger(\beta_{k^-})|0\rangle_{p,L}^n \\
&= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{s=0}^{\infty}\sum_{\{J_i^\pm\}} \frac{1}{s!(s+q)!}\, e^{i\ell\left(\sum_{i=1}^{q+s}P(\tilde\theta_i)+\sum_{i=1}^{s}P(\tilde\beta_i)-\sum_{i=1}^{k^+}P(\theta_i)-\sum_{i=1}^{k^-}P(\beta_i)\right)} \\
&\quad\times {}_{p,L}^n\langle 0|\mathfrak{a}_p(\theta_1)\dots\mathfrak{a}_p(\theta_k^+)\mathfrak{b}_p(\beta_1)\dots\mathfrak{b}_p(\beta_{k^-})\mathcal{T}_{p+\alpha}(0)\mathfrak{a}_p^\dagger(\tilde\theta_1)\dots\mathfrak{a}_p^\dagger(\tilde\theta_{q+s})\mathfrak{b}_p^\dagger(\tilde\beta_1)\dots\mathfrak{a}_p^\dagger(\tilde\beta_s)|0\rangle_{p,L}^n \\
&\quad\times {}_{p,L}^n\langle 0|\mathfrak{a}_p(\tilde\theta_1)\dots\mathfrak{a}_p(\tilde\theta_{q+s})\mathfrak{b}_p(\tilde\beta_1)\dots\mathfrak{b}_p(\tilde\beta_s)\mathcal{T}_{-p-\alpha}(0)\mathfrak{a}_p^\dagger(\theta_1)\dots\mathfrak{a}_p^\dagger(\theta_{k^+})\mathfrak{b}_p^\dagger(\beta_1)\dots\mathfrak{b}_p^\dagger(\beta_{k^-})|0\rangle_{p,L}^n. \qquad (2.207)
\end{aligned}
$$

In the expansion above we assumed the total charge of the excited state to be positive, $q := k^+ - k^- > 0$. However, the computation steps are unchanged if one assumes $q < 0$, the only difference being in the structure of the resolution of the identity. Denoting $\hat{x} := x + i\pi$, the infinite-volume form factor corresponding to the first matrix element reads:

$$
F^{p+\alpha,n}_{q+2s+k}(\tilde\theta_1,\dots,\tilde\theta_{q+s},\hat\beta_1,\dots,\hat\beta_{k^-};\tilde\beta_1,\dots,\tilde\beta_s,\hat\theta_1,\dots,\hat\theta_{k^+}), \qquad (2.208)
$$

where the total charge conservation is ensured by the equality $q+s+k^- = s+k^+$. When turning to the finite-volume, one needs to divide the previous infinite-volume form factor by a quantity:

$$
\left[\prod_{i=1}^{q+s} LE(\tilde\theta_i)\prod_{i=1}^{k^-} LE(\beta_i)\prod_{i=1}^{s} LE(\tilde\beta_i)\prod_{i=1}^{k^+} LE(\theta_i)\right]^{\frac{1}{2}}. \qquad (2.209)
$$

Taking into account also the contribution coming from the other form factor, this results into a factor of order $\mathcal{O}(L^{-q-2s-k})$ for every term in the expansion (2.207), and the latter reads:

$$
\begin{aligned}
&\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{s=0}^{\infty}\sum_{\{J_i^\pm\}} \frac{1}{s!(s+q)!}\, e^{i\ell\left(\sum_{i=1}^{q+s}P(\tilde\theta_i)+\sum_{i=1}^{s}P(\tilde\beta_i)-\sum_{i=1}^{k^+}P(\theta_i)-\sum_{i=1}^{k^-}P(\beta_i)\right)} \\
&\times \frac{|F^{p+\alpha,n}_{q+2s+k}(\tilde\theta_1,\dots,\tilde\theta_{q+s},\hat\beta_1,\dots,\hat\beta_{k^-};\tilde\beta_1,\dots,\tilde\beta_s,\hat\theta_1,\dots,\hat\theta_{k^+})|^2}{\prod_{i=1}^{q+s} LE(\tilde\theta_i)\prod_{i=1}^{k^-} LE(\beta_i)\prod_{i=1}^{s} LE(\tilde\beta_i)\prod_{i=1}^{k^+} LE(\theta_i)}. \qquad (2.210)
\end{aligned}
$$

In the large-volume limit, the leading contributions are those coming from simultaneous contractions in both form factors. In turn, in each form factor the simple poles arise from the pairings of the rapidities $\hat\theta_i$ with $\tilde\theta_i$ (these are at most k^+ contractions) and from the pairings of the rapidities $\hat\beta_i$ with $\tilde\beta_i$ (these are at most k^- contractions). Again, these pairings have to be made simultaneously. The terms with $s < k^-$ (or equivalently $q+s < k^+$) do not contribute in this limit, as they contain some extra factors of L in the denominator. On the other hand, the terms with $s > k^-$ contain a sum of

$k^+!k^-!$ products of the form:

$$|f^n_{p+\alpha}(\tilde\theta_i-\hat\theta_j)f^n_{p+\alpha}(\hat\beta_{i'}-\tilde\beta_{j'})|^2\times\text{ residual form factors.}$$

By making use of (2.201), the simultaneous expansion around the poles leads to the following leading contribution:

$$\begin{aligned}
&\sum_{J_i^+,J_{j'}^-}\frac{|f^n_{p+\alpha}(\tilde\theta_i-\hat\theta_j)f^n_{p+\alpha}(\hat\beta_{i'}-\tilde\beta_{j'})|^2}{\cosh\tilde\theta_i\cosh\theta_i\cosh\beta_{i'}\cosh\tilde\beta_{j'}}\,e^{i\ell[P(\tilde\theta_i)-P(\theta_j)+P(\tilde\beta_{j'})-P(\beta_{i'})]}\\
&\simeq(mL)^4\sum_{J_i^+\in\mathbb{Z}}\frac{\sin^2\frac{\pi(p+\alpha)}{n}}{\pi^2}\frac{e^{2\pi ir(J_i^+-I_j+\frac{p+\alpha}{n})}}{(J_i^+-I_j+\frac{p+\alpha}{n})^2}\sum_{J_{j'}^-\in\mathbb{Z}}\frac{\sin^2\frac{\pi(p+\alpha)}{n}}{\pi^2}\frac{e^{2\pi ir(J_{j'}^--I_{i'}-\frac{p+\alpha}{n})}}{(J_{j'}^--I_{i'}-\frac{p+\alpha}{n})^2}\\
&=(mL)^4g^n_{p+\alpha}(r)g^n_{-p-\alpha}(r).
\end{aligned}\tag{2.211}$$

Since each of the $k^+!k^-!$ terms in the sum contains exactly k^+ functions $g^n_{p+\alpha}$ and k^- functions $g^n_{-p-\alpha}$ we have, relabelling $m=s-k^-$:

$$\begin{aligned}
&{}^n_L\langle 1^{\epsilon_1}1^{\epsilon_2}\dots1^{\epsilon_k}|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|1^{\epsilon_1}1^{\epsilon_2}\dots1^{\epsilon_k}\rangle^n_L\\
&=\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{m=0}^{\infty}\sum_{\{J_i^\pm\}}\frac{1}{(m!)^2}(g^n_{p+\alpha}(r))^{k^+}(g^n_{-p-\alpha}(r))^{k^-}\\
&\times e^{i\ell\sum_{i=1}^m(P(\tilde\theta_i)+P(\tilde\beta_i))}\,\frac{|F^{p+\alpha,n}_{2m}(\tilde\theta_1,\dots,\tilde\theta_m;\tilde\beta_1,\dots,\tilde\beta_m)|^2}{\prod_{i=1}^m LE(\tilde\theta_i)LE(\tilde\beta_i)}\\
&=\left(\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}g^n_{p+\alpha}(r)\right)^{k^+}\left(\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}g^n_{-p-\alpha}(r)\right)^{k^-}{}^n_L\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|0\rangle^n_L,
\end{aligned}\tag{2.212}$$

and therefore, thanks to (2.22), the ratio of charged moments is:

$$M_n^{1^\pm1^\pm\dots1^\pm}(r;\alpha)=((1-r)^n+e^{2\pi i\alpha}r^n)^{k^+}((1-r)^n+e^{-2\pi i\alpha}r^n)^{k^-}.\tag{2.213}$$

2.C Finite-volume two-point function in the ground state

In this appendix we investigate the large-volume expansion of the correlator

$${}^n_L\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(r\,L)|0\rangle^n_L,\tag{2.214}$$

which is the denominator of the ratio of charged moments $M_n^\Psi(r;\alpha)$ in (2.18). We show that, as expected, the leading contribution in the limit (2.1) is given by the squared modulus of the vacuum expectation value of the composite twist field $\mathcal{T}^\alpha$, and we compute the first finite-volume corrections

to this quantity. The calculations are carried out in the fermionic case, but they apply to the free boson case with few changes.

The first step in the evaluation of (2.214) is as usual the insertion of a projection onto asymptotic states:

$$\begin{aligned}
{}^n_L\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|0\rangle^n_L &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} {}^n_{p,L}\langle 0|\mathcal{T}_{p+\alpha}(0)\ \mathcal{T}_{-p-\alpha}(\ell)\ |0\rangle^n_{p,L} \\
&= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{s=0}^{\infty} \sum_{\{J_i^\pm\}} \frac{1}{(s!)^2} {}^n_{p,L}\langle 0|\mathcal{T}_{p+\alpha}(0)\ \mathfrak{a}_p^\dagger(\theta_1)\dots\mathfrak{a}_p^\dagger(\theta_s)\mathfrak{b}_p^\dagger(\beta_1)\dots\mathfrak{b}_p^\dagger(\beta_s)|0\rangle^n_{p,L} \\
&\quad \times {}^n_{p,L}\langle 0|\mathfrak{a}_p\ (\theta_1)\ \dots\mathfrak{a}_p(\theta_s)\mathfrak{b}_p(\beta_1)\dots\mathfrak{b}_p(\beta_s)\mathcal{T}_{-p-\alpha}(0)|0\rangle^n_{p,L} e^{i\ell\sum_{i=1}^s (P(\theta_i)+P(\beta_i))} \\
&= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \sum_{s=0}^{\infty} \sum_{\{J_i^\pm\}} \frac{|F_{2s}^{p+\alpha,n}(\theta_1,\dots,\theta_s;\beta_1,\dots,\beta_s)|^2}{(s!)^2\prod_{i=1}^s L^2E(\theta_i)E(\beta_i)} e^{i\ell\sum_{i=1}^s (P(\theta_i)+P(\beta_i))}. \qquad (2.215)
\end{aligned}$$

Notice that the equal number of particles of the two types is dictated by the fact that twist fields preserve the total charge of the state. In a free fermion theory, the infinite-volume form factor of an even number of particle is given by Wick's theorem:

$$F_{2s}^{p+\alpha,n}(\theta_1,\dots,\theta_s;\beta_1,\dots,\beta_s) = \sum_{\sigma\in\mathcal{P}_s} \tau_{p+\alpha}\ \mathrm{sgn}\,\sigma \prod_{i=1}^s f_{p+\alpha}^n(\theta_{\sigma(i)}-\beta_i), \qquad (2.216)$$

where the normalised fermionic two-particle form factor is obtained diving (2.45) by the VEV $\tau_{p+\alpha}$. Thus, the squared modulus of the $2s$-particle form factor is a sum of $(s!)^2$ terms, each of which is a product of s terms of the type

$$f_{p+\alpha}^n(\theta_i-\beta_j)f_{p+\alpha}^n(\theta_k-\beta_j)^* = \sin^2\frac{\pi(p+\alpha)}{n}\frac{e^{\frac{p+\alpha}{n}(\theta_i+\theta_k-2\beta_j)}}{\cosh\frac{\theta_i-\beta_j}{2}\cosh\frac{\theta_k-\beta_j}{2}}. \qquad (2.217)$$

We can therefore explicitly rewrite (2.215) as:

$$\begin{aligned}
{}^n_L\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|0\rangle^n_L &= \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} |\tau_{p+\alpha}|^2 \sum_{s=0}^{\infty} \frac{\sin^{2s}\frac{\pi(p+\alpha)}{n}}{(s!)^2(mL)^{2s}} \sum_{\{J_i^\pm\}}\sum_{\sigma,\omega\in\mathcal{P}_s} \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\omega \\
&\quad \times \prod_{i=1}^s \frac{e^{i\ell(P(\theta_i)+P(\beta_i))}}{\cosh\theta_i\cosh\beta_i} \frac{e^{\frac{p+\alpha}{n}(\theta_{\sigma(i)}+\theta_{\omega(i)}-2\beta_i)}}{\cosh\frac{\theta_{\sigma(i)}-\beta_i}{2}\cosh\frac{\theta_{\omega(i)}-\beta_i}{2}}. \qquad (2.218)
\end{aligned}$$

From this expression we easily see that the vacuum expectation value is corrected by contributions of multi-particle states, and that in general every $2s$-particle state (containing s particles with positive charge and s particles with negative charge) contributes with a leading large-volume term $\mathcal{O}((mL)^{-2s})$. Further corrections can be obtained by working out how the product in the second line of (2.218)

depends on L. This is done by solving the Bethe equations for $\sinh\theta_i$ and $\sinh\beta_i$:

$$\sinh\theta_i = \frac{2\pi\left(J_i^+ + \frac{p+\alpha}{n}\right)}{mL} := \frac{c_i^+}{mL}, \tag{2.219}$$

$$\sinh\beta_i = \frac{2\pi\left(J_i^- - \frac{p+\alpha}{n}\right)}{mL} := \frac{c_i^-}{mL}. \tag{2.220}$$

Where either $J_i^+, J_i^- \in \mathbb{Z}$ or $J_i^+, J_i^- \in \mathbb{Z} + \frac{1}{2}$. Let us consider in detail the expansion up to $s = 1$ terms, assuming that $J_1^+, J_1^- \in \mathbb{Z}$. Some elementary algebra shows that:

$$\frac{e^{2(\theta_1-\beta_1)\frac{p+\alpha}{n}}}{\cosh^2\frac{\theta_1-\beta_1}{2}} = 1 + 2\left(\frac{p+\alpha}{n}\right)\frac{c_1^+ - c_1^-}{mL} + \mathcal{O}\left(\frac{1}{(mL)^2}\right) \tag{2.221}$$

and:

$$\frac{e^{i\ell(P(\theta_1)+P(\beta_1))}}{\cosh\theta_1\cosh\beta_1} = e^{2\pi i r(J_1^+ + J_1^-)}\left(1 - \frac{(c_1^+)^2 + (c_1^-)^2}{2(mL)^2} + \mathcal{O}\left(\frac{1}{(mL)^4}\right)\right). \tag{2.222}$$

Thus we have:

$$\begin{aligned}
&{}_L^n\langle 0|\mathcal{T}^\alpha(0)\tilde{\mathcal{T}}^\alpha(\ell)|0\rangle_L^n = \\
&\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} |\tau_{p+\alpha}|^2 \left[1 + \frac{\sin^2\frac{\pi(p+\alpha)}{n}}{(mL)^2}\sum_{J_1^+,J_1^-\in\mathbb{Z}} e^{2\pi i r(J_1^+ + J_1^-)}\left(1 + 2\left(\frac{p+\alpha}{n}\right)\frac{c_1^+ - c_1^-}{mL} + \mathcal{O}\left(\frac{1}{(mL)^2}\right)\right)\right].
\end{aligned} \tag{2.223}$$

We immediately notice that there is no contribution of order $\mathcal{O}((mL)^{-2})$, as we can regularise the non convergent double sum by introducing two small real parameters ε, δ:

$$\begin{aligned}
&\sum_{J_1^+,J_1^-\in\mathbb{Z}} e^{2\pi i r(J_1^+ + J_1^-)} \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to 0}} \left(-1 + \sum_{J\geq 0} e^{2\pi i J(r+i\varepsilon)} + \sum_{J\geq 0} e^{-2\pi i J(r-i\varepsilon)}\right)\left(-1 + \sum_{J\geq 0} e^{2\pi i J(r+i\delta)} + \sum_{J\geq 0} e^{-2\pi i J(r-i\delta)}\right) = 0.
\end{aligned} \tag{2.224}$$

The term of order $\mathcal{O}((mL)^{-3})$ can similarly be regularised to zero. Therefore, the first finite-volume correction to the ratio of moments of the SREE is (at least) of order $\mathcal{O}((mL)^{-4})$. However, at that order the computation becomes more involved as there are contributions coming from terms with $s > 1$ in (2.218).

2.D Trace calculations via wave-functional method

In this appendix we present explicit calculations of the ratio of charged moments for excited states in a complex free boson theory using the wave-functional method introduced in Section 2.5.2. The discretisation of these results leads to the formulae for the complex harmonic chain presented in Subsection 2.5.2.2.

2.D.1 Zero flux

We wish to compute the ratio of charged moments

$$\frac{\mathrm{Tr}_A\left(\rho_A^n\, e^{2\pi i\alpha Q_A}\right)}{\mathrm{Tr}_A\left(\rho_{0,A}^n\, e^{2\pi i\alpha Q_A}\right)}, \tag{2.225}$$

where ρ_A and $\rho_{0,A}$ are the reduced density matrices of the excited and ground states, respectively.

We define restricted wave-functionals which take as arguments (complex) fields that either have support on region $A := [0,\ell)$ or on $\bar{A} := [\ell, L]$. Leaving the dependence on the conjugate fields $\phi^\dagger$ implicit, we write:

$$\Phi(x)\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}] = \left(\delta_{x\in A}\phi(x) + \delta_{x\in\bar{A}}\phi'(x)\right)\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}], \tag{2.226}$$

$$i\Pi(x)\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}] = \left(\delta_{x\in A}\frac{\delta}{\delta\phi(x)} + \delta_{x\in\bar{A}}\frac{\delta}{\delta\phi'(x)}\right)\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}], \tag{2.227}$$

and similarly for the action of $\Phi^\dagger(x)$, $i\Pi^\dagger(x)$. The action of the charge operator $\hat{Q}_A$ on the wave-functional is simply:

$$\hat{Q}_A\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}] = \int_A \mathrm{d}x\, \left(\phi^\dagger(x)\frac{\delta}{\delta\phi^\dagger(x)} - \phi(x)\frac{\delta}{\delta\phi(x)}\right)\mathbf{\Psi}[\phi_A,\phi'_{\bar{A}}], \tag{2.228}$$

while from (2.168) it follows that

$$e^{2\pi i\alpha\hat{Q}_A}|\phi_A,\phi'_{\bar{A}}\rangle = |e^{2\pi i\alpha}\phi_A,\phi'_{\bar{A}}\rangle. \tag{2.229}$$

The reduced density matrix then admits the functional representation

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho = \int \mathcal{D}\phi_{\bar{A}}\langle\phi_{\bar{A}}|\rho|\phi_{\bar{A}}\rangle \Rightarrow \langle\phi'_A|\rho_A|\phi''_A\rangle = \int \mathcal{D}\phi_{\bar{A}}\mathbf{\Psi}[\phi'_A,\phi_{\bar{A}}]\mathbf{\Psi}[\phi''_A,\phi_{\bar{A}}]^*, \tag{2.230}$$

where $\mathcal{D}\phi$ is shorthand for $\mathcal{D}\phi\mathcal{D}\phi^\dagger$.

We start with the simple case where there is no symmetry resolution, that is, we compute (2.225)

for $\alpha = 0$, following [115]. For n integer, we insert the resolution of the identity n times and obtain:

$$\begin{aligned}\mathrm{Tr}_A(\rho_{0,A}^n) &= \int \mathcal{D}\phi_{1A}\dots\mathcal{D}\phi_{nA}\langle\phi_{1A}|\rho_{\mathrm{vac},A}|\phi_{2A}\rangle\dots\langle\phi_{nA}|\rho_{\mathrm{vac},A}|\phi_{1A}\rangle \\ &= \int \mathcal{D}\phi_1\dots\mathcal{D}\phi_n \prod_{i=1}^n \mathbf{\Psi}_{\mathrm{vac}}[\phi_{i,A},\phi_{i,\bar{A}}]\prod_{i=1}^n \mathbf{\Psi}_{\mathrm{vac}}[\phi_{i+1,A},\phi_{i,\bar{A}}]^*. \end{aligned} \tag{2.231}$$

The product of the diagonal terms, i.e. those in which the fields in subsystems A and $\bar{A}$ are in the same copy, is

$$\prod_{i=1}^n \mathbf{\Psi}_{\mathrm{vac}}[\phi_{i,A},\phi_{i,\bar{A}}] = \exp\left[-\sum_{i=1}^n \int_{x,y\in A\cup\bar{A}} \mathrm{d}x\mathrm{d}y\, \phi_i^\dagger(x)K(x-y)\phi_i(y)\right], \tag{2.232}$$

with $K(x)$ defined in (2.161). For the non-diagonal terms we notice that since $K(x)^* = K(-x)$ we have $\mathbf{\Psi}_{\mathrm{vac}}[\phi_{i+1,A},\phi_{i,\bar{A}}]^* = \mathbf{\Psi}_{\mathrm{vac}}[\phi_{i+1,A},\phi_{i,\bar{A}}]$ and:

$$\prod_{i=1}^n \mathbf{\Psi}_{\mathrm{vac}}[\phi_{i+1,A},\phi_{i,\bar{A}}] = \exp\left\{-\sum_{i=1}^n \left[\int_{\substack{x\in A\\ y\in A}} \mathrm{d}x\mathrm{d}y\, \phi_{i+1}^\dagger(x)K(x-y)\phi_{i+1}(y)\right.\right. \tag{2.233}$$

$$\left.\left. + \int_{\substack{x\in \bar{A}\\ y\in \bar{A}}} \mathrm{d}x\mathrm{d}y\, \phi_i^\dagger(x)K(x-y)\phi_i(y) + \left(\int_{\substack{x\in A\\ y\in \bar{A}}} \mathrm{d}x\mathrm{d}y\, \phi_{i+1}^\dagger(x)K(x-y)\phi_i(y) + \text{c.c.}\right)\right]\right\}. \tag{2.234}$$

Putting all the terms together, we end up with the Gaussian measure:

$$\mathrm{Tr}_A(\rho_{0,A}^n) = \int \mathcal{D}\phi_1\dots\mathcal{D}\phi_n \exp\left[-\mathcal{G}\right], \tag{2.235}$$

where

$$\begin{aligned}\mathcal{G} &= \sum_{i=1}^n 2\left(\int_{\substack{x\in A\\ y\in A}} + \int_{\substack{x\in \bar{A}\\ y\in \bar{A}}}\right)\mathrm{d}x\mathrm{d}y\, \phi_i^\dagger(x)K(x-y)\phi_i(y) \\ &\quad + \sum_{i=1}^n \left(\int_{\substack{x\in A\\ y\in \bar{A}}} \mathrm{d}x\mathrm{d}y\, (\phi_i(x)+\phi_{i+1}(x))^\dagger K(x-y)\phi_i(y) + \text{c.c.}\right). \end{aligned} \tag{2.236}$$

Let us now compute the numerator of (2.225). For a state of the form (2.164) we have:

$$\mathrm{Tr}_A\left(\rho_A^n\right) = \int \mathcal{D}\phi_1\dots\mathcal{D}\phi_n \prod_{i=1}^n \mathbf{\Psi}_{\{p_{j^+},p_{j^-}\}}[\phi_{i,A},\phi_{i,\bar{A}}]\prod_{i=1}^n \mathbf{\Psi}_{\{p_{j^+},p_{j^-}\}}[\phi_{i+1,A},\phi_{i,\bar{A}}]^*, \tag{2.237}$$

with:

$$\prod_{i=1}^{n}\mathbf{\Psi}_{\{p_{j^+},p_{j^-}\}}[\phi_{i,A},\phi_{i,\bar{A}}]=\left(\prod_{j^+}\frac{2E_{p_{j^+}}}{L}\prod_{j^-}\frac{2E_{p_{j^-}}}{L}\right)^{\frac{n}{2}}\prod_{i=1}^{n}\prod_{j^+}\int_{A\cup\bar{A}}\mathrm{d}x e^{ip_{j^+}x}\phi_i^\dagger(x)$$
$$\times\prod_{j^-}\int_{A\cup\bar{A}}\mathrm{d}x e^{ip_{j^-}x}\phi_i(x)\prod_{i=1}^{n}\mathbf{\Psi}_{\mathrm{vac}}[\phi_{i,A},\phi_{i,\bar{A}}], \tag{2.238}$$

$$\prod_{i=1}^{n}\mathbf{\Psi}_{\{p_{j^+},p_{j^-}\}}[\phi_{i+1,A},\phi_{i,\bar{A}}]^*=\left(\prod_{j^+}\frac{2E_{p_{j^+}}}{L}\prod_{j^-}\frac{2E_{p_{j^-}}}{L}\right)^{\frac{n}{2}}$$
$$\times\prod_{i=1}^{n}\prod_{j^+}\left(\int_A\mathrm{d}x e^{-ip_{j^+}x}\phi_{i+1}(x)+\int_{\bar{A}}\mathrm{d}x e^{-ip_{j^+}x}\phi_i(x)\right)$$
$$\times\prod_{j^-}\left(\int_A\mathrm{d}x e^{-ip_{j^-}x}\phi_{i+1}^\dagger(x)+\int_{\bar{A}}\mathrm{d}x e^{-ip_{j^-}x}\phi_i^\dagger(x)\right)\prod_{i=1}^{n}\mathbf{\Psi}_{\mathrm{vac}}[\phi_{i+1,A},\phi_{i,\bar{A}}]. \tag{2.239}$$

Putting everything together, we obtain

$$\frac{\mathrm{Tr}_A\left(\rho_A^n\right)}{\mathrm{Tr}_A(\rho_{0,A}^n)}=\left(\prod_{j^+}\frac{2E_{p_{j^+}}}{L}\prod_{j^-}\frac{2E_{p_{j^-}}}{L}\right)^n\langle\prod_{i=1}^{n}\prod_{j^+}U_i^+(p_{j^+})V_i^+(p_{j^+})\prod_{j^-}U_i^-(p_{j^-})V_i^-(p_{j^-})\rangle \tag{2.240}$$

where the correlation function is defined with respect to the Gaussian measure:

$$\langle\mathcal{O}[\phi_1,\phi_1^\dagger\ldots,\phi_n,\phi_n^\dagger]\rangle:=\frac{\int\mathcal{D}\phi_1\mathcal{D}\phi_1^\dagger\ldots\mathcal{D}\phi_n\mathcal{D}\phi_n^\dagger\mathcal{O}[\phi_1,\phi_1^\dagger\ldots,\phi_n,\phi_n^\dagger]\exp\left[-\mathcal{G}\right]}{\int\mathcal{D}\phi_1\mathcal{D}\phi_1^\dagger\ldots\mathcal{D}\phi_n\mathcal{D}\phi_n^\dagger\exp\left[-\mathcal{G}\right]}, \tag{2.241}$$

and the operators are

$$U_i^+(p):=\int_{A\cup\bar{A}}\mathrm{d}x\,e^{ipx}\phi_i^\dagger(x),\quad U_i^-(p):=\int_{A\cup\bar{A}}\mathrm{d}x\,e^{ipx}\phi_i(x) \tag{2.242}$$

$$V_i^+(p):=\int_A\mathrm{d}x\,e^{-ipx}\phi_{i+1}(x)+\int_{\bar{A}}\mathrm{d}x\,e^{-ipx}\phi_i(x), \tag{2.243}$$
$$V_i^-(p):=\int_A\mathrm{d}x\,e^{-ipx}\phi_{i+1}^\dagger(x)+\int_{\bar{A}}\mathrm{d}x\,e^{-ipx}\phi_i^\dagger(x). \tag{2.244}$$

If the excited state is of the form (2.165), the result (2.240) is minimally modified. The terms inside the correlator are exactly the same and the prefactor is modified to:

$$\left(\prod_{j^+=1}^{m^+}\frac{1}{k_{j^+}^+!}\left(\frac{2E_{p_{j^+}}}{L}\right)^{k_{j^+}^+}\prod_{j^-=1}^{m^-}\frac{1}{k_{j^-}^-!}\left(\frac{2E_{p_{j^-}}}{L}\right)^{k_{j^-}^-}\right)^n. \tag{2.245}$$

2.D.2 Non-trivial flux insertion

We now come to the quantity (2.225) with $\alpha \neq 0$, assuming the excited state to be of the form (2.164). Let us consider the denominator first. Because of the flux insertion to the right of the nth operator $\rho_{0,A}$, everything is the same as in (2.231) except for the last resolution of the identity, which produces a term:

$$\langle \Psi | e^{2\pi i \alpha \hat{Q}_A} | \phi_{1A}, \phi_{n\bar{A}} \rangle = \langle \Psi | e^{2\pi i \alpha} \phi_{1A}, \phi_{n\bar{A}} \rangle = \mathbf{\Psi}[\phi_{1A} e^{2\pi i \alpha}, \phi_{n\bar{A}}]^*. \tag{2.246}$$

Thus, the Gaussian measure in the presence of the charge is modified as follows:

$$\mathrm{Tr}_A(\rho_{0,A}^n e^{2\pi i \alpha \hat{Q}_A}) = \int \mathcal{D}\phi_1 \dots \mathcal{D}\phi_n \exp\left[-\mathcal{G}_\alpha\right], \tag{2.247}$$

with

$$\begin{aligned} \mathcal{G}_\alpha = \sum_{i=1}^{n} \Bigg[2 \left(\int_{\substack{x \in A \\ y \in A}} + \int_{\substack{x \in \bar{A} \\ y \in \bar{A}}} \right) \mathrm{d}x\mathrm{d}y \, \phi_i^\dagger(x) K(x-y) \phi_i(y) \\ + \left(\int_{\substack{x \in A \\ y \in \bar{A}}} \mathrm{d}x\mathrm{d}y \, \left(\phi_i^\dagger(x) + \phi_{i+1}^\dagger(x) e^{-2\pi i \alpha \delta_{i,n}} \right) K(x-y) \phi_i(y) + \text{c.c.} \right) \Bigg]. \end{aligned} \tag{2.248}$$

As for the numerator of (2.225), we have similarly

$$\begin{aligned} \mathrm{Tr}_A(\rho_A^n) = \int \mathcal{D}\phi_1 \dots \mathcal{D}\phi_n \prod_{i=1}^{n-1} \mathbf{\Psi}_{\{p_{j^+}, p_{j^-}\}}[\phi_{iA}, \phi_{i\bar{A}}] \mathbf{\Psi}_{\{p_{j^+}, p_{j^-}\}}[\phi_{i+1,A}, \phi_{i\bar{A}}]^* \\ \times \mathbf{\Psi}_{\{p_{j^+}, p_{j^-}\}}[\phi_{nA}, \phi_{n\bar{A}}] \mathbf{\Psi}_{\{p_{j^+}, p_{j^-}\}}[\phi_{1,A} e^{2\pi i \alpha}, \phi_{n\bar{A}}]^*. \end{aligned} \tag{2.249}$$

The product in the first line yields the same quantity as in the $\alpha = 0$ case, with the replica index taking values up to $n-1$ only. The product of the last two functionals gives:

$$\begin{aligned} &\left(\prod_{j^+} \alpha_{p_{j^+}}[\phi_n^\dagger] \alpha_{p_{j^+}}[\phi_{1A}^\dagger e^{-2\pi i \alpha}, \phi_{n\bar{A}}^\dagger]^* \prod_{j^-} \beta_{p_{j^-}}[\phi_n] \beta_{p_{j^-}}[\phi_{1A} e^{2\pi i \alpha}, \phi_{n\bar{A}}]^* \right) \\ &\times \mathbf{\Psi}_{\mathrm{vac}}[\phi_{nA}, \phi_{n\bar{A}}] \mathbf{\Psi}_{\mathrm{vac}}[\phi_{1,A} e^{2\pi i \alpha}, \phi_{n\bar{A}}]^*. \end{aligned} \tag{2.250}$$

and from the definitions (2.162) we obtain our final result:

$$\begin{aligned} \frac{\mathrm{Tr}_A \left(\rho_A^n \, e^{2\pi i \alpha \hat{Q}_A} \right)}{\mathrm{Tr}_A \left(\rho_{0,A}^n \, e^{2\pi i \alpha \hat{Q}_A} \right)} = \left(\prod_{j^+} \frac{2E_{p_{j^+}}}{L} \prod_{j^-} \frac{2E_{p_{j^-}}}{L} \right)^n \\ \times \langle \prod_{i=1}^{n} \prod_{j^+} U_i^+(p_{j^+}) V_i^+(p_{j^+}) \prod_{j^-} U_i^-(p_{j^-}) V_i^-(p_{j^-}) \rangle_\alpha, \end{aligned} \tag{2.251}$$

where $\langle \dots \rangle_\alpha$ is the average with Gaussian measure (2.248). The quantities $U_i^\pm(p)$ are defined exactly as in (2.242), while the functions $V_i^\pm(p)$ are now modified by the presence of the charge as follows:

$$V_i^+(p) = e^{2\pi i \alpha \delta_{i,n}} \int_A \mathrm{d}x\, e^{-ipx} \phi_{i+1}(x) + \int_{\bar{A}} \mathrm{d}x\, e^{-ipx} \phi_i(x), \tag{2.252}$$

$$V_i^-(p) = e^{-2\pi i \alpha \delta_{i,n}} \int_A \mathrm{d}x\, e^{-ipx} \phi_{i+1}^\dagger(x) + \int_{\bar{A}} \mathrm{d}x\, e^{-ipx} \phi_i^\dagger(x). \tag{2.253}$$

If the excited state is given by (2.165), the right-hand side of (2.251) is changed exactly as in the case $\alpha = 0$.

CHAPTER

THREE

SYMMETRY-RESOLVED NEGATIVITY OF EXCITED STATES

In the previous chapter, we studied the symmetry-resolved Rényi entropies of quasi-particle excited states in QFTs with $U(1)$ symmetry. We found that the entropies display some model-independent features which we characterised using different approaches. In this chapter, based on [3], we extend this line of investigation by providing analytical and numerical evidence that a similar universal behavior arises for the symmetry-resolved negativity. In particular, we compute the ratio of charged moments of the partially transposed reduced density matrix. These charged ratios are given by expectation values of the composite twist operators introduced in the previous chapter: their use allows us to perform the computation in an arbitrary number of spacial dimensions. We show that, in the large-volume limit, only the commutation relations between the twist operators and local fields matter, and computations reduce to a purely combinatorial problem. We address some specific issues regarding fermionic excitations, whose treatment requires the notion of partial time-reversal transformation, and we discuss the differences with their bosonic counterpart. We find that although the operation of partial transposition requires a redefinition for fermionic theories, the ratio of the negativity moments between an excited state and the ground state is universal and identical for fermions and bosons, as it is found by performing computations on QFT states as well as simple qubit states. Our predictions are tested numerically on a 1D Fermi chain.

3.1 Introduction

Over the past two decades, entanglement measures have been widely studied in the context of low-dimensional QFT, starting with several seminal works [11–16, 48] which focused on one measure (the entanglement entropy [17]) and on one type of theory, largely 1+1D CFT and its discrete

counterpart, critical spin chains. From these papers sprang several important ideas and techniques which have been extensively exploited thereafter. Notable among them is the numerical and analytical observation that the entanglement entropy exhibits universal properties, i.e. properties that depend only on the theory's universality class characterised by the central charge c. As already discussed in the previous chapters, from a computational point of view, an important idea to emerge from [12, 48] and later reinterpreted and generalised to non-critical theories in [20] is that entanglement measures can be written in terms of correlation functions of semi-local fields of a replica version of the QFT under study. One particular development of these ideas has been the proposal and study of new measures of entanglement, each tailored to capturing particular features of entanglement and/or of the state whose entanglement is being measured. One such new measure is the (logarithmic) negativity [56, 57, 176–180] which we defined in (1.16) for a tripartite system consisting of subsystems A, B and $C := \overline{A \cup B}$ and Hilbert space given by (1.14).

An interesting issue that is specific to the logarithmic negativity is the fact that the definition of $\mathcal{E}$ in terms of the partial transpose $\rho_{A\cup B}^{T_B}$, whose matrix elements are given in (1.15), directly apply to spin chains or bosonic systems, but it is ill-suited for fermionic systems. The reason for this is rather technical and can be explained in different ways. Unlike for bosons, the partial transpose of the Gaussian density matrix of a free fermion state is not Gaussian, which makes the computation of the negativity spectrum particularly difficult. There have been several proposals as to how to modify the definition of $\mathcal{E}$ in a way that is better adapted to deal with fermionic degrees of freedom. The first definition of partial transposition specifically modified for fermionic states was introduced in [181]. However, in [182] it was proved that, because of the anticommuting nature of the fermionic degrees of freedom, two of the standard requirements of a partial transposition operation, namely that if $\rho \equiv \rho_{A\cup B}$ then

$$(\rho^{T_A})^{T_B} = \rho^T \quad \text{and} \quad \rho_1^{T_A} \otimes \cdots \otimes \rho_n^{T_A} = (\rho_1 \otimes \cdots \otimes \rho_n)^{T_A}\,, \tag{3.1}$$

with T representing transposition over the total space (here $T = T_{A\cup B}$), may not hold with the definition given in [181]. On the other hand, these properties are satisfied with the definition introduced in [182]: this is the *time-reversal* (or fermionic) negativity, which accounts for the locality properties of fermions [182, 183] and which we present below. Following [183], we now take $\mathcal{H}_A \otimes \mathcal{H}_B$ to be a Fock space associated to fermionic degrees of freedom in $\ell_A + \ell_B$ sites and choose an occupation number basis:

$$\begin{aligned} |\{n_j\}_A, \{n_j\}_B\rangle &:= |n_1, \ldots, n_{\ell_A}, n_{\ell_A+1}, \ldots, n_{\ell_A+\ell_B}\rangle \\ &= (f_1^\dagger)^{n_1} \ldots (f_{n\ell_A}^\dagger)^{n_{\ell_A}} (f_{n\ell_A+1}^\dagger)^{n_{\ell_A+1}} \ldots (f_{n\ell_A+\ell_B}^\dagger)^{n_{\ell_A+\ell_B}} |0\rangle, \end{aligned} \tag{3.2}$$

such that all the $n_j \in \{0,1\}$ and $f_j^\dagger$ is a fermionic creation operator at site j. Then, given the RDM

$$\rho_{A\cup B} = \sum_{\substack{\{n_j\}_A,\{n_j\}_B \\ \{n'_j\}_A,\{n'_j\}_B}} |\{n_j\}_A,\{n_j\}_B\rangle\langle\{n_j\}_A,\{n_j\}_B|\rho_{A\cup B}|\{n'_j\}_A,\{n'_j\}_B\rangle\langle\{n'_j\}_A,\{n'_j\}_B|\,, \tag{3.3}$$

we define the fermionic partial transposition as

$$\rho_{A\cup B}^{R_B} := \sum_{\substack{\{n_j\}_A,\{n_j\}_B \\ \{n'_j\}_A,\{n'_j\}_B}} i^{\phi(\{n_j\},\{n'_j\})}|\{n_j\}_A,\{n'_j\}_B\rangle\langle\{n_j\}_A,\{n_j\}_B|\rho_{A\cup B}|\{n'_j\}_A,\{n'_j\}_B\rangle\langle\{n'_j\}_A,\{n_j\}_B|\,, \tag{3.4}$$

where $\phi(\{n_j\},\{n'_j\})$ is given by

$$\phi(\{n_j\},\{n'_j\}) = (\tau_B+\tau'_B)(\mathrm{mod}\,2) + 2(\tau_A+\tau'_A)(\tau_B+\tau'_B)\,, \tag{3.5}$$

and $\tau_{A/B} = \sum_{j\in A/B} n_j, \tau'_{A/B} = \sum_{j\in A/B} n'_j$ are the numbers of occupied states in each subsystem. Thus, the novelty of the fermionic partial transposition (3.4), as compared to (1.15), is the presence of an additional phase shift which depends on the number of fermions. The operation R_B defined in (3.4) is also called partial time reversal, as in a path integral formalism for spinless fermions it implements time reversal only in the subsystem B [184]. While in general $\rho_{A\cup B}^{R_B}$ is no longer Hermitian for fermionic systems, one can still use the definition (1.16) for fermionic logarithmic negativity, but where now

$$|\rho_{A\cup B}^{R_B}| = \sqrt{(\rho_{A\cup B}^{R_B})^\dagger \rho_{A\cup B}^{R_B}}\,, \tag{3.6}$$

which is a positive semi-definite matrix.

Let us now add the final layer of definitions by introducing symmetry-resolved negativities. As seen in the previous chapters, symmetry-resolved entanglement measures have become very popular in the past few years and extend the standard definitions by exploiting the presence of internal symmetries. While the earliest studies focused on the entanglement entropies (see [19, 94] for the CFT/QFT and quantum spin chain constructions), more recently also the logarithmic negativity has been generalised in a similar fashion [96, 120, 185, 186]. Let us consider a theory with a global $U(1)$ symmetry (i.e. a complex free boson/fermion). In that case, a global $U(1)$ charge $\hat{Q}_{A\cup B} = \hat{Q}_A \oplus \hat{Q}_B$ commutes with the state $\rho_{A\cup B}$:

$$[\rho_{A\cup B}, \hat{Q}_A \oplus \hat{Q}_B] = 0\,. \tag{3.7}$$

Then, it has been shown in [96] that the charge imbalance operator $\hat{Q}_A - \hat{Q}_B^T$, defined in (1.51), commutes with $\rho_{A\cup B}^{T_B}$

$$[\rho_{A\cup B}^{T_B}, \hat{Q}_A - \hat{Q}_B^T] = 0, \tag{3.8}$$

and it generates a $U(1)$ symmetry for the (bosonic or fermionic) partial transpose. The matrix representation of the charge imbalance operator is basis-dependent, and in the occupation number basis $\hat{Q}_B^T = \hat{Q}_B$ so we will drop the transposition on $\hat{Q}_B$ from now on. At this point, it is natural to consider the charged moments of the partial transpose. These are defined as

$$\mathrm{Tr}\left((\rho_{A\cup B}^{T_B})^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right), \quad \alpha\in[-1/2,1/2]\,, \tag{3.9}$$

and, generalising the standard moments (1.30), when integrated over α they provide the symmetry-resolved Rényi negativities. For fermions, the definition above is changed to

$$\mathrm{Tr}\left(|\rho_{A\cup B}^{R_B}|^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right), \quad \alpha\in[-1/2,1/2]\,. \tag{3.10}$$

The computation of charged moments of the partial transpose was performed in [120] in the ground and thermal state of massless free fermions in 1+1 dimensions, where the universal UV divergences were captured by the underlying CFT. They have also been measured in an experimental setup in [23]. In this chapter, we are interested in zero-density quasi-particle states of (massive) QFT, obtained as excitations of the ground state with finite number of particles at given momenta. These are the same states we considered in the previous chapter, and that were already used in [114–116, 144] for the standard (uncharged) entanglement measures. We aim to compute the contribution to the charged moments given by the quasi-particles, which arises in addition to the zero-point fluctuations.

We will now briefly state the main results of this chapter. Let us consider a QFT in $D = d+1$ dimensions carrying a global $U(1)$ symmetry and two non-complementary spacial regions A and B. Let $|0\rangle$ be the vacuum state in the Hilbert space (1.14), and let $|\Psi^{(k)}\rangle$ be an excited state containing k identical excitations with unit charge. We construct the associated reduced density matrices (RDM) over $A\cup B$ as

$$\rho_{A\cup B,0} := \mathrm{Tr}_C|0\rangle\langle 0|, \quad \rho_{A\cup B} := \mathrm{Tr}_C|\Psi^{(k)}\rangle\langle\Psi^{(k)}|, \quad \text{with} \quad C := \overline{A\cup B}\,. \tag{3.11}$$

Then, generalising what we did in the previous chapter, we consider the limit in which the generalised volume V_A, V_B, V_C of each region goes to infinity while the ratios

$$r_A := \frac{V_A}{V}, \quad r_B := \frac{V_B}{V} \quad \text{and} \quad r := \frac{V_C}{V} = 1 - r_A - r_B\,, \tag{3.12}$$

are finite and $V = V_A + V_B + V_C$. We then define the ratio of charged moments:

$$\mathcal{R}_k^n(r_A, r_B, r; \alpha) := \frac{\mathrm{Tr}\left((\rho_{A\cup B}^{T_B})^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right)}{\mathrm{Tr}\left((\rho_{A\cup B,0}^{T_B})^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right)}\,. \tag{3.13}$$

We find this ratio to be universal in the large-volume limit, and for a single particle excitation ($k = 1$) it is given by

$$\mathcal{R}_1^n(r_A, r_B, r; \alpha) = e^{2\pi i\alpha} r_A^n + e^{-2\pi i\alpha} r_B^n + \left(\frac{r + \sqrt{r^2 + 4r_A r_B}}{2}\right)^n + \left(\frac{r - \sqrt{r^2 + 4r_A r_B}}{2}\right)^n . \quad (3.14)$$

We notice that, because $r_A, r_B, r \in (0, 1)$, the last term in (3.14) is positive (negative) when n is an even (odd) integer and therefore one needs to consider two distinct analytic continuations over the even and odd integers. In particular, the analytic continuation from n even to $n = 1$ gives

$$\lim_{n \to \frac{1}{2}} \mathcal{R}_1^{2n}(r_A, r_B, r; \alpha) = e^{2\pi i\alpha} r_A + e^{-2\pi i\alpha} r_B + \sqrt{r^2 + 4r_A r_B} \, . \quad (3.15)$$

Comparing these expression to the $\alpha = 0$ results found in [116, 144], we see that the phases $e^{\pm 2\pi i\alpha}$ appear only in the first two terms of (3.14), whereas the other terms are unchanged. This provides a useful hint as to how more complicated formulae for multi-particle states will generalise to the symmetry-resolved measure, namely by the substitutions $r_A \mapsto e^{\frac{2\pi i\alpha}{n}} r_A$ and $r_B \mapsto e^{-\frac{2\pi i\alpha}{n}} r_B$. Therefore, we can write:

$$\mathcal{R}_1^n(r_A, r_B, r; \alpha) = \mathcal{R}_1^n \left(e^{\frac{2\pi i\alpha}{n}} r_A, e^{-\frac{2\pi i\alpha}{n}} r_B, r; 0 \right) . \quad (3.16)$$

The generalisation to states of many distinct quasiparticles is straightforward, and each particle contributes independently (multiplicatively) to the ratio of charged moments, similar to the structure found in the previous chapter for the charged Rényi entropies. We highlight that the result (3.14) holds also for fermions with the definitions (3.6) and (3.10) when n is even. This is the case we will consider in the following when treating the fermionic case. For bosonic systems, states of multiple identical excitations can also be considered. The total (uncharged) negativity of these states was obtained in [116, 144], and for the ratio of charged moments in this case we find:

$$\mathcal{R}_k^n(r_A, r_B, r; \alpha) = \sum_{p=-k}^{k} \sum_{q=\max(0,-np)}^{[\frac{n}{2}(k-p)]} \mathcal{A}_{p,q} r_A^{np+q} r^{n(k-p)-2q} r_B^q \, e^{2\pi i\alpha p}, \quad (3.17)$$

where $[\cdot]$ represents the integer part and

$$\mathcal{A}_{p,q} := \sum_{\{k_1,\dots,k_n\} \in P_n(q)} \prod_{j=1}^{n} \frac{k!}{(p + k_j)!(k - p - k_j - k_{j+1})! k_j!}, \quad (3.18)$$

are combinatorial factors, with the sum running over the set $P_n(q)$ of partitions of $q \in \mathbb{N}_0$ into n non-negative integers.

We structure this chapter as follows. In Section 3.2 we analyse in detail a simplified model consisting

of a state of few qubits. In spite of the simplicity of these states, their symmetry-resolved negativity moments capture the main universal features found in QFT. In Section 3.3 we give a field-theoretic formulation of the charged moments of the partially transposed RDM, employing the notion of twist operators. The difference between fermionic and bosonic particles is thoroughly discussed in this context, and the evaluation of the moments is shown to reduce to a combinatorial problem which we solve exactly for single and multiple distinct excitations. Some details of our combinatorial arguments are left to Appendix 3.A, while in Appendix 3.B we prove some useful identities that we use in Section 3.3. We check numerically our predictions on a 1D Fermi chain in Section 3.4 and find good agreement. We conclude in Section 3.5.

3.2 Qubit computation

In this section we derive the main formulae for the charged replica negativities in a multi-qubit system. This toy model was already employed in [114–116], as well as in the previous chapter, following the observation that even if multi-qubit states are much simpler than the excited states of a QFT, they both produce the same universal contribution to entanglement entropies and negativities. The advantage of this picture is that the ground state of a multi-qubit system is trivial from the point of view of the entanglement content, which means that the excess of entropy, or the excess of negativity, of an excited state with respect to the ground state effectively reduces to the entropy or negativity of the excited state. Analogously, the ratio of charged moments of the partial transpose coincides with the charged moment of the partial transpose of the excited state. The notion of charge imbalance in a qubit setup was introduced in [96] and the notion of fermionic partial transposition in the same setup was later used in [120].

The main result of this section is to show that equation (3.14) for a state consisting of a single excitation can be derived employing either the bosonic or the fermionic notion of partial transposition. Interestingly, even if the intermediate steps of the computation are different in the two cases, the final result is still the same. For bosonic theories, the result can be generalised to multiple identical excitations to give (3.17).

3.2.1 Single bosonic excitation

Proceeding exactly as in Section 2.3, let us assume that a single bosonic excitation of charge[1] $+1$ is localised in space according to a uniform probability distribution, so that r_A, r_B and r can be regarded as the probabilities for the excitation to be found in regions A, B, C respectively. Then the state in

[1] Without loss of generality, in this section we take all the quasiparticle excitations to be positively charged. In this way the action of the $U(1)$ charge $\hat{Q}$ on the multi-qubit states is identical to that of the number operator $\hat{N}$.

$\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ representing a single excitation can be written as

$$|\Psi^{(1)}_{\rm qb}\rangle = \sqrt{r_A}|100\rangle + \sqrt{r_B}|010\rangle + \sqrt{r}|001\rangle, \tag{3.19}$$

where the values 0 (1) represent the absence (presence) of the excitation and the coefficients can be interpreted as probabilities of finding the excitation in a particular region. Here and in the following we omit the tensor products in writing qubit states, identifying $|k_1k_2k_3\rangle := |k_1\rangle_A \otimes |k_2\rangle_B \otimes |k_3\rangle_C$. The RDM $\rho_{A\cup B}$ is obtained taking the trace over $\mathcal{H}_C$:

$$\begin{aligned}\rho_{A\cup B} &= {\rm Tr}_C |\Psi^{(1)}_{\rm qb}\rangle\langle\Psi^{(1)}_{\rm qb}| \\ &= r_A|10\rangle\langle 10| + r_B|01\rangle\langle 01| + \sqrt{r_Ar_B}(|01\rangle\langle 10| + |10\rangle\langle 01|) + r|00\rangle\langle 00|,\end{aligned} \tag{3.20}$$

or in matrix form

$$\rho_{A\cup B} = \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & r & 0 & 0 & 0 \\ 01 & 0 & r_B & \sqrt{r_Ar_B} & 0 \\ 10 & 0 & \sqrt{r_Ar_B} & r_A & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array}\right). \tag{3.21}$$

This matrix has a block-diagonal structure with respect to the charge operator $\hat{Q}_{A\cup B} = \hat{Q}_A \oplus \hat{Q}_B$:

$$(\hat{Q}_A \oplus \hat{Q}_B)|k_Ak_B\rangle = (k_A + k_B)|k_Ak_B\rangle\,, \quad k_A, k_B \in \{0,1\}, \tag{3.22}$$

as indeed we can decompose it according to the eigenspaces of $\hat{Q}_{A\cup B}$, with eigenvalues $Q = 0, 1, 2$:

$$\rho_{A\cup B} = (r)_{Q=0} \oplus \begin{pmatrix} r_B & \sqrt{r_Ar_B} \\ \sqrt{r_Ar_B} & r_A \end{pmatrix}_{Q=1} \oplus (0)_{Q=2}. \tag{3.23}$$

Let us now come to the partially transposed matrix $\rho^{T_B}_{A\cup B}$. From the definition (1.15), it follows:

$$\begin{aligned}\rho^{T_B}_{A\cup B} &= r_A|10\rangle\langle 10| + r_B|01\rangle\langle 01| + \sqrt{r_Ar_B}(|00\rangle\langle 11| + |11\rangle\langle 00|) + r|00\rangle\langle 00| \\ &= \left(\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & r & 0 & 0 & \sqrt{r_Ar_B} \\ 01 & 0 & r_B & 0 & 0 \\ 10 & 0 & 0 & r_A & 0 \\ 11 & \sqrt{r_Ar_B} & 0 & 0 & 0 \end{array}\right),\end{aligned} \tag{3.24}$$

which is a block-diagonal matrix with respect to the charge imbalance operator $\hat{Q}_A - \hat{Q}_B$:

$$(\hat{Q}_A - \hat{Q}_B)|k_A k_B\rangle = (k_A - k_B)|k_A k_B\rangle\,, \quad k_A, k_B \in \{0,1\}, \tag{3.25}$$

and denoting the eigenvalues of the charge imbalance by $\Delta Q = 1, 0, -1$ we can write:

$$\rho_{A\cup B}^{T_B} = (r_A)_{\Delta Q=1} \oplus \begin{pmatrix} r & \sqrt{r_A r_B} \\ \sqrt{r_A r_B} & 0 \end{pmatrix}_{\Delta Q=0} \oplus (r_B)_{\Delta Q=-1}\,. \tag{3.26}$$

The partial transpose $\rho_{A\cup B}^{T_B}$ has four distinct eigenvalues, one of which is negative:

$$\sigma\left(\rho_{A\cup B}^{T_B}\right) = \left\{ r_A,\, r_B,\, \frac{r + \sqrt{r^2 + 4 r_A r_B}}{2},\, \frac{r - \sqrt{r^2 + 4 r_A r_B}}{2} \right\}, \tag{3.27}$$

and therefore this system has non-vanishing negativity. From the block-diagonal structure of $\rho_{A\cup B}^{T_B}$ it is immediate to construct the matrix $(\rho_{A\cup B}^{T_B})^n e^{2\pi i \alpha(\hat{Q}_A - \hat{Q}_B)}$ and we finally obtain the expected result:

$$\mathrm{Tr}\left((\rho_{A\cup B}^{T_B})^n e^{2\pi i \alpha(\hat{Q}_A - \hat{Q}_B)}\right) = \mathcal{R}_1^n(r_A, r_B, r; \alpha)\,, \tag{3.28}$$

with $\mathcal{R}_1^n(r_A, r_B, r; \alpha)$ given by (3.14). We highlight that the knowledge of the spectrum (3.27) allows for a direct evaluation of the logarithmic negativity (1.16) without the need to obtain the moments first.

3.2.2 Single fermionic excitation

In this section we show that the result (3.14) holds for n even also if we adopt the fermionic prescription (3.4) for the partial transposition. With that definition, the matrix $\rho_{A\cup B}^{R_B}$ differs from $\rho_{A\cup B}^{T_B}$ as given in (3.26) only because there is now an extra phase in the off-diagonal elements:

$$(|10\rangle\langle 01|)^{R_B} = -i|11\rangle\langle 00|\,, \quad (|01\rangle\langle 10|)^{R_B} = -i|00\rangle\langle 11|, \tag{3.29}$$

while the diagonal elements are not modified. It follows that $\rho_{A\cup B}^{R_B}$ is still block-diagonal with respect to the imbalance operator:

$$\rho_{A\cup B}^{R_B} = (r_A)_{\Delta Q=1} \oplus \begin{pmatrix} r & -i\sqrt{r_A r_B} \\ -i\sqrt{r_A r_B} & 0 \end{pmatrix}_{\Delta Q=0} \oplus (r_B)_{\Delta Q=-1}\,. \tag{3.30}$$

The two eigenvalues in the sector $\Delta Q = 0$ can now be imaginary, depending on the values of r_A and r_B. However, we are eventually interested in the evaluation of the charged moments $\mathrm{Tr}\left(|\rho_{A\cup B}^{R_B}|^n e^{i2\pi\alpha(\hat{Q}_A - \hat{Q}_B)}\right)$ for n even, which requires only the knowledge of the eigenvalues of

$|\rho_{A\cup B}^{R_B}|^2$. From the definition (3.6), and making use of the above block diagonal decomposition we can write:

$$(\rho_{A\cup B}^{R_B})^\dagger \rho_{A\cup B}^{R_B} = (r_A^2)_{\Delta Q=1} \oplus \begin{pmatrix} r^2 + r_A r_B & ir\sqrt{r_A r_B} \\ -ir\sqrt{r_A r_B} & r_A r_B \,, \end{pmatrix}_{\Delta Q=0} \oplus (r_B^2)_{\Delta Q=-1}, \tag{3.31}$$

and

$$\sigma\left((\rho_{A\cup B}^{R_B})^\dagger \rho_{A\cup B}^{R_B}\right) = \left\{ r_A^2,\, r_B^2,\, \left(\frac{r+\sqrt{r^2+4r_A r_B}}{2}\right)^2,\, \left(\frac{r-\sqrt{r^2+4r_A r_B}}{2}\right)^2 \right\}. \tag{3.32}$$

Since these eigenvalues are the squares of the eigenvalues of $\rho_{A\cup B}^{T_B}$, the result (3.14) is recovered here for n even.

3.2.3 Multiple distinct excitations

We now consider states containing k distinct excitations. Among these states, let us focus on those which are tensor products of the linear combination (3.19), that is, on states of the form:

$$|\Psi_{\rm qb}^{(1,\dots,1)}\rangle := |\Psi_{\rm qb}^{(1)}\rangle^{\otimes k}. \tag{3.33}$$

For $k = 2$ we have for instance

$$\begin{aligned} |\Psi_{\rm qb}^{(1,1)}\rangle = {} & r_A|100\rangle\otimes|100\rangle + r_B|010\rangle\otimes|010\rangle + r|001\rangle\otimes|001\rangle \\ & + \sqrt{r_A r_B}(|100\rangle\otimes|010\rangle + |010\rangle\otimes|100\rangle) \\ & + \sqrt{r_A r}(|100\rangle\otimes|001\rangle + |001\rangle\otimes|100\rangle) \\ & + \sqrt{r_B r}(|010\rangle\otimes|001\rangle + |001\rangle\otimes|010\rangle)\,. \end{aligned} \tag{3.34}$$

For a tensor product state the density matrix is a tensor product of single-particle matrices, and this applies to the RDM and to its partial transpose as well:

$$\rho^{(1,\dots,1)} = \rho^{\otimes k}, \quad \rho_{A\cup B}^{(1,\dots,1)} = (\rho_{A\cup B})^{\otimes k}, \quad \rho_{A\cup B}^{(1,\dots,1),T_B} = (\rho_{A\cup B}^{T_B})^{\otimes k}, \tag{3.35}$$

with $\rho_{A\cup B}$ and $\rho_{A\cup B}^{T_B}$ given by (3.21) and (3.24) respectively. Decomposing the charge imbalance as $\hat{Q}_A - \hat{Q}_B = \bigoplus_{j=1}^k (\hat{Q}_A^{(j)} - \hat{Q}_B^{(j)})$, we can compute the charged replica negativities using the bosonic partial transposition (1.15):

$$\mathrm{Tr}\left(\left(\rho_{A\cup B}^{(1,\dots,1),T_B}\right)^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right) = \prod_{j=1}^k \mathrm{Tr}\left(\left(\rho_{A\cup B}^{T_B}\right)^n e^{2\pi i\alpha(\hat{Q}_A^{(j)}-\hat{Q}_B^{(j)})}\right) = (\mathcal{R}_1^n(r_A, r_B, r; \alpha))^k\,. \tag{3.36}$$

The result above is not surprising and it is a consequence of the choice of the state: if the multi-particle state is a tensor product then there is no correlation between different particles and the total charged moment is the product of the single-particle ones. As we shall see below, this is not the case when the particles are indistinguishable.

3.2.4 Multiple identical excitations

In this section we consider a state consisting of k positively charged indistinguishable excitations. Its associated qubit state can be written as:

$$|\Psi_{\rm qb}^{(k)}\rangle = \sum_{k_A,k_B,k_C\in\mathbb{N}_0} c_{k_A,k_B,k_C}\,|k_A k_B k_C\rangle, \tag{3.37}$$

where the coefficient

$$c_{k_A,k_B,k_C} := \sqrt{\frac{k! r_A^{k_A} r_B^{k_B} r^{k_C}}{k_A! k_B! k_C!}}\,\delta_{k_A+k_B+k_C,k}, \tag{3.38}$$

is the square root of the probability of finding k_A identical particles in region A, k_B identical particles in region B and the remaining k_C particles in region C. From $r_A + r_B + r = 1$ it is easy to show that if the vectors $|k_A k_B k_C\rangle$ form an orthonormal set, then also $\langle\Psi_{\rm qb}^{(k)}|\Psi_{\rm qb}^{(k)}\rangle = 1$.

From this expression it is then possible to explicitly construct the matrix elements of the (bosonic) partially transposed density matrix as:

$$\langle k_A^1 k_B^1|\rho_{A\cup B}^{T_B}|k_A^2 k_B^2\rangle = \sum_{k_C\in\mathbb{N}_0} c_{k_A^1 k_B^2 k_C} c_{k_A^2 k_B^1 k_C}, \tag{3.39}$$

where the sum represents taking the trace over the degrees of freedom in C and the partial transposition exchanges the indices k_B^1 and k_B^2 in the coefficients. The matrix elements of the nth power can then be computed inserting n resolutions of the identity in $\mathcal{H}_{A\cup B}$:

$$\begin{aligned}\langle k_A^1 k_B^1|\left(\rho_{A\cup B}^{T_B}\right)^n|k_A^{n+1} k_B^{n+1}\rangle &= \sum_{\substack{k_A^s,k_B^s\in\mathbb{N}_0\,;\,s=2,\dots,n\\ k_C^r\in\mathbb{N}_0\,;\,r=1,\dots,n}} \prod_{j=1}^{n} c_{k_A^j k_B^{j+1} k_C^j}\, c_{k_A^{j+1} k_B^j k_C^j} \\ &= \sum_{\substack{k_A^s,k_B^s\in\mathbb{N}_0\,;\,s=2,\dots,n\\ k_C^r\in\mathbb{N}_0\,;\,r=1,\dots,n}} \prod_{j=1}^{n} \frac{k! r_A^{k_A^j} r^{k_C^j} r_B^{k_B^j}}{k_A^j! k_C^j! k_B^j!}\delta_{k_A^j+k_B^{j+1}+k_C^j,k}\,\delta_{k_A^{j+1}+k_B^j+k_C^j,k},\end{aligned} \tag{3.40}$$

While expression (3.40) was already presented in [116], its symmetry-resolved version is new and can

be easily written by introducing phase factors in the sum above. We have

$$\begin{aligned}
&\langle k_A^1 k_B^1| \left(\rho_{A\cup B}^{T_B}\right)^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)} |k_A^{n+1} k_B^{n+1}\rangle \\
&= \sum_{\substack{k_A^s,k_B^s\in\mathbb{N}_0\,;\,s=2,\dots,n \\ k_C^r\in\mathbb{N}_0\,;\,r=1,\dots,n}} \prod_{j=1}^{n} \frac{k!\, r_A^{k_A^j} r^{k_C^j} r_B^{k_B^j}}{k_A^j! k_C^j! k_B^j!}\, e^{\frac{2\pi i\alpha}{n}(k_A^j-k_B^j)} \delta_{k_A^j+k_B^{j+1}+k_C^j,k}\, \delta_{k_A^{j+1}+k_B^j+k_C^j,k}.
\end{aligned} \tag{3.41}$$

Starting with this result, the derivation of equations (3.17) and (3.18) is identical to that presented in [116, 144]. The idea is to employ all the delta-function constraints on the values of k_A^j, k_B^j and k_C^j in order to reduce the number of terms in the sum. After all the constraints have been implemented, equation 3.41 reproduces the result (3.17) with (3.18).

Before concluding the section let us analyse the simplest case, $k = 1$. For a single excitation, the right-hand side of (3.17) is given by:

$$\mathcal{A}_{-1,n}\, r_B^n\, e^{-2\pi i\alpha} + \mathcal{A}_{1,0}\, r_A^n\, e^{2\pi i\alpha} + \sum_{q=0}^{[n/2]} \mathcal{A}_{0,q} (r_A r_B)^q r^{n-2q}. \tag{3.42}$$

By looking at the definition (3.18) one immediately gets $\mathcal{A}_{-1,n} = \mathcal{A}_{1,0} = 1$. On the other hand, $\mathcal{A}_{0,q} = \sum_{\{k_1,\dots,k_n\}\in P_n(q)} 1$, where each $k_j \in \{0,1\}$ and whenever $k_j = 1$ then $k_{j+1} = 0$. Counting the number of sequences $(k_1, \dots, k_n)$ that satisfy these constraints is a combinatorial problem identical to the one we solve in the next section. The number of these sequences is $\frac{n}{n-q}\binom{n-q}{n}$. As we explain in the next section, this implies that expression (3.42) exactly reproduces (3.14). More generally, looking at the coefficients (3.18) it is clear that even for $k > 1$ the computation has an underlying combinatorial interpretation. For the $\alpha = 0$ case this has been established by reinterpreting the sum (3.17) as a partition function for a certain class of graphs [144]. A combinatorial picture will emerge again in the next section in a related context: the computation using twist operators.

3.3 Twist operator approach

In this section, we provide a field-theoretic description of the charged moments of the partially transposed RDM, valid in principle for any local QFT and in any spacetime dimensions. To do so, we employ the replica construction and the formalism of twist operators that we introduced in Section 2.4. There, we showed that it is possible to compute entanglement entropies (or at least charged ratios) of quasi-particle states just by relying on a few algebraic properties: the exchange relations between local fields and twist operators and the operator product expansion (OPE) of the local fields. On the other hand, most of the theory-dependent features are hidden in the zero-point fluctuations (ground state entanglement), which factors out. In this section we show that the same applies to the logarithmic negativity.

Proceeding as we did in the previous section, here we compute the charged moments of the partial transpose in a bosonic and a fermionic theory, starting with single-particle states and generalising the result to excitations containing many quasiparticles. In the fermionic case, we discuss how to modify the algebra of twist operators and fermionic fields and, as a byproduct, we perform a simpler derivation valid for free fermions in any dimension.

3.3.1 Single bosonic excitation

We consider now a bosonic QFT, described by its algebra of local observables $\mathcal{A}$, acting on the Hilbert space $\mathcal{H}$, and $|0\rangle \in \mathcal{H}$ is the ground state of the theory. We then consider the replica version of this theory, consisting of n non-interacting copies of the same model. The algebra of observables is now denoted by $\mathcal{A}^n$, so that $\mathbb{Z}_n$ becomes an internal (global) symmetry, which includes cyclic permutation symmetry among copies [58, 59, 187, 188]. We also assume that the QFT carries an additional global $U(1)$ symmetry. This procedure allows us to introduce a set of twist operators, supported on extended spacial regions, which mix cyclic permutation and internal $U(1)$ symmetries, and generalise the notion of composite twist fields of 1+1D QFT [19, 107, 117, 156–158].

Let $\mathcal{O}^j(\mathbf{x}) \in \mathcal{A}^n$ be a local bosonic field of the j-th replica ($j = 1, \dots, n$) with $U(1)$ charge $\kappa_{\mathcal{O}}$. We consider a region A, and we associate to the twist operator T_A^α which satisfies equations (2.120). That is, the action of T_A^α is non-trivial only in the region A, where it consists of a replica shift $j \to j+1$ followed by the insertion of a $U(1)$ flux between the n-th and the first copy. The operator T_A^α is sufficient for the computation of moments of the RDM in case of a spacial bipartition. However, here we need to define a conjugate twist operator $\tilde{T}_A^\alpha$ so that

$$\tilde{T}_A^\alpha \mathcal{O}^j(\mathbf{x}) = \begin{cases} e^{-2\pi i \kappa_{\mathcal{O}} \alpha \delta_{j,1}} \mathcal{O}^{j-1}(\mathbf{x}) \tilde{T}_A^\alpha & \mathbf{x} \in A\,, \\ \mathcal{O}^j(\mathbf{x}) \tilde{T}_A^\alpha & \mathbf{x} \notin A, \end{cases} \tag{3.43}$$

and we can identify $\tilde{T}_A^\alpha = (T_A^\alpha)^\dagger$. These operators will now allow us to develop a field-theoretic formulation of the symmetry-resolved negativity and its moments. Let A and B be two non-complementary connected regions which share at most a boundary, $|\Psi\rangle \in \mathcal{H}$ a zero-density state and $\rho_{A\cup B}$ its RDM over $A \cup B$. In 1+1D QFT one can interpret the moments of the partial transpose as functions over a n-sheeted Riemann surface with two branch-cuts in A and B, connecting the replicas in two opposite directions [182]. A similar construction in the presence of fluxes has been proposed in [96, 120]. In analogy with the works above, we establish the following relation between the charged moments and the twist operators:

$$\mathrm{Tr}\left((\rho_{A\cup B}^{T_B})^n e^{2\pi i \alpha (\hat{Q}_A - \hat{Q}_B)} \right) = \frac{{}^n\langle \Psi | T_A^\alpha \tilde{T}_B^\alpha | \Psi \rangle^n}{{}^n\langle \Psi | \Psi \rangle^n}\,, \tag{3.44}$$

where $|\Psi\rangle^n$ represents the replicated version of the state $|\Psi\rangle$.

Following the construction of the previous chapter, we consider a one-particle excitation of fixed momentum $\mathbf{p}$ and charge $+1$, which is created by a field $\mathcal{O}$ acting on the ground state as

$$|\Psi^{(1)}\rangle = \mathcal{O}(\mathbf{p})|0\rangle\,. \tag{3.45}$$

Here $\mathcal{O}(\mathbf{p})$ is the Fourier transform of $\mathcal{O}(\mathbf{x})$

$$\mathcal{O}(\mathbf{p}) = \int_{\mathcal{M}} \mathrm{d}^d x e^{-i\mathbf{p}\cdot\mathbf{x}}\,\mathcal{O}(\mathbf{x})\,, \tag{3.46}$$

and $\mathcal{M}$ is the whole space, which, for simplicity, we take to be a d-dimensional torus. We note that $\mathcal{O}^\dagger(-\mathbf{p}) = [\mathcal{O}(\mathbf{p})]^\dagger$ which will be important for later computations. Our aim, as in the previous section, is to compute the ratio of charged moments, which up to a normalisation constant is given by

$$\frac{{}^n\langle\Psi^{(1)}|T_A^\alpha \tilde{T}_B^\alpha|\Psi^{(1)}\rangle^n}{{}^n\langle 0|T_A^\alpha \tilde{T}_B^\alpha|0\rangle^n}\,. \tag{3.47}$$

We define the projection of $\mathcal{O}(\mathbf{p})$ over a generic region A as the restricted integral

$$\mathcal{O}_A(\mathbf{p}) = \int_A \mathrm{d}^d x e^{-i\mathbf{p}\cdot\mathbf{x}}\mathcal{O}(\mathbf{x})\,, \tag{3.48}$$

then we can write

$$\frac{{}^n\langle\Psi^{(1)}|T_A^\alpha \tilde{T}_B^\alpha|\Psi^{(1)}\rangle^n}{{}^n\langle\Psi^{(1)}|\Psi^{(1)}\rangle^n} = \frac{{}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})T_A^\alpha \tilde{T}_B^\alpha \mathcal{O}^1(\mathbf{p})\dots\mathcal{O}^n(\mathbf{p})|0\rangle^n}{{}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})\mathcal{O}^1(\mathbf{p})\dots\mathcal{O}^n(\mathbf{p})|0\rangle^n}\,. \tag{3.49}$$

We point out that bosonic creation operators, whether on the same or on different copies, commute with each other, therefore the order of a string of operators $\mathcal{O}^j(\boldsymbol{p})$ is irrelevant. We now observe that

$$\mathcal{O}^j(\mathbf{p}) = \mathcal{O}^j_A(\mathbf{p}) + \mathcal{O}^j_B(\mathbf{p}) + \mathcal{O}^j_C(\mathbf{p}), \tag{3.50}$$

which, when inserted into (3.49), leads to a large numbers of terms both in the numerator and the denominator. The key idea is that in the infinite volume limit many of these terms are subleading and the leading contribution is the one which reproduces the expected result. We now present the details of the calculation.

Employing the exchange relations (2.120) and (3.43), we can bring all the bosonic fields $\mathcal{O}^j(\mathbf{p})$ to

the left of $T_A^\alpha \tilde{T}_B^\alpha$ in (3.49). This gives

$$\begin{aligned} {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})T_A^\alpha\tilde{T}_B^\alpha\mathcal{O}^1(\mathbf{p})\dots\mathcal{O}^n(\mathbf{p})|0\rangle^n = \\ {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})(\mathcal{O}_A^2(\mathbf{p})+\mathcal{O}_B^n(\mathbf{p})e^{-2\pi i\alpha}+\mathcal{O}_C^1(\mathbf{p}))\cdots\times \\ (\mathcal{O}_A^1(\mathbf{p})e^{2\pi i\alpha}+\mathcal{O}_B^{n-1}(\mathbf{p})+\mathcal{O}_C^n(\mathbf{p}))T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n, \end{aligned} \tag{3.51}$$

where the phases are attached only to the fields $\mathcal{O}_A^1$ and $\mathcal{O}_B^n$. To proceed with the evaluation of the expectation value, we focus on the large-volume behavior. In the previous chapter, we argued that the leading terms come from the contractions of fields belonging to the same replica, which amount to the following formal replacement inside the correlation function:

$$(\mathcal{O}^\dagger)^j_{A'}(-\mathbf{p})\mathcal{O}^j_A(\mathbf{p}) \to \langle 0|(\mathcal{O}^\dagger)^j_{A'}(-\mathbf{p})\mathcal{O}^j_A(\mathbf{p})|0\rangle \propto V_{A\cap A'}, \tag{3.52}$$

with A, $A' \subseteq \mathcal{M}$ generic spacial regions. Proportionality to the volume is valid in the large-volume limit and the proportionality constant can in principle be absorbed in the normalisation of the field $\mathcal{O}$ and does not affect the final result. We remarked in Section 2.4 that the accuracy of the approximation (3.52) is supported when the correlation length or the de Broglie length (represented by $|\mathbf{p}|^{-1}$) are significantly smaller than the size of the subsystem. This condition is an essential requirement for the excited states we aim to describe. In this regime, the vacuum expectation value of the twist operators also factors out:

$$\begin{aligned} {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})(\mathcal{O}_A^2(\mathbf{p})+\mathcal{O}_B^n(\mathbf{p})e^{-i2\pi\alpha}+\mathcal{O}_C^1(\mathbf{p}))\cdots\times \\ (\mathcal{O}_A^1(\mathbf{p})e^{i2\pi\alpha}+\mathcal{O}_B^{n-1}+\mathcal{O}_C^n(\mathbf{p}))T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n \simeq \\ {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})(\mathcal{O}_A^2(\mathbf{p})+\mathcal{O}_B^n(\mathbf{p})e^{-2\pi i\alpha}+\mathcal{O}_C^1(\mathbf{p}))\cdots\times \\ (\mathcal{O}_A^1(\mathbf{p})e^{2\pi i\alpha}+\mathcal{O}_B^{n-1}(\mathbf{p})+\mathcal{O}_C^n(\mathbf{p}))|0\rangle^n \times {}^n\langle 0|T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n, \end{aligned} \tag{3.53}$$

which means that the charge moments of the ground state factor out and will subsequently be cancelled in the ratio (3.47). While the formula above is already a large-volume approximation, when the sums are expanded and each individual term considered, many terms that are subleading for large volume still appear. To make things clear, consider the terms

$${}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})\mathcal{O}_A^2(\mathbf{p})\dots\mathcal{O}_A^n(\mathbf{p})\mathcal{O}_A^1(\mathbf{p})|0\rangle^n e^{2\pi i\alpha} \simeq V_A^n e^{2\pi i\alpha}\,, \tag{3.54}$$

and

$${}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})\mathcal{O}_B^n(\mathbf{p})\mathcal{O}_B^1(\mathbf{p})\dots\mathcal{O}_B^{n-1}(\mathbf{p})|0\rangle^n e^{-2\pi i\alpha} \simeq V_B^n e^{-2\pi i\alpha}\,. \tag{3.55}$$

These are the terms that generate the highest powers in the volume of regions A and B respectively. Among the other terms that are generated, the leading ones at large volume are those containing a

string of operators $\mathcal{O}^j(\mathbf{p})$ and their daggered versions all inserted at different replicas, just as in the examples above. If that is not the case, there is at least a pair of operators that can not be contracted as in (3.52) and the term is subleading.

We now proceed with the systematic evaluation and counting of all the leading terms generated in the expansion (3.53). We introduce the following notation to identify each term

$$(A_1 \dots A_n) := {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p}) \dots (\mathcal{O}^\dagger)^1(-\mathbf{p})\mathcal{O}^{j_1}_{A_1}(\mathbf{p}) \dots \mathcal{O}^{j_n}_{A_n}(\mathbf{p})|0\rangle^n, \tag{3.56}$$

where $A_i \in \{A, B, C\}$ and $j_i \in \{1, \dots, n\}$, $\forall i = 1, \dots, n$. We observe that, once the sequence of regions $(A_1 \dots A_n)$ is identified, the sequence of replica indices $(j_1 \dots j_n)$ is fixed unambiguously, as in fact

$$j_i = \begin{cases} i+1\,, & \text{if } A_i = A \\ i\,, & \text{if } A_i = C \\ i-1\,, & \text{if } A_i = B \end{cases}\,, \tag{3.57}$$

hence the choice of notation above. Moreover, due to the contraction rules discussed above, only the terms for which $(j_1 \dots j_n) = (\sigma(1), \dots, \sigma(n))$, with σ a permutation of the indices $\{1, \dots, n\}$ are non-vanishing. As a consequence, one can show (See Appendix 3.A.1) that the only possible non-vanishing terms fit into one of these two categories:

- Either $(A_1 \dots A_n) = (A \dots A)$ or $(A_1 \dots A_n) = (B \dots B)$, the two cases which have been already discussed in equations (3.54), (3.55),
- Or, whenever A appears in $(A_1 \dots A_n)$, it has to be followed by B. Similarly, if B appears in $(A_1 \dots A_n)$, then it has to be preceded by A.

We focus on the second set of terms. It is convenient to split this into two additional subsets, which we call type-I and type-II

- Type-I: $(A_1 \dots A_n) = (B\ A_2 \dots A_{n-1}\ A)$,
- Type-II: $(A_1 \dots A_n)$ with $A_1 \neq B$, $A_n \neq A$.

Thus, each string in both these subsets contains a number $k \geq 0$ of pairs AB and $n - 2k$ C's, and according to (3.52) each of them will yield a term proportional to $(V_A V_B)^k V_C^{n-2k}$. Due to the balance of A's and B's there is no phase present in these terms (no α dependence). We now just need to count how many strings of each type we have.

Among the strings of type-I, there is always at least one pair AB and there are at most $[n/2] - 1$ additional pairs AB that can be present. The number of such strings is precisely (a proof is given in

3.A.2):

$$\binom{n-k-1}{k-1} \quad \text{for} \quad k=0,\dots[n/2]. \tag{3.58}$$

Similarly, one can show that the number of type-II strings consisting of k pairs of consecutive A and B is:

$$\binom{n-k}{k} \quad \text{for} \quad k=0,\dots[n/2]. \tag{3.59}$$

In summary,

$$\begin{aligned} {}^n\langle 0|(\mathcal{O}^\dagger)^n(-\mathbf{p})\dots(\mathcal{O}^\dagger)^1(-\mathbf{p})(\mathcal{O}_A^2(\mathbf{p})+\mathcal{O}_B^n(\mathbf{p})e^{-i2\pi\alpha}+\mathcal{O}_C^1(\mathbf{p}))\cdots\times \\ (\mathcal{O}_A^1(\mathbf{p})e^{i2\pi\alpha}+\mathcal{O}_B^{n-1}(\mathbf{p})e^{i2\pi\alpha}+\mathcal{O}_C^n(\mathbf{p}))|0\rangle^n \simeq \\ V_A^n e^{2\pi i\alpha}+V_B^n e^{-2\pi i\alpha}+\sum_{k=0}^{[\frac{n}{2}]}\frac{n}{n-k}\binom{n-k}{k}(V_AV_B)^k(V_C)^{n-2k}, \end{aligned} \tag{3.60}$$

where we used the identity

$$\binom{n-k}{k}+\binom{n-k-1}{k-1}=\frac{n}{n-k}\binom{n-k}{k}. \tag{3.61}$$

The denominator in (3.49) can be fully contracted and yields V^n (up to a non-universal normalisation constant). Therefore, when properly normalised, the ratio (3.47) becomes a function of the variables r_A, r_B and r defined in (3.12) and we we obtain

$$\begin{aligned} \mathcal{R}_1^n(r_A,r_B,r;\alpha) &= \frac{{}^n\langle\Psi^{(1)}|T_A^\alpha\tilde{T}_B^\alpha|\Psi^{(1)}\rangle^n}{{}^n\langle\Psi^{(1)}|\Psi^{(1)}\rangle^n}\,\frac{{}^n\langle 0|0\rangle^n}{{}^n\langle 0|T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n} \\ &= r_A^n e^{2\pi i\alpha}+r_B^n e^{-2\pi i\alpha}+\sum_{k=0}^{[\frac{n}{2}]}\frac{n}{n-k}\binom{n-k}{k}(r_Ar_B)^k r^{n-2k}\,, \end{aligned} \tag{3.62}$$

which is the main result of this section. Note that although this formula looks different from (3.14), they are in fact equivalent. That is

$$\left(\frac{r+\sqrt{r^2+4r_Ar_B}}{2}\right)^n+\left(\frac{r-\sqrt{r^2+4r_Ar_B}}{2}\right)^n=\sum_{k=0}^{[\frac{n}{2}]}\frac{n}{n-k}\binom{n-k}{k}(r_Ar_B)^k r^{n-2k}\,. \tag{3.63}$$

This relation was used in [116, 144] without a proof. The proof is indeed quite involved, and can be performed using properties of the generalised Lucas' polynomials. This is presented in Appendix 3.B.1, where we also derive two interesting corollaries. The equality (3.63) is particularly interesting because it shows that the result is always a polynomial in integer powers of r_A, r_B, r for n positive, even or odd. However, its analytic continuation from n even to $n=1$ does contain a square root, as seen in (3.15).

3.3.2 Single fermionic excitation

Let us consider a theory for which the algebra $\mathcal{A}$ contains fermionic observables. In other words, we assume that $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra (superalgebra) generated by bosonic/fermionic fields, which are even/odd with respect to the $\mathbb{Z}_2$ fermionic parity. Here, to generalise properly the twist operator construction, one must take care of the fermionic nature of the fields. Indeed, two such fields sitting at distinct points, say $\Psi(\mathbf{x})$ and $\Psi(\mathbf{x}')$ will now anticommute

$$\Psi(\mathbf{x})\Psi(\mathbf{x}') = -\Psi(\mathbf{x}')\Psi(\mathbf{x}). \tag{3.64}$$

Moreover, when the replica construction is performed, we require that fermionic fields on distinct replicas also anticommute

$$\Psi^j(\mathbf{x})\Psi^{j'}(\mathbf{x}') = -\Psi^{j'}(\mathbf{x}')\Psi^j(\mathbf{x}). \tag{3.65}$$

As a result, the algebra of the replica theory $\mathcal{A}^n$ is not a conventional tensor product.

As before, we assume that an additional $U(1)$ symmetry is present in the theory. Let A be a spacial region, and we associate to it a twist operator T_A^α which shifts the replica index and appends a $U(1)$ flux to the field. The natural generalisation of (2.120) for a fermionic field Ψ of charge κ_Ψ is:

$$T_A^\alpha \Psi^j(\mathbf{x}) = \begin{cases} (-1)^{(n-1)\delta_{j,n}} e^{2\pi i \kappa_\Psi \alpha \delta_{j,n}} \Psi^{j+1}(\mathbf{x}) T_A^\alpha & \mathbf{x} \in A, \\ \Psi^j(\mathbf{x}) T_A^\alpha & \mathbf{x} \notin A. \end{cases} \tag{3.66}$$

We point out that the only difference with respect to (2.120) is the presence of an additional flux $(-1)^{n-1}$ between the n-th and the first replica, a factor that was already introduced in [162] and employed for instance in [20] in the calculation of the vacuum expectation value (VEV) of the Ising twist field. A derivation of (3.66) for $\alpha = 0$ in 1+1D using the coherent state representation of fermionic density matrices can be found in [182].

For fermionic theories we need to define another twist operator which implements explicitly the fermionic partial transposition, and from now on we only consider n even. It has been shown in [182] that the effect of the partial transposition on the fermions gives rise to an additional insertion of a flux (-1) among any pair of consecutive replicas, in addition to the usual replica shift. To implement this construction, we define a twist operator $\tilde{T}_A^\alpha$ satisfying

$$\tilde{T}_A^\alpha \Psi^j(\mathbf{x}) = \begin{cases} -(-1)^{(n-1)\delta_{j,1}} e^{-2\pi i \kappa_\Psi \alpha \delta_{j,1}} \Psi^{j-1}(\mathbf{x}) \tilde{T}_A^\alpha & \mathbf{x} \in A, \\ \Psi^j(\mathbf{x}) \tilde{T}_A^\alpha & \mathbf{x} \notin A. \end{cases} \tag{3.67}$$

We are now ready to compute the ratio of charged moments, along the same lines of the previous

computation. Namely, given a fermionic field $\Psi(\mathbf{x})$ with $U(1)$ charge $+1$, we consider the state

$$|\Psi^{(1)}\rangle = \Psi(\mathbf{p})|0\rangle, \tag{3.68}$$

and its replicated version

$$|\Psi^{(1)}\rangle^n = \Psi^1(\mathbf{p})\dots\Psi^n(\mathbf{p})|0\rangle. \tag{3.69}$$

Given two regions A and B defined as in the previous section, we express the ratio of charged moments of the partial transpose also in this case as (up to a normalisation constant)

$$\frac{{}^n\langle\Psi^{(1)}|T_A^\alpha\tilde{T}_B^\alpha|\Psi^{(1)}\rangle^n}{{}^n\langle 0|T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n}\,. \tag{3.70}$$

We then expand the expectation value of the twist operators as follows:

$$\begin{aligned}
&{}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p})\dots(\Psi^\dagger)^1(-\mathbf{p})(\Psi_A^2(\mathbf{p})+\Psi_B^n(\mathbf{p})e^{-2\pi i\alpha}+\Psi_C^1(\mathbf{p}))\cdots\times\\
&\qquad(-\Psi_A^1(\mathbf{p})e^{2\pi i\alpha}-\Psi_B^{n-1}(\mathbf{p})e^{2\pi i\alpha}+\Psi_C^n(\mathbf{p}))T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n\simeq\\
&{}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p})\dots(\Psi^\dagger)^1(-\mathbf{p})(\Psi_A^2(\mathbf{p})+\Psi_B^n(\mathbf{p})e^{-2\pi i\alpha}+\Psi_C^1(\mathbf{p}))\cdots\times\\
&\qquad(-\Psi_A^1(\mathbf{p})e^{2\pi i\alpha}-\Psi_B^{n-1}(\mathbf{p})+\Psi_C^n(\mathbf{p}))|0\rangle^n\times{}^n\langle 0|T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n.
\end{aligned} \tag{3.71}$$

As in the bosonic case, many terms are generated from the expansion of the above expression, and we can apply similar considerations as to which of these terms are leading and which are sub-leading in the large-volume limit. However, since the fermionic fields anticommute, we need to pay attention to the order of the fields. For example, following the notation (3.56), the term $(A\ A\dots A)$ can be evaluated to

$$\begin{aligned}
&{}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p})\dots(\Psi^\dagger)^1(-\mathbf{p})\Psi_A^2(\mathbf{p})\dots\Psi_A^1(\mathbf{p})|0\rangle^n(-e^{2\pi i\alpha})=\\
&{}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p})\dots(\Psi^\dagger)^1(-\mathbf{p})\Psi_A^1(\mathbf{p})\Psi_A^2(\mathbf{p})\dots\Psi_A^n(\mathbf{p})|0\rangle^n e^{2\pi i\alpha}\simeq e^{2\pi i\alpha}V_A^n,
\end{aligned} \tag{3.72}$$

where $\Psi_A^1(\mathbf{p})$ has been recast in the first position after crossing $n-1$ (odd) fermions, thus acquiring an additional phase -1. Similarly, it is easy to show that

$${}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p})\dots(\Psi^\dagger)^1(-\mathbf{p})\Psi_B^n(\mathbf{p})\dots\Psi_B^{n-1}(\mathbf{p})|0\rangle^n(-e^{-2\pi i\alpha})\simeq V_B^n e^{-2\pi i\alpha}. \tag{3.73}$$

In general, each term of the expansion is weighted with a phase which arises from the commutation relations between twist operators and fermions and from those between fermions and fermions. This is the crucial difference with respect to the calculation presented for the boson. We can summarise

the total contribution to the phase for a generic term[2]

$$(A_1 \dots A_n) := {}^n\langle 0|(\Psi^\dagger)^n(-\mathbf{p}) \dots (\Psi^\dagger)^1(-\mathbf{p})\Psi^{j_1}_{A_1}(\mathbf{p}) \dots \Psi^{j_n}_{A_n}(\mathbf{p})|0\rangle^n, \tag{3.74}$$

as follows:

- If $j_n = 1$ and $A_n = A$ there is a $-e^{2\pi i\alpha}$ phase. Similarly, if $j_1 = 1$, and $A_1 = B$, there is a contribution of $-e^{-2\pi i\alpha}$.
- In addition to the previous phase, an additional -1 is present for each B which appears in the string $(A_2 \ \dots \ A_n)$. This is due to the fermionic partial transposition over B.
- Once the contraction is performed, there is a sign coming from the order of the fields. Given $(j_1 \ \dots \ j_n) = (\sigma(1) \dots \sigma(n))$, with σ a permutation of the replica indices, one can show that the sign appearing after the contraction is $\mathrm{sign}(\sigma)$.

Having suitably modified the definition of the twist operators for the fermion, the phase of each term appearing in (3.71) is the same as the one for the corresponding term in (3.53), leading to the same result for fermions and bosons. To see this, let us first analyse a term of type-I:

$$(A_1, \dots, A_n) = (B, A_2, \dots, A_{n-1}, A). \tag{3.75}$$

The phases coming from the last A and the first B cancel each other. Let $A_{i+1} = B$: then this must be preceded by $A_i = A$. The resulting replica indices at the corresponding positions are $j_{i+1} = i$ and $j_i = i + 1$. In other words, there is an exchange of the replica indices i and $i + 1$, which changes the sign of the permutation σ and it contributes as -1 after the contraction, but it is compensated by the -1 due to the presence of a B. Similar considerations apply straightforwardly to the strings of type-II. Putting everything together, the same formula (3.14) is obtained again, which is the final result. We emphasise that this derivation relies on the assumption that n is even, something that was not necessary for a single bosonic excitation.

3.3.2.1 Single fermionic excitation via replica diagonalisation

In this section, we show that the result just derived for fermions in full generality can also be obtained via replica diagonalisation in the free case in 1+1D. This is essentially the procedure employed in Section 2 of the previous chapter, but the twist operator approach, together with the simple assumptions on the large-volume behaviour of the OPE, allows us to bypass the lengthy form factor calculations. The key observation is that, as show in Ref. [182], the fermionic partial transpose of a Gaussian state is still Gaussian. This allows to simplify the analysis of the replica theory, reducing it

[2] given a sequence $(A_1 \dots A_n)$, the indices j_i are fixed by (3.57) also for the fermion.

to a single-replica model in the presence of proper fluxes. For instance, given a Gaussian state and its RDM, and taking again n even, one can show that the following factorisation [182] holds:

$$\mathrm{Tr}\left(|\rho_{A\cup B}^{R_B}|^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right)=\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\mathrm{Tr}\left(\rho_{A\cup B}e^{\frac{2\pi i(\alpha+p)}{n}(\hat{Q}_A-\hat{Q}_B)+i\pi\hat{Q}_B}\right). \tag{3.76}$$

Each term appearing inside the product is a single-copy charged partition function with fluxes along A and B. Thus, we can evaluate the ratio of charged moments of the partial transpose as a product of ratios of single-copy correlators of twist operators, the only difference being that now we need to shift the flux α in T_A^α:

$$\mathcal{R}_1^n(r_A,r_B,r;\alpha)=\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\frac{\mathrm{Tr}\left(\rho_{A\cup B}e^{\frac{2\pi i(\alpha+p)}{n}(\hat{Q}_A-\hat{Q}_B)+i\pi\hat{Q}_B}\right)}{\mathrm{Tr}\left(\rho_{A\cup B,0}e^{\frac{2\pi i(\alpha+p)}{n}(\hat{Q}_A-\hat{Q}_B)+i\pi\hat{Q}_B}\right)}\propto\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\frac{\langle\Psi|T_A^{\alpha+p}T_B^{-\alpha-p+\frac{1}{2}}|\Psi\rangle}{\langle 0|T_A^{\alpha+p}T_B^{-\alpha-p+\frac{1}{2}}|0\rangle}, \tag{3.77}$$

where the twist operators are defined by (3.66) with $n=1$ and the proportionality constant is fixed by normalising the state. A similar calculation as in the previous sections for a one-particle states gives

$$\begin{aligned}\langle\Psi^{(1)}|T_A^\alpha T_B^{-\alpha+\frac{1}{2}}|\Psi^{(1)}\rangle=\langle 0|\Psi^\dagger(\mathbf{p})T_A^\alpha T_B^{-\alpha+\frac{1}{2}}\Psi(\mathbf{p})|0\rangle=\\ \langle 0|\Psi^\dagger(\mathbf{p})(\Psi_A(\mathbf{p})e^{2\pi i\alpha}-\Psi_B(\mathbf{p})e^{-2\pi i\alpha}+\Psi_C(\mathbf{p}))T_A^\alpha T_B^{-\alpha+\frac{1}{2}}|0\rangle\simeq\\ \langle 0|T_A^\alpha T_B^{-\alpha+\frac{1}{2}}|0\rangle(V_Ae^{2\pi i\alpha}-V_Be^{-2\pi i\alpha}+V_C),\end{aligned} \tag{3.78}$$

so that:

$$\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}\frac{\langle\Psi^{(1)}|T_A^{\alpha+p}T_B^{-\alpha-p+\frac{1}{2}}|\Psi^{(1)}\rangle}{\langle 0|T_A^{\alpha+p}T_B^{-\alpha-p+\frac{1}{2}}|0\rangle}\frac{\langle 0|0\rangle}{\langle\Psi^{(1)}|\Psi^{(1)}\rangle}=\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}(r_Ae^{\frac{2\pi i(\alpha+p)}{n}}-r_Be^{-\frac{2\pi i(\alpha+p)}{n}}+r). \tag{3.79}$$

This product can be shown yet again to be equal to (3.14), although the proof requires some mathematical identities that we present in Appendix (3.B.2).

3.3.3 Bosonic state with multiple distinct excitations

Consider now a k-particle bosonic state where all particles have the same (unitary) charge and momentum $\mathbf{p}$. In order to ensure the presence of $U(1)$ symmetry we consider the complex free boson, described by a field $\mathcal{O}$ which satisfies Wick's theorem in the vacuum state. We thus describe the excited state as

$$|\Psi^{(k)}\rangle=(\mathcal{O}(\mathbf{p}))^k|0\rangle. \tag{3.80}$$

Before entering the core of the computation in the replica model, it is convenient to slightly modify the definition (2.120) of the twist operators as follows

$$T_A^\alpha \mathcal{O}^j(\mathbf{x}) = \begin{cases} e^{\frac{2\pi i\alpha}{n}} \mathcal{O}^{j+1}(\mathbf{x}) T_A^\alpha & \mathbf{x} \in A, \\ \mathcal{O}^j(\mathbf{x}) T_A^\alpha & \mathbf{x} \notin A. \end{cases} \tag{3.81}$$

This amounts to distributing the total flux $e^{2\pi i\alpha}$ in equal part among all copies, rather than inserting it between the n-th and the first replica only. The expectation values are not affected by this choice, which is however computationally useful. An analogous fractionalisation will be considered for $\tilde{T}_A^\alpha$. We can now evaluate

$$\frac{{}^n\langle \Psi^{(k)} | T_A^\alpha \tilde{T}_B^\alpha | \Psi^{(k)} \rangle^n}{{}^n\langle \Psi^{(k)} | \Psi^{(k)} \rangle^n} = \frac{{}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k T_A^\alpha \tilde{T}_B^\alpha \left(\mathcal{O}^1(\mathbf{p})\right)^k \dots \left(\mathcal{O}^n(\mathbf{p})\right)^k |0\rangle^n}{{}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k \left(\mathcal{O}^1(\mathbf{p})\right)^k \dots \left(\mathcal{O}^n(\mathbf{p})\right)^k |0\rangle^n}. \tag{3.82}$$

The denominator, required to ensure normalisation, can be computed using Wick's theorem, which gives the expectation value as a sum over the possible contractions:

$$\begin{aligned} {}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k \left(\mathcal{O}^1(\mathbf{p})\right)^k \dots \left(\mathcal{O}^n(\mathbf{p})\right)^k |0\rangle^n = \\ \left[\langle 0 | \left((\mathcal{O}^\dagger)(-\mathbf{p})\right)^k \left(\mathcal{O}(\mathbf{p})\right)^k |0\rangle\right]^n \simeq V^{nk}(k!)^n. \end{aligned} \tag{3.83}$$

The numerator in (3.82) can be manipulated in a way analogous to the single-particle state, now using the exchange relation (3.81) :

$$\begin{aligned} {}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k T_A^\alpha \tilde{T}_B^\alpha \left(\mathcal{O}^1(\mathbf{p})\right)^k \dots \left(\mathcal{O}^n(\mathbf{p})\right)^k |0\rangle^n = \\ {}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k \left(\mathcal{O}_A^2(\mathbf{p}) e^{\frac{2\pi i\alpha}{n}} + \mathcal{O}_B^n(\mathbf{p}) e^{-\frac{2\pi i\alpha}{n}} + \mathcal{O}_C^1(\mathbf{p})\right)^k \cdots \times \\ \left(\mathcal{O}_A^1(\mathbf{p}) e^{\frac{2\pi i\alpha}{n}} + \mathcal{O}_B^{n-1}(\mathbf{p}) e^{-\frac{2\pi i\alpha}{n}} + \mathcal{O}_C^n(\mathbf{p})\right)^k T_A^\alpha \tilde{T}_B^\alpha |0\rangle^n \simeq \\ {}^n\langle 0 | \left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k \dots \left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k \left(\mathcal{O}_A^2(\mathbf{p}) e^{\frac{2\pi\alpha}{n}} + \mathcal{O}_B^n(\mathbf{p}) e^{-\frac{2\pi i\alpha}{n}} + \mathcal{O}_C^1(\mathbf{p})\right)^k \cdots \times \\ \left(\mathcal{O}_A^1(\mathbf{p}) e^{\frac{2\pi i\alpha}{n}} + \mathcal{O}_B^{n-1}(\mathbf{p}) e^{-\frac{2\pi i\alpha}{n}} + \mathcal{O}_C^n(\mathbf{p})\right)^k |0\rangle^n \times {}^n\langle 0 | T_A^\alpha \tilde{T}_B^\alpha | 0 \rangle^n. \end{aligned} \tag{3.84}$$

The evaluation of the previous expression in the general case is a hard combinatorial task, but many crucial features are already apparent. Namely, it is clear that 3^{nk} terms are generated simply from the expansion of all the products. Each of these terms is a string containing nk daggered operators $(\mathcal{O}^\dagger)^j(-\mathbf{p})$, where every $j \in \{1, \dots, n\}$ appears exactly k times, followed by nk operators $\mathcal{O}_{A_i}^j(\mathbf{p})$: to have a non vanishing expectation value, also in these latter operators every $j \in \{1, \dots, n\}$ must appear exactly k times. Each of the non vanishing terms can then be evaluated via Wick's theorem and will give rise to precisely $(k!)^n$ identical contractions, thus canceling the combinatorial factor in Eq. (3.83). Moreover, whenever the restriction of $\mathcal{O}_A^j(\mathbf{p})$ over the region A appears, a factor $V_A e^{\frac{2\pi i\alpha}{n}}$

is present after the Wick contraction; similarly, a factor $V_B e^{-\frac{2\pi i\alpha}{n}}$ appears with every B and a factor V_C for every C. Putting everything together, we can infer the general structure

$$\begin{aligned} {}^n\langle 0|\left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k\dots\left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k\left(\mathcal{O}_A^2(\mathbf{p})e^{2\pi i\alpha/n}+\mathcal{O}_B^n(\mathbf{p})e^{-2\pi i\alpha/n}+\mathcal{O}_C^1(\mathbf{p})\right)^k\cdots\times \\ \left(\mathcal{O}_A^1(\mathbf{p})e^{2\pi i\alpha/n}+\mathcal{O}_B^{n-1}(\mathbf{p})e^{-2\pi i\alpha/n}+\mathcal{O}_C^n(\mathbf{p})\right)^k|0\rangle^n \simeq \\ (k!)^n\sum_{k_A,k_B}C_{k_A,k_B}V_A^{k_A}V_B^{k_B}(V_C)^{nk-k_A-k_B}e^{\frac{2\pi i\alpha(k_A-k_B)}{n}}, \end{aligned} \tag{3.85}$$

so that the expectation value is a homogeneous polynomial of degree nk in V_A, V_B, V_C, and C_{k_A,k_B} is a combinatorial coefficient. Thus, generalising (3.62), the ratio of charged moments is:

$$\begin{aligned} \mathcal{R}_k^n(r_A,r_B,r;\alpha) &= \frac{{}^n\langle\Psi^{(1)}|T_A^\alpha\tilde{T}_B^\alpha|\Psi^{(1)}\rangle^n}{{}^n\langle\Psi^{(1)}|\Psi^{(1)}\rangle^n}\,\frac{{}^n\langle 0|0\rangle^n}{{}^n\langle 0|T_A^\alpha\tilde{T}_B^\alpha|0\rangle^n} \\ &= \sum_{k_A,k_B}C_{k_A,k_B}{r_A}^{k_A}{r_B}^{k_B}r^{nk-k_A-k_B}e^{\frac{2\pi i\alpha(k_A-k_B)}{n}}, \end{aligned} \tag{3.86}$$

valid as usual in the infinite volume limit. The closed formula for the combinatorial coefficient C_{k_A,k_B} at any k and n is difficult to obtain by this method, but it has been obtained earlier for simpler qubit states. Indeed, The numbers C_{k_A,k_B} are nothing but the coefficients $\mathcal{A}_{p,q}$ in (3.18). Rather then giving a complete proof of this claim, in the next section we study in detail a simple example.

3.3.3.1 An example: $k = n = 2$

Let us consider the example $n = 2$ and $k = 2$ to get an idea of how the combinatorics of (3.85) works in this case. First, we define the symbol

$$\left(\begin{smallmatrix} A_1 & \dots & A_{nk} \\ j_1 & \dots & j_{nk}\end{smallmatrix}\right) :=^n \langle 0|\left((\mathcal{O}^\dagger)^n(-\mathbf{p})\right)^k\dots\left((\mathcal{O}^\dagger)^1(-\mathbf{p})\right)^k\mathcal{O}_{A_1}^{j_1}(\mathbf{p})\dots\mathcal{O}_{A_{kn}}^{j_{kn}}(\mathbf{p})|0\rangle^n. \tag{3.87}$$

Among the strings which are generated, we only keep those for which any replica index j appears exactly k times among $(j_1 \dots j_{nk})$, as all others will vanish after Wick contractions. For $n = k = 2$ the length of the strings above is $nk = 4$. We notice that, unlike the case of a single excitation, the indices j_i in (3.87) are not uniquely fixed by the corresponding A_i's: in particular, there may be different permutations of the A_i's corresponding to the same sequence of j_i's. For any given string, there are others that can be obtained via the following permutations of the A_i indices:

$$\begin{aligned} &(A_1\ A_2\ A_3\ A_4)\to(A_2\ A_1\ A_3\ A_4),\quad (A_1\ A_2\ A_3\ A_4)\to(A_1\ A_2\ A_4\ A_3), \\ &(A_1\ A_2\ A_3\ A_4)\to(A_3\ A_4\ A_1\ A_2), \end{aligned} \tag{3.88}$$

corresponding to a swap of the two operators in the first copy, a swap of the two operators in the second copy and to a cyclic permutation of the copies respectively. All these terms contribute equally.

Furthermore, as noticed in the previous section, each non-vanishing string yields a combinatorial factor $(k!)^n = 4$. These considerations allow us to slightly simplify the combinatorial counting, and we only list the terms coming from strings up to the transformations generated by Eq. (3.88), taking care of the degeneracy for each representative distinct string. Up to the combinatorial factors, the contributing terms are:

- $(A\ A\ A\ A)$: it yields $(V_A e^{\frac{2\pi i\alpha}{2}})^4$.
- $(B\ B\ B\ B)$: it yields $(V_B e^{-\frac{2\pi i\alpha}{2}})^4$.
- $(A\ A\ B\ B)$ and $(A\ B\ A\ B)$: they yield $6V_A^2V_B^2$.
- $(B\ A\ A\ A)$: it yields $4(V_A e^{\frac{2\pi i\alpha}{2}})^3(V_B e^{-\frac{2\pi i\alpha}{2}})$.
- $(A\ B\ B\ B)$: it yields $4(V_B e^{-2\pi i\alpha/2})^3(V_A e^{2\pi i\alpha/2})$.
- $(A\ C\ B\ C)$: it yields $8V_AV_BV_C^2$.
- $(C\ C\ C\ C)$: it yields V_c^4.
- $(A\ C\ A\ C)$: it yields $4V_c^2(V_A e^{\frac{2\pi i\alpha}{2}})^2$.
- $(B\ C\ B\ C)$: it yields $4V_C^2(V_B e^{-\frac{2\pi i\alpha}{2}})^2$.

Putting all these pieces together and diving the result by (3.83), we obtain

$$\begin{aligned}\mathcal{R}_2^2(r_A, r_B, r; \alpha) = r_A^4 e^{4\pi i\alpha} + r_B^4 e^{-4\pi i\alpha} + 6r_A^2 r_B^2 + 4r_A^3 r_B e^{2\pi i\alpha} + 4r_B^3 r_A e^{-2\pi i\alpha} + \\ 8r_A r_B r^2 + r^4 + 4r_A^2 r^2 e^{2\pi i\alpha} + 4r_B^2 r^2 e^{-2\pi i\alpha}.\end{aligned} \tag{3.89}$$

This result is consistent with the one obtained from a direct evaluation of the right hand side of (3.17) for $n = 2$, $k = 2$.

3.4 Numerics

In this section, we present numerical results for a 1D lattice Fermi gas, the same model we already considered in Section 2.5.1. In particular, we consider the ground state at vanishing chemical potential, which is a Fermi sea and has critical features. This Fermi sea is then excited through the insertion of an additional particle above the Fermi energy at large momentum. We aim to compute the ratio of charged moments for a state of a single excitation and show the validity of result (3.14) numerically. Agreement with the latter confirms the claim made earlier in this chapter and in the previous one, namely that while the ground state exhibits theory-dependent, highly non-trivial behavior (in this case, captured by a free fermion CFT [120]), the contribution given by the excitation is universal.

3.4.1 The method

Let us consider a Fermi chain of length L described by the fermionic operators $\{f_j, f_j^\dagger\}_{j=1,\dots,L}$ satisfying the standard anticommutation relations

$$\{f_j, f_{j'}\} = \{f_j^\dagger, f_{j'}^\dagger\} = 0, \qquad \{f_j, f_{j'}^\dagger\} = \delta_{jj'}. \tag{3.90}$$

We choose a Gaussian state with a given number of particles and consider its correlation matrix, denoted by

$$C(j, j') = \langle f_j^\dagger f_{j'} \rangle, \quad j, j' = 1, \dots, L. \tag{3.91}$$

Let us further define the $L \times L$ covariance matrix

$$\Gamma = 1 - 2C. \tag{3.92}$$

Given any two disjoint spacial subsystems A and B, of length ℓ_A, ℓ_B respectively, the restriction of Γ over $A \cup B$ is a $(\ell_A + \ell_B) \times (\ell_A + \ell_B)$ matrix defined by

$$\Gamma_{A\cup B} = \begin{pmatrix} \Gamma_{AA} & \Gamma_{AB} \\ \Gamma_{BA} & \Gamma_{BB} \end{pmatrix}, \tag{3.93}$$

Following [189] one can show that, if $\rho_{A\cup B}$ is the RDM for this system, the fermionic partial transposition $\rho_{A\cup B}^{R_B}$ is Gaussian. Moreover, since in general $\rho_{A\cup B}^{R_B}$ is not Hermitian, it is convenient to introduce a matrix $\rho^\times$ defined as

$$\rho^\times = \frac{(\rho_{A\cup B}^{R_B})(\rho_{A\cup B}^{R_B})^\dagger}{\mathrm{Tr}(\rho_{A\cup B}^2)}. \tag{3.94}$$

From the definition of partial time-reversal transposition, it is possible to show [190] that $\rho^\times$ has unit trace. If one interprets $\rho^\times$ as an unphysical mixed state of $A \cup B$, its associated covariance matrix is[3] (See [189, 191])

$$\Gamma_{A\cup B}^\times \equiv \frac{2}{1 + \Gamma_{A\cup B}^2} \begin{pmatrix} \Gamma_{AA} & 0 \\ 0 & -\Gamma_{BB} \end{pmatrix}. \tag{3.95}$$

One can then express the even charged moments of the partially transposed RDM in terms of the eigenvalues of $\Gamma_{A\cup B}$ and $\Gamma_{A\cup B}^\times$ as (See also [120])

$$\begin{aligned} \log \mathrm{Tr}\left(|\rho_{A\cup B}^{R_B}|^n e^{2\pi i \alpha (\hat{Q}_A - \hat{Q}_B)}\right) = \mathrm{Tr} \log \left(\left(\frac{1 - \Gamma_{A\cup B}^\times}{2} \right)^{\frac{n}{2}} e^{2\pi i \alpha} + \left(\frac{1 + \Gamma_{A\cup B}^\times}{2} \right)^{\frac{n}{2}} \right) \\ + \frac{n}{2} \mathrm{Tr} \log \left(\left(\frac{1 + \Gamma_{A\cup B}}{2} \right)^2 + \left(\frac{1 - \Gamma_{A\cup B}}{2} \right)^2 \right). \end{aligned} \tag{3.96}$$

[3] Strictly speaking, the correlation matrix is unitarily equivalent to the one in Eq. (3.95), as shown in [190]. However, this is not important for our purpose, as we are interested on its spectrum only.

We stress that Eq. (3.96) makes sense also if n is not an even integer, and it naturally provides the analytic continuation over n for the even charged moments.

3.4.2 Lattice Fermi gas

For our numerics we take the Hamiltonian of a lattice free Fermi gas on a ring of length L

$$H = -\frac{1}{2}\sum_j f^\dagger_{j+1} f_j + f^\dagger_j f_{j+1}\,. \tag{3.97}$$

Its ground state is a Fermi sea, with Fermi momentum $k_F = \pi/2$, and its correlation matrix is

$$C_0(j,j') := \langle f^\dagger_j f_{j'}\rangle_0 = \frac{\sin k_F(j-j')}{L\sin\frac{\pi(j-j')}{L}}\,. \tag{3.98}$$

We then consider the excited state obtained via the insertion of a particle at momentum

$$k = k_F + \frac{\pi}{2} - \frac{\pi}{L}, \tag{3.99}$$

above the Fermi sea, whose correlation matrix is

$$C(j,j') = C_0(j,j') + \frac{1}{L}e^{-i(k_F+\frac{\pi}{2}-\frac{\pi}{L})(j-j')}. \tag{3.100}$$

Notice that this state differs from the particle-hole excitation considered in 2.5.1. While the specific choice of k is irrelevant for our purpose, it is important to require that $k - k_F$ is finite in the thermodynamic limit. We then consider the subsystems

$$A = \{1,\dots,\ell_A\}, \quad B = \{\ell_A+1,\dots,\ell_A+\ell_B\}, \tag{3.101}$$

and we fix the size of $A\cup B$ to be half the subsystem size:

$$\frac{\ell_A+\ell_B}{L} = \frac{1}{2}. \tag{3.102}$$

Finally, by using (3.96) we evaluate numerically the difference of charged Rényi negativities

$$\mathcal{E}_n(\alpha) - \mathcal{E}_{n,0}(\alpha) := \log\frac{\mathrm{Tr}\left(|\rho^{R_B}_{A\cup B}|^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right)}{\mathrm{Tr}\left(|\rho^{R_B}_{A\cup B,0}|^n e^{2\pi i\alpha(\hat{Q}_A-\hat{Q}_B)}\right)}, \tag{3.103}$$

for some values of the flux α as a function of $r_A = \ell_A/L$, and we compare it with the prediction (3.14) (with $r_B = \ell_B/L = 1/2 - r_A$). In figures 3.1 and 3.2 we show the results for a chain of length $L = 400$ and given values of n and α, while varying the value of r_A from 0 to 1/2. We consider also non-even

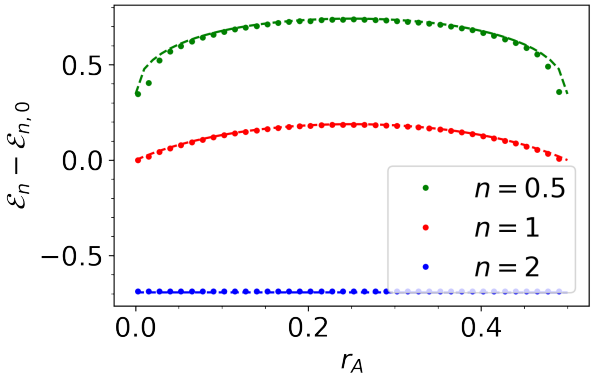


Figure 3.1 Difference of (uncharged, $\alpha = 0$) Rényi negativities for the one-particle state at $n = 0.5, 1, 2$. Note that although we derived equation (3.14) for n even via the replica approach, we can analytically continue the latter to any value of n. The theoretical results are the dashed lines, and the numerical values are the dots.

values of n, and we compare the numerics with the analytic continuation (over the even integers) of our predictions. The agreement between the numerical results and equation (3.14) is good, even if there are small discrepancies for small values of ℓ_A or ℓ_B (corresponding to $r_A \simeq 0$ and $r_A \simeq 0.5$) due to finite-size effects which vanish in the large-volume limit.

3.5 Concluding remarks

In this chapter we concluded the investigation of $U(1)$ symmetry-resolved entanglement measures in zero-density excited states. Zero-density here means that volume is taken to infinity, while the number of excitations above the ground state (which may be trivial, as for a qubit state, or highly non-trivial as in QFT) is kept fixed and finite. The results presented in this chapter and in the previous one extend work on entanglement measures for zero-density excited states carried out in [114–116, 144]. Other important contributions to this research field are [146, 148–150, 152, 192, 193]. In line with the results of the previous chapter for the Rényi entropies, we expected and indeed found that the contribution of a finite number of excitations to the symmetry-resolved (logarithmic) negativity is given by a simple formula, a polynomial on the variables r_A, r_B and $r = 1 - r_A - r_B$, which represent the relative sizes of two subsystems A and B and their complement, respectively. For the symmetry-resolved moments of the negativity, this polynomial depends also on a parameter α related to the internal $U(1)$ symmetry of the theory. The formulae that we obtained generalise the results for the uncharged moments in a simple way and are consistent with numerical results. However, some of the methods that we have employed to obtain these results are quite new and have potential for further use.

The method of twist operators, suggested by the computation of entanglement measures in d-dimensional free bosonic theories [144], was introduced in the previous chapter and here we provided a very non-trivial check of its validity. From this method alone, we can claim that our

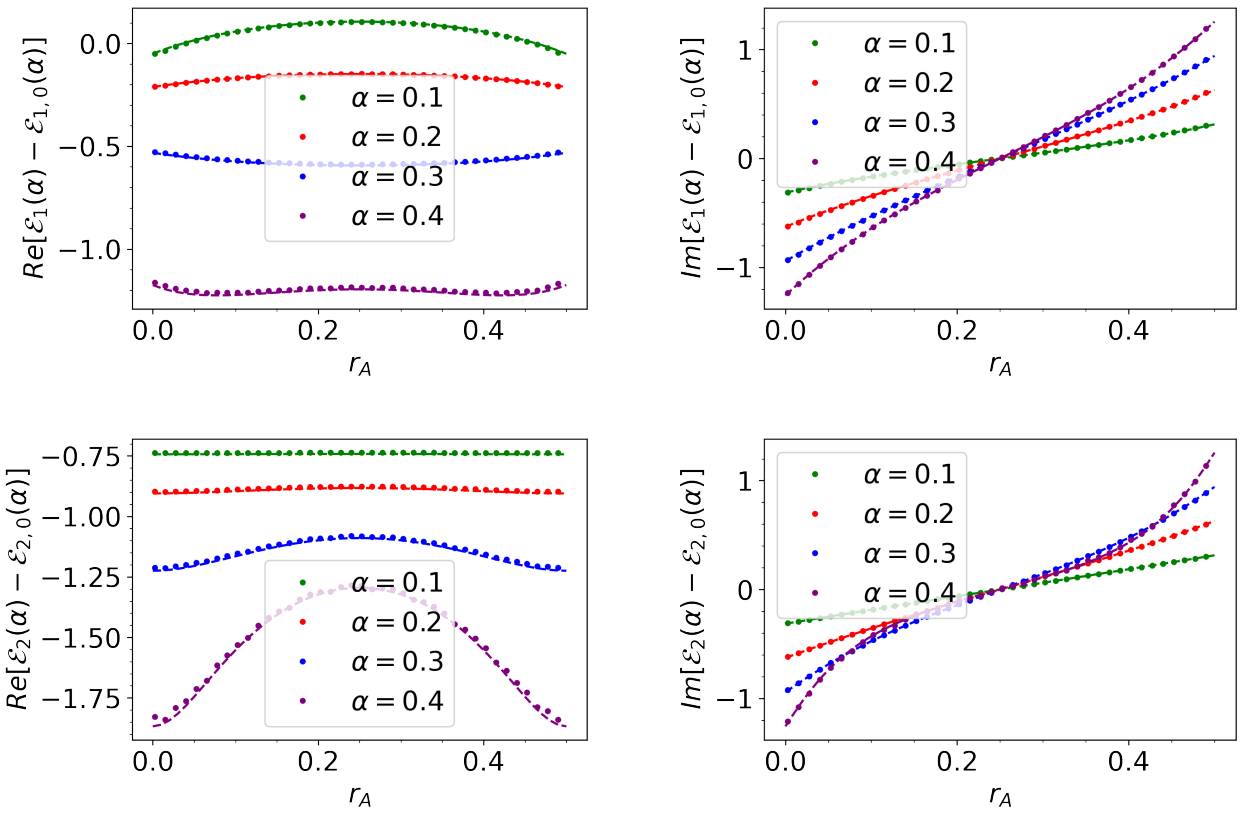


Figure 3.2 Difference of charged Rényi negativities for the one-particle state at flux $\alpha = 0.1, 0.2, 0.3, 0.4$ and $n = 1, 2$ evaluated numerically (dots) versus the analytical predictions (3.14) (dashed lines). The left (right) panels show the real (imaginary) part of $\mathcal{E}_n(\alpha) - \mathcal{E}_{n,0}(\alpha)$. The size of the chain is $L = 400$, and we plot the results as functions of $r_A \in (0, 1/2)$.

formulae should be valid in any dimensionality and in the presence of short-range interactions as well. Compared to a computation based on branch-point twist fields for free QFT, as performed for the negativity in [116], the use of twist operators captures the same universal result through a significantly simpler computation. A further application of twist operators and one of the most interesting and novel results of this chapter is the fact these operators can be easily adapted to treat particles with both fermionic and bosonic statistics. In particular, it has been known for some time that the negativity of fermionic theories requires a redefinition of the operation of partial transposition [181, 182]. Here we find that, first, this redefinition is easy to implement in the context of twist operators and, second, that once implemented it leads to a result which is the same as for bosons. This ties in well with the idea that the universal part of the entanglement associated with these types of excitation has a semiclassical interpretation (as recently explored in [152]), thus the statistics of excitations plays no role at leading order in the large-volume expansion.

Looking ahead, there are many directions to explore in relation to the role of quasiparticle excitations in the context of symmetry-resolved entanglement measures. Via twist operators, any new measures of charged entanglement should be computable by a suitable redefinition of the

operators. In particular, a quantum distance such as the relative entropy [113] can be computed to detect a difference in the entanglement content of zero-density states which have the same symmetry-resolved entropy and/or negativity. The symmetry-resolved relative entropy of excited states in 1+1D CFT was studied in [97].

Equally interesting would be the investigation of the crossover from low- to high-energy states in CFT, from the model-dependent predictions of [154] to the sort of universal results obtained in [1, 2, 114–116, 144] and here. The results of [154] apply to low-lying excited states of CFT, whereas the universal formulae obtained for zero-density excited states apply for large momentum/energy. This suggests that there must be a crossover between these two behaviours, which could be understood using CFT arguments.

Finally, twist operators seem to be a promising approach to computing entanglement measures in limiting cases where many details of the interaction can be neglected, i.e. the semiclassical limit. A field where these ideas can be applied is that of out-of-equilibrium protocols [136, 194]. In this context, characterising the entanglement growth for free or interacting theories in any dimension via the twist operator approach constitutes an interesting challenge. For some protocols, such as a global quench, we expect that the linear growth of entanglement may be captured by a semiclassical approximation of correlation functions, similar to what we considered in this chapter and in the previous one.

3.A Combinatorics

3.A.1 Non-vanishing strings

Here, we show which terms in (3.53) give rise to non-vanishing contractions in the large-volume limit, following the notation introduced in (3.56). As noted after equation (3.56), the strings that produce leading powers of the volume are those where the sequence of indices $(j_1, \dots, j_n)$ is a permutation $(\sigma(1), \dots, \sigma(n))$, and the j_i are further constrained by (3.57).

Now suppose that $A_i = A$ for one value of i. this means that $j_i = i + 1$ and thus it can be either $j_{i+1} = i + 2$, which fixes $A_{i+1} = A$ or $j_{i+1} = i$, which fixes $A_{i+1} = B$. In the first case the fact that $A_i = A_{i+1} = A$ selects all the other $A_i = A$, while in the second case we simply have a pair AB. Analogously, we can start with $A_i = B$, $j_i = i - 1$. Now the two possibilities are $j_{i-1} = i$, that is $A_{i-1} = A$ and thus again we have a pair AB, or $j_{i-1} = i - 2$, that is $A_{i-1} = B$ and thus $A_i = B$ for every i. To summarise, there are only three possibilities: $A_i = A$ for every i, $A_i = B$ for every i, and finally whenever $A_i = A$, $A_{i+1} = B$. This is the claim presented after equation (3.57).

3.A.2 Combinatorial counting of strings

In this appendix we count the number of non-vanishing strings

$$(A_1 \dots A_n), \tag{3.104}$$

containing k pairs of consecutive A's and B's. We first focus on the type-I strings

$$(B\ A_2 \dots\dots A_{n-1}\ A). \tag{3.105}$$

The number of strings that satisfy the constraints derived in Section (3.3.1) is given by all the possible ways one can insert sequences of C's among any pair of A's and B's. In other words, the generic string will look like

$$(B\ C\ C \dots C\ A\ B\ C\ C \dots\dots C\ A), \tag{3.106}$$

where k sequences of C's of length $\{x_i\}_{i=1,\dots,k}$ are present, and $x_i \geq 0$ are integer numbers. As the length of the total string is n, the $\{x_i\}_{i=1,\dots,k}$ satisfy the following constraint

$$x_1 + \dots + x_k = n - 2k. \tag{3.107}$$

We now make use of a remarkable mathematical result, namely that the number of non-negative integer solutions of $x_1 + \dots + x_k = n$, that is the number of non-negative integer partitions of n into k parts is $\binom{n+k-1}{n}$ [195]. As a consequence, the number of type-I strings satisfying the previous constraints is

$$\binom{n-k-1}{k-1}. \tag{3.108}$$

Similarly, we consider now the type-II strings, having the following structure

$$(C\ C \dots C\ A\ B\ C\ C \dots\dots C). \tag{3.109}$$

In this case, when there are k pairs of consecutive A's and B's, there are $k+1$ sequences of consecutive C's. Thus, we now have to count the number of non-negative integer solutions of

$$x_1 + \dots + x_{k+1} = n - 2k, \tag{3.110}$$

which is

$$\binom{n-k}{k}. \tag{3.111}$$

Summing up the contribution of both type of strings, we get precisely

$$\binom{n-k-1}{k-1} + \binom{n-k}{k} \tag{3.112}$$

as the total number of strings containing k pairs of consecutive A's and B's, and this proves the result (3.60). We notice that in type-I strings there is always at least one pair of consecutive A's and B's, given by $A_n = A$ and $A_1 = B$. Then, if $k = 0$ in (3.108) there are no strings satisfying the constraints, which is compatible with the convention $\binom{n-1}{-1} = 0$.

As a final remark, we stress that the problem of counting the number of type-I and type-II strings containing k pairs of consecutive A's and B's is identical to the problem mentioned after equation (3.42), namely counting the number of sequences $(k_1, \dots, k_n)$ with $k_i \in \{0, 1\}$ and $k_{i+1} = 0$ whenever $k_i = 0$.

3.B Mathematical identities

3.B.1 Generalised Lucas polynomials and a proof of equation (3.63)

In this appendix we prove the identity (3.63) by taking advantage of some properties of generalised Lucas polynomials. These are polynomials $V(x, y)$ defined by the recurrence relation [196, 197]:

$$V_{n+2}(x, y) = xV_{n+1}(x, y) + yV_n(x, y)\,, \quad n \in \mathbb{N}_0, \tag{3.113}$$

the first two polynomials being $V_0(x, y) = 2$, $V_1(x, y) = x$. The proof of (3.63) is based on the fact that the two sides of the equation are precisely two equivalent closed formulae for the n-th Lucas polynomial, with $x = r$, $y = r_A r_B$. In fact, we will now prove the following two statements:

1. For all integers $n \geq 0$, one has a generalised Binet formula

$$V_n(x, y) = \alpha^n + \beta^n\,, \quad \alpha = \frac{x + \sqrt{x^2 + 4y}}{2}\,, \quad \beta = \frac{x - \sqrt{x^2 + 4y}}{2}\,. \tag{3.114}$$

   This is immediate to prove, as (3.113) holds by inspection for $n = 0$, $n = 1$ and furthermore $\alpha^2 = x\alpha + y$, $\beta^2 = x\beta + y$, which implies that $\alpha^{n+2} = x\alpha^{n+1} + y\alpha^n$ and $\beta^{n+2} = x\beta^{n+1} + y\beta^n$. This means that $\alpha^n + \beta^n$ satisfies the relation (3.113) for all $n \geq 0$.

2. For all integers $n \geq 1$ another explicit formula for $V_n(x, y)$ is given by

$$V_n(x, y) = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k\,. \tag{3.115}$$

We prove this by showing again that the recurrence relation is satisfied. For $n = 1$, $n = 2$ it is immediate to see that this reproduces the correct polynomials. For $n \geq 3$, we can make use of the identity

$$\frac{n}{n-k}\binom{n-k}{k} = \frac{n-1}{n-k-1}\binom{n-k-1}{k} + \frac{n-2}{n-k-1}\binom{n-k-1}{k-1}, \tag{3.116}$$

and we adopt the convention that $\binom{n}{k} = 0$ if $k > n$ or $k < 0$. Let us now consider $n = 2m$, the case of n odd being completely analogous. If $n = 2m$, $[n/2] = m$, $[(n-1)/2] = [(n-2)/2] = m-1$. From (3.115) and (3.116) we have

$$\begin{aligned} V_n(x,y) &= \sum_{k=0}^{m} \frac{2m}{2m-k}\binom{2m-k}{k} x^{2m-2k} y^k \\ &= x \sum_{k=0}^{m} \frac{2m-1}{2m-1-k}\binom{2m-1-k}{k} x^{2m-1-2k} y^k \\ &+ y \sum_{k=0}^{m} \frac{2m-2}{2m-1-k}\binom{2m-1-k}{k-1} x^{2m-2k} y^{k-1}. \end{aligned} \tag{3.117}$$

The first sum in the right-hand side vanishes if $k = m$, so that this term is $xV_{n-1}(x,y)$. The second sum on the other hand vanishes if $k = 0$, so we can shift the summation variable and we see that this term reproduces $yV_{n-2}(x,y)$. Hence the recurrence relation is proved.

Equation (3.63) follows from the identity of expressions (3.114) and (3.115), and it has two interesting implications. The first one comes from a direct expansion of the Binet formula using the binomial theorem:

$$\begin{aligned} &\left(\frac{x-\sqrt{x^2+4y}}{2}\right)^n + \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^n \\ &= \frac{1}{2^n}\sum_{j=0}^{n}\left[(-1)^k\binom{n}{j}x^{n-j}(x^2+4y)^{j/2} + \binom{n}{j}x^{n-j}(x^2+4y)^{j/2}\right] \\ &= 2^{1-n}\sum_{j=0}^{[n/2]}\left[\binom{n}{2j}x^{n-2j}(x^2+4y)^j\right] \\ &= 2^{1-n}\sum_{j=0}^{[n/2]}\binom{n}{2j}x^{n-2j}\sum_{k=0}^{j}\binom{j}{k}x^{2j-2k}2^{2k}y^k = 2^{1-n}\sum_{j=0}^{[n/2]}\sum_{k=0}^{j}\binom{n}{2j}\binom{j}{k}x^{n-2k}2^{2k}y^k \\ &= 2^{1-n}\sum_{k=0}^{[n/2]}x^{n-2k}2^{2k}y^k\sum_{j=k}^{[n/2]}\binom{n}{2j}\binom{j}{k} = \sum_{k=0}^{[n/2]}\left(2^{1-n+2k}\sum_{j=k}^{[n/2]}\binom{n}{2j}\binom{j}{k}\right)x^{n-2k}y^k\,, \end{aligned} \tag{3.118}$$

where in the last line we rearranged the sums over j and k. This quantity equals (3.115), which implies the non-trivial combinatorial identity:

$$\frac{n}{n-k}\binom{n-k}{k} = 2^{1-n+2k}\sum_{j=k}^{[n/2]}\binom{n}{2j}\binom{j}{k}. \tag{3.119}$$

To the best of our knowledge, this identity was only proved for n odd in [196].

The other interesting implication is obtained for $x = y = 1$. In this case, the Lucas polynomials (3.113) reduce to the Lucas numbers:

$$L_n = L_{n-1} + L_{n-2}\,, \quad n \geq 2, \tag{3.120}$$

with $L_0 = 2$, $L_1 = 1$. The recurrence formula is the same defining the Fibonacci sequence, except for the different initial values. Equation (3.114) with $x = y = 1$ gives a closed formula for the Lucas numbers, and thus we have, for $n \geq 1$:

$$\sum_{k=0}^{[n/2]}\frac{n}{n-k}\binom{n-k}{k} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{3.121}$$

The quantity on the right-hand side is $\phi^n + (1-\phi)^n$, with ϕ the golden ratio, and it is always a positive integer. On the other hand, the quantity on the left hand side is the number of non-vanishing strings of type-I and type-II (out of a total of 3^n possible strings) obtained via the contraction methods discussed in Section 3.3.

3.B.2 Product formulae

Here we point out two useful identities which are employed to obtain the free fermion result in Subsection 3.3.2.1. The first relation, which is immediate to check, is:

$$\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}}(x + e^{\frac{2\pi i p}{n}}y) = x^n + y^n, \quad n \in \mathbb{N}, \tag{3.122}$$

where the product is performed over p integer (resp. semi-integer) when n is odd (resp. even). A consequence of this identity is:

$$\begin{aligned}
&\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} (xe^{\frac{2\pi ip}{n}} + ye^{-\frac{2\pi ip}{n}} + z) \\
&= x^n \prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} \left[1 + \frac{z - \sqrt{z^2 - 4yx}}{2x} e^{-\frac{2\pi ip}{n}}\right] \left[1 + \frac{z + \sqrt{z^2 - 4yx}}{2x} e^{-\frac{2\pi ip}{n}}\right] \\
&= x^n \left[1 + \left(\frac{z - \sqrt{z^2 - 4yx}}{2x}\right)^n\right] \left[1 + \left(\frac{z + \sqrt{z^2 - 4yx}}{2x}\right)^n\right] \\
&= x^n + y^n + \left(\frac{z - \sqrt{z^2 - 4yx}}{2}\right)^n + \left(\frac{z + \sqrt{z^2 - 4yx}}{2}\right)^n .
\end{aligned}$$

Finally, the equation above can be employed to evaluate

$$\begin{aligned}
&\prod_{p=-\frac{n-1}{2}}^{\frac{n-1}{2}} (r_A e^{\frac{2\pi i}{n}(\alpha+p)} - r_B e^{-\frac{2\pi i}{n}(\alpha+p)} + r) \\
&= e^{2\pi i\alpha} r_A^n + e^{-2\pi i\alpha} r_B^n + \left(\frac{r + \sqrt{r^2 + 4r_A r_B}}{2}\right)^n + \left(\frac{-r + \sqrt{r^2 + 4r_A r_B}}{2}\right)^n ,
\end{aligned} \tag{3.123}$$

valid if n is an even integer [4].

[4] It is important here that n is even, as $(-r_B)^n = r_B^n$.

PART II : TWIST OPERATORS AND ENTANGLEMENT MEASURES IN THE ISING FIELD THEORY

CHAPTER

FOUR

TWO-POINT FUNCTION OF COMPOSITE TWIST FIELDS IN THE DISORDERED PHASE OF THE ISING FIELD THEORY

All standard measures of bipartite entanglement in one-dimensional QFT can be expressed in terms of correlators of the branch-point twist fields (BPTF) $\mathcal{T}$ and $\tilde{\mathcal{T}}$. These are symmetry fields associated to cyclic permutation symmetry in a replica theory and having the smallest conformal dimension at the critical point. As seen in the previous chapters, the composite twist fields (CTF), typically of higher dimension, play a role in the study of symmetry-resolved measures of entanglement. In this chapter, based on [4], we give an exact expression for the two-point function of a CTF that arises in the Ising field theory. In doing so we extend the techniques originally developed for the standard BPTF in free theories as well as an existing computation [107] of the same two-point function which focussed on the leading large-distance contribution. We study the ground state two-point function of the composite twist field $\mathcal{T}_\mu$ and its conjugate $\tilde{\mathcal{T}}_\mu$. At criticality, this field can be defined as the leading field in the operator product expansion of $\mathcal{T}$ and the disorder field μ. We find a general formula for the logarithm of $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle$ and for (the derivative of) its analytic continuation to positive real replica numbers greater than 1. We check our formula for consistency by showing that at short distances it exactly reproduces the expected conformal dimension

4.1 Introduction

It is well known that the 1+1D Ising field theory, obtained as the continuum limit of the quantum Ising chain, is described near the critical point by the action of a free Majorana fermion [60, 61, 74, 198]:

$$\mathcal{S} = \int \mathrm{d}z\mathrm{d}\bar{z}\left(\psi(z)\bar{\partial}\psi(z) + \bar{\psi}(\bar{z})\partial\bar{\psi}(\bar{z}) + m\bar{\psi}(\bar{z})\psi(z)\right), \tag{4.1}$$

where ψ, $\bar{\psi}$ are the two components of the Majorana field Ψ and the sign of the mass term indicates whether the theory is in the ordered or in the disordered phase (the critical point being at $m = 0$). This theory has an internal $\mathbb{Z}_2$ symmetry, as the action is invariant under $\Psi \to -\Psi$. This symmetry is associated to the order operator σ (the spin field) and the disorder operator μ (disorder operator), which are semi-local with respect to the fermion field Ψ, so that the three fields can be characterised by their mutual equal-time exchange relations [60, 61, 81]:

$$\Psi(x)\sigma(y) = \begin{cases} \sigma(y)\Psi(x) & y > x \\ \sigma(y)\Psi(x) & y < x \end{cases} \quad \text{and} \quad \Psi(x)\mu(y) = \begin{cases} -\mu(y)\Psi(x) & y > x \\ \mu(y)\Psi(x) & y < x \end{cases} . \tag{4.2}$$

In the replica version of the theory, the fields above acquire an index $\{\mu_j, \sigma_j, \Psi_j\}$ with $j = 1, \dots, n$, running over the copy numbers. The resulting model possesses a larger amount of symmetry, as the $\mathbb{Z}_2$ symmetry on each copy is now enhanced with symmetry under the exchange of any copies. As discussed in Section 1.2, the latter symmetry plays a fundamental role in computations of the entanglement entropy and other measures of entanglement [11, 12, 20].

In [20] the branch-point twist fields $\mathcal{T}$ and its conjugate $\tilde{\mathcal{T}}$ were defined as the symmetry fields associated with cyclic permutation symmetry of copies in a replica theory. These fields too are characterised by their exchange relations with respect to the fermions, which are identical to those in (1.26):

$$\Psi_j(x)\mathcal{T}(y) = \begin{cases} \mathcal{T}(y)\Psi_{j+1}(x) & y > x \\ \mathcal{T}(y)\Psi_j(x) & y < x \end{cases} \quad \text{and} \quad \Psi_j(x)\tilde{\mathcal{T}}(y) = \begin{cases} \tilde{\mathcal{T}}(y)\Psi_{j-1}(x) & y > x \\ \tilde{\mathcal{T}}(y)\Psi_j(x) & y < x \end{cases} \tag{4.3}$$

for $j = 1, \dots, n$ and $n + 1 \equiv 1$. These relations can be written for any 1+1D QFT, however, in the context of massive integrable theories they provide -together with the two-body scattering matrix- all the information needed to compute correlation functions and matrix elements of $\mathcal{T}$. These computations have now been carried out for many theories and entanglement measures (see e.g. [63, 117, 199, 200]) revealing many new insights into the universal properties of entanglement at near-critical points.

In recent years, it has been shown that also the fields resulting from the conformal OPE of $\mathcal{T}$ with other fields of the Ising field theory can be of interest in the context of entanglement [19, 94, 156–158, 201]. In particular, the correlation functions of the leading field in the OPE of $\mathcal{T}$ and $\sum_j \mu_j$, denoted by $\mathcal{T}_\mu$, are related to the $\mathbb{Z}_2$-resolved entanglement entropy [19, 94, 107]. $\mathcal{T}_\mu$ satisfies exchange relations which combine those for $\mathcal{T}$ and μ as seen above:

$$\Psi_j(x)\mathcal{T}_\mu(y) = \begin{cases} -\mathcal{T}_\mu(y)\Psi_{j+1}(x) & y > x \\ \mathcal{T}_\mu(y)\Psi_j(x) & y < x \end{cases} \quad \text{and} \quad \Psi_j(x)\tilde{\mathcal{T}}_\mu(y) = \begin{cases} -\tilde{\mathcal{T}}_\mu(y)\Psi_{j-1}(x) & y > x \\ \tilde{\mathcal{T}}_\mu(y)\Psi_j(x) & y < x \end{cases} . \tag{4.4}$$

The computation of the symmetry-resolved entanglement provides strong motivation to study correlators of $\mathcal{T}_\mu$ and this is the focus of the present chapter. Using techniques of integrable QFT, we find an exact analytic expression for the (logarithm of the) two-point function $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle$ in the disordered phase of the model. The applications of such a result in the context of entanglement will not be discussed here, but they follow quite straightforwardly from existing literature. In particular, the form factors of $\mathcal{T}_\mu$ and the leading contribution to its two-point function were computed in [107]. The present work is an extension of those results to include higher particle contributions and to show how non-trivial resummation identities allow for relatively simple closed formulae for all correlation function cumulants.

Correlation functions of composite twist fields have been studied in a number of works both for the Ising field theory and other, interacting models. Most of these results build upon the form factor program for the matrix elements of $\mathcal{T}$ [20] and its extension to composite twist fields [107]. In [105, 123, 124] free theories were studied, whereas interacting IQFTs such as the Ising and sinh-Gordon models (both with discrete $\mathbb{Z}_2$ symmetry), the sine-Gordon model (with continuous $U(1)$ symmetry) and the 3-state Potts model (with discrete $\mathbb{Z}_3$ symmetry) were studied in [106, 107] and [108], respectively. It is also possible to study composite twist fields where $\mathcal{T}$ is composed with a local field *not* associated with an internal symmetry. Such composite fields are associated with cyclic permutation symmetry too and have a conformal dimension which is distinct from that of $\mathcal{T}$. In particular, for theories whose UV fixed point is described by a non-unitary CFT, it is possible to construct composite twist fields whose dimension is lower than that of $\mathcal{T}$ and they play a critical role in describing the usual measures of entanglement [158]. This happens for instance for the Lee-Yang theory both at and away from criticality. The form factors and two-point functions of the BPTF and CTF for this theory were studied in [201]. The expectation values of composite twist fields involving the energy field in the Ising field theory were studied in [156, 157].

The structure of this chapter is as follows: In Section 4.2 we review form factor results for the order and disorder fields in the Ising field theory as well as for $\mathcal{T}$ and $\mathcal{T}_\mu$. We then present the cumulant expansion of two-point functions and introduce an example of the type of convergence issues that arise in the cumulant expansion of $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle/\langle \mathcal{T}_\mu\rangle^2$. In Section 4.3 we find closed formulae for all higher cumulants, leading to a close-form expression for the two-point function. In Section 4.4 we test this expression by obtaining the exact conformal dimension of $\mathcal{T}_\mu$ from resummation of leading terms in the short-distance expansion of the cumulants. We show that the normalised two-point function $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle/\langle \mathcal{T}_\mu\rangle^2$ is in fact proportional to the normalised two-point function $\langle \mu(0)\mu(\ell)\rangle/\langle \mu\rangle^2$, thus it factorises into n-dependent and n-independent parts. In Section 4.5 we show how to analytically continue the cumulant expansion from n integer and greater than 1 to n real. This allows us to write a formula for the n-derivative of the two-point function at $n = 1$, a quantity that typically plays a

role in entanglement measures. We conclude in Section 4.6.

4.2 Field content of the Ising model and form factors

The correlation functions and form factors of the fields σ, μ in the disordered phase can be obtained via form factor bootstrap [77, 78] and were studied in the seminal papers [62, 81]. Form factors of descendent fields (in the CFT sense) of the energy field ε were studied in [80] and shown to match in number and spin the field content of the corresponding Verma module in the underlying Ising CFT. Starting from the relations (4.2) the form factor equations can be written and solved for matrix elements of σ, μ and these were found to take an extremely simple form [62], namely (the factor i^k is needed to satisfy the kinematic residue equation):

$$F^{\mu}_{2k}(\theta_1, \dots, \theta_{2k}) = i^k \langle \mu \rangle \prod_{1 \le i < j \le 2k} \tanh \frac{\theta_{ij}}{2}, \tag{4.5}$$

$$F^{\sigma}_{2k+1}(\theta_1, \dots, \theta_{2k+1}) = i^k F^{\sigma}_1 \prod_{1 \le i < j \le 2k+1} \tanh \frac{\theta_{ij}}{2}, \tag{4.6}$$

with $\theta_{ij} := \theta_i - \theta_j$ and $\langle \mu \rangle$ and F^{σ}_1 normalisation constants which can be identified with the vacuum expectation value of μ and the one-particle form factor of σ, respectively. In general, as discussed in Section 1.2, form factors are characterised by a set of quantum numbers specifying the particle types in the asymptotic state, but in the Ising model there is a single particle type so these do not need to be specified. For the field μ the products above can be rewritten as a Pfaffian of a $2k \times 2k$ antisymmetric matrix $\mathcal{A}$ with entries $\mathcal{A}_{ij} = \tanh \frac{\theta_{ij}}{2}$. In particular this means that the vacuum expectation value of μ is non-vanishing, whereas it is vanishing for σ. This is a consequence of the fact that in the disordered phase the field σ (μ) is odd (even) with respect to the $\mathbb{Z}_2$ action on the the fermion field, which creates the asymptotic states, and hence σ (μ) will have a non vanishing correlator only with an odd (even) number of particles.

The form factors of the BPTFs $\mathcal{T}$ and $\tilde{\mathcal{T}}$ in the (replica) Ising model have been known for some time [20, 63] and due to the free nature of the model they can also be expressed in terms of a Pfaffian

$$F^{\mathcal{T}|11\dots1}_{2k}(\theta_1, \dots, \theta_{2k}; n) = \langle \mathcal{T} \rangle \mathrm{Pf}(K), \qquad \mathrm{Pf}(K) = \sqrt{\det K}, \tag{4.7}$$

where n labels the number of replicas,

$$K_{ij} := k(\theta_{ij}) = \frac{\sin \frac{\pi}{n}}{2n \sinh\left(\frac{i\pi - \theta_{ij}}{2n}\right) \sinh\left(\frac{i\pi + \theta_{ij}}{2n}\right)} \frac{\sinh \frac{\theta_{ij}}{2n}}{\sinh \frac{i\pi}{2n}}, \quad \text{with} \quad i, j = 1, \dots, 2k, \tag{4.8}$$

and the superindices $11\dots1$ indicate that all particles are in the same copy 1. From this representation

we also see that all form factors are functions of rapidity differences only, a property that holds for all spinless fields in relativistic QFT. The two-particle form factor is simply $F_2^{\mathcal{T}|11}(\theta_1,\theta_2;n)=\langle\mathcal{T}\rangle k(\theta_{12})$. Form factors for particles in copies $j_1\dots j_{2k}$ can be obtained from the above using the standard form factor equations presented in [20]

$$F_{2k}^{\mathcal{T}|j_1\dots j_{2k}}(\theta_1,\dots,\theta_{2k};n)=F_{2k}^{\mathcal{T}|1\dots1}(\theta_1^{j_1-1},\dots,\theta_{2k}^{j_k-1};n),\qquad\text{for}\qquad j_1\geq j_2\cdots\geq j_{2k},\tag{4.9}$$

with

$$\theta^j:=\theta+2\pi ij.\tag{4.10}$$

The form factors of the composite twist field $\mathcal{T}_\mu$ where first obtained in [107] and have again the Pfaffian structure typical of the Ising model, that is

$$F_{2k}^{\mathcal{T}_\mu|11\dots1}(\theta_1,\dots,\theta_{2k};n)=\langle\mathcal{T}_\mu\rangle\mathrm{Pf}(W),\tag{4.11}$$

with

$$W_{ij}:=w(\theta_{ij})=\frac{\sin\frac{\pi}{n}}{2n\sinh\left(\frac{i\pi-\theta_{ij}}{2n}\right)\sinh\left(\frac{i\pi+\theta_{ij}}{2n}\right)}\frac{\sinh\frac{\theta_{ij}}{n}}{\sinh\frac{i\pi}{n}}.\tag{4.12}$$

As we can see, this function differs from $k(\theta)$ above only because n is replaced by $n/2$ in the minimal part of the form factor (i.e. the part that does not contain kinematic poles). However, this small change leads to some important differences, the main one being the asymptotic properties

$$\lim_{\theta\to\infty}k(\theta)=0,\qquad\text{and}\qquad\lim_{\theta\to\pm\infty}w(\theta)=\pm\frac{i}{n},\tag{4.13}$$

as well as

$$\lim_{n\to1}k(\theta)=0,\qquad\text{and}\qquad\lim_{n\to1}w(\theta)=i\tanh\frac{\theta}{2}.\tag{4.14}$$

Note that the last equality simply shows that the two-particle form factor of $\mathcal{T}_\mu$ reduces to that of μ for $n=1$, as expected. This extends to higher-particle form factors too. It is known from the study of many models and arguments such as those presented in [202] that the asymptotics of two particle form factors should be related to the value of a one-particle form factor. This is a consequence of so-called cluster decomposition in momentum space. In simple theories, as assumed in [202], this one-particle form factor would be that of the same field. However, in the Ising model, due to $\mathbb{Z}_2$ symmetry there is a mixing between form factors of μ and σ and also those of $\mathcal{T}_\sigma$ (defined as the composition of $\mathcal{T}$ and $\sum_j\sigma_j$) and $\mathcal{T}_\mu$, in such a way that:

$$\lim_{\theta\to\pm\infty}w(\theta)=\pm\tau^2,\tag{4.15}$$

where $\tau:=F_1^{\mathcal{T}_\sigma|1}$ is the one-particle form factor of $\mathcal{T}_\sigma$, which by relativistic invariance does not depend

on θ. Combining (4.15) with (4.13) we have that

$$|F_1^{\mathcal{T}_\sigma|1}|^2 = |\tau|^2 = \frac{1}{n}. \tag{4.16}$$

Higher form factors of $\mathcal{T}_\sigma$ can also be related to Pfaffians by employing a more general version of the cluster decomposition property. Namely

$$\lim_{\theta_{2k+2}\to\infty} \langle \mathcal{T}_\mu \rangle^{-1} F_{2k+2}^{\mathcal{T}_\mu}(\theta_1, \dots, \theta_{2k+2}; n) = \tau F_{2k+1}^{\mathcal{T}_\sigma}(\theta_1, \dots, \theta_{2k+1}; n). \tag{4.17}$$

Note that the prefactor $\langle \mathcal{T}_\mu \rangle^{-1}$ ensures that when $k = 0$ both sides of the equation become τ^2. In this way, the form factors $F_{2k+1}^{\mathcal{T}_\sigma}(\theta_1, \dots, \theta_{2k+1}; n)$ can be computed systematically and it is easy to show that they can be written as sums of Pfaffians involving $2k$ variables. In fact, we can show that

$$F_{2k+1}^{\mathcal{T}_\sigma}(\theta_1, \dots, \theta_{2k+1}; n) = \frac{\tau}{\langle \mathcal{T}_\mu \rangle} \sum_{j=1}^{2k+1} (-1)^{j+1} F_{2k}^{\mathcal{T}_\mu}(\theta_1, \dots, \bar{\theta}_j, \dots \theta_{2k+1}; n), \tag{4.18}$$

where the sign depends on the position of the variable θ_j and can be worked out by counting Wick contractions. Similarly, the symbol $\bar{\theta}_j$ means that this variable is removed, hence this is a sum over $2k$-particle form factors depending on a subset of the variables $\{\theta_1, \dots, \theta_{2k+1}\}$. For instance

$$\begin{aligned} F_3^{\mathcal{T}_\sigma}(\theta_1, \theta_2, \theta_3; n) &= \tau(w(\theta_{12}) - w(\theta_{13}) + w(\theta_{23})) \\ &= \frac{\tau}{\langle \mathcal{T}_\mu \rangle}(F_2^{\mathcal{T}_\mu}(\theta_1, \theta_2; n) - F_2^{\mathcal{T}_\mu}(\theta_1, \theta_3; n) + F_2^{\mathcal{T}_\mu}(\theta_2, \theta_3; n)). \end{aligned} \tag{4.19}$$

The formula (4.18) is, to the best of our knowledge, new and first presented here. However, this structure is the same relating the form factors (4.5) and (4.6) of the fields μ and σ which are obtained in the limit $n = 1$, and ultimately relies on the factorisation properties of the tanh function.

In [107] it was also shown that the form factor (4.12) gives the correct conformal dimension of $\mathcal{T}_\mu$ via the Δ-sum rule [202]. This dimension is [12, 19, 58, 59, 156]

$$\Delta_{\mathcal{T}_\mu} = \Delta_{\mathcal{T}} + \frac{\Delta_\mu}{n} = \frac{n}{48} + \frac{1}{24n}, \qquad \text{with} \qquad \Delta_{\mathcal{T}} = \frac{1}{48}\left(n - \frac{1}{n}\right), \quad \Delta_\mu = \frac{1}{16}. \tag{4.20}$$

We observe that, since $\Delta_\mu = \Delta_\sigma$, it also holds $\Delta_{\mathcal{T}_\mu} = \Delta_{\mathcal{T}_\sigma}$.

4.2.1 Two-point function and cumulant expansion

In this chapter, we are interested in the two-point function of the field $\mathcal{T}_\mu$ in the ground state of the disordered phase. Our goal is to write down an expansion of the form

$$\log\left(\frac{\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle}{\langle \mathcal{T}_\mu\rangle^2}\right) = \sum_{k=1}^{\infty} c_k^{\mathcal{T}_\mu}(\ell;n) \overset{m\ell\ll 1}{\simeq} -4\Delta_{\mathcal{T}_\mu}\log(m\ell) - K_{\mathcal{T}_\mu}, \tag{4.21}$$

where the sum is over functions $c_k^{\mathcal{T}_\mu}(\ell;n)$ known as cumulants and $K_{\mathcal{T}_\mu}$ is a constant that depends on the vacuum expectation value $\langle \mathcal{T}_\mu\rangle$. The structure of the right-hand side in the above equation is dictated by the form of correlators of primary fields in CFT. The cumulant expansion of the two-point function of the standard BPTF in the ground state of the free massive boson was performed in [117], and we borrow ideas from that work. The cumulants are multiple integrals of linear combinations of products of form factors. More precisely, we have the following structure

$$c_k^{\mathcal{T}_\mu}(\ell;n) = \frac{1}{k!(2\pi)^k}\sum_{j_1,\dots,j_k=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\theta_k\; h_k^{\mathcal{T}_\mu|j_1\dots j_k}(\theta_1,\cdots,\theta_k,n)e^{-m\ell\sum_{i=1}^k\cosh\theta_i}, \tag{4.22}$$

where the functions $h_k^{\mathcal{O}|j_1\dots j_k}(\theta_1,\cdots,\theta_k,n)$ are given in terms of the form factors of the field involved, and j_i are the copy numbers. For example:

$$\begin{aligned}
h_2^{\mathcal{T}_\mu|j_1j_2}(\theta_1,\theta_2,n) &= \langle\mathcal{T}_\mu\rangle^{-2}\left|F_2^{\mathcal{T}_\mu|j_1j_2}(\theta_1,\theta_2,n)\right|^2,\\
h_4^{\mathcal{T}_\mu|j_1j_2j_3j_4}(\theta_1,\theta_2,\theta_3,\theta_4,n) &= \langle\mathcal{T}_\mu\rangle^{-2}\left|F_4^{\mathcal{T}_\mu|j_1j_2j_3j_4}(\theta_1,\theta_2,\theta_3,\theta_4,n)\right|^2\\
&\quad -h_2^{\mathcal{T}_\mu|j_1j_2}(\theta_1,\theta_2,n)h_2^{\mathcal{T}_\mu|j_3j_4}(\theta_3,\theta_4,n)\\
&\quad -h_2^{\mathcal{T}_\mu|j_1j_3}(\theta_1,\theta_3,n)h_2^{\mathcal{T}_\mu|j_2j_4}(\theta_2,\theta_4,n)\\
&\quad -h_2^{\mathcal{T}_\mu|j_1j_4}(\theta_1,\theta_4,n)h_2^{\mathcal{T}_\mu|j_2j_3}(\theta_2,\theta_3,n),
\end{aligned} \tag{4.23}$$

and so on, whereas all odd particle terms are zero. Similar formulae can be written for the cumulants of $\mathcal{T}_\sigma$ where only odd particle cumulants are non-vanishing. A general diagrammatic construction of the functions appearing in a cumulant expansion can be found for instance in [203]. For a generic local field $\mathcal{O}$, it is standard to require that

$$h_k^{\mathcal{O}|j_1\dots j_k}(\theta_1,\cdots,\theta_k)\sim e^{-\theta_i}, \tag{4.24}$$

as $\theta_i\to\infty$. Given the properties of the form factors presented in the previous section, we see that this asymptotic behaviour is not satisfied for the cumulants of the two-point function of $\mathcal{T}_\mu$, or indeed for the cumulants of the two-point function of μ as shown in [62]. In fact, the cumulant expansion is still convergent in both cases, but the leading behaviour for small $m\ell$ is harder to extract than in theories

where (4.24) holds.

4.2.2 Two-particle contribution

One of the simplest ways to check the validity of the two-point function expansion consists in recovering the conformal dimension of the field by exact resummation of all terms which are proportional to $\log(m\ell)$ for $m\ell \ll 1$, that is the first term in (4.21). Let us start by considering the simplest contribution to the connected part of the two-point function $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle/\langle \mathcal{T}_\mu\rangle^2$, which has already been studied in the literature [107]. The first non-vanishing contribution to the cumulant expansion comes from $h_2^{\mathcal{T}_\mu|j_1 j_2}(\theta_1,\theta_2,n)$, which is nothing but the normalised squared modulus of the two-particle form factor. Using (4.9), the latter can be rewritten as

$$\sum_{i,j=1}^{n}\left|F_2^{\mathcal{T}_\mu|ij}(\theta_1,\theta_2)\right|^2 = n\sum_{j=0}^{n-1}\left|F_2^{\mathcal{T}_\mu|11}(\theta_1+2\pi i j,\theta_2)\right|^2 = n\langle \mathcal{T}_\mu\rangle^2\sum_{j=0}^{n-1} w((-\theta_{12})^j)w(\theta_{12}^j). \tag{4.25}$$

Thus we have

$$\begin{aligned} c_2^{\mathcal{T}_\mu}(\ell;n) &= n\sum_{j=0}^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\mathrm{d}\theta_2}{2(2\pi)^2}\, w((-\theta_{12})^j)w(\theta_{12}^j)\, e^{-m\ell\cosh\theta_1 - m\ell\cosh\theta_2} \\ &= \frac{n}{(2\pi)^2}\sum_{j=0}^{n-1}\int_{-\infty}^{\infty}\mathrm{d}\theta\, w((-\theta)^j)w(\theta^j)\, K_0(2m\ell\cosh\frac{\theta}{2}), \end{aligned} \tag{4.26}$$

where the second line is obtained by performing a change of variable $\theta=\theta_1-\theta_2$, $\Theta=\frac{\theta_1+\theta_2}{2}$ and using the integral representation of the Bessel function [91]

$$K_0(z) = \int_0^\infty \mathrm{d}x\, e^{-z\cosh x}. \tag{4.27}$$

The sum over the copy index j was computed in [107] via contour integration and it is given by

$$\sum_{j=0}^{n-1} w((-\theta)^j)w(\theta^j) = -i\tanh\frac{\theta}{2}(w(2\theta+i\pi)+w(2\theta-i\pi)) - \frac{1}{n}. \tag{4.28}$$

This function tends asymptotically to the value $\frac{1}{n}$ for $|\theta|\to\infty$. This means that the usual procedure consisting of expanding the Bessel function for $m\ell \ll 1$ and isolating the $\log(m\ell)$ leading term, thus effectively removing the Bessel function from the integrand in (4.26), now leads to a divergent integral. However, we can rewrite (4.26) as

$$\begin{aligned} c_2^{\mathcal{T}_\mu}(\ell;n) &= \frac{n}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}\theta\left[\sum_{j=0}^{n-1} w((-\theta)^j)w(\theta^j) - \frac{1}{n}\right] K_0(2m\ell\cosh\frac{\theta}{2}) \\ &+ \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}\theta\, K_0(2m\ell\cosh\frac{\theta}{2}). \end{aligned} \tag{4.29}$$

In this form, the integral in the first line can be approximated for $m\ell \ll 1$ by expanding the Bessel function, giving a leading contribution which is proportional to $\log(m\ell)$, while the integral in the second line can be computed exactly to

$$\int_{-\infty}^{\infty} \mathrm{d}\theta\, K_0(2m\ell \cosh\frac{\theta}{2}) = 2K_0(m\ell)^2 \overset{m\ell\ll 1}{\simeq} -2(\log(m\ell))^2, \tag{4.30}$$

so that, in this case, the leading small $m\ell$ contribution diverges as $(\log(m\ell))^2$. Thus, although the cumulant (4.26) is still well-defined, its leading small $m\ell$ behaviour is now dominated by $(\log(m\ell))^2$ instead of $\log(m\ell)$. This is a consequence of the property (4.24) not holding in this case. Nonetheless, terms of order $(\log(m\ell))^2$ should cancel out when including further contributions in the form factor series as one expects to recover the $1/r^{4\Delta_{\mathcal{T}_\mu}}$ behaviour of the two-point function at short distances. In the next sections we will show that this is indeed the case, providing a way to recover the expected scaling (4.21) from our cumulant expansion.

4.3 Higher particle contributions: closed formulae

Existing studies of the branch-point twist field two-point function for free fermions [63] and bosons [117] have revealed that the form of higher cumulants can be considerably simplified. This is because under sum over particle types and integration over the rapidities, many of the terms in the cumulant either cancel each other out or can be shown to be identical. In fact, it is possible to show that just as for the standard BPTF, and for the same reasons already discussed in [63, 117] the cumulants of the two-point function of $\mathcal{T}_\mu$ take the generic form

$$\begin{aligned} c_{2k}^{\mathcal{T}_\mu}(\ell;n) = \frac{n}{2k(2\pi)^{2k}} \sum_{j_1,\dots,j_{2k-1}=0}^{n-1} \left[\prod_{i=1}^{2k}\int_{-\infty}^{+\infty} \mathrm{d}\theta_i\, e^{-m\ell\cosh\theta_i}\right] \\ \times (-1)^k \left(w(\theta_{12}^{-j_1})\prod_{i=1}^{k-1} w(\theta_{2i+1\,2i+2}^{j_{2i}-j_{2i+1}})\right)\left(w(\theta_{1\,2k}^{j_{2k-1}})\prod_{i=1}^{k-1} w(\theta_{2i\,2i+1}^{-j_{2i-1}+j_{2i}})\right). \end{aligned} \tag{4.31}$$

By using the fact that $w(\theta^{-j}) = -w((-\theta)^j)$, we can change the sign of half of the factors in the second line, cancelling out the factor $(-1)^k$, so that the integrand becomes:

$$\sum_{j_1,\dots,j_{2k-1}=0}^{n-1} w((-\theta_{12})_1^j) w(\theta_{1\,2k}^{j_{2k-1}}) \prod_{i=1}^{k-1} w(\theta_{2i+1\,2i+2}^{j_{2i}-j_{2i+1}}) w((-\theta_{2i\,2i+1})^{j_{2i-1}-j_{2i}}). \tag{4.32}$$

In order to evaluate the integrals (4.31), it is convenient to perform a change of variables whereby we first change the sign of all the rapidities θ_i with i even, without any change in the integration measure.

Then, defining $\hat{\theta}_{ij} \equiv \theta_i + \theta_j$ the integrand becomes a function of rapidity *sums* only:

$$\sum_{j_1,\dots,j_{2k-1}=0}^{n-1} w((-\hat{\theta}_{12})^{j_1}) w(\hat{\theta}_{23}^{j_1-j_2}) w(\hat{\theta}_{34}^{j_2-j_3}) \dots w(\hat{\theta}_{2k-1\,2k}^{j_{2k-2}-j_{2k-1}}) w(\hat{\theta}_{1\,2k}^{j_{2k-1}}). \tag{4.33}$$

We will refer to this as a fully connected sum, meaning that all terms are cyclically "connected" both at the level of the rapidities and the summation indices.

4.3.1 Recursive formulae

The sum (4.33) can be computed recursively, leading to generalisations of the following result:

$$f_1(x,y;n) := \sum_{j=0}^{n-1} w((-x)^j) w(y^j) = -\frac{i}{2} \frac{\sinh\left(\frac{x+y}{2}\right)}{\cosh\frac{x}{2}\cosh\frac{y}{2}} [w(x+y+i\pi) + w(x+y-i\pi)] - \frac{1}{n}, \tag{4.34}$$

which is presented here for the first time, although the case $x = y$ was obtained in [107] and has already been reported in (4.28). It is also useful to know that

$$\sum_{j=0}^{n-1} w(x^j) = i \tanh\frac{x}{2}. \tag{4.35}$$

A derivation of formulae (4.34), (4.35) and their generalisations to multiple sums (see below) is presented in Appendix 4.A.

For the branch-point twist field of free fermions and bosons [63, 117] a formula almost identical to (4.34) also holds, albeit without the term $-\frac{1}{n}$. This term in fact makes the generalisation of (4.34) to multiple sums more complex for $\mathcal{T}_\mu$ than it is for $\mathcal{T}$. It can nonetheless be done as follows. Let us consider, as an example, the next sum in the series, namely a sum of the form

$$\sum_{j_1,j_2=0}^{n-1} w((-x)^{j_1}) w(y^{j_1-j_2}) w(z^{j_2}) = \sum_{j=0}^{n-1} f_1(x,y^{-j},n) w(z^j). \tag{4.36}$$

Repeated use of (4.34) and (4.35) leads to

$$\begin{aligned}\sum_{j_1,j_2=0}^{n-1} w((-x)^{j_1}) w(y^{j_1-j_2}) w(z^{j_2}) &= -\frac{i}{n}\left(\tanh\frac{x}{2} + \tanh\frac{y}{2} + \tanh\frac{z}{2}\right) \\ &+ \frac{1}{4}\frac{\cosh\left(\frac{x+y+z}{2}\right)}{\cosh\frac{x}{2}\cosh\frac{y}{2}\cosh\frac{z}{2}} \left[2w(x+y+z) + w(x+y+z+2i\pi) + w(x+y+z-2i\pi)\right].\end{aligned} \tag{4.37}$$

This special case gives a good indication of the kind of structures that emerge. We observe that the contribution in the second line of the above expression has exactly the same structure as found for the BPTF in the free fermion theory [63]. The terms in the first line form a symmetric polynomial on the

variables $\tanh\frac{x}{2}, \tanh\frac{y}{2}, \tanh\frac{z}{2}$. The general structure for higher sums goes as follows. let us define:

$$f_k(x_1,\dots,x_{2k},n) := \frac{2i(-1)^k \sinh\frac{x}{2}}{\prod_{i=1}^{2k} 2\cosh\frac{x_i}{2}} \mathcal{F}_k(x;n), \tag{4.38}$$

$$g_k(x_1,\dots,x_{2k+1},n) := \frac{2(-1)^{k+1} \cosh\frac{x}{2}}{\prod_{i=1}^{2k+1} 2\cosh\frac{x_i}{2}} \mathcal{G}_k(x;n), \tag{4.39}$$

where $x := \sum_i x_i$ and

$$\mathcal{F}_k(x;n) := \sum_{j=1}^{k} \binom{2k-1}{k-j} \left[w(x^{j-\frac{1}{2}}) + w(x^{-j+\frac{1}{2}}) \right], \tag{4.40}$$

$$\mathcal{G}_k(x;n) := \binom{2k}{k} w(x) + \sum_{j=1}^{k} \binom{2k}{k-j} \left[w(x^{j}) + w(x^{-j}) \right], \tag{4.41}$$

with

$$\lim_{|x|\to\infty} \mathcal{F}_k(x\,;n) = \mathrm{sgn}(x)\frac{i}{n}2^{2k-1}, \quad \lim_{|x|\to\infty} \mathcal{G}_k(x\,;n) = \mathrm{sgn}(x)\frac{i}{n}2^{2k}, \tag{4.42}$$

and

$$\mathcal{F}_k(x\,;1) = 2^{2k-1} i \coth\frac{x}{2}, \quad \mathcal{G}_k(x\,;1) = 2^{2k-1} i \tanh\frac{x}{2}. \tag{4.43}$$

We can then compute the sum (4.33) to

$$\begin{aligned} &\sum_{j_1,\dots,j_{2k-1}=0}^{n-1} w((-x_1)^j) w(x_2^{j_1-j_2}) \dots w(x_{2k-1}^{j_{2k-2}-j_{2k-1}}) w(x_{2k}^{j_{2k-1}}) \\ &\quad = f_k(x_1,\dots,x_{2k},n) + \frac{(-1)^k}{n} \sum_{j=0}^{k-1} \sigma_{2j}^{(2k)} \left(\tanh\frac{x_1}{2}, \dots, \tanh\frac{x_{2k}}{2} \right), \end{aligned} \tag{4.44}$$

whereas a similar sum involving an even number of indices can be evaluated to

$$\begin{aligned} &\sum_{j_1,\dots,j_{2k}=0}^{n-1} w((-x_1)^j) w(x_2^{j_1-j_2}) \dots w(x_{2k}^{j_{2k-1}-j_{2k}}) w(x_{2k+1}^{j_{2k}}) \\ &\quad = g_k(x_1,\dots,x_{2k+1},n) + i\frac{(-1)^k}{n} \sum_{j=0}^{k-1} \sigma_{2j+1}^{(2k+1)} \left(\tanh\frac{x_1}{2}, \dots, \tanh\frac{x_{2k+1}}{2} \right). \end{aligned} \tag{4.45}$$

In both formulae, $\sigma_j^{(k)}(a_1,\dots,a_k)$ is the elementary symmetric polynomial of order j in k variables, defined as

$$\sigma_0^{(k)}(a_1,\dots,a_k) = 1, \qquad \text{and} \qquad \sigma_j^{(k)}(a_1,\dots,a_k) = \sum_{1\le i_1<i_2<\dots<i_j\le k} a_{i_1} a_{i_2} \cdots a_{i_j}. \tag{4.46}$$

Equations (4.44) and (4.45) can be proven by induction in k, following a procedure already employed for the standard BPTF of the free fermion in [63]. The proofs are presented in Appendix 4.A.

An interesting property of the formula (4.44) and a consistency check of its validity is the fact that the cumulant expansion of $\langle\mu(0)\mu(\ell)\rangle/\langle\mu\rangle^2$ is recovered for $n=1$. Indeed, from (4.43) it follows that

$$f_k(x_1,\dots,x_{2k},1)=\frac{(-1)^{k+1}\cosh\frac{x}{2}}{\prod_{i=1}^{2k}\cosh\frac{x_i}{2}}=(-1)^{k+1}\sum_{j=0}^{k}\sigma_{2j}^{(2k)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2}). \tag{4.47}$$

Then, in the limit $n\to 1$ the only term remaining from the sum (4.44) is the symmetric polynomial $\sigma_{2k}^{(2k)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2})$ which is just the product of its arguments. This exactly reproduces the cumulant expansion of $\log\langle\mu(0)\mu(\ell)\rangle$ given in [62], formula (3.12a). Similarly, it can be shown that

$$g_k(x_1,\dots,x_{2k+1},1)=\frac{i(-1)^{k+1}\sinh\frac{x}{2}}{\prod_{i=1}^{2k+1}\cosh\frac{x_i}{2}}=i(-1)^{k+1}\sum_{j=0}^{k}\sigma_{2j+1}^{(2k+1)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k+1}}{2}). \tag{4.48}$$

4.3.2 Main result of this section

Putting together the expression (4.31) with the sum formula (4.44), we have the following exact result for the cumulant expansion of the logarithm of the ground state two-point function of the CTF $\mathcal{T}_\mu$ in the disordered phase of the Ising model:

$$\begin{aligned}\log\left(\frac{\langle\mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle}{\langle\mathcal{T}_\mu\rangle^2}\right) &= \sum_{k=1}^{\infty}c_{2k}^{\mathcal{T}_\mu}(\ell;n)\\ &=\sum_{k=1}^{\infty}\frac{n}{2k(2\pi)^{2k}}\left[\prod_{i=1}^{2k}\int_{-\infty}^{+\infty}\mathrm{d}\theta_i\,e^{-m\ell\cosh\theta_i}\right]\Big[f_k(\hat\theta_{12},\dots,\hat\theta_{2k-1\,2k},\hat\theta_{1\,2k},n)\\ &+\frac{(-1)^k}{n}\sum_{j=0}^{k-1}\sigma_{2j}^{(2k)}\left(\tanh\frac{\hat\theta_{12}}{2},\dots,\tanh\frac{\hat\theta_{2k-1\,2k}}{2},\tanh\frac{\hat\theta_{1\,2k}}{2}\right)\Big].\end{aligned} \tag{4.49}$$

We now proceed to test the validity of this expression by examining its leading short-distance behaviour beyond the two-particle contribution of Section 4.2.2.

4.4 Conformal dimension from the cumulant expansion

The conformal dimension of the field $\mathcal{T}_\mu$ was already recovered by Δ-sum rule in [107], but the computation in that case only involved two-particle form factors. Below we provide a more extensive test of all the form factors by obtaining the conformal dimension from the complete cumulant expansion (4.49). Each cumulant is expected to contain a leading contribution which is proportional to $\log m\ell$ whereas other divergent terms should cancel out, so that the overall sum gives (4.21) with dimension given by (4.20).

First, let us return to the fully connected sum (4.33) and change variables once more. We define

$$x_i = \hat{\theta}_{i\,i+1} \quad \text{for} \quad i = 1, \dots, 2k-1, \quad x_{2k} = \theta_{2k}, \tag{4.50}$$

so that:

$$\theta_i = \sum_{j=i}^{2k} (-1)^{j-i} x_j, \quad \sum_{i=1}^{2k} \theta_i = \sum_{i=1}^{k} x_{2i-1}, \quad \hat{\theta}_{1\,2k} = \sum_{i=1}^{2k-1} (-1)^{i-1} x_i. \tag{4.51}$$

The Jacobian of the transformation from the θ variables to the x variables is an upper triangular matrix with the diagonal terms being all $+1$, so the measure acquires no extra factor. By applying this change of variables to (4.33) and expressing the result in the new variables (4.50), we obtain:

$$\begin{aligned}
&\sum_{j_1,\dots,j_{2k-1}=0}^{n-1} w((-x_1)^j) w(x_2^{j_1-j_2}) \dots w(x_{2k-1}^{j_{2k-2}-j_{2k-1}}) w\left(\left(\sum_{i=1}^{2k-1} (-1)^{i-1} x_i\right)^{j_{2k-1}}\right) \\
&= (-1)^k \frac{2i \sinh\left(\sum_{i=1}^{k} x_{2i-1}\right)}{2\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right) \prod_{i=1}^{2k-1} 2\cosh\left(\frac{x_i}{2}\right)} \mathcal{F}_k\left(2\sum_{i=1}^{k} x_{2i-1}\,; n\right) \\
&+ \frac{(-1)^k}{n} \sum_{j=0}^{k-1} \sigma_{2j}^{(2k)} \left(\tanh\frac{x_1}{2}, \dots, \tanh\frac{x_{2k-1}}{2}, \tanh\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right),
\end{aligned} \tag{4.52}$$

with $\mathcal{F}_k(x;n)$ the function defined by (4.40). Furthermore, recalling equation (4.47), it is possible to express the sum over symmetric polynomials in terms of products of hyperbolic functions as

$$\begin{aligned}
&\sum_{j=0}^{k-1} \sigma_{2j}^{(2k)} \left(\tanh\frac{x_1}{2}, \dots, \tanh\frac{x_{2k-1}}{2}, \tanh\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right) \\
&\qquad = \frac{\cosh\left(\sum_{i=1}^{k} x_{2i-1}\right)}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right) \prod_{i=1}^{2k-1} \cosh\left(\frac{x_i}{2}\right)} - \tanh\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2} \prod_{i=1}^{2k-1} \tanh\frac{x_i}{2}.
\end{aligned} \tag{4.53}$$

This rewriting will prove useful later on.

4.4.1 Exponential factors

Now let us look at the exponential factors in the integrand of (4.31) and see what they look like in terms of the new variables x_i. From the first relation in (4.51) one has:

$$\sum_{j=i}^{2k-1} (-1)^{j-i} x_j = \begin{cases} \theta_i - \theta_{2k} & \text{for } i \text{ even} \\ \theta_i + \theta_{2k} & \text{for } i \text{ odd} \end{cases}, \tag{4.54}$$

so that

$$\begin{aligned}
&\sum_{i=1}^{2k}\cosh\theta_i = \cosh\theta_{2k} + \sum_{i=1}^{2k-1}\cosh(\theta_i-\theta_{2k}+\theta_{2k}) \\
&= \cosh\theta_{2k} + \cosh\theta_{2k}\left(\sum_{i\,\text{even}}\cosh(\theta_i-\theta_{2k}) + \sum_{i\,\text{odd}}\cosh(\theta_i+\theta_{2k})\right) \\
&+ \sinh\theta_{2k}\left(\sum_{i\,\text{even}}\sinh(\theta_i-\theta_{2k}) - \sum_{i\,\text{odd}}\sinh(\theta_i+\theta_{2k})\right) \\
&= \cosh x_{2k}\left[1+\sum_{i=1}^{2k-1}\cosh\left(\sum_{j=i}^{2k-1}(-1)^{j-i}x_j\right)\right] + \sinh x_{2k}\left[\sum_{i=1}^{2k-1}(-1)^i\sinh\left(\sum_{j=i}^{2k-1}(-1)^{j-i}x_j\right)\right]. \quad (4.55)
\end{aligned}$$

Therefore, since none of the functions in (4.52) depends on x_{2k} the integral on this variable can be carried out by making use of the identity[1]

$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\exp(-A\cosh t - B\sinh t) = 2K_0\left(\sqrt{A^2-B^2}\right), \quad \text{for } A>B, \quad (4.56)$$

giving

$$\int_{-\infty}^{+\infty}\mathrm{d}x_{2k}\,e^{-m\ell\sum_{i=1}^{2k}\cosh\theta_i} = 2K_0(m\ell d_{2k-1}), \quad (4.57)$$

with

$$d_{2k-1}^2 = \left[1+\sum_{i=1}^{2k-1}\cosh\left(\sum_{j=i}^{2k-1}(-1)^{j-i}x_j\right)\right]^2 - \left[\sum_{i=1}^{2k-1}(-1)^i\sinh\left(\sum_{j=i}^{2k-1}(-1)^{j-i}x_j\right)\right]^2. \quad (4.58)$$

The $m\ell \ll 1$ expansion of the modified Bessel function is:

$$K_0(m\ell d_{2k-1}) = -\log m\ell + \log 2 - \ln d_{2k-1} - \gamma + o(m\ell d_{2k-1}), \quad (4.59)$$

from which the leading short-distance contributions to the cumulant expansion can be obtained. It is worth mentioning that one could also resum contributions proportional to the constant term $\log 2-\gamma$ in (4.59) and those should contribute to the $K_{\mathcal{T}_\mu}$-term in (4.21), that is to the logarithm of $\langle\mathcal{T}_\mu\rangle$. A similar computation was carried out in [117] for $\langle\mathcal{T}\rangle$ in the free boson theory.

4.4.2 Short-distance behaviour of the cumulant expansion

Putting together (4.52), (4.53) and (4.57) in (4.31) we can split the cumulant into three contributions

$$c_{2k}^{\mathcal{T}_\mu}(\ell;n) = c_{2k}^{(1)}(\ell;n) + c_{2k}^{(2)}(\ell) + c_{2k}^{\mu}(\ell). \quad (4.60)$$

[1] This identity is obtained by rewriting $A\cosh t + B\sinh t = \sqrt{A^2-B^2}\cosh\left(t+\ln\sqrt{\frac{A+B}{A-B}}\right)$ and then shifting the integration variable t.

We will define these contributions as follows. First:

$$c_{2k}^{(1)}(\ell;n) = \frac{2(-1)^k in}{k(4\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{2k-1}\, K_0(m\ell d_{2k-1}) \times \frac{\sinh\left(\sum_{i=1}^{k} x_{2i-1}\right)}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)} \hat{\mathcal{F}}_k\left(2\sum_{i=1}^{k} x_{2i-1}\,;n\right), \tag{4.61}$$

with

$$\hat{\mathcal{F}}_k(x;n) := \mathcal{F}_k(x;n) - \mathrm{sgn}(x)\frac{i}{n}2^{2k-1}. \tag{4.62}$$

This shift is motivated by the asymptotics (4.42) and ensures that the function $\hat{\mathcal{F}}_k(x;n)$ goes to zero for $|x|$ large. This in turn ensures the convergence of the integrals even when the Bessel function is approximated by its leading short-distance contribution $-\log(m\ell)$.

The next contribution is then a combination of the first term in (4.53) and the term introduced by the shift (4.62):

$$c_{2k}^{(2)}(\ell) = \frac{(-1)^k}{k(2\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{2k-1}\, K_0(m\ell d_{2k-1}) \times \left[\frac{\cosh\left(\sum_{i=1}^{k} x_{2i-1}\right) - \sinh\left(\sum_{i=1}^{k} x_{2i-1}\right)\mathrm{sgn}\left(\sum_{i=1}^{k} x_{2i-1}\right)}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)}\right]. \tag{4.63}$$

Note that this contribution is n-independent. Finally, the contribution $c_{2k}^{\mu}(\ell)$ is nothing but the cumulant of the expansion of $\langle\mu(0)\mu(\ell)\rangle/\langle\mu\rangle^2$ resulting from the last term (the product of tanh functions) in (4.53):

$$c_{2k}^{\mu}(\ell) = \frac{(-1)^{k+1}}{k(2\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{2k-1}\, K_0(m\ell d_{2k-1}) \times \tanh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\tanh\left(\frac{x_i}{2}\right), \tag{4.64}$$

as it indeed coincides with the cumulant presented in [62] when expressed in terms of the variables (4.50).

4.4.3 Leading contribution to $c_{2k}^{(1)}(\ell;n)$

In order to evaluate the integral (4.61) we can perform yet another change of variables:

$$y = \sum_{i=1}^{k} x_{2i-1} \quad \Rightarrow \quad x_{2k-1} = y - \sum_{i=1}^{k-1} x_{2i-1}, \quad \sum_{i=1}^{2k-1}(-1)^{i-1}x_i = y - \sum_{i=1}^{k-1} x_{2i}, \tag{4.65}$$

so that, at short distances $\sum_k c_{2k}^{(1)}(\ell;n) \simeq -z_n \log(m\ell)$ with

$$\begin{aligned} z_n =& \sum_{k=1}^{\infty} \frac{(-1)^k 2ni}{k(4\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}y\, \sinh y\, \hat{\mathcal{F}}_k(2y;n) \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{2k-2} \\ & \left[\operatorname{sech}\left(\frac{y - \sum_{i=1}^{k-1} x_{2i-1}}{2} \right) \prod_{i=1}^{k-1} \operatorname{sech}\left(\frac{x_{2i-1}}{2}\right) \right] \left[\operatorname{sech}\left(\frac{y - \sum_{i=1}^{k-1} x_{2i}}{2} \right) \prod_{i=1}^{k-1} \operatorname{sech}\left(\frac{x_{2i}}{2}\right) \right] \\ =& \sum_{k=1}^{\infty} \frac{(-1)^k 2ni}{k(4\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}y\, \sinh y\, \hat{\mathcal{F}}_k(2y;n) G_k^2(y), \end{aligned} \tag{4.66}$$

where, exactly as in [63, 117]:

$$G_k(y) = \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{k-1} \left[\operatorname{sech}\left(\frac{y - \sum_{i=1}^{k-1} x_i}{2} \right) \prod_{i=1}^{k-1} \operatorname{sech}\left(\frac{x_i}{2}\right) \right] = \int_{-\infty}^{+\infty} \mathrm{d}a \frac{(2\pi)^{k-1} e^{iay}}{\cosh^k \pi a}. \tag{4.67}$$

The functions $G_k(y)$ can be evaluated explicitly to

$$G_k(y) = \frac{(2\pi)^{k-1}}{(k-1)!} \begin{cases} \frac{y}{\pi \sinh \frac{y}{2}} \prod_{j=1}^{\frac{k}{2}-1} \left(\frac{y^2}{\pi^2} + (2j)^2\right) & \text{for k even} \\ \frac{1}{\cosh \frac{y}{2}} \prod_{j=1}^{\frac{k-1}{2}} \left(\frac{y^2}{\pi^2} + (2j-1)^2\right) & \text{for k odd} \end{cases}. \tag{4.68}$$

By replacing $\mathcal{F}_k(x;n)$ with $\hat{\mathcal{F}}_k(x;n)$ in (4.61), we have ensured that the integrals (4.66) are convergent since $G_k^2(y)\sinh y$ is asymptotically polynomial in y and $\hat{\mathcal{F}}_k(2y;n)$ is exponentially decaying. They can be evaluated with great precision and fitted to the function

$$z_n = \frac{1}{12}\left(n - \frac{1}{n}\right) + \frac{1}{4n} + z' = 4\Delta_{\mathcal{T}_\mu} + z', \tag{4.69}$$

with $z' = -0.217(4)$. This additional constant should be cancelled by contributions coming from $c_{2k}^{(2)}(\ell) + c_{2k}^{\mu}(\ell)$.

Numerical results for z_n are shown in Fig. 4.1. It is interesting to observe that there is very good agreement with the formula (4.69) for n integer and also for n not integer, greater than 2. However for $1 < n < 2$ the numerical data differ from (4.69), suggesting that the analytic continuation of (4.66) to $n = 1$ from n real greater than 1 is non-trivial. This is in agreement with results found in [107] where the limit $n \to 1$ of the two-particle form factor contribution produced a delta-function term.

4.4.4 Leading contribution to $c_{2k}^{(2)}(\ell) + c_{2k}^{\mu}(\ell)$

We now consider the leading contribution to the second term in the cumulant (4.60). This is independent of n, and employing the same change of variables as above it is easy to write an expression which is given by a convergent integral involving the functions $G_k(y)$. Letting

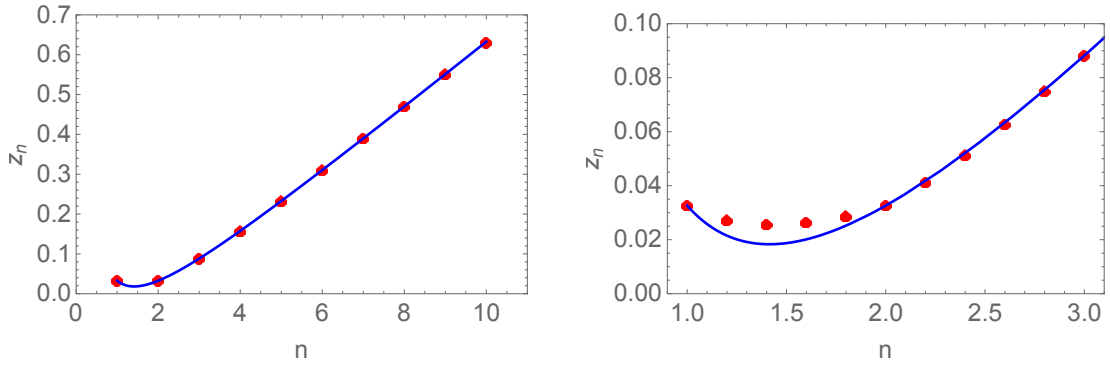


Figure 4.1 Left: The function z_n evaluated numerically through the sum (4.66) for integer values of $n = 1, \ldots, 10$ (red squares) against the formula (4.69) (blue solid line). Right: The same comparison for $n \in [1, 3]$ including non integer values. When evaluating the sum (4.66) numerically we truncate at some value of k. This value of k is different for each value of n and is chosen so that the sum is stable up to 5 decimal digits.

$\sum_k c^{(2)}_{2k}(\ell) \simeq -z'' \log(m\ell)$, we obtain

$$z'' = \sum_{k=1}^{\infty} \frac{2(-1)^k}{k(2\pi)^{2k}} \int_0^{+\infty} \mathrm{d}y\, e^{-y}\, G_k^2(y) = -0.0326(1), \tag{4.70}$$

and we note that

$$z' + z'' = -0.250(0) \simeq -\frac{1}{4}. \tag{4.71}$$

Remarkably, this value is precisely what we need to recover the correct dimension of the field $\mathcal{T}_\mu$. This is because [62]

$$\sum_{k=1}^{\infty} c^{\mu}_{2k}(\ell) \simeq -\frac{1}{4} \log(m\ell), \tag{4.72}$$

as this is the sum over cumulants corresponding to the two-point function $\langle \mu(0)\mu(\ell)\rangle / \langle \mu\rangle^2$ and μ has dimension $4\Delta_\mu = 1/4$. Therefore, the overall leading short-distance behaviour of the $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle / \langle \mathcal{T}_\mu\rangle^2$ cumulants correctly predicts the conformal dimension (4.20). This highly non-trivial result provides strong support for the formula (4.49). In addition, the structure of the cumulants means that we can also write

$$\frac{\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle}{\langle \mathcal{T}_\mu\rangle^2} = \mathcal{R}(\ell; n)\, \frac{\langle \mu(0)\mu(\ell)\rangle}{\langle \mu\rangle^2}, \tag{4.73}$$

where $\mathcal{R}(\ell; n) := \prod_{k=1}^{\infty} e^{c^{(1)}_{2k}(\ell;n)} e^{c^{(2)}_{2k}(\ell)}$ has the property $\mathcal{R}(\ell; 1) = 1$.

Recalling the observation of Section 4.2.2, namely that the cumulant expansion of $\mathcal{T}_\mu$ posed some convergence issues, we note that those issues did not feature in the computations of this section. This is because by writing the cumulant as we have done, all convergence issues have been "hidden" in the contribution $c^{\mu}_{2k}(\ell)$. Indeed, a naive expansion of the Bessel function in (4.64) leads to a divergent

integral. Nonetheless, as shown in [62], the short-distance limit of this quantity can be obtained via a semiclassical approach and it ultimately leads to the expected result (4.72).

4.5 Analytic continuation to $n \in \mathbb{R}^{\geq 1}$

All results obtained so far are valid for $n \in \mathbb{N}$. This is always the case in the replica picture where n represents a replica number. However, the entanglement measures that our two-point function describes are typically defined for generic positive n. Therefore it is useful to write an expression for the correlation function which is valid for $n \in \mathbb{R}^{\geq 1}$. Let us start by studying the analytic continuation of the leading short-distance terms.

4.5.1 Analytic continuation of leading short-distance contributions

The plot in figure 4.1 (right) strongly suggests that our formula needs to be analytically continued in the region $1 < n < 2$. A similar problem was addressed in [63, 117], where it was shown that as n approaches 1 from $n > 1$ some of the poles of the cumulants will cross or pinch the real line and provide additional contributions to the cumulant expansion which are non-vanishing for $n \in \mathbb{R}$ and need to be added. The correct analytic continuation is obtained when these contributions are correctly accounted for. The discussion is nearly identical as for the free boson case [117], albeit involving different functions.

As we have seen, only the contribution $c_{2k}^{(1)}(\ell; n)$ to the cumulant is n-dependent. Therefore we only need to analytically continue the coefficient of the leading short-distance contribution to this term, that is the quantity z_n defined in (4.66). For non-integer n larger than 1, z_n picks up additional contributions which account for the residues of the poles of $\hat{\mathcal{F}}_k(2y; n)$ that cross the real axis as $n \to 1^+$. The sum (4.40) in the function $\hat{\mathcal{F}}_k(2y, n)$ has poles at[2]

$$2y \pm (2j-1)i\pi = (2mn+1)i\pi \quad \text{and} \quad 2y \pm (2j-1)i\pi = (2mn-1)i\pi \quad \text{for} \quad m \in \mathbb{Z}, \tag{4.74}$$

for every $j = 1, \dots, k$. These are due to the kinematic poles of the two-particle form factor (4.12) at $i\pi \pm \theta = 2mni\pi$. Solving the above equation for y gives rise to four families of poles

$$y_1 = (mn+1-j)i\pi, \qquad y_2 = (mn-j)i\pi, \qquad m \in \mathbb{Z}, \tag{4.75}$$

$$y_3 = (mn-1+j)i\pi, \qquad y_4 = (mn+j)i\pi, \qquad m \in \mathbb{Z}, \tag{4.76}$$

[2]The twist field approach assumes n integer larger than 1 (since n is a copy number). For that reason it is natural to look for an analytic continuation to $n = 1$ from $n > 1$. However, once found, the analytic continuation is unique and thus valid for all n.

with corresponding residues of the function inside the sum (4.66) given by:

$$R_1(k,j,m,n) = \frac{n(-1)^{k+j}}{k(4\pi)^{2k}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi mn) G_k^2((nm-j+1)i\pi), \tag{4.77}$$

$$R_2(k,j,m,n) = -\frac{n(-1)^{k+j}}{k(4\pi)^{2k}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi mn) G_k^2((nm-j)i\pi), \tag{4.78}$$

$$R_3(k,j,m,n) = \frac{n(-1)^{k+j}}{k(4\pi)^{2k}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi mn) G_k^2((nm+j-1)i\pi), \tag{4.79}$$

$$R_4(k,j,m,n) = -\frac{n(-1)^{k+j}}{k(4\pi)^{2k}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi mn) G_k^2((nm+j)i\pi). \tag{4.80}$$

These functions are all zero for n integer but they contribute for non-integer n. Let us now investigate which of these poles cross the real line in the limit $n \to 1^+$.

Since there are many indices involved, let us start by considering just one example: $n = \frac{4}{3}$ and up to $k = 2$ in the sum (4.66). According to the formula (4.20) $4\Delta_{\mathcal{T}_\mu} = 0.236111$ in this case but the numerical evaluation of (4.66), after subtracting the constant z', gives the value 0.243211 which slightly overestimates the result. The disagreement is not simply due to numerical imprecision. The function $\hat{\mathcal{F}}_3(2y, 4/3)$ has poles that cross the integration line as $n \to 4/3$. From (4.76) and the definition (4.40) we see that for $k = 1$ the sum runs only over the value $j = 1$. For $j = 1$ the four families of poles labeled by the integer m are:

$$y_1 = imn\pi, \qquad y_2 = (mn-1)i\pi, \qquad m \in \mathbb{Z}, \tag{4.81}$$

$$y_3 = imn\pi, \qquad y_4 = (mn+1)i\pi, \qquad m \in \mathbb{Z}. \tag{4.82}$$

It is clear that all these poles are always above the real line (for $m > 0$) or below the real line (for $m < 0$), that is they never cross the real line, as n approaches $\frac{4}{3}$. Therefore, there is no correction coming from the $k = 1$ contribution. Let us consider $k = 2$. Now $j = 1, 2$. For $j = 1$ the poles are the same as above and never cross the real line. For $j = 2$ we have the following four families:

$$y_1 = i(mn-1)\pi, \qquad y_2 = (mn-2)i\pi, \qquad m \in \mathbb{Z} \tag{4.83}$$

$$y_3 = i(mn+1)\pi, \qquad y_4 = (mn+2)i\pi, \qquad m \in \mathbb{Z}. \tag{4.84}$$

We have already seen above that the poles y_1 and y_3 never cross the real line, so we can only have some contributions from y_2 and y_4. For $m > 0$ and n positive and large both families of poles are above the real line. However, for $n \to \frac{4}{3}$ we see that the pole $(mn-2)i\pi$ crosses the real line for $m = 1$. Similarly, for $m < 0$ and n positive and large all poles are in the lower half plane but the pole

$(mn+2)i\pi$ crosses the real line for $n \to \frac{4}{3}$ and $m = -1$.

In summary, there are two poles for $j = 2$ located at $\pm\frac{2\pi i}{3}$. The corresponding residue contributions are

$$2\pi i(R_2(2,2,1,4/3) - R_4(2,2,-1,4/3)) = -0.00680653. \tag{4.85}$$

Therefore, the addition of the residua of these two poles improves the estimate of the conformal dimension from $4\Delta_{\mathcal{T}_\mu} = 0.243211$ to $4\Delta_\mu = 0.243211 - 0.00680653 = 0.236404$ which is much closer to the exact value (note that the formula (4.66) gives -$4\Delta_{\mathcal{T}_\mu}$, hence the minus sign of (4.85)). The addition of poles for higher values of k, and hence of j, will bring this value ever closer to formula (4.20) as shown in Fig 4.2. In the general n case, in order to fully identify those poles that will cross

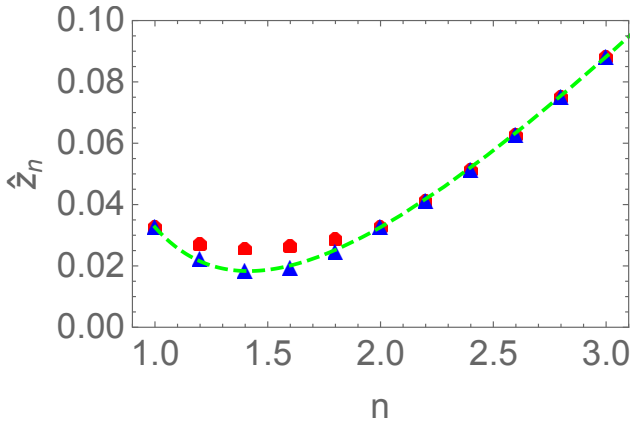


Figure 4.2 The function z_n evaluated numerically through the sum (4.66) for $n \in [1,3]$ (red squares) against the formula (4.69) (green dashed line) and its analytically continued values (blue triangles) given by (4.87).

the real line we find once more four cases:

$$\begin{aligned}
y_1 : mn + 1 - j < 0 \quad &\Rightarrow \quad 1 \le m < \frac{j-1}{n}, \\
y_2 : mn - j < 0 \quad &\Rightarrow \quad 1 \le m < \frac{j}{n}, \\
y_3 : mn - 1 + j < 0 \quad &\Rightarrow \quad -\frac{j-1}{n} < m \le -1, \\
y_4 : mn + j < 0 \quad &\Rightarrow \quad -\frac{j}{n} < m \le -1,
\end{aligned} \tag{4.86}$$

This gives the analytically continued values $\hat{z}_n$

$$\begin{aligned}\hat{z}_n &= z_n + \sum_{k=1}^{\infty}\sum_{j=1}^{k}\sum_{m=1}^{[\frac{j-1}{n}]-q_1} \frac{in(-1)^{k+j+1}}{k(4\pi)^{2k-1}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi nm)\, G_k^2\left((nm-j+1)\,i\pi\right) \\ &\quad + \sum_{k=1}^{\infty}\sum_{j=1}^{k}\sum_{m=1}^{[\frac{j}{n}]-q_2} \frac{in(-1)^{k+j+1}}{k(4\pi)^{2k-1}} \begin{pmatrix} 2k-1 \\ k-j \end{pmatrix} \sinh(i\pi nm)\, G_k^2\left((nm-j)\,i\pi\right), \end{aligned} \tag{4.87}$$

where we used the fact that the residues $R_2(k,j,m,n) = -R_4(k,j,m,n)$ and $R_1(k,j,m,n) = -R_3(k,j,m,n)$ (which produces a factor 2) and are multiplied by a factor $2\pi i$ as required by the residue theorem. The shifts q_1, q_2 take the value 1 when $n[\frac{j-1}{n}] = j-1$ and $n[\frac{j}{n}] = j$, respectively and are zero otherwise (they can be removed by requiring n to be non-integer). Here the symbol [.] represents the integer part. Fig. 4.2 shows the same functions as in Fig. 4.1 (right) plus an additional set of values, which are the analytically continued values of z_n (in blue). As we can see these now agree perfectly with the fit (4.69), even for non-integer n between 1 and 2.

4.5.2 Analytic continuation of the n-derivative

Applications of the correlation function (4.49) in the context of entanglement measures frequently require the computation of its derivative with respect to n followed by the limit $n \to 1$. As discussed in [20, 63] and [107] the derivative with respect to n of the function (4.38) has a discontinuity. More precisely, as n approaches 1 and poles cross the real line, the derivative is not uniformly convergent as a function of θ and this leads to terms involving δ-functions. The simplest examples of this phenomenon are seen for the two-particle contribution to the two-point function of $\mathcal{T}$ [20] and of $\mathcal{T}_\mu$ [107]. Here we show how this generalises to the whole cumulant sum. Notice that we only need to consider the contribution from the function $c_{2k}^{(1)}(\ell;n)$ in (4.61) since all other terms are independent of n and so the derivative is zero. For this term, we actually only need to consider $\mathcal{F}_k(x;n)$ as the additional term in $\hat{\mathcal{F}}_k(x;n)$ version is also n-independent. Thus, we define

$$\begin{aligned} s_{2k}^{\mathcal{T}_\mu}(\ell) &:= -\lim_{n\to 1}\frac{d}{dn} c_{2k}^{(1)}(\ell;n) \\ &= \frac{2(-1)^{k+1}i}{k(4\pi)^{2k}} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_{2k-1}\, K_0(m\ell d_{2k-1}) \\ &\quad \times \frac{\sinh\left(\sum_{i=1}^{k} x_{2i-1}\right)}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)} \lim_{n\to 1}\frac{d}{dn}\left[n\mathcal{F}_k\left(2\sum_{i=1}^{k} x_{2i-1}\,;n\right)\right]. \end{aligned} \tag{4.88}$$

One way to treat the derivative is to recall the $k=1$ result that was derived in [107], namely

$$\begin{aligned}\lim_{n\to 1}\frac{d}{dn}nf_1(x,x,n) &= -\frac{i}{2}\frac{\sinh x}{\cosh^2\frac{x}{2}}\lim_{n\to 1}\frac{d}{dn}n[w(2x+i\pi)+w(2x-i\pi)] \\ &= \frac{x}{\cosh^2\frac{x}{2}\sinh x}-\frac{\pi^2}{2}\delta(x),\end{aligned} \tag{4.89}$$

that is, there is a finite part and a distribution part that accounts for the behaviour around $x=0$. Recall that the function $f_1(x,y,n)$ is defined in (4.34). This extends to higher cumulants in similar ways, so that we can write

$$s_{2k}^{\mathcal{T}_\mu}(\ell)=s_{2k}^{\text{fin}}(\ell)+s_{2k}^{\delta}(\ell), \tag{4.90}$$

where the two contributions represent the finite and δ-function contributions. The finite part can be easily computed by noting that

$$\lim_{n\to 1}\frac{d}{dn}n\sinh x\mathcal{F}(2x;n)=\frac{i2^{2k-1}x}{\sinh x}, \tag{4.91}$$

and therefore

$$\begin{aligned}s_{2k}^{\text{fin}}(\ell) &= \frac{(-1)^k}{k(2\pi)^{2k}}\int_{-\infty}^{+\infty}\mathrm{d}x_1\cdots\int_{-\infty}^{+\infty}\mathrm{d}x_{2k-1}\,K_0(m\ell d_{2k-1}) \\ &\quad\times\frac{\sum_{i=1}^k x_{2i-1}}{\sinh\left(\sum_{i=1}^k x_{2i-1}\right)\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)}.\end{aligned} \tag{4.92}$$

The δ-function contribution is a generalisation of the $k=1$ case seen above and can be obtained by identical arguments as those presented in [63]. In fact, the result is also identical to formula (4.6) in [63], that is,

$$\begin{aligned}s_{2k}^{\delta}(\ell) &= \frac{\pi^2(-1)^k}{k(4\pi)^{2k}}\int_{-\infty}^{+\infty}\mathrm{d}x_1\cdots\int_{-\infty}^{+\infty}\mathrm{d}x_{2k-1}\,\delta(\sum_{i=1}^k x_{2i-1}) \\ &\quad\times\left[\binom{2k-2}{k-1}\frac{2K_0(2m\ell d_{2k-1})}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)}\right] \\ &\quad-\frac{\pi^2(-1)^k}{k(4\pi)^{2k}}\int_{-\infty}^{+\infty}\mathrm{d}x_1\cdots\int_{-\infty}^{+\infty}\mathrm{d}x_{2k}\,\delta(\sum_{i=1}^k x_{2i-1}) \\ &\quad\times\sum_{j=1}^{k}\sum_{m=1}^{j-1}\sum_{q=\pm}\left[\binom{2k-1}{k-j}(-1)^j\frac{\prod_{i=1}^{2k}e^{-rm\cosh\left(\sum_{j=1}^{2k}(-1)^{j-i}x_i+i\pi q\frac{j-m}{2k}\right)}}{\cosh\left(\frac{\sum_{i=1}^{2k-1}(-1)^{i-1}x_i}{2}\right)\prod_{i=1}^{2k-1}\cosh\left(\frac{x_i}{2}\right)}\right].\end{aligned} \tag{4.93}$$

4.6 Concluding remarks

In this paper we studied the normalised two-point function $\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle/\langle \mathcal{T}_\mu\rangle^2$ of the composite twist field $\mathcal{T}_\mu$ and its conjugate. The motivation to study this object comes from recent investigations of symmetry-resolved entanglement entropy in QFTs possessing an internal $\mathbb{Z}_2$ symmetry [19, 94, 107]. More fundamentally, our work contributes to developing the understanding of correlation functions in the replica Ising field theory, a theory that contains a large number of symmetry fields or twist fields which are not present in the standard, non-replicated model.

We employed traditional IQFT techniques, namely the the form factor bootstrap program adapted to composite twist fields [107], to expand the logarithm of the correlation function into a series of cumulants. The main result of this chapter consists in deriving closed-form expressions for these cumulants which result from a number of multiple sum formulae involving the two-particle form factors of the field $\mathcal{T}_\mu$, presented in Appendix 4.A.

Employing the cumulant expansion we found the following structure

$$\frac{\langle \mathcal{T}_\mu(0)\tilde{\mathcal{T}}_\mu(\ell)\rangle}{\langle \mu(0)\mu(\ell)\rangle}\frac{\langle\mu\rangle^2}{\langle \mathcal{T}_\mu\rangle^2} = \mathcal{R}(\ell;n) \qquad \text{with} \qquad \mathcal{R}(\ell;1)=1, \tag{4.94}$$

where μ is the disorder field of the Ising field theory, and we provided an explicit expression for $\mathcal{R}(\ell;n)$. By exact resummation of leading contributions to the cumulant expansion, we showed that at short distances this two-point function scales as a power law in r with exponent consistent with the CFT dimension (4.20). Furthermore, we provided the analytic continuation of our formulae to real replica number, generalising results found in [107] and [20]. As a byproduct of our investigation, we have also showed how the form factors of the composite field $\mathcal{T}_\sigma$ can be obtained from those of $\mathcal{T}_\mu$ via clustering in momentum space, in much the same way as the form factors of the fields σ and μ are related.

As mentioned above, our result has applications in the context of the symmetry-resolved entanglement entropy and directly leads to a more complete formula for the latter in the Ising model. We further expect the results of this investigation to apply with some modifications to other composite fields, at least for free theories, for instance those associated with $U(1)$ symmetry in doubled free models which were studied in [105, 123, 124].

4.A Summation formulae

In this appendix we derive the identities presented in Section 4.3.1. Let us first prove equations (4.35) and (4.34), which are both obtained via contour integration. To show (4.35), consider the

integral in the complex plane:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z\, \pi \cot(\pi z) w(x^z), \tag{4.95}$$

where $\mathcal{C}$ is the rectangular contour with vertices $-\epsilon + iL$, $-\epsilon - iL$, $n - \epsilon - iL$, $n - \epsilon + iL$. The vertical contributions cancel off because the integrand is invariant under the shift $z \to z + n$. The same holds for the horizontal contributions in the large L limit, as

$$\lim_{L\to\infty} \cot \pi(t \pm iL) = \mp i, \quad \lim_{L\to\infty} w\left(x^{t\pm iL}\right) = \mp \frac{i}{n}, \quad t \in \mathbb{R}, \tag{4.96}$$

and therefore the sum of the residues must vanish. Within the integration contour, the function $\pi \cot(\pi z)$ has simple poles at $z = 0, 1, \dots n-1$ with unite residue. The kinematic poles of $w(x^z)$ are at $z = \frac{1}{2} - \frac{x}{2\pi i}$, $z = n - \frac{1}{2} - \frac{x}{2\pi i}$, with residue $\frac{1}{2\pi}$. At both these points, $\cot(\pi z) = -i \tanh \frac{x}{2}$. Putting all the pieces together, one has therefore:

$$0 = \sum_{j=0}^{n-1} w(x^j) - i \tanh \frac{x}{2}. \tag{4.97}$$

Using the very same strategy, one can prove (4.34). The integral to evaluate is now

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z\, \pi \cot(\pi z) w((-x)^z) w(y^z), \tag{4.98}$$

along the same contour $\mathcal{C}$ as before. In this case, however, the horizontal contributions do not cancel off, as

$$\lim_{L\to\infty} w((-x)^{t\pm iL})) w(y^{t\pm iL}) = -\frac{1}{n^2}, \quad t \in \mathbb{R}, \tag{4.99}$$

and thus the integral evaluates to $-\frac{1}{n}$ in the large L limit. Summing over the residues of the poles of $\pi \cot(\pi z)$ gives the left-hand side of (4.34), while the kinematic poles are now at $z = \frac{1}{2} + \frac{x}{2\pi i}$, $-\frac{1}{2} + n + \frac{x}{2\pi i}$ and $z = \frac{1}{2} - \frac{y}{2\pi i}$, $-\frac{1}{2} + n - \frac{y}{2\pi i}$, with residues:

$$\operatorname*{Res}_{z=\frac{1}{2}+\frac{x}{2\pi i}} w((-x)^z) w(y^z) = \frac{1}{2\pi} w(x + y + i\pi), \qquad \operatorname*{Res}_{z=n-\frac{1}{2}+\frac{x}{2\pi i}} w((-x)^z) w(y^z) = \frac{1}{2\pi} w(x + y - i\pi),$$
$$\operatorname*{Res}_{z=\frac{1}{2}-\frac{y}{2\pi i}} w((-x)^z) w(y^z) = -\frac{1}{2\pi} w(x + y - i\pi), \qquad \operatorname*{Res}_{z=n-\frac{1}{2}-\frac{y}{2\pi i}} w((-x)^z) w(y^z) = -\frac{1}{2\pi} w(x + y + i\pi).$$

By evaluating the cotangent at the kinematic poles and putting all the pieces together, we obtain

$$\begin{aligned} -\frac{1}{n} &= \sum \mathrm{Res}[\pi \cot(\pi z) w((-x)^z) w(y^z)] \\ &= \sum_{j=0}^{n-1} w((-x)^j) w(y^j) + \frac{i}{2} \frac{\sinh\left(\frac{x+y}{2}\right)}{\cosh\left(\frac{x}{2}\right)\cosh\left(\frac{y}{2}\right)} [w(x + y + i\pi) + w(x + y - i\pi)], \end{aligned}$$

which is indeed (4.34).

Using the above results, it is possible to prove (4.44) and (4.45) by induction. It is useful to observe beforehand that the following expansions hold:

$$\frac{\sinh\left(\sum_{i=1}^{k} x_i\right)}{\prod_{i=1}^{k}\cosh x_i} = \sum_{j=0}^{[\frac{k-1}{2}]} \sigma_{2j+1}^{(k)}(\tanh x_1,\dots,\tanh x_k), \tag{4.100}$$

$$\frac{\cosh\left(\sum_{i=1}^{k} x_i\right)}{\prod_{i=1}^{k}\cosh x_i} = \sum_{j=0}^{[\frac{k}{2}]} \sigma_{2j}^{(k)}(\tanh x_1,\dots,\tanh x_k). \tag{4.101}$$

Following the procedure employed in [63, 117], we will prove that (4.44) implies (4.45). If (4.44) holds, than we can shift x_{2k} by $-2i\pi p$, multiply the left-hand side by a factor $w(x_{2k+1}^p)$ and sum over p to obtain:

$$\begin{aligned}&\sum_{j_1,\dots,j_{2k-1},p=0}^{n-1} w((-x_1)^j)w(x_2^{j_1-j_2})\dots w(x_{2k-1}^{j_{2k-2}-j_{2k-1}})w(x_{2k}^{j_{2k-1}-p})w(x_{2k+1}^p)\\ &=\sum_{p=0}^{n-1} f_k(x_1,\dots,x_{2k}^{-p},n)w(x_{2k+1}^p)+\sum_{p=0}^{n-1}\frac{(-1)^k}{n}\sum_{j=0}^{k-1}\sigma_{2j}^{(2k)}\left(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}^{-p}}{2}\right)w(x_{2k+1}^p),\end{aligned} \tag{4.102}$$

Let us focus on the first term in the second line, which yields two contributions due to the presence of a constant term in the right-hand side of (4.34). Indeed, defining $x=\sum_{i=1}^{2k}x_i$, we obtain:

$$\begin{aligned}&\sum_{p=0}^{n-1} f_k(x_1,\dots,x_{2k}^{-p},n)w(x_{2k+1}^p)\\ =&\frac{2i(-1)^k\sinh\frac{x}{2}}{\prod_{i=1}^{2k}2\cosh\frac{x_i}{2}}\sum_{p=0}^{n-1}\sum_{j=1}^{k}\binom{2k-1}{k-j}\left[w\left(x^{j-p}-i\pi\right)+w\left(x^{-j-p}+i\pi\right)\right]w(x_{2k+1}^p)\\ =&g_k(x_1,\dots,x_{2k+1},n)+\frac{4i(-1)^k\sinh\frac{x}{2}}{n\prod_{i=1}^{2k}2\cosh\frac{x_i}{2}}\sum_{j=1}^{k}\binom{2k-1}{k-j}\\ =&g_k(x_1,\dots,x_{2k+1},n)+\frac{i(-1)^k}{n}\sum_{j=0}^{k-1}\sigma_{2j+1}^{(2k)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2}).\end{aligned} \tag{4.103}$$

The emergence of the function g_k in going from the second to the third line was already proved in Appendix A of [63]. In going from the third to the fourth line we used the identity (4.100) and $\sum_{j=1}^{k}\binom{2k-1}{k-j}=2^{2k-2}$. We now consider the second term in the second line of (4.102): using the fact that $\tanh\frac{x\pm p}{2}=\tanh\frac{x}{2}$ and the sum (4.35), we have:

$$\begin{aligned}&\sum_{p=0}^{n-1}\frac{(-1)^k}{n}\sum_{j=0}^{k-1}\sigma_{2j}^{(2k)}\left(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}^{-p}}{2}\right)w(x_{2k+1}^p)\\ &=\frac{i(-1)^k}{n}\sum_{j=0}^{k-1}\sigma_{2j}^{(2k)}\left(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2}\right)\tanh\frac{x_{2k+1}}{2}.\end{aligned} \tag{4.104}$$

Now we observe that the elementary symmetric polynomial of degree j in k variables can be

decomposed as $\sigma_j^{(k)}(a_1,\dots,a_k) = \sigma_j^{(k-1)}(a_1,\dots,a_{k-1}) + \sigma_{j-1}^{(k-1)}(a_1,\dots,a_{k-1})\,a_k$, hence:

$$\begin{aligned}&\sum_{j=0}^{k-1}\left[\sigma_{2j+1}^{(2k)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2}) + \sigma_{2j}^{(2k)}\left(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k}}{2}\right)\tanh\frac{x_{2k+1}}{2}\right]\\ =&\sum_{j=0}^{k-1}\sigma_{2j+1}^{(2k+1)}(\tanh\frac{x_1}{2},\dots,\tanh\frac{x_{2k+1}}{2}).\end{aligned} \tag{4.105}$$

Thus the sum of (4.103) and (4.104) yields (4.45). In an analogous way it is possible to prove (4.44) starting from (4.45).

CHAPTER

FIVE

ENTANGLEMENT ASYMMETRY IN THE ORDERED PHASE OF THE ISING FIELD THEORY

Global symmetries of quantum many-body systems can be spontaneously broken. Whenever this mechanism happens, the ground state is degenerate and one encounters an ordered phase. In this chapter, based on [5], we investigate this phenomenon by examining the entanglement asymmetry of a region in a one-dimensional many-body system in its ordered phase. This quantity has been recently introduced in the context of $U(1)$ symmetry breaking, and we extend its definition to encompass arbitrary finite groups G. We also establish a field-theoretic framework in the replica theory using twist operators. We explicitly demonstrate our construction in the ordered phase of the Ising field theory in 1+1 dimensions, where a $\mathbb{Z}_2$ symmetry is spontaneously broken, and we employ a form factor bootstrap approach to characterise a family of composite twist fields. Analytical predictions are provided for the entanglement asymmetry of an interval in the Ising model as the length of the interval becomes large. We also propose a general conjecture relating the entanglement asymmetry and the number of degenerate vacua, which we expect to be valid for a large class of states, and we prove it explicitly in a simple case.

5.1 Introduction

Symmetry is nowadays considered a cornerstone of modern Physics. Its breaking is responsible for a plethora of interesting phenomena, such as ferromagnetism [204], superconduction, superfluidity [205]. Spontaneous symmetry breaking is the phenomenon by which a symmetry possessed by a quantum system, described in terms of its Hamiltonian/Lagrangian or its equations of motion, is not mirrored by the ground state of the system. For instance, it is very well known that at low-enough temperatures

some ferromagnetic materials magnetise spontaneously in a given direction, which depends solely on the way those materials are cooled down. A similar mechanism has been observed at zero temperature in frustrated quantum systems, e.g. quantum spin chains, where spontaneous symmetry breaking arises if an external parameter, such as a magnetic field or a chemical potential, is varied [206].

Zero-temperature phases with different symmetries can be separated by a quantum phase transition [207]. Close to the transition, quantum correlations are dominant, giving rise to a large amount of entanglement among the regions of the systems. In the last decade, much attention has been devoted to the relation between symmetries and entanglement [19, 92, 94], especially in the context of zero-temperature states close to phase transitions, which can be investigated quantitatively via QFT [104, 124–126]. As a result of this interest, a notion of symmetry-resolved entanglement for one-dimensional quantum many-body systems was introduced in [19, 92, 94] and further developed in [96, 97, 121, 185, 194, 208].

In contrast, until recently, little research focused on exploring the relationship between symmetry breaking and entanglement. In [29] a new measure of entanglement, dubbed *entanglement asymmetry*, was introduced to probe symmetry breaking in many-body systems. Originally, this quantity was used in the context of quench dynamics to analyse the restoration of a $U(1)$ symmetry in a symmetry-breaking state of a quantum spin chain, evolved using a symmetric Hamiltonian [29–31]. Following these initial works, entanglement asymmetry was investigated in several other models, as we reviewed in Section 1.3. However, a general framework for the study of entanglement asymmetry in quantum field theories is still missing. Therefore, the purpose of this work is to apply the proposed approach of [29] to characterise the spontaneous symmetry-breaking pattern in equilibrium quantum many-body systems that can be described by 1+1D QFT.

We initiate this program by giving a definition of entanglement asymmetry that can be applied to any finite or compact Lie group, extending the construction already provided for $U(1)$ in [29] and for $\mathbb{Z}_N$ in [112]. Moreover, we provide a field-theoretic treatment of the one-dimensional quantum Ising model via form factor bootstrap. Our approach combines the expression of the Rényi entropies in terms of twist fields via the replica trick [18, 20] and its extension in the presence of additional Aharonov–Bohm fluxes [19], which stems from the action of the group and gives rise to composite (charged) twist fields [157]. A vast literature regarding integrable field theories where similar fields were considered is present, and we refer the reader to [1–4, 105–108] for further details. However, most of these works refer to paramagnetic phases of field theories, where a single symmetric vacuum is present: there, different ways of inserting the same total Aharonov–Bohm flux among the replicas give rise to the same result (see e.g [105, 107, 108]). An example of this is provided by complex free theories with $U(1)$ symmetry, in which the flux can be equivalently inserted between the nth and the first replica or fractionalised and uniformly distributed among all replicas. These distinct

choices correspond to the insertion of distinct operators, and symmetry-broken states explicitly spot these differences. This mechanism, which is new to our knowledge, lies at the core of entanglement asymmetry, as we will show.

Before delving into the main content of this work, we first provide a definition of Rényi entanglement asymmetry, inspired by [29], which applies to any finite group G. Let us consider a (possibly mixed) state ρ of a bipartite system $A \cup \bar{A}$, described by the Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}. \tag{5.1}$$

We assume that a finite group G acts unitarily on $\mathcal{H}$ via a linear map $G \ni g \mapsto \hat{g} \in \mathrm{End}(\mathcal{H})$ which satisfies

$$\hat{g} = \hat{g}_A \otimes \hat{g}_{\bar{A}} \in \mathrm{End}\left(\mathcal{H}_A\right) \otimes \mathrm{End}\left(\mathcal{H}_{\bar{A}}\right). \tag{5.2}$$

That is, A and $\bar{A}$ are not mixed by G, which plays the role of a global symmetry for the system (see also [95]). Given ρ, we construct the reduced density matrix over A in the usual way as

$$\rho_A := \mathrm{Tr}_{\bar{A}}\left(\rho\right), \tag{5.3}$$

and we aim to understand whether ρ_A is symmetric under the group G. This is equivalent to asking whether the equality

$$\rho_A = \hat{g}_A \rho_A \hat{g}_A^{-1}, \tag{5.4}$$

always holds or it is violated for some $g \in G$, thus signaling a breaking of the symmetry. To do so, we introduce a fictitious density matrix $\tilde{\rho}_A$ defined as

$$\tilde{\rho}_A := \frac{1}{|G|} \sum_{g \in G} \hat{g}_A \rho_A \hat{g}_A^{-1}, \tag{5.5}$$

where $|G|$ denotes the order of the group G. The quantity $\tilde{\rho}_A$ can be regarded as the *symmetrisation* of ρ_A under the adjoint action of the group (see e.g. a standard textbook in linear representations of finite groups [209]), as it is easy to show that $\tilde{\rho}_A$ is symmetric under G:

$$\hat{g}_A \tilde{\rho}_A \hat{g}_A^{-1} = \tilde{\rho}_A, \quad \forall g \in G. \tag{5.6}$$

Moreover, it is not difficult to check that

$$\tilde{\rho}_A = \rho_A \quad \text{iff} \quad \rho_A = \hat{g}_A \rho_A \hat{g}_A^{-1}, \quad \forall g \in G. \tag{5.7}$$

Therefore, following the logic of [29], it is rather natural to compare the two states ρ_A and $\tilde{\rho}_A$ in order

to probe (spontaneous or explicit) symmetry breaking at the level of the subsystem A. We do so via the introduction of the *Rényi entanglement asymmetry*, defined as

$$\Delta S_n := \frac{1}{1-n} \log \mathrm{Tr}\,(\tilde{\rho}_A^n) - \frac{1}{1-n} \log \mathrm{Tr}\,(\rho_A^n)\,, \tag{5.8}$$

that is the difference of Rényi entropies of the two states. Similarly, in the limit $n \to 1$ we get the difference of von Neumann entropies

$$\Delta S_1 := -\mathrm{Tr}\,(\tilde{\rho}_A \log \tilde{\rho}_A) + \mathrm{Tr}\,(\rho_A \log \rho_A)\,, \tag{5.9}$$

and we refer to ΔS_1 as the *entanglement asymmetry*. We mention that the definitions above can be generalised to compact Lie groups in a straightforward way [210]. For instance, given the (normalised) Haar measure $\int_G \mathrm{d}g$ of a compact Lie Group G [209], it is sufficient to replace (5.5) with

$$\tilde{\rho}_A := \int_G \mathrm{d}g\, \hat{g}_A \rho_A \hat{g}_A^{-1}, \tag{5.10}$$

that is compatible with the original formulation of asymmetry valid for $U(1)$ [29]. While most of the theory we discuss in this work is unchanged for continuous groups, finite groups are the only relevant ones in the context of spontaneous symmetry breaking in zero-temperature one-dimensional systems, due to the Mermin-Wagner theorem[211].

In this work we analyse in detail the Ising field theory [212, 213] in its ferromagnetic phase, probing the symmetry-breaking pattern

$$\mathbb{Z}_2 \to \{1\}, \tag{5.11}$$

via the (Rényi) entanglement asymmetry associated with the group $G = \mathbb{Z}_2$. In particular, we consider one (of the two) spontaneously broken ground states, and we compute the asymmetry of an interval of size ℓ which is large compared to the correlation length $\sim m^{-1}$ of the model. We do so with a replica trick, relating the Rényi entanglement asymmetry for $n \geq 2$ integer to the expectation values of some $\mathbb{Z}_2$ composite twist fields. We describe systematically the form factors of these fields, and we obtain analytical results for their correlation functions in the two-particle approximation. The main result for the Ising QFT can be summarised by the following expression:

$$\Delta S_n \simeq \log 2, \quad m\ell \gg 1, \tag{5.12}$$

which is valid for any integer $n \geq 2$ up to exponentially small corrections which we compute at first order.

We structure this chapter as follows. In Section 5.2 we provide an explicit construction of the

composite twist operators, valid for finite-dimensional Hilbert spaces, and we relate their expectation values to the asymmetry of a subsystem. In Section 5.3, we review the scattering properties of the Ising field theory, and we characterise the form factors of the standard twist field in the ferromagnetic phase. These form factors are eventually employed in the computation of the Rényi entropies in this phase. Then, in Section 5.4, the core of our work, we extend our analysis to a family of composite $\mathbb{Z}_2$ twist fields. In particular, we focus on those fields with vanishing net $\mathbb{Z}_2$ flux across the replicas, and we establish a connection between their form factors and the ones of the standard twist fields. This analysis allows us to compute the entanglement asymmetry of a large interval. Finally, in Section 5.5 we identify a fundamental mechanism behind the large-volume behavior of the entanglement asymmetry of arbitrary (clustering) states, and we provide a general conjecture valid for any finite group G. We leave conclusions and outlook to Section 5.6. Appendix 5.A contains the details of some two-particle form factor calculations, while in Appendix 5.B we provide some useful bounds on the Rényi asymmetries for finite groups. Finally, in Appendix 5.C we discuss some results on the entanglement entropy in the ordered and disordered phase of the transverse-field Ising chain.

5.2 Twist operators and entanglement asymmetry

In this section, following the ideas of [214, 215], we introduce a family of operators in the replica theory, the twist operators, which allow us to express the entanglement measures of interest as expectation values. The branch-point twist fields in 1+1D replica theories, together with their composite versions and their generalisations to higher-dimensional theories, have been widely used throughout this thesis. However, the point of view we adopt here is slightly different. While in the previous chapters we defined the twist operators via their equal-time commutation relations with local fields of the theory, in this chapter we give a rigorous characterisation of these operators for finite-dimensional Hilbert spaces, extending the analysis of [214, 215] with the additional introduction of Aharonov-Bohm fluxes arising in the presence of the action of a group G. While the technical details of our construction do not immediately apply to infinite chains or quantum field theories, the main important properties are expected to remain valid in the continuum limit, as explained in [214].

5.2.1 Characterisation of the twist operators

Let us consider a finite-dimensional Hilbert space $\mathcal{H}$ describing a bipartition $A \cup \bar{A}$ as in (5.1). We take n copies of the above, so that the total Hilbert space of the replica model is $\mathcal{H}^{\otimes n}$. We aim to define a twist operator $\mathcal{T}_A$, associated with a cyclic permutation among the replicas restricted to the

subsystem A. We do so by requiring that on any factorised state of the replica model

$$|v_1,\dots,v_n\rangle\otimes|\bar v_1,\dots,\bar v_n\rangle := \left(\otimes_{j=1}^n|v_j\rangle\right)\otimes\left(\otimes_{j=1}^n|\bar v_j\rangle\right), \quad |v_j\rangle\in\mathcal{H}_A, |\bar v_j\rangle\in\mathcal{H}_{\bar A}, \tag{5.13}$$

the action of $\mathcal{T}_A$ is[1]

$$\mathcal{T}_A\left(|v_1,v_2,\dots,v_n\rangle\otimes|\bar v_1,\dots,\bar v_n\rangle\right) := |v_n,v_1,\dots,v_{n-1}\rangle\otimes|\bar v_1,\dots,\bar v_n\rangle. \tag{5.14}$$

Physically, $\mathcal{T}_A$ implements the permutation $j\to j+1$ on A, where j is a replica index.

In the presence of a global symmetry associated with a group G, it is possible to "charge"the twist operator defined above via the action of G. Following the terminology of [19], this corresponds to the additional insertion of Aharonov–Bohm fluxes between the replicas. The action of G on $\mathcal{H}$, defined in (5.2), is naturally extended to the replica model $\mathcal{H}^{\otimes n}$. Thus, we can construct a composite twist operator $\mathcal{T}_A^{\{g_1,\dots,g_n\}}$ obtained as the combination of the replica shift $j\to j+1$ and the insertion of a flux g_j between the jth and the $(j+1)$th replicas. We do that by defining

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}} := \mathcal{T}_A\circ(\hat g_{1,A}\otimes\cdots\otimes\hat g_{n,A}\otimes\mathbb{1}_{\bar A}^{\otimes n}), \tag{5.15}$$

from which it follows that

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}}\left(|v_1,\dots,v_n\rangle\otimes|\bar v_1,\dots,\bar v_n\rangle\right) = \hat g_{n,A}|v_n\rangle\otimes\hat g_{1,A}|v_1\rangle\otimes\cdots\otimes|\bar v_1,,\dots,\bar v_n\rangle, \tag{5.16}$$

where $\hat g_{j,A}$ refers to the action of g_j restricted to A and $\mathbb{1}$ is the action of the identity element $1\in G$. An operator is therefore associated to any n-tuple $\{g_1,\dots,g_n\}$ and, in particular, $\mathcal{T}_A$ is recovered for $\{g_1,\dots,g_n\}=\{1,\dots,1\}$.

In the remaining part of this section, we investigate some useful properties of the composite twist operators. Namely, we relate them to specific traces (charged moments) that appear in the computation of the Rényi entanglement asymmetry. Then, we show that distinct twist operators can be related to each other via global unitary transformations induced by the group elements. Finally, we present the mutual locality relations of twist operators with local observables.

1. **Computation of traces.** Let $\rho\in\mathrm{End}(\mathcal{H})$ be the density matrix of a (possibly mixed) state and $\rho_A=\mathrm{Tr}_{\mathcal{H}_{\bar A}}\rho$. We show that the following identities hold:

$$\mathrm{Tr}_{\mathcal{H}^{\otimes n}}\left(\rho^{\otimes n}\mathcal{T}_A\right) = \mathrm{Tr}_{\mathcal{H}_A}\rho_A^n, \tag{5.17}$$

[1] By linearity this definition allows us to express the action of $\mathcal{T}_A$ on any vector in $\mathcal{H}^{\otimes n}$.

$$\mathrm{Tr}_{\mathcal{H}^{\otimes n}}\left(\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_n\}}\right)=\mathrm{Tr}_{\mathcal{H}_A}\left(\rho_A\hat{g}_{n,A}\rho_A\hat{g}_{n-1,A}\dots\rho_A\hat{g}_{1,A}\right). \tag{5.18}$$

These are relations between the (charged) moments of ρ_A and the expectation values of the (composite) twist operators in the replica model. Let us stress that in the left-hand side the traces are taken over $\mathcal{H}^{\otimes n}$, while in the right-hand side the traces are only over one copy of the subsystem A. Since the first relation is a special case of the second one, obtained when all the $g_j=1$, we focus on (5.18).

A straightforward proof can be provided if one picks two orthonormal bases for $\mathcal{H}_A,\mathcal{H}_{\bar{A}}$ and compute the trace in those bases. We denote by $|e_j\rangle,|\bar{e}_j\rangle$ the generic basis elements of $\mathcal{H}_A,\bar{\mathcal{H}}_A$ in the jth replica respectively, and we simply expand

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}^{\otimes n}}\left(\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_n\}}\right)\\
=&\sum_{\substack{e_1,\dots,e_n\\ \bar{e}_1,\dots,\bar{e}_n}}\langle e_1,\dots,e_n|\otimes\langle\bar{e}_1,\dots,\bar{e}_n|\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_n\}}|e_1,\dots,e_n\rangle\otimes|\bar{e}_1,\dots,\bar{e}_n\rangle\\
=&\sum_{e_1,\dots,e_n}\langle e_1|\rho_A\hat{g}_{n,A}|e_n\rangle\langle e_2|\rho_A\hat{g}_{1,A}|e_1\rangle\dots\langle e_n|\rho_A\hat{g}_{n-1,A}|e_{n-1}\rangle\\
=&\mathrm{Tr}_{\mathcal{H}_A}\left(\rho_A\hat{g}_{n,A}\rho_A\hat{g}_{n-1,A}\dots\hat{g}_{2,A}\rho_A\hat{g}_{1,A}\right),
\end{aligned} \tag{5.19}$$

which proves equation (5.18).

2. Unitary transformations

An important observation is that different composite twist operators $\mathcal{T}_A^{\{g_1,\dots,g_n\}}$, corresponding to different choices of $\{g_1,\dots,g_n\}$, can be related to each other via global unitary transformations. Specifically, given $h_j\in G$, $j=1,\dots,n$, we show that

$$\left(\hat{h}_1\otimes\dots\otimes\hat{h}_n\right)\mathcal{T}_A^{\{g_1,\dots,g_n\}}\left(\hat{h}_1\otimes\dots\otimes\hat{h}_n\right)^{-1}=\mathcal{T}_A^{\{g'_1,\dots,g'_n\}},\quad g'_j:=h_{j+1}g_jh_j^{-1}. \tag{5.20}$$

In the relation above, the action of h_j is not restricted to the subsystem A. Indeed, the region $\bar{A}$ is not affected by the twist operator appearing in Eq. (5.20), and the combined action of h_j,h_j^{-1} gives the identity on that subsystem.

To prove (5.20) it is sufficient to show that the left- and right-hand side act in the same way on

factorised vectors of $\mathcal{H}^{\otimes n}$. Therefore, for a given state (5.13) we just compute

$$\begin{aligned}
&\left(\hat{h}_1 \otimes \cdots \otimes \hat{h}_n\right) \mathcal{T}_A^{\{g_1,\dots,g_n\}} \left(\hat{h}_1 \otimes \cdots \otimes \hat{h}_n\right)^{-1} |v_1,\dots,v_n\rangle \otimes |\bar{v}_1,\dots,\bar{v}_n\rangle \\
=&\left(\hat{h}_1 \otimes \cdots \otimes \hat{h}_n\right) \left(\hat{g}_{n,A}\hat{h}_{n,A}^{-1}|v_n\rangle \otimes \cdots \otimes \hat{g}_{n-1,A}\hat{h}_{n-1,A}^{-1}|v_{n-1}\rangle \otimes \hat{h}_{1,\bar{A}}^{-1}|\bar{v}_1\rangle \otimes \cdots \otimes \hat{h}_{n,\bar{A}}^{-1}|\bar{v}_n\rangle\right) \\
=&\left(\hat{h}_{1,A}\hat{g}_{n,A}\hat{h}_{n,A}^{-1}|v_n\rangle \otimes \cdots \otimes \hat{h}_{n,A}\hat{g}_{n-1,A}\hat{h}_{n-1,A}^{-1}|v_{n-1}\rangle\right) \otimes |\bar{v}_1,\dots,\bar{v}_n\rangle \\
=&\mathcal{T}_A^{\{h_2 g_1 h_1^{-1},\dots,h_1 g_n h_n^{-1}\}}|v_1,\dots,v_n\rangle \otimes |\bar{v}_1,\dots,\bar{v}_n\rangle,
\end{aligned} \tag{5.21}$$

that is an elementary proof of (5.20).

We now discuss a number of remarkable consequences of the above equation. First, the relation $g'_j = h_{j+1} g_j h_j^{-1}$ implies that

$$g'_n\, g'_{n-1} \dots g'_1 = 1 \quad \text{iff} \quad g_n\, g_{n-1} \dots g_1 = 1, \tag{5.22}$$

regardless of the choice of $\{h_j\}$. Physically, this means that a twist operator $\mathcal{T}_A^{\{g_1,\dots,g_n\}}$ with a vanishing net Aharonov-Bohm flux accumulated across the replicas is unitarily equivalent via (5.20) to all and only the other operators $\mathcal{T}_A^{\{g'_1,\dots,g'_n\}}$ with the same property. To characterise this unitary transformation it is sufficient to show that for any $\{g'_i\}$ satisfying $g'_n \dots g'_1 = 1$ there is a n-tuple $\{g_j\}$ such that

$$g'_j = g_{j+1} g_j^{-1}. \tag{5.23}$$

A solution to (5.23) is indeed

$$g_1 = 1, \quad g_2 = g'_1, \quad g_3 = g'_2 g'_1, \quad \dots \quad g_n = g'_{n-1} \dots g'_1. \tag{5.24}$$

Therefore, with the previous choices of $\{g_j\}, \{g'_j\}$, it holds

$$\left(\hat{g}_1 \otimes \cdots \otimes \hat{g}_n\right) \mathcal{T}_A \left(\hat{g}_1 \otimes \cdots \otimes \hat{g}_n\right)^{-1} = \mathcal{T}_A^{\{g'_1,\dots,g'_n\}}, \tag{5.25}$$

which means that $\mathcal{T}_A$ is always unitarily equivalent to a given composite twist operator with a vanishing net flux.

Finally, we notice that distinct unitary transformations can give rise to the same twist operator, since equation (5.23) has multiple solutions. For example, it is easy to show that

$$\left(\hat{g}^{\otimes n}\right) \mathcal{T}_A \left(\hat{g}^{\otimes n}\right)^{-1} = \mathcal{T}_A, \tag{5.26}$$

for any choice of $g \in G$, which means that the standard twist operator $\mathcal{T}_A$ is neutral under a global unitary transformation. More generally, equation (5.23) has precisely $|G|$ solutions:

indeed, if $\{g_j\}$ is a solution of (5.23) for a given n-tuple $\{g'_j\}$, then also $\{g_j g\}$ satisfies the same relation for any $g \in G$.

3. **Action on local observables**

The last properties we show are the commutation relations between the twist operators and the local observables of the system. This is particularly useful to make an explicit connection between the construction proposed above and the defining properties of twist operators in QFT based on the algebra of local observables. Let $\mathcal{O} \in \mathrm{End}(\mathcal{H}_A)$ be an observable of the subsystem A. $\mathcal{O}$ can be naturally embedded in the space of observables of the whole system $A \cup \bar{A}$ by mapping it to $\mathcal{O} \otimes \mathbb{1}_{\bar{A}} \in \mathrm{End}(\mathcal{H})$. Similarly, we can embed $\mathcal{O}$ in the jth replica of the replica model by defining

$$\mathcal{O}^j := \underbrace{\mathbb{1}_A \otimes \dots \mathcal{O}}_{j} \otimes \dots \mathbb{1}_A \otimes \mathbb{1}_{\bar{A}}^{\otimes n} \in \mathrm{End}(\mathcal{H}^{\otimes n}). \tag{5.27}$$

As a consequence of the definitions, it is not difficult to show that

$$\mathcal{T}_A \mathcal{O}^j = \mathcal{O}^{j+1} \mathcal{T}_A, \tag{5.28}$$

and we say that the twist operator acts as a replica shift $j \to j+1$ on the observable $\mathcal{O}$ of A. Similarly, one has

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}} \mathcal{O}^j = \left(\hat{g}_j^{\otimes n}\right) \mathcal{O}^{j+1} \left(\hat{g}_j^{\otimes n}\right)^{-1} \mathcal{T}_A^{\{g_1,\dots,g_n\}}. \tag{5.29}$$

Physically, this means that the action of the composite twist operators on a local observable of A amounts to the usual replica shift followed by the action of the group element g_j on $\mathcal{O}$. Vice versa, local observables of $\bar{A}$ are not affected by the presence of the twist operators. For instance, given $\mathcal{O} \in \mathrm{End}\left(\mathcal{H}_{\bar{A}}\right)$, and $\mathcal{O}^j$ its embedding in the replica model, one easily shows

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}} \mathcal{O}^j = \mathcal{O}^j \mathcal{T}_A^{\{g_1,\dots,g_n\}}. \tag{5.30}$$

Therefore, as expected, the action of the twist operator on A commutes with the observables of the complement $\bar{A}$.

5.2.2 Entanglement asymmetry via twist operators

In this section, we make a connection between the twist operators introduced previously and the definition of the Rényi entanglement asymmetry in Eq. (5.8). To do so, we characterise the Rényi asymmetry of a state $\rho \in \mathrm{End}\left(\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}\right)$ for an integer $n \geq 2$. First, we recall that (5.17) allows us to express the Rényi entropy of ρ_A via the twist operator. Then, by making use of the definition of

$\tilde{\rho}_A$ in Eq. (5.5) and after a change of variable, we obtain

$$\begin{aligned}\mathrm{Tr}\,(\tilde{\rho}_A^n) &= \frac{1}{|G|^n}\sum_{g_1,\dots,g_n\in G}\mathrm{Tr}\left(\hat{g}_{1,A}\rho_A\hat{g}_{1,A}^{-1}\hat{g}_{2,A}\rho_A\hat{g}_{2,A}^{-1}\cdots\hat{g}_{n,A}\rho_A\hat{g}_{n,A}^{-1}\right)\\ &= \frac{1}{|G|^{n-1}}\sum_{g'_1,\dots,g'_{n-1}\in G}\mathrm{Tr}\left(\rho_A\hat{g}'_{1,A}\cdots\rho_A\hat{g}'_{n-1,A}\rho_A(\hat{g}'_{1,A}\cdots\hat{g}'_{n-1,A})^{-1}\right).\end{aligned} \tag{5.31}$$

Here, the second line follows from the change of variable $g'_j = g_j^{-1}g_{j+1}$. Notably, the g'_j are not independent since they satisfy the constraint $\prod_j g'_j = 1$. This allows us to express $\mathrm{Tr}(\tilde{\rho}_A^n)$ as a sum of $|G|^{n-1}$ terms rather than $|G|^n$ terms in the second line. Finally, we employ the property (5.18) and we express the moments of $\tilde{\rho}_A$ using the composite twist operators as

$$\mathrm{Tr}\,(\tilde{\rho}_A^n) = \frac{1}{|G|^{n-1}}\sum_{g_1,\dots,g_{n-1}\in G}\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1\dots g_{n-1})^{-1}\}}\right), \tag{5.32}$$

and, eventually, from (5.8) we get

$$\Delta S_n = \log|G| + \frac{1}{1-n}\log\left(\sum_{g_1,\dots,g_{n-1}\in G}\frac{\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1\dots g_{n-1})^{-1}\}}\right)}{\mathrm{Tr}\,(\rho^{\otimes n}\mathcal{T}_A)}\right). \tag{5.33}$$

We point out that all the possible $|G|^{n-1}$ composite twist operators with a vanishing net Aharonov–Bohm flux appear explicitly[2] in the sum in (5.33).

5.3 Form factor program for the twist fields in the Ising ferromagnetic phase

In this section, our focus is on the integrable QFT of the Ising model in 1+1 dimensions in its ferromagnetic phase. We aim to provide an explicit characterisation of the twist operators introduced in Section 5.2 by means of form factor bootstrap. In particular, we consider n replicas of the Ising model, and we investigate a family of branch-point twist fields (BPTF). This allows us to obtain the Rényi entropy of the symmetry-broken phase, which is different from the one of the disordered phase investigated in [20].

Before delving into the details of the Ising model, it is worth making a connection between the twist operators of Section 5.2 and the BPTF widely used in 1+1D quantum field theories (see e.g. [18] for CFTs, and [20] for massive integrable QFTs). For any half-line $A = (x, \infty)$, we regard the

[2]Namely, the operators appearing in the sum are all and only the ones with zero net flux across all replicas. This condition is required for the consistency of our construction. Indeed, an operator with a net flux different from zero would signal that the state has a definite charge value under the symmetry G, but this would in turn imply vanishing entanglement asymmetry.

corresponding twist operator as a (semi-local) field inserted at x and we denote it by $\mathcal{T}(x)$, namely

$$\mathcal{T}_A \sim \mathcal{T}(x), \quad A = (x, \infty), \tag{5.34}$$

up to a non-universal proportionality constant that has been neglected. These fields are the building blocks needed to reconstruct the entanglement properties of any region, and through their insertion at different points, one can express the twist operator of any union of disjoint intervals [18]. A paradigmatic example is the case of a single finite interval, where it is possible to express

$$\mathcal{T}_A \sim \mathcal{T}(0)\mathcal{T}^\dagger(\ell), \quad A = (0, \ell), \tag{5.35}$$

with $\mathcal{T}, \mathcal{T}^\dagger$ a pair of Hermitian conjugated twist fields. This correspondence was generalised to $U(1)$ composite twist fields in [2, 3].

Having made the identification (5.34) or (5.35), it is worth noting that the expression for ΔS_n in (5.33) is invariant after rescaling of the space coordinate. Indeed, under a transformation $x \to \lambda x$ the twist operators transform with the same scaling dimension d:

$$\mathcal{T}_A \to \lambda^{-d}\mathcal{T}_A, \quad \mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}\}} \to \lambda^{-d}\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}\}}. \tag{5.36}$$

This observation is fundamental for quantum field theories in which spontaneous symmetry breaking occurs, and it can be argued from the fact that the two fields are related by a global unitary transformation. That is, equation (5.25) implies that the charged and uncharged twist operators must have the same (UV) scaling dimension. Indeed, in the limit of a small region their correlation functions are expected to be the same as those computed at the UV critical point. But because the critical point is invariant under the symmetry group G, these expectation values are equal, and so must be the scaling dimensions of the two fields. Furthermore, because of equation (5.25), the normalisation of $\mathcal{T}_A$ unambiguously determines the normalisation of $\mathcal{T}_A^{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}}$. Consequently, the ratio appearing in (5.33), and therefore the entanglement asymmetry, is universal. It is important to emphasise that this is not the case for entanglement entropy (see for instance [10, 216]) due to the presence of UV divergences.

Despite the extensive literature existing on twist fields in 1+1D integrable QFT, the vast majority of those works [18, 21, 200, 217, 218] focus on the entanglement properties in the the paramagnetic (disordered) phase only, where a single vacuum is present. However, in the presence of spontaneous symmetry breaking, multiple vacua are present, and one must be cautious when specifying them. Moreover, as there might be elementary excitations (kinks) interpolating between different vacua, analytic continuations in rapidity space may result in a mixing of the corresponding form factors, a

mechanism that is well understood for the q-state Potts field theory [219], which describes the Ising model for $q = 2$. To the best of our knowledge, the consequence of vacuum degeneracy in the context of entanglement for field theories has not been explored yet. Our objective is to address this gap by examining the simplest, yet non-trivial case: the Ising model. In particular, in this section we aim to formulate and solve the form factor bootstrap equations for the standard twist fields in the ferromagnetic case. These results will be eventually employed to compute the Rényi entropy of a symmetry-broken ground state.

We first review some basic properties of the massive Ising field theory [60–62, 81, 212, 213], and then we discuss its n-replica model, where the twist fields arise. The theory can be regarded as a relevant perturbation of the Ising CFT [166, 198] with Euclidean action[3]

$$\mathcal{S} = \mathcal{S}_{\text{CFT}} + \lambda \int \mathrm{d}x\mathrm{d}\tau \; \varepsilon(x,\tau). \tag{5.37}$$

Here $\mathcal{S}_{\text{CFT}}$ is the action at criticality, associated to a CFT with central charge $c = 1/2$, λ is a parameter with the dimension of a mass and $\varepsilon(x,\tau)$ is a scalar field of dimension 1 representing an energy density. Depending on the sign of λ, the theory is in its paramagnetic (disordered, $\lambda < 0$) or ferromagnetic (ordered, $\lambda > 0$) phase. We focus on the ferromagnetic phase where spontaneous symmetry breaking arises, and two degenerate vacua $|+\rangle, |-\rangle$ related by $\mathbb{Z}_2$ symmetry are present. The elementary excitations above the ground state(s) are multi-kink configurations interpolating between the two vacua [219]. In particular, a single-particle (kink) state with rapidity θ, denoted by

$$|K_{+-}(\theta)\rangle, \tag{5.38}$$

interpolates between $|+\rangle$ at $-\infty$ and $|-\rangle$ at ∞. To make contact with the quantum Ising chain, one should regard a one-kink state with definite momentum as the $L \to \infty$ limit of the plane-wave superposition:

$$|K_{+-}(p)\rangle = \mathcal{N} \sum_{j=-L+1}^{L} e^{ipj} |\tilde{K}_{+-}(j)\rangle, \tag{5.39}$$

where p is the quantised momentum, $\mathcal{N}$ is a normalisation factor and $|\tilde{K}_{+-}(j)\rangle$ is the domain wall created by the lattice disorder operator $\mu_j = \prod_{j\le i} \sigma_i^z$ on the state $\otimes_{i=-L}^{L} |+\rangle_i$:

$$|\tilde{K}_{+-}(j)\rangle = \mu_j \otimes_{i=-L}^{L} |+\rangle_i = |+\rangle_{-L} \cdots \otimes |+\rangle_{j-1} \otimes |-\rangle_j \cdots \otimes |-\rangle_L. \tag{5.40}$$

In the continuum limit, the momentum p is parametrised by the rapidity as $p = m \sinh\theta$. In general,

[3] After changing coordinates from Euclidean to holomorphic and expressing the energy density in terms of the fermion field, equation (5.37) reduces to the Majorana action (4.1).

a configuration with N kinks has the form

$$|K_{\alpha_1\alpha_2}(\theta_1)K_{\alpha_2\alpha_3}(\theta_2)\dots K_{\alpha_N\alpha_{N+1}}(\theta_N)\rangle, \tag{5.41}$$

where the indices $\alpha_j \in \{+,-\}$ label the vacua and $\theta_j \in \mathbb{R}$, $j = 1,\dots,N$ are the rapidities of the excitations. We say that the configuration is neutral, or topologically trivial, if $\alpha_1 = \alpha_{N+1}$. It is convenient to identify four topological sectors, labelled by the values of (α_1, α_{N+1}): physically, these sectors correspond to different choices of the boundary conditions (i.e. spin up or spin down) at $x = -\infty$ and $x = +\infty$, and they can be regarded as constraints for the states interpolating between them.

The multi-kink configurations are the (asymptotic) excited states of the theory, and their energy-momentum is

$$(E,P) = \left(m\sum_{j=1}^{N}\cosh\theta_j, m\sum_{j=1}^{N}\sinh\theta_j \right). \tag{5.42}$$

Here $m = |\lambda|$ is the mass of each kink, and it plays the role of inverse correlation length away from criticality. When two kinks scatter, they acquire a scattering phase $e^{i\delta} = -1$ which does not depend on the rapidity difference, a property ultimately related to the equivalence (see for instance [198]) between the Ising model and free fermionic Majorana field theory[4].

For any field $\mathcal{O}(x)$ one can define the form factors, i.e. the matrix elements of $\mathcal{O}$ between the vacua and the multi-kink states, as

$$\langle \pm|\mathcal{O}(0)|K_{\alpha_1\alpha_2}(\theta_1)K_{\alpha_2\alpha_3}(\theta_2)\dots K_{\alpha_N\alpha_{N+1}}(\theta_N)\rangle. \tag{5.43}$$

In principle, depending on the field $\mathcal{O}$, there may be several non-vanishing form factors; however, their number is drastically reduced by topological constraints, as discussed in [219] for the q-state Potts model. In the Ising field theory, for the order field σ only the form factors with neutral states may be non-vanishing, and they are

$$\langle \pm|\sigma(0)|K_{\pm\alpha_2}(\theta_1)K_{\alpha_2\alpha_3}(\theta_2),\dots K_{\alpha_N\pm}(\theta_N)\rangle, \tag{5.44}$$

together with the vacuum expectation values (VEVs) $\langle\pm|\sigma(0)|\pm\rangle$. In contrast, the disorder field $\mu(0)$ is topologically non-trivial, as it introduces a spin-flip along the half-line $[0,+\infty)$ that mixes the two

[4]There are some technical differences between the Ising model and the field theory of free Majorana fermions, which can be traced back to the presence of distinct spin sectors. However, since this feature does not play any role in this work, we do not discuss it further.

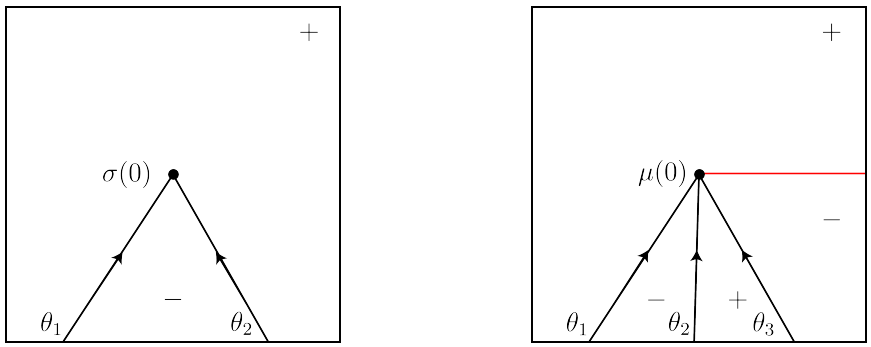


Figure 5.1 (Left) Order field $\sigma(x)$: the two-particle form factor $\langle +|\sigma(0)|K_{+-}(\theta_1)K_{-+}(\theta_2)\rangle$ is shown. (Right) Disorder field $\mu(x)$: the three-particle form factor $\langle +|\mu(0)|K_{+-}(\theta_1)K_{-+}(\theta_2)K_{+-}(\theta_3)\rangle$ is represented. The red line corresponds to a spin-flip exchanging the two vacua. Space is in the horizontal direction and time in the vertical direction.

vacua, and the form factors which are not ruled out by topological constraints are

$$\langle \pm|\mu(0)|K_{\pm\alpha_2}(\theta_1)K_{\alpha_2\alpha_3}(\theta_2)\dots K_{\alpha_N\mp}(\theta_N)\rangle. \tag{5.45}$$

In Figure 5.1 we represent pictorially the form factors of the order and disorder field. While $\sigma(x)$ is a local field, $\mu(x)$ is semi-local, and we depict this semi-locality with a red line in Figure 5.1. The locality properties of the two fields fix the boundary conditions at infinity for incoming states interpolating with the outgoing vacuum. In particular, only an even/odd number of kinks gives rise to non-vanishing form factors for $\sigma(x), \mu(x)$ respectively.

In the next sections, we first extend the analysis above to the twist fields of the replica model, identifying their topological constraints. Then, via the bootstrap program, we characterise analytically their non-vanishing form factors, and we give an analytical expression for the two-particle ones.

5.3.1 Standard twist fields

The replica theory of the Ising model in the ordered phase is obtained by taking n decoupled copies of the model and considering the resulting theory itself as a QFT. The (asymptotic) spectrum of the replica QFT is expressed in terms of single-replica vacua and kink states, as no correlations are present between distinct copies. In particular, in the ordered phase, there are 2^n degenerate vacua which are obtained as tensor products of single-replica vacua:

$$|+\rangle^{\otimes n},\ |+\rangle^{\otimes n-1}\otimes|-\rangle,\ \dots,\ |-\rangle^{\otimes n}. \tag{5.46}$$

The excitations can be described via the insertion of kinks in each replica, similarly to what happens with quasi-particle excitations in the paramagnetic phase (see [20]). Moreover, the scattering matrix

between two kinks at replicas i and j with rapidities θ_i, θ_j respectively is

$$S_{ij}(\theta_i - \theta_j) = \begin{cases} -1 & i = j, \\ 1 & i \neq j, \end{cases} \tag{5.47}$$

since no interactions between distinct replicas are present. This further implies that two kinks on the same replica satisfy the Faddeev-Zamolodchikov algebra:

$$K_{\pm\mp}(\theta_1)K_{\mp\pm}(\theta_2) = -K_{\pm\mp}(\theta_2)K_{\mp\pm}(\theta_1), \tag{5.48}$$

which is consistent with the algebra of the scattering kinks in the q-state Potts model [219, 220] for $q = 2$.

The standard BPTF is as usual introduced as a semi-local field of the replica model implementing the replica cyclic permutation $j \to j + 1$. In particular, from the commutation relations (5.28), (5.30) and the identification (5.35) we obtain

$$\mathcal{T}(x)\mathcal{O}_j(y) = \begin{cases} \mathcal{O}_{j+1}(y)\mathcal{T}(x) & x < y, \\ \mathcal{O}_j(y)\mathcal{T}(x) & \text{otherwise}, \end{cases} \tag{5.49}$$

for any local field[5] $\mathcal{O}_j(x)$ inserted in the jth replica. In particular, this reproduces the commutation relation (4.3) between the BPTF and the fermion field in the paramagnetic phase. According to this commutation relation (see Refs [18, 20]), we say that $\mathcal{T}(x)$ inserts a branch cut on the semi-infinite line (x, ∞), which connects the jth replica with the $(j + 1)$th. It is important to stress that Eq. (5.49), rather than unambiguously identifying a single field $\mathcal{T}(x)$, selects a space of fields with given monodromy properties (details can be found in [64]). However, we are interested in the lightest fields satisfying (5.49), which correspond to a deformation of the primary twist fields of CFTs [18]. As seen in the previous chapters, these are scalar fields with conformal dimension $\Delta_{\mathcal{T}}$ given by (1.27).

In the remaining part of the section, we study the form factors of branch-point twist fields in the ferromagnetic phase of the Ising model. These are matrix elements of $\mathcal{T}$, $\mathcal{T}^\dagger$ between the multi-kink state and one of the 2^n vacua. For our purposes, the most interesting form factors are the ones with the vacuum $\langle + |^{\otimes n}$. However, as we will see, other form factors are generated from the latter using the bootstrap equations.

[5] One should be careful about the definition of local fields, and more in general local observables. Here, we just mean that $\mathcal{O}$ belongs to the algebra generated by the fields ε (the energy density) and the order operator σ. This requirement is very natural in the lattice counterpart, as it corresponds to the usual notion of locality in the computational basis of the quantum spin chain.

5.3.1.1 Zero-particle form factors

The simplest form factor that is expected to be non-vanishing is the VEV

$$ {}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}. \tag{5.50}$$

We represent it pictorially in Figure 5.2: the n replicas are connected together along the branch cuts inserted by the twist fields $\mathcal{T}$ at the origin of space-time, the state $|+\rangle^{\otimes n}$ (${}^{\otimes n}\langle +|$) is the incoming (outgoing) state represented below (above) the origin in each replica. In particular, since the standard twist fields exchange replicas but do not act as spin-flips, if $\pm$ is present in the jth replica above the branch cut, then $\pm$ has to be present also in the $(j+1)$th replica just below the branch cut. In other words, fixing the boundary condition of the outgoing state at spacial infinity unambiguously fixes the boundary conditions of the incoming state. Specifically, the boundary conditions at $x=-\infty$ remain the same for both outgoing and incoming states, while the boundary conditions at $x=+\infty$ are connected through a replica shift $j\to j+1$. For a similar discussion concerning the (single-replica) q-state Potts model, we refer the reader to [219].

According to these considerations, it is not difficult to show that the only other non-vanishing zero-particle form factor is

$$ {}^{\otimes n}\langle -|\mathcal{T}(0)|-\rangle^{\otimes n}. \tag{5.51}$$

Its value can be related to the one of ${}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}$ via a global $\mathbb{Z}_2$ symmetry. Indeed, let us consider the global unitary transformation μ [6] which generates the $\mathbb{Z}_2$ symmetry exchanging the two vacua. In the replica theory, it holds

$$ \mu^{\otimes n}|\pm\rangle^{\otimes n}=|\mp\rangle^{\otimes n}. \tag{5.52}$$

Since the twist field is neutral with respect to the action of the global spin-flip $\mu^{\otimes n}$ (see Eq. (5.26)), this immediately implies that the two zero-particle form factors above are equal: this value is referred to as

$$ \tau := {}^{\otimes n}\langle \pm|\mathcal{T}(0)|\pm\rangle^{\otimes n}. \tag{5.53}$$

5.3.1.2 Vanishing one-particle form factors and further constraints

Here, we show that the one-particle form factors are all vanishing. To do that, it is sufficient to observe that these form factors are not compatible with the topological constraints arising from the boundary conditions at spacial infinity.

[6] Notice that the operator μ acts as a spin-flip everywhere while the field $\mu(x)$ acts on the half-line $(x,+\infty)$. Roughly, the relation between these operators is (up to a proportionality constant) $\mu\propto\mu(-\infty)\mu^\dagger(+\infty)$.

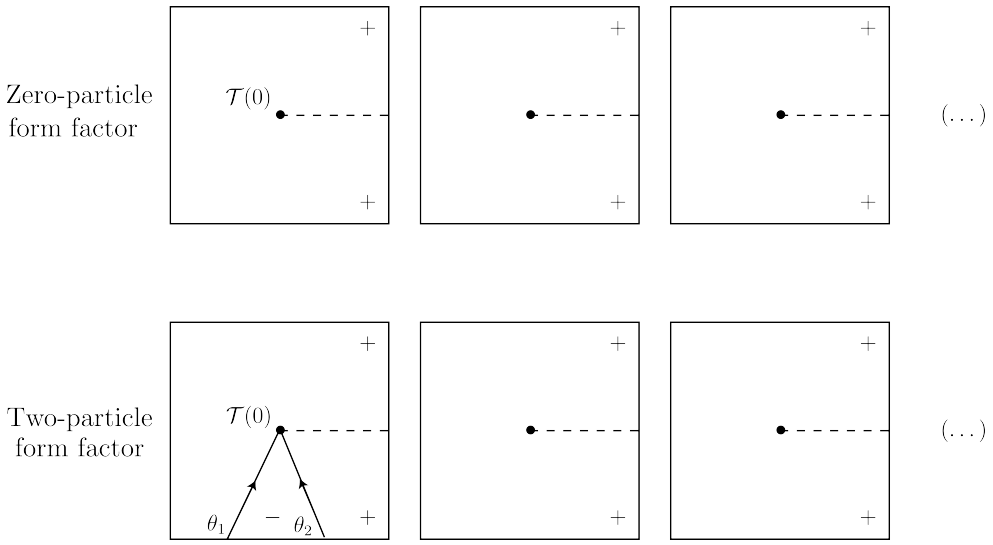


Figure 5.2 The form factors of the twist field $\mathcal{T}(0)$. Consecutive replicas are connected to each other via the branch cut, represented by a dashed line starting at $x = 0$. Top: vacuum expectation value ${}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}$. Bottom: Two-particle form factors ${}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1)K_{-+}(\theta_2), +, \dots, +\rangle$ with two kinks in the same replica.

Let us consider a generic outgoing vacuum state $\langle \alpha_1, \dots, \alpha_n|$, with $\alpha_j = \pm$. If $\mathcal{T}(0)$ interpolates between a multi-particle incoming state and the vacuum above, we can immediately obtain the boundary conditions for the incoming state. Indeed, it is easy to show that the boundary conditions for the latter in the jth replica at $x = -\infty$, $x = \infty$ have to be α_j, α_{j-1} respectively. This property rules out the presence of matrix elements with an odd number of kinks having $\langle \alpha_1, \dots, \alpha_n|$ as outgoing state, and, in particular, the one-particle form factor.

Further constraints are also implied, and, for the sake of concreteness, we only discuss them for the outgoing vacuum $\langle +, \dots, +|$. Since $+$ should be present at $x = \pm\infty$ of the incoming state in the jth replica, only an even number of kinks has to be present for any replica. For example, two-particle states with two kinks inserted at different replicas have vanishing form factors. It is worth noting that this property is specific to the ferromagnetic phase of the Ising model, while the same mechanism does not arise in the paramagnetic one.

5.3.2 Two-particle form factors

The two-particle (or two-kink) form factors of $\mathcal{T}(x)$ are responsible for the leading corrections to the saturation value of the Rényi entropy. They are generally non-vanishing, as they are not ruled out by symmetry or topological constraints, and can be exactly determined via the bootstrap equations. We perform an analysis similar to the one in [20], where the paramagnetic phase was considered: many analogies arise due to the same scattering properties in the two phases, but there are also important conceptual differences stemming from the vacuum degeneracy. The two-kink form factors can be determined by their monodromy conditions and the residues at their kinematic poles. We shall first

derive the monodromy equations, and then move to the equations for the kinematic poles.

Let us focus on the following two-kink form factor

$$F_{11}^{(n)}(\theta_1-\theta_2) := {}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1)K_{-+}(\theta_2),+,\dots,+\rangle, \tag{5.54}$$

which depends on the difference $\theta_1-\theta_2$ only, due to relativistic invariance. In general, this is expected to be non-vanishing, since, as argued in Section 5.3.1.2, two kinks are presented in the same replica (the first). We give a pictorial representation of $F_{11}^{(n)}(\theta_1-\theta_2)$ in Fig. 5.2. As a consequence of replica symmetry there is no explicit dependence on the replica j where the pair of kinks is inserted, and we have

$$F_{jj}^{(n)}(\theta_1-\theta_2) := {}^{\otimes n}\langle +|\mathcal{T}(0)|\underbrace{+,\dots,+,K_{+-}(\theta_1)K_{-+}(\theta_2)}_{j},+,\dots,+\rangle = F_{11}^{(n)}(\theta_1-\theta_2), \tag{5.55}$$

with $j=1,\dots,n$.

Let us discuss the monodromy equations of $F_{11}^{(n)}(\theta)$. The rapidity shift $\theta_1\to\theta_1+2\pi i$ in Eq. (5.54) has the effect of moving the corresponding kink from the first to the second replica. In addition, the topological sectors are mixed with each other, and we get

$$\begin{aligned}&{}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1+2\pi i)K_{-+}(\theta_2),+,\dots,+\rangle\\ =&\langle -,+,\dots,+|\mathcal{T}(0)|K_{-+}(\theta_2),K_{+-}(\theta_1),+,\dots,+\rangle.\end{aligned} \tag{5.56}$$

This means that the outgoing vacuum is modified under the above rapidity shift. This is not particularly surprising, since we already argued in Section 5.3.1.2 that the two-particle form factors of $\langle +,\dots,+|$ with kinks in distinct replicas vanish. Similarly, for $j=1,\dots,n-1$, we have

$$\begin{aligned}&{}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1+2\pi ij)K_{-+}(\theta_2),+,\dots,+\rangle\\ =&\langle\underbrace{-,\dots,-}_{j},+,\dots,+|\mathcal{T}(0)|\underbrace{K_{-+}(\theta_2),-,\dots,-,K_{+-}(\theta_1)}_{j+1},+,\dots,+\rangle,\end{aligned} \tag{5.57}$$

and the kink moves from the first to the $(j+1)$th replica under $\theta_1\to\theta_1+2\pi ij$. Under the shift $\theta_1\to\theta_1+2\pi in$, the kink goes back to the first replica, but all the signs are exchanged, namely

$$\begin{aligned}&{}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1+2\pi in)K_{-+}(\theta_2),+,\dots,+\rangle\\ =&\,{}^{\otimes n}\langle -|\mathcal{T}(0)|K_{-+}(\theta_2)K_{+-}(\theta_1),-,\dots,-\rangle.\end{aligned} \tag{5.58}$$

This expression can be further manipulated by using the $\mathbb{Z}_2$ neutrality of the twist fields together with

the (fermionic) scattering properties (5.48), yielding:

$$\begin{aligned} &{}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1+2\pi i n)K_{-+}(\theta_2),+,\dots,+\rangle \\ &= -\,{}^{\otimes n}\langle +|\mathcal{T}(0)|K_{+-}(\theta_1)K_{-+}(\theta_2),+,\dots,+\rangle, \end{aligned} \tag{5.59}$$

that is $F_{11}^{(n)}(\theta+2in\pi) = -F_{11}^{(n)}(\theta)$. The monodromy equations thus lead to an anti-periodicity condition on the fundamental two-kink form factor.

Following closely the arguments in [21], we argue that the form factor $F_{11}^{(n)}(\theta)$ has two kinematic poles at $\theta = i\pi$ and $\theta = 2\pi i n - i\pi$. Their residues are

$$\mathrm{Res}_{\theta=i\pi}F_{11}^{(n)}(\theta) = i\tau, \quad \mathrm{Res}_{\theta=(2n-1)i\pi}F_{11}^{(n)}(\theta) = -i\tau, \tag{5.60}$$

with τ the vacuum expectation value (5.53). Combining all these results, we conclude that the monodromy and the bootstrap equations of $F_{11}^{(n)}(\theta)$ are the same encountered in the paramagnetic phase [20]. Therefore, a solution[7] to these equations can be easily provided:

$$F_{11}^{(n)}(\theta) = \frac{i\tau\,\cos\left(\frac{\pi}{2n}\right)\sinh\left(\frac{\theta}{2n}\right)}{n\,\sinh\left(\frac{\theta-i\pi}{2n}\right)\sinh\left(\frac{\theta+i\pi}{2n}\right)}. \tag{5.61}$$

5.3.3 Rényi entropy and its analytic continuation

With the previous results, we can finally present a form-factor expansion for the two-point function of the twist fields, valid in the limit $m\ell \gg 1$. This expansion is then used to obtain the Rényi entropies. Eventually, we analytically continue the replica index $n \to 1$ to get the entanglement entropy.

Starting from the replica state $|+\rangle^{\otimes n}$, the quantity we need to compute is

$$\mathrm{Tr}\left(\rho^{\otimes n}\mathcal{T}_A\right) = {}^{\otimes n}\langle +|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|+\rangle^{\otimes n}. \tag{5.62}$$

We insert a resolution of the identity between the branch-point twist fields and, at the two-particle approximation, we get

$${}^{\otimes n}\langle +|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|+\rangle^{\otimes n} \simeq |\tau|^2 + \sum_{j=1}^{n}\int_{\theta_1>\theta_2}\frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{(2\pi)^2}|F_{jj}^{(n)}(\theta_1-\theta_2)|^2 e^{-m\ell(\cosh\theta_1+\cosh\theta_2)}. \tag{5.63}$$

In deriving the expression above, we used the fact that the only vacuum that can be connected to ${}^{\otimes n}\langle +|$ by $\mathcal{T}(0)$ is $|+\rangle^{\otimes n}$. Moreover, only pairs of kinks inserted in the same replica j give a non-vanishing

[7] It is well known that the bootstrap equations have multiple solutions, but the Δ-theorem [202] provides a severe additional constraint. Here, as in [20], the solution with the mildest growth for $|\theta| \to \infty$ is given. This one is expected to describe the twist field with the lowest scaling dimension, corresponding to a primary field of the associated CFT.

contribution in the expansion above.

By performing the usual change of variable $\theta = \theta_1 - \theta_2$, $\Theta = \frac{\theta_1+\theta_2}{2}$ and after an integration over Θ and a sum over the copy index j, which trivially follows from $|F^{(n)}_{jj}(\theta)|^2 = |F^{(n)}_{11}(\theta)|^2$, the above equation reduces to

$$ {}^{\otimes n}\langle +|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|+\rangle^{\otimes n} \simeq |\tau|^2 + \frac{n}{4\pi^2}\int_{-\infty}^{+\infty} \mathrm{d}\theta\, K_0\left(2m\ell\cosh\left(\frac{\theta}{2}\right)\right)|F^{(n)}_{11}(\theta)|^2. \tag{5.64}$$

In the large volume limit $m\ell \to \infty$ the correlation function factorises and the square of the vacuum expectation value $|\tau|^2$ is recovered: the first non-trivial correction, which decays exponentially to zero as $\sim e^{-2m\ell}$, comes from the second term of equation (5.64) and is due to the pairs of kinks. This result should be compared to the (two-particle approximation of) the two-point function obtained in the paramagnetic phase [20]. The crucial difference is that in the paramagnetic phase form factors with single-particle excitations inserted in distinct replicas are allowed, which amounts to replacing

$$ |F^{(n)}_{11}(\theta)|^2 \to \sum_{j=0}^{n-1} |F^{(n)}_{11}(-\theta + 2\pi i j)|^2 \tag{5.65}$$

in equation (5.64). As we show below, this difference has important consequences in the replica limit $n \to 1$.

Now, we can finally express the Rényi entropy of an interval in the state $|+\rangle$ as

$$\begin{aligned} S_n &= \frac{1}{1-n}\log {}^{\otimes n}\langle +|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|+\rangle^{\otimes n} + \mathcal{C}_n \\ &\simeq \frac{1}{1-n}\log\left(1 + \frac{n}{4\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\theta\, K_0\left(2m\ell\cosh\left(\frac{\theta}{2}\right)\right)\frac{|F^{(n)}_{11}(\theta)|^2}{|\tau|^2}\right) + \mathcal{C}_n, \end{aligned} \tag{5.66}$$

where the constant $\mathcal{C}_n$, which contains the non-universal twist field normalisation and the short-distance cut-off, will not play any role in the following. In the large $m\ell$ limit, the support of the integral in equation (5.66) is localised near $\theta \simeq 0$, and a saddle-point analysis reveals that the two-particle contribution decreases exponentially, scaling as $\sim e^{-2m\ell}$. While these considerations hold for integer values of $n \geq 2$, the analytical continuation $n \to 1$ is more subtle, as we explain below.

Indeed, we observe that the ratio $\frac{|F^{(n)}_{11}(\theta)|^2}{|\tau|^2}$ does not explicitly depend on the field normalisation, and it is fixed by bootstrap. Moreover, since for $n = 1$ the twist field becomes the identity operator [20], its two-particle form factors vanish. This suggests that under analytic continuation over n, $F^{(n)}_{11}(\theta)$ converges to zero uniformly in the distributional sense (for θ real) as $n \to 1$ (see [20]), a property that can be explicitly checked from the expression (5.61). Therefore, $\frac{|F^{(n)}_{11}(\theta)|^2}{|\tau|^2}$ behaves as $\mathcal{O}((n-1)^2)$ for $(n-1)$ small and, accordingly, the two-particle contribution to the von Neumann entropy, obtained as

the limit $n \to 1$ in equation (5.66), vanishes identically. This circumstance is quite unexpected and, to the best of our knowledge, it has not been previously observed. Indeed, a two-particle contribution is always present in the paramagnetic phase of a large class of theories with a single vacuum, as shown in [21].

Finally, we stress that our analysis does not imply that the entanglement entropy is exactly independent of ℓ. In fact, it is reasonable to expect a non-vanishing contribution from the four-kink form factors, thus yielding

$$S_1 \simeq \text{const.} + O(e^{-4m\ell}). \tag{5.67}$$

However, the investigation of higher-kink form factors is beyond the purpose of this work.

5.4 Form factors for the composite twist fields and entanglement asymmetry in the ferromagnetic phase of Ising QFT

In this section, we analyse the $\mathbb{Z}_2$ composite twist fields that arise in the computation of the entanglement asymmetry. These are the building blocks used to reconstruct the composite twist operators introduced in Section 5.2, and, as for the standard twist fields, the correspondence goes as

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}} \sim \mathcal{T}^{\{g_1,\dots,g_n\}}(x), \quad A = (x, \infty), \tag{5.68}$$

$$\mathcal{T}_A^{\{g_1,\dots,g_n\}} \sim \mathcal{T}^{\{g_1,\dots,g_n\}}(0) \left(\mathcal{T}^{\{g_1,\dots,g_n\}}\right)^\dagger (\ell), \quad A = (0, \ell), \tag{5.69}$$

and similarly for any union of disjoint intervals. Physically, $\mathcal{T}^{\{g_1,\dots,g_n\}}(x)$ introduces a branch cut along (x, ∞) which connects the jth and $(j+1)$th replicas via the additional insertion of an Aharonov–Bohm flux g_j. In particular, to compute the entanglement asymmetry, we only need those composite twist fields with vanishing net flux (satisfying $g_n \cdots g_1 = 1$). A fundamental observation pointed out in Section 5.2 is that such composite twist fields can be related to the standard one via global unitary transformations induced by the symmetry. In particular, given the form factors of $\mathcal{T}(x)$, the ones of $\mathcal{T}^{\{g_1,\dots,g_n\}}(x)$ can be easily reconstructed once the action of the symmetry on the multi-particle states is known. Before entering the core of this section, we recall that $\mathbb{Z}_2$ composite twist fields with non-vanishing net flux have been characterised in the paramagnetic phase of the Ising model [4, 107] and used to compute the symmetry-resolved entanglement entropy: those fields are not related by global unitary transformations to $\mathcal{T}(x)$ and the discussion of this section does not apply to them.

Our analysis refers to the ferromagnetic phase of the Ising field theory, where the group is $G = \mathbb{Z}_2$ which corresponds to a spin-flip exchanging the two vacua. We parametrise the elements of $\mathbb{Z}_2$ as

$$\mathbb{Z}_2 = \{1, \mu\}, \tag{5.70}$$

with $\mu^2 = 1$, and in the forthcoming discussion, with a slight abuse of notation, we do not distinguish the elements of $\mathbb{Z}_2$ from the corresponding unitary operators (i.e. we drop the "hat" notation that we adopted in Section 5.2). It is easy to show that the n-tuples $\{g_j\}$ with vanishing net flux are the ones for which an even number of elements μ is present. Since $|G| = 2$, there are 2^{n-1} of those n-tuples (see Eq. (5.32)), and, for the sake of completeness, we list them explicitly in a few cases

- $n = 2$: $\{g_1, g_2\}$ can take the values $\{1, 1\}$ and $\{\mu, \mu\}$.
- $n = 3$: $\{g_1, g_2, g_3\}$ can take the values $\{1, 1, 1\}$, $\{1, \mu, \mu\}$ and its two cyclic permutations.
- $n = 4$: $\{g_1, g_2, g_3, g_4\}$ can take the values $\{1, 1, 1, 1\}$, $\{\mu, \mu, \mu, \mu\}$, $\{1, \mu, 1, \mu\}$ and its cyclic permutation, $\{1, 1, \mu, \mu\}$ and its three cyclic permutations.

We first discuss the zero-particle form factors of the composite twist fields. We anticipate that $\mathcal{T}^{\{g_1,\dots,g_n\}}(0)$ does not interpolate between ${}^{\otimes n}\langle +|$ and any other vacuum (except in the trivial case $\{g_j = 1\}$), a property which is crucial for the large-distance behavior of the entanglement asymmetry. Then, we move to the analysis of the two-particle form factors, providing exact analytical results. Finally, we collect these results together and give an expression for the Rényi entanglement asymmetry of a large interval.

5.4.1 Zero-particle form factors

We first recall the relation (5.25) and apply it to the composite twist fields, obtaining

$$\left(g'_1 \otimes \cdots \otimes g'_n\right) \mathcal{T}(0) \left(g'_1 \otimes \cdots \otimes g'_n\right)^{-1} = \mathcal{T}^{\{g_1,\dots,g_n\}}(0), \quad g_j = g'_{j+1}(g'_j)^{-1}, \tag{5.71}$$

which is valid for $g_n \cdots g_1 = 1$. As previously established, the only non-vanishing vacuum expectation values of $\mathcal{T}(0)$ (see Eq. (5.53)) are ${}^{\otimes n}\langle \pm|\mathcal{T}(0)|\pm\rangle^{\otimes n}$. Similarly, thanks to (5.71), the composite twist fields have only two non-vanishing zero-particle form factors: these are overlaps between the same vacuum, which however is in general different from $|\pm\rangle^{\otimes n}$, as we show below.

If $g'_j = 1$ or $g'_j = \mu$ for every $j = 1, \dots, n$ in Eq. (5.71), we have $\mathcal{T}^{\{g_1,\dots,g_n\}}(0) = \mathcal{T}(0)$. In all the other cases, namely if at least two elements of the n-tuple $\{g'_j\}$ are different, we get a non-trivial composite twist field $\mathcal{T}^{\{g_1,\dots,g_n\}}(x)$, i.e. at least one of the $g_j = \mu$. For these non-trivial cases, it holds

$${}^{\otimes}\langle \pm|\mathcal{T}^{\{g_1,\dots,g_n\}}(0)|\pm\rangle^{\otimes n} = {}^{\otimes n}\langle \pm| \left(g'_1 \otimes \cdots \otimes g'_n\right) \mathcal{T}(0) \left(g'_1 \otimes \cdots \otimes g'_n\right)^{-1} |\pm\rangle^{\otimes n} = 0, \tag{5.72}$$

since $\left(g'_1 \otimes \cdots \otimes g'_n\right)^{-1} |\pm\rangle^{\otimes n}$ is always different from $|\pm\rangle^{\otimes n}$. This is one of the main results we will need later.

Other vacua have nevertheless non-vanishing amplitudes. Even if the latter do not appear in the

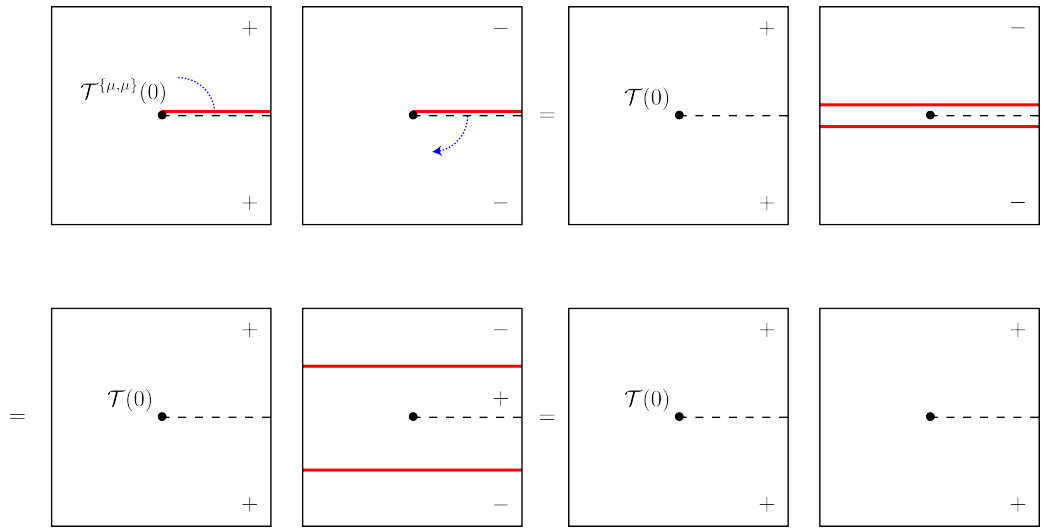


Figure 5.3 Pictorial representation of the form factor $\langle +,-|\mathcal{T}^{\{\mu,\mu\}}(0)|+,-\rangle$. The blue dotted arrow in the top left figure shows that the second replica is reached from the first one when the branch cut, depicted by a dashed black line, is crossed clockwise. The red line represents the spin-flip, which exchanges $+$ and $-$. $\mathcal{T}^{\{\mu,\mu\}}(0)$ can be equivalently obtained via global spin-flips acting on $\mathcal{T}(0)$, as shown in the top right figure. Since the global spin-flip lines commute with the Hamiltonian, they can be translated and positioned at infinity, so that they act on the ingoing/outgoing states. This equivalence allows us to establish the relation $\langle +,-|\mathcal{T}^{\{\mu,\mu\}}(0)|+,-\rangle = \langle +,+|\mathcal{T}(0)|+,+\rangle$.

computation of the entanglement asymmetry (because in equation (5.33) we take $\rho = |+\rangle\langle +|$), it is useful to analyse a simple case explicitly. We consider for $n = 2$ the twist field $\mathcal{T}^{\{\mu,\mu\}}(0)$. This is related to the standard twist field via the transformation (5.71), with $\{g_1' = 1, g_2' = \mu\}$. Therefore, from (5.71) we get

$$\langle \pm,\mp|\mathcal{T}^{\{\mu,\mu\}}(0)|\pm,\mp\rangle = \langle \pm,\pm|\mathcal{T}(0)|\pm,\pm\rangle, \tag{5.73}$$

and these are the only non-vanishing vacuum expectation values of $\mathcal{T}^{\{\mu,\mu\}}(x)$.

We can interpret this result as follows. If the boundary condition $+$ is chosen in the first replica for both the outgoing and the incoming states then, due to the spin-flip induced by the field, the boundary condition $-$ has to be present in the second replica. We represent this mechanism in Figure 5.3, which shows pictorially the relation between $\mathcal{T}(x)$ and $\mathcal{T}^{\{\mu,\mu\}}(x)$.

5.4.2 Two-particle form factors

The analysis for the two-particle form factors is slightly more involved. In principle, one can just use the form factors of Section 5.3.2, with vacua as outgoing states and pairs of kinks as incoming ones, use the relation (5.71) and systematically reconstruct every non-vanishing two-particle form factor. In practice, this procedure becomes rather cumbersome as the number of composite twist operators grows exponentially with n. For our purpose, it is more convenient to focus on the outgoing state

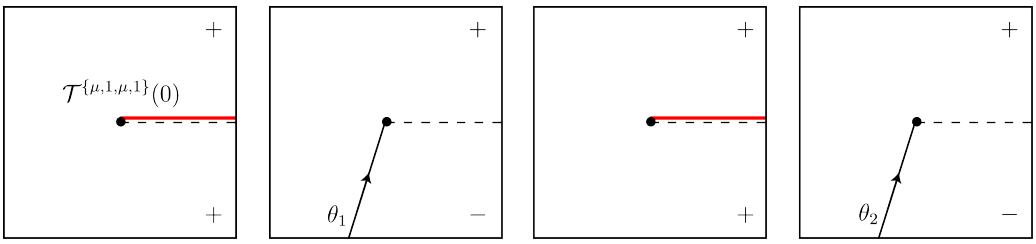


Figure 5.4 Two-particle form factor ${}^{\otimes 4}\langle +|\mathcal{T}^{\{\mu,1,\mu,1\}}(0)|+, K_{+-}(\theta_1), +, K_{+-}(\theta_2)\rangle$. Since a single kink K_{+-} is present in the second replica, a spin-flip, depicted as a red line, has to be inserted in the first replica to get a non-vanishing form factor. Similarly, the presence of a kink in the fourth replica implies the spin-flip in the third replica.

${}^{\otimes n}\langle +|$ and the incoming states

$$|+, \dots, K_{+-}(\theta_1), \dots, K_{+-}(\theta_2), \dots, +\rangle, \tag{5.74}$$

with the kinks $K_{+-}(\theta_1)$, $K_{+-}(\theta_2)$ inserted in the replicas j and j' respectively and $+$ in every other replica. We do so as these are the only contributions entering our computation of the entanglement asymmetry. Then, we investigate which n-tuples $\{g_j\}$ give rise to a non-vanishing form factor of $\mathcal{T}^{\{g_1,\dots,g_n\}}(x)$ for the states above. We also explicitly assume $j \neq j'$ (say $j < j'$), since, for $j = j'$, that is when the two kinks are in the same replica, only the trivial twist field $\mathcal{T}(0)$ can interpolate between this state and ${}^{\otimes n}\langle +|$, a case already discussed in Section 5.3.2. Let us now discuss the consequences of the presence of a single kink K_{+-} in the jth replica. Since the boundary conditions are $\pm$ for $x = \mp\infty$, a spin-flip has to connect the $(j-1)$th and the jth replica. This is the only way to match the $+$ above the branch cut in the $(j-1)$th replica and the $-$ below the branch cut in the jth one. Vice versa, if no kinks are present in the jth replica, no spin-flips between the $(j-1)$th and the jth replicas can be present. An example of the mechanism above is shown in Fig. 5.4. As a result, the only composite twist fields having non-vanishing form factors with two kinks in different replicas are precisely those with two spin-flips. Namely, the matrix element

$${}^{\otimes n}\langle +|\mathcal{T}(0)^{\{1,\dots,\mu,\dots,\mu,\dots,1\}}|+, \dots, K_{+-}(\theta_1), \dots, K_{+-}(\theta_2), \dots, +\rangle, \tag{5.75}$$

with μ inserted in the $(j-1)$th, $(j'-1)$th replicas and $K_{+-}(\theta_1)$, $K_{+-}(\theta_2)$ in the jth and j'th replica respectively, is non-vanishing. Furthermore, it is not difficult to show that the value of (5.75) is precisely $F_{11}^{(n)}(2\pi i(j'-j) - \theta_1 + \theta_2)$, and we leave some details to Appendix 5.A.

5.4.3 Entanglement asymmetry

In this section we put together all the results obtained so far and we compute the entanglement asymmetry of an interval of length ℓ for the state $|+\rangle$. We focus on the limit $m\ell \gg 1$ keeping only the two-particle contribution. In the two-particle approximation, the only twist fields appearing

in the computation are the standard twist field $\mathcal{T}(x)$ and the composite fields with two spin-flips $\mathcal{T}^{\{1,\dots,\mu,\dots,\mu,\dots,1\}}(x)$. The reason is that the other fields do not have non-vanishing two-particle form factors with ${}^{\otimes n}\langle+|$, and their contribution is therefore subleading.

For $A=(0,\ell)$ and $\rho=|+\rangle\langle+|$ we can write

$$\begin{aligned}
&\sum_{g_1,\dots,g_{n-1}\in\{1,\mu\}} \mathrm{Tr}\left(\rho^{\otimes n}\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}\}}\right)\\
&=\sum_{g_1,\dots,g_{n-1}\in\{1,\mu\}} {}^{\otimes n}\langle+|\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}\}}|+\rangle^{\otimes n} \qquad (5.76)\\
&\simeq {}^{\otimes n}\langle+|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|+\rangle^{\otimes n}+\sum_{1\le j<j'\le n} {}^{\otimes n}\langle+|\mathcal{T}^{\{1,\dots,\mu,\dots,\mu,\dots,1\}}(0)\left(\mathcal{T}^{\{1,\dots,\mu,\dots,\mu,\dots,1\}}\right)^\dagger(\ell)|+\rangle^{\otimes n},
\end{aligned}$$

where the spin-flips μ are inserted at positions j and j'. In the large-volume limit $m\ell\to\infty$ the two-point function of the standard twist field converges to $|\tau|^2$ (the square modulus of the vacuum expectation value). In contrast, the one of any composite twist field goes to zero exponentially fast as $\sim e^{-2m\ell}$, because these fields have vanishing vacuum expectation value and the first non-trivial contribution comes from the two-particle form factors. Expanding in the basis of the multi-kink states, we arrive at

$$\sum_{g_1,\dots,g_{n-1}\in\{1,\mu\}} \frac{{}^{\otimes n}\langle+|\mathcal{T}_A^{\{g_1,\dots,g_{n-1},(g_1,\dots,g_{n-1})^{-1}\}}|+\rangle^{\otimes n}}{{}^{\otimes n}\langle+|\mathcal{T}_A|+\rangle^{\otimes n}} \qquad (5.77)$$

$$\simeq 1+\frac{n}{4\pi^2}\sum_{j=1}^{n-1}\int_{\mathbb{R}}\mathrm{d}\theta\, K_0\left(2m\ell\cosh\left(\frac{\theta}{2}\right)\right)\frac{|F_{11}^{(n)}(2\pi ij-\theta)|^2}{|\tau|^2}. \qquad (5.78)$$

Finally, we employ the definition of Rényi asymmetry entanglement, and from (5.33) we get

$$\Delta S_n\simeq\log 2-\frac{1}{n-1}\log\left(1+\frac{n}{4\pi^2}\sum_{j=1}^{n-1}\int_{\mathbb{R}}\mathrm{d}\theta\, K_0\left(2m\ell\cosh\left(\frac{\theta}{2}\right)\right)\frac{|F_{11}^{(n)}(2\pi ij-\theta)|^2}{|\tau|^2}\right), \qquad (5.79)$$

an expression valid for any integer $n\ge 2$. Before discussing the analytic continuation over n, some comments are in order. First, the Rényi entanglement asymmetry is a universal quantity as it does not depend on the normalisation of the twist fields: this is manifest in the appearance of the two-particle form factor $F_{11}^{(n)}(\theta)$ through its ratio with the VEV only. Moreover, for large $m\ell$, the quantity ΔS_n approaches its asymptotic value $\log 2$ from below, and this limit is compatible with the inequality

$$0\le\Delta S_n\le\log|G|, \qquad (5.80)$$

which we expect to have general validity, and we prove in some simple cases in Appendix 5.B. These conclusions appear to be quite general, as they do not depend on the explicit expression of the

two-particle form factors in Eq. (5.61): instead, they strongly rely on the fact that the VEV of composite twist fields vanish, a property which results from topological constraints only.

In order to obtain the entanglement asymmetry we need to take the limit $n \to 1^+$ of ΔS_n, which requires to analytically continue the sum

$$\sum_{j=1}^{n-1} |F_{11}^{(n)}(2i\pi j - \theta)|^2, \tag{5.81}$$

a task that has been accomplished in [20]. Following the same steps and employing the notation thereof, we define the function

$$f(\theta, n) = \frac{1}{|\tau|^2} \sum_{j=0}^{n-1} |F_{11}^{(n)}(2\pi i j - \theta)|^2, \tag{5.82}$$

and we denote with $\tilde{f}(\theta, n)$ its analytic continuation from $n \in \mathbb{N} \setminus \{1\}$ to $n \in [1, \infty)$. In [20] it was shown that

$$\lim_{n\to 1^+} \tilde{f}(\theta, n) = 0, \quad \lim_{n\to 1^+} \frac{\partial \tilde{f}(\theta, n)}{\partial n} = \frac{\pi^2}{2}\delta(\theta), \tag{5.83}$$

a relation valid in the distributional sense over real values of θ. Here, we are interested in the sum

$$g(\theta, n) = \frac{1}{|\tau|^2} \sum_{j=1}^{n-1} |F_{11}^{(n)}(2\pi i j - \theta)|^2 = f(\theta, n) - \frac{|F_{11}^{(n)}(\theta)|^2}{|\tau|^2}, \tag{5.84}$$

namely, the term with $j = 0$ in Eq. (5.82) is not present in our calculation. Since, as we showed in Section 5.3.3, $|F_{11}^{(n)}(\theta)| \sim O((n-1)^2)$, the same properties established above for $\tilde{f}(\theta, n)$ are valid for $\tilde{g}(\theta, n)$, including its analytic continuation. In particular, we have

$$\lim_{n\to 1^+} \tilde{g}(\theta, n) = 0, \quad \lim_{n\to 1^+} \frac{\partial \tilde{g}(\theta, n)}{\partial n} = \frac{\pi^2}{2}\delta(\theta). \tag{5.85}$$

Inserting this result in (5.79) we finally get

$$\begin{aligned} \Delta S_1 &= \log 2 - \frac{1}{4\pi^2} \lim_{n\to 1^+} \int_{\mathbb{R}} \mathrm{d}\theta \, K_0\left(2m\ell \cosh\left(\frac{\theta}{2}\right)\right) \frac{\partial}{\partial n}(n\tilde{g}(\theta, n)) + O(e^{-4m\ell}) \\ &= \log 2 - \frac{K_0(2m\ell)}{8} + \mathcal{O}(e^{-4m\ell}), \end{aligned} \tag{5.86}$$

that is the main result of this section. As a concluding remark, we observe that from the large-z asymptotics of the modified Bessel function [91], that is $K_0(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}$, one immediately sees that the two-particle contribution to the entanglement asymmetry is exponentially suppressed in the size of the region.

5.5 Further generalisations

In this section, we propose a general conjecture regarding the entanglement asymmetry for any finite group G. We believe that our finding $\Delta S_n \simeq \log 2$ is not specific to the integrability features of the Ising field theory, and it mostly relies on the symmetry-breaking pattern $\mathbb{Z}_2 \to \{1\}$ of this model. We also provide a paradigmatic case in which our conjecture can be proven with elementary techniques, that is, the case of zero-entanglement states.

Let us first recall that a state ρ is symmetric under $g \in G$ if the following equality holds

$$\rho = g\rho g^{-1}. \tag{5.87}$$

Clearly, the set of $g \in G$ leaving ρ invariant is a group itself. Therefore, we define

$$H := \{h \in G | \rho = h\rho h^{-1}\} \subset G, \tag{5.88}$$

and we say that the symmetry-breaking pattern

$$G \to H, \tag{5.89}$$

arises for the state ρ. We conjecture that the Rényi asymmetry of a large subsystem A in the state ρ is

$$\Delta S_n \simeq \log \frac{|G|}{|H|}. \tag{5.90}$$

In particular, for the Ising model $G = \mathbb{Z}_2$, $H = \{1\}$ (the trivial group), and our conjecture is compatible with the main result (5.12) of this paper. In general, $|G|/|H|$ is precisely the number of ground states (vacua) in the ordered phase of a theory. For example, for the q-state Potts model [219], q distinct vacua are present and our conjecture (5.90) gives $\Delta S_n \simeq \log q$. We remark that equation (5.90) refers to finite groups only, and it does not apply to the case of $U(1)$ that has been previously studied in Ref. [29] (see also Appendix 5.B and [210]).

While we do not have a rigorous general proof of (5.90), we think that a few physical hypotheses should be sufficient to prove it:

- Homogeneity: a finite region A has to encode the global symmetries of the total system, that is the case for homogeneous systems.
- Short-range correlations: we require that correlations among points of the region A decay fast enough. This is the case for ground states, and equilibrium states of short-range Hamiltonians. We believe that this hypothesis is sufficient to prove the existence of a large-volume limit for the

entanglement asymmetry.

Before discussing an explicit case where the conjecture (5.90) can be proven, we would like to point out the intuition behind it. Let us first express, for $n \geq 2$

$$\frac{\mathrm{Tr}(\tilde{\rho}_A^n)}{\mathrm{Tr}(\rho_A^n)} = \frac{1}{|G|^{n-1}} \sum_{g_1,\dots,g_{n-1}\in G} \frac{\mathrm{Tr}(\rho_A g_1 \rho_A \dots g_{n-1}\rho_A (g_1\dots g_{n-1})^{-1})}{\mathrm{Tr}(\rho_A^n)}, \tag{5.91}$$

where both the index A and the hat have been dropped from the operators for notational convenience. Whenever $g_j \in H$ in the sum above, that is

$$\rho g_j = g_j \rho, \quad j = 1, \dots, n-1, \tag{5.92}$$

one can commute the elements inside the trace in the numerator, obtaining

$$\frac{\mathrm{Tr}(\rho_A g_1 \rho_A \dots g_{n-1}\rho_A (g_1\dots g_{n-1})^{-1})}{\mathrm{Tr}(\rho_A^n)} = 1, \quad g_j \in H. \tag{5.93}$$

There are precisely $|H|^{n-1}$ ways one can choose such $(n-1)$-tuples $\{g_1, \dots, g_{n-1}\}$ such that every $g_j \in H$. The other terms in the sum (5.91) are expected to vanish in the large volume limit. Indeed, if $[\rho_A, g_j] \neq 0$ the eigenspaces of g_j and ρ_A are distinct: in the large volume limit, we expect that the eigenspaces above are at "generic positions" with respect to each other at large size, leading to a fast decay of the traces in which ρ_A and g_j are inserted consecutively: a related mechanism is known in the context of *free independence* between random matrices, and we refer the interested reader to [221] for a review. The considerations above lead to (5.90) after simple manipulations.

5.5.1 Zero-entanglement states

For zero-entanglement states, the conjecture (5.90) can be proven for any finite group G, as we show below. While this example may not be a realistic description of physical states with short-range correlations, it correctly captures the important features we mentioned above.

Let us consider the pure state

$$\rho = |0\dots 0\rangle_A \otimes |0\dots 0\rangle_{\bar{A}}, \tag{5.94}$$

in which $|0\rangle$ is an on-site configuration belonging to a finite-dimensional Hilbert space. The reduced density matrix of the state above is

$$\rho_A = |0\dots 0\rangle_A \, {}_A\langle 0\dots 0|, \tag{5.95}$$

and, since ρ_A is pure, no entanglement between A and $\bar{A}$ is present. In particular, this means $\mathrm{Tr}\,(\rho_A^n) = 1$. Let us then assume that a finite group G acts unitarily as a global symmetry of the system. It is

not difficult to show that the subgroup H that leaves the state invariant is:

$$H = \{h \in G \text{ s.t } |\langle 0|h|0\rangle| = 1\}, \tag{5.96}$$

namely $|0\rangle$ and $h|0\rangle$ are proportional through a phase. In contrast, whenever $g \notin H$, one has

$$|\langle 0|g|0\rangle| < 1. \tag{5.97}$$

By considering a generic n-tuple $\{g_j \in G\}_{j=1,\dots,n}$, we can compute

$$\mathrm{Tr}\,(\rho_A g_1 \dots \rho_A g_n) = \prod_{j=1}^{n} {}_A\langle 0\dots 0|g_j|0\dots 0\rangle_A = \prod_{j=1}^{n} \langle 0|g_j|0\rangle^{|A|}, \tag{5.98}$$

with $|A|$ the number of sites of A. As long as at least one element satisfies $g_j \notin H$, one has

$$\lim_{|A|\to\infty} |\langle 0|g_j|0\rangle|^{|A|} = 0, \tag{5.99}$$

which trivially implies

$$\lim_{|A|\to\infty} \mathrm{Tr}\,(\rho_A g_1 \dots \rho_A g_n) = 0. \tag{5.100}$$

On the other hand, for n-tuples of elements of H, equation (5.93) holds. Putting together these results with (5.91), one finally gets (5.90) for the case of a large subsystem A.

5.6 Concluding remarks

In this paper we generalised the notion of Rényi entanglement asymmetry, first proposed in [29], to include the action of an arbitrary finite group G, and we characterised its value in the symmetry-broken ground state of the Ising field theory. In particular, we employed the replica trick to describe the quantities of interest as expectation values of composite twist operators, and we provided analytical expressions of their form factors using integrable bootstrap.

In addition, we proposed a general conjecture (5.90), which we expect to hold for a large class of quantum states and for any finite group G. Remarkably, if our conjecture is correct, the entanglement asymmetry of a large but finite region becomes a useful tool to "count" the number of ground states in a spontaneously symmetry-broken phase via a local probe.

Many interesting directions are left to explore. It would be interesting to investigate other integrable QFTs using the approach we developed in this paper. For instance, the q-state Potts model in its ferromagnetic phase [219] stands out as the simplest generalisation of the Ising model. Moreover, one could analyse massless flows [222] in which the symmetry is broken explicitly along

the renormalisation group flow, and distinct symmetries are present in the IR and UV fixed points. It may also be worth considering field theories where the symmetry breaking arises due to the presence of boundary conditions or impurities (see e.g. [223, 224]) and translational invariance is absent.

Furthermore, the relation between the paramagnetic and the ferromagnetic phase of the Ising model is, from the point of view of entanglement content, quite puzzling. As we have shown, there are formal analogies in the form factor bootstrap of the twist fields, but the relation is subtle and, in general, the entropies are different in the two phases (see Appendix 5.C). It is well-known that the Kramers-Wannier duality [225], which is nowadays understood in the language of non-invertible defects [226], relates the ordered and the disordered phase non-locally: this duality might be a good candidate to explain the relationship between the twist fields in the two phases. In particular, it would be interesting to understand what is the "dual"of the entanglement of a region under the Kramers-Wannier duality.

We also consider the study of low-energy states in the ferromagnetic phase of one-dimensional quantum systems (i.e. domain walls interpolating between distinct vacua) to be promising. Indeed, it is not obvious a priori whether the results found in [114–116, 144] for quasi-particles should also hold for these topological excitations typical of the ordered phase.

Finally, a general, comprehensive proof of conjecture (5.90) is still missing. Recently, its validity has been rigorously shown for matrix product states (see e.g. [227]) in [210], where the authors also proposed a generalisation of equation (5.90) for compact Lie groups. However, many physically relevant states, such as (generalised) Gibbs ensembles or regularised boundary states of CFT [228] are yet to be studied. Moreover, it would be interesting to understand whether Eq. (5.90) is also valid for the steady states arising in the long-time dynamics, which is the main motivation behind the original formulation of the entanglement asymmetry [29].

5.A Two-particle form factors of composite twist fields

In this appendix we show in some simple cases the validity of the relation

$$ {}^{\otimes n}\langle +|\mathcal{T}(0)^{\{1,\dots,\mu,\dots,\mu,\dots,1\}}|+,\dots,K_{+-}(\theta_1),\dots,K_{+-}(\theta_2),\dots,+\rangle = F_{11}^{(n)}(2\pi i(j'-j)-\theta_1+\theta_2), \quad (5.101) $$

which was presented at the end of Section 5.4.2. A proof of the equation above for generic values of n, j, j' is easy to obtain by making use of the monodromy equation, global spin-flip invariance and the algebra (5.48). However, the general mechanism is better highlighted by looking at some explicit examples.

Let us consider $n = 2, 3, 4$ and the following fields (see Section 5.2)

$$\mathcal{T}^{\{\mu,\mu\}}(x) = (1 \otimes \mu)\mathcal{T}(x)(1 \otimes \mu) \quad \text{for } n = 2, \tag{5.102}$$

$$\mathcal{T}^{\{1,\mu,\mu\}}(x) = (1 \otimes 1 \otimes \mu)\mathcal{T}(x)(1 \otimes 1 \otimes \mu) \quad \text{for } n = 3, \tag{5.103}$$

$$\mathcal{T}^{\{\mu,1,\mu,1\}}(x) = (1 \otimes \mu \otimes \mu \otimes 1)\mathcal{T}(x)(1 \otimes \mu \otimes \mu \otimes 1) \quad \text{for } n = 4. \tag{5.104}$$

For $n = 2$ we have

$$\begin{aligned}
&\langle +,+|\mathcal{T}^{\{\mu,\mu\}}(0)|K_{+-}(\theta_1), K_{+-}(\theta_2)\rangle \\
=&\langle +,+|(1 \otimes \mu)\mathcal{T}(0)(1 \otimes \mu)|K_{+-}(\theta_1), K_{+-}(\theta_2)\rangle \\
=&\langle +,-|\mathcal{T}(0)|K_{+-}(\theta_1), K_{-+}(\theta_2)\rangle \\
=&\langle -,-|\mathcal{T}(0)|K_{-+}(\theta_2 + 2\pi i)K_{+-}(\theta_1), -\rangle \\
=&\langle +,+|\mathcal{T}(0)|K_{+-}(\theta_2 + 2\pi i)K_{-+}(\theta_1), +\rangle = F_{11}^{(2)}(2\pi i + \theta_2 - \theta_1),
\end{aligned} \tag{5.105}$$

where we used the monodromy equation for $\mathcal{T}(x)$ (discussed in Section 5.3) in going from the third to the fourth line and applied an overall spin-flip transformation to each replica to obtain the last line. Similar calculations are shown below for $n = 3$:

$$\begin{aligned}
&\langle +,+,+|\mathcal{T}^{\{1,\mu,\mu\}}(0)|K_{+-}(\theta_2), +, K_{+-}(\theta_1)\rangle \\
=&\langle +,+,+|(1 \otimes 1 \otimes \mu)\mathcal{T}(0)(1 \otimes 1 \otimes \mu)|K_{+-}(\theta_2), +, K_{+-}(\theta_1)\rangle \\
=&\langle +,+,-|\mathcal{T}(0)|K_{+-}(\theta_2), +, K_{-+}(\theta_1)\rangle \\
=&\langle -,-,-|\mathcal{T}(0)|K_{-+}(\theta_1 + 4\pi i)K_{+-}(\theta_1), -, -\rangle \\
=&\langle +,+,+|\mathcal{T}(0)|K_{+-}(\theta_1 + 4\pi i)K_{-+}(\theta_2), +, +\rangle \\
=&F_{11}^{(3)}(4\pi i + \theta_2 - \theta) = F_{11}^{(3)}(2\pi i - \theta_1 + \theta_2).
\end{aligned} \tag{5.106}$$

Finally, for $n = 4$ one has

$$\begin{aligned}
&\langle +,+,+,+|\mathcal{T}^{\{\mu,1,\mu,1\}}(0)|+, K_{+-}(\theta_1), +, K_{+-}(\theta_2)\rangle \\
=&\langle +,+,+,+|(1 \otimes \mu \otimes \mu \otimes 1)\mathcal{T}(0)(1 \otimes \mu \otimes \mu \otimes 1)|+, K_{+-}(\theta_1), +, K_{+-}(\theta_2)\rangle \\
=&\langle +,-,-,+|\mathcal{T}(0)|+, K_{-+}(\theta_1), -, K_{+-}(\theta_2)\rangle \\
=&\langle +,+,+,+|\mathcal{T}(0)|+, K_{+-}(\theta_2 + 4\pi i)K_{-+}(\theta_1), +, +\rangle \\
=&F_{11}^{(4)}(4\pi i - \theta_1 + \theta_2).
\end{aligned} \tag{5.107}$$

5.B Useful inequalities

In this appendix, we provide some upper and lower bounds for the Rényi entanglement asymmetry of finite groups. In particular, we focus on $n = 2$, leaving some considerations for the other values of n at the end of this section. The main result, proved below, is

$$0 \leq \Delta S_2 \leq \log |G|, \tag{5.108}$$

with $\Delta S_2 = 0$ iff $\rho_A = \tilde{\rho}_A$.

We first express ΔS_2, defined in (5.8), as (see also equation (5.33))

$$\Delta S_2 = \log |G| - \log \left(\sum_{g \in G} \frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)} \right). \tag{5.109}$$

Therefore, to bound ΔS_2 it is sufficient to analyse the possible values taken by

$$\frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)}, \tag{5.110}$$

as a function of $g \in G$. We observe that, since ρ_A and $g\rho_A g^{-1}$ are both positive semi-definite, one can easily show that[8]

$$\frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)} \geq 0, \quad \forall g \in G. \tag{5.111}$$

Furthermore, because the ratio takes the value 1 for $g = 1$, we can write

$$\log \left(\sum_{g \in G} \frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)} \right) = \log \left(1 + \sum_{g \neq 1} \frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)} \right) \geq 0, \tag{5.112}$$

which, from equation (5.109), implies $\Delta S_2 \leq \log |G|$.

To prove $\Delta S_2 \geq 0$, we apply the von Neumann's trace inequality [229] to ρ_A and $g\rho_A g^{-1}$, obtaining

$$\frac{\mathrm{Tr}\left(\rho_A g \rho_A g^{-1}\right)}{\mathrm{Tr}\left(\rho_A^2\right)} \leq 1, \tag{5.113}$$

and the bound is saturated iff ρ_A and $g\rho_A g^{-1}$ share the same eigenvectors (which means they have to be equal, as their eigenvalues are always the same). Inserting the result above in equation (5.109), we obtain

$$\Delta S_2 \geq 0, \tag{5.114}$$

where the equality holds iff $\rho_A = g\rho_A g^{-1}$ for any $g \in G$. Therefore, from the definition (5.5), we have

[8] Given A, B positive semi-definite matrices, one has $\mathrm{Tr}\,(AB) = \mathrm{Tr}\left((\sqrt{A}\sqrt{B})(\sqrt{A}\sqrt{B})^\dagger\right) \geq 0$.

$\Delta S_2 = 0$ iff $\rho_A = \tilde{\rho}_A$, which proves the main result (5.108) of this appendix.

It is worth considering whether comparable strategies can be used to establish the inequality $0 \leq \Delta S_n \leq \log|G|$ for other values of n. The first issue we encounter is that

$$\frac{\mathrm{Tr}\left(\rho_A \cdot g_1 \rho_A g_1^{-1} \cdot g_2 \rho_A g_2^{-1} \dots g_{n-1} \rho_A g_{n-1}^{-1}\right)}{\mathrm{Tr}\left(\rho_A^n\right)}, \tag{5.115}$$

is not necessarily real for $n \geq 3$. For example, the matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_3 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \tag{5.116}$$

are positive semi-definite, having 0 and 1 as eigenvalues, but $\mathrm{Tr}\left(M_1 M_2 M_3\right) = \frac{1+i}{4} \notin \mathbb{R}$. This implies that the previous technique, employed for $n = 2$ to show $\Delta S_2 \leq \log|G|$, does not apply directly to $n \geq 3$.

On the other hand, we can provide a simple proof of $\Delta S_n \geq 0$ valid for any, possibly non-integer, $n \geq 1$. The key idea is that the symmetrised state $\tilde{\rho}_A$ is generically more mixed (less pure) than ρ_A, and, therefore, it has more entropy. More precisely, one can show from (5.5) that the superoperator Φ, defined as

$$\Phi(\rho_A) := \tilde{\rho}_A, \tag{5.117}$$

is completely positive and trace-preserving (CPT). Then, we use the monotonicity of the sandwiched Rényi divergence [230], valid for CPT maps, and we apply it directly to ρ_A and the infinite-temperature state $\frac{\mathbb{1}_A}{\mathrm{Tr}(\mathbb{1}_A)}$. As an immediate consequence, we get

$$\Delta S_n \geq 0, \quad n \geq 1. \tag{5.118}$$

However, with this approach it is not possible to claim that, for a generic value of n, $\Delta S_n = 0$ implies $\rho_A = \tilde{\rho}_A$.

As a last remark, we point out that for a compact Lie group the number of elements is infinite and the Rényi entanglement asymmetry does not have an upper bound. For instance, in a simple example analysed in [29], the $U(1)$ asymmetry grows logarithmically in the subsystem size. A detailed exploration of the entanglement asymmetry for compact Lie groups can be found in [210].

5.C Exact results for the Rényi entropies via corner transfer matrix and Kramers-Wannier duality

In this appendix, we review some exact results available for the entanglement of the transverse-field Ising chain, following closely [231]. Let us consider the quantum one-dimensional Ising model, described by the Hamiltonian

$$H = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z\right), \tag{5.119}$$

with σ_j^x, σ_j^z the Pauli matrices at position j. This model displays a quantum critical point at $h = 1$ separating a ferromagnetic phase ($h < 1$) from a paramagnetic one ($h > 1$). It is well known that Kramers-Wannier duality [225] relates the spectra of the Hamiltonian at h and h^{-1}, and, more in general, local and semi-local observables are mapped into each other by this duality. The same mechanism is observed in the underlying Ising field theory, where one can relate the paramagnetic and the ferromagnetic phase at the same value of the mass m. However, as shown in [231], the entropies at dual points differ explicitly. In particular, the entanglement of the half-chain has been characterised analytically in the infinite-volume limit via the Corner Transfer Matrix (CTM) method [232]. Here, we only report and discuss the final result found in [231].

Let us define the parameters

$$k := \min(h, h^{-1}), \quad \epsilon = \pi \frac{K(\sqrt{1-k^2})}{K(k)}, \tag{5.120}$$

with $K(z)$ the complete elliptic integral of the first kind [91]. Then, given

$$\epsilon_j := \begin{cases} (2j+1)\epsilon & h > 1, \\ 2j\epsilon & h < 1, \end{cases} \tag{5.121}$$

for $j = 0, 1, \dots$ one can express the nth Rényi entropy of the half chain at field h as

$$S_n(h) = \frac{1}{1-n} \sum_{j=0}^{\infty} \log \frac{1 + e^{-\epsilon_j n}}{(1 + e^{-\epsilon_j})^n}. \tag{5.122}$$

Equation (5.120) clearly shows that the value of the parameter ϵ is the same at the dual points h, h^{-1}, even though the "single-particle eigenvalues" ϵ_j are different.

We emphasise that for $h < 1$, corresponding to the ferromagnetic phase, the ground state is degenerate and one should be careful. For instance, the result (5.122) refers to the symmetric ground

state (often dubbed GHZ state), which in our notation corresponds to

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right). \tag{5.123}$$

To obtain the entropies of the symmetry-broken ground state $|\pm\rangle$ one has to explicitly remove the zero-mode $j = 0$ from the sum in (5.122), as explained in [231], thus getting

$$S_n(h<1) = \frac{1}{1-n}\sum_{j=1}^{\infty} \log\frac{1+e^{-\epsilon_j n}}{(1+e^{-\epsilon_j})^n} = \frac{1}{1-n}\sum_{j=0}^{\infty} \log\frac{1+e^{-\epsilon_j n}}{(1+e^{-\epsilon_j})^n} - \log 2, \tag{5.124}$$

which means that the Rényi entropy of the doubly degenerate state $|\pm\rangle$ is smaller than that of the GHZ state (5.122), as one would intuitively expect. Remarkably, in the limit $h \to 1$ one has $\epsilon \to 0$ and, accordingly, the Rényi entropy diverges. The origin of this divergence is universal, and it is traced back to the properties of the underlying CFT (see [231] for further details): in particular, the divergence is the same if the critical magnetic field $h = 1$ is approached from above or below.

Discrepancies between dual points can nevertheless be spotted if one takes into account also the finite terms arising in (5.122) when taking the limit $h \to 1$. In particular, using the results of [231] one can show that the entropy difference between the paramagnetic phase and the ferromagnetic phase (in the GHZ state) satisfies

$$\lim_{h\to 1^+}\left[S_n(h) - S_n(h^{-1})\right] = -\frac{\log(2)}{2}. \tag{5.125}$$

The origin of the constant $-\frac{\log(2)}{2}$ is much more subtle than the $\log(2)$ term in (5.124): indeed, the former appears only close to the critical point, and a full explanation based on the formal analogy between (5.122) and the thermal entropy of a CFT has been provided in [231] (see also [63] for a related discussion).

Below, we interpret these results in the language of the Ising field theory. Let us focus on a given value of the mass m and let $|0\rangle$ be the paramagnetic ground state, $|\pm\rangle$ the symmetry-broken ferromagnetic ground state, and $|\mathrm{GHZ}\rangle$ the state defined by (5.123). To compute the Rényi entropy of a given state in the half-infinite chain, say $A = (0, \infty)$, we have to evaluate the twist field $\mathcal{T}(0)$ over the corresponding replica state. In particular, the difference of Rényi entropy between $|\mathrm{GHZ}\rangle$ and $|+\rangle$ would be just

$$\frac{1}{1-n}\log\frac{{}^{\otimes n}\langle \mathrm{GHZ}|\mathcal{T}(0)|\mathrm{GHZ}\rangle^{\otimes n}}{{}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}} = \log 2. \tag{5.126}$$

Here, to obtain (5.126) we only used the definition (5.123) and the fact that the only non-vanishing VEVs of $\mathcal{T}(0)$ over replica vacua are ${}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n} = {}^{\otimes n}\langle -|\mathcal{T}(0)|-\rangle^{\otimes n}$ (see Section 5.3). This result is compatible with (5.124), and reasonably a similar argument can be employed even beyond the field theoretic regime, when $|h-1|$ is finite.

Instead, a different mechanism arises if we try to compare $|+\rangle$ and $|0\rangle$. We first notice that the ratio

$$\frac{{}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}}{{}^{\otimes n}\langle 0|\mathcal{T}(0)|0\rangle^{\otimes n}}, \tag{5.127}$$

is dimensionless, i.e. it is a universal number which does not depend on m (the only mass scale). Moreover, according to the lattice result (5.125) valid near the critical regime where the field theory is predictive, we expect

$$\frac{{}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}}{{}^{\otimes n}\langle 0|\mathcal{T}(0)|0\rangle^{\otimes n}} = g^{n-1}, \quad g = \sqrt{2}, \tag{5.128}$$

so that the difference of entropy between $|+\rangle$ and $|0\rangle$ would be just[9]

$$\frac{1}{1-n}\log\frac{{}^{\otimes n}\langle +|\mathcal{T}(0)|+\rangle^{\otimes n}}{{}^{\otimes n}\langle 0|\mathcal{T}(0)|0\rangle^{\otimes n}} = -\frac{\log 2}{2}. \tag{5.129}$$

We believe that (5.128) can be proven, once the normalisation of the twist field is fixed, via a form factor approach similar to that of [63]. We also conjecture that the value of the constant g in Eq. (5.128) comes precisely from Affleck-Ludwig boundary entropy [233]: however, it is not clear to us whether an explicit relation between the off-critical paramagnetic/ferromagnetic phases and a boundary CFT can be provided. We only point out that if (5.128) is correct, then it cannot be a trivial consequence of the Kramers-Wannier duality, which holds exactly on the lattice, as the relation (5.125) is only valid close to the critical point.

[9] Equation (5.125) compares the GHZ state and the paramagnetic ground state. If we take $|+\rangle$ instead of the GHZ state, we get $\lim_{h\to 1^+} S_n(h) - S_n(h^{-1}) = +\frac{\log(2)}{2}$.

CHAPTER

SIX

OUTLOOK

The results we presented in this thesis contribute to expand the knowledge in the field of symmetry-resolved entanglement measures in quantum many-body systems, an area which has been the object of intense activity in the past five years. One of the main theoretical tools we employed throughout this work is that of composite twist operators: these are operators that are defined in a very simple way by means of their algebraic relations with the local observables of a theory in a generic D-dimensional QFT. In the context of charged zero-density excited states of a QFT, by making some simple approximations, composite twist operators allow us to reproduce, at leading order in the system size, the results which can be obtained by form factor calculations and generalise those to higher-dimensional theories. Furthermore, crucially, we used twist operators to derive a new, general way of computing entanglement asymmetry in a symmetry-breaking ground state of a QFT, paving the way for a deeper investigation of entanglement properties in the ordered phase of a field theory.

Since the notion of symmetry resolution of entanglement was introduced, different techniques have been employed to establish exact results as well as asymptotic behaviours of the symmetry-resolved bipartite and multipartite entanglement measures in one-dimensional systems. These include, most notably, spin chains, conformal field theories, integrable quantum field theories, and quantum cellular automata. The most prominent result that has emerged from these extensive investigations is the equipartition of symmetry-resolved entropy at leading order in the system size, that is, the property that different charge sectors equally contribute to the total entanglement. Our main contribution to the field of symmetry-resolved entanglement consists in obtaining exact and universal leading-order results for the $U(1)$ symmetry-resolved entanglement of zero-density excited states in massive QFTs. Our results exclusively rely on the locality of the excitations, and bridge a significant gap in the

existing literature. Next-to-leading-order contributions to the entanglement will reveal how the latter depends on the microscopic details of the theory; however, a perturbative study of the large-volume limit of entanglement measures generally requires the knowledge of finite-volume form factors, and the development of a form factor program for non-local fields at finite volume is a challenging and -so far- virtually untackled task.

Employing a notion of entanglement asymmetry to investigate the properties of symmetry-broken phases in one-dimensional systems is an idea that made its appearence in the community of quantum many-body physics very recently. Prior to our work, only few papers had appeared, primarily focussing on the evolution of $U(1)$ entanglement asymmetry after a quantum quench. In our work, we looked at the ferromagnetic phase of the Ising field theory to propose an algebraic approach for the computation of entanglement asymmetry in field theories, based on composite twist operators. Furthermore, we conjectured a formula for the entanglement asymmetry of a state displaying partial symmetry breaking which, to the best of our knowledge, was verified in any system studied to date.

We now highlight some research directions that emerge from our work. A first topic that deserves further investigation is that of entanglement properties of localised quasi-particle excitations. Indeed, the universality of entanglement entropy (total and symmetry-resolved) of excited states of gapped theories poses a question: how do we distinguish between states at different energy, from the point of view of the entanglement content? A possible way to tackle the problem is by looking at relative entropy as a measure of distinguishability among excited states with the same energy and same entanglement entropy. Moreover, form factor expansions allows us to obtain finite-volume corrections to the correlation functions, and beyond leading order the energy of the excitations (as well as the integrability features of the theory) becomes relevant.

On the other hand, the entanglement entropy of descendant states of a CFT has been well studied in the past decade. It is known that the low-lying and the high-lying states in the spectrum have entropies that depend on the details of the system in functionally different ways. The high-energy states reproduce, in certain limits, the universal result found in gapped theories. How does this crossover between the CFT and massive QFT results happens physically? Specific limits can be taken but a general explanation is missing. Holography may be a key, as high-energy CFT states may be obtained by looking at (2+1)d in a small coupling regime.

Two physical questions of the utmost relevance can be approached using integrability techniques, namely: the interplay between symmetries and boundary effects, and the dynamics of entanglement in QFT. Indeed, on the one hand the twist field and twist operator techniques we developed in [1, 2, 5] pave the way for a study of correlation functions and symmetry-resolved entanglement measures in theories with boundaries, thus addressing the question of whether -or to which extent- entropy

equipartition is spoiled by boundary effects. On the other hand, the very same techniques allow us to investigate what happens to entanglement-related physical observable of a QFT after a quench in the mass or in other parameters of the theory.

Finally, the entanglement properties specific to the ordered phase of a many-body system are particularly interesting. In the recent work [8], we observed that the entanglement content of kink-like excitations in the ferromagnetic phase of the Ising model is not captured by the universal results for the entanglement of quasi-particle excitations. This is due to the different localisation properties of domain walls, which induce topological effects. The algebraic framework we developed in [5] and [8] can be employed for the investigation of entanglement properties in the ordered phase of a QFT, where excitations can be described by kinks interpolating between vacua.

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