Free Vibration of Axially Loaded Axial-Bending Coupled Timoshenko-Ehrenfest Beams

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Abstract

The free vibration analysis of an axially loaded Timoshenko-Ehrenfest beam coupled in axial and bending deformations is carried out in this paper for different boundary conditions. First, the governing differential equations of motion in free vibration are developed using Hamilton's principle and then they are solved in closed algebraic form for axial displacement, bending displacement and bending rotation. The expressions for axial force, shear force and bending moment are also obtained in explicit algebraic form. Finally, by applying the boundary conditions, the natural frequencies and mode shapes of the axially loaded axial-bending coupled Timoshenko-Ehrenfest beam are computed for an illustrative example with clamped-free (C-F), pinned-pinned (P-P) and clamped-clamped (C-C) supports at the ends. The results are discussed, and some conclusions are drawn.

1. Introduction

The free vibration behaviour of axial-bending coupled beams using Bernoulli-Euler and Timoshenko-Ehrenfest theories has been investigated by several authors [1-6], but these publications do not generally account for the case when the beam carries an axial load whose effect on the beam's free vibration characteristics can be significant. For an axial-bending coupled Timoshenko-Ehrenfest beam exhibiting free vibration, the inclusion of an axial load increases the level of complexity greatly. The problem does not appear to have been adequately dealt with in the literature. The present paper addresses this problem.

2. Theory

Figure 1 shows a uniform axial-bending coupled Timoshenko-Ehrenfest beam of length L in a right-handed Cartesian coordinate system with the Y-axis coinciding with the beam elastic axis which is the locus of shear centres of the beam cross-sections. A compressive axial load (P) considered to be positive, is assumed to act through the elastic axis of the beam as shown. Note that P can be positive so that tension is included in the theory. The coupling between axial and bending displacements will occur in a beam of this type because of the eccentricity between the centroid (G_c) and shear centre (E_s) of the beam cross-section, see Figure 1. There are many practical cross-sections for which the centroid and shear centre are non-coincident (see Figure 2 of [4]), but the inverted T section is shown in Figure 1 only for convenience. The mass axis which is the locus of the centroid of the beam cross-sections is separated by a distance z_α from the elastic axis, as shown. Now, if v_0, w_0 and θ are axial displacement, bending displacement and bending rotation of a point on the elastic axis at a distance y from the origin in the coordinate system of Figure 1, the governing differential equations of motion in free vibration of the axially loaded axial-bending coupled Timoshenko-Ehrenfest beam can be obtained by applying Hamilton's principle and they are in the usual notation, given by

$$EAv_0'' - EAz_\alpha\theta'' - \rho A\ddot{v}_0 + \rho Az_\alpha\ddot{\theta} = 0 \quad (1)$$

$$EI_e\theta'' - \rho I_e\ddot{\theta} + \rho Az_\alpha\ddot{v}_0 - EAz_\alpha v_0'' + kAG(w_0' - \theta) = 0 \quad (2)$$

$$kAG(w_0'' - \theta') - Pw_0'' - \rho A\ddot{w}_0 = 0 \quad (3)$$

where EA, EI_e and kAG are axial, bending (about the elastic axis) and shear rigidities of the beam, ρA, ρI_e are mass per unit length and mass moment of inertia per unit length (about the elastic axis) and a prime and an over dot denote partial differentiation with respect to length y and time t, respectively.

The expressions for axial force (f), bending moment (m) and shear force (s) which result from the natural boundary conditions of the Hamiltonian formulation are given by

$$f = -EAv_0' + EAz_\alpha\theta' \tag{4}$$

$$m = -EI_e\theta' + EAz_\alpha v_0' \tag{5}$$

$$s = -kAG(w_0' - \theta) + Pw_0' \tag{6}$$

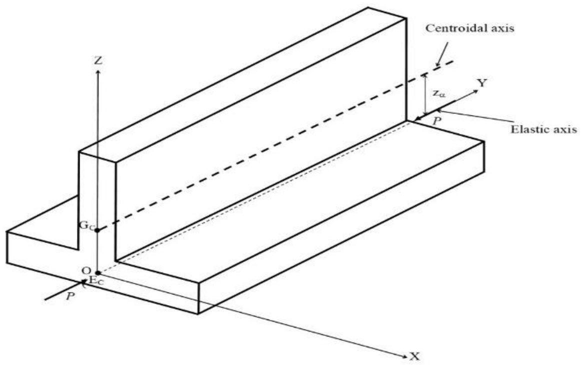

Figure 1. Coordinate system and notation for an axially loaded axial-bending coupled Timoshenko-Ehrenfest beam.

For harmonic oscillation with circular or angular frequency ω, and by introducing the non-dimensional length parameter $\xi = y/L$, Equations (1)-(3) can be solved for the amplitudes of axial displacement (V), bending displacement (W) and bending rotation (Θ) in terms of integration constants A_1-A_6 to give

$$V(\xi) = \mu k_\alpha A_1 \sinh\alpha\xi + \mu k_\alpha A_2 \cosh\alpha\xi + \mu k_\beta A_3 \sin\beta\xi - \mu k_\beta A_4 \cos\beta\xi + A_5 \sin\gamma\xi + A_6 \cos\gamma\xi \tag{7}$$

$$W(\xi) = A_1 \cosh\alpha\xi + A_2 \sinh\alpha\xi + A_3 \cos\beta\xi + A_4 \sin\beta\xi \tag{8}$$

$$\Theta(\xi) = A_1 \frac{k_\alpha}{L} \sinh\alpha\xi + A_2 \frac{k_\alpha}{L} \cosh\alpha\xi + A_3 \frac{k_\beta}{L} \sin\beta\xi - A_4 \frac{k_\beta}{L} \cos\beta\xi \tag{9}$$

where α, β, γ, k_α, and k_β are given by

$$\alpha = \sqrt{-\frac{C_1}{2} + \frac{\sqrt{C_1^2 + 4C_2}}{2}};\ \beta = \sqrt{\frac{C_1}{2} + \frac{\sqrt{C_1^2 + 4C_2}}{2}};\ \gamma = \sqrt{\frac{\omega^2 \rho A L^2}{EA}}\ k_\alpha = \frac{b^2 s^2 + \alpha^2 \lambda^2}{\alpha};\ k_\beta = \frac{b^2 s^2 - \beta^2 \lambda^2}{\beta} \tag{10}$$

with

$$C_1 = \frac{(a^2 - \mu^2 b^2)\{b^2(r^2 + s^2)(a^2 - \mu^2 b^2) + a^2 p^2 - b^2 r^2 s^2 (a^2 - \mu^2 b^2)\}}{(1 - p^2 s^2)(a^2 - \mu^2 b^2)};\quad C_2 = \frac{\{a^2 b^2 - b^4 r^2 s^2 (a^2 - \mu^2 b^2)\}}{(1 - p^2 s^2)(a^2 - \mu^2 b^2)} \tag{11}$$

$$a^2 = \frac{\omega^2 \rho A L^2}{EA};\ b^2 = \frac{\omega^2 \rho A L^4}{EI_e};\ p^2 = \frac{PL^2}{EI}\ r^2 = \frac{EI_e}{EAL^2};\ s^2 = \frac{EI_e}{kAGL^2};\ \mu^2 = \frac{z_\alpha^2}{L^2};\ \lambda^2 = 1 - p^2 s^2 \tag{12}$$

Similarly, the expressions for the amplitudes of axial force (*F*), shear force (*S*) and bending moment (*M*) for harmonic oscillation can be obtained as

$$F(\xi) = -\frac{EA}{L}\left(\frac{dV}{d\xi} - \mu\frac{d\Theta}{d\xi}\right) = -\frac{EA}{L}\ \gamma(A_5 \cos\gamma\xi - A_6 \sin\gamma\xi) \tag{13}$$

$$S(\xi) = \frac{EI_e}{L^3}\left(A_1 g_\alpha \sinh\alpha\xi + A_2 g_\alpha \cosh\alpha\xi +\ A_3 g_\beta \sin\beta\xi - A_4 g_\beta \cos\beta\xi\right) \tag{14}$$

$$M(\xi) = -\frac{EI}{L^2}\left(A_1 h_\alpha \cosh\alpha\xi + A_2\, h_\alpha \sinh\alpha\xi + A_3 h_\beta \cos\beta\xi + A_4 h_\beta \sin\beta\xi - A_5\, h_\gamma \cos\gamma\xi + A_6 h_\gamma \sin\gamma\xi\right) \tag{15}$$

where

$$g_\alpha = \frac{b^2}{\alpha};\quad g_\beta = \frac{b^2}{\beta};\ h_\alpha = \alpha k_\alpha(1 - \mu^2 b^2/a^2);\ \ h_\beta = \beta k_\beta(1 - \mu^2 b^2/a^2);\ h_\gamma = \gamma\mu b^2/a^2 \tag{16}$$

Now, Equations (7)-(9) and Equations (13)-(15) can be used to apply boundary conditions for displacements and rotations, as well as for forces and moments, respectively, to eliminate the constants A_1-A_6 and arrive at the frequency equation which yields the natural frequencies of the axially loaded axial-bending coupled Timoshenko-Ehrenfest beam. The mode shapes can be recovered by assigning a chosen value of one the constants and determining the rest of the constants in terms of the chosen one.

3. Discussion of results and conclusions

To demonstrate the application of the developed theory, an axially loaded coupled axial-bending Timoshenko-Ehrenfest beam made of aluminium and with the inverted T cross-section shown Figure 2 which is that of [5] is now analysed for its free vibration characteristics. The dimensions used for the cross-section (see Figure 2) are b = 40 mm, t = 4 mm and the length of the beam L is taken as 1 m. The distance between the shear centre and the centroid of the cross-section is worked out to be z_α = 9.474 mm. The material properties used in the analysis are the Young's modulus E = 70 GPa, the shear modulus G = 26.92 GPa and the density ρ = 2700 kg/m^3. The shear correction factor (also known as the shape factor) k is taken to be 2/3. Using the above data, the stiffness and mass properties of the section are calculated as axial stiffness (EA) = 2.128×10^7 N, (ii) bending stiffness (EI_e) = 5135.57 Nm^2, (iii) shear stiffness (kAG) = 5.4564 ×10^6 N, (iv) mass per unit length (ρA) = 0.8208 kg/m and (v) the mass moment of inertia (rotatory) per unit length (ρI_e) = 0.001981 kgm.

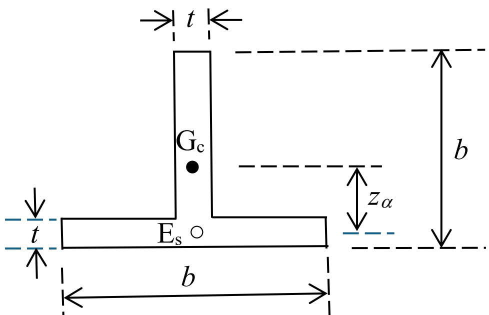

Figure 2. Cross-sectional details of an axially loaded coupled axial-bending Timoshenko-Ehrenfest beam, mass axis (centroid): G_c, elastic axis: E_s.

The critical buckling loads (P_{cr}) of the axial-bending coupled Timoshenko-Ehrenfest beam for clamped-Free (C-F), Pinned-Pinned (P-P) and clamped-clamped (C-C) boundary conditions were established at 7.9471 kN, 45.055kN and 124.44 kN, respectively by using the theory published by Banerjee [6]. (Note that the P-P boundary condition prevents axial motion at the ends.) Next, the first five natural frequencies of the beam for C-F, P-P and C-C boundary conditions were computed using the current theory considering the axial load 0.0, 0.5P_{cr} and -0.5P_{cr}, respectively, and the results are shown in Table 1. These results were checked using the computer program BUNVIS-RG [7] which has the capability to connect eccentrically an axially loaded beam to nodes at the centroid of the cross-section to idealise an axially loaded axial-bending coupled beam, giving approximate, but sufficiently accurate results.

Table 1. Natural frequencies of an axially loaded axial-bending coupled Timoshenko-Ehrenfest beam for different boundary conditions.

Frequency Number (i)	Natural frequencies ω_i (rad/s)								
	C-F			P-P			C-C		
	Axial Load (P)			Axial Load (P)			Axial Load (P)		
	0.0	$0.5P_{cr}$	$-0.5P_{cr}$	0.0	$0.5P_{cr}$	$-0.5P_{cr}$	0.0	$0.5P_{cr}$	$-0.5P_{cr}$
1	220.036	158.654	264.873	736.381	520.752	901.802	1381.23	985.952	1678.63
2	1364.98	1306.85	1420.42	2431.11	2200.10	2641.98	3735.55	3243.68	4165.44
3	3761.76	3712.99	3809.90	5510.47	5288.36	5723.95	7147.55	6620.77	7636.82
4	7210.10	7163.85	7256.04	9214.82	8992.90	9430.97	11478.1	10923.3	12006.6
5	7998.11	7998.11	7998.11	14275.8	14046.7	14501.1	15996.2	15996.2	15996.2

The results shown in Table 1 indicate as expected that the effect of the compressive axial load (P=0.5P_{cr}) is to reduce the natural frequencies whereas the corresponding effect of a tensile load (P=-0.5P_{cr}) is to increase the natural frequencies. It should be noted that the fifth natural frequency for the C-F and C-C boundary conditions shown in Table 1 is unaltered because it corresponds to a pure axial mode for which the axial load is not expected to have any major effect. The theory developed and the results presented demonstrate the importance of axial-bending coupling effects on the free vibration characteristics of axially loaded axial-bending coupled Timoshenko-Ehrenfest beams.

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