On the blocks of the walled Brauer algebra

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1. Introduction

The representation theories over $\mathbb{C}$ of the symmetric group Σ_r and the general linear group $\mathrm{GL}_n(\mathbb{C})$ are related by Schur-Weyl duality. This is the observation that the rth tensor product $V^{\otimes r}$ of the natural representation V of $\mathrm{GL}_n(\mathbb{C})$ has actions both by $\mathrm{GL}_n(\mathbb{C})$ and by Σ_r (the latter by place permutation of the tensor factors) such that the image of each group algebra under its action can be identified with the centraliser algebra of the other [Wey46].

The Brauer algebra $B_r(\delta)$ is an extension of $\mathbb{C}\Sigma_r$ introduced to play the role of $\mathbb{C}\Sigma_r$ in corresponding dualities for symplectic and orthogonal groups [Bra37]. When δ is a positive integer the duality is with $\mathrm{O}_\delta(\mathbb{C})$; when δ is a negative even integer the duality is with $\mathrm{Sp}_{-\delta}(\mathbb{C})$. However, the algebra $B_r(\delta)$ itself is defined for all choices of $\delta \in \mathbb{C}$.

The walled Brauer algebra (also known as the rational Brauer algebra) $B_{r,s}(\delta)$ arises from a third version of Schur-Weyl duality. For $\delta \in \mathbb{N}$, consider the mixed tensor product $V^{\otimes r} \otimes (V^*)^{\otimes s}$ of the natural representation (and its dual) for $\mathrm{GL}_\delta(\mathbb{C})$. There is a subalgebra $B_{r,s}(\delta)$ of the Brauer algebra which acts on this product to give a Schur-Weyl duality with the $\mathrm{GL}_\delta(\mathbb{C})$ action. This algebra was studied by Turaev [Tur89], Koike [Koi89], and Benkart et al [BCH$^+$94].

The symmetric group, Brauer, and walled Brauer algebras may be considered over arbitrary fields. It is known that the respective Schur-Weyl dualities continue to hold in types A and C (when the field is infinite) and are expected to hold for other types. This is well known in type A; see [DDH08] for type C.

Over $\mathbb{C}$ the Brauer and walled Brauer algebras are isomorphic to the corresponding centraliser algebra when $|\delta| >> 0$, and hence must be semisimple. However, for small values of δ the centraliser algebra is only a quotient of the original algebra, and non-semisimple cases can occur.

Until relatively recently the representation theory of the Brauer algebra was ill understood. A precise semisimplicity criterion was given by Rui [Rui05] only very recently. However, in recent work [CDM05,CDM06] it has been shown that the non-semisimple cases have a rich combinatorial structure, which is controlled by the type D Weyl group; at present a structural explanation for this phenomena is lacking. The goal of the present paper is to analyse the representation theory of the walled Brauer algebra in the same manner. This will combine an application of the towers of recollement formalism from [CMPX06], generalised to the cellular setting, with analogues of various explicit calculations for the Brauer algebra in [DWH99]. Then we will introduce a geometric formulation of the combinatorics obtained.

Section 2 introduces the walled Brauer algebra, and starts to show how it is compatible with a cellular version of the basic towers of recollement machinery. This is completed in Section 3, where the cell modules are introduced.

In Section 4 we give a necessary condition (for arbitrary fields and parameter choices) for two simple modules to be in the same block by considering the action of certain central elements in $B_{r,s}$. This gives a necessary condition for semisimplicity; this is shown to give a sufficient condition (except in obviously non-semisimple cases) in Sections 5 and 6 by constructing certain homomorphisms between cell modules. Thus we are able to give a complete semisimplicity criterion in Theorem 6.3.

In Section 7 we are able to refine these results to give a complete description of when two simple modules are in the same block in characteristic zero.

For the Brauer algebra the combinatorial description of blocks in [CDM05] has been reinterpreted [CDM06] in terms of an action of the Weyl group of type D as in Lie theory. In Section 8 we will provide a similar reinterpretation for the walled Brauer, in terms of the Weyl group of type A, but with an unusual choice of dominant weights.

Simple modules for the walled Brauer algebra $B_{r,s}$ can be labelled by certain bipartitions (λ^L, λ^R), and we choose to identify these with elements of $\mathbb{Z}^{r+s}$ (with a fixed origin) by embedding λ^R normally but reflecting λ^L about the two axes passing through the origin. These dominant weights are thus generalised partitions with Young diagram of the general form shown in Figure 1. As in Lie theory, the natural action of Σ_{r+s} on the set of weights has to be shifted, but in this case the shift also depends on the parameter δ.

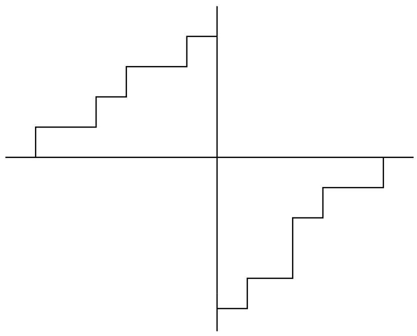

Fig. 1. A dominant weight for $B_{r,s}$

In Section 10 we give a necessary condition for two weights to be in the same block in positive characteristic (i.e. a linkage principle) by replacing W by the corresponding affine Weyl group. This is exactly analogous to the Brauer algebra case.

Although the combinatorics is a little more intricate (as the role of the symmetric group for the Brauer algebra is replaced by a product of symmetric groups), the actual proofs are rather simpler for the walled Brauer algebra than for the Brauer algebra. This paper is also almost entirely self-contained, requiring only a result of Halverson (on the symmetric group content of walled Brauer modules) [Hal96] from the existing walled Brauer literature. Thus it may also be read as an introduction to the methods used in the Brauer algebra papers [CDM05] and [CDM06].

2. The walled Brauer algebra

Fix an algebraically closed field k of characteristic $p \geq 0$, and $\delta \in k$. For $r, s \in \mathbb{N}$, the walled Brauer algebra $B_{r,s}(\delta)$ can be defined as a subalgebra of the ordinary Brauer algebra $B_{r+s}(\delta)$ in the following manner.

Recall that for $n \in \mathbb{N}$ the *Brauer algebra* $B_n(\delta)$ can be defined in terms of a basis of partitions of $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ into pairs. The product AB of two basis elements A and B is obtained by representing each by a graph on $2n$ points, and identifying the vertices $\overline{1}, \ldots, \overline{n}$ of A with the vertices $1, \ldots, n$ of B respectively. This produces a new graph on the vertices $1, \ldots, n$ of A and $\overline{1}, \ldots, \overline{n}$ of B, possibly together with some number (t say) of connected components not connected to any of these vertices. The product AB is then defined to be $\delta^t C$, where C is the basis element corresponding to the graph obtained by removing these connected components.

It is usual to represent basis elements graphically by means of diagrams with n northern nodes numbered 1 to n from left to right, and n southern nodes numbered $\overline{1}$ to $\overline{n}$ from left to right, where each node is connected to precisely one other by a line. Edges connecting a northern and a southern node are called *propagating lines*, and the remainder are called *northern* or *southern arcs*.

It is now easy to realise the *walled Brauer algebra* $B_{r,s}(\delta)$ as a subalgebra of the Brauer algebra $B_{r+s}(\delta)$. Partition the basis diagrams with a wall separating the first r northern nodes and first r southern nodes from the remainder. Then the walled Brauer algebra is the subalgebra with basis those Brauer diagrams such that no propagating edge crosses the wall, and every northern or southern arc does cross the wall. It is easy to verify that the space spanned by such diagrams is indeed a subalgebra. Note that $B_{0,n}(\delta) \cong B_{n,0}(\delta) \cong k\Sigma_n$, the group algebra of the symmetric group Σ_n on n letters. An example of two walled Brauer diagrams and their product is given in Figure 2.

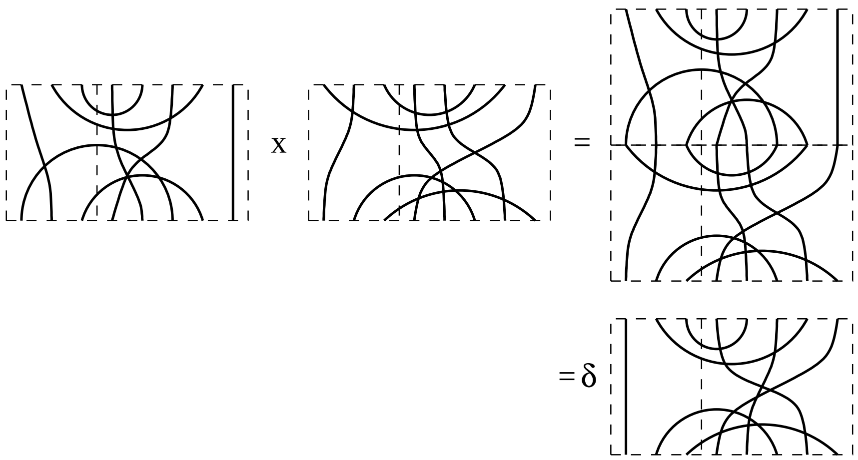

Fig. 2. The product of two basis elements in $B_{3,5}(\delta)$.

We will show that the algebras $B_{r,s}(\delta)$ form cellular analogues of the towers of recollement introduced in[CMPX06]. Roughly, such a tower consists of a family of cellular algebras related by inclusions and idempotent embeddings in a compatible way, such that restriction and induction of cell modules is well-behaved in the tower. For general k and δ we will obtain a similar formalism with quasiheredity replaced by cellularity.

More precisely, there are six conditions labelled (A1–6) in [CMPX06] which are required for a tower of recollement, and we will consider each of these in turn. All but (A2) (concerning quasi-heredity) will turn out to hold (if suitably interpreted) for arbitrary k and δ, and in the general case we will also be able to replace (A2) by a cellular analogue. Henceforth we will suppress all δs in our notation when no ambiguity can occur.

Suppose that k is arbitrary, with $r, s > 0$ and $\delta \neq 0$, and let $e_{r,s} \in B_{r,s}$ be δ^{-1} times the diagram with one northern arc connecting r and $r+1$, one southern arc connecting $\overline{r}$ and $\overline{r+1}$ and all remaining edges being propagating lines from i to $\overline{i}$. (The element $e_{3,5}$ is illustrated in the left-hand side of Figure 3; the element $e_{3,5,2}$ in the right-hand side of the Figure will be discussed later.) Clearly $e_{r,s}$ is an idempotent in $B_{r,s}$.

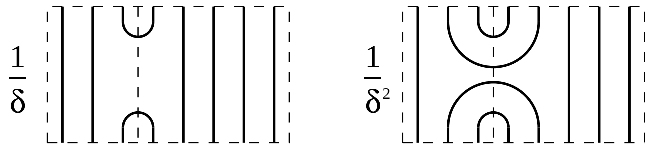

Fig. 3. The elements $e_{3,5}$ and $e_{3,5,2}$ in $B_{3,5}$.

If $\delta = 0$ then we cannot define the idempotent $e_{r,s}$ as above. However, if r or s is at least 2 then we can define an alternative idempotent $\tilde{e}_{rs}$ as illustrated in Figure 4.

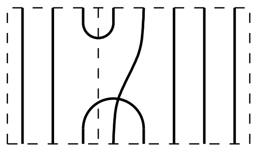

Fig. 4. The element $\tilde{e}_{3,5}$ in $B_{3,5}$.

Our first result is

Proposition 2.1 (A1) *If $\delta \neq 0$ then for each $r, s > 0$ we have an algebra isomorphism*

$$\Phi_{r,s} : B_{r-1,s-1} \longrightarrow e_{r,s} B_{r,s} e_{r,s}.$$

If $\delta = 0$ and $r \geq 2$ or $s \geq 2$ we have an algebra isomorphism

$$\tilde{\Phi}_{r,s} : B_{r-1,s-1} \longrightarrow \tilde{e}_{r,s}B_{r,s}\tilde{e}_{r,s}.$$

PROOF. We prove the first statement, the second is very similar.

Given a diagram D in $B_{r-1,s-1}$ we define a new diagram D' in $B_{r,s}$ by adding two propagating lines immediately before and after the wall in D, so that r is connected to $\overline{r}$ and $r+1$ to $\overline{r+1}$. It is clear that the map taking D to $e_{r,s}D'e_{r,s}$ (as illustrated in Figure 5) is an injective algebra homomorphism, and it is easy to verify that the image is precisely $e_{r,s}B_{r,s}e_{r,s}$.

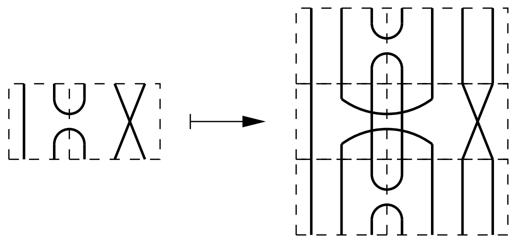

Fig. 5. An example of the action of the map $\Phi_{r,s}$.

Remark 2.2 *The roles of $e_{r,s}$ and $\tilde{e}_{r,s}$ are very similar, and so we will henceforth write $e_{r,s}$ for both types of idempotent (and similarly write Φ for the isomorphisms in (A1)). This will allow us to deal with the cases $\delta = 0$ and $\delta \neq 0$ simultaneously. In proofs we will work with the original idempotent; the obvious (trivial) modifications are left to the reader.*

We wish to define a sequence of idempotents $e_{r,s,i}$ in $B_{r,s}$. Set $e_{r,s,0} = 1$, and for $1 \leq i \leq \min(r,s)$ set $e_{r,s,i} = \Phi_{r,s}(e_{r-1,s-1,i-1})$. Note that when $\delta = 0$ and $r = s$ the element $e_{r,r,r}$ is not defined.

To these elements we associate quotients $B_{r,s,i} = B_{r,s}/B_{r,s}e_{r,s,i}B_{r,s}$. When $\delta \neq 0$ we can give an alternative description of the $e_{r,s,i}$ (via our explicit description of $\Phi_{r,s}$) as δ^{-i} times the diagram with i northern (respectively southern) arcs connecting $r-t$ to $r+1+t$ (respectively $\overline{r-t}$ to $\overline{r+1+t}$ for $0 \leq t \leq i = 1$ and the remaining edges all propagating lines connecting u to $\overline{u}$ for some u. The element $e_{3,5,2}$ is illustrated in the right-hand side of Figure 3. A similar description can be given in the case $\delta = 0$.

We define the *propagating vector* of a diagram D to be the pair (a,b) where D has a propagating lines to the left of the wall, and b to the right. Note that if we multiply two diagrams with propagating vectors (a_1,b_1) and (a_2,b_2) then the result must have propagating vector (a,b) with $a \leq \min(a_1,a_2)$ and $b \leq \min(b_1,b_2)$.

Set $J_i = B_{r,s}e_{r,s,i}B_{r,s}$ and consider the sequence of ideals

$$\cdots \subset J_2 \subset J_1 \subset J_0 = B_{r,s}. \tag{1}$$

Proposition 2.3 *The ideal J_i has a basis of all diagrams with propagating vector (a,b) for some $a \leq r-i$ and $b \leq s-i$. In particular the section*

$$J_i/J_{i+1}$$

in the filtration (1) has a basis of all diagrams with propagating vector $(r-i, s-i)$.

PROOF. This is a routine exercise (see for example [MS94, Corollary 1.1]).

In particular we have that

$$B_{r,s}/J_1 \cong k(\Sigma_r \times \Sigma_s). \tag{2}$$

We will denote $\Sigma_r \times \Sigma_s$ by $\Sigma_{r,s}$.

We will need some basic facts about symmetric group representations; details can be found in [Jam78]. For each partition λ of n, we can define a *Specht module* S^λ for $k\Sigma_n$. We say that a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is *p-regular* if either $p > 0$ and there is no $1 \leq i \leq r$ such that

$$\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p}$$

or $p = 0$. Then the heads D^λ of the Specht modules S^λ for λ p-regular form a complete set of inequivalent simple $k\Sigma_n$-modules.

As k is algebraically closed (and so certainly a splitting field for Σ_r and Σ_s), the simple modules for $k\Sigma_{r,s}$ are precisely those modules of the form $D \boxtimes D'$ where D is a simple $k\Sigma_r$-module, and D' a simple $k\Sigma_s$-module [CR81, (10.33) Theorem], and so can be labelled by pairs (λ^L, λ^R) where λ^L is a p-regular partition of r and λ^R is a p-regular partition of s. We will denote the set of such pairs by $\Lambda^{r,s}_{reg}$.

If $p = 0$ or $p > \max(r,s)$ then the group algebra $k\Sigma_{r,s}$ is semisimple, and $\Lambda^{r,s}_{reg}$ consists of all pairs of partitions of r and s. If (λ^L, λ^R) is such a pair we will denote this by $(\lambda^L, \lambda^R) \vdash (r,s)$, and denote the set of such by $\Lambda^{r,s}$. We will call elements of $\Lambda^{r,s}$ *weights*. We will say that k is $\Sigma_{r,s}$-*semisimple* (or just Σ-*semisimple* whenever this does not cause confusion) when $p = 0$ or $p > \max(r,s)$.

Let $\Lambda_{r,s}$ denote an indexing set for the simple $B_{r,s}$-modules. From Proposition 2.1 (A1) we have an exact *localisation* functor

$$F_{r,s} : B_{r,s}\text{-mod} \longrightarrow B_{r-1,s-1}\text{-mod}$$

coming from the relevant idempotent, which takes a $B_{r,s}$-module M to $e_{r,s}M$. There is a corresponding right exact *globalisation* functor $G_{r-1,s-1}$ in the opposite direction which takes a $B_{r-1,s-1}$-module N to $B_{r,s}e_{r,s}\otimes_{e_{r,s}B_{r,s}e_{r,s}} N$. By standard properties of localisation functors [Gre80] and (2) we have for $r,s>0$ that

$$\Lambda_{r,s} = \Lambda_{r-1,s-1} \sqcup \Lambda_{reg}^{r,s}.$$

As $B_{r,0} \cong B_{0,r} \cong k\Sigma_r$ we deduce

Proposition 2.4 *If $\delta \neq 0$ or $r \neq s$, then*

$$\Lambda_{r,s} = \coprod_{i=0}^{\min(r,s)} \Lambda_{reg}^{r-i,s-i}.$$

It will be convenient to have the following alternative way of describing walled Brauer diagrams, in terms of partial one-row diagrams (confer for example [GL96]). Given a walled Brauer diagram $D \in B_{r,s}$ with t northern and t southern arcs, we will write $D = X_{v,w,\sigma}$ in the following manner. Let v represent the configuration of northern arcs in D, and w represent the configuration of southern arcs. Then D is uniquely specified by giving $\sigma \in \Sigma_{r-t,s-t}$ (regarded as a subset of Σ_{r+s} in the obvious way) such that $\sigma(i) = j$ if the ith northern node on a propagating line is connected to node j. We denote the set of elements v arising thus by $\mathcal{V}_{r,s,t}$ (and by abuse of notation use the same set to refer to the elements w that arise), and call this the set of *partial one-row (r,s,t) diagrams*, or just *partial one-row diagrams* when r, s, and t are clear from context.

The tower of recollement formalism in [CMPX06] is realised in the context of quasihereditary algebras. However, it is easy to recast it in the more general cellular algebra setting, albeit at the expense of some additional hypotheses.

The notion of a cellular algebra was introduced by Graham and Lehrer [GL96] in terms of an involution and a basis with very special properties. However, for our purposes the alternative (equivalent) definition given later by König and Xi [KX99] in terms of ideals and iterated inflations (together with an involution) will ease our exposition. (These two approaches also have a hybrid version in the tabular framework [GM07], although we will not consider this here.)

Given a k-algebra C, a k-vector space V, and a bilinear form $\phi : V\otimes V \longrightarrow C$, König and Xi define a (possibly nonunital) algebra structure on $A_{C,V}^{\phi} = V\otimes V\otimes C$ by setting the product of two basis elements to be

$$(a\otimes b\otimes x).(c\otimes d\otimes y) = a\otimes d\otimes x\phi(b,c)y.$$

If i is an involution on C with $i(\phi(v, w)) = \phi(v, w)$ then there is an involution j on $A^{\phi}_{C,V}$ given by

$$j(a \otimes b \otimes x) = b \otimes a \otimes i(x).$$

The algebra $A^{\phi}_{C,V}$ is called the *inflation of C along V*.

König and Xi also need to define algebra structures on sums of the form $C \oplus D$ where C is a (possibly nonunital) algebra and D is a unital algebra, extending the two algebra structures and any involutions which they possess in a compatible way. This is elementary but rather involved; details can be found in [KX99,KX01]. Iterating such constructions forms *iterated inflations*. The key result is that the inflation of a cellular algebra is again cellular [KX99, Proposition 3.3] In fact, carrying out this construction on full matrix algebras gives precisely the class of cellular algebras [KX99, Theorem 4.1].

In [KX01, Section 5] these constructions were used to give a simple proof that the Brauer algebra is cellular, by constructing it as an iterated inflation of symmetric group algebras. We will modify this argument to sketch a proof of a similar result for the walled Brauer algebras involving the group algebras $k\Sigma_{m,n}$. This result has been proved using tabular methods in [GM07]. An explicit construction of a cellular basis can be found in [Eny02].

There is an obvious involution i on $B_{r,s}$ given by inverting diagrams (so that northern nodes become southern nodes and vice versa).

Lemma 2.5 *For $l \geq 0$ the algebra J_l/J_{l+1} is isomorphic to an inflation*

$$V_l \otimes V_l \otimes k\Sigma_{r-l,s-l}$$

of $k\Sigma_{r-l,s-l}$ along a free k-module V_l of rank $|\mathcal{V}_{r,s,l}|$, with respect to some bilinear form ϕ (described below).

PROOF. This is a very slight modification of the corresponding proof for Brauer algebras in [KX01, Lemma 5.3]. Let V_l have basis $\mathcal{V}_{r,s,l}$, and let the map

$$V_l \otimes V_l \otimes k\Sigma_{r-l,s-l} \longrightarrow J_l/J_{l+1}$$

be given by $v \otimes w \otimes \sigma \longmapsto X_{v,w,\sigma}$. To define the value of $\phi(v, w)$ consider a product $X_{u,v,\sigma_1}X_{w,x,\sigma_2}$ for some $u, x \in \mathcal{V}_{r,s,l}$ and $\sigma_1, \sigma_2 \in \Sigma_{r-l,s-l}$. If this product does not have propagating vector $(r - l, s - l)$ then set $\phi(v, w) = 0$. Otherwise $\phi(v, w) = \delta^t\sigma$ where t is the number of closed loops in $X_{u,v,\sigma_1}X_{w,x,\sigma_2}$, and σ is the unique permutation such that

$$X_{u,v,\sigma_1}X_{w,x,\sigma_2} = \delta^t X_{u,x,\sigma_1\sigma\sigma_2}.$$

Note that this definition is independent of the choice of u, x, σ_1, σ_2. It is now easy to verify that we have the desired algebra isomorphism.

Arguing exactly as in [KX01] one can then show that

Proposition 2.6 *The walled Brauer algebra $B_{r,s}$ is an iterated inflation of group algebras of the form $\Sigma_{r-l,s-l}$ for $0 \leq l \leq \min(r,s)$ along V_l.*

The group algebras of the symmetric groups are cellular [GL96] (indeed they were the motivating example for cellularity), with cell modules given by the Specht modules S^λ. From this follows

Theorem 2.7 *(i) The walled Brauer algebra $B_{r,s}$ is cellular with a cell module $\Delta_{r,s}(\lambda^L, \lambda^R)$ for each $(\lambda^L, \lambda^R) \in \Lambda^{r-l,s-l}$ with $0 \leq l \leq \min(r,s)$.*
(ii) If $\delta \neq 0$ or $r \neq s$ then the simple modules are indexed by all pairs $(l, \lambda^L, \lambda^R)$ where $0 \leq l \leq \min(r,s)$ and $(\lambda^L, \lambda^R) \in \Lambda^{r-l,s-l}_{reg}$.
(iii) If $\delta = 0$ and $r = s$ we get the same indexing set for simples as in (ii), but with the single simple corresponding to $l = \min(r,s)$ omitted.

PROOF. From the basis definition in [GL96] (or see [KX99, Proposition 6.15]) it is clear that a cell basis for $k\Sigma_{r,s}$ can be obtained as a product of cell bases for $k\Sigma_r$ and $k\Sigma_s$, and hence $k\Sigma_{r,s}$ is cellular with cell modules of the form $M \boxtimes N$ where M, N are cell modules for $k\Sigma_r$, $k\Sigma_s$, respectively. Part (i) now follows from Proposition 2.6.

Part (ii) has already been shown in Proposition 2.4. The modification needed when $\delta = 0$ and $r = s$ follows as in [KX01, Corollary 5.8], or from the known structure of the cellular algebra $B_{1,1}$, as this is identical to the Temperley-Lieb algebra $\mathrm{TL}_2(0)$.

Corollary 2.8 (A2) *If k is Σ-semisimple, and either $\delta \neq 0$ or $\delta = 0$ and $r \neq s$, then the algebra $B_{r,s}$ is quasi-hereditary, with heredity chain induced by the idempotent $e_{r,s,i}$. In all other cases $B_{r,s}$ is not quasihereditary.*

PROOF. This follows immediately from the fact that a cellular algebra is quasihereditary precisely when there are the same number of simples as cell modules. (The quasi-hereditary structure can also be proved directly as in [CMPX06, Proposition 2.10].)

When $(\lambda^L, \lambda^R) \in \Lambda^{r-l,s-l}_{reg}$ for some $l \geq 0$ (with the same exception as in Theorem 2.7(iii)) we shall denote the corresponding simple $B_{r,s}$-module by

$L_{r,s}(\lambda^L, \lambda^R)$. By standard cellular theory this arises as the head of the cell module $\Delta_{r,s}(\lambda^L, \lambda^R)$.

The tower of recollement formalism relies on the interplay between two different ways of relating algebras: localisation/globalisation and induction/restriction. Thus we also need a way of identifying one walled Brauer algebra as a subalgebra of another. We will do this in two different (but closely related) ways.

For $r > 0$ we may identify $B_{r-1,s}$ as a subalgebra of $B_{r,s}$, and similarly for $B_{r,s-1}$ if $s > 0$. There are a variety of ways of doing this, but we will use

Lemma 2.9 (A3) *The map Ψ_L (respectively Ψ_R) obtained by inserting a propagating line immediately to the left (respectively right) of the wall in a $B_{r-1,s}$ (respectively $B_{r,s-1}$) diagram extends to an algebra inclusion of $B_{r-1,s}$ (respectively of $B_{r,s-1}$) inside $B_{r,s}$.*

We thus have two restriction functors, $\mathrm{res}^L_{r,s}$ from $B_{r,s}$-mod to $B_{r-1,s}$-mod and $\mathrm{res}^R_{r,s}$ from $B_{r,s}$-mod to $B_{r,s-1}$-mod, and the corresponding right adjoint induction functors $\mathrm{ind}^L_{r,s}$ from $B_{r,s}$-mod to $B_{r+1,s}$-mod and $\mathrm{ind}^R_{r,s}$ from $B_{r,s}$-mod to $B_{r,s+1}$-mod. We will often omit the subscripts from these functors when this is unambiguous. Our choice of algebra inclusions is motivated by the following compatibility relation between restriction and localisation.

Proposition 2.10 (A4) *For all k with $r, s > 0$ (and $\delta \neq 0$ if $r = s = 1$) we have that*

$$B_{r,s}e_{r,s} \cong B_{r-1,s}$$

as a $(B_{r-1,s}, B_{r-1,s-1})$-bimodule, where the right action of $B_{r-1,s-1}$ on $B_{r,s}e_{r,s}$ is given via the isomorphism in (A1), and the left action of $B_{r-1,s}$ is given via the map Ψ_L. There is a similar isomorphism

$$B_{r,s}e_{r,s} \cong B_{r,s-1}$$

as a $(B_{r,s-1}, B_{r-1,s-1})$-bimodule replacing Ψ_L by Ψ_R.

PROOF. We will consider the first case, the second is similar. Consider a diagram D in $B_{r,s}e_{r,s}$. As a $(B_{r-1,s}, B_{r-1,s-1})$-bimodule this can be represented schematically as in the left-hand diagram in Figure 6, where the shaded area above the diagram indicates the northern nodes acted on via $B_{r-1,s}$ and below indicates the southern nodes acted on via $B_{r-1,s-1}$. Note that the node r (marked A) in the diagram is not acted on from above. We can convert D into a diagram for $B_{r-1,s}$ by removing the southern arc shown, and deforming the line terminating at A so that it terminates at the point B in the right-hand diagram in Figure 6. It is easy to verify that this new diagram is indeed a walled Brauer diagram, and lies in $B_{r-1,s}$. This gives the desired bimodule isomorphism.

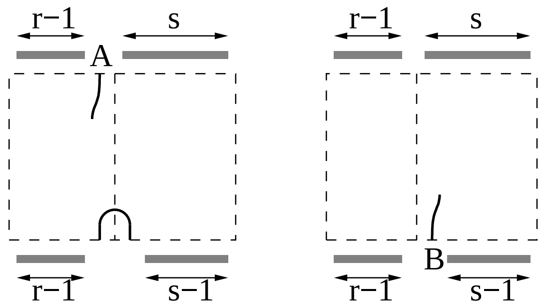

Fig. 6. Realising the bimodule isomorphism between $B_{r,s}e_{r,s}$ and $B_{r-1,s}$.

Note that we have a choice of many different towers in our construction, because at each stage we can chose either of the two inclusions.

3. Cell modules for the walled Brauer algebra

To complete our verification of the tower of recollement axioms, and their cellular analogues, we next analyse further the structure of the cell modules.

We start by giving an explicit construction of the cell modules (given a corresponding construction of Specht modules for the symmetric groups). For concreteness we will fix $\mathcal{B}(\lambda^L, \lambda^R)$ to be the tensor product of the integral bases for S^{λ^L} and S^{λ^R} given in [JK81].

Proposition 3.1 *If* $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$ *then the module* $\Delta_{r,s}(\lambda^L, \lambda^R)$ *has a basis given by*

$$\{X_{v,1,id} \otimes x : v \in \mathcal{V}_{r,s,t},\ x \in \mathcal{B}(\lambda^L, \lambda^R)\}$$

where $X_{v,1,id}$ *is a diagram with* $r+s$ *northern and southern nodes, and* 1 *denotes the (fixed) southern half-diagram with arcs configured as in the southern half of* $e_{r,s,t}$.

PROOF. This follows from the definition of cell ideals [KX98, Definition 3.2 and the following proof] and the construction of cell ideals in inflations [KX99, Section 3.3], together with the explicit description of the inflations in Lemma 2.5.

We will denote the space spanned by the elements $X_{v,1,id}$ with $v \in \mathcal{V}_{r,s,t}$ by $I^t_{r,s}$.

Note that the action of a walled Brauer diagram on this basis is by concatenation from above, except that products with too few propagating lines are set equal to zero. As the action of the diagram may induce a permutation $\sigma \in \Sigma_{r,s}$ of the propagating lines this must be removed, and this is done by passing σ through the tensor product to act on the basis element x of $S^{\lambda^L} \boxtimes S^{\lambda^R}$ in the natural manner. So if $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$ then

$$\Delta_{r,s}(\lambda^L, \lambda^R) \cong I^t_{r,s} \otimes_{\Sigma_{r-t,s-t}} (S^{\lambda^L} \boxtimes S^{\lambda^R}).$$

Corollary 3.2 *For $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$ the cell module $\Delta_{r,s}(\lambda^L, \lambda^R)$ can be identified with the module*

$$B_{r,s}e_{r,s,t} \otimes_{e_{r,s,t}B_{r,s}e_{r,s,t}} S^{\lambda^L} \boxtimes S^{\lambda^R}$$

(when $e_{r,s,t}$ exists).

PROOF. This follows as in the proof of [CMPX06, Proposition 2.10].

We say that $(\lambda^L, \lambda^R) \leq (\mu^L, \mu^R)$ if $(\lambda^L, \lambda^R) = (\mu^L, \mu^R)$ or $(\lambda^L, \lambda^R) \in \Lambda^{a,b}$ and $(\mu^L, \mu^R) \in \Lambda^{a-t,b-t}$ for some $0 \leq t < \min(a, b)$. As this ordering is compatible with the cellular structure, all composition factors of $\Delta_{r,s}(\lambda^L, \lambda^R)$ are labelled by weights (μ^L, μ^R) with $(\mu^L, \mu^R) \leq (\lambda^L, \lambda^R)$. It follows from the construction that

$$\Delta_{r,s}(\lambda^L, \lambda^R) \cong S^{\lambda^L} \boxtimes S^{\lambda^R}$$

if $(\lambda^L, \lambda^R) \in \Lambda^{r,s}$, the lift of a Specht module for the quotient algebra

$$B_{r,s}/J_1 \cong k\Sigma_{r,s}.$$

As our globalisation and localisation functors (when they exist) are compatible with the cell chain, we have by Corollary 3.2 that

$$G_{r,s}(\Delta_{r,s}(\lambda^L, \lambda^R)) \cong \Delta_{r+1,s+1}(\lambda^L, \lambda^R) \tag{3}$$

for all $(\lambda^L, \lambda^R) \in \Lambda_{r,s}$ and

$$F_{r,s}(\Delta_{r,s}(\lambda^L, \lambda^R)) \cong \begin{cases} \Delta_{r-1,s-1}(\lambda^L, \lambda^R) & \text{if } (\lambda^L, \lambda^R) \in \Lambda_{r-1,s-1} \\ 0 & \text{if } (\lambda^L, \lambda^R) \in \Lambda^{r,s}. \end{cases}$$

As $F_{r,s}$ is exact we also have that

$$F_{r,s}(L_{r,s}(\lambda^L, \lambda^R)) \cong \begin{cases} L_{r-1,s-1}(\lambda^L, \lambda^R) & \text{if } (\lambda^L, \lambda^R) \in \Lambda_{r-1,s-1} \\ 0 & \text{if } (\lambda^L, \lambda^R) \in \Lambda^{r,s}. \end{cases}$$

The compatibility of induction/restriction with localisation/globalisation given in Proposition 2.10 (A4) immediately implies that

$$\operatorname{res}^{\dagger}(G_{r,s}(\Delta_{r,s}(\lambda^L,\lambda^R)) \cong \operatorname{ind}^{\ddagger}\Delta_{r,s}(\lambda^L,\lambda^R) \tag{4}$$

for all $(\lambda^L,\lambda^R)\in\Lambda_{r,s}$ where $(\dagger,\ddagger)$ represents either (L,R) or (R,L).

Note that the only case where localisation and globalisation functors do not exist occurs when $\delta=0$ and $r=s$. In this case we do not have $F_{1,1}$ and $G_{0,0}$.

The remaining two axioms (A5) and (A6) concern the behaviour of cell modules under induction and restriction.

We first consider the restriction rules for cell modules. For this we need to consider the action of certain special elements in the walled Brauer algebra. Write $E_{i,j}$ for the walled Brauer diagram with edges between t and $\bar{t}$ for $t\neq i,j$, and arcs between i and j, and $\bar{i}$ and $\bar{j}$. Note that $B_{r,s}$ is generated by the elements $E_{i,j}$ (with $1\leq i\leq r$ and $r+1\leq j\leq r+s$) and the group $\Sigma_{r,s}$ (identified with the set of diagrams with no northern or southern arcs). (In fact, $B_{r,s}$ is generated by $\Sigma_{r,s}$ together with just one $E_{i,j}$.)

Consider the action of $E_{i,j}$ on an element $X_{w,1,id}\otimes x\in\Delta_{r,s}(\lambda^L,\lambda^R)$. There are four possible cases:

(a) i and j are connected in w. In this case the action of $E_{i,j}$ creates a closed loop while leaving the underlying diagram unchanged. Hence

$$E_{i,j}(X_{w,1,id}\otimes x)=\delta(X_{w,1,id}\otimes x). \tag{5}$$

(b) i and j are free vertices in w. In this case the action of $E_{i,j}$ creates an extra northern arc, hence reducing the number of propagating lines by two. Hence

$$E_{i,j}(X_{w,1,id}\otimes x)=0. \tag{6}$$

(c) One of the vertices i and j is free, and the other is joined to some vertex m in w. Suppose that i is the free vertex. Then the action of $E_{i,j}$ is illustrated schematically in Figure 7 (where we have omitted all lines which do not concern us). From this it is clear that

$$E_{i,j}(X_{w,1,id}\otimes x)=(i,m)(X_{w,1,id}\otimes x) \tag{7}$$

where (i,m) denotes the transposition swapping i and m in $\Sigma_{r,s}\subset B_{r,s}$. A similar result holds if we reverse the roles of i and j.

(d) i and j are unconnected, but not free vertices in w. Say i is joined to l and j is joined to m. This case is illustrated schematically in Figure 8. From this it is clear that

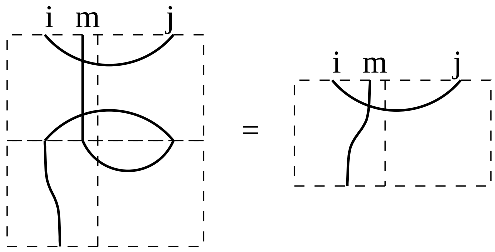

Fig. 7. A schematic example of the action of $E_{i,j}$ in case (c)

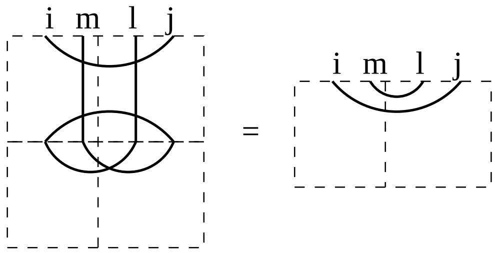

Fig. 8. A schematic example of the action of $E_{i,j}$ in case (d)

$$E_{i,j}(X_{w,1,id} \otimes x) = (i,m)(X_{w,1,id} \otimes x) = (j,l)(X_{w,1,id} \otimes x). \tag{8}$$

We will denote the Young diagram associated to a partition λ by $[\lambda]$. For a partition λ, recall that the set of *removable boxes* are those which can be removed (singly) from $[\lambda]$ such that the result is the Young diagram of a partition. Similarly the set of *addable boxes* are those which can be added (singly) to $[\lambda]$ such that the result is the Young diagram of a partition. We denote these sets by $\mathrm{rem}(\lambda)$ and $\mathrm{add}(\lambda)$ respectively, and given a box in $\mathrm{rem}(\lambda)$ or $\mathrm{add}(\lambda)$ denote the associated partition obtained by addition or subtraction by $\lambda \pm \Box$.

Given two partitions λ and μ of n, we say that μ is *dominated* by λ (written $\mu \lhd \lambda$) if for all $i \geq 1$ we have

$$\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_j.$$

That is, the Young diagram for λ can be obtained from that for μ by moving some of the boxes to earlier rows in the diagram. We will extend this to give a partial order on pairs of partitions in a very restricted form by saying that $(\lambda^L, \lambda^R) \lhd^L (\mu^L, \mu^R)$ if $\lambda^R = \mu^R$ and $\lambda^L \lhd \mu^L$, and similarly for $\lhd^R$ reversing the roles of L and R.

Given a family of modules M_i we will write $\biguplus_i M_i$ to denote some module with a filtration whose quotients are exactly the M_i, each with multiplicity one. This is not uniquely defined as a module, but the existence of a module with such a filtration will be sufficient for our purposes.

We can now prove

Theorem 3.3 *(i) Suppose that* $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$. *If* $t = 0$ *then*

$$\operatorname{res}^L_{r,s} \Delta_{r,s}(\lambda^L, \lambda^R) \cong \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R).$$

If $t > 0$ *then we have a short exact sequence*

$$0 \longrightarrow \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R) \longrightarrow \operatorname{res}^L_{r,s} \Delta_{r,s}(\lambda^L, \lambda^R)$$
$$\longrightarrow \biguplus_{\Box \in \operatorname{add}(\lambda^R)} \Delta_{r-1,s}(\lambda^L, \lambda^R + \Box) \longrightarrow 0$$

where the left-hand sum equals 0 *if* $\lambda^L = \emptyset$.
(ii) In each of the filtered modules which arise in (i), the filtration can be chosen so that the weights labelling successive quotients are ordered by $\lhd^L$ *or* $\lhd^R$*, with the top quotient maximal among these. When* k *is* Σ*-semisimple the* $\uplus$ *all become direct sums.*
(iii) There is a similar result for $\operatorname{res}^R_{r,s}$ *replacing* $\operatorname{rem}(\lambda^L)$ *by* $\operatorname{rem}(\lambda^R)$ *and* $\operatorname{add}(\lambda^R)$ *by* $\operatorname{add}(\lambda^L)$.

PROOF. We prove the result for $\operatorname{res}^L_{r,s}$; the right-hand case is similar. Our proof is very similar to that for the ordinary Brauer algebra in [DWH99, Theorem 4.1].

Let W be the subspace of $\Delta_{r,s}(\lambda^L, \lambda^R)$ spanned by elements of the form $X_{w,1,id} \otimes x$ where the node in $X_{w,1,id}$ numbered r is on a propagating line. Recall our realisation of $B_{r-1,s}$ inside $B_{r,s,}$. It is clear that the elements of $\Sigma_{r-1,s}$, and the $E_{i,j}$ with $1 \leq i < r$ and $r+1 \leq j \leq r+s$ preserve the space W, and hence W is a $B_{r-1,s}$-submodule.

We will show that

$$W \cong \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R).$$

When $t = 0$ the space W is the whole of $\Delta_{r,s}(\lambda^L, \lambda^R)$, and so this will complete the proof of (i) in that case. We have

$$I^t_{r-1,s} \otimes_{\Sigma_{r-1-t,s-t}} \operatorname{res}^{\Sigma_{r-t,s-t}}_{\Sigma_{r-t-1,s-t}}(S^{\lambda^L} \boxtimes S^{\lambda^R})$$
$$\cong I^t_{r-1,s} \otimes_{\Sigma_{r-1-t,s-t}} \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} S^{\lambda^L - \Box} \boxtimes S^{\lambda^R}$$
$$\cong \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} I^t_{r-1,s} \otimes_{\Sigma_{r-1-t,s-t}} S^{\lambda^L - \Box} \boxtimes S^{\lambda^R} \cong \biguplus_{\Box \in \operatorname{rem}(\lambda^L)} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R). \tag{9}$$

By [Jam78, Theorem 9.3], the restriction of a Specht module satisfies condition (ii) above, which will thus be inherited by our filtered module. Thus it is enough to show that W is isomorphic to

$$I^t_{r-1,s} \otimes_{\Sigma_{r-1-t,s-t}} \operatorname{res}^{\Sigma_{r-t,s-t}}_{\Sigma_{r-t-1,s-t}}(S^{\lambda^L} \boxtimes S^{\lambda^R}).$$

Given a diagram $d = X_{w,1,id}$ with a propagating line from node r, let $\bar{\phi}(d)$ be the diagram obtained by deleting this line. We claim that the map

$$\phi : W \longrightarrow I^t_{r-1,s} \otimes_{\Sigma_{r-1-t,s-t}} \operatorname{res}^{\Sigma_{r-t,s-t}}_{\Sigma_{r-t-1,s-t}}(S^{\lambda^L} \boxtimes S^{\lambda^R})$$

given by $d \otimes x \longmapsto \bar{\phi}(d) \otimes x$ provides the desired isomorphism. Note that as vector spaces the isomorphism is clear, as $\bar{\phi}$ is a bijection. Thus it is enough to show that ϕ commutes with the action of $\Sigma_{r-1,s}$ and $E_{i,j}$ with $1 \leq i < r$ and $r+1 \leq j \leq r+s$. That ϕ commutes with the $\Sigma_{r-1,s}$ action is clear, and by our discussion of the cases (a)–(d) above, this is also the case for the $E_{i,j}$ action.

It remains to show that when $t > 0$ the quotient

$$V = \Delta_{r,s}(\lambda^L, \lambda^R)/W \cong \biguplus_{\Box \in \operatorname{add}(\lambda^R)} \Delta_{r-1,s}(\lambda^L, \lambda^R + \Box).$$

Arguing as in (9) (using [Jam78, 17.14] which gives (ii) for induction of Specht modules), it is easy to see that it is enough to show that

$$V \cong I^{t-1}_{r-1,s} \otimes_{\Sigma_{r--t,s-t+1}} \operatorname{ind}^{\Sigma_{r-t,s-t+1}}_{\Sigma r-t,s-t}(S^{\lambda^L} \boxtimes S^{\lambda^R}).$$

We will need the following explicit realisation of the induced module. Let $a = r-t$ and $b = s-t$. For $a+1 \leq i \leq a+b$ let τ_i be the transposition $(i, a+b+1)$, and let $\tau_{a+b+1} = 1$. These elements form a set of coset representatives for $\Sigma_{a,b+1}/\Sigma_{a,b}$. If A is a basis for $S^{\lambda^L} \boxtimes S^{\lambda^R}$ then $A \times \{a+1, \ldots, a+b+1\}$ can be regarded as a basis for $\operatorname{ind}^{\Sigma_{a,b+1}}_{\Sigma_{a,b}}(S^{\lambda^L} \boxtimes S^{\lambda^R})$. The action of $\sigma \in \Sigma_{a,b+1}$ on such a basis element (v, j) is given by

$$\sigma(v,j) = ((\tau_l \sigma \tau_j)v, l) \tag{10}$$

where l is the unique value such that $\tau_l \sigma \tau_j \in \Sigma_{a,b}$.

Consider the group algebra $k\Sigma_{a,b+1}$ as a subset of the walled Brauer algebra $B_{a,b+1}$ in the usual way, and suppose that we use a diagrammatic notation for representing the action of $\Sigma_{a,b}$ on $S^{\lambda^L} \boxtimes S^{\lambda^R}$ where the action of the group $\Sigma_{a,b}$ is via the $a+b$ propagating lines. We will wish to represent the action of $\Sigma_{a,b+1}$ on $\operatorname{ind} S^{\lambda^L} \boxtimes S^{\lambda^R}$ in a similar manner, with the aid of a dummy node at the right-hand end of the southern edge of a diagram. Given a basis element v for $S^{\lambda^L} \boxtimes S^{\lambda^R}$, denote the element (v, j) by a diagram of the form shown in Figure 9, where node $*$ denotes the jth northern node (from the left) on a propagating line, all northern arcs have been suppressed for simplicity, and the only lines that cross occur when a line crosses the line from $*$. The basis element of v is symbolically attached to the first $a + b$ southern nodes on propagating lines as usual, and the final node is a dummy node. It is now a routine exercise to verify that this representation coincides with the action in (10) under concatenation of diagrams, and so is a diagrammatic realisation of the induced module.

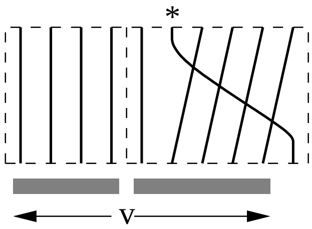

Fig. 9. Realising the element (v, j) as a diagram.

Given a diagram $d = X_{w,1,id}$ where w has a northern arc connecting node r to some other node i say, we form a new diagram $\psi(d)$ by deleting this arc, and replacing it by a propagating line from i to a new node at the right-hand end of the southern edge. Note that this new edge may cross over some of the original propagating lines, and that there is a unique permutation in $\Sigma_{a,b+1}$ which transforms it into an element of the form $X_{w,1,id}$. We claim that the map

$$\psi : V \longrightarrow I^{t-1}_{r-1,s} \otimes_{\Sigma_{r-t,s-t+1}} \operatorname{ind}^{\Sigma_{a,b+1}}_{\Sigma a,b}(S^{\lambda^L} \boxtimes S^{\lambda^R})$$

given by

$$d \otimes x \longmapsto \psi(d) \otimes (x, a + b + 1)$$

(where we represent $(x, a + b + 1)$ diagrammatically as above) gives the desired $B_{r-1,s}$-isomorphism. First note that as $\bar{\psi}$ is a $(b + 1)$ to 1 map, and

$$\dim \operatorname{ind}^{\Sigma_{a,b+1}}_{\Sigma a,b}(S^{\lambda^L} \boxtimes S^{\lambda^R}) = (b + 1) \dim(S^{\lambda^L} \boxtimes S^{\lambda^R})$$

the map ψ is between two spaces of the same dimension.

We next show that ψ is onto. A basis for $I^{t-1}_{r-1,s} \otimes \operatorname{ind}^{\Sigma_{a,b+1}}_{\Sigma_{a,b}}(S^{\lambda^L} \boxtimes S^{\lambda^R})$ is given by $\{X_{w,1,id} \otimes (x,j)\}$ where x runs over a basis for $S^{\lambda^L} \boxtimes S^{\lambda^R}$. From the explicit description of our action on the induced module, we can easily see that there exists some second basis element x' for $S^{\lambda^L} \boxtimes S^{\lambda^R}$ such that $\sigma_j(x,j) = (x', a+b+1)$, as $\tau_{a+b+1}\sigma_j\tau_j \in \Sigma_{a,b}$. Therefore

$$X_{w,1,id} \otimes (x,j) = X_{w,1,id} \otimes \sigma_j(x', a+b+1) = X_{w,1,\rho} \otimes (x', a+b+1).$$

But it is easy to construct an element d such that $\psi(d) = X_{w,1,\rho}$, and hence $X_{w,1,id} \otimes (x,j)$ is in the image of ψ.

Thus it remains to shown that ψ is a $B_{r-1,s}$-homomorphism. By considering the diagrammatic realisation of the induced module, it is easy to see that this is a $\Sigma_{r-1,s}$-homomorphism, so it will be enough to show that the action of the $E_{i,j}$ commutes with ψ.

Consider a basis element $b \otimes x$ in V, and an element $E_{i,j}$ with $1 \leq i < r$ and $r+1 \leq j \leq r+s$. The northern node r in b is connected to some northern node t with $r+1 \leq t \leq r+s$. If $j \neq t$ then it is clear that the action of $E_{i,j}$ commutes with ψ. If $j = t$ and the northern node i in b is on a northern arc then the result is also clear.

Finally, suppose that $j = t$ and the northern node i in b is on a propagating line. Then in $E_{i,j}(b \otimes x)$ the northern node r is on a propagating line, and hence this product is in W (and so is 0 in V). The action of $E_{i,j}$ on $\psi(d) \otimes (x, a+b+1)$ is clearly 0 as the diagram obtained has an extra arc, and hence too few propagating lines. Thus $E_{i,j}$ commutes with the action of ψ in all cases, and so we are done.

Remark 3.4 *In the non-Σ-semisimple cases, we get partial direct sum decompositions for the filtered modules in Theorem 3.3 associated to the corresponding block decompositions for the symmetric group. This follows from (9) and its analogue for induction.*

If we are in the quasi-hereditary case then we have that

$$\Lambda_{r,s} = \coprod_{t=0}^{\min(r,s)} \Lambda^{r-t,s-t}.$$

We will write $\Lambda^m_{r,s}$ for the set $\Lambda^{r-t,s-t}$ with $m = r+s-2t$ when regarded as a subset of $\Lambda_{r,s}$ in this way. (Note that $m = |\mu^L| + |\mu^R|$ for $(\mu^L, \mu^R) \in \Lambda^{r-t,s-t}$.) Given a $B_{r,s}$-module M with a filtration by cell modules (which we call a *cell filtration*), we would like to be able to define the *support* $\operatorname{supp}_{r,s}(M)$ of M to be the set of labels of cell modules in such a filtration. In the quasi-hereditary case this is well-defined as standard modules form a basis for the

Grothendieck group, however for general cellular algebras it may depend on the filtration chosen. However, for our purposes we will only ever need that there exists some filtration with a certain support.

We start with a weaker version of Theorem 3.3, verifying the fifth tower axiom:

Corollary 3.5 (A5) *For each* $(\lambda^L, \lambda^R) \in \Lambda^m_{r,s}$ *there is a cell filtration of* $\mathrm{res}^L(\Delta_{r,s}(\lambda^L, \lambda^R))$ *such that*

$$\mathrm{supp}(\mathrm{res}^L(\Delta_{r,s}(\lambda^L, \lambda^R))) \subseteq \Lambda^{m-1}_{r-1,s} \sqcup \Lambda^{m+1}_{r-1,s}$$

and a similar result for res^R*.*

From (A4) we have that

$$\mathrm{res}^{\dagger} G(\Delta_{r,s}(\lambda^L, \lambda^R)) \cong \mathrm{ind}^{\ddagger} \Delta_{r,s}(\lambda^L, \lambda^R)$$

where $(\dagger, \ddagger)$ equals (L, R) or (R, L). Combining this with (A5) and (3) we obtain the analogue of Theorem 3.3 for induction:

Corollary 3.6 *Suppose that* $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$*. Then we have a short exact sequence*

$$0 \longrightarrow \biguplus_{\Box \in \mathrm{rem}(\lambda^L)} \Delta_{r,s+1}(\lambda^L - \Box, \lambda^R) \longrightarrow \mathrm{ind}^R_{r,s} \Delta_{r,s}(\lambda^L, \lambda^R)$$
$$\longrightarrow \biguplus_{\Box \in \mathrm{add}(\lambda^R)} \Delta_{r,s+1}(\lambda^L, \lambda^R + \Box) \longrightarrow 0$$

where the left-hand sum equals 0 *if* $\lambda^L = \emptyset$*. Each of the filtered modules arising has filtration compatible with the* $\lhd^L$ *or* $\lhd^R$ *order as in Theorem 3.3, and when* k *is* Σ*-semisimple the* $\uplus$ *all become direct sums. There is a similar result for* $\mathrm{ind}^L_{r,s}$ *replacing* $\mathrm{rem}(\lambda^L)$ *by* $\mathrm{rem}(\lambda^R)$ *and* $\mathrm{add}(\lambda^R)$ *by* $\mathrm{add}(\lambda^L)$*.*

Note that the roles of R and L are reversed for induction and restriction rules. We also have

Corollary 3.7 *For each* $(\lambda^L, \lambda^R) \in \Lambda^m_{r,s}$ *we have that* $\mathrm{ind}^R(\Delta_{r,s}(\lambda^L, \lambda^R))$ *has a cell filtration, such that*

$$\mathrm{supp}(\mathrm{ind}^R(\Delta_{r,s}(\lambda^L, \lambda^R))) \subseteq \Lambda^{m-1}_{r,s+1} \sqcup \Lambda^{m+1}_{r,s+1}$$

and a similar result for ind^L*.*

Finally note that it is clear from the precise form of the induction rules that

Corollary 3.8 (A6) *If* $r > 0$ *then for each* $(\lambda^L, \lambda^R) \in \Lambda^{r+s}_{r,s}$ *there exists* $(\mu^L, \mu^R) \in \Lambda^{r+s-1}_{r-1,s}$*, and a cell filtration of* $\mathrm{ind}^L_{r-1,s} \Delta_{r-1,s}(\mu^L, \mu^R)$*, such that*

$$(\lambda^L, \lambda^R) \in \mathrm{supp}(\mathrm{ind}^L_{r-1,s} \Delta_{r-1,s}(\mu^L, \mu^R)).$$

There is a similar result if $s > 0$ *involving* $\mathrm{ind}^R_{r,s-1}$.

The tower of recollement machinery was introduced in [CMPX06] in part as an organisational tool to reduce the analysis of representations of towers of algebras to certain special cases. To this extent the adaptations above with cellularity instead of quasi-heredity are sufficient.

We would like to have a tower of recollement (or cellular analogue) containing $B_{r,s}$ in the sense of [CMPX06]. As stated there the tower depends on one indexing label rather than two, and so we will need to make choices for our algebra inclusions in a consistent way. Suppose that $r \geq s$ (the case $r < s$ is similar). Let $t = r - s$, and set $A_0 = B_{t,0}$. For $n > 0$ set

$$A_n = \begin{cases} B_{t+u,u+1} & \text{if } n = 2u+1 \\ B_{t+u,u} & \text{if } n = 2u. \end{cases}$$

The functors F and G corresponding to each choice of A_n are compatible with this choice of algebras, and give functors F_n from A_n-mod to A_{n-2}-mod and G_n from A_n-mod to A_{n+2}-mod. Modules in A_n-mod will be identified by the subscript n instead of the corresponding pair r, s. Choosing ind_n and res_n alternately left and right as n increases (so that they go from A_{n-1}-mod to A_n-mod and vice versa) it is easy to check that in the quasi-hereditary case (A1–6) now follow exactly as in [CMPX06], with $\Lambda_n = \Lambda_{a,b}$ and $\Lambda_n^m = \Lambda_{a,b}^{m-t}$ if $A_m = B_{a,b}$.

We have the following cellular version of [CMPX06, Theorem 3.7] for $B_{r,s}$.

Theorem 3.9 *(i) For all* $(\lambda^L, \lambda^R) \in \Lambda_n^m$ *and* $(\mu^L, \mu^R) \in \Lambda_n^l$ *we have*

$$\mathrm{Hom}(\Delta_n(\lambda^L, \lambda^R), \Delta_n(\mu^L, \mu^R)) \cong \begin{cases} \mathrm{Hom}(\Delta_m(\lambda^L, \lambda^R), \Delta_m(\mu^L, \mu^R)) & \text{if } l \leq m \\ 0 & \text{if } l > m. \end{cases}$$

(ii) The algebras A_n *are semisimple for* $0 \leq n \leq N$ *if and only if they are quasi-hereditary and for all* $0 \leq n \leq N$ *and pairs of weights* $(\lambda^L, \lambda^R) \in \Lambda_n^n$ *and* $(\mu^L, \mu^R) \in \Lambda_n^{n-2}$ *we have*

$$\mathrm{Hom}(\Delta_n(\lambda^L, \lambda^R), \Delta_n(\mu^L, \mu^R)) \cong 0.$$

PROOF. Part (i) follows exactly as in the proof of [CMPX06, Theorem 1.1]. For part (ii), note that if a cellular algebra is not quasi-hereditary then it cannot be semisimple [GL96, (3.8) Theorem], so it is enough to consider the quasi-hereditary case. This has already been proved in [CMPX06, Theorem 1.1].

4. A necessary condition for blocks

The principal aim of this section is to give a necessary condition for two weights to label simple $B_{r,s}$ modules in the same block. (We will abuse terminology and say that the weights themselves are in the same block.) This is closely modelled on a similar result for the Brauer algebra in [DWH99, Theorem 3.3] (as interpreted in [CDM05, Proposition 4.2]). Throughout this section k and δ are arbitrary.

Let λ be a partition. For a box d in row i and column j of the Young diagram $[\lambda]$ we set $c(d) = j - i$, the *content* of d. We set

$$T_{r,s} = \sum_{\substack{1\leq i\leq r,\\ r+1\leq j\leq r+s}} E_{i,j}.$$

Lemma 4.1 *Let $(\lambda^L, \lambda^R) \in \Lambda^{r-t,s-t}$. For all $y \in \Delta_{r,s}(\lambda^L, \lambda^R)$ we have that*

$$T_{r,s}y = \Big(t\delta - \sum_{d\in[\lambda^L]} c(d) - \sum_{d\in[\lambda^R]} c(d) + \sum_{1\leq i<m\leq r} (i,m) + \sum_{r<i<m\leq r+s} (i,m)\Big)y$$

where (i,m) denotes the element of $\Sigma_{r,s}$ which transposes i and m.

PROOF. First suppose that $k = \mathbb{C}$. It is enough to consider the case where $y = X_{w,1,id} \otimes x$ where $w \in \mathcal{V}_{r,s,t}$ and $x \in S^{\lambda^L} \boxtimes S^{\lambda^R}$. Recall that the action of $E_{i,j}$ on such an element is given by one of the four cases (a)–(d) given by equations (5–8).

We need to determine the contribution of each of the four cases to our final sum. As there are t northern arcs, there are precisely t distinct pairs (i,j) in case (a), and so they contribute a total of $t\delta(X_{w,1,id}\otimes x)$ to the sum. Clearly case (b) makes no contribution.

From case (c) we obtain a total contribution of

$$\sum_{\substack{1\leq i\leq r\\ i \text{ free in } w}} \sum_{\substack{1\leq m\leq r\\ m \text{ joined in } w}} (i,m)(X_{w,1,id}\otimes x) + \sum_{\substack{r<j\leq r+s\\ j \text{ free in } w}} \sum_{\substack{r<l\leq r+s\\ l \text{ joined in } w}} (j,l)(X_{w,1,id}\otimes x). \tag{11}$$

From case (d), note that each contribution occurs twice, as it comes from the action of $E_{i,j}$ and of $E_{l,m}$. Therefore the total contribution can be obtained by summing over all pairs which are not free to the left of the wall, together with the corresponding pairs to the right of the wall. This corresponds to

$$\Big(\sum_{1\le i<m\le r}(i,m)+\sum_{r<j<l\le r+s}(j,l)-\sum_{\substack{1\le i<m\le r\\ i,m \text{ free in } w}}(i,m)-\sum_{\substack{r<j<l\le r+s\\ j,l \text{ free in } w}}(j,l)$$
$$-\sum_{\substack{1\le i\le r\\ i \text{ free in } w}}\ \sum_{\substack{1\le m\le r\\ m \text{ joined in } w}}(i,m)-\sum_{\substack{r<j\le r+s\\ j \text{ free in } w}}\ \sum_{\substack{r<l\le r+s\\ l \text{ joined in } w}}(j,l)\Big)(X_{w,1,id}\otimes x). \tag{12}$$

Adding the four cases we see that the two double summations in (12) are cancelled out by the terms in (11), and hence we see that

$$T_{r,s}(X_{w,1,id}\otimes x)=$$
$$\Big(t\delta+\sum_{1\le i<m\le r}(i,m)+\sum_{r<j<l\le r+s}(j,l)-\sum_{\substack{1\le i<m\le r\\ i,m \text{ free in } w}}(i,m)-\sum_{\substack{r<j<l\le r+s\\ j,l \text{ free in } w}}(j,l)\Big)(X_{w,1,id}\otimes x).$$

By our identifications, the third sum in this equation corresponds to the standard action of $\sum_{1\le i<m\le r}(i,m)$ in Σ_r on the first Specht module S^{λ^L}. But this sum is a central element in Σ_r acting on an irreducible module, and hence acts as a scalar. By [Dia88, Chapter 1] this scalar is given by $\sum_{d\in[\lambda^L]}c(d)$. Similarly the final sum corresponds to the action of the sum of all transpositions in Σ_s on S^{λ^R}, and hence acts as the scalar $\sum_{d\in[\lambda^R]}c(d)$. Substituting for these scalars gives the desired result when $k=\mathbb{C}$.

For general k note that the cell modules are all defined over $\mathbb{Z}[\delta]$. Thus our result must be true over $\mathbb{Z}[\delta]$, and hence by base change over any field.

Theorem 4.2 *Suppose that* $[\Delta_{r,s}(\mu^L,\mu^R):L_{r,s}(\lambda^L,\lambda^R)]\neq 0$. *Then either* $(\lambda^L,\lambda^R)=(\mu^L,\mu^R)$ *or* $(\lambda^L,\lambda^R)\in\Lambda^{r-a,s-a}$ *and* $(\mu^L,\mu^R)\in\Lambda^{r-b,s-b}$ *for some* $b-a=t\ge 0$. *Further we must have that*

$$t\delta+\sum_{d\in[\lambda^L]}c(d)+\sum_{d\in[\lambda^R]}c(d)-\sum_{d\in[\mu^L]}c(d)-\sum_{d\in[\mu^R]}c(d)=0.$$

PROOF. A slightly stronger version of the first part of the theorem (with $t>0$) has already been noted in the quasi-hereditary case. The weaker version here holds in all cases by the cellular structure of $B_{r,s}$. For the second part, it follows from the exactness of the localisation functor that

$$[\Delta_{r,s}(\mu^L,\mu^R):L_{r,s}(\lambda^L,\lambda^R)]=[\Delta_{r-a,s-a}(\mu^L,\mu^R):L_{r-a,s-a}(\lambda^L,\lambda^R)]$$

and hence we may assume that $(\lambda^L,\lambda^R)\in\Lambda^{r,s}$. If $k=\mathbb{C}$ we have that $L_{r,s}(\lambda^L,\lambda^R)=\Delta_{r,s}(\lambda^L,\lambda^R)\cong S^{\lambda^L}\boxtimes S^{\lambda^R}$, the lift of the irreducible for $k\Sigma_{r,s}$, and so any walled Brauer diagram having fewer than $r+s$ propagating lines must act as zero. In particular this includes the action of the element $T_{r,s}$. By base change via $\mathbb{Z}[\delta]$ this must hold over any field.

By assumption there exists a $B_{r,s}$-submodule $M \leq \Delta_{r,s}(\mu^L, \mu^R)$ and a $B_{r,s}$-homomorphism

$$\phi : L_{r,s}(\lambda^L, \lambda^R) \longrightarrow \Delta_{r,s}(\mu^L, \mu^R)/M.$$

If $k = \mathbb{C}$ then the action of

$$\sum_{1\leq i<m\leq r} (i,m) + \sum_{r<i<m\leq r+s} (i,m)$$

in the centre of $\mathbb{C}\Sigma_{r,s}$ on $S^{\lambda^L} \boxtimes S^{\lambda^R}$ must be by a scalar, and by another application of [Dia88, Chapter 1] this equals

$$\sum_{d\in[\lambda^L]} c(d) + \sum_{d\in[\lambda^R]} c(d).$$

The same is true for general fields exactly as before. Hence

$$\Big(\sum_{1\leq i<m\leq r} (i,m) + \sum_{r<i<m\leq r+s} (i,m)\Big)\phi(x) = \Big(\sum_{d\in[\lambda^L]} c(d) + \sum_{d\in[\lambda^R]} c(d)\Big)\phi(x)$$

for all $x \in \Delta_{r,s}(\lambda^L, \lambda^R)$, and so for all $y + M \in \operatorname{im}\phi$ we must have

$$T_{r,s}(y+M) = \Big(t\delta + \sum_{d\in[\lambda^L]} c(d) + \sum_{d\in[\lambda^R]} c(d) - \sum_{d\in[\mu^L]} c(d) - \sum_{d\in[\mu^R]} c(d)\Big)(y+M)$$

by Lemma 4.1. But $T_{r,s}$ must act by zero, and so the result follows.

By standard cellular arguments [GL96, (3.9.8)] we deduce

Corollary 4.3 *Suppose that $(\lambda^L, \lambda^R) \in \Lambda^{r-a,s-a}$ and $(\mu^L, \mu^R) \in \Lambda^{r-b,s-b}$ for some $b - a = t \geq 0$. If $L_{r,s}(\lambda^L, \lambda^R)$ and $L_{r,s}(\mu^L, \mu^R)$ are in the same block then*

$$t\delta + \sum_{d\in[\lambda^L]} c(d) + \sum_{d\in[\lambda^R]} c(d) - \sum_{d\in[\mu^L]} c(d) - \sum_{d\in[\mu^R]} c(d) = 0.$$

We are now able to give a complete description of the blocks of $B_{r,s}$ when δ in not an integer.

Theorem 4.4 *Suppose that $\delta \notin \mathbb{Z}$ and k is arbitrary. Then two simple $B_{r,s}$ modules $L(\lambda^L, \lambda^R)$ and $L(\mu^L, \mu^R)$ are in the same block if and only if $|\lambda^L| = |\mu^L|$ (and hence $|\lambda^R| = |\mu^R|$) and the corresponding simple $k\Sigma_{|\lambda^L|,|\lambda^R|}$-modules are in the same block.*

In particular, $B_{r,s}$ is semisimple if $\delta \notin \mathbb{Z}$ and k is Σ-semisimple.

PROOF. The result follows immediately from Corollary 4.3, and the fact that via localisation any two modules in the same block must both arise as lifts from the given group algebra.

5. A sufficient condition for semisimplicity

In the preceding section we saw that if k is Σ-semisimple then $B_{r,s}$ is semisimple for $\delta \notin \mathbb{Z}$, and it also followed that over such fields $B_{r,s}$ was semisimple for $|\delta| >> 0$. We would like to give a stronger semisimplicity criterion, which we will achieve by refining our condition for the existence of a (nonzero) homomorphism between cell modules. In this section we assume that k is Σ-semisimple; clearly $B_{r,s}$ must be non-semisimple in the other cases, by the non-semisimplicity of $\Sigma_{r,s}$. Throughout this section we leave to the reader the easy modifications to the proofs required for the case $r = s$ with $\delta = 0$.

Given two partitions λ and μ we write $\lambda \subseteq \mu$ if λ is a subpartition of μ (i.e. $\lambda_i \leq \mu_i$ for all i). For a module M we call the largest semisimple quotient of M the *head* of M, and denote this by $\operatorname{hd} M$.

Proposition 5.1 *Suppose that $(\lambda^L, \lambda^R) \vdash (a, b)$ and $(\mu^L, \mu^R) \vdash (a-t, b-t)$ for some $t \geq 0$ are such that*

$$\operatorname{Hom}(\Delta_{r,s}(\lambda^L, \lambda^R), \Delta_{r,s}(\mu^L, \mu^R)) \neq 0. \tag{13}$$

Then we must have $\mu^L \subseteq \lambda^L$ and $\mu^R \subseteq \lambda^R$ (which we will write as $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$).

PROOF. By Theorem 3.9(i) we may assume that $(a, b) = (r, s)$. We proceed by induction on t and on $r + s$. The result is clear when $t = 0$ by the quasihereditary structure of $B_{r,s}$; note that the base cases where r or s equal 1 for the induction on $r+s$ will be considered as part of the inductive argument.

Suppose $t > 0$ (and hence $r > 0$ and $s > 0$). By Corollary 3.6 and Σ-semisimplicity we have for any removable box $\Box \in \lambda^L$ that

$$\Delta_{r,s}(\lambda^L, \lambda^R) \subseteq \operatorname{hd}(\operatorname{ind}^L_{r-1,s} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R)).$$

Therefore our assumption (13) implies that

$$\operatorname{Hom}(\operatorname{ind}^L_{r-1,s} \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \Delta_{r,s}(\mu^L, \mu^R)) \neq 0$$

and by Frobenius reciprocity we deduce that

$$\operatorname{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \operatorname{res}^L_{r,s} \Delta_{r,s}(\mu^L, \mu^R)) \neq 0. \tag{14}$$

By Theorem 3.3 we have a short exact sequence

$$0 \longrightarrow \biguplus_{\Box' \in \mathrm{rem}(\mu^L)} \Delta_{r-1,s}(\mu^L - \Box', \mu^R) \longrightarrow \mathrm{res}^L_{r,s} \Delta_{r,s}(\mu^L, \mu^R)$$
$$\longrightarrow \biguplus_{\Box'' \in \mathrm{add}(\mu^R)} \Delta_{r-1,s}(\mu^L, \mu^R + \Box'') \longrightarrow 0$$

where the left-hand term is zero if $\mu^L = \emptyset$, and hence from (14) we must have either

$$\mathrm{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \Delta_{r-1,s}(\mu^L - \Box', \mu^R)) \neq 0 \tag{15}$$

for some removable box $\Box'$ for μ^L or

$$\mathrm{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \Delta_{r-1,s}(\mu^L, \mu^R + \Box'')) \neq 0 \tag{16}$$

for some addable box $\Box''$ for μ^R.

In the case (16) we have that $(\lambda^L - \Box, \lambda^R) \vdash (a-1, b)$ and $(\mu^L, \mu^R + \Box'') \vdash (a-1-(t-1), b-(t-1))$ and so $\mu^L \subseteq \lambda^L - \Box$ and $\mu^R + \Box'' \subseteq \lambda^R$ by the inductive hypothesis on t. But clearly this implies that $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$.

In the case (15) (which cannot occur when $r = 1$) we have that $(\lambda^L - \Box, \lambda^R) \vdash (a-1, b)$ and $(\mu^L - \Box', \mu^R) \vdash (a-1-t, b-t)$ and so $\mu^L - \Box' \subseteq \lambda^L - \Box$ and $\mu^R \subseteq \lambda^R$ by the inductive hypothesis on $r+s$. This only implies that $\mu^R \subseteq \lambda^R$, however repeating the above argument with ind^R and res^R instead of ind^L and res^L also gives that $\mu^L \subseteq \lambda^L$, and so we are done.

Corollary 5.2 *The algebra $B_{r,s}(\delta)$ is semisimple if k is Σ-semisimple and $|\delta| \geq r+s-1$.*

PROOF. By Theorem 3.9(ii) it is enough to show that there are no homomorphisms from $\Delta_{r,s}(\lambda^L, \lambda^R)$ to $\Delta_{r,s}(\mu^L, \mu^R)$ with $(\lambda^L, \lambda^R) \vdash (r, s)$ and $(\mu^L, \mu^R) \vdash (r-1, s-1)$. By Proposition 5.1, the existence of such a homomorphism implies that $\lambda^L = \mu^L + \Box$ and $\lambda^R = \mu^R + \Box'$ for some addable boxes $\Box$ and $\Box'$.

As $\lambda^L \vdash r$ and $\lambda^R \vdash s$ we must have $|c(\Box)| \leq r-1$ and $|c(\Box')| \leq s-1$. But Theorem 4.2 implies that

$$\delta + c(\Box) + c(\Box') = 0$$

which then gives the desired result.

6. A semisimplicity criterion

In this section we will complete the classification of semisimple walled Brauer algebras by constructing non-zero homomorphisms between cell modules whose weights differ by two boxes (and satisfy the content condition in Theorem 4.2), when k is Σ-semisimple. These form a very special class of homomorphisms, where the two weights are as near as is possible for such a non-trivial homomorphism to exist, but it will turn out that these will also be sufficient to determine the blocks in characteristic zero.

Throughout this section we will assume (unless otherwise stated) that k is Σ-semisimple.

Let $c^{\lambda}_{\mu\nu}$ be the Littlewood-Richardson coefficient denoting the multiplicity of S^λ in the module $\mathrm{ind}^{\mathbb{C}\Sigma_{r+s}}_{\mathbb{C}\Sigma_{r,s}}(S^\mu \boxtimes S^\nu)$. We will need the following result of Halverson [Hal96, Corollary 7.24] (an analogue of [HW90, Theorem 4.1]), describing the decomposition of cell modules for $B_{r,s}$ when regarded as $\Sigma_{r,s}$-modules, in terms of the $c^{\lambda}_{\mu\nu}$. Although stated only for $k=\mathbb{C}$ in [Hal96], it clearly holds whenever k is Σ-semisimple.

Theorem 6.1 *Suppose that k is Σ-semisimple and $(\mu^L,\mu^R)\in\Lambda^{r-t,s-t}$. Then*

$$[\mathrm{res}_{k\Sigma_{r,s}}\,\Delta_{r,s}(\mu^L,\mu^R) : S^{\lambda^L}\boxtimes S^{\lambda^R}] = \sum_{\tau\vdash t} c^{\lambda^L}_{\mu^L\tau}c^{\lambda^R}_{\mu^R\tau}.$$

We note that Halverson's result can also be used to give an alternative proof of Proposition 5.1.

We say that $(\mu^L,\mu^R)\lhd(\lambda^L,\lambda^R)$ if (μ^L,μ^R) can be obtained from (λ^L,λ^R) by removing a box from λ^L and from λ^R. Then we have in the case $t=1$ that

$$\mathrm{res}_{k\Sigma_{r,s}}\,\Delta_{r,s}(\mu^L,\mu^R)\cong\bigoplus_{(\mu^L,\mu^R)\lhd(\lambda^L,\lambda^R)} S^{\lambda^L}\boxtimes S^{\lambda^R}. \tag{17}$$

We will now deduce the existence of the desired two box homomorphisms. The proof is similar to that of [CDM05, Theorem 5.2]. For two partitions λ and μ, we denote by λ/μ the skew partition whose skew Young diagram $[\lambda/\mu]$ consists of those boxes in $[\lambda]$ which are not in $[\mu]$. Given $\mu\subset\lambda$ with $|\lambda|-|\mu|=1$ we denote by $c(\lambda/\mu)$ the content of the unique box in $[\lambda/\mu]$.

Theorem 6.2 *Let k be Σ-semisimple. Suppose that $(\lambda^L,\lambda^R)\in\Lambda^{r-t,s-t}$, and $(\mu^L,\mu^R)\in\Lambda^{r-t-1,s-t-1}$. If we have that $(\mu^L,\mu^R)\lhd(\lambda^L,\lambda^R)$ and*

$$c(\lambda^L/\mu^L)+c(\lambda^R/\mu^R)+\delta=0$$

then

$$\mathrm{Hom}_{B_{r,s}}(\Delta_{r,s}(\lambda^L,\lambda^R),\Delta_{r,s}(\mu^L,\mu^R)) \cong k.$$

PROOF. By the exactness of the localisation functor we may assume that $(\lambda^L,\lambda^R) \vdash (r,s)$, and hence that

$$\Delta_{r,s}(\lambda^L,\lambda^R) \cong S^{\lambda^L} \boxtimes S^{\lambda^R}$$

(where all Brauer diagrams with propagating vector other than (r,s) act as zero on the righthand side).

We will fix a labelling of the boxes of λ^L and λ^R. Number the boxes of λ^L with $1,2,\ldots,r$ along the rows from left to right, and from top to bottom. Similarly number the boxes of λ^R with $r+1,\ldots,r+s$. Using the natural identifications of Σ_r with $\mathrm{Sym}\{1,2,\ldots r\}$ and Σ_s with $\mathrm{Sym}\{r+1,r+2,\ldots r+s\}$ we can define elements e_{λ^L} and e_{λ^R} by setting

$$e_{\lambda^L} = \frac{f^{\lambda^L}}{r!}\sum_{\sigma_L\in C(\lambda^L)}\sum_{\tau_L\in R(\lambda^L)} \mathrm{sgn}(\sigma_L)\sigma_L\tau_L$$

and

$$e_{\lambda^R} = \frac{f^{\lambda^R}}{s!}\sum_{\sigma_R\in C(\lambda^R)}\sum_{\tau_R\in R(\lambda^R)} \mathrm{sgn}(\sigma_R)\sigma_R\tau_R$$

where $f^\lambda = \dim S^\lambda$ and $C(\lambda)$ and $R(\lambda)$ are respectively the column and row stabilisers of λ. These are idempotents, such that $e_{\lambda^L}\mathbb{C}\Sigma_r \cong a_\lambda S^{\lambda^L}$ and $e_{\lambda^R}\mathbb{C}\Sigma_s \cong b_\lambda S^{\lambda^R}$ where the multiplicities a_λ and b_λ will not concern us (see for example [Ful97, Chapter 7]). (It is clear that the denominators in each idempotent are non-zero when k is Σ-semisimple. The numerator is also non-zero as dimensions of Specht modules can be written as products of hook lengths, and hence cannot involve factors bigger than the total number of boxes.) Note that the idempotent

$$e_\lambda = e_{\lambda^L}e_{\lambda^R} = e_{\lambda^R}e_{\lambda^L} \in k\Sigma_{r,s}$$

is such that for any $\Sigma_{r,s}$-module M, $e_\lambda M$ is the $S^{\lambda^L}\boxtimes S^{\lambda^R}$ isotypic component of M.

Let $W = e_\lambda\Delta_{r,s}(\mu^L,\mu^R)$. By (17) we know that

$$W \cong S^{\lambda^L}\boxtimes S^{\lambda^R}$$

as a $\Sigma_{r,s}$-module. To show that this is a $B_{r,s}$-submodule of $\Delta_{r,s}(\mu^L,\mu^R)$, it is enough to show that $E_{i,j}W = 0$ for all $1 \le i \le r < j \le r+s$. Indeed, it is enough to show that this holds for a single choice of i and j, as

$$\sigma E_{i,j}\sigma^{-1} = E_{\sigma(i),\sigma(j)}$$

for all $\sigma \in \Sigma_{r,s}$.

Fix $1 \leq i \leq r < j \leq r+s$ and consider the map

$$E_{i,j} : \Delta_{r,s}(\mu^L, \mu^R) \longrightarrow \Delta_{r,s}(\mu^L, \mu^R).$$

This is a $k\Sigma_{r-1,s-1}$-homomorphism, where we identify

$$\Sigma_{r-1,s-1} \cong \mathrm{Sym}(\{1, 2, \ldots r\}\backslash\{i\}) \times \mathrm{Sym}(\{r+1, r+2, \ldots r+s\}\backslash\{j\}).$$

Note that $E_{i,j}(\Delta_{r,s}(\mu^L, \mu^R)) \subseteq U$ where U is the span of all elements of the form $X_{w_0,1,id} \otimes x$ where w_0 has an arc between i and j and $x \in S^{\mu^L} \boxtimes S^{\mu^R}$, and

$$\mathrm{res}_{k\Sigma r-1,s-1}\, U \cong S^{\mu^L} \boxtimes S^{\mu^R} \tag{18}$$

By (17) we have that

$$\mathrm{res}_{k\Sigma r-1,s-1}\, W \cong \bigoplus_{(\nu^L,\nu^R)\lhd(\lambda^L,\lambda^R)} S^{\nu^L} \boxtimes S^{\nu^R}$$

and by (18) every summand in this direct sum must be sent by $E_{i,j}$ to zero except possibly for $V = S^{\mu^L} \boxtimes S^{\mu^R}$. Write $\mathrm{res}_{k\Sigma r-1,s-1}\, W = V \oplus Y$.

Up until this point i and j have been arbitrary. Henceforth we will consider the case where i is the number labelling the box in λ^L/μ^L and j is the number labelling the box in λ^R/μ^R. For example, if $\lambda^L = (3, 2, 1)$ and $\lambda^R = (2, 2)$, with $\mu^L = (2, 2, 1)$ and $\mu^R = (2, 1)$, then $i = 3$ and $j = 10$ as illustrated in Figure 10.

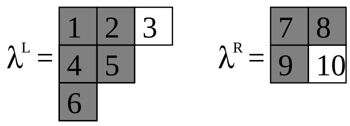

Fig. 10. $\lambda^L = (3, 2, 1)$ and $\lambda^R = (2, 2)$, with $\mu^L = (2, 2, 1)$ and $\mu^R = (2, 1)$ shaded.

Writing $e_\lambda(X_{w_0,1,id} \otimes x) = v + y$ with $v \in V$ and $y \in Y$ (which is independent of δ) the remarks above imply that

$$E_{i,j}e_\lambda(X_{w_0,1,id} \otimes x) = E_{i,j}v.$$

We claim that the coefficient of $X_{w_0,1,id} \otimes x$ in $E_{i,j}v$ is a non-zero multiple of

$$\delta + c(\lambda^L/\mu^L) + c(\lambda^R/\mu^R).$$

This would imply that $v \neq 0$, and that for

$$\delta + c(\lambda^L/\mu^L) + c(\lambda^R/\mu^R) = 0$$

we would have $E_{i,j}V = 0$ (as V is a simple module), and hence that $E_{i,j}W = 0$ as required. Thus it is enough to prove the claim.

We have that

$$E_{i,j}e_\lambda(X_{w_0,1,id}\otimes x) =$$
$$\frac{f^{\lambda^L}}{r!}\frac{f^{\lambda^R}}{s!}\sum_{\substack{\sigma_L\in C(\lambda^L)\\ \sigma_R\in C(\lambda^R)}}\ \sum_{\substack{\tau_L\in R(\lambda^L)\\ \tau_R\in R(\lambda^R)}} \mathrm{sgn}(\sigma_L\sigma_R)E_{i,j}\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id}\otimes x)$$

and we wish to find the coefficient of $(X_{w_0,1,id}\otimes x)$ in this sum.

In order to keep track of the various cases, it will be convenient to have a graphical notation for the partial one row diagrams arising as configurations of northern arcs in the summands of this expression. We will represent elements of $\mathcal{V}_{r,s,t}$ by adding ties to the double Young tableau joining each pair of nodes connected by an arc, omitting any such ties which do not play a role in the calculation. For example, the element w_0 can be represented by the diagram in Figure 11.

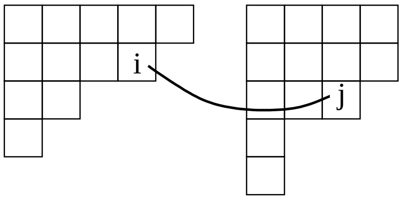

Fig. 11. A graphical representation of the arc ij in w_0

Case 1: Suppose that $\sigma_L\sigma_R\tau_L\tau_R X_{w_0,1,id}$ has an edge between i and j. This occurs if and only if all of σ_L, σ_R, τ_L and τ_R fix i and j. In this case

$$E_{i,j}\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id}\otimes x) = \delta\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id}\otimes x).$$

For $\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id}\otimes x)$ to be in the span of $X_{w_0,1,id}\otimes S^{\lambda^L}\boxtimes S^{\lambda^R}$ we must have

$$\begin{aligned}\tau_L &\in R(\mu^L)\subset R(\lambda^L)\\ \sigma_L &\in C(\mu^L)\subset C(\lambda^L)\\ \tau_R &\in R(\mu^R)\subset R(\lambda^R)\\ \sigma_R &\in C(\mu^R)\subset C(\lambda^R).\end{aligned}$$

As any such quartet of elements fixes i and j, the contribution to our sum in this case equals

$$\frac{f^{\lambda^L}}{r!}\frac{f^{\lambda^R}}{s!}\sum_{\substack{\sigma_L\in C(\mu^L)\\ \sigma_R\in C(\mu^R)}}\ \sum_{\substack{\tau_L\in R(\mu^L)\\ \tau_R\in R(\mu^R)}} \delta\,\mathrm{sgn}(\sigma_L\sigma_R)E_{i,j}\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id}\otimes x)$$

which equals

$$\delta \frac{f^{\lambda^L}}{r!} \frac{f^{\lambda^R}}{s!} \frac{(r-1)!}{f^{\mu^L}} \frac{(s-1)!}{f^{\mu^R}} X_{w_0,1,id} \otimes e_\mu x.$$

Now $e_\mu(x) = x$ for all $x \in S^{\mu^L} \boxtimes S^{\mu^R}$, and hence we obtain the contribution

$$\delta \frac{f^{\lambda^L}}{f^{\mu^L}} \frac{f^{\lambda^R}}{f^{\mu^R}} \frac{1}{rs} X_{w_0,1,id} \otimes x$$

from the terms arising in Case 1.

Case 2: Suppose that neither i nor j is part of an arc in $\sigma_L\sigma_R\tau_L\tau_R X_{w_0,1,id}$. Then

$$E_{i,j}\sigma_L\sigma_R\tau_L\tau_R(X_{w_0,1,id} \otimes x) = 0$$

so such terms contribute nothing to our sum.

Case 3: The only remaining case is when exactly one of i or j is part of an arc in $\sigma_L\sigma_R\tau_L\tau_R X_{w_0,1,id}$. (Note that they cannot both be, as there is only one arc in such a term.) We will consider the subcase where the arc connects i to some element k with $r+1 \leq k \leq r+s$ (the subcase of an edge between j and some element k with $1 \leq k \leq r$ is similar). We must have

$$\sigma_L \in C(\mu^L) \qquad \text{and} \qquad \tau_L \in R(\mu^L) \tag{19}$$

There are two possibilities: (a) k is in the same column as j, or (b) k is in a column to the left of the column containing j

Subcase 3(a): Suppose that k is in the same column as j, as illustrated in Figure 12.

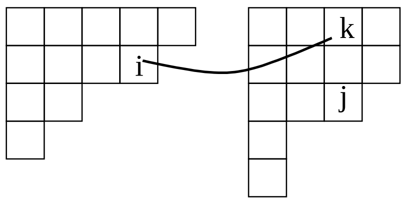

Fig. 12. The subcase 3(a)

We must have

$$\tau_R \in R(\mu^R) \qquad \text{and} \qquad \sigma_R \in (k,j)C(\mu^R) \tag{20}$$

and every quartet satisfying (19) and (20) will arise in this way. Writing $\sigma_R = (k,j)\sigma'_R$ with $\sigma'_R \in C(\mu^R)$, and noting that $\operatorname{sgn}(\sigma'_R) = -\operatorname{sgn}(\sigma_R)$, we see that we obtain a contribution of

$$\sum_{k \text{ above } j} \frac{f^{\lambda^L}}{r!} \frac{f^{\lambda^R}}{s!} \sum_{\substack{\sigma_L \in C(\mu^L) \\ \sigma'_R \in C(\mu^R)}} \sum_{\substack{\tau_L \in R(\mu^L) \\ \tau_R \in R(\mu^R)}} -\operatorname{sgn}(\sigma_L\sigma'_R) E_{i,j}\sigma_L(k,j)\sigma'_R\tau_L\tau_R(X_{w_0,1,id} \otimes x).$$

Arguing as in Case 1 we see that this equals

$$-\sum_{k \text{ above } j} \frac{f^{\lambda^L} f^{\lambda^R}}{f^{\mu^L} f^{\mu^R}} \frac{1}{rs} E_{i,j}(k,j) X_{w_0,1,id} \otimes e_\mu x = -\operatorname{abv}_{\lambda^R}(j) \frac{f^{\lambda^L} f^{\lambda^R}}{f^{\mu^L} f^{\mu^R}} \frac{1}{rs} X_{w_0,1,id} \otimes x$$

where $\operatorname{abv}_{\lambda^R}(j)$ denotes the number of boxes above j in λ^R.

Subcase 3(b): Suppose that k is in a column to the left of the column containing j, with l as illustrated in Figure 13.

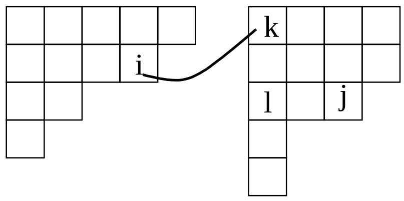

Fig. 13. The subcase 3(b)

We must have

$$\tau_R \in (j,l)R(\mu^R) \qquad \text{and} \qquad \sigma_R \in C(\mu^R) \tag{21}$$

(as k is an arbitrary element in the column containing l) and every quartet satisfying (19) and (21) will arise in this way. Arguing as in Subcase 3(a), but now with no sgn modifications, we see that we obtain a contribution of

$$\operatorname{lft}_{\lambda^R}(j) \frac{f^{\lambda^L}}{f^{\mu^L}} \frac{f^{\lambda^R}}{f^{\mu^R}} \frac{1}{rs} X_{w_0,1,id} \otimes x$$

where $\operatorname{lft}_{\lambda^R}(j)$ denotes the number of boxes to the left of j in λ^R.

Combining Subcase 3(a) and 3(b), and the corresponding versions where the arc connects to j rather than i, we obtain a total contribution in Case 3 of

$$\Big(\operatorname{lft}_{\lambda^R}(j) - \operatorname{abv}_{\lambda^R}(j) + \operatorname{lft}_{\lambda^L}(i) - \operatorname{abv}_{\lambda^L}(i)\Big) \frac{f^{\lambda^L}}{f^{\mu^L}} \frac{f^{\lambda^R}}{f^{\mu^R}} \frac{1}{rs} X_{w_0,1,id} \otimes x.$$

As

$$\operatorname{lft}_{\lambda^R}(j) - \operatorname{abv}_{\lambda^R}(j) + \operatorname{lft}_{\lambda^L}(i) - \operatorname{abv}_{\lambda^L}(i) = c(\lambda^L/\mu^L) + c(\lambda^R/\mu^R)$$

combining Cases 1–3 now gives that the coefficient of $X_{w_0,1,id} \otimes x$ in $E_{i,j}v$ equals

$$\frac{f^{\lambda^L}}{f^{\mu^L}} \frac{f^{\lambda^R}}{f^{\mu^R}} \frac{1}{rs} (\delta + c(\lambda^L/\mu^L) + c(\lambda^R/\mu^R))$$

which is a nonzero multiple of $\delta + c(\lambda^L/\mu^L) + c(\lambda^R/\mu^R)$ as required.

Combining the last result with the semisimplicity results in Section 5 we obtain

Theorem 6.3 *The walled Brauer algebra $B_{r,s}(\delta)$ is semisimple if and only if k is Σ-semisimple and one of the following conditions holds:*
(i) $\delta \notin \mathbb{Z}$, or
(ii) $|\delta| > r+s-2$, or
(iii) $r=0$ or $s=0$, or
(iii) $\delta = 0$ and $(r,s) = (1,2)$, $(1,3)$, $(2,1)$, or $(3,1)$.

PROOF. Clearly $B_{r,s}$ cannot be semisimple if k is not Σ-semisimple. If k is Σ-semisimple and r or s equals zero then $B_{r,s}$ must be semisimple (by the definition of Σ-semisimplicity as it is just the group algebra of the symmetric group $\Sigma_{r,s}$). By Theorem 4.4 and Corollary 5.2 the only other cases where $B_{r,s}$ can be non-semisimple occur when $\delta \in \mathbb{Z}$ and $|\delta| \leq r+s-2$. By Theorem 3.9 it is enough to classify exactly those pairs (r,s) for which there exists $(\lambda^L, \lambda^R) \vdash (r,s)$ and $(\mu^L, \mu^R) \vdash (r-1, s-1)$ with a non-zero homomorphism from $\Delta_{r,s}(\lambda^L, \lambda^R)$ to $\Delta_{r,s}(\mu^L, \mu^R)$.

We may assume that $\delta \in \mathbb{Z}$ with $|\delta| \leq r+s-2$. We write $\delta = \delta_0 + \delta_1$ with $|\delta_0| \leq r-1$ and $|\delta_1| \leq s-1$. First suppose that we can choose such a decomposition with both δ_0 and δ_1 non-zero. Then we can construct explicitly a quartet (λ^L, λ^R) and (μ^L, μ^R) with a nonzero homomorphism in the following manner. If $\delta_0 > 0$ set

$$\lambda^L = ((\delta_0+1), 1^{r-\delta_0-1}) \supset \mu^L = (\delta_0, 1^{r-\delta_0-1})$$

and if $\delta_0 < 0$ take the transpose of this pair for $-\delta_0$. Similarly construct λ^R and μ^R in terms of δ_1. In each case the resulting bipartitions satisfy the conditions of Theorem 6.2 and so we have a non-zero homomorphism.

The only cases where we are forced to take δ_0 or δ_1 equal to zero occur when $r=1$ or $s=1$. We consider the case $s=1$, the other is similar. Clearly $\lambda^R = (1)$ has a removable box of content zero, so we just have to determine when there exists $\lambda^L \vdash r$ with a removable box of content $-\delta$. If $\delta \neq 0$ this is always possible: if $\delta > 0$ then we take $\lambda^L = (r-\delta, 1^{\delta-1})$, while for $\delta < 0$ we take the transpose of this partition.

Finally we are left with the case where $\delta = 0$, and so require $\lambda^L \vdash r$ with a removable box of content 0. Such a box exists when $\lambda^L = (1)$ or $\lambda^L = (2,2,1^{r-4})$, but not when $r=2$ or $r=3$. This (and the corresponding argument when $r=1$) provides the exceptional semisimple cases listed above.

7. The blocks of the walled Brauer algebra in characteristic zero

In this section we will determine the blocks of the walled Brauer in the case when k is Σ-semisimple. Our approach is modelled on that for the corresponding result for the Brauer algebra in [CDM05].

We begin by giving a refinement of Theorem 4.2.

Proposition 7.1 *Let k be Σ-semisimple. Suppose that*

$$[\Delta_{r,s}(\mu^L, \mu^R) : L_{r,s}(\lambda^L, \lambda^R)] \neq 0.$$

Then $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$, and there exists a pairing of the boxes in λ^L/μ^L with those in λ^R/μ^R such that the sum of the contents of the boxes in each pair equals $-\delta$ in k.

PROOF. First note that the condition $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$ follows immediately from Theorem 6.1. If $r = 0$ or $s = 0$ then $B_{r,s}$ is just the group algebra of the symmetric group σ_{r+s}, and the result follows from the definition of Σ-semisimplicity.

We will proceed by induction on $r + s$. The case $r = s = 1$ follows from Corollary 4.3, as the only allowable bipartitions are $(1, 1)$ and $(0, 0)$, and hence the result is true if $r + s = 2$. Thus we assume that the result is true for all $B_{a,b}$ with $a + b = n - 1$ and will show that it is also true for $B_{r,s}$ with $r + s = n$ and $r, s \neq 0$.

If $[\Delta_{r,s}(\mu^L, \mu^R) : L_{r,s}(\lambda^L, \lambda^R)] \neq 0$ then by the above remarks and Corollary 4.3 we must have $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$ and

$$t\delta + \sum_{d \in [\lambda^L/\mu^L]} c(d) + \sum_{d \in [\lambda^R/\mu^R]} c(d) = 0 \tag{22}$$

where $t = |\lambda^L - \mu^L| = |\lambda^R - \mu^R|$. By localising we may assume that $(\lambda, \mu) \vdash (r, s)$, so that $L_{r,s}(\lambda^L, \lambda^R) = \Delta_{r,s}(\lambda^L, \lambda^R)$. Thus $\Delta_{r,s}(\mu^L, \mu^R)$ has a submodule M such that there is an injection

$$\Delta_{r,s}(\lambda^L, \lambda^R) \hookrightarrow \Delta_{r,s}(\mu^L, \mu^R)/M.$$

By our assumption that r is nonzero there exists a removable box $\square$ in λ^L, and by Corollary 3.6 and Σ-semisimplicity there exists a surjection

$$\operatorname{ind}^L_{r-1,s} \Delta_{r-1,s}(\lambda^L - \square, \lambda^R) \longrightarrow \Delta_{r,s}(\lambda^L, \lambda^R).$$

Hence we have

$$\operatorname{Hom}(\operatorname{ind}^L_{r-1,s} \Delta_{r-1,s}(\lambda^L - \square, \lambda^R), \Delta_{r,s}(\mu^L, \mu^R)/M) \neq 0$$

and so by Frobenius reciprocity we have

$$\mathrm{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \mathrm{res}^L_{r,s}\left(\Delta_{r,s}(\mu^L, \mu^R)/M\right)) \neq 0.$$

This implies that

$$L_{r-1,s}(\lambda^L - \Box, \lambda^R) = \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R)$$

is a composition factor of $\mathrm{res}^L_{r,s}(\Delta_{r,s}(\mu^L, \mu^R)$. By Theorem 3.3 we see that either (i)

$$[\Delta_{r-1,s}(\mu^L - \Box', \mu^R) : L_{r-1,s}(\lambda^L - \Box, \lambda^R)] \neq 0$$

for some $\Box' \in \mathrm{rem}(\mu^L)$, or (ii)

$$[\Delta_{r-1,s}(\mu^L, \mu^R + \Box') : L_{r-1,s}(\lambda^L - \Box, \lambda^R)] \neq 0$$

for some $\Box' \in \mathrm{add}(\mu^L)$. We consider each case in turn.

In case (i), our inductive hypothesis implies that $\mu^L - \Box' \subseteq \lambda^L - \Box$ and

$$t\delta + \Bigg(\sum_{d \in [\lambda^L/\mu^L]} c(d) + \sum_{d \in [\lambda^R/\mu^R]} c(d) \Bigg) - c(\Box) + c(\Box') = 0.$$

Comparing with (22) we see that $c(\Box) = c(\Box')$. By induction we know that there is a pairing of the boxes in $(\lambda^L - \Box)/(\mu^L - \Box')$ with those in λ^R/μ^R such the contents of each pair sum to $-\delta$. But as multisets, the set of contents in $\lambda^L - \Box/\mu^L - \Box'$ and in λ^L/μ^L are equal, and hence there is such a pairing between λ^L/μ^L and λ^R/μ^R as required.

Next we consider case (ii). By induction we must have that $\mu^L \subseteq \lambda^L - \Box$ and $\mu^R + \Box' \subseteq \lambda^R$ with

$$(t-1)\delta + \Bigg(\sum_{d \in [\lambda^L/\mu^L]} c(d) + \sum_{d \in [\lambda^R/\mu^R]} c(d) \Bigg) - c(\Box) - c(\Box') = 0.$$

Comparing with (22) we see that $c(\Box) + c(\Box') = -\delta$, and by induction we know that there is a pairing of the boxes in $(\lambda^L - \Box)/\mu^L$ with those in $\lambda^R/(\mu^R + \Box')$ such the contents of each pair sum to $-\delta$. But then extending this pairing to one between λ^L/μ^L and λ^R/μ^R by adding the paired boxes $\Box$ and $\Box'$ gives the desired result.

Given two partitions λ and μ, we denote by $\lambda \cap \mu$ the partition whose corresponding Young diagram is the intersection of those for λ and μ.

Definition 7.2 *We will say that (λ^L, λ^R) and (μ^L, μ^R) are* δ-balanced *(or just* balanced *when this will not cause confusion) if there is a pairing of the boxes in $[\lambda^L/(\lambda^L \cap \mu^L)]$ with those in $[\lambda^R/(\lambda^R \cap \mu^R)]$ and of the boxes in $[\mu^L/(\lambda^L \cap \mu^L)]$ with those in $[\mu^R/(\lambda^R \cap \mu^R)]$ such that the contents of each pair sum to $-\delta$ in k.*

Just as for Corollary 4.3, we deduce from Proposition 7.1 the following partial block result.

Corollary 7.3 *Let k be Σ-semisimple. If (λ^L, λ^R) and (μ^L, μ^R) are in the same block for $B_{r,s}(\delta)$ then they are δ-balanced.*

We will show that this is in fact a necessary and sufficient condition for block membership when k is Σ-semisimple. Given a partition $\mu \subset \lambda$, we denote by $\mathrm{rem}(\lambda/\mu)$ the set of boxes in $\mathrm{rem}(\lambda)$ which are not in μ.

Definition 7.4 *Suppose that $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$ is a balanced pair. For each $\Box_i \in \mathrm{rem}(\lambda^L/\mu^L)$ we wish to consider $(\mu^L, \mu^R)^i$, the* i-maximal balanced sub-bipartition between (λ^L, λ^R) and (μ^L, μ^R). *This is the maximal bipartition in (λ^L, λ^R) not containing $\Box_i$ such that (λ^L, λ^R) and $(\mu^L, \mu^R)^i$ is δ-balanced.*

We can give an explicit recursive construction of $(\mu^L, \mu^R)^i$. Given two boxes $\Box$ and $\Box'$ with the same content in a partition λ we will say that $\Box$ is *larger* than $\Box'$ if $\Box$ appears on a later row than $\Box'$. Suppose that $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$ is a balanced pair with $\Box_i \in \mathrm{rem}(\lambda^L/\mu^L)$. By the balanced pair condition there exists a largest box $\Box_i'$ in λ^R/μ^R such that $c(\Box_i) + c(\Box_i') = -\delta$. Let $[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_0 = \{\Box_i, \Box_i'\}$. Given $[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_m$ we set

$$[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_{m+1} = [(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_m \cup A_{m+1} \cup A'_{m+1}$$

where A_{m+1} is the set of boxes in λ^L or λ^R which are to the right of or below a box in $[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_m$, and A'_{m+1} is the set of boxes $\Box'$ in $(\lambda^L/\mu^L, \lambda^R/\mu^R)$ whose content satisfies $c(\Box) + c(\Box') = -\delta$ for some $\Box \in A_{m+1}$ with $\Box$ and $\Box'$ not both in the same partition which are largest with such content. Let $(\mu^L, \mu^R)^i_m$ be the sub-bipartition of (λ^L, λ^R) with complement $[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_m$.

This iterative process will eventually stabilise to produce a δ-balanced sub-bipartition $(\mu^L, \mu^R)^i$ of (λ^L, λ^R), obtained by removing a strip of boxes one box wide from the edges of each of λ^L and λ^R. To see this, first note that the construction of each $[(\lambda^L, \lambda^R)/(\mu^L, \mu^R)]_m$ clearly only involves boxes from the edges of λ^L and λ^R, and so produces a strip in each at most one box wide (as each box in a given strip has different content, and is the largest with such). Second, the only way in which the process could terminate without producing a balanced sub-bipartition would be if one or other of the two strips ended with one of the boxes at the end of the first row or first column of λ^L or λ^R, without the other strip being removable. But this would contradict the fact that (λ^L, λ^R) and (μ^L, μ^R) are δ-balanced. It is also easy to see that $(\mu^L, \mu^R)^i$ is maximal in (λ^L, λ^R) with this property.

Example 7.5 *We will illustrate the above construction with an example.*

Let $(\lambda^L, \lambda^R) = ((4^3, 1^3), (5^2, 2^3))$ and $(\mu^L, \mu^R) = ((2, 1), (4))$ as in Figure 14. These form a balanced pair with $\delta = +1$.

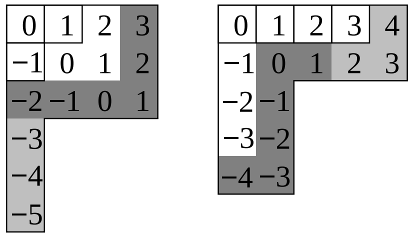

Fig. 14. Two examples of the $(\mu^L, \mu^R)^i$ construction.

If $\square_i$ is the largest box in λ^L with content -5, then the associated maximal balanced sub-bipartition is obtained by removing the three lightly shaded boxes in each partition. If $\square_i$ is the other removable box in λ^L (with content 1) then the associated sub-bipartition is obtained by removing all *shaded boxes from each partition.*

As the above example illustrates, some of the removable strips so far constructed may contain others. Partially order the removable strips obtained from the above construction by inclusion. Then we define a *maximal balanced sub-bipartition $(\mu^L_\lambda, \mu^R_\lambda)$ between (λ^L, λ^R) and (μ^L, μ^R)* to be any balanced sub-bipartition for which the associated removable strip is minimal. Thus in the above example there is a unique choice of $(\mu^L_\lambda, \mu^R_\lambda) = ((4^3), (3^2, 2^3))$ given by removing the three lightly shaded boxes. In general the choice will not be unique.

Note that if k is Σ-semisimple then for each equivalence class of integers mod p there is at most one member of that class which occurs as a content in a partition λ of r or s. Thus throughout the following proof, it is unambiguous to regard all contents mod p.

Theorem 7.6 *Let k be Σ-semisimple. If $(\mu^L, \mu^R) \subset (\lambda^L, \lambda^R)$ is a balanced pair then for any maximal balanced sub-bipartition $(\mu^L_\lambda, \mu^R_\lambda)$ we have*

$$\mathrm{Hom}(\Delta_{r,s}(\lambda^L, \lambda^R), \Delta_{r,s}(\mu^L_\lambda, \mu^R_\lambda))) \neq 0.$$

PROOF. As usual, we may assume that $(\lambda^L, \lambda^R) \vdash (r, s)$. If $(\mu^L_\lambda, \mu^R_\lambda) \vdash (r-1, s-1)$ then we are done by Theorem 6.2. For the remaining cases, pick $\square \in \mathrm{rem}(\lambda^L/\mu^L_\lambda)$ with $|c(\square) + \frac{\delta}{2}|$ maximal, and suppose that $\square'$ is the corresponding box in λ^R/μ^R_λ with $c(\square) + c(\square') = -\delta$.

Note that by the maximality of $\square$ and the construction of $(\mu^L_\lambda, \mu^R_\lambda)$ there is no box of content $c(\square)$ in $\mathrm{rem}(\mu^L_\lambda)$, and exactly one box of content $c(\square')$

in add(μ_λ^R) (namely $\Box'$ itself). Clearly $(\lambda^L - \Box, \lambda^R)$ and $(\mu_\lambda^L, \mu_\lambda^R + \Box')$ is a balanced pair; we claim that in fact $(\mu_\lambda^L, \mu_\lambda^R + \Box')$ is also a maximal balanced sub-bipartition for this balanced pair. But this is also obvious, as any larger balanced sub-bipartition would give rise to a corresponding balanced sub-bipartition between (μ^L, μ^R) and (λ^L, λ^R), which would contradict the maximality of $(\mu_\lambda^L, \mu_\lambda^R)$.

If $\Box' \in \mathrm{rem}(\lambda^R)$ then $(\mu_\lambda^L, \mu_\lambda^R) = (\lambda^L - \Box, \lambda^R - \Box')$ by maximality, and so we are done. Otherwise by our initial assumptions there is no removable box in λ^R with content $c(\Box')$. Therefore by Frobenius reciprocity, Corollary 7.3, Corollary 3.6, and Σ-semisimplicity we have

$$\begin{aligned}
&\mathrm{Hom}(\Delta_{r,s}(\lambda^L, \lambda^R), \Delta_{r,s}(\mu_\lambda^L, \mu_\lambda^R)) \\
&\qquad \cong \mathrm{Hom}(\mathrm{ind}_{r-1,s}^L \Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \Delta_{r,s}(\mu_\lambda^L, \mu_\lambda^R)) \\
&\qquad\qquad \cong \mathrm{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \mathrm{res}_{r,s}^L \Delta_{r,s}(\mu_\lambda^L, \mu_\lambda^R)).
\end{aligned}$$

By the remarks above and Theorem 3.3 this final Hom-space is isomorphic to

$$\mathrm{Hom}(\Delta_{r-1,s}(\lambda^L - \Box, \lambda^R), \Delta_{r-1,s}(\mu_\lambda^L, \mu_\lambda^R + \Box'))$$

which is non-zero by induction.

Corollary 7.7 *Let k be Σ-semisimple. Two weights (λ^L, λ^R) and (μ^L, μ^R) are in the same block of $B_{r,s}$ if and only if they are balanced. Each block contains a unique minimal weight.*

PROOF. One implication was proved in Corollary 7.3. For the reverse implication we proceed by induction. If (λ^L, λ^R) contains a smaller balanced weight (μ^L, μ^R) then by Theorem 7.6 there exists some $(\mu_\lambda^L, \mu_\lambda^R) \subset (\lambda^L, \lambda^R)$ with a non-zero homomorphism from $\Delta_{r,s}(\lambda^L, \lambda^R)$ to $\Delta_{r,s}(\mu_\lambda^L, \mu_\lambda^R)$, and hence and $(\mu_\lambda^L, \mu_\lambda^R)$ will lie in the same block. By induction we also have that (λ^L, λ^R) and (μ^L, μ^R) lie in the same block. Thus it is enough to show that there is a unique minimal weight in the set of weights balanced with (λ^L, λ^R).

But given two such minimal weights (μ^L, μ^R) and (ν^L, ν^R), set $\eta^L = \mu^L \cap \nu^L$ and $\eta^R = \mu^R \cap \nu^R$. Clearly (η^L, η^R) is a weight, and forms a balanced pair with both (μ^L, μ^R) and (ν^L, ν^R) (and hence with (λ^L, λ^R)). But this contradicts the minimality of (μ^L, μ^R) and (ν^L, ν^R).

In particular, we have now determined the blocks of the walled Brauer algebra in characteristic zero.

8. An alcove geometry for the walled Brauer algebra

We would like to have an alcove geometry, coming from some suitable reflection group, which controls the representation theory of the walled Brauer algebra in the non-semisimple cases. This is typically regarded as a Lie theoretic phenomenon, but has been shown to exist for the Brauer algebra in [CDM06].

By Theorem 6.3 we may assume that $\delta \in \mathbb{Z}$. In this case we will show that there is such a geometry for the walled Brauer algebra, associated to Σ_{r+s}, the Weyl group of type A_{r+s}. However, we will need a new notion of dominant weights (and a modified group action) to realise this.

Let $\{\epsilon_{-r}, \epsilon_{-(r-1)}, \dots, \epsilon_{-1}, \epsilon_1, \epsilon_2, \dots, \epsilon_s\}$ be a set of formal symbols. We set

$$X = X_{r,s} = \bigoplus_{i=-r}^{-1} \mathbb{Z}\epsilon_i \oplus \bigoplus_{i=1}^{s} \mathbb{Z}\epsilon_i$$

which will be our weight lattice. We will denote an element

$$\lambda = \lambda_{-r}\epsilon_{-r} + \cdots + \lambda_{-1}\epsilon_{-1} + \lambda_1\epsilon_1 + \cdots + \lambda_s\epsilon_s$$

in X by $(\lambda_{-r}, \lambda_{-(r-1)}, \dots, \lambda_{-1}; \lambda_1, \dots, \lambda_s)$. The set of dominant weights in X is defined to be

$$X^+ = \{\lambda \in X : 0 \geq \lambda_{-r} \geq \lambda_{-(r-1)} \geq \cdots \geq \lambda_{-1} \text{ and } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0\}.$$

(This is not a standard choice of dominant weights, but will be justified by our labelling conventions for the walled Brauer algebra.) Define an inner product on $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ by setting

$$(\epsilon_i, \epsilon_j) = \delta_{ij}$$

for all nonzero i, j with $-r \leq i, j \leq s$, and extending by linearity.

There is a root system of type A given by

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) : -r \leq i < j \leq s, \quad i, j \neq 0\}.$$

For each root $\beta \in \Phi$ we define a reflection s_β on E by

$$s_\beta(\lambda) = \lambda - \frac{2(\lambda, \beta)}{(\beta, \beta)}\beta = (\lambda, \beta)\beta$$

for all $\lambda \in E$, and let W be the group generated by these reflections. Then W is just the Weyl group of type A, which can be identified with Σ_{r+s}.

In algebraic Lie theory it is convenient to shift the action of the Weyl group on weights relative to some fixed vector ρ. While the same will be true here also, our choice of ρ is rather different. Fix $\delta \in \mathbb{Z}$ and define $\rho = \rho(\delta) \in E$ by

$$\rho = (r, r-1, \dots, 1; \delta, \delta-1, \dots, \delta-s+1).$$

We consider the dot action of W on E given by

$$w.\lambda = w(\lambda+\rho) - \rho$$

for all $w \in W$ and $\lambda \in E$. Note that this preserves the lattice X inside E.

A pair of partitions (λ^L, λ^R) with at most r and s parts respectively will be identified with a dominant weight $\lambda \in X^+$ via the map

$$(\lambda^L, \lambda^R) \longmapsto (\bar{\lambda}^L; \lambda^R) = (-\lambda^L_r, -\lambda^L_{r-1}, \dots - \lambda^L_1; \lambda^R_1, \lambda^R_2, \dots, \lambda^R_s). \qquad (23)$$

It will also be convenient to have a graphical representation of elements of X. We will represent any $\lambda \in X$ by a sequence of $r+s$ rows of boxes (to be defined shortly), with r rows above and s below some fixed horizontal bar. The ith row below this bar will be called row i, and the ith row above this bar will be called row $-i$. Columns will be labelled in increasing order from left to right by elements of $\mathbb{Z}$, and there will be a vertical bar between columns 0 and 1. (Note that there is a column 0, but no row 0.) With these conventions, row i in the representation of λ will contain all boxes to the left of column λ_i inclusive.

We have already defined the content of a box in a pair of partitions (λ^L, λ^R). Via the identification in (23) this corresponds to setting the content of a box in row i and column j of λ to be $j-i$ if $i>0$ and $1+i-j$ if $i<0$. For example, when $(r,s) = (3,4)$ the element $(5,-1,2;3,2,-3,0)$ (and the contents of its boxes) is illustrated in Figure 15.

So far our choices of dominant weights, ρ, and of contents seems rather artificial. However, we will see that with these identifications, the blocks of the walled Brauer algebra for k Σ-semisimple have a very simple description in terms of W. Before doing this, we will need the following elementary observation concerning contents of δ-balanced bipartitions. We denote the number of boxes in a partition λ of content i by $c_i(\lambda)$.

Lemma 8.1 *Two bipartitions (μ^L, μ^R) and (λ^L, λ^R) are δ-balanced if and only if*

$$c_i(\lambda^L) - c_i(\mu^L) = c_{-\delta-i}(\lambda^R) - c_{-\delta-i}(\mu^R). \qquad (24)$$

PROOF. We begin by noting that the result is obvious if $(\mu^L, \mu^R) \subseteq (\lambda^L, \lambda^R)$ (or vice versa). In general, if we have a δ-balanced pair then (24) is also clearly satisfied. Thus it is enough to show that the content condition (24) implies δ-balanced.

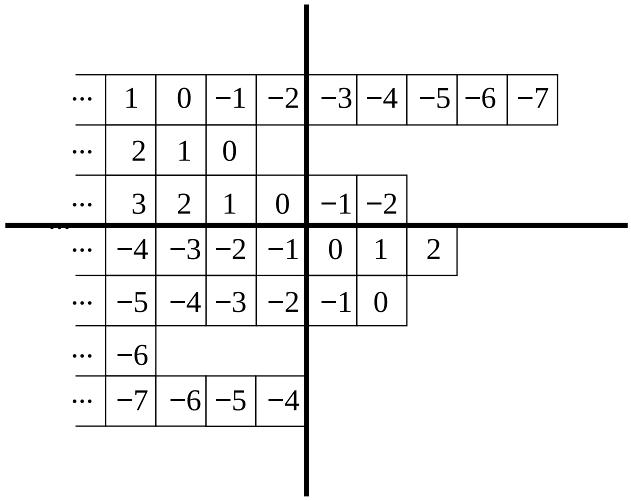

Fig. 15. The element $(5,-1,2;3,2,-3,0)$ and its associated contents

Suppose that (μ^L,μ^R) and (λ^L,λ^R) satisfy (24). Let $\tau^L = \mu^L \cap \lambda^L$ and $\tau^R = \mu^R \cap \lambda^R$. By the above remarks it is enough to show that

$$c_i(\lambda^L) - c_i(\tau^L) = c_{-\delta-i}(\lambda^R) - c_{-\delta-i}(\tau^R) \tag{25}$$

and

$$c_i(\mu^L) - c_i(\tau^L) = c_{-\delta-i}(\mu^R) - c_{-\delta-i}(\tau^R). \tag{26}$$

Note that $c_i(\tau^L) = \min(c_i(\lambda^L), c_i(\mu^L))$ and $c_i(\tau^R) = \min(c_i(\lambda^R), c_i(\mu^R))$. Further by (24)

$$c_i(\tau^L) = c_i(\lambda^L) \quad \text{if and only if} \quad c_{-\delta-i}(\tau^R) = c_{-\delta-i}(\lambda^R). \tag{27}$$

Hence exactly one of (25) and (26) becomes the trivial equality $0 = 0$.

Now (24) implies that

$$[c_i(\lambda^L) - c_i(\tau^L)] - [c_i(\mu^L) - c_i(\tau^L)] = [c_i(\lambda^R) - c_i(\tau^R)] - [c_i(\mu^R) - c_i(\tau^R)].$$

As we already know that one of (25) and (26) holds, the other is now obvious.

Lemma 8.1 will allow us to extend the notion of δ-balanced to arbitrary pairs of elements in X. We cannot count the number of boxes of a given content in a composition, as there is no natural point at which to stop including boxes on the left of the diagram. However we can sensibly extend the notation to arbitrary differences of compositions as follows. Suppose that $(\lambda^l;\lambda^r)$ and $(\mu^l;\mu^r)$ are both in X (where we use lower case superscripts to emphasise that these need not be bipartitions). We define $c_i(\lambda^r - \mu^r)$ (and $c_i(\lambda^l - \mu^l)$) in the following manner. For $1 \leq j \leq s$ let $c_{i,j}(\lambda^r - \mu^r)$ be the number of boxes of content i in row j between columns $\min(\lambda^r_j, \mu^r_j) + 1$ and $\max(\lambda^r_j, \mu^r_j)$ inclusive, and let $\epsilon_{i,j}(\lambda^r - \mu^r)$ be $+1$ if $\lambda^r_j > \mu^r_j$ and -1 otherwise. Then we set

$$c_i(\lambda^r - \mu^r) = \sum_{j=1}^{s} \epsilon_{i,j}(\lambda^r - \mu^r) c_{i,j}(\lambda^r - \mu^r).$$

Similarly, for $-r \leq j \leq -1$ let $c_{i,j}(\lambda^l - \mu^l)$ be the number of boxes of content i in row j between columns $\min(\lambda^l_j, \mu^l_j) + 1$ and $\max(\lambda^l_j, \mu^l_j)$ inclusive, and let $\epsilon_{i,j}(\lambda^l - \mu^l)$ be -1 if $\lambda^l_j > \mu^l_j$ and $+1$ otherwise. (Note that this is the opposite of the previous choice.) Then we set

$$c_i(\lambda^l - \mu^l) = \sum_{j=-r}^{-1} \epsilon_{i,j}(\lambda^l - \mu^l) c_{i,j}(\lambda^l - \mu^l).$$

With these conventions, and using the identification of bipartitions with X^+ in (23), we see that the condition (24) for bipartitions (λ^L, λ^R) and (μ^L, μ^R) is equivalent to

$$c_i(\bar{\lambda}^L - \bar{\mu}^L) = c_{-\delta-i}(\lambda^R - \mu^R). \tag{28}$$

We are now in a position to replace our δ-balanced condition for bipartitions with an orbit condition from the action of the Weyl group. We proceed in two stages.

Proposition 8.2 *Suppose that (λ^L, λ^R) and (μ^L, μ^R) are bipartions such that*

$$(\bar{\mu}^L; \mu^R) = w.(\bar{\lambda}^L; \lambda^R).$$

Then (λ^L, λ^R) and (μ^L, μ^R) are δ-balanced.

PROOF. By the above remarks it will be enough to show that (28) holds for any elements of X when w is one of the generators s_β. Such a generator will only change $\lambda = (\bar{\lambda}^L; \lambda^R)$ in two rows; there are two cases to consider depending on whether the two rows are on opposite sides of the horizontal bar.

From (28) it will be enough to show (i) that in the case where the two rows are on the same side of the bar the action of s_β corresponds to replacing boxes in one row by boxes with the same content in the other, and (ii) that in the case where the two rows are on opposite sides of the bar the action of s_β corresponds to replacing boxes in one row by boxes in the other such that the two sets can be paired up with contents in each pair summing to $-\delta$.

Consider a general element $\lambda \in X$. Note that the content of the last box in row $i > 0$ (reading from left to right) equals $\lambda_i - i$, while the content of the last box in row $-j < 0$ is $-\lambda_{-j} - j + 1$. Now consider the action of the generators of W on λ.

First suppose that $i, j > 0$, and consider $s_{\epsilon_i - \epsilon_j}.\lambda$. We have

$$\begin{aligned} s_{\epsilon_i-\epsilon_j}.\lambda &= \lambda - (\lambda_i - \lambda_j + (\delta - i + 1) - (\delta - j + 1))(\epsilon_i - \epsilon_j) \\ &= \lambda - (\lambda_i - \lambda_j + j - i)(\epsilon_i - \epsilon_j). \end{aligned}$$

If $(\lambda_i - \lambda_j + j - i) > 0$ then the action of $s_{\epsilon_i-\epsilon_j}$ removes $(\lambda_i - \lambda_j + j - i)$ boxes from row i and adds the same number to row j. In row i the boxes to be removed have contents

$$\lambda_i - i - (\lambda_i - \lambda_j + j - i) + 1, \ldots \lambda_i - i - 1, \lambda_i - i$$

and on row j the boxes added have contents

$$\lambda_j - j + 1, \lambda_j - j + 2, \ldots, \lambda_j - j + (\lambda_i - \lambda_j + j - i).$$

Simplifying both theses expressions for the contents we arrive at the same list:

$$\lambda_j - j + 1, \ldots, \lambda_i - i - i, \lambda_i - i$$

and so the number of boxes of each content has remained unchanged. The case where $(\lambda_i - \lambda_j + j - i) < 0$ is similar.

Next suppose that $i, j > 0$, and consider $s_{\epsilon_{-i}-\epsilon_{-j}}.\lambda$. We have that

$$s_{\epsilon_{-i}-\epsilon_{-j}}.\lambda = \lambda - (\lambda_{-i} - \lambda_{-j} + i - j)(\epsilon_{-i} - \epsilon_{-j}).$$

Arguing as above we see that if $(\lambda_{-i} - \lambda_{-j} + i - j) > 0$ then this number of boxes are removed from row $-i$ and added to row $-j$, with both the removed and added sets having contents

$$-\lambda_{-j} - j, \ldots, -\lambda_{-i} - i + 1.$$

Thus we again see that the number of boxes of each content remains unchanged. As above, the case where $(\lambda_{-i} - \lambda_{-j} + j - i) < 0$ is similar.

Finally, suppose that $i, j > 0$, and consider $s_{\epsilon_i-\epsilon_{-j}}.\lambda$. We have that

$$s_{\epsilon_i-\epsilon_{-j}}.\lambda = \lambda - (\lambda_i - \lambda_{-j} + (\delta - i + 1) - j)(\epsilon_i - \epsilon_{-j}).$$

Suppose that $\lambda_i - \lambda_{-j} + (\delta - i + 1) - j > 0$ (the other case is similar). Then $s_{\epsilon_i-\epsilon_{-j}}$ removes $\lambda_i - \lambda_{-j} + (\delta - i + 1) - j$ boxes from row i and adds the same number to row $-j$. In row $-j$ the added contents are

$$-\lambda_{-j} - j, -\lambda_{-j} - j - 1, \ldots, -\lambda_{-j} - j - (\lambda_i - \lambda_{-j} + (\delta - i + 1) - j) + 1$$

and in row i the removed contents are

$$\lambda_i - i - (\lambda_i - \lambda_{-j} + (\delta - i + 1) - j + 1, \ldots, \lambda_i - i - 1, \lambda_i - i.$$

Simplifying we see that the added contents are

$$-\lambda_{-j} - j, -\lambda_{-j} - j - 1, \ldots, -\lambda_i - \delta + i$$

and the removed contents are

$$\lambda_{-j} - \delta + j, \ldots, \lambda_i - i - 1, \lambda_i - i.$$

Comparing the corresponding entries in each of these last two expressions, we see that in each case they sum to $-\delta$, as required.

We next consider the reverse implication.

Proposition 8.3 *If (λ^L, λ^R) and (μ^L, μ^R) are δ-balanced then there exists $w \in W$ such that*

$$(\bar{\mu}^L; \mu^R) = w.(\bar{\lambda}^L; \lambda^R).$$

PROOF. It is enough to consider the case where $(\lambda^L, \lambda^R) \supseteq (\mu^L, \mu^R)$. We will proceed by induction on $|\lambda^L/\mu^L| = |\lambda^L/\mu^L|$, and write λ and μ for the corresponding elements of X.

If $|\lambda^L/\mu^L| = 1$ and under our identification the unique box in λ^L/μ^L (respectively in λ^R/μ^R) is in row $-j$ (respectively row i) then it is easy to verify that

$$s_{\epsilon_i - \epsilon_{-j}}.\lambda = \mu$$

and so we are done.

Next suppose that $|\lambda^L/\mu^L| > 1$. We define the edge of a skew partition τ to be those boxes in τ such that the box diagonally below and to the right is not in τ. Let ϵ be the box of maximal content in the edge of λ^R/μ^R. Suppose that the corresponding box in λ is in row i. By the δ-balanced condition there is a unique box ϵ' in the edge of λ^L/μ^L such that $c(\epsilon) + c(\epsilon') = -\delta$. Suppose that the corresponding box in λ is in row $-j$.

Let α be the first box in row $-j$ *not* in λ. As α is in the edge of λ^L/μ^L we can find a matching box α' on the edge of λ^R/μ^R. Say that α' is in row l; we have that $l \geq i$. This configuration is illustrated schematically in Figure 16 with the edge of the two skew partitions shaded grey, and μ denoted by the curved lines.

If $i = l$ then define

$$\lambda' = s_{\epsilon_i - \epsilon_{-j}}.\lambda.$$

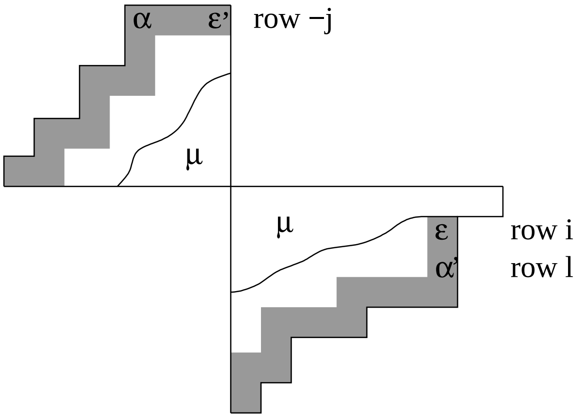

Fig. 16. Dominant weights λ and μ with associated edge strip shaded.

If $i < l$ define

$$\lambda' = (s_{\epsilon_i - \epsilon_{i+1}} \dots s_{\epsilon_i - \epsilon_{l-1}} s_{\epsilon_i - \epsilon_l} s_{\epsilon_i - \epsilon_{-j}}).\lambda.$$

In both cases λ' is obtained from λ by removing the boxes on the edge of λ^L/μ^L (respectively of λ^R/μ^R) between ϵ' and α (respectively between ϵ and α') inclusive.

If λ' is in X^+ then we are done by induction. If not, then either there is a box in the edge of λ^L/μ^L directly below α, or there is a box in the edge of λ^R/μ^R directly below α'. In these cases we repeat the above process replacing i with $l+1$ and $-j$ with $-j+1$.

Eventually this process will terminate, as if the whole of the edge of λ/μ is removed then the result will be in X^+. The result now follows by induction.

Combining Propositions 8.2 and 8.3 with Corollary 7.7 we obtain

Corollary 8.4 *Let k be Σ-semisimple with $\delta \in \mathbb{Z}$. Two weights λ and μ in X^+ are in the same block of $B_{r,s}$ if and only if $\lambda = w.\mu$ for some $w \in W$.*

9. Comparing blocks in characteristic zero and characteristic p

In this section we will recall some standard results relating blocks for algebras in different characteristics coming from a common integral form. This will motivate the results in the following section.

Let $A_{\mathbb{Z}}$ be an integral form giving rise to a corresponding algebra A_k over a field k. Suppose also that we have a family of integral forms for modules $M_{\mathbb{Z}}(\lambda)$ such that over any given field k, all simples can be realised as quotient modules of the corresponding modules $M_k(\lambda)$ (and are labelled by the corresponding λ). We will have in mind the case when $A_{\mathbb{Z}}$ is the integral form for a Brauer or walled Brauer algebra (where δ has been specialised to a fixed integer) and the $M_k(\lambda)$ are cell modules.

Let $1 = \sum_i e_i$ be a primitive central idempotent decomposition in the algebra $A_{\mathbb{F}_p}$ defined over the finite field with p elements. This corresponds to the block decomposition of $A_{\mathbb{F}_p}$. The idempotent decomposition lifts to a decomposition over $\mathbb{Z}_p$ by the lifting theorem (see for example [Ben91, Theorem 1.9.4]). These idempotents pass injectively to $\mathbb{Q}_p$, and hence to $\mathbb{C}_p$ which is isomorphic to $\mathbb{C}$.

This final decomposition may no longer be primitive, but can be refined into a primitive decomposition (and hence the blocks over $\mathbb{C}$ will in general be smaller). As the labelling scheme for simple modules in each algebra has been chosen in a consistent manner (via the integral forms) this proves that if λ and μ are in the same block over $\mathbb{C}$, then they are also in the same block over $\mathbb{F}_p$.

Combining this with our characteristic zero block results for the Brauer algebra [CDM06, Theorem 4.2] and walled Brauer algebra (Corollary 8.4) we obtain

Proposition 9.1 *Let λ and μ be weights for the Brauer (or walled Brauer) algebra $A(\delta)$ with $\delta \in \mathbb{Z}$. Let W be the Weyl group corresponding to A. If there exists $\delta' \in \mathbb{Z}$ with $\delta' \equiv \delta \mod p$ such that $\lambda = w.\mu$ for some $w \in W$, where the dot action is with respect to $\rho(\delta')$, then λ and μ are in the same block for $A(\delta)$ over $\mathbb{F}_p$.*

10. A linkage principle in positive characteristic

So far we have given a complete description of the blocks of $B_{r,s}$ when either k is Σ-semisimple or $\delta \notin \mathbb{Z}$. For the remaining cases we have a necessary conditions for two weights to be in the same block coming from Corollary 4.3. We will strengthen this into a linkage principle, using orbits of the affine Weyl group of type A. We will assume throughout this section that $\delta \in \mathbb{Z}$.

Consider the (type A) affine Weyl group W_p, the group generated by the affine reflections

$$s_{\beta,rp}(\lambda) = \lambda - ((\lambda, \beta) - rp)\beta$$

where $\beta \in \Phi$ and $r \in \mathbb{Z}$. Just as for W, this acts on X via the dot action

$$w.\lambda = w(\lambda + \rho) - \rho.$$

It is easy to verify (as in [CDM06, Section 5]) that the dot action of W_p on X is generated by the various dot actions of W with respect to $\rho(\delta + rp)$ for $r \in \mathbb{Z}$. Thus from Proposition 9.1 we might expect this affine Weyl group action to control in large part the block structure in positive characteristic. We will see that it does in fact give a necessary condition for two weights to be in the same block. For this it will be convenient to have the following simple combinatorial description of when two elements of X are in the same W_p-orbit.

Identify Σ_{r+s} with the group of permutations of $\{-r, \ldots, -1, 1, 2, \ldots, s\}$. Given an element $\sigma \in \Sigma_{r+s}$ we define an element $\mathrm{s}(\sigma)_i$ for each $-r \leq i \leq s$ with $i \neq 0$ by

$$\mathrm{s}(\sigma)_i = \begin{cases} -1 \text{ if } i < 0 \text{ and } \sigma(i) > 0 \\ +1 \text{ if } i > 0 \text{ and } \sigma(i) < 0 \\ \;\;0 \text{ otherwise.} \end{cases}$$

We also generalise the notion of degree from partitions to elements of X by setting

$$|\lambda| = \sum_{i=-r}^{s} \lambda_i.$$

Lemma 10.1 *Suppose that $\lambda, \mu \in X$. Then $\mu \in W_p.\lambda$ if and only if $|\lambda| = |\mu|$ and there exists $\sigma \in \Sigma_{r+s}$ such that for all $-r \leq i \leq s$ with $i \neq 0$ we have*

$$\mu_i - i = \lambda_{\sigma(i)} - \sigma(i) + \mathrm{s}(\sigma)_i(\delta + 1) \mod p.$$

PROOF. We have $\mu \in W_p.\lambda$ if and only if

$$\mu + \rho = w(\lambda + \rho) + p\nu$$

for some $w \in W$ and $\nu \in \mathbb{Z}\Phi$. Considering each component of μ in turn and substituting the corresponding values for ρ we obtain the congruences given in the lemma. The additional condition that $|\lambda| = |\mu|$ follows by summing over the expressions for each μ_i, and using the fact that $\nu \in \mathbb{Z}\phi$ implies that $|\nu| = 0$.

Note that under our correspondence with bipartitions, the condition that $|\lambda| = |\mu|$ is equivalent to the condition that

$$\lambda^R - \lambda^L = \mu^R - \mu^L$$

which we already know is a requirement for two bipartitions to be labels of $B_{r,s}$. Thus when λ and μ both come from bipartitions for $B_{r,s}$ it is enough to check the congruences in Lemma 10.1 to determine if they are in the same W_p-orbit.

Theorem 10.2 *Suppose that $\delta \in \mathbb{Z}$, and that (λ^L, λ^R) and (μ^L, μ^R) are bipartitions. If there exists $M \leq \Delta_{r,s}(\mu^L, \mu^R)$ with*

$$\mathrm{Hom}(\Delta_{r,s}(\lambda^L, \lambda^R), \Delta_{r,s}(\mu^L, \mu^R)/M) \neq 0 \tag{29}$$

then $(\bar{\mu}^L; \mu^R) \in W_p.(\bar{\lambda}^L; \lambda^R)$.

PROOF. By the cellular structure of $B_{r,s}$, if (29) holds then we must have $(\lambda^L, \lambda^R) \in \Lambda^a_{r,s}$ and $(\mu^L, \mu^R) \in \Lambda^b_{r,s}$ for some $b \leq a$. We will proceed by induction on $r + s$.

First suppose that $s = 0$. Then $B_{r,s} \cong \Sigma_r$, and we have

$$\Delta_{r,s}(\lambda^L, \lambda^R) \cong S^{\lambda^L} \quad \text{and} \quad \Delta_{r,s}(\mu^L, \mu^R) \cong S^{\mu^L}.$$

By the block result for symmetric groups (see for example the formulation in [Don94]) there exists $v \in W_p^{A_r}$, the affine Weyl group of type A_r such that $\mu^L = v.\lambda^L$. But this affine Weyl group is a subgroup of W_p, and taking w to be the corresponding element in W_p we have that $\lambda = w.\mu$. A similar argument holds when $r = 0$.

Now suppose that λ^L and λ^R are both nonempty partitions. By localising we may assume that $(\lambda^L, \lambda^R) \in \Lambda^{r,s}$. Possibly by enlarging M, we may also assume that our non-zero homomorphism kills everything in $\Delta_{r,s}(\lambda^L, \lambda^R)$ except for one simple in the head. Such a simple is labelled by some weight $(\tau^L, \tau^R) \in \Lambda^{r,s}_{reg}$, and by the block result for symmetric groups, this weight is in the same W_p orbit as (λ^L, λ^R). Clearly there will also be a homomorphism from $\Delta_{r,s}(\tau^L, \tau^R)$ into $\Delta_{r,s}(\mu^L, \mu^R)/M$, and so we may assume that $(\lambda^L, \lambda^R) \in \Lambda^{r,s}_{reg}$.

Choose the highest removable box $\Box$ in λ^R. The partition $\lambda^R - \Box$ cannot have a higher addable box of the same content as $\Box$ by our assumption on (λ^L, λ^R). Therefore by the block result for symmetric groups together with Corollary 3.6 (and Remark 3.4) we have a surjection

$$\mathrm{ind}^R\, \Delta_{r,s-1}(\lambda^L, \lambda^R - \Box) \longrightarrow \Delta_{r,s}(\lambda^L, \lambda^R) \longrightarrow 0$$

and so by (29) we have

$$\mathrm{Hom}(\mathrm{ind}^R\, \Delta_{r,s-1}(\lambda^L, \lambda^R - \Box), \Delta_{r,s}(\mu^L, \mu^R)/M) \neq 0.$$

Applying Frobenius reciprocity we see that

$$\mathrm{Hom}(\Delta_{r,s-1}(\lambda^L, \lambda^R - \Box), \mathrm{res}^R(\Delta_{r,s}(\mu^L, \mu^R)/M)) \neq 0.$$

We will set $\lambda'^L = \lambda^L$ and $\lambda'^R = \lambda^R - \Box$. By Theorem 3.3 we must have either

$$\operatorname{Hom}(\Delta_{r,s-1}(\lambda'^L, \lambda'^R), \Delta_{r,s-1}(\mu^L + \Box', \mu^R)/N) \neq 0 \tag{30}$$

for some addable box $\Box'$ for μ^L and some $N < \Delta_{r,s-1}(\mu^L + \Box', \mu^R)$ or

$$\operatorname{Hom}(\Delta_{r,s-1}(\lambda'^L, \lambda'^R), \Delta_{r,s}(\mu^L, \mu^R - \Box')/N) \neq 0 \tag{31}$$

for some removable box $\Box'$ in μ^R and some $N < \Delta_{r,s-1}(\mu^L, \mu^R - \Box')$.

First suppose that we are in the situation in (30), and set $(\tau^L, \tau^R) = (\mu^L + \Box', \mu^R)$. The condition in Corollary 4.3 applies both to the weights (λ^L, λ^R) and (μ^L, μ^R) and to the weights (λ'^L, λ'^R) and (τ^L, τ^R). Comparing the resulting expressions we see that we must have

$$c(\Box) + c(\Box') + \delta = 0 \mod p.$$

Also by induction there exists $w \in W_p$ such that $\tau = w.\lambda'$

Next, suppose that we are in the situation in (31), and set $(\tau^L, \tau^R) = (\mu^L, \mu^R - \Box')$. As in the preceding case, we deduce from Corollary 4.3 that

$$c(\Box) = c(\Box') \mod p.$$

Also by induction there exists $w \in W_p$ such that $\tau = w.\lambda'$.

In both cases, suppose that $\Box$ is in row i of λ and $\Box'$ is in row j of τ, and that σ is the element of W corresponding to w as in Lemma 10.1. Let $\sigma(j) = t$ and $\sigma(u) = j$ for some t and u. Define σ' by setting $\sigma'(j) = i$, $\sigma'(u) = t$, and $\sigma'(v) = \sigma(v)$ for all $v \neq j, u$. It is easy to check that σ' satisfies the conditions in Lemma 10.1, and hence that $\mu = w'.\lambda$ for some $w' \in W_p$ as required.

An immediate consequence of this (and the cellularity of $B_{r,s}$) is

Corollary 10.3 *Suppose that $\delta \in \mathbb{Z}$. Then $L_{r,s}(\lambda^L, \lambda^R)$ and $L_{r,s}(\mu^L, \mu^R)$ are in the same block only if $(\bar{\mu}^L; \mu^R) \in W_p.(\bar{\lambda}^L; \lambda^R)$.*

11. Concluding remarks

We have given a complete characterisation of the blocks of the walled Brauer algebra in characteristic zero, and a linkage principle in characteristic p. In general the positive characteristic result cannot be strengthened to give the full blocks as orbits of the affine Weyl group, as this is not true for the special case of the ordinary Brauer algebra (see [CDM06, Theorem 7.2]).

The geometric description of these results depends on our choice of embedding of dominant weights inside a larger weight space. This has two aspects: the use of 'negative' partitions and the relative positions of the left and right hand components of a bipartition.

Negative partitions are used so that the natural action of the symmetric group (which would normally preserve the total number of boxes) now correspond to adding or removing boxes when acting on rows from both parts of the bipartition.

The choice of relative positions of the two parts is slightly more arbitrary. In particular, [DD08, Section 4.5] adopts an alternative convention of placing the negative partition below the usual partition. If $\delta > r + s$ then this can be done in such a way that the Weyl group action we describe corresponds to the standard choice of ρ from Lie theory (i.e. without a shift by δ). However this is precisely the case where the block result is trivial, as all blocks consist of singletons. In general there is no way to position negative partitions so that ρ is the standard shift from Lie theory, as for small values of δ the rows would have to overlap.

The convention used in this paper also has the advantage that the embedding of $B_{r,s}$ into $B_{r+t,s+t}$ by globalising is compatible with the natural embedding of weight spaces for each algebra.

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