Comonotonic approximations to quantiles of life annuity conditional expected present values: extensions to general ARIMA models and comparison with the bootstrap

M. Denuit, S. Haberman and A.E. Renshaw

Abstract
This paper aims to provide accurate approximations for the quantiles of the conditional expected present value of the payments made by the annuity provider, given the future path of the Lee-Carter time index. Conditional cohort and period life expectancies are also considered. The paper also addresses some associated simulation issues, which, hitherto, have been unresolved.

Key words and phrases: Life annuity, life expectancy, mortality projection, Lee-Carter model, comonotonicity, simulation.

1 Lee-Carter stochastic modelling for dynamic mortality

1.1 Motivation

In this paper, we consider present values of life annuity benefits as functions of the unknown life table applying in the future to the policyholders of a portfolio when death rates are described by the Lee-Carter model. Deriving the exact distribution for this random variable requires extensive simulations or numerical evaluations. Therefore, we take the comonotic approximations proposed by Denuit & Dhaene (2007) and Denuit (2007) in the random walk with drift case and extend these to general ARIMA models. This helps avoid the requirement to conduct simulations within simulations in, for instance, Solvency 2 reserving calculations. Numerical illustrations show that the comonotonic approximations perform well, which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates.

1.2 Log-bilinear model for mortality projection

We recall the basic features of the classical Lee-Carter approach. In this framework, the population central death rate at age *x* in year *t*, denoted as $m_x(t)$, is of the form

$$\ln m_x(t) = \alpha_x + \beta_x \kappa_t . \qquad (1.1)$$

Interpretation of the parameters involved in model (1.1) is quite simple. The value of α_x is an average of $\ln m_x(t)$ over time *t* so that $\exp \alpha_x$ represents the general shape of the age-specific mortality profile. The actual forces of mortality change according to an overall mortality index κ_t which is modulated by an age response variable β_x. β_x

indicates the sensitivity of different ages to the time trend so that the shape of the β_x profile tells which rates decline rapidly and which slowly over time in response to changes in κ_t.

An appropriate error structure has to be specified in order to estimate the parameters involved in (1.1). Lee & Carter (1992) opted for Normal disturbances and an estimation procedure based on Singular Value Decomposition whereas the Authors propose Binomial, Poisson or Negative Binomial regression models. Note that the results derived in this paper apply whatever the statistical model used for estimation purposes. In the empirical illustrations, a Poisson regression model will be used.

1.3 Stochastic modelling of the time index

In order to make forecasts, Lee & Carter (1992) assume that the α_x and β_x remain constant over time and forecast future values of κ_t using a standard univariate time series model. After testing several specifications, they found that a random walk with drift was the most appropriate model for their data. They also make clear that other ARIMA models might be preferable for different data sets.

Here, we assume that the κ_t obey an ARIMA(p,1,q) model, with arbitrary values of p and q, which are to be determined. Furthermore, we assume that the κ_t are positively dependent, in the sense that the covariance between any pair $(\kappa_{t_1}, \kappa_{t_2})$ of time indices is non-negative. Since the κ_t are multivariate normal, this ensures that the κ_t are positively associated, that is, the inequality

$$Cov\left[\Psi_1\left(\kappa_{t_1}, \kappa_{t_2}, ..., \kappa_{t_n}\right), \Psi_2\left(\kappa_{t_1}, \kappa_{t_2}, ..., \kappa_{t_n}\right)\right] \geq 0$$

is valid for all values $t_1 < t_2 < ... < t_n$ and for all choices of the non-decreasing functions Ψ_1 and Ψ_2 such that the covariance exists.

2 Life annuity and life expectancy

2.1 Life annuity conditional expected present value

Let us denote as ${}_dP_x(t_0 \mid \boldsymbol{\kappa})$ the random d-year survival probability for an individual aged x in year t_0, that is, the conditional probability that this individual reaches age $x + d$ in year $t_0 + d$, given the vector $\boldsymbol{\kappa}$ of the κ_t. It is formally defined as

$${}_dP_x(t_0 \mid \boldsymbol{\kappa}) = \exp\left\{-\sum_{j=0}^{d-1} \exp\left(\alpha_{x+j} + \beta_{x+j}\kappa_{t_0+j}\right)\right\}.$$

Let us consider a basic life annuity contract paying 1 unit of currency at the end of each year, as long as the annuitant survives. The random life annuity single premium, that is, the conditional expectation of the payments made to an annuitant aged x in the year t_0 given the time index, is

$$a_x(t_0 \mid \boldsymbol{\kappa}) = \sum_{d \geq 1} {}_dP_x(t_0 \mid \boldsymbol{\kappa}) v(0,d),$$

where $v(.,.)$ is the discount factor (precisely, $v(s,t)$ is the present value at time s of a unit payment made at time t). Note that $a_x(t_0 \mid \boldsymbol{\kappa})$ corresponds to the generation aged x in calendar year t_0, and accounts for future mortality improvements experienced by this particular cohort. Clearly, $a_x(t_0 \mid \boldsymbol{\kappa})$ is a random variable that depends on the future trajectory of the time index (that is, on $\kappa_{t_0}, \kappa_{t_0+1}, \kappa_{t_0+2}, \ldots$). An analytical computation of the distribution function of $a_x(t_0 \mid \boldsymbol{\kappa})$ seems to be out of reach.

The random variable $a_x(t_0 \mid \boldsymbol{\kappa})$ can be regarded as the residual risk per annuity contract in a sufficiently large portfolio. Indeed, let us consider a group of annuitants who are all aged x in year t_0, with respective remaining life times $T_1, T_2, T_3, \ldots$. Given the time index, these random variables are assumed to be independent and identically distributed, with common conditional d-year survival probability ${}_dP_x(t_0 \mid \boldsymbol{\kappa})$. Formally,

$$P[T_1 > d_1, \ldots, T_n > d_n \ ] = E\left[\prod_{i=1}^{n} P\left[T_i > d_i \middle| \kappa\right]\right] = E\left[\prod_{i=1}^{n} {}_{d_i}P_x\left(t_0 \middle| \kappa\right)\right].$$

Let us denote as $[\xi]$ the integer part of ξ, and as

$$a_{\overline{T_i}|} = \sum_{d=1}^{[T_i]} v(0,d)$$

the present value of the payments made to annuitant i (with the convention that the empty sum is zero). Now, since the $a_{\overline{T_i}|}$ are exchangeable, we have from Proposition 1.1 in Denuit & Vermandele (1998) that the stochastic inequality

$$a_x(t_0 \mid \boldsymbol{\kappa}) = E\left[a_{\overline{T_i}|} \middle| \kappa\right] \leq_{CX} \ldots \leq_{CX} \frac{\sum_{i=1}^{n+1} a_{\overline{T_i}|}}{n+1} \leq_{CX} \frac{\sum_{i=1}^{n} a_{\overline{T_i}|}}{n} \leq_{CX} \ldots \leq_{CX} a_{\overline{T_1}|},$$

is valid for any n, where $\leq_{CX}$ denotes the convex order, defined for random variables X and Y as X$\leq_{CX}$Y if $E[g(X)]\leq E[g(Y)]$ for all the convex functions g for which the expectations exist. In words, X$\leq_{CX}$Y means that X is less variable, or less dangerous than

Y. Increasing the size of the portfolio makes the average payment per annuity less variable (in the $\leq_{CX}$-sense), but this average remains random whatever the number of policies comprising the portfolio, being bounded from below by $a_x(t_0 \mid \boldsymbol{\kappa})$ in the $\leq_{CX}$-sense. It is interesting to note that, even if $T_1, T_2, T_3, ...$ are positively dependent, some diversification remains as long as the economic capital is computed from a risk measure agreeing with $\leq_{CX}$.

2.2 Period life expectancies

Demographic indicators can be calculated in two ways. Period indicators are worked out using age-specific mortality rates for a given year, with no allowance for any later actual or projected changes in mortality. Cohort indicators are worked out using age-specific mortality rates which allow for known or projected changes in mortality in later years. In this section, we consider the period life expectancy, computed from the set of death rates corresponding to a given calendar year.

Let us denote as $e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k})$ the period conditional life expectancy at age x in year $t_0 + k$, given κ_{t_0+k}. Assuming that the deaths are uniformly distributed over the calendar year, this demographic indicator is given by

$$e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k}) = \frac{1}{2} + \sum_{d \geq 1} \exp\left\{ -\sum_{j=0}^{d-1} \exp\left(\alpha_{x+j} + \beta_{x+j}\kappa_{t_0+k} \right) \right\}.$$

The superscript $\uparrow$ is used to indicate that we work along a vertical band in the Lexis diagram. Henceforth, we denote the distribution function of $e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k})$ by $F_{e_x^{\uparrow}(t_0+k|\kappa_{t_0+k})}$. Note that computation of life annuity values in a period setting cannot be justified when computation in the cohort setting is possible, since this approach underestimates the liabilities of the annuity provider when mortality declines.

In many applications of the Lee-Carter model, we find that all of the β_{x+j} typically have the same sign. It is then easy to see that $e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k})$ appears as a 1-1 monotone function of κ_{t_0+k} (and only depends on a single time index). Let us assume that all of the β_{x+j} are positive. Then, $e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k})$ is a decreasing function of the time index κ_{t_0+k}. The quantile function of $e_x^{\uparrow}(t_0 + k \mid \kappa_{t_0+k})$ is then given by

$$F^{-1}_{e_x^{\uparrow}(t_0+k|\kappa_{t_0+k})}(z) = \frac{1}{2} + \sum_{d \geq 1} \exp\left[-\sum_{j=0}^{d-1} \exp\left\{ \alpha_{x+j} + \beta_{x+j} \left(E[\kappa_{t_0+k}] + \sqrt{Var[\kappa_{t_0+k}]}.\Phi^{-1}(1-z) \right) \right\} \right] \quad (2.1)$$

Where the expectation and variance are conditional to past values of the time index and Φ^{-1} is the quantile function of $N(0,1)$.

2.3 Cohort life expectancies

Cohort life expectancies forecast the expected remaining lifetime taking into account future changes in mortality. They are usually computed at the end of the observation period (at time t_0). Specifically, $e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa})$ is the expected remaining lifetime of an individual aged x in year t. Keeping the assumption that deaths are uniformly distributed over each calendar year, this demographic indicator is given by

$$e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa}) = \frac{1}{2} + \sum_{d\geq 1} {}_dP_x(t_0\,|\,\boldsymbol{\kappa}).$$

We use the $\nearrow$ superscript to indicate that we work along a diagonal band in the Lexis diagram. Note that $e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa})$ is a random variable that depends on the future trajectory of the κ_t's (and not on a single time index, as period life expectancies). Except for the additive constant $1/2$, $e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa})$ coincides with $a_x(t_0\,|\,\boldsymbol{\kappa})$ if we let the interest rate tend to zero. As was the case for $a_x(t_0\,|\,\boldsymbol{\kappa})$, an analytic computation of the distribution function of $e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa})$ thus seems to be out of reach.

3 Comonotonic approximations

3.1 Comonotonic approximations to life annuity conditional expected present value

Assuming a random walk with drift model for the κ_t's, Denuit & Dhaene (2007) have proposed comonotonic approximations for the quantiles of the random survival probabilities ${}_dP_x(t_0\,|\,\boldsymbol{\kappa})$. Since the expression for $a_x(t_0\,|\,\boldsymbol{\kappa})$ involves the weighted sum of the ${}_dP_x(t_0\,|\,\boldsymbol{\kappa})$'s, Denuit (2007) supplemented this first comonotonic approximation with a second one. Here, we extend these results to general ARIMA dynamics for the κ_t's.

Let us define

$$S_d = \sum_{j=0}^{d-1} \exp\left(\alpha_{x+j} + \beta_{x+j}\kappa_{t_0+j}\right) = \sum_{j=0}^{d-1} \delta_j \exp\left(Z_j\right),$$

where $\delta_j = \exp\left(\alpha_{x+j}\right) > 0$ and $Z_j = \beta_{x+j}\kappa_{t_0+j}$. Clearly ${}_dP_x(t_0\,|\,\boldsymbol{\kappa}) = \exp(-S_d)$. Conditional on κ_{t_0}, it follows that $Z_j \sim N(\mu_j, \sigma_j^2)$ with

$$\mu_j = \beta_{x+j} E\left[\kappa_{t_0+j}\right] \text{ and } \sigma_j^2 = \left(\beta_{x+j}\right)^2 Var\left[\kappa_{t_0+j}\right]$$

subject to the convention that a Normally distributed random variable with zero variance is constantly equal to its mean (note that the mean and variance are taken conditionally on past values of the time index).

Approximating S_d by a sum of perfectly dependent random variables, with the same marginal distributions, gives the approximation

$$S_d \approx S_d^u = \sum_{j=o}^{d-1} \delta_j \exp(\mu_j + \sigma_j Z), \text{ with } Z \sim N(0,1).$$

Since S_d^u is a sum of comonotonic random variables, its quantile function is additive. The quantile function $F_{S_d^u}^{-1}$ of S_d^u is given by

$$F_{S_d^u}^{-1}(z) = \sum_{j=0}^{d-1} \delta_j \exp\left(\mu_j + \sigma_j \Phi^{-1}(z)\right), \tag{3.1}$$

where, as above, Φ^{-1} is the quantile function of $N(0,1)$.

Another approximation of S_d is $S_d^l = E\left[S_d \mid \Lambda_d\right]$, where Λ_d is taken as the first-order approximation of S_d, that is, $\Lambda_d = \sum_{j=0}^{d-1} \delta_j \exp(\mu_j) Z_j$. It is expected that S_d and S_d^l are "close" to each other. A straightforward computation gives

$$S_d^l = \sum_{j=0}^{d-1} \delta_j \exp\left(\mu_j + r_j(d)\sigma_j Z + \frac{1}{2}(1-(r_j^2(d))\sigma_j^2\right)$$

where $r_i(d),\ i = 0,1,\ldots,d-1$, is the correlation coefficient between Λ_d and Z_i, that is

$$r_i(d) = \frac{Cov[Z_i, \Lambda_d]}{\sigma_i \sigma_{\Lambda_d}} = \frac{\sum_{j=0}^{d-1} \delta_j \exp(\mu_j) \beta_{x+i} \beta_{x+j} Cov[\kappa_{t_0+i}, \kappa_{t_0+j}]}{\sigma_i \sqrt{\sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \delta_j \delta_k \exp(\mu_j + \mu_k) \beta_{x+i} \beta_{x+j} Cov[\kappa_{t_0+i}, \kappa_{t_0+j}]}}. \tag{3.2}$$

In the applications we have in mind, β_{x+i} and β_{x+j} typically have the same sign so that all of the r_i's are non-negative. This means that the S_d^l's are sums of comonotonic random variables and allows us to take advantage of the quantile additivity. Specifically, the quantile function of S_d^l is given by

$$F_{S_d^l}^{-1}(z) = \sum_{j=0}^{d-1} \delta_j \exp\left(\mu_j + r_j(d)\sigma_j \Phi^{-1}(z) + \frac{1}{2}(1-(r_j^2(d))\sigma_j^2\right). \tag{3.3}$$

From the approximations S_d^u and S_d^l derived for S_d, we get the following approximations for the random survival probabilities

$$ {}_dP_x(t_0\,|\,\boldsymbol{\kappa}) \approx \exp\left(-F^{-1}_{S_d^u}(1-U)\right) \text{ and } {}_dP_x(t_0\,|\,\boldsymbol{\kappa}) \approx \exp\left(-F^{-1}_{S_d^l}(1-U)\right) $$

where U is uniformly distributed on the interval (0,1). Note that the same random variable U is used for all of the values of d, making the approximations to the conditional survival probabilities comonotonic. Hence, we obtain the following approximations for $a_x(t_0\,|\,\boldsymbol{\kappa})$

$$ a_x(t_0\,|\,\boldsymbol{\kappa}) \approx \sum_{d\geq 1} \exp\left(-F^{-1}_{S_d^u}(1-U)\right) v(0,d) $$

and

$$ a_x(t_0\,|\,\boldsymbol{\kappa}) \approx \sum_{d\geq 1} \exp\left(-F^{-1}_{S_d^l}(1-U)\right) v(0,d)\,. $$

Since these approximations are sums of comonotonic random variables, their quantile functions are additive. We then get the following approximations for the quantile function $F^{-1}_{a_x(t_0|\boldsymbol{\kappa})}$ of $a_x(t_0\,|\,\boldsymbol{\kappa})$

$$ F^{-1}_{a_x(t_0|\boldsymbol{\kappa})}(z) \approx \sum_{d\geq 1} \exp\left(-F^{-1}_{S_d^u}(1-z)\right) v(0,d) $$

where $F^{-1}_{S_d^u}$ is given in (3.1), and

$$ F^{-1}_{a_x(t_0|\boldsymbol{\kappa})}(z) \approx \sum_{d\geq 1} \exp\left(-F^{-1}_{S_d^l}(1-z)\right) v(0,d) $$

where $F^{-1}_{S_d^l}$ is given in (3.3).

3.2 Comonotonic approximation for the cohort conditional life expectancy

From the approximations S_d^u and S_d^l derived for S_d, we get the approximations for $e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa})$

$$ e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa}) \approx \frac{1}{2} + \sum_{d\geq 1}\exp(-S_d^u) \text{ or } e_x^{\nearrow}(t_0\,|\,\boldsymbol{\kappa}) \approx \frac{1}{2} + \sum_{d\geq 1}\exp(-S_d^l)\,. $$

Since the S_d^u's are sums of comonotonic random variables, their quantile functions are additive. Moreover, the z th quantile of $\exp\left(-S_d^u\right)$ is $\exp\left(-F_{S_d^u}^{-1}(1-z)\right)$. This provides the following approximations for the quantile function $F_{e_x^{\nearrow}(t_0|\boldsymbol{\kappa})}^{-1}(z)$ of $e_x^{\nearrow}(t_0 \mid \boldsymbol{\kappa})$

$$F_{e_x^{\nearrow}(t_0|\boldsymbol{\kappa})}^{-1}(z) \approx \frac{1}{2} + \sum_{d \geq 1} \exp\left(-F_{S_d^u}^{-1}(1-z)\right)$$

where $F_{S_d^u}^{-1}$ is given by (3.1). Now, assuming that the S_d^l's are comonotonic, we get

$$F_{e_x^{\nearrow}(t_0|\boldsymbol{\kappa})}^{-1}(z) \approx \frac{1}{2} + \sum_{d \geq 1} \exp\left(-F_{S_d^l}^{-1}(1-z)\right)$$

where $F_{S_d^l}^{-1}$ is given by (3.3).

4 Associated simulation methods

4.1 Background

Consider a rectangular mortality data array $\left(d_{xt}, e_{xt}\right)$, comprising the numbers of deaths, d_{xt}, with matching (central) exposures to the risk of death e_{xt}. We model the numbers of deaths as independent Poisson responses in combination with the log-bilinear structure (1.1), to target the central death rate (or force of mortality). Let $\widehat{d}_{xt} = e_{xt} \exp\left(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t\right)$ and r_{xt} denote the respective fitted values and deviance residuals. Model extrapolation is subsequently achieved by applying the most appropriate $ARIMA\left(p,1,q\right)$ model to $\left\{\hat{\kappa}_t\right\}$ and then the indices of interest are computed. These include life expectancy and fixed rate annuities, computed either by the cohort or period approach, involving future predicted central rates of mortality.

In a comparative study of various proposed simulation approaches for constructing prediction intervals of future life expectancy using the log-bilinear structure (1.1) in combination with an $ARIMA\left(0,1,0\right)$ time series

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t, \ \xi_t \sim N(0, \sigma^2), \ i.i.d.,$$

Renshaw & Haberman (2008) include a report of their findings on applying the following algorithm to the UK male pensioners' mortality experience (collected by the Continuous Mortality Investigation Bureau):

Algorithm

For $m = 1, 2, 3, \ldots, M$

1. simulate responses $d^*_{xt,m}$, (preserving any empty data cells), either
   (a) by sampling $Poi\left(\hat{d}_{xt}\right)$, or
   (b) by randomly sampling $\left\{r^*_{xt,m}\right\}$ from $\left\{r_{xt}\right\}$ with replacement and mapping $r^*_{xt,m} \mapsto d^*_{xt,m}$
2. obtain estimates $\hat{\alpha}^*_x, \hat{\beta}^*_x, \hat{\kappa}^*_t$ by fitting the log-bilinear structure to $d^*_{xt,m}$
3. obtain estimates $\hat{\theta}^*_m, \left(\hat{\sigma}^{2*}_m\right)$ by fitting the $ARIMA(0,1,0)$ time series $\left\{\hat{\kappa}^*_t\right\}$
4. for $k = 0,1,...,K$
   set $\kappa^*_{t_0+k,m} = E\left[\kappa^*_{t_0+k,m}\right] \left(= \kappa_{t_0} + k\hat{\theta}^*_m\right)$
5. compute the statistics of interest.

Such simulation algorithms were originally proposed in the belief that both the log-bilinear model fitting error and time series forecast error were captured in Step 2 and Step 3 respectively (Brouhns *et al.* (2002)). However, two key inter-related unresolved issues arising from the Renshaw & Haberman (2008) study concern (i) the general narrowness of the prediction intervals for future life expectancies and therefore annuity values, and (ii) the failure of these algorithms to capture the full magnitude of the forecast error in the time series. We address this issue next.

4.2 Bootstrapping the forecast error in the ARIMA time series

Bootstrapping is possible either by ignoring the error in the log-bilinear model and formulating:

Algorithm A1

For $m = 1,2,3,...,M$

1. for $k = 0,1,...,K$
   (i) randomly sample z^*_m from N(0,1)
   (ii) set $\kappa^*_{t_0+k,m} = E\left[\kappa_{t_0+k,m}\right] + \sqrt{Var\left[\kappa_{t_0+k,m}\right]}.z^*_m$, for the same ARIMA model
2. compute the statistics of interest.

or, by additionally allowing for the error in the log-bilinear model and formulating:

Algorithm A2

For $m = 1,2,3,...,M$

1. simulate responses $d^*_{xt,m}$ either
   a. by sampling $Poi\left(\hat{d}_{xt}\right)$, or
   b. by randomly sampling $\left\{r^*_{xt,m}\right\}$ from $\left\{r_{xt}\right\}$ with replacement and mapping $r^*_{xt,m} \mapsto d^*_{xt,m}$

2. obtain estimates $\hat{\alpha}_x^*, \hat{\beta}_x^*, \hat{\kappa}_t^*$ by fitting the log-bilinear structure to $d_{xt,m}^*$
3. obtain the same *ARIMA* parameter estimates by fitting the time series $\left\{\hat{\kappa}_t^*\right\}$

For $n = 1, 2, ..., N$

4. for $k = 0,1,...,K$

   (i) randomly sample z_{mn}^* from N(0,1)

   (ii) set $\kappa_{t_0+k,mn}^* = E\left[\kappa_{t_0+k,m}^*\right] + \sqrt{Var\left[\kappa_{t_0+k,m}^*\right]}.z_{mn}^*$
5. compute the statistics of interest.

We stress the difference between these two approaches, with A1 merely replicating the prediction (forecast) error in the time series while conditioning on the fitted log-bilinear structure and associated parameter estimates throughout the simulation process. This contrasts with A2, which additionally includes provision for the error in the log-bilinear model.

5 An application

5.1 UK male pensioner 1983-2004 experience: ARIMA(1,1,0)

We consider the UK male pensioner 1983-2004 mortality experience (ages 60-99): this is an updated version of the 1983-2003 experience which was reported in Renshaw & Haberman (2008). For these data, we depict the results on fitting the Poisson log-bilinear model structure (1.1) in Fig 1. In addition to plotting the parameter estimates (Fig 1 (a),(c),(d)), the deviance residual plots (Fig 1 (e)) show that the log-bilinear structure adequately captures the main age-period effects, while confirming the absence of any residual systematic cohort effect. We remark that the appearance of the discontinuity in the residual plot against year of birth (lower right frame) coincides with the 1919 influenza pandemic. The irregularities in the $\hat{\alpha}_x$ and $\hat{\beta}_x$ plots, in particular at the extremities of the age range, are due to the paucity of exposure at these extreme ages. For the purpose of this study, we choose not to apply smoothing, (illustrated in Fig 1), since it does not contribute anything additional to the comparative aspects of the prediction intervals reported in this study. We note that the diagnostic plot (Fig 1 (b)), displaying the annual differences in the actual and fitted total deaths, is also pattern free.

The time index $\{\kappa_t\}$ is modelled as an $ARIMA(1,1,0)$ process, for which

$$y_t = \kappa_t - \kappa_{t-1},\ y_t = \theta + \phi y_{t-1} + \xi_t;\ \xi_t \sim N\left(0, \sigma^2\right)\ i.i.d.$$

with forecasts $\kappa_{t_0+k} : k = 1, 2, 3, ...$, where

$$E\left[\kappa_{t_0+k}\right] = \kappa_{t_0} + \sum_{j=1}^{k}\left\{\mu + \phi^j\left(y_{t_0} - \mu\right)\right\},\ \mu = \frac{\theta}{1-\phi}$$

$$Var\left[\kappa_{t_0+k}\right] = \left\{1 + \left(1+\phi\right)^2 + \left(1+\phi+\phi^2\right)^2 + \ldots + \left(1+\phi+\phi^2+\ldots+\phi^{k-1}\right)^2\right\}\sigma^2$$

(e.g. Section 15.3, pp 438-444, Hamilton 1994). As noted by Lee and Carter (1992), it is necessary to impose 2 constraints on the parameters in order to ensure that the model is identifiable. As in Renshaw and Haberman (2008), we adopt the following choice of constraints

$$\sum_x \beta_x = 1 \text{ and } \kappa_{t_o} = 0$$

Numerical investigations (not reported here) show that adopting the standard constraints advocated by Lee and Carter (1992) viz

$$\sum_x \beta_x = 1 \text{ and } \quad \sum_t \kappa_t = 0$$

would lead to identical numerical answers.

Details of the parameter estimates (with standard errors in brackets) are as follows:

$\hat{\theta}$	$\hat{\phi}$	$\hat{\sigma}^2$
-1.2785	-0.4702	1.3397
(0.3180)	(0.2084)	

with

$$\hat{\mu} = -0.8696 \quad \kappa_{t_0} = 0 \quad y_{t_0} = -0.5384$$

for fitting by least squares: applied consistently throughout the subsequent application of of simulation algorithm A2. Again, the use of more sophisticated methods of fitting is not essential, given the comparative nature of the study.

Prediction intervals based on the comonotonic approximations to the quantile function derived in Section 3 (called henceforth theoretical prediction intervals) and simulated prediction intervals for (a) life expectancy and (b) a 4% fixed rate annuity are depicted in Figs 2a&b respectively. For computation of the cohort-based values (upper frames), for period 2004 coupled with ages 65, 70, 75, 80, 85, the l-type theoretical intervals (lower continuous lines) are computed using

$$F^{-1}_{e_x^{\nearrow}(t_0|\boldsymbol{\kappa})}(z) \approx \frac{1}{2} + \sum_{d\geq 1} \exp\left[-\sum_{j=0}^{d-1} \delta_j \exp\left\{\mu_j + \frac{1}{2}\left(1-r_j^2(d)\right)\sigma_j^2 + r_j(d)\sigma_j \Phi^{-1}(1-z)\right\}\right]$$

$$F^{-1}_{a_x(t_0|\boldsymbol{\kappa})}(z) \approx \sum_{d\geq 1} \exp\left[-\sum_{j=0}^{d-1} \delta_j \exp\left\{ \mu_j + \frac{1}{2}\left(1 - r_j^2(d)\right)\sigma_j^2 + r_j(d)\sigma_j \Phi^{-1}(1-z) \right\} \right] v\left(0,d\right)$$

based on Sections 3.1 & 3.2 above, and the u-type intervals computed on setting $r_j(d) = 1 \; \forall \, j,d$ in the above relationships. For the $ARIMA\left(1,1,0\right)$ time series, the evaluation of $r_i\left(d\right)$, expression (3.2), requires the $d \times d$ matrix of co-variances

$$\left[Cov\left(\kappa_{t_0+i}, \kappa_{t_0+j} \right) \right] = \mathbf{T.A.T}' \left(1 + \frac{\phi^2}{1-\phi^2} \right) \sigma^2 ; \; i,j = 0,1,2,...,d-1 \qquad (5.1)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & - & 0 \\ 0 & 1 & \phi & \phi^2 & - & \phi^{d-2} \\ 0 & \phi & 1 & \phi & - & \phi^{d-3} \\ 0 & \phi^2 & \phi & 1 & - & \phi^{d-4} \\ - & - & - & - & - & - \\ 0 & \phi^{d-2} & \phi^{d-3} & \phi^{d-4} & - & 1 \end{bmatrix}, \; \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & - & 0 \\ 1 & 1 & 0 & 0 & - & 0 \\ 1 & 1 & 1 & 0 & - & 0 \\ 1 & 1 & 1 & 1 & - & 0 \\ - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & 1 \end{bmatrix}$$

(e.g. using Brockwell and Davis (2002), Section 3.2), and reducing to

$$\left[Cov\left(\kappa_{t_0+i}, \kappa_{t_0+j} \right) \right] = \left[\min\left\{ i,j \right\} \right] \sigma^2 ; \; i,j = 0,1,2,...,d-1$$

when $\phi = 0$ for $ARIMA\left(0,1,0\right)$.

For computation of the period-based values (lower frames), for age 65 coupled with periods 2008, 2012, 2016, 2020, the life expectancy theoretical intervals use (2.1), while, for completeness, we also depict the period-based theoretical annuity intervals using:

$$F^{-1}_{a_x^{\uparrow}(t_0+k|\kappa_{t_0+k})}(z) = \sum_{d\geq 1} \exp\left[-\sum_{j=0}^{d-1} \exp\left\{ \alpha_{x+j} + \beta_{x+j} \left(E\left[\kappa_{t_0+k} \right] + \sqrt{Var\left[\kappa_{t_0+k} \right]} . \Phi^{-1}(1-z) \right) \right\} \right] v\left(0,d\right)$$

.

The simulated prediction intervals involve a total of M = 5,000 simulations in the case of algorithm A1, and M = 75, N = 75, and a total of 5625 simulations in the case of algorithms A2 a & b.

Although not strictly justified on the basis of the exploratory time series analysis of the period index κ_t, we repeat the theoretical and simulated prediction intervals, computed with the $ARIMA(0,1,0)$ process (random walk) replacing the $ARIMA\left(1,1,0\right)$

time series, and these results are depicted in Figs 3a&b, for which $\hat{\theta} = -0.8698(0.2715)$, $\hat{\sigma}^2 = 1.5482$.

5.2 Results

First, we acknowledge the relative short time span of this data set, thus compromising the full potential of time series methods to some extent. However, we justify their use on the basis of our primary aim, which is to conduct a comparative study of the choices involved: while the data are shown to fit adequately the Lee-Carter model structure.

In conclusion, referring to Figs 2a&b and Figs 3a&b, we note the following:

- The close vertical alignment of the medians, for each batch of results (*viz.* fixed x and t), within each frame.
- The close agreement of matching simulated and theoretical u-type prediction interval widths, depicted above the theoretical l-type prediction interval in the upper frames throughout. In this respect, neither the simulated intervals nor the theoretical u-type prediction intervals make use of the co-variance terms (5.1).
- The impact of the co-variance terms (5.1) in reducing the width of the theoretical u-type prediction intervals (upper frames).
- The dominance of the (correctly) simulated forecast error in the time index, over the log-bilinear simulated model fitting error. This is implied by the close agreement of the widths of simulated prediction intervals using algorithm A1 compared with both versions of algorithm A2. This finding is consistent with that of Lee and Carter (1992) (Appendix B), based on a different simulation approach, who conclude 'that for life expectancy forecasts, it is reasonable to restrict attention to the errors in forecasting the [time] index and to ignore those in fitting the [bilinear structure], even for short run forecasts'. This extends to fixed rate annuity forecasts on the basis of the evidence provided here. We note the relative simplicity of A1 over A2a&b which has implications for forecasts using an age-period-cohort parametric model (Renshaw and Haberman (2006), (2009)), where model fitting is slow to converge: this would be compounded by repeated application of simulation algorithm A2 but would be avoided under algorithm A1 or by theory.
- The more focused nature of the prediction intervals under $ARIMA(1,1,0)$ time series modelling (Figs 2a&b) compared with $ARIMA(0,1,0)$ time series modelling (Figs 3a&b), while the central point predictions are essentially the same under the two different time series models.

As expected, the small differences between prediction intervals simulated from A1 and A2 shows that the uncertainty is mainly due to the future path of the time index. These computations also show that the u-type theoretical prediction intervals based on the comonotonic approximation (3.1) gives a very accurate approximation to the simulated prediction intervals. This suggests that we could resort to this approximation in actuarial applications. The comparison between prediction intervals obtained from ARIMA(1,1,0) and ARIMA(0,1,0) dynamics stresses the importance of selecting the appropriate order of the ARIMA model. Routinely using a random walk with drift produces wider prediction intervals compared to the ARIMA(1,1,0) model, which is optimal in this case.

6. Discussion

In this paper, we have studied the accuracy of the comonotonic approximations to prediction intervals for cohort life expectancies and life annuity premiums viewed as functions of future death rates in the Lee-Carter model. Our main finding is that the u-type approximation seems to be efficient for actuarial purposes.

The comonotonic approximations used in this paper are derived for the single-factor Lee-Carter model. They could nevertheless be extended to models with multiple sources of randomness such as those by Renshaw and Haberman (2003), Cairns, Blake & Dowd (2006) or by Plat (2009), for instance. Thinking about sums of conditional survival probabilities (i.e. conditional life expectancies) or weighted sums of such probabilities (i.e. life annuity premiums as functions of the life table), the basic idea of the comonotonic approximation considered in this paper is to take the one-year survival conditional probabilities for a given cohort as comonotonic random variables. In the Lee-Carter case, this means that the future κ_t are taken to be comonotonic with a marginal distribution that is inherited from the ARIMA dynamics. The same idea should provide good results in multi-factor models, too, providing that the factors are strongly correlated for each given calendar year and also strongly correlated over time. Then, taking all the random variables to be comonotonic might give a reasonable approximation.

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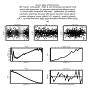
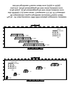


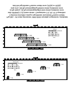
(a) (b)

(c) (d)



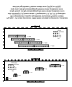
(a) computations by cohort, period 2004, various ages (x).

(b) computations by period, age 65, various periods (t > 2004).



(a) computations by cohort, period 2004, various ages (x).

(b) computations by period, age 65, various periods (t > 2004).



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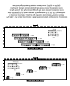

(a) computations by cohort, period 2004, various ages (x).

(b) computations by period, age 65, various periods (t > 2004).