Noncommutative quantum mechanics in a time-dependent background

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ABSTRACT: We investigate a quantum mechanical system on a noncommutative space for which the structure constant is explicitly time-dependent. Any autonomous Hamiltonian on such a space acquires a time-dependent form in terms of the conventional canonical variables. We employ the Lewis-Riesenfeld method of invariants to construct explicit analytical solutions for the corresponding time-dependent Schrödinger equation. The eigenfunctions are expressed in terms of the solutions of variants of the nonlinear Ermakov-Pinney equation and discussed in detail for various types of background fields. We utilize the solutions to verify a generalized version of Heisenberg's uncertainty relations for which the lower bound becomes a time-dependent function of the background fields. We study the variance for various states including standard Glauber coherent states with their squeezed versions and Gaussian Klauder coherent states resembling a quasi-classical behaviour. No type of coherent states appears to be optimal in general with regard to achieving minimal uncertainties, as this feature turns out to be background field dependent.

1. Introduction

The study of quantum mechanics and quantum field theories on noncommutative space-time structures is motivated by the fact that it achieves gravitational stability [1] in almost all currently known approaches to quantum gravity, such as string theory [2, 3, 4] or loop quantum gravity [5, 6]. In a quantum mechanical setting the most commonly studied version of these space-time structures consists of replacing the standard set of commutation relations for the canonical coordinates x^μ by noncommutative versions, such as $[x^\mu, x^v] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is taken to be a constant antisymmetric tensor. More interesting structures, leading for instance to minimal length and generalized versions of Heisenberg's uncertainty relations, are obtained when $\theta^{\mu\nu}$ is taken to be a function of the momenta and coordinates, e.g. [7, 8, 9, 10, 11]. In addition, one may of course also introduce an explicit time-dependence in $\theta^{\mu\nu}$. Although various effective Lagrangians for such type of noncommutative field theories have been derived, e.g. [12], little is known about explicit quantum theories on such type of spaces, one of the reasons being that they are far more difficult to solve.

Here our aim is to find explicit solutions for a simple prototype quantum mechanical model on a time-dependent background and study the physical consequences such a space will imply. We focus here on the particular two-dimensional space with nonvanishing commutators for the coordinates X, Y and momenta P_x, P_y

$$[X,Y] = i\theta(t), \qquad [P_x,P_y] = i\Omega(t), \qquad [X,P_x] = [Y,P_y] = i\hbar + i\frac{\theta(t)\Omega(t)}{4\hbar}, \tag{1.1}$$

where the noncommutative structure constants $\theta(t)$ and $\Omega(t)$ are taken to be real valued functions of time t. Of course a multitude of other possibilities exists. The specific form presented here allows for an elegant representation, as we shall see in detail below. When considering representations for these phase-space variables one is inevitably lead to time-dependent Hamiltonians $H(X,Y,P_x,P_y) \to H(t)$.

We will employ here the method of invariants, introduced originally by Lewis and Riesenfeld [13], to solve the time-dependent Schrödinger equation

$$i\hbar\partial_t |\psi_n\rangle = H(t) |\psi_n\rangle, \tag{1.2}$$

for the time-dependent or dressed states $|\psi_n\rangle$ associated to the Hamiltonian $H(t)$.

Let us briefly describe the key steps of the method for future reference. The initial step in that approach consists of constructing a Hermitian time-dependent invariant $I(t)$ from the evolution equation

$$\frac{dI(t)}{dt} = \partial_t I(t) + \frac{1}{i\hbar}[I(t),H(t)] = 0. \tag{1.3}$$

In the next step one needs to solve the corresponding eigenvalue system involving the invariant

$$I(t) |\phi_n\rangle = \lambda |\phi_n\rangle, \tag{1.4}$$

for real and time-independent eigenvalues λ and for time-dependent states $|\phi_n\rangle$. It was shown in [13] that the states

$$|\psi_n\rangle = e^{i\alpha(t)} |\phi_n\rangle \tag{1.5}$$

satisfy the time-dependent Schrödinger equation (1.2) provided that the real function $\alpha(t)$ in (1.5) obeys

$$\frac{d\alpha(t)}{dt} = \frac{1}{\hbar} \langle\phi_n| i\hbar\partial_t - H(t) |\phi_n\rangle. \tag{1.6}$$

For more details on the derivation of these key equations we refer the reader to [13].

Having obtained the explicit solutions for the wavefunctions one is in the position to compute expectation values for any desired observable. Of special interest is to investigate the modified version of Heisenberg's uncertainty relations resulting from non-vanishing commutation relations (1.1). Following standard arguments, the uncertainty for the simultaneous measurement of the observables A and B has to obey the inequality

$$\Delta A \Delta B|_{\psi} \geq \frac{1}{2} \left| \langle\psi| \left[A,B\right] |\psi\rangle \right|, \tag{1.7}$$

with $\Delta A|_{\psi}^{2} = \langle\psi| A^2 |\psi\rangle - \langle\psi| A |\psi\rangle^2$ and similarly for B for any state $|\psi\rangle$. Evidently, for instance the first relation in (1.1) implies that the uncertainty for the simultaneous measurement of X and Y is greater than the function of time $|\theta(t)|/2$ rather than simply being greater than a constant. Of special interest is to see whether the time-dependent bound can be saturated by the use of various types of coherent states in (1.7).

Our manuscript is organized as follows: In section 2 we construct the time-dependent invariant $I(t)$ for the two dimensional harmonic oscillator on the background described by (1.1). We compute its time-dependent eigenfunctions $|\phi_n\rangle$, determine the phase $\alpha(t)$ thereafter and hence the eigenstates $|\psi_n\rangle$ of $H(t)$. As all solutions are dependent on the solutions of the nonlinear Ermakov-Pinney equation we devote section 3 to a discussion of its solutions. In section 4 we assemble the solutions from section 2 and 3 to investigate the validity and quality of a generalized version of Heisenberg's uncertainty relations. Particular focus is placed on the study of the uncertainty relations when computed with regard to standard Glauber coherent states, including their squeezed versions and also Gaussian Klauder coherent states. In section 5 we state our conclusions.

2. The 2D harmonic oscillator in a time-dependent background

The main features of models on time-dependent backgrounds can be explained by considering simple two dimensional models. Therefore we will examine here as prototype two dimensional model the harmonic oscillator of the form

$$H(X,Y,P_x,P_y) = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{m\omega^2}{2}(X^2+Y^2), \tag{2.1}$$

on the noncommutative space (1.1). From the many possibly representations, we choose here a Hermitian one obtained from standard Bopp-shifts in the conventional canonical variables x, y, p_x and p_y, with nonvanishing commutators $[x,p_x] = [y,p_y] = i\hbar$, as

$$X = x - \frac{\theta(t)}{2\hbar}p_y, \quad Y = y + \frac{\theta(t)}{2\hbar}p_x, \quad P_x = p_x + \frac{\Omega(t)}{2\hbar}y, \quad P_y = p_y - \frac{\Omega(t)}{2\hbar}x. \tag{2.2}$$

As anticipated, when converting the Hamiltonian in (2.1) to the standard variables it becomes explicitly time-dependent

$$H(t) = \frac{1}{2}a(t)\left(p_x^2+p_y^2\right) + \frac{1}{2}b(t)\left(x^2+y^2\right) + c(t)\left(p_x y - x p_y\right) \tag{2.3}$$

with coefficients

$$a(t) = \frac{1}{m} + \frac{m\omega^2}{4\hbar^2}\theta^2(t), \quad b(t) = m\omega^2 + \frac{\Omega^2(t)}{4m\hbar^2}, \quad c(t) = \frac{m\omega^2\theta(t)}{2\hbar} + \frac{\Omega(t)}{2\hbar m}. \tag{2.4}$$

We notice that for $\theta(t) = 0$ we can view this Hamiltonian with an appropriate identification of the remaining functions as describing a particle with mass m moving in an axially symmetric electromagnetic field, see section IV in [13]. It should also be noted that with a re-definition of the time-dependent coefficient attempts to solve the eigenvalue problem related to (2.3) can be found in the literature [14, 15]. Unfortunately the solutions provided are partly incorrect or not useful for our purposes as we shall be commenting on below in more detail.

The quantum equations of motion for the canonical variables associated to the Hamiltonian (2.3) are simply

$$\dot{x} = \frac{1}{i\hbar}[x, H] = a(t)p_x + c(t)y, \qquad \dot{y} = \frac{1}{i\hbar}[y, H] = a(t)p_y - c(t)x, \tag{2.5}$$

$$\dot{p}_x = \frac{1}{i\hbar}[p_x, H] = -b(t)x + c(t)p_y, \qquad \dot{p}_y = \frac{1}{i\hbar}[p_y, H] = -b(t)y - c(t)p_x, \tag{2.6}$$

where we adopt the usual convention for the time derivative $\partial_t f =: \dot{f}$.

2.1 Construction of time-dependent invariants

A non-Hermitian invariant is constructed right away, by following the argumentation already provided in [13]. Defining the non-canonical variables

$$Q := (x + iy)e^{i\int^t c(s)ds} \qquad \text{and} \qquad P := (p_x + ip_y)e^{i\int^t c(s)ds}, \tag{2.7}$$

satisfying $[Q, P] = 0$, we find with (2.5) and (2.6) the same equations of motion for these variables

$$\dot{Q} = a(t)P \qquad \text{and} \qquad \dot{P} = -b(t)Q. \tag{2.8}$$

as for the harmonic oscillator with a time-dependent mass term [16]. This is all that matters for the identification of a formal invariant $\tilde{I}(t)$ in terms of the variables Q and P

$$\tilde{I}(t) = \frac{1}{2}\left[\frac{\tau}{\sigma^2}Q^2 + (\sigma P - \frac{\dot{\sigma}}{a}Q)^2\right] \neq \tilde{I}^\dagger(t), \tag{2.9}$$

since we may simply take the expression from the literature and adapt the relevant quantities appropriately. Here σ is a new auxiliary quantity that has to satisfy a nonlinear Ermakov-Pinney (EP) [17, 18] equations including a dissipative term

$$\ddot{\sigma} - \frac{\dot{a}}{a}\dot{\sigma} + ab\sigma = \tau\frac{a^2}{\sigma^3}, \tag{2.10}$$

with integration constant τ. It is well-known that variations of this equation are ubiquitous in this context of solving time-dependent Hamiltonian systems, see for instance equation (5) in [19], which reduces exactly to (2.10) for $A \to a$, $B \to 0$ and $C \to \tau$ and [21, 22, 23, 24] for variations of this equation. Note that $\sigma = 0$ implies that $a = 0$, which is impossible according to (2.4), such that we can devide by σ without any further concern.

In principle the fact that $\tilde{I}$ in (2.9) is an invariant means $\tilde{I}\tilde{I}^\dagger$ or $\tilde{I}^\dagger\tilde{I}$ constitute Hermitian invariants. However, since they will be quartic in the canonical variables and not directly suitable to an operator approach to find the corresponding eigensystems we seek an additional one of lower order in the canonical variables, having however equation (2.10) in common.

The symmetry of the Hamiltonian suggest to carry out a quantum canonical transformation using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, which indeed turns out to be very suitable. The canonical coordinates and momenta are then $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$ and $p_r = (xp_x + yp_y)/r - i\hbar/(2r)$, $p_\theta = xp_y - yp_x$, such that the canonical commutation relations are $[r, p_r] = [\theta, p_\theta] = i\hbar$. The last term in p_r is not essential for the canonical commutation relations, but its inclusion ensures the Hermiticity of p_r and leads to the convenient identity $p_x^2 + p_y^2 = p_r^2 + p_\theta^2/r^2 - \hbar^2/(4r^2)$ allowing to convert the Hamiltonian (2.3) into the form

$$H(t) = \frac{1}{2}a(t)\left(p_r^2 + \frac{p_\theta^2}{r^2} - \frac{\hbar^2}{4r^2}\right) + \frac{1}{2}b(t)r^2 - c(t)p_\theta. \tag{2.11}$$

Applying now the Lewis-Riesenfeld method of invariants and construct a Hermitian time-dependent invariant $I(t)$ by using (1.3), we commence with the standard assumption that the invariant is of the same order and form in the canonical variables as the original Hamiltonian. Similarly as the Hamiltonian, we assume here that also the invariant does not depend explicitly on θ and take it to be of the general form

$$I(t) = \alpha(t)p_r^2 + \beta(t)r^2 + \gamma(t)\{r, p_r\} + \delta(t)\frac{p_\theta^2}{r^2} + \varepsilon(t)\frac{p_\theta}{r^2} + \phi(t)\frac{1}{r^2}, \tag{2.12}$$

with unknown time-dependent coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$ etc. The substitution of (2.12) into (1.3) then yields the following constraints on these coefficients

$$\dot{\alpha} = -2a\gamma, \quad \dot{\beta} = 2b\gamma, \quad \dot{\gamma} = b\alpha - a\beta, \tag{2.13}$$

$$\dot{\delta}p_\theta^2 + \dot{\varepsilon}p_\theta + \dot{\phi} = \hbar^2 a\gamma - 2a\gamma p_\theta^2, \qquad (\delta - \alpha)\, p_\theta^2 + \varepsilon p_\theta + \phi + \frac{\alpha\hbar^2}{4} = 0. \tag{2.14}$$

We observe that the equations in (2.13) take on the same form as the equations underlying the explicit construction for the time-dependent harmonic oscillator [16]. They can be solved by parameterizing $\alpha(t) = \sigma^2(t)$ and after one integration we are led exactly to the nonlinear Ermakov-Pinney equations (2.10) underlying the solution for our non-Hermitian invariant $\tilde{I}(t)$. The remaining equations (2.14) are consistently solved by

$$\delta = \alpha, \qquad \varepsilon = 0, \quad \text{and} \quad \phi = -\frac{\alpha\hbar^2}{4}. \tag{2.15}$$

Assembling everything, the Hermitian invariant $I(t)$ for the time-dependent Hamiltonian (2.3) then acquires the form

$$I(t) = \frac{\tau}{\sigma^2}r^2 + \left(\sigma p_r - \frac{\dot{\sigma}}{a}r\right)^2 + \frac{\sigma^2 p_\theta^2}{r^2} - \frac{\sigma^2\hbar^2}{4r^2}, \tag{2.16}$$

with $\sigma(t)$ determined by the Ermakov-Pinney equation (2.10). As argued already in [13] the arbitrary constant τ may be scaled away, thus that from now on we simply set it to 1 for convenience without introducing a new quantity.

Next we solve the eigenvalue equation (1.4) by expressing the invariant $I(t)$ in terms of time-dependent creation and annihilation operators

$$\hat{a}(t) = \frac{1}{2\sqrt{\hbar}}\left[\left(\sigma p_r - \frac{\dot{\sigma}}{a}r\right) - i\left(\frac{r}{\sigma} + \frac{\sigma}{r}(p_\theta + \frac{\hbar}{2})\right)\right]e^{-i\theta}, \tag{2.17}$$

$$\hat{a}^\dagger(t) = \frac{1}{2\sqrt{\hbar}}e^{i\theta}\left[\left(\sigma p_r - \frac{\dot{\sigma}}{a}r\right) + i\left(\frac{r}{\sigma} + \frac{\sigma}{r}(p_\theta + \frac{\hbar}{2})\right)\right], \tag{2.18}$$

satisfying $[\hat{a}, \hat{a}^\dagger] = 1$, by means of the identity

$$\hbar\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) - p_\theta = \frac{1}{4}I(t) - \frac{1}{2}p_\theta =: \hat{I}(t). \tag{2.19}$$

Clearly $\hat{I}(t)$ is also an invariant, where the factor 1/4 simply amounts to a new value for the integration constant τ and p_θ may be added to $I(t)$ since $[H(t), p_\theta] = 0$.

2.2 Eigensystem for the time-dependent invariant

We can now employ the standard argumentation from [13] to construct the eigenstates and eigenfunctions for the invariant $\hat{I}(t)$. Noting first that $[\hat{I}(t), p_\theta] = 0$, one concludes that $\hat{I}(t)$ and p_θ possess simultaneous eigenvectors, say $|n, \ell\rangle$, with

$$\hat{I}\,|n,\ell\rangle = \hbar\left(n + \frac{1}{2}\right)|n,\ell\rangle, \qquad p_\theta\,|n,\ell\rangle = \hbar\ell\,|n,\ell\rangle, \qquad \langle n,\ell\,|n,\ell\rangle = 1. \tag{2.20}$$

Computing therefore $\langle n,\ell|\,\hat{a}^\dagger\hat{a}\,|n,\ell\rangle = n + \ell \geq 0$ implies that for given n we have $\ell \in \{-n, \ldots, 0, 1, 2, \ldots\}$. The eigenstates of this sequence therefore obey

$$\hat{a}\,|n,-n\rangle = 0, \qquad |n, m-n\rangle = \frac{1}{\sqrt{m!}}\left(\hat{a}^\dagger\right)^m |n,-n\rangle, \qquad \text{with } n, m \in \mathbb{N}_0. \tag{2.21}$$

For all observables that can be expressed in terms of the time-dependent creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, we can simply use operator techniques to compute their expectation values. However, the former is not possible for our observables X, Y, P_x and P_y. We therefore use the explicit representations in coordinate space $p_\theta = -i\hbar\partial_\theta$ and $p_r = -i\hbar[\partial_r + 1/(2r)]$ to compute the eigenstates. Assuming now $\langle r,\theta\,|n,\ell\rangle = \psi_{n,\ell}(r,\theta) = \varphi_n(r)e^{i\ell\theta}$ we have the desired property $p_\theta\psi_{n,\ell}(r,\theta) = \hbar\ell\psi_{n,\ell}(r,\theta)$. For given n, the lowest states are then found from solving the differential equation $\hat{a}\psi_{n,-n}(r,\theta) = 0$, that is

$$\frac{ie^{-i\theta - i\theta n}}{2ar\sigma\sqrt{\hbar}}\left[\left(a\hbar n\sigma^2 - ar^2 + ir^2\sigma\dot{\sigma}\right)\varphi(r) - a\hbar r\sigma^2\partial_r\varphi(r)\right] = 0. \tag{2.22}$$

The solution to (2.22) is then easily found to be

$$\psi_{n,-n}(r,\theta) = \lambda_n r^n e^{-\frac{r^2(a - i\sigma\dot{\sigma})}{2a\hbar\sigma^2}}e^{-i\theta n}, \qquad \lambda_n^2 = \frac{1}{\pi n!(\hbar\sigma^2)^{(1+n)}}. \tag{2.23}$$

We have fixed here the constant of integration by demanding the ground state to be normalized. Subsequently we construct the normalized excited states from the second relation in (2.21) to

$$\psi_{n,m-n}(r,\theta) = \lambda_n\frac{\left(i\hbar^{1/2}\sigma\right)^m}{\sqrt{m!}}r^{n-m}e^{i\theta(m-n) - \frac{a - i\sigma\dot{\sigma}}{2a\hbar\sigma^2}r^2}U\left(-m, 1-m+n, \frac{r^2}{\hbar\sigma^2}\right), \tag{2.24}$$

with $U(a,b,z)$ denoting the confluent hypergeometric function. The orthonormality relation $\int_0^{2\pi} d\theta \int_0^{\infty} dr\ r\psi^*_{n,m-n}(r,\theta)\psi_{n',m'-n'}(r,\theta) = \delta_{nn'}\delta_{mm'}$ is verified by using the standard properties of the latter function.

It should be noted here that our solution differs from those found in the literature [14, 15]. As was pointed out in [15] the solutions provided in [14] are incorrect as they lead to time-dependent eigenvalues and thus contradict the basic foundations of the Lewis-Riesenfeld theory, i.e. equation (1.4). Our solution differs also slightly from those in [15]. Moreover, in [15] the normalization constant was left undetermined, which is, however, crucial in concrete computations following below.

2.3 Eigensystem for the Hamiltonian

The last step in the Lewis-Riesenfeld procedure consists of computing the phase $\alpha(t)$ in (1.5) by solving the equation

$$\dot{\alpha}_{n,\ell} = \frac{1}{\hbar} \langle n,\ell| i\hbar\partial_t - H |n,\ell\rangle . \tag{2.25}$$

As already argued in [13], this may be achieved by constructing a recursive equation for the right hand side of (2.25), computing some explicit expectation values, using the freedom to choose the phase for the vacuum state and a subsequent integration.

We commence by simply replacing $|n,\ell\rangle = \hat{a}^\dagger/\sqrt{n+\ell}\,|n,\ell-1\rangle$ in (2.25), obtaining

$$\langle n,\ell| i\hbar\partial_t - H |n,\ell\rangle = \langle n,\ell-1| i\hbar\partial_t - H |n,\ell-1\rangle + \frac{1}{n+\ell} \langle n,\ell-1| [\hat{a}, i\hbar\partial_t - H]\,\hat{a}^\dagger |n,\ell-1\rangle . \tag{2.26}$$

Using next the expression (2.17) for the annihilation operator and the Hamiltonian in polar coordinates (2.11), we compute

$$[\hat{a}, i\hbar\partial_t - H] = \hbar \left(c(t) - \frac{a(t)}{\sigma^2(t)} \right) \hat{a}, \tag{2.27}$$

upon replacing $\ddot{\sigma}$ by means of the EP-equation in the form (2.10). Substitution of (2.27) into (2.26) allows for the computation of the expectation value, thus leading to the recursive equation

$$\langle n,\ell| i\hbar\partial_t - H |n,\ell\rangle = \langle n,\ell-1| i\hbar\partial_t - H |n,\ell-1\rangle + \hbar \left(c(t) - \frac{a(t)}{\sigma^2(t)} \right) . \tag{2.28}$$

We may now iterate this equation until we reach the expectation values for vacuum state $\langle n,-n| i\hbar\partial_t - H |n,-n\rangle$. As argued in [13], the matrix element $\langle n,-n| \partial_t |n,-n\rangle$ involves an arbitrary constant, which we conveniently choose to set to $\langle n,-n| \partial_t |n,-n\rangle = \langle n,-n| H |n,-n\rangle$. Therefore we obtain the expectation value

$$\langle n,\ell| i\hbar\partial_t - H |n,\ell\rangle = (n+\ell)\hbar \left(c(t) - \frac{a(t)}{\sigma^2(t)} \right) , \tag{2.29}$$

allowing us to compute the phase to

$$\alpha_{n,\ell}(t) = (n+\ell) \int^t \left(c(s) - \frac{a(s)}{\sigma^2(s)} \right) ds. \tag{2.30}$$

Our result for $\alpha_{n,\ell}(t)$ differs from the phase computed in [15], where the $c(s)$-term is absent.

We have now obtained explicit eigenfunctions for the Hamiltonian (2.1) for any time-dependent background field in terms of the solutions of the EP-equation. Mostly in the literature the analysis is abandoned at this stage and the invariants and wavefunctions are simply expressed in terms of the yet to be determined solution to the EP-equation. However, for concrete computations of measurable quantities one needs to address the auxiliary problem and solve the equations explicitly for the time-dependent functions appearing in the Hamiltonian. Surprisingly little attention has been paid to this problem in the context of solving time-dependent Hamiltonian systems and therefore we will discuss the solutions of our auxiliary equation (2.10) in the next subsection.

3. The Ermakov-Pinney equation

The simplest special solution arises when taking $\theta(t) = const$, such that $\dot{a} = 0$ and consequently the dissipative term vanishes. For this case particular solutions were already found by Pinney [18]

$$\sigma = \sqrt{u_1^2 + \tau a^2 \frac{u_2^2}{W^2}}, \tag{3.1}$$

where u_1, u_2 are the two linearly independent solutions of the equation

$$\ddot{u} + ab(t)u = 0, \tag{3.2}$$

and $W = u_1\dot{u}_2 - \dot{u}_1 u_2$ is the corresponding Wronskian.

When $\dot{a} \neq 0$ no general solution to (2.10) is known, although one can construct a variety of explicit solutions following the procedure proposed in [25, 26]. We briefly outline the method and use it to construct some new solutions, which we employ later on. We start by considering the ordinary differential equation of the general form

$$\frac{d^2\sigma}{dt^2} + g(\sigma)\frac{d\sigma}{dt} + h(\sigma) = 0, \tag{3.3}$$

for which the EP-equation can be seen as a special case with the appropriate choices for $g(\sigma)$ and $h(\sigma)$. Introducing the new quantity $\eta(\sigma) := d\sigma/dt$, the equation (3.3) is easily converted into the first order differential equation

$$\eta\frac{d\eta}{d\sigma} + g(\sigma)\eta + h(\sigma) = 0. \tag{3.4}$$

This implies that when having solved (3.4), a solution to the original equation (3.3) can be obtained simply from inverting $\int^\sigma \eta^{-1}(s)ds = t$. It can be shown by direct substitution that (3.4) admits the solution

$$\eta(\sigma) = \lambda_\kappa \frac{h(\sigma)}{g(\sigma)} \qquad \text{with } \lambda_\kappa^\pm = \frac{-1 \pm \sqrt{1-4\kappa}}{2\kappa}, \tag{3.5}$$

if the Chiellini integrability condition [27]

$$\frac{d}{d\sigma}\left(\frac{h(\sigma)}{g(\sigma)}\right) = \kappa g(\sigma), \tag{3.6}$$

with $\kappa \in \mathbb{R}$ holds. Based on this we may then find exact analytical solutions for instance by starting with a given $g(\sigma)$ and subsequently compute

$$\eta(\sigma) = \kappa\lambda_\kappa \int^\sigma g(s)ds \qquad \text{and} \qquad h(\sigma) = \kappa g(\sigma) \int^\sigma g(s)ds, \tag{3.7}$$

or by starting with a given $h(\sigma)$ and subsequently evaluate

$$\eta(\sigma) = \pm\lambda_\kappa \sqrt{2\kappa \int^\sigma h(s)ds} \qquad \text{and} \qquad g(\sigma) = \frac{h(\sigma)}{\sqrt{2\kappa \int^\sigma h(s)ds}}. \tag{3.8}$$

Following this solution procedure means of course that we are not pre-selecting our background fields $\theta(t)$ and $\Omega(t)$, but instead we determine them by primarily insisting on the integrability of the EP-equation. Comparing (3.4) with the EP-equation (2.10) we identify

$$g(\sigma) = -\frac{\dot{a}}{a} = -\partial_t \ln a = -\frac{2m^2\omega^2\theta\dot{\theta}}{4\hbar^2 + m^2\omega^2\theta^2}, \tag{3.9}$$

$$h(\sigma) = ab\sigma - \tau\frac{a^2}{\sigma^3} = \frac{\left(4\hbar^2 + m^2\omega^2\theta^2\right)\left[m^2\omega^2\left(4\hbar^2\sigma^4 - \tau\theta^2\right) - 4\hbar^2\tau + \sigma^4\Omega^2\right]}{16\hbar^4 m^2\sigma^3}. \tag{3.10}$$

The virtue of this method is that it leads to exact solutions. Nonetheless, one might also be interested in concrete types of background fields for which the integrability condition (3.6) does not hold, in which case we will resort to a numerical analysis.

3.1 Non-dissipative solutions

For the special case $\theta(t) = const$, i.e. $\dot{a} = 0$ we can simply pre-select any explicit form for $\Omega(t)$, and thereby $b(t)$, to construct the solutions from the general formula (3.1). For instance for $a(t) = \alpha$ and $b(t) = \beta e^{\gamma t}$, $\alpha, \beta, \gamma \in \mathbb{R}$, i.e. $\theta(t) = \pm 2\hbar/m\omega\sqrt{m\alpha - 1}$ and $\Omega(t) = \pm 2\hbar\sqrt{m\beta e^{\gamma t} - m^2\omega^2}$, we solve (3.2) in terms of Bessel functions and subsequently obtain the particular solution by means of (3.1)

$$\sigma(t) = \sqrt{\frac{\pi^2\alpha^2\tau}{\gamma^2 c_1^2} Y_0^2\left(\frac{2\sqrt{\alpha\beta}e^{\gamma t/2}}{\gamma}\right) + c_1^2 J_0^2\left(\frac{2\sqrt{\alpha\beta}e^{\gamma t/2}}{\gamma}\right)}, \tag{3.11}$$

with integration constant $c_1 \in \mathbb{R}$ and J_0, Y_0 denoting the Bessel functions of first and second kind, respectively. Similarly different solutions are easily constructed for any other explicit choice of $b(t)$ for which (3.2) admits a solution.

3.2 Exponentially decaying solutions

Let us now switch on the dissipative term and take $\dot{a} \neq 0$ by making the additional assumption $g(\sigma) = \gamma \in \mathbb{R}$. Then the second equation in (3.7) together with the explicit form of $h(\sigma)$ from (2.10) yields the consistency equation

$$\kappa\gamma^2\sigma = ab\sigma - \tau\frac{a^2}{\sigma^3}, \tag{3.12}$$

from which we deduce that $ab = const$ and $a \sim \sigma^2$. Since we may find $a(t)$ simply from $-\dot{a}/a = \gamma$, all other functions follow from the proportionality relations. We find exponentially decaying and increasing background fields $\theta(t) = \pm 2\hbar/m\omega\sqrt{m\alpha e^{-\gamma t} - 1}$ and $\Omega(t) = \pm 2\hbar\sqrt{m\beta e^{\gamma t} - m^2\omega^2}$ corresponding to exponentially decaying solutions of the EP-equation

$$a(t) = \alpha e^{-\gamma t}, \quad b(t) = \beta e^{\gamma t}, \quad \text{and} \quad \sigma(t) = \mu e^{-\gamma t/2}, \tag{3.13}$$

with α, β, $\gamma \in \mathbb{R}$, together with the constraint $\mu^4 = \tau\alpha^2/(\alpha\beta - \kappa\gamma^2)$ resulting from (3.12). The Chiellini constant κ is not fixed at this point, but simply determined by substituting the expressions from (3.13) into (2.10), leading to $\kappa = 1/4$. A special case of our solution corresponds to the one reported in [19] where the EP-equation of the type (2.10) appears as an auxiliary equation in the solution procedure for the Caldirola-Kanai Hamiltonian [28, 29].

Notice that for our background fields the requirement that $\theta(t), \Omega(t) \in \mathbb{R}$ implies that this solution leads to cutoff times t_c after which the background field needs to be vanishing, that is $t < t_c = \ln(m\alpha)/\gamma$ for $\alpha, \gamma > 0$. It should also be noted that the constraint on the constants is quite severe and one might change the overall qualitative behaviour of the solution from a decaying solution to an oscillatory behaviour when relaxing the integrability condition.

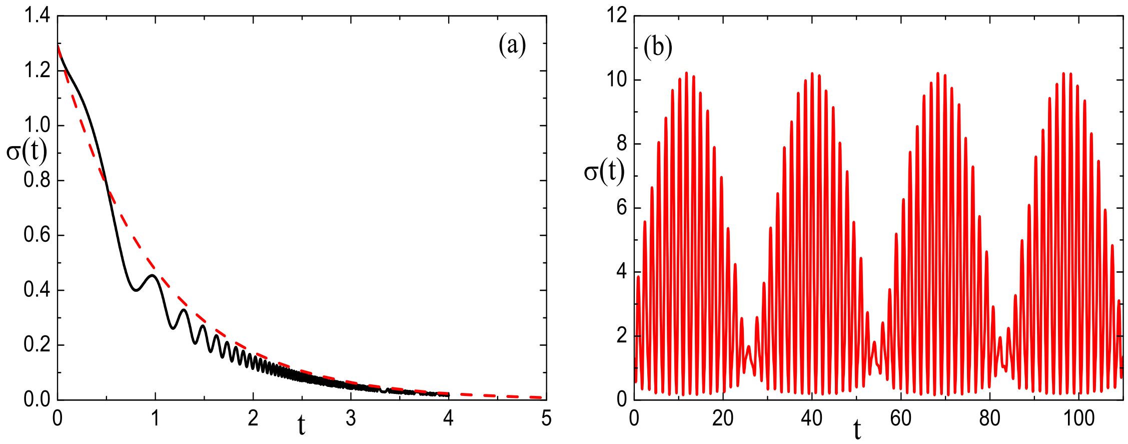

Figure 1: (a) Exactly integrable solution (3.13) (red, dashed) versus a non-Chiellini integrable solution for pre-selected exponential backgrounds $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ (black, solid). (b) Non-Chiellini integrable solution for pre-selected sinusoidal background $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

3.3 Rationally decaying solutions

Next we assume $g(\sigma) = \gamma\sigma^n$ with $n \in \mathbb{N}$. The consistency equation then reads

$$\kappa\gamma^2 \frac{\sigma^{2n+1}}{n+1} = ab\sigma - \tau\frac{a^2}{\sigma^3}, \tag{3.14}$$

which implies that $ab \sim \sigma^{2n}$ and $a \sim \sigma^{n+2}$. Determining $a(t)$ simply from $-\dot{a}/a = \gamma\sigma^n$, we compute all other functions from the proportionality relations. We find rational solutions to the background fields and the EP-equation

$$a(t) = \frac{\alpha \left(\frac{n+2}{n}\right)^{\frac{n+2}{n}}}{(\gamma t - \mu)^{(n+2)/n}}, \quad b(t) = \frac{\beta \left(\frac{n}{n+2}\right)^{\frac{2}{n}-1}}{(\gamma t - \mu)^{1-\frac{2}{n}}}, \quad \text{and} \quad \sigma(t) = \frac{\left(\frac{n+2}{n}\right)^{\frac{1}{n}}}{(\gamma t - \mu)^{1/n}}, \tag{3.15}$$

with constraint $\gamma^2 = (n+1)(\alpha\beta - \tau\alpha^2)/\kappa$. The Chiellini constant is subsequently fixed to $\kappa = (n+1)/(n+2)^2$. To maintain real solutions requires here a cutoff time $t < t_c = \mu/\gamma$ for $\gamma, \mu > 0$.

3.4 Non-Chiellini integrable solutions with pre-selected background

As pointed out, the solutions constructed in the previous subsections are special in the sense that the Chiellini integrability has been superimposed onto them. Nonetheless, given a specific background we may always find numerical solutions. In figure 1 we depict some solutions for exponential and sinusoidal background fields which we shall employ below in our solutions for the time-dependent wavefunctions.

4. The generalized uncertainty relations

4.1 The generalized uncertainty relations for eigenstates

We have assembled now all the necessary ingredients for the explicit computation of expectation values. We are therefore in the position to test the generalized uncertainty relations (1.7). Having obtained explicit expressions for the wavefunctions in coordinate space, we simply use the representation in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $p_x = -i\hbar\cos\theta\partial_r + i\hbar/r\sin\theta\partial_\theta$, $p_y = -i\hbar\sin\theta\partial_r - i\hbar/r\cos\theta\partial_\theta$ and the corresponding relations for the operators in (2.2) to compute the relevant matrix elements. We comence with the verification of the standard uncertainty relations for the auxiliary variables x, y, p_x, p_y. By evaluating the explicit integrals we obtain their matrix elements

$$\langle n, m-n| \, x \left|n, m'-n\right\rangle = i\frac{\sqrt{\hbar}}{2}\sigma\left(\sqrt{m'}e^{i\alpha_{0,1}}\delta_{m',m+1} - \sqrt{m}e^{-i\alpha_{0,1}}\delta_{m,m'+1}\right), \tag{4.1}$$

$$\langle n, m-n| \, y \left|n, m'-n\right\rangle = -\frac{\sqrt{\hbar}}{2}\sigma\left(\sqrt{m'}e^{i\alpha_{0,1}}\delta_{m',m+1} + \sqrt{m}e^{-i\alpha_{0,1}}\delta_{m,m'+1}\right), \tag{4.2}$$

$$\langle n, m-n| \, p_x \left|n, m'-n\right\rangle = \frac{\sqrt{\hbar}}{2}\left[\chi_+\sqrt{m'}e^{i\alpha_{0,1}}\delta_{m',m+1} + \chi_-\sqrt{m}e^{-i\alpha_{0,1}}\delta_{m,m'+1}\right], \tag{4.3}$$

$$\langle n, m-n| \, p_y \left|n, m'-n\right\rangle = i\frac{\sqrt{\hbar}}{2}\left[\chi_+\sqrt{m'}e^{i\alpha_{0,1}}\delta_{m',m+1} - \chi_-\sqrt{m}e^{-i\alpha_{0,1}}\delta_{m,m'+1}\right], \tag{4.4}$$

and

$$
\begin{aligned}
\langle n, m-n| \, x^2, y^2 \, |n, m'-n\rangle &= \frac{\hbar}{2}(n+m+1)\sigma^2\delta_{m,m'} \mp \frac{\hbar\sigma^2}{2\sqrt{2}}\mu(m,m')e^{i\alpha_{0,2}}\delta_{m',m+2} \\
&\mp \frac{\hbar\sigma^2}{2\sqrt{2}}\mu(m',m)e^{-i\alpha_{0,2}}\delta_{m,m'+2}, \qquad (4.5) \\
\langle n, m-n| \, p_x^2, p_y^2 \, |n, m'-n\rangle &= \frac{\hbar}{2}(n+m+1)\chi_+\chi_-\delta_{m,m'} \pm \frac{\hbar\chi_+^2}{2\sqrt{2}}\mu(m,m')e^{i\alpha_{0,2}}\delta_{m',m+2} \\
&\pm \frac{\hbar\chi_-^2}{2\sqrt{2}}\mu(m',m)e^{-i\alpha_{0,2}}\delta_{m,m'+2}, \qquad (4.6) \\
\langle n, m-n| \, xp_y \, |n, m'-n\rangle &= \frac{\hbar}{2}(m-n)\delta_{m,m'} - \frac{\hbar\sigma\chi_+}{2\sqrt{2}}\mu(m,m')e^{i\alpha_{0,2}}\delta_{m',m+2} \\
&- \frac{\hbar\sigma\chi_-}{2\sqrt{2}}\mu(m',m)e^{-i\alpha_{0,2}}\delta_{m,m'+2}, \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
\langle n, m-n| \, yp_x \, |n, m'-n\rangle &= \frac{\hbar}{2}(n-m)\delta_{m,m'} - \frac{\hbar\sigma\chi_+}{2\sqrt{2}}\mu(m,m')e^{i\alpha_{0,2}}\delta_{m',m+2} \\
&- \frac{\hbar\sigma\chi_-}{2\sqrt{2}}\mu(m',m)e^{-i\alpha_{0,2}}\delta_{m,m'+2}, \qquad (4.8)
\end{aligned}
$$

where we abbreviated $\chi_\pm := \frac{1}{\sigma} \pm i\frac{\dot{\sigma}}{a}$ and $\mu(x,y) := \sqrt{\left(\frac{x}{2}+1\right)(y-1)}$.

Using the above expressions the relevant variances are computed to

$$
\begin{aligned}
\Delta x|^2_{\psi_{n,m-n}} &= \Delta y|^2_{\psi_{n,m-n}} = \frac{\hbar}{2}(n+m+1)\sigma^2, \qquad (4.9) \\
\Delta p_x|^2_{\psi_{n,m-n}} &= \Delta p_y|^2_{\psi_{n,m-n}} = \frac{\hbar}{2}(n+m+1)\left(\frac{1}{\sigma^2}+\frac{\dot{\sigma}^2}{a^2}\right). \qquad (4.10)
\end{aligned}
$$

It is then easy to verify that the standard uncertainty relations indeed hold

$$
\begin{aligned}
\Delta x\Delta p_x|_{\psi_{n,m-n}} &= \Delta y\Delta p_y|_{\psi_{n,m-n}} = \frac{\hbar}{2}(n+m+1)\sqrt{1+\frac{\sigma^2\dot{\sigma}^2}{a^2}} \geq \frac{\hbar}{2}, \qquad (4.11) \\
\Delta x\Delta y|_{\psi_{n,m-n}} &= \frac{\hbar}{2}(n+m+1)\sigma^2 \geq 0, \qquad (4.12) \\
\Delta p_x\Delta p_y|_{\psi_{n,m-n}} &= \frac{\hbar}{2}(n+m+1)\left(\frac{1}{\sigma^2}+\frac{\dot{\sigma}^2}{a^2}\right) \geq 0. \qquad (4.13)
\end{aligned}
$$

However, for our model (2.1) these quantities are mere auxiliary objects. Therefore, we need to compute the corresponding relations for the noncommutative quantities in our original system (2.1) on the time-dependent background. In the light of (1.1) and (1.7) they should produce a generalized version of the uncertainty relations with a time-dependent lower bound. We find $\langle n, m-n| \, \mathcal{O} \, |n, m-n\rangle = 0$ for $\mathcal{O} = X, Y, P_x, P_y$, not reported here, and afterwards

$$
\begin{aligned}
\Delta X|^2_{\psi_{n,m-n}} &= \Delta Y|^2_{\psi_{n,m-n}} = \Delta x|^2_{\psi_{n,m-n}} + \frac{n-m}{2}\theta(t) + \frac{n+m+1}{8\hbar}\left(\frac{1}{\sigma^2}+\frac{\dot{\sigma}^2}{a^2}\right)\theta^2(t), \qquad (4.14) \\
\Delta P_x|^2_{\psi_{n,m-n}} &= \Delta P_y|^2_{\psi_{n,m-n}} = \Delta p_x|^2_{\psi_{n,m-n}} + \frac{n-m}{2}\Omega(t) + \frac{n+m+1}{8\hbar}\sigma^2\Omega^2(t), \qquad (4.15)
\end{aligned}
$$

from which we deduce the generalized version of the uncertainty relations

$$\Delta X \Delta Y|_{\psi_{n,m-n}} = \frac{n-m}{2}\theta(t) + \frac{n+m+1}{8\hbar}\left[4\hbar\sigma^2 + \left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right)\theta^2(t)\right] \geq \frac{\theta(t)}{2}, \quad (4.16)$$

$$\Delta P_x \Delta P_y|_{\psi_{n,m-n}} = \frac{\hbar}{2}(n+m+1)\left[\frac{\sigma^2\Omega^2(t)}{4} + \left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right)\right] + \frac{n-m}{2}\Omega(t) \geq \frac{\Omega(t)}{2}, \quad (4.17)$$

$$\Delta X \Delta P_x|_{\psi_{n,m-n}} = \Delta Y \Delta P_y|_{\psi_{n,m-n}} \geq \frac{\hbar}{2} + \frac{\theta(t)\Omega(t)}{8\hbar}. \quad (4.18)$$

To prove the validity of these inequalities we note for instance that the smallest value for the left hand side of (4.16) results from $\Delta X \Delta Y|_{\psi_{0,0}}$. Therefore demonstrating that the quantity $f[\theta(t)] := \Delta X \Delta Y|_{\psi_{0,0}} - \theta(t)/2$ is always nonnegative will establish (4.16). Noting for this purpose that $f[0] = \hbar\sigma^2/2$, $\lim_{\theta(t)\to\infty} f[\theta(t)] \to \infty$ and that the local minimum at $\theta_{\min}(t) = 2\hbar\sigma^2 a^2/(a^2 + \sigma^2\dot{\sigma}^2)$ acquires the value $f[\theta_{\min}(t)] = \hbar\sigma^4\dot{\sigma}^2/(2a^2 + 2\sigma^2\dot{\sigma}^2) \geq 0$ guarantees that $f[\theta(t)] \geq 0$ and therefore the validity of (4.16). One may argue similarly for (4.17) and (4.18), which we will not present here.

In order to display the deviation from the lower bound we depict in figure 2-4 the uncertainty for backgrounds corresponding to the solutions of the EP-equation displayed in figure 1. As expected from our analytical expressions in (4.16) and previous results, the smallest uncertainties are observed for the smaller quantum numbers.

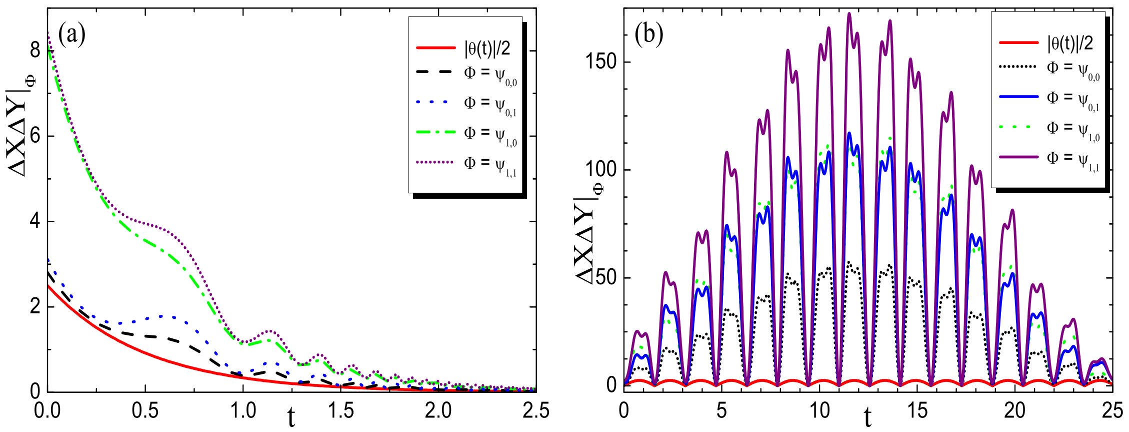

Figure 2: Uncertainties $\Delta X \Delta Y|_{\psi_{n,m-n}}$ versus the generalized lower bound (a) for background fields $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ and (b) for background fields $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

4.2 The generalized uncertainty relation for coherent states

As is well known coherent states are convenient to use in a number of fields of quantum theory, especially in quantum optics, because of the fact that by definition they constitute the transition from a classical to a quantum mechanical formulation of a given system. Starting with Schrödinger's investigations [30], the first systematic and formal way was developed by

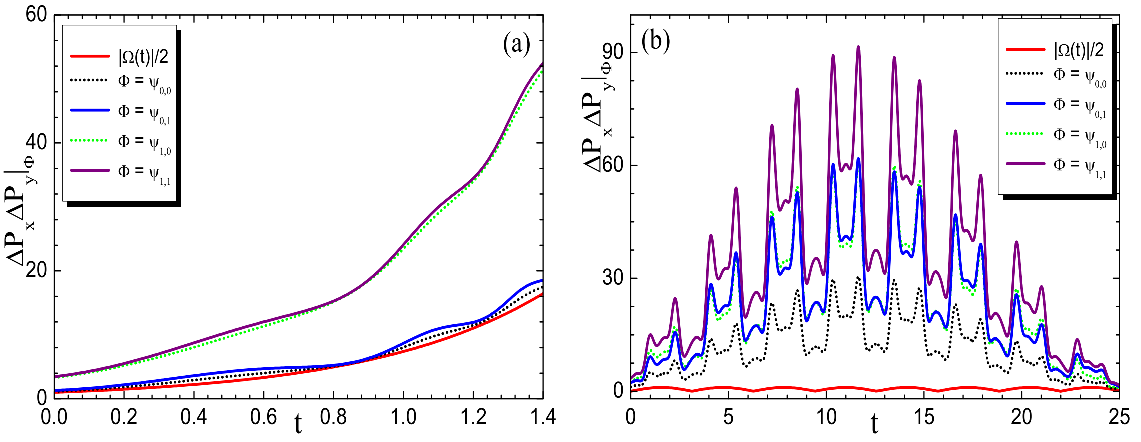

Figure 3: Uncertainties $\Delta P_x \Delta P_y|_{\psi_{n,m-n}}$ versus the generalized lower bound (a) for background fields $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ and (b) for background fields $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

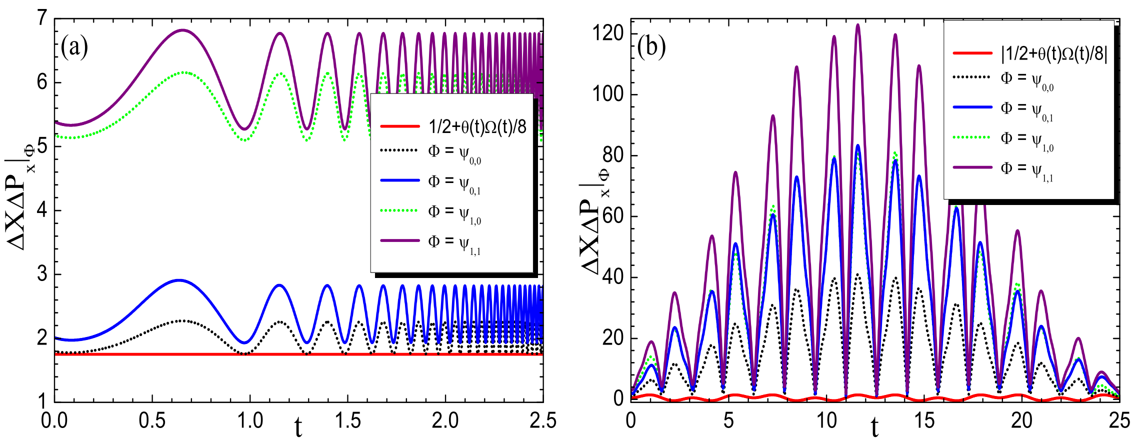

Figure 4: Uncertainties $\Delta X \Delta P_x|_{\psi_{n,m-n}}$ versus the generalized lower bound (a) for background fields $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ and (b) for background fields $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

Glauber [31], who also coined the term coherent states. Since some of properties are very specific to the harmonic oscillator several types and generalizations of coherent states have been proposed thereafter to accommodate different types of situations, see for instance [32] for a review on the developments up to 2001. Fo instance, so-called Klauder [33, 34] and Gazeau-Klauder [35] cherent states, for which the quantum classical correspondence was recently investigated in [36, 37], are extremely useful.

Even though the model under consideration here is of course not the harmonic oscillator, we still have the invariant $I(t)$ expressed in terms of the time-dependent creation and annihilation operators. This enables us to employ techniques used for the construction of Glauber coherent states [31]. Defining therefore the coherent states by means of the time-dependent displacement operator $D(\alpha, t)$ as

$$|\alpha, t\rangle := D(\alpha, t)\,|0,0\rangle\,, \quad \text{with} \;\; D(\alpha, t) := e^{\alpha\hat{a}^\dagger(t) - \alpha^*\hat{a}(t)}, \tag{4.19}$$

it is immediately verified that they constitute eigenstates of the annihilation operator $\hat{a}(t)$, i.e. $\hat{a}(t)\,|\alpha, t\rangle = \alpha\,|\alpha, t\rangle$. Using the matrix elements for the expectation values with respect to the eigenfunction (4.1)-(4.8), we compute the expectation values with respect to the Glauber coherent states

$$\langle\alpha, t|\,x\,|\alpha, t\rangle = -\sqrt{\hbar}\sigma\,\mathrm{Im}\,\alpha, \quad \langle\alpha, t|\,x^2\,|\alpha, t\rangle = \hbar\sigma^2\left(\frac{1}{2} + \mathrm{Im}^2\,\alpha\right), \tag{4.20}$$

$$\langle\alpha, t|\,y\,|\alpha, t\rangle = -\sqrt{\hbar}\sigma\,\mathrm{Re}\,\alpha, \quad \langle\alpha, t|\,y^2\,|\alpha, t\rangle = \hbar\sigma^2\left(\frac{1}{2} + \mathrm{Re}^2\,\alpha\right), \tag{4.21}$$

$$\langle\alpha, t|\,p_x\,|\alpha, t\rangle = \sqrt{\hbar}\left(\frac{\mathrm{Re}\,\alpha}{\sigma} - \frac{\dot{\sigma}\,\mathrm{Im}\,\alpha}{a}\right), \quad \langle\alpha, t|\,p_x^2\,|\alpha, t\rangle = \frac{\hbar}{2}\left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right) + \langle\alpha, t|\,p_x\,|\alpha, t\rangle^2,$$

$$\langle\alpha, t|\,p_y\,|\alpha, t\rangle = -\sqrt{\hbar}\left(\frac{\mathrm{Im}\,\alpha}{\sigma} + \frac{\dot{\sigma}\,\mathrm{Re}\,\alpha}{a}\right), \quad \langle\alpha, t|\,p_y^2\,|\alpha, t\rangle = \frac{\hbar}{2}\left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right) + \langle\alpha, t|\,p_y\,|\alpha, t\rangle^2,$$

such that

$$\Delta x|^2_{|\alpha,t\rangle} = \Delta y|^2_{|\alpha,t\rangle} = \frac{\hbar\sigma^2}{2}, \qquad \Delta p_x|^2_{|\alpha,t\rangle} = \Delta p_y|^2_{|\alpha,t\rangle} = \frac{\hbar}{2}\left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right). \tag{4.22}$$

Notice that the uncertainties are the same as those computed with respect to the ground state $\psi_{0,0}$. Likewise we compute

$$\Delta X|^2_{|\alpha,t\rangle} = \Delta Y|^2_{|\alpha,t\rangle} = \Delta X|^2_{\psi_{0,0}}, \qquad \Delta P_x|^2_{|\alpha,t\rangle} = \Delta P_y|^2_{|\alpha,t\rangle} = \Delta P_x|^2_{\psi_{0,0}}, \tag{4.23}$$

such that the uncertainty relations are identical to those in (4.16)-(4.18) with $\psi_{0,0}$ replaced by $|\alpha, t\rangle$. The crucial difference is of course that $\psi_{0,0}$ is annihilated by $a(t)$, whereas $|\alpha, t\rangle$ constitutes an eigenstate for $\hat{a}(t)$.

Having creation and annihilation operators at our disposal we can use standard techniques from quantum optics to construct squeezed states [38] and improve on the uncertainties obtained so far. Employing for this purpose the so-called squeezing operator $S(\beta, t)$ by defining

$$|\alpha, \beta, t\rangle := S(\beta, t)D(\alpha, t)\,|0,0\rangle\,, \quad \text{with} \;\; S(\beta, t) := e^{\frac{\beta}{2}[\hat{a}^2(t) - \hat{a}^{\dagger 2}(t)]}, \tag{4.24}$$

we may compute the relevant matrix elements for these states, not reported here. Using those we may subsequently deduce the uncertainties for the auxiliary variables to

$$\Delta x|^2_{|\alpha,\beta,t\rangle} = \Delta y|^2_{|\alpha,-\beta,t\rangle} = \frac{\hbar}{2}\sigma^2 e^{\beta}\cosh\beta, \tag{4.25}$$

$$\Delta p_x|^2_{|\alpha,\beta,t\rangle} = \Delta p_y|^2_{|\alpha,-\beta,t\rangle} = \frac{\hbar}{2}\left(\frac{1}{\sigma^2}e^{-\beta} + \frac{\dot{\sigma}^2}{a^2}e^{\beta}\right)\cosh\beta, \tag{4.26}$$

and for our noncommutative variables to

$$\Delta X|^2_{|\alpha,\beta,t\rangle} = \Delta Y|^2_{|\alpha,-\beta,t\rangle} = \frac{\hbar}{2}\left[\sigma^2 e^{\beta} + \frac{\theta^2(t)}{4\hbar^2}\left(\frac{1}{\sigma^2}e^{\beta} + \frac{\dot{\sigma}^2}{a^2}e^{-\beta}\right)\right]\cosh\beta + \frac{\theta(t)}{4}(1-e^{2\beta}),$$

$$\Delta P_x|^2_{|\alpha,\beta,t\rangle} = \Delta P_y|^2_{|\alpha,-\beta,t\rangle} = \frac{\hbar}{2}\left[\frac{1}{\sigma^2}e^{-\beta} + \frac{\dot{\sigma}^2}{a^2}e^{\beta} + \frac{\Omega^2(t)}{4\hbar^2}\sigma^2 e^{-\beta}\right]\cosh\beta + \frac{\Omega(t)}{4}(1-e^{2\beta}).$$

As expected these expressions reduce to (4.22) and (4.23) when $\beta \to 0$.

We can now use the freedom to choose the function $\beta(t)$ to minimize the uncertainties further. For instance, it is easily found that the uncertainty $\Delta x\ \Delta p_x|_{|\alpha,\beta,t\rangle}$ is minimal for $\beta(t) = \beta_{\min}(t) = 1/2\ln\left[\left(a\sqrt{a^2+8\sigma^2\dot{\sigma}^2} - a^2\right)/(4\sigma^2\dot{\sigma}^2)\right]$. Thus taking this value we should find $\Delta x\ \Delta p_x|_{|\alpha,\beta_{\min},t\rangle} < \Delta x\ \Delta p_x|_{|\alpha,t\rangle}$, which is indeed confirmed in figure 5, where we observe that squeezing leads to a considerable reduction in the uncertainties.

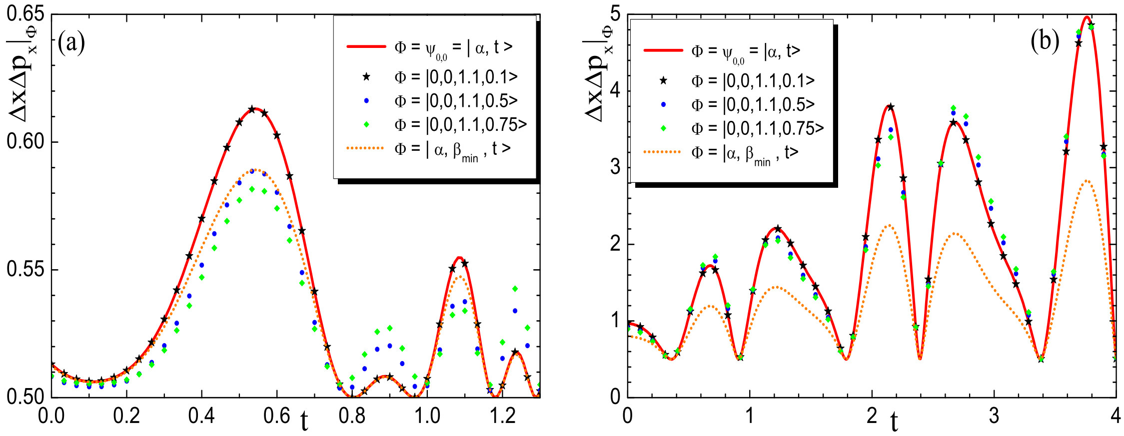

Figure 5: Uncertainties with respect to Glauber coherent states versus squeezed Glauber coherent states and Gaussian Klauder coherent states for the auxiliary variables x, p_x, $\Delta x\Delta p_x|_{|\alpha,t\rangle}$ versus $\Delta x\Delta p_x|_{|\alpha,\beta,t\rangle}$ versus $\Delta x\Delta p_x|_{|GK>}$ (a) for background fields $\theta(t) = \alpha e^{-\gamma t}$ and $\Omega(t) = \beta e^{\gamma t}$ and (b) for background fields $\theta(t) = \alpha\sin(\gamma t)$ and $\Omega(t) = \beta\sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

The minimization for the uncertainties involving our noncommutative variables is less obvious. Due to the complexity of the expressions we can not perform this task for generic $\beta(t)$, but only for specific instances in time. For instance, we find numerically the minimum for $\Delta X\ \Delta P_x|_{|\alpha,\beta,t=4\rangle}$ at $\beta = -1.88203$. Indeed, as seen in figure 6 panel (a), at $t = 4$ this value leads to a reduction in the uncertainties when compared to $\Delta X\ \Delta P_x|_{|\alpha,t=4\rangle}$.

However, for different values of time the uncertainties have grown considerably. It appears that the squeezing works only well for momentum-coordinate uncertainties as for instance $\Delta X\ \Delta Y|_{|\alpha,\beta,t\rangle}$ is always minimal at $\beta(t) = 0$, such that the squeezing does not lead to any reduction in these uncertainties. Figure 6 panel (b) exhibits these findings.

Let us next compare our findings with the uncertainties computed with respect to Gaussian Klauder coherent states defined as [39, 40, 41]

$$|GK\rangle = |n, m_0, \phi_0, s\rangle := \frac{1}{\sqrt{N(m_0)}}\sum_{m=0}^{\infty}\exp\left[-\frac{(m-m_0)^2}{4s^2}\right]e^{im\phi_0}\,|n, m-n\rangle\,, \qquad (4.27)$$

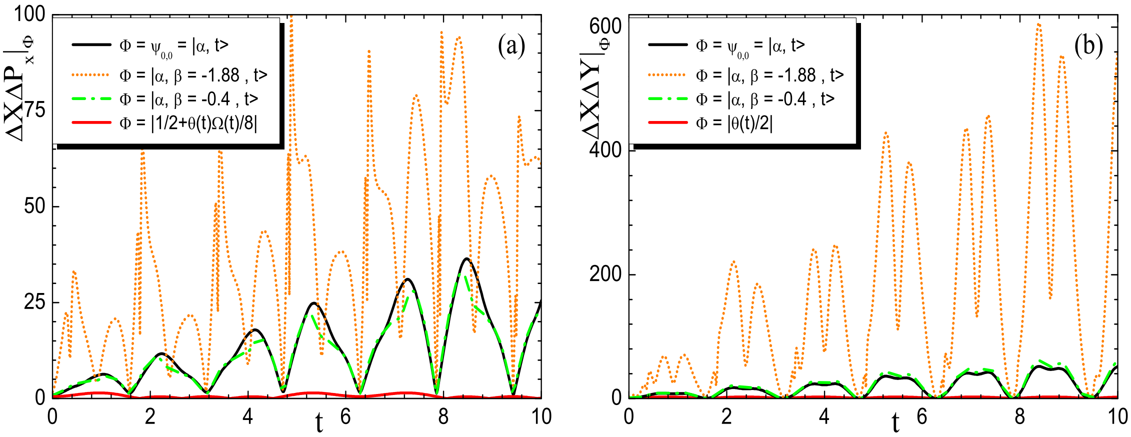

Figure 6: Uncertainties with respect to Glauber coherent states versus squeezed Glauber coherent states for the noncommutative variables X, Y, P_x for background fields $\theta(t) = \alpha \sin(\gamma t)$ and $\Omega(t) = \beta \sin(\gamma t/2)$. In both panels the constants are $\alpha = 5$, $\beta = 2$, $\gamma = 2$, $m = \hbar = \tau = \omega = 1$, $\kappa = 1/4$ and $\mu = \sqrt{5/3}$.

with normalization factor $N(m_0) := \sum_{m=0}^{\infty} \exp\left[-(m - m_0)^2/(2s^2)\right]$, initial phase factor ϕ_0 and Gaussian standard deviation s. Using the matrix elements (4.1)-(4.8) we readily compute the expectation values with respect to these states

$$\langle GK| x |GK\rangle = -\frac{\sqrt{\hbar}}{N(m_0)} \sigma \sin(\phi_0 + \alpha_{01}) S_1(m_0), \tag{4.28}$$

$$\langle GK| y |GK\rangle = -\frac{\sqrt{\hbar}}{N(m_0)} \sigma \cos(\phi_0 + \alpha_{01}) S_1(m_0), \tag{4.29}$$

$$\langle GK| p_x |GK\rangle = \frac{\sqrt{\hbar}}{N(m_0)} \left[\frac{1}{\sigma} \cos(\phi_0 + \alpha_{01}) - \frac{\dot{\sigma}}{a} \sin(\phi_0 + \alpha_{01})\right] S_1(m_0), \tag{4.30}$$

$$\langle GK| p_y |GK\rangle = -\frac{\sqrt{\hbar}}{N(m_0)} \left[\frac{1}{\sigma} \sin(\phi_0 + \alpha_{01}) + \frac{\dot{\sigma}}{a} \cos(\phi_0 + \alpha_{01})\right] S_1(m_0), \tag{4.31}$$

and

$$\langle GK| x^2, y^2 |GK\rangle = \frac{\hbar \sigma^2}{2N(m_0)} \left[S_2(n+1, m_0) \mp \sqrt{2} \cos(2\phi_0 + \alpha_{02}) S_3(m_0)\right], \tag{4.32}$$

$$\begin{aligned}\langle GK| p_x^2, p_y^2 |GK\rangle = \frac{\hbar}{2N(m_0)} \Bigg\{ &\left(\frac{1}{\sigma^2} + \frac{\dot{\sigma}^2}{a^2}\right) S_2(n+1, m_0) \\ &\pm\sqrt{2}\left[\left(\frac{1}{\sigma^2} - \frac{\dot{\sigma}^2}{a^2}\right) \cos(2\phi_0 + \alpha_{02}) - 2\frac{\dot{\sigma}}{a\sigma} \sin(2\phi_0 + \alpha_{02})\right] S_3(m_0)\Bigg\},\end{aligned} \tag{4.33}$$

$$\begin{aligned}\langle GK| x p_y, y p_x |GK\rangle = \frac{\hbar}{2N(m_0)} \Bigg\{ &\sqrt{2}\left[\frac{\dot{\sigma}\sigma}{a} \sin(2\phi_0 + \alpha_{02}) - \cos(2\phi_0 + \alpha_{02})\right] S_3(m_0) \\ &\pm S_2(-n, m_0)\Bigg\}.\end{aligned} \tag{4.34}$$

We abbreviated $G(m, m_0) := \exp\left[-(m-m_0)^2/(4s^2)\right]$ and the sums

$$S_1(y) := \sum_{k=0}^{\infty} \sqrt{k+1} G(k,y) G(k+1,y), \tag{4.35}$$

$$S_2(x,y) := \sum_{k=0}^{\infty} (k+x) G^2(k,y), \tag{4.36}$$

$$S_3(y) := \sum_{k=0}^{\infty} \mu(k,k+2) G(k,y) G(k+2,y). \tag{4.37}$$

One could make some approximations here for the sums by replacing them with Gaussian integrals, as for instance in [40, 42]. However, these sums converge very fast with only some of the initial terms taken into account and therefore it suffices here for our purposes to present numerical values. When the Gaussian enveloping function is very sharp we notice that the main contribution simply results from the center of the Gaussian. For instance, for $s = 0.1$, we compute $S_1(0) < 10^{-10}$, $S_2(n,0) = n$, $S_3(0) < 10^{-10}$ and $N(0) = 1$, such that

$$\Delta o|^2_{\psi_{0,0}} = \Delta o|^2_{|\alpha,t\rangle} = \Delta o|^2_{|GK\rangle} \quad \text{for } o = x, y, p_x, p_y. \tag{4.38}$$

This behaviour is clearly observable in figure 5. For a broader Gaussian enveloping function other modes start to contribute. For instance, for $s = 0.5$ we compute $S_1(0) = 0.3774$, $S_2(0,0) = 0.1360$, $S_2(1,0) = 1.2717$, $S_3(0) = 0.0184$ and $N(0) = 1.1357$ and for $s = 0.75$ we find $S_1(0) = 0.7998$, $S_2(0,0) = 1.9092$, $S_2(1,0) = 0.4693$, $S_3(0) = 0.1897$ and $N(0) = 1.4400$. For these values the uncertainties for the auxiliary variables are depicted in figure 5 for two different types of background fields. We observe that depending on the instance of time the uncertainties might be lowered or increased.

When comparing with the uncertainties for the squeezed coherent states it appears that optimal minimum is dependent on the type of background field. We observe in figure 5 that for sinusoidal background fields the squeezed Glauber coherent states lead to minimal uncertainties which can not be undercut when using Gaussian Klauder coherent states instead, whereas for exponential backgrounds Gaussian Klauder coherent states allow for a further minimization.

5. Conclusions

We have formulated and investigated a prototype model on a time-dependent background. For an explicit representation of the underlying noncommutative algebra the Hamiltonian naturally acquire a time-dependent form. Using the Lewis-Riesenfeld method of invariants we constructed the time-dependent invariants together with their eigensystem. Following the standard procedure allowed to compute the eigenfunctions for the original Hamiltonian. As common in the context of the invariant method all solutions are expressed in terms the solutions of the nonlinear Ermakov-Pinney equation and variations thereof. In general this auxiliary problem is not dealt with in this context and all expressions are left as still dependent on an unknown function, $\sigma(t)$ in our case. In order to make the solutions more explicit and to allow also for numerical studies thereafter, we have included here a detailed discussion of some solutions.

Our explicit solutions then allow for a analysis of the generalized uncertainty relations for which the lower bounds become time-dependent functions. Since our invariants are expressed in terms of time-dependent creation and annihilation operators, standard Glauber coherent states were constructed by means of the displacement operator in a straightforward manner. We found that the uncertainties for these states are identical to those of the ground state annihilated by $a(t)$. By constructing the so-called squeezing operator we demonstrated that these uncertainties can be further minimized for momentum-coordinate uncertainties, where the absolute lower bound was only be reached for certain instances in time. For coordinate-coordinate uncertainties the minimal uncertainties were already reached by the Glauber coherent states and squeezing does not lead to any further improvement. We compared these findings with an analysis for so-called Gaussian Klauder coherent states. A major difference towards the forgoing computations is that the phase $\alpha_{n,\ell}(t)$ becomes a relevant quantity. While in the computation of expectation values for eigenstates the phase always cancels due to the sum in $|GK\rangle$ it leads here to interferences. We observe that also for the Gaussian Klauder coherent states the uncertainties resulting from the computations for the ground state and the nonsqueezed Glauber coherent state can be undercut. The answer to the question which type of the coherent states is optimal appears to be background field dependent. The time-dependent lowest bounds are well respected for all investigated scenarios.

There remain a multitude of challenges. First of all it would be highly desirable to investigate models on different types of time-dependent backgrounds rather than (1.1), possibly even those leading to minimal length. As always the study of different types of models will complete and enrich the understanding. The interesting question in all these different types of scenarios is whether they still allow for explicit solvability, which is one of the main virtue of our investigations, or if one needs to resort to additional approximations.

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