The blocks of the q-Schur algebra

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In [10], Donkin determined the blocks of the classical Schur algebras in positive characteristic from the blocks of the corresponding general linear group. We show in this paper that an analogous result for the q-Schur algebras $S_q(n,d)$ (when q is a primitive lth root of unity) can be derived in the same way from the blocks of an appropriate quantum general linear group.

There are a number of different quantisations of the general linear group. We shall mainly consider that due to Dipper and Donkin, though the Manin quantisation will also prove important. After some preliminary sections, the first main result is a proof of the strong linkage principle for our quantum group. This follows the classical proof due to Andersen [1], and is based on that for the Manin quantisation in [21]. For technical reasons, the proof in [21] requires l to be odd, but we can remove this restriction for both quantisations.

Using the Strong Linkage Principle, we are then able to partially determine the blocks of our quantum group. The rest of this paper then verifies that the argument of [10] holds essentially unchanged for the quantum case. From this, it is then straightforward to complete our determination of the blocks of the quantum group.

For $d \leq n$, the blocks of the q-Schur algebra have already been determined by James and Mathas in [18], while the blocks of the quantum group when $n = 2$ have also been calculated in [3]. It should also be noted that in [24] Thams has already determined the blocks of the quantum enveloping algebra (from which the blocks of our group could be derived) but under the additional assumptions that q is a primitive lth root of unity with l odd and greater than the Coxeter number for the group.

1 Two quantum general linear groups

In this section we introduce the two quantisations of the general linear group that we shall need, and recall a result from [14] which will allow us to transfer results between them. Our two quantisations are most easily introduced as certain special cases of a more general construction due to Takeuchi [23], which we begin by describing.

Henceforth we shall denote by k an algebraically closed field of characteristic $p \geq 0$. Following Parshall and Wang [21], we regard the category of quantum groups as the dual of the category of k-Hopf algebras, and identify a quantum group's module category with the comodule category of the corresponding Hopf algebra.

Fix $\alpha, \beta \in k \backslash \{0\}$, and define $A_{\alpha,\beta}(n)$ to be the k-algebra generated by the n^2 indeterminates c_{ij}, with $1 \leq i, j \leq n$, subject to the relations

$$\begin{array}{lll} c_{ij}c_{ir} & = \alpha c_{ir}c_{ij} & \text{for } j > r, \\ c_{ji}c_{ri} & = \beta c_{ri}c_{ji} & \text{for } j > r, \\ c_{ij}c_{rs} & = \alpha^{-1}\beta c_{rs}c_{ij} & \text{for } i > r \text{ and } j < s, \\ c_{ij}c_{rs} & = (\alpha^{-1} - \beta)c_{is}c_{rj} + c_{rs}c_{ij} & \text{for } i < r \text{ and } j < s. \end{array}$$

As noted in [23], there exist comultiplication and counit maps δ and ϵ respectively giving $A_{\alpha,\beta}(n)$ the structure of a bialgebra. Further, after localising at a certain quantum determinant, this can be given a Hopf algebra structure. We denote the corresponding quantum group by $G^{\alpha,\beta}$.

For fixed $q \in k\backslash\{0\}$, the Dipper–Donkin quantisation [4] corresponds to the case $\alpha = 1$ and $\beta = q$, while the Manin quantisation [20] corresponds to the case $\alpha = \beta = q$. In these cases we will denote $G^{\alpha,\beta}$ by q-$\mathrm{GL}(n, k)$ (or just G) and $\mathrm{GL}_q(n, k)$ respectively. When $\alpha = \beta = 1$ we recover the classical coordinate algebra of $\mathrm{GL}(n, k)$, and in this case we shall often write x_{ij} for the c_{ij}.

The main result relating these two quantisations is

Theorem 1.1 *There is a coalgebra isomorphism between $k[q^2\text{-}\mathrm{GL}(n, k)]$ and $k[\mathrm{GL}_q(n, k)]$, and hence the categories $\mathrm{Mod}(q^2\text{-}\mathrm{GL}(n, k))$ and $\mathrm{Mod}(\mathrm{GL}_q(n, k))$ are isomorphic.*

Proof: See [14, Proposition 2.1 and Theorem 2.4].

As observed in [14, Remark 2.7], we should note that this is not necessarily an isomorphism of tensor categories.

Remark 1.2 We will often wish to use the last result to translate between the Dipper–Donkin and Manin quantisations. When translating results across that depend on the value of l, it should be noted that these results may change. In particular, if q is a primitive lth root of unity with l even, then q^2 is a primitive $l/2$th root of unity. Thus results that for q-$\mathrm{GL}(n, k)$ depend on l will translate across to $\mathrm{GL}_q(n, k)$ as results that depend either on l, if l odd, or $l/2$ if l even.

Finally in this section, we relate our quantum groups to the q-Schur algebra $S_q(n,d)$ of Dipper and James [5, 6], originally defined as the centralising algebra of a certain action of the Hecke algebra $\mathcal{H}_q(d)$. Let $A_{\alpha,\beta}(n,d)$ be the subcoalgebra of $A_{\alpha,\beta}(n)$ consisting of homogeneous polynomials of degree d in the generators c_{ij}. If $\alpha\beta = q$ then the dual algebra $A_{\alpha,\beta}(n,d)^*$ is isomorphic to $S_q(n,d)$ (see [14, Theorem 5.5]).

2 Preliminaries

In this section we shall briefly review the representation theory developed for the Dipper–Donkin quantisation in [12] and [11]. A similar theory is developed for the Manin quantisation in [21]. When q is a root of unity we also consider certain related infinitesimal subgroups, as in [12]. If further our field k has positive characteristic, we then generalise this procedure to give a family of such infinitesimal subgroups.

Just as in the classical case, we can define certain subgroups of G corresponding to the Borel subgroup of lower triangular matrices and the torus of diagonal matrices. We shall denote these by B and T respectively. T is just the ordinary (classical) n-dimensional torus, with character group $X(T) \cong \mathbb{Z}^n$.

We make the usual choice of root system (see [19, II 1.21]), and denote the set of roots (respectively positive roots) by Φ (respectively Φ^+). The simple roots will be denoted $\alpha_i = \epsilon_i - \epsilon_{i+1}$, with $1 \leq i < n$. We shall also need to consider the standard basis of $X(T)$ consisting of the fundamental dominant weights $\varpi_i = \epsilon_1 + \cdots + \epsilon_i$, and will occasionally denote ϖ_n just by ϖ. There is a $\mathbb{Z}$-bilinear form $\langle -, - \rangle$ on $X(T)$ satisfying $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. We will also use the usual dominance partial order on $X(T)$.

The symmetric group Σ_n acts on $X(T)$, as an analogue of the classical Weyl group. When considered thus, we shall denote Σ_n by W, and the element of Σ_n corresponding to the transposition $(i, i+1)$ by s_{α_i}. As well as the usual action of W on the character group, we also have the 'dot' action $w.\lambda = w(\lambda + \rho) - \rho$, where $\rho = (n-1, n-2, \ldots, 0)$. With this we can define the affine Weyl group, W_l, associated to G. This is the transformation group on $X(T)$ generated by w with the dot action, and the translations $\lambda \longmapsto \lambda + l\alpha$ for all $\alpha \in \Phi$. Occasionally we shall also need to consider $\bar{\rho}$ which will equal half the sum of the positive roots.

Given any quantum group H with subgroup K, and a K-module V, we shall denote the H-module induced up from V by $\mathrm{ind}_K^H(V)$. The ith right derived functor of induction will in such cases be denoted by $R^i\mathrm{ind}_K^H(V)$. To each element λ of $X(T)$ there corresponds a one-dimensional B-module k_λ of weight λ. We denote the module obtained by inducing this up to G by $\nabla(\lambda)$ (or occasionally $\nabla_n(\lambda)$), and more generally for a B-module V we denote $R^i\mathrm{ind}_B^G(V)$ by $H^i(V)$. When considering both the classical and quantum case simultaneously, we shall sometimes denote the classical induced module by $\bar{\nabla}(\lambda)$.

The weights λ for which $\nabla(\lambda)$ is non-zero form the dominant weights $X(T)^+$. Each $\nabla(\lambda)$ with λ dominant has a simple socle $L(\lambda)$, and these form a complete set of inequivalent simple G-modules. The set of dominant polynomial weights of degree d in turn parameterises the simple $S_q(n,d)$-modules, and will be denoted by $\Lambda^+(n,d)$. Given a module V, we denote the composition multiplicity of $L(\lambda)$ in V by $[V : L(\lambda)]$.

The one-dimensional induced module $\nabla(\varpi)$ corresponds to the determinant representation, and will be denoted q-det (or $\det_q$ for the Manin quantisation). For $\lambda = (lp^{r-1}-1)\rho$, we have $L(\lambda) = \nabla(\lambda)$, and we will denote this module by St_r, the rth *Steinberg module.*

A G-module V is said to have a *good filtration* if there is a G-module filtration $0 = V_0 \leq V_1 \leq \ldots$ with $V = \cup_{i\geq 0} V_i$, such that each quotient V_i/V_{i-1} is either 0 or isomorphic to $\nabla(\lambda_i)$ for some dominant weight λ_i. The multiplicity of $\nabla(\lambda)$ in such a filtration is independent of the choice of filtration, and we denote it by $(V : \nabla(\lambda))$.

We shall denote the injective hull of $L(\lambda)$ as a G-module, and as an $S_q(n,d)$-module, by $I(\lambda)$ and $I_S(\lambda)$ respectively. When determining the blocks of both G and $S_q(n,d)$, the following result from [11, Section 4(6)] will prove useful.

Proposition 2.1 *For $\lambda \in X(T)^+$ (respectively $\Lambda^+(n,d)$), the module $I(\lambda)$ (respectively $I_S(\lambda)$) has a good filtration with multiplicities given by*

$$(I(\lambda) : \nabla(\mu)) = [\nabla(\mu) : L(\lambda)]$$

for $\mu \in X(T)^+$ (respectively $\Lambda^+(n,d)$). Thus two elements λ, $\mu \in X(T)^+$ belong to the same block of G (respectively $S_q(n,d)$) if, and only if, there exists a chain $\lambda = {}_1\lambda, \ldots, {}_t\lambda = \mu$ of elements of $X(T)^+$ (respectively $\Lambda^+(n,d)$) such that for each $1 \leq i < t$ we have either $[\nabla({}_i\lambda) : L({}_{i+1}\lambda)] \neq 0$ or $[\nabla({}_{i+1}\lambda) : L({}_i\lambda)] \neq 0$.

Proof: This follows just as in the classical case (see [8, Theorem 2.6]).

3 Infinitesimal subgroups of q-GL(n, k)

In this section we shall consider certain infinitesimal subgroups of our quantum group. A quantum analogue of the first Frobenius kernel is defined in [12, Section 3.1], and also of the corresponding Jantzen subgroup. When the field k has positive characteristic, this definition can be generalised to give the higher kernels, as outlined in this section.

Throughout this section we take q to be a primitive lth root of unity, and when considering the higher kernels further require that the characteristic p of our field is strictly positive. Note that for this to be possible, we must have $(l,p) = 1$. By [14, (3.1) Theorem, taking $\alpha = 1$ and $\beta = q$] we have that there exists a Hopf algebra homomorphism

$$\hat{F} : k[\mathrm{GL}(n,k)] \longrightarrow k[q\text{-}\mathrm{GL}(n,k)]$$

taking x_{ij} to c_{ij}^{l}. Thus we define the *Frobenius morphism* $F : q\text{-}\mathrm{GL}(n,k) \longrightarrow \mathrm{GL}(n,k)$ to be the morphism of quantum groups with associated comorphism $\hat{F}$. If $p > 0$, then we also have the usual Frobenius map F on $\mathrm{GL}(n,k)$ associated to the comorphism taking x_{ij} to x_{ij}^{p}, and so we may consider the composition of these maps. Henceforth we will abuse notation and write F^r for $\mathrm{F}^{r-1}F$.

Following [11], we say that a quantum group $\bar{H}$ is a *factor group* of a quantum group H if $k[\bar{H}]$ is a subHopf algebra of $k[H]$. Given a factor group $\bar{H}$ of H whose coordinate algebra is central in $k[H]$, we obtain a subgroup H_1 of H whose defining ideal is given by $I_{H_1} = k[H].(\ker(\epsilon_H) \cap k[\bar{H}])$.

Consider the subHopf algebra of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i, j \leq n$, and $d_q^{-lp^{r-1}}$ (where if $r > 1$ we assume that $p > 0$). This is isomorphic to $k[\mathrm{GL}(n,k)]$ via F^r. The corresponding factor group will be denoted $\bar{G}^r$, or just $\bar{G}$ in the case $r = 1$. Then by the previous paragraph, there is a subgroup of G with defining ideal generated by the elements $c_{ij}^{lp^{r-1}} - \delta_{ij}$ for $1 \leq i, j \leq n$, and $d_q^{-lp^{r-1}} - 1$. This subgroup will be denoted G_r, and called the *rth Frobenius kernel*. We can also define infinitesimal analogues of B and T; respectively $B_r = B \bigcap G_r$, and $T_r = T \bigcap G_r$.

Finally, we introduce quantum analogues of the *Jantzen subgroups*, which can be regarded as infinitesimal thickenings of the Frobenius kernels by the torus. Consider the ideal of $k[G]$ generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i \neq j \leq n$. (Again, if $r > 1$ then we assume that $p > 0$.) This is clearly a biideal, and by the isomorphism of $\bar{G}^r$ with $\mathrm{GL}(n,k)$ above, along with the description of the antipode in [4, Lemmas 4.2.20 and 4.2.12], it is easy to verify that it is in fact a Hopf ideal. We denote the subgroup of G with this as defining ideal by G_rT, and the corresponding intersection with B by B_rT. Similarly one can show that the ideal generated by the elements $c_{ij}^{lp^{r-1}}$ for $1 \leq i < j \leq n$ is a Hopf ideal of $k[G]$; we denote the subgroup corresponding to this by G_rB.

The basic representation theory of these infinitesimal subgroups has been developed in [12, Sections 3.1 and 3.2]. Unfortunately, [12] only considers the case $r = 1$; however the arguments given there all still hold, *mutatis mutandis*, in the general case. For most of our purposes the $r = 1$ theory will suffice, but we shall also require the following two more general results.

We define the set of *lp^{r-1}-restricted weights*

$$X_r(T) = \{\lambda \in X(T) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq lp^{r-1} - 1 \text{ for } 1 \leq i \leq n\},$$

where we set $\lambda_{n+1} = 0$ (with our usual requirement for p if $r > 1$). Then we have

Lemma 3.1 *For $\lambda \in X_r(T)$, the simple module $L(\lambda)$ remains simple on restriction to G_r.*

Proof: This follows as in [12, 3.2(3)].

We shall also need

Lemma 3.2 *Let V, $Z \in \text{mod } G$ be such that $\mathrm{res}_{G_r}V$ is absolutely indecomposable, G_r acts trivially on Z, and Z is absolutely indecomposable as a $\bar{G}^r$-module. Then $V \otimes Z$ is an absolutely indecomposable G-module.*

Proof: This follows by the same arguments as in [12, 3.3(5)] (or [7, Section 2, Lemma] in the classical case).

Finally, we consider induction in the infinitesimal case. By arguing as in [21, (9.6.1–2)] we see that for $r = 1$ or $p > 0$ we have

Lemma 3.3 *The induction functor $\mathrm{ind}_B^{G_rB}$ is exact.*

For $\lambda \in X(T)$ define $\hat{Z}_r(\lambda) = \mathrm{ind}_{B_rT}^{G_rT}(k_\lambda)$, and $\tilde{Z}_r(\lambda) = \mathrm{ind}_B^{G_rB}(k_\lambda)$, where as usual if $r > 1$ we assume that $p > 0$. The basic properties of $\hat{Z}(\lambda)$ have been developed in [12] (at least for $r = 1$), but similar arguments also hold for $\tilde{Z}_r(\lambda)$. In particular, we have the following result as in [21, (9.6.5)] (compare with [12, 3.1(21–2)]).

Theorem 3.4 *For any $\lambda \in X(T)$, we have:*

i) $\tilde{Z}_r(\lambda + lp^{r-1}\mu) \cong \tilde{Z}_r(\lambda) \otimes k_{lp^{r-1}\mu}$;

ii) $\tilde{Z}_r(\lambda)^* \cong \tilde{Z}_r(2(lp^{r-1}-1)\bar{\rho} - \lambda)$;

iii) $\tilde{Z}_r(\lambda)$ *has an irreducible head, isomorphic to* $\tilde{L}_r(2(lp^{r-1}-1)\bar{\rho} - \lambda)$;

iv) $res_{G_rT}\tilde{Z}_r(\lambda) \cong \hat{Z}(\lambda)$,

where $\tilde{L}_r(\lambda)$ *is the simple* G_rB*-module of highest weight* λ*, and* $r > 1$ *or* $p > 0$.

Finally, we should remark that in the case where q is a primitive odd root of unity, a similar infinitesimal theory can also be developed for the Manin quantisation. In the case $r = 1$, this has been studied in [21].

4 The strong linkage principle

In this section we will prove the strong linkage principle for q-GL(n, k), when q is a root of unity. This is proved for the Manin quantisation (for q a primitive lth root of unity with l odd) in [21, (10.3.5)] and hence, by the isomorphism of module categories in (1.1), for our chosen quantisation (for odd l). However we will show that the restriction on l is unnecessary in both cases.

Before we can state the main result of this section, we begin with a pair of technical lemmas that will be needed later. We recall from [11, Section 2] that to any composition $\mathbf{a}$ of n, we can associate a corresponding parabolic subgroup $P(\mathbf{a})$. We consider those $\mathbf{a}$ of the form $(1, \ldots, 1, 2, 1, \ldots, 1)$, where the 2 lies in the jth position, and denote the corresponding parabolic just by P_j. For a simple B-module k_λ, we will write $H^i_j(\lambda)$ for $R^i\mathrm{ind}_B^{P_j}(k_\lambda)$. Then we have

Lemma 4.1 *Let* $\lambda = \sum_{i=1}^n t_i\varpi_i \in X(T)$ *with* $t_j > 0$ *for some* $j < n$*. Then there exist the following short exact sequences of* B*-modules:*

i) $0 \longrightarrow K^\lambda_j \longrightarrow H^0_j(\lambda) \longrightarrow k_\lambda \longrightarrow 0$;

ii) $0 \longrightarrow s_{\alpha_j}(\lambda) \longrightarrow K^\lambda_j \longrightarrow V^\lambda_j \longrightarrow 0$;

iii) $0 \longrightarrow C^\lambda_j \longrightarrow V^\lambda_j \longrightarrow F^\lambda_j \longrightarrow 0$;

iv) $0 \longrightarrow F^\lambda_j \longrightarrow H^0_j(\lambda - \alpha) \longrightarrow D^\lambda_j \longrightarrow 0$.

Moreover, the set of weights of both C^λ_j *and* D^λ_j *is* $\{s_{\alpha_j}(\lambda) + tl\alpha_j : 0 < t < t_j/l\}$.

Proof: This corresponds to [21, (10.2.1)], and as the proof given there is valid for all l, the isomorphism in (1.1) gives the result for our quantisation. Alternatively, the argument given can be rederived directly for our quantisation using the explicit description of the B-module structure of the symmetric powers given in [25].

Corollary 4.2 *If $\lambda = \sum_{i=1}^{n} t_i \varpi_i \in X(T)$ with $t_j \geq 0$ for some $j < n$, then there exist two long exact sequences:*

i) $\cdots \to H^r(s_{\alpha_j}.\lambda) \to H^{r-1}(\lambda) \to H^r(V_j^{\lambda+\rho}(-\rho)) \to H^{r+1}(s_{\alpha_j}.\lambda) \to \cdots;$

ii) $\cdots \to H^r(C_j^{\lambda+\rho}(-\rho)) \to H^r(V_j^{\lambda+\rho}(-\rho)) \to H^{r-1}(D_j^{\lambda+\rho}(-\rho)) \to H^{r+1}(C_j^{\lambda+\rho}(-\rho)) \to \cdots,$

where for a B-module X we write $X(-\rho)$ for $X \otimes k_{-\rho}$.

Proof: With the preceding lemma, and [11, Lemma 3.2], we obtain this just as in [21, (10.2.2)].

Just as in [21], we define a strong linkage relation on $X(T)$ with respect to the dot action of the affine Weyl group. In particular, a weight λ is *strongly linked to* μ, written $\lambda \uparrow \mu$, if $\lambda = \mu$ or there exists a finite sequence of weights $\mu = {}_0\mu, {}_1\mu, \ldots, {}_t\mu = \lambda$ such that for $i = 0, \ldots, t-1$,

$${}_{i+1}\mu = s_{\alpha_i} \cdot {}_i\mu + m_i l \alpha_i$$

for α_i a positive root and m_i a non-negative integer with $\langle {}_i\mu + \rho, \alpha_i \rangle \geq m_i l$. The main result in this section is

Theorem 4.3 (The strong linkage principle) *Let $\lambda \in X(T)^+$ and $\mu \in X(T)$ with $\mu + \rho \in X(T)^+$. If $L(\lambda)$ is a composition factor of $H^r(w.\mu)$ for some $w \in W$ and $r \in \mathbb{N}$ then $\lambda \uparrow \mu$.*

Proof: As noted in [21], the result follows just as in [1] provided certain preliminary results hold. We merely verify that each of these results holds just as for the Manin quantisation. Given the previous two lemmas, the result will follow from Serre duality, and the Borel–Weil–Bott theorem for small dominant weights. So we will be done if we can prove the two theorems below.

A dominant weight λ is called *small* if either $\lambda = \sum_{i=1}^{n} r_i \varpi_i$ with $\sum_{i=1}^{n-1} r_i \leq l + 1 - n$, or λ is a minimal dominant weight. We have the following result (compare with [11, Theorem 3.9]).

Theorem 4.4 (Borel–Weil–Bott) *Let $\lambda \in X(T)$ be a small dominant weight. Then*

$$H^r(w.\lambda) = \begin{cases} \nabla(\lambda) & \text{if } r = l(w), \\ 0 & \text{otherwise,} \end{cases}$$

where l is the usual length function on W.

Proof: As we have the Grothendieck vanishing theorem (see [11, Proposition 3.10]), this follows just as in [21, (10.2.3)], noting that the other results used there have already been verified above.

So it remains to prove that Serre duality holds, which shall take the rest of this section. We first give an alternative description of induction for quantum groups.

Given K a subgroup of a quantum group H, and V a K-module, we define a map

$$\Theta : V \otimes k[H] \longrightarrow V \otimes k[K] \otimes k[H]$$

as follows. We will use the convention that we suppress certain summations, indicated by primes, in a similar manner to Sweedler's notation (see [22]). Thus we shall write the structure map τ of V as $v \longmapsto v' \otimes g''$, and comultiplication δ in $k[H]$ by $f \longmapsto f' \otimes f''$. Denoting the antipode in $k[K]$ by σ, and the image of $f \in k[H]$ in $k[K]$ by $\bar{f}$, we define Θ on elements of the form $v \otimes f$ by $v \otimes f \longmapsto v' \otimes g''\sigma(\bar{f}') \otimes f''$, and extend by linearity. We define the fixed points under this map to be those elements $\sum_i v_i \otimes f_i$ satisfying $\Theta(\sum_i v_i \otimes f_i) = \sum_i v_i \otimes 1 \otimes f_i$, and denote the set of these by $(V \otimes k[H])^K$. Then we have

Proposition 4.5 *Given H, K and V as above, we have*

$$\mathrm{ind}_K^H(V) = (V \otimes k[H])^K \,.$$

Proof: We first show $\mathrm{ind}_K^H(V) \subseteq (V \otimes k[H])^K$. Consider $\sum_i v_i \otimes f_i \in \mathrm{ind}_K^H(V)$. Now

$$\Theta\Big(\sum_i v_i \otimes f_i\Big) = (\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \bar{\delta})\Big(\sum_i v_i' \otimes g_i'' \otimes f_i\Big),$$

where $\bar{\delta} = (^{-} \otimes \mathrm{id})\delta$ and m is the usual multiplication map. By the definition of induction (see [11, Section 1]) we have

$$\sum_i v_i' \otimes g_i'' \otimes f_i = \sum_i v_i \otimes \bar{f}_i' \otimes f_i'',$$

and so by applying our alternative description of Θ to this we obtain

$$\Theta\Big(\sum_i v_i \otimes f_i\Big) = \sum_i v_i \otimes \bar{f}_i' \sigma(\bar{f}_i'') \otimes f_i'''.$$

As $\delta(\bar{f}) = \overline{\delta(f)}$, we have by the relations for a Hopf algebra that this equals $\sum_i v_i \otimes 1 \otimes f_i$ as required. Next we consider the reverse inclusion. As

$$(\mathrm{id} \otimes m \otimes \bar{\delta})\Big(\sum_i v_i \otimes 1 \otimes f_i\Big) = \sum_i v_i \otimes \bar{f}_i' \otimes f_i'',$$

we have by our alternative description of Θ above that it is enough to show that

$$(\mathrm{id} \otimes m \otimes \bar{\delta})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \bar{\delta}) = \mathrm{id}.$$

So consider the left-hand side acting on some element $v \otimes a \otimes b$. The image of this is

$$v \otimes a\sigma(\bar{b}')\bar{b}'' \otimes b''' = v \otimes a \otimes b$$

as required, and the result now follows.

With this last result we can now prove

Lemma 4.6 *For all $i \geq 0$, $r \geq 1$ (with $p > 0$ if $r > 1$), $B(n,k)$-modules M and G-modules V, we have*

$$R^i \operatorname{ind}_{G_r B}^{G}(V \otimes M^{F^r}) \cong V \otimes (R^i \operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}.$$

Proof: By the generalised tensor identity [11, Proposition 1.3(ii)], it is enough to show that

$$R^i \operatorname{ind}_{G_r B}^{G}(M^{F^r}) \cong (R^i \operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}.$$

We first consider the case $i = 0$. Let us denote $G_r B$ by H. Now G_r is a subgroup of H, and we shall denote the corresponding factor group by $\bar{H}^r$. Then by the last proposition we have

$$\begin{aligned} \operatorname{ind}_H^G(M^{F^r}) &\cong (M^{F^r} \otimes k[G])^H \\ &\cong ((M^{F^r} \otimes k[G])^{G_r})^{\bar{H}^r} \\ &\cong (M^{F^r} \otimes k[\bar{G}^r])^{\bar{H}^r} \\ &\cong \operatorname{ind}_{\bar{H}^r}^{\bar{G}^r} M^{F^r}, \end{aligned}$$

using for the two intermediate steps [11, Proposition 1.5]. Now we have $\bar{H}^r \cong \mathrm{B}(n,k)$ and $\bar{G}^r \cong \mathrm{GL}(n,k)$, both via F^r. Hence $\operatorname{ind}_H^G(M^{F^r}) \cong (\operatorname{ind}_{\mathrm{B}(n,k)}^{\mathrm{GL}(n,k)} M)^{F^r}$ as required.

The argument for the general case now proceeds much as in [19, I 6.11]. We replace appeals to [19, I 4.5(c)] by [11, Proposition 1.2], and note that induction is exact where required by arguments as in [21, Sections 7.3–4].

We define $N = \binom{n}{2}$. By (4.4), we have that $H^N(-2\bar{\rho}) \cong k$, and so by [21, (10.3.1)] we obtain for every finite dimensional B-module V a pairing

$$H^{N-i}(-2\bar{\rho} \otimes V^*) \otimes H^i(V) \longrightarrow H^N(-2\bar{\rho}) \cong k$$

of rational G-modules. With this we can now prove

Theorem 4.7 (Serre duality) *For any finite dimensional rational B-module V,*

$$H^{N-r}(-2\bar{\rho} \otimes V^*) \cong (H^r(V))^*.$$

Proof: Using (3.3) and (3.4), we obtain [21, (10.3.3)] by the arguments given there. Then, as we have the generalised tensor identities (see [11, Proposition 1.3(ii)]), the proof now follows as in [21, (10.3.4)].

This concludes the proof of the strong linkage principle. Note that, via our usual isomorphism (1.1) this also gives the result for the Manin quantisation without restriction on l (with the appropriate modifications — see (1.2)).

5 The blocks of the q-Schur algebra

The main result in this section is a determination of the block structure of the q-Schur algebra, and hence of q-GL(n, k). If q is not a root of unity then the q-Schur algebra is semisimple by [12, 4.3(7)(i)], so henceforth we assume that q is a primitive lth root of unity. We first consider the blocks of G, using an easy argument based on the strong linkage principle and the following lemma.

Lemma 5.1 *For any dominant weight λ and $r \geq 1$ (with $p > 0$ if $r \neq 1$),*

$$St_r \otimes \bar{\nabla}(\lambda)^{F^r} \cong \nabla((lp^{r-1} - 1)\rho + lp^{r-1}\lambda). \tag{1}$$

Proof: Set $\lambda' = (lp^{r-1} - 1)\rho + lp^{r-1}\lambda$. As both sides of (1) have the same character, by the universal property of ∇'s it is enough to show that

$$\mathrm{soc}(St_r \otimes \bar{\nabla}(\lambda)^{F^r}) \cong L(\lambda').$$

For $0 \leq \alpha < lp^{r-1}$, consider

$$\mathrm{Hom}_G(L(\alpha) \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}) \cong \mathrm{Hom}_{G_r}(L(\alpha) \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r})^{\bar{G}^r}.$$

As $\bar{L}(\beta)^{F^r}$ and $\bar{\nabla}(\lambda)^{F^r}$ are both trivial as G_r-modules, the terms in the right-hand side are isomorphic to direct sums of $L(\alpha)$'s and St_r's. But St_r and $L(\alpha)$ are simple as G_r-modules by (3.1), so for a non-zero homomorphism to exist we must have $L(\alpha) \cong St_r$.

By Schur's lemma, $\mathrm{Hom}_{G_r}(St_r, St_r) = k$, and we have an injection

$$\mathrm{Hom}_k(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r}) \longrightarrow \mathrm{Hom}_{G_r}(St_r \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}),$$

taking θ to $1 \otimes \theta$. By dimensions this is an isomorphism. Hence

$$\begin{aligned} \mathrm{Hom}_G(St_r \otimes \bar{L}(\beta)^{F^r}, St_r \otimes \bar{\nabla}(\lambda)^{F^r}) &\cong \mathrm{Hom}_k(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r})^{\bar{G}^r} \\ &\cong \mathrm{Hom}_{\bar{G}^r}(\bar{L}(\beta)^{F^r}, \bar{\nabla}(\lambda)^{F^r}) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_n}(\bar{L}(\beta), \bar{\nabla}(\lambda)) \\ &\cong \begin{cases} k & \text{if } \beta = \lambda, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $\mathrm{soc}(St_r \otimes \bar{\nabla}(\lambda)^{F^r})$ consists of copies of $St_r \otimes \bar{L}(\lambda) \cong L(\lambda')$. But $\dim \bar{\nabla}(\lambda)_\lambda = 1$ and all other weights are less than λ, so only one such copy can occur, which gives the result.

Consider $\lambda \in X(T)$, not equal to $-\rho$. If $p > 0$ then we define $m(\lambda)$ to be the least positive integer such that there exists an $\alpha \in \Phi^+$ with $\langle \lambda + \rho, \alpha \rangle \notin lp^{m(\lambda)}\mathbb{Z}$. If $p = 0$ then we define $m(\lambda)$ to be 0 if there exists an $\alpha \in \Phi^+$ with $\langle \lambda + \rho, \alpha \rangle \notin l\mathbb{Z}$ and 1 otherwise. Our first partial result on the blocks of G is

Proposition 5.2 *If $\lambda \in X(T)^+$, then $(W.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap X(T)^+$ is a union of blocks for G. If further $p = 0$ and $m(\lambda) = 1$ then λ is the unique element in its block.*

Proof: First consider the case $m(\lambda) = 0$. It is enough to check that if τ, ν are dominant, with $\tau \in W.\lambda + l\mathbb{Z}\Phi$ and $L(\nu)$ a composition factor of $\nabla(\lambda)$, then $\nu \in W.\lambda + l\mathbb{Z}\Phi$. But this is an easy consequence of the strong linkage principle (4.3).

Now suppose that λ is any dominant weight, with $m(\lambda) = m > 0$. Again, it is enough to show that if τ, ν are dominant, with $\tau \in W.\lambda + lp^m\mathbb{Z}\Phi$ and $L(\nu)$ a composition factor of $\nabla(\tau)$ then $\nu \in W.\lambda + lp^m\mathbb{Z}\Phi$, or if $p = 0$ that $\nu = \tau$. We first note that we have

$$\tau = (lp^{m-1} - 1)\rho + a\varpi + lp^{m-1}\tau',$$

with $\tau' \in X^+$ and $0 \leq a < lp^{m-1}$ (c.f. the definition of *normal form* in [10, Section 1]). So by the preceding lemma, $\nabla(\tau - a\varpi) \cong St_m \otimes \bar{\nabla}(\tau')^{F^m}$, and hence $\nabla(\tau) \cong (q\text{-det})^a \otimes St_m \otimes \bar{\nabla}(\tau')^{F^m}$ (as both sides have a simple socle and the same character). It is easy to see, by Steinberg's Tensor Product Theorem, that any composition factor of this module is of the form $L(\nu) \cong (q\text{-det})^a \otimes St_m \otimes \bar{L}(\nu')^{F^m}$, and (decomposing λ in the same way as for τ) that $\tau' \in W.\lambda' + p\mathbb{Z}\Phi$. So it is enough to show that $\nu' \in W.\lambda' + p\mathbb{Z}\Phi$, or if $p = 0$ that $\nu'=\lambda'$. But this follows for $p > 0$ from the Strong Linkage Principle for $\mathrm{GL}(n, k)$ [1, Theorem 1], and for $p = 0$ from the semisimplicity of $\mathrm{Mod}(\mathrm{GL}(n, k))$.

We will show later that the sets described in the theorem above are in fact precisely the blocks of G. This will follow from the following description of the blocks of the q-Schur algebra, which most of the rest of this section is taken up with proving.

Theorem 5.3 *For any $\lambda \in \Lambda^+(n, d)$, the $S_q(n, d)$-block containing λ is*

$$\begin{array}{ll} (W.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap \Lambda^+(n, d) & \text{if } p > 0 \text{ or } m(\lambda) = 0, \\ \{\lambda\} & \text{otherwise.} \end{array}$$

In what follows, it will be convenient to call a weight λ *primitive* if $m(\lambda) = 0$. By the last result, it makes sense to define a *primitive block* as one consisting of primitive elements. We first deal with the non-primitive blocks, as for these the result can be easily deduced from the classical case.

Proposition 5.4 *For $d \geq 0$, $m \geq 0$ (with $p > 0$ if $m \neq 0$), $0 \leq a < lp^m$ and $\mathcal{B}$ a block of $S(n, d)$, the set*

$$\mathcal{B}^\dagger = \{(lp^m - 1)\rho + a\varpi + lp^m\mu : \mu \in \mathcal{B}\}$$

is a block of $S_q(n, e)$, where $e = (lp^m - 1)|\rho| + na + lp^m d$.

Proof: Define $\Phi : \mathrm{mod}\ \mathrm{GL}(n, k) \longrightarrow \mathrm{mod}\ G$ by

$$\begin{array}{ll} \Phi(\bar{V}) & = (q\text{-det})^a \otimes St_{m+1} \otimes \bar{V}^{F^{m+1}}, \\ \Phi(\theta) & = 1 \otimes 1 \otimes \theta. \end{array}$$

Now $(q\text{-det})^a \otimes St_{m+1} \cong \nabla(\sigma)$ where $\sigma = (lp^m - 1)\rho + a\varpi$, as both sides have a simple socle and the same character. The result now follows just as in [10, Section 4, Theorem], noting that if $\bar{V}$ is indecomposable then so is $\nabla(\sigma) \otimes \bar{V}^{F^{m+1}}$ by (3.2).

With the above proposition, the main theorem now follows for λ non-primitive from the description of the blocks for $\mathrm{S}(n,d)$ given in [10, Section 4, Corollary] if $p>0$, and from the semisimplicity of $\mathrm{Mod}(S(n,d))$ otherwise.

So it remains to prove the theorem when λ is a primitive weight. Recall that to each partition λ we can associate a corresponding l-core, whose Young diagram is obtained from that of λ by removing skew l-hooks. We shall call a subset of $\Lambda^+(n,d)$ a *core class* if it consists of all elements of $\Lambda^+(n,d)$ with some fixed l-core. We note that the remarks in [10, page 405] concerning p-cores all hold when p is replaced by l, and so (given our partial result on the blocks of G) we have the following result.

Lemma 5.5 *For primitive dominant weights $\lambda=(\lambda_1,\ldots,\lambda_n)$ and $\mu=(\mu_1,\ldots,\mu_n)$, if λ and μ belong to the same G-block then there exists a $\pi\in\Sigma_n$ such that*

$$\lambda_i - i \equiv \mu_{\pi(i)} - \pi(i) \pmod{l}$$

for all $1\leq i\leq n$. Further, for any λ, $\mu\in\Lambda^+(n,d)$ there exists such a $\pi\in\Sigma_n$ if, and only if, λ and μ have the same l-core.

Remark 5.6 When $d\leq n$ the blocks of the q-Schur algebra have already been determined by James and Mathas [18, Theorem 4.24]. To see that our main result is consistent with this, note that in this case all of the elements $\lambda\in\Lambda^+(n,d)$ are primitive. Hence (5.3) and the last lemma give that λ and μ lie in the same block for $S_q(n,d)$ if, and only if, they have the same l-core.

The remainder of this section is essentially devoted to verifying that the proof given in the classical case for primitive blocks in [10, Section 3, Theorem] holds (with the obvious modifications) in this setting. Examination of the proof given there gives that this will be the case provided [10, Section 3 (1–6), Section 1 (5,8) and Section 2 (3, Proposition)] all hold.

To a partition λ we can associate a *hook tableau* $h_{ij}(\lambda)$, which is the λ-tableau whose (i,j)th entry is the hook length $\lambda_i+\lambda'_j-i-j+1$. Here λ' is the conjugate partition to λ. Further, given two partitions $\lambda\in\Lambda^+(n,d)$ and $\mu\in\Lambda^+(m,e)$ with $\lambda_n\geq\mu_1$, we shall write $(\lambda|\mu)$ for the partition in $\Lambda^+(n+m,d+e)$ obtained by concatenation.

An element $\lambda\in\Lambda^+(n,d)$ is called *row l-regular* if there does not exist an i with $0\leq i\leq n-l$ such that $\lambda_{i+1}=\lambda_{i+2}=\cdots=\lambda_{i+l}>0$. The set of row l-regular elements in

$\Lambda^+(n,d)$ is denoted $\Lambda^+(n,d)_{\text{row}}$. We will also need to consider the Schur functor $f = f_{n,d}$: mod $S_q(n,d) \longrightarrow$ mod $\mathcal{H}_q(d)$, defined when $d \leq n$. This is analogous to the usual Schur functor defined in [15, Chapter 6], and its basic properties are outlined in [12, Section 2.1].

The six results in Section 3, and Section 2(3), are all straightforward, and summarised in the following proposition.

Proposition 5.7 *i) (Carter's Criterion) Let $\lambda \in \Lambda^+(n,d)$ with $d \leq n$. Then λ is row l-regular and $f\nabla(\lambda)$ is irreducible if, and only if, the l-adic valuation $\nu_l : \mathbb{Z} \longrightarrow \mathbb{N}$ is constant on the columns of the hook tableau $(h_{ij}(\lambda))$.*

ii) A partition λ is an l-core if, and only if, every entry of the hook tableau is not divisible by l.

iii) Every core class in $\Lambda^+(n,d)$ is a union of $S_q(n,d)$-blocks, and if $\lambda \in \Lambda^+(n,d)$ is primitive then its core class consists of primitive elements.

iv) Elements λ and μ of $\Lambda^+(n,d)$ belong to the same core class if, and only if, $\lambda + \varpi$ and $\mu + \varpi$ belong to the same core class in $\Lambda^+(n, d+n)$

v) If λ, $\mu \in \Lambda^+(n,d)$ are in the same $S_q(n,d)$-block, then $\lambda + \varpi$, $\mu + \varpi \in \Lambda^+(n, d+n)$ are in the same $S_q(n, d+n)$-block

vi) Suppose that $(\lambda|\mu) \in \Lambda^+(n+m, d+e)$, with μ being the unique maximal element in its $S_q(m,e)$-block and $\tau \in \Lambda^+(m,e)$ belonging to the same block as μ. Then $(\lambda|\tau)$ is in the same $S_q(n+m, d+e)$-block as $(\lambda|\mu)$.

vii) If λ, $\mu \in \Lambda^+(n,d)$ belong to the same block of $S_q(n,d)$, then they belong to the same block of $S_q(m,d)$ for all $m \geq n$.

Proof: A quantum Carter's criterion is proved in [18, 4.15] for q-Specht modules, and hence by the identification of these in [12, 4.5h] with the $f\nabla(\lambda)$'s we obtain (i). For (ii) the classical criterion for λ to be a p-core in [10] can be seen from [17, 2.7.40] not to require p prime (provided we replace "coprime to p" by "not divisible by p"). The result on core classes (iii) follows directly from the previous lemma and proposition, while (iv) is immediate. Both (v) and (vi) — consequences of James' results on row and column removal and decomposition numbers — hold with the same proofs as given, but replacing references to [9, Theorems 1 and 2] by [12, 4.2(9) and 4.2(15)] respectively. Finally, the last part is clear from [12, 4.2(6)]

The next result is an analogue of [10, Section 1(5)].

Proposition 5.8 *Let $\lambda \in X(T)^+$. Suppose that λ is primitive and $\nabla(\lambda)$ is irreducible. Then we have $\langle \lambda, \alpha \rangle < l$ for all $\alpha \in \Pi$.*

Proof: Consider the Manin quantisation. Now Steinberg's tensor product theorem holds for l odd by [21, (9.4.1)], and for l even by [2, Theorem, and concluding remarks] (but replacing l in this case by $l/2$ as remarked in (1.2)). Hence it also holds for $\mathrm{SL}_q(n,k)$. Now [16, Section 2.5 Theorem] clearly holds with l or $l/2$ replacing p, and hence we have [16, Section 2.5 Corollary], possibly with l modified. The result now follows just as in [10] (possibly with modified l). Again, tensoring with $\det_q$ will not affect reducibility, giving the result for $\mathrm{GL}_q(n,k)$. The usual category isomorphism (1.1) now gives the result for our quantisation, and corrects any modifications to l introduced during the Manin stage.

To start the induction off in the proof of the main theorem, we need to check some small cases. It will be convenient for this to define, as in [19, II 5.7], an Euler characteristic for any given finite dimensional B-module M by

$$\chi(M) := \sum_{i \geq 0} (-1)^i \mathrm{ch} H^i(M).$$

As usual, we write $\chi(\lambda)$ for $\chi(k_\lambda)$, and then Kempf's vanishing theorem [11, Theorem 3.4] gives

$$\chi(\lambda) = \mathrm{ch}\nabla(\lambda) \quad \text{for all } \lambda \in X(T)^+,$$

so our notation agrees with that in [11]. Just as in the classical case we have

Lemma 5.9 *i) The characters $\mathrm{ch}L(\lambda)$ with $\lambda \in X(T)^+$ form a basis of $\mathbb{Z}[X]^W$.*

ii) For all $\lambda \in X$ and $\sum_\mu a(\mu)e(\mu) \in \mathbb{Z}[X]^W$,

$$\chi(\lambda) \sum_\mu a(\mu)e(\mu) = \sum_\mu a(\mu)\chi(\lambda + \mu).$$

iii) For all $w \in W$ and $\lambda \in X$ we have $\chi(w.\lambda) = (\mathrm{sgn}\ w)\chi(\lambda)$.

Proof: The first two parts follow just as in [19, II 5.8 Lemma], using [11, Theorem 2.10 and Lemma 3.1], while the last part follows just as in [19, II 5.9] from [11, Lemmas 2.12 and 3.1].

We are now able to check the necessary small cases, corresponding to those in [10, Section 1(8)].

Proposition 5.10 *Let $n = 1$,2 or 3, and $\lambda \in X(T)^+$ be primitive. Then*

i) the module $\nabla(\lambda)$ is irreducible if, and only if, λ is minimal in its block;

ii) each primitive G-block contains a unique minimal element.

Proof: The case $n = 1$ is clear, while the $n = 2$ case follows from [3, Theorem 2.1 and Corollary 2.2]. Consider $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ primitive. Without loss of generality we may assume that $\lambda_3 = 0$ (as tensoring up with an appropriate power of q-det will give the general case in what follows). By (5.8), if λ is primitive and $\nabla(\lambda)$ is irreducible then $0 \leq \lambda_1 - \lambda_2, \lambda_2 \leq l-1$ and $(\lambda_1 - \lambda_2, \lambda_2) \neq (l-1, l-1)$. Suppose $(\lambda_1 - \lambda_2) + \lambda_2 > l$, and $\lambda_1 - \lambda_2, \lambda_2 < l-1$ (this cannot arise when $l = 2$). Then $\lambda_1 - \lambda_2 = l-1-a$ and $\lambda_2 = l-1-b$ with a, $b > 0$ and $a+b < l-2$. Consider $\nabla(l-2-b, l-1-a-b, 0)$. This is minimal in its block, and hence simple, so by Steinberg's Tensor Product Theorem we see that $\bar{\nabla}(1,0,0)^F \otimes \nabla(l-2-b, l-1-a-b, 0)$ is also simple, isomorphic to $L(\tau)$ for some τ.

Now, using the previous proposition, we have

$$\begin{aligned} \mathrm{ch}L(\tau) &= (e(l,0,0) + e(0,l,0) + e(0,0,l))\chi(l-2-b, l-1-a-b, 0) \\ &= \chi(\tau) + \chi(l-2-b, 2l-1-a-b, 0) + \chi(l-2-b, l-1-a-b, l) \\ &= \chi(\tau) - \chi(2l-2-a-b, l-1-b, 0) + \chi(l-2, l-1-b, l-a-b). \end{aligned}$$

After rearranging, and noting that the central term on the right is just $\mathrm{ch}\nabla(\lambda_1, \lambda_2, \lambda_3)$, we see that

$$\mathrm{ch}\nabla(\tau) = \mathrm{ch}L(\tau) + \mathrm{ch}\nabla(\lambda) - \mathrm{ch}\nabla(l-2, l-1-b, l-a-b),$$

which implies that $\nabla(\lambda)$ is not simple. After tensoring with q-det we see that the primitive weights λ with $\nabla(\lambda)$ simple are a subset of

$$\begin{gathered} \{(\lambda_1, \lambda_2, \lambda_3) \in X(T)^+ : 0 \leq \lambda_1 - \lambda_3 \leq l-2\} \\ \cup\{(l-1+a+\lambda_3, a+\lambda_3, \lambda_3), (l-1+a+\lambda_3, l-1+\lambda_3, \lambda_3) : 0 \leq a \leq l-2\}. \end{gathered}$$

But all these elements are minimal in their corresponding blocks, and as any minimal element must be simple this gives the result.

As this last result corresponds to [10, Section 2(3)], it just remains to check the following proposition (corresponding to that in [10, Section 2]), and most of the rest of this section will be devoted to this. For the rest of this section we assume that $d \leq n$.

Proposition 5.11 *Let $\lambda \in \Lambda^+(n,d)$ be such that $[\nabla(\lambda) : L(\mu)] = 0$ for all $\mu \in \Lambda^+(n,d)$ with $\mu > \lambda$. Then* *i) for all $N \geq n$, the $S_q(N,d)$-module $\nabla_N(\lambda)$ is injective;*

ii) λ is row l-regular;

iii) for $N \geq n$, the $k\Sigma_N$-module $f_{N,d}\nabla_N(\lambda)$ is irreducible.

We say that $\lambda \in \Lambda^+(n,d)$ is *column l-regular* if $\lambda_i - \lambda_{i+1} < l$ for all $1 \leq i \leq n$, and denote the set of these by $\Lambda^+(n,d)_{\mathrm{col}}$. With this notation we have the following result from [12, 4.4(4)(ii)].

Lemma 5.12 *Suppose that $d \leq n$. Then $\{fL(\lambda) : \lambda \in \Lambda^+(n,d)_{\mathrm{col}}\}$ is a complete set of inequivalent irreducible $\mathcal{H}_q(d)$-modules.*

As noted in [12, 4.3(10)(ii)], there is a bijection $i : \Lambda^+(n,d)_{\mathrm{row}} \longrightarrow \Lambda^+(n,d)_{\mathrm{col}}$ such that, for $\lambda \in \Lambda^+(n,d)_{\mathrm{row}}$, we have $I_S(\lambda) \cong T(i(\lambda))$, the indecomposable tilting module of highest weight $i(\lambda)$. Hence we obtain

Lemma 5.13 *Suppose $d \leq n$ and $\lambda \in \Lambda^+(n,d)_{\mathrm{row}}$. Then $i(\lambda)$ is the unique highest element in the set $\mathcal{D}(\lambda) = \{\mu \in \Lambda^+(n,d) : [\nabla(\mu) : L(\lambda)] \neq 0\}$, and further $[\nabla(i(\lambda)) : L(\lambda)] = 1$.*

We also have a notion of contravariant duality (see [12, Remarks before 4.1d]), and we shall denote the contravariant dual of a module V by V^0. This, combined with the results above, allows us to prove the following analogue of [10, 2(5)].

Lemma 5.14 *Suppose $d \leq n$, and let $f = f_{n,d}$. Then we have:*

i) $E^{\otimes d} \cong \bigoplus_{\lambda \in \Lambda^+(n,d)_{row}} I_S(\lambda)^{d(\lambda)}$, where $d(\lambda) = \dim fL(\lambda)$;

ii) $I_S(\lambda)^0 \cong I_S(\lambda)$ for $\lambda \in \Lambda^+(n,d)_{\mathrm{row}}$, and $I_S(\lambda)$ has unique highest weight $i(\lambda)$;

iii) $(E^{\otimes d} : \nabla(\mu)) = \dim f\nabla(\mu)$.

Proof: We have that $E^{\otimes d}$ is injective by [12, 2.1(8)], and the rest of part (i) follows just as in the classical case, using [12, 4.3(9)] instead of [15, (6.4b)]. Part (ii) follows from the arguments above and [12, 4.3(10)(i)]. Finally part (iii) follows much as in the original case, but replacing reduction to characteristic zero by reduction to the case q a non-root of unity, and then using that the corresponding q-Schur algebra is semi-simple (see [12, 4.3(7)(i)]).

We are now almost in a position to prove (5.11). The one outstanding fact needed is an analogue of [15, (6.4c) Theorem] giving a basis of $f\nabla(\lambda)$. But, using the identification given in [12, 4.5h] of $f\nabla(\lambda)$ with the Specht module of Dipper and James, this follows from [6, 8.1], as noted in [13, Remark after Theorem 1.5]. Now (5.11) follows just as in [10], which then gives the main result.

Finally in this section, we use (5.3) to determine precisely the blocks of G.

Theorem 5.15 *For $\lambda \in X(T)^+$, the G-block containing λ is*

$$\begin{array}{ll} (W.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap X(T)^+ & \text{if } p > 0 \text{ or } m(\lambda) = 0 \\ \{\lambda\} & \text{otherwise.} \end{array}$$

Proof: Clearly, by (5.2), it is enough to show that any two elements of the above set are in the same block. So assume that $p > 0$ or $m(\lambda) = 0$, and that τ, $\mu \in (W.\lambda + lp^{m(\lambda)}\mathbb{Z}\Phi) \cap X(T)^+$. If these lie in $\Lambda^+(n,d)$ for some d then we are done, as they are then in the same $S_q(n,d)$ block, and the result follows from [11, 4(5)]. Otherwise there exists an e such that $\tau' = \tau + e\varpi$ and $\mu' = \mu + e\varpi$ lie in $\Lambda^+(n,d)$ for some d. As these then lie in the same block of G by the above argument, there exists a sequence of weights, say $\tau' = {}_1\tau', \ldots, {}_t\tau' = \mu'$, with $[\nabla({}_i\tau') : L({}_{i+1}\tau')] \neq 0$ or $[\nabla({}_{i+1}\tau') : L({}_i\tau')] \neq 0$ for $1 \leq i < t$. Setting ${}_i\tau = {}_i\tau' - e\varpi$, we note that $\nabla({}_i\tau') \cong \nabla({}_i\tau) \otimes (q\text{-det})^e$ and $L({}_i\tau') \cong L({}_i\tau) \otimes (q\text{-det})^e$. Thus the sequence $\tau = {}_1\tau, \ldots, {}_t\tau = \mu$ is such that $[\nabla({}_i\tau) : L({}_{i+1}\tau)] \neq 0$ or $[\nabla({}_{i+1}\tau) : L({}_i\tau)] \neq 0$ for $1 \leq i < t$. Hence τ and μ lie in the same G-block.

I am grateful to Stephen Donkin for various helpful comments and suggestions. This work was supported by the EPSRC.

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