The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case

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Abstract

In this paper, we extend the analysis of the behaviour of pension contracts with guaranteed annuity conversion options (as presented in Ballotta and Haberman, 2003) to the case in which mortality risk is incorporated via a stochastic model for the evolution over time of the underlying hazard rates. The pricing framework makes also use of a Black-Scholes/Heath-Jarrow-Morton economy in order to obtain an analytical solution to the fair valuation problem of the liabilities implied by these particular pension policies. The solution is not in closed form and therefore we resort to Monte Carlo simulation. Numerical results are investigated and the sensitivity of the price of the option to changes in the key parameters from the financial and mortality models is also analyzed.

Keywords: fair value, guaranteed annuity options, incomplete markets, stochastic mortality.

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Insurance branch codes: IB10

Subject codes: IM10

1 Introduction

A large number of products offered by life insurance companies involve a range of complex contingent claims involving equity risk, interest rate risk and mortality risk. The presence of these products in the life insurance and pension liabilities of insurance companies has given rise to an increasing focus on the issues of capital adequacy and solvency requirements. As regulation in this area makes use of accounting information as a starting point, the fair valuation methodologies promoted by the International Accounting Standards Board (IASB) in the newly issued International Financial Reporting Standards (IFRS), are going to be adopted for the construction of consolidated financial statements, in an

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attempt to make the assessment process of companies' financial performance much more realistic and reliable.

An example of such a complex contract, which also created instability in the life insurance market, is the so-called guaranteed annuity option (GAO). A GAO is a design feature attached to individual pension policies which provides the policyholder with the right to receive at retirement either a cash payment or an annuity which would be payable throughout the policyholder's remaining lifetime and which is calculated at a guaranteed rate, depending on which has the greater value. This guaranteed conversion rate between cash and pension was a common feature of individual pension policies sold in the UK during the (late) 1970s and 1980s, with more than 40 companies involved in this market.

Until the early 1990s, the UK experience has been that the cash benefit was more valuable than the guaranteed annuity payment since a higher pension could be obtained by using the cash to buy the best annuity rates available in the market (the so-called "open market option"). Since the late 1990s, reductions in market interest rates and unanticipated falls in mortality rates at the oldest ages have meant that the position has changed and the guaranteed annuity has tended to be worth more than the cash benefit. As a result of these two combined effects, many UK insurance companies (which have sold policies with guaranteed annuity options) have experienced solvency problems, requiring the setting up of extra reserves, and leading one large mutual life insurer (Equitable Life, the world's oldest life insurance company) to be closed to new business in 2000. Although pension policies with these guarantees are no longer being sold in the UK, these are a common feature of corresponding policies in other countries, for example the US. Thus, currently in the US variable annuity market, there are guaranteed annuity rate (GAR) contracts and guaranteed minimum income benefit (GMIB) contracts. A GAR contract is identical to a GAO. A GMIB contract includes the additional feature that the cash benefit available at retirement is guaranteed to be at least a pre-specified amount.[1].

Although the new accounting directives promoted by IASB mentioned above focus essentially on the financial risk affecting life insurance contracts, the UK historical experience shows that very long term products like GAOs are significantly exposed to unanticipated changes over time in the mortality rates of the reference population (mortality risk). This means that the fair valuation techniques proposed in the IFRS need to be integrated with an accurate assessment of future mortality rates. Hence, in this paper, we propose a possible integrated framework for the market consistent valuation of GAOs, which incorporates mortality risk as well by means of a stochastic model for the evolution of mortality rates over time.

In this paper, we focus on unit-linked deferred annuity contracts purchased originally by a single premium. For simplicity, we ignore insurance company expenses, taxes, profit and pre-retirement death benefits in order to concentrate on the GAO. The analytical approach that we adopt follows the financial economics literature and exploits the well-known option valuation theory in order to obtain results for the pricing, reserving and hedging of the GAO. In particular, we follow Ballotta and Haberman (2003) and we use a single-factor Heath-Jarrow-Morton framework for the term structure of interest rates.

[1]At the time of writing, GARs are an almost universal feature of variable annuity contracts, and because of the low nominal interest rate guaranteed, are mainly "out of the money".

This choice is justified by the need to avoid dependence of the model on the market price of interest rate risk, which usually implies an arbitrary specification of the model parameters leading to arbitrage opportunities (Heath, Jarrow and Morton, 1992).

An alternative approach based on modelling the dynamics of the annuity price, rather than the underlying term structure of interest rates, has been used by Bezooyen et al. (1998), Pelsser (2003) and Wilkie et al. (2003). However, we argue that a methodology based on the term structure of interest rates is more sound in that, on one hand, it relies on quantities, like zero coupon bonds, that are fully traded in the financial market, and, on the other hand, it facilitates the analysis of the effect on the GAO of changes in market interest rates and their term structure. In this respect, we believe that the model proposed in this paper is more consistent with the recommendation from IASB that the valuation techniques used to estimate fair values should maximize the use of market inputs (see also Jørgensen, 2004, for a more detailed discussion an accounting standards for life insurance liabilities).

Under the additional assumption of a mortality risk that is independent of the financial risk, a general pricing model is proposed and a numerical procedure, using Monte Carlo techniques, for the estimation of the value of the guaranteed annuity option is implemented. Numerical results are investigated and the sensitivity of the price of the option to changes in the key parameters is also analyzed.

The paper is organized as follows: section 2 develops the framework for the valuation of guaranteed annuity conversion options. In section 3, we introduce a stochastic model for the mortality risk; section 4 provides a model for the financial market and a pricing formula for the guaranteed annuity option. In section 5 we discuss the numerical evidence produced and concluding remarks are offered in section 6.

2 A valuation approach for guaranteed annuity options

A guaranteed annuity option (GAO) provides the holder of the contract the right to receive at retirement, at time T, either a cash benefit (equal to the current value of the reference portfolio, S_T), or an annuity which would be payable throughout his/her remaining lifetime and which is calculated at a guaranteed rate, g, depending on which has the greater value.

Hence, if the policyholder is aged x_0 at time 0, when the contract is initiated, and N is the normal retirement age, the GAO pays out at maturity $T = N - x_0$:

$$\begin{aligned} C_T &= \left(gS_T a_{x_0+T} - S_T\right)^+ \\ &= gS_T \left(a_{x_0+T} - K\right)^+, \end{aligned} \tag{1}$$

where $K = 1/g$ and a_{x_0+T} represents the "annuity factor", i.e. the expected present value at time T of a life annuity which pays £1 per year throughout the remaining lifetime of the policyholder.

Consider a competitive market with continuous trading and assume that the market is frictionless, i.e. that there are no taxes, no transaction costs, no restrictions on borrowing or short sales, all securities are perfectly divisible, and that the price process of the equity fund S follows an adapted, càdlàg and strictly positive semimartingale. Let r_t be the stochastic short rate; applying risk neutral valuation, the fair value at time T of the annuity can be calculated as

$$a_{x_0+T} = \hat{\mathbb{E}}\left[\left.\sum_{j=0}^{w-(T+x_0)} e^{-\int_T^{T_j} r_u du} 1_{\left(\tau_{x_0+T} > T_j - T\right)}\right| \mathcal{F}_T\right], \tag{2}$$

where $\hat{\mathbb{E}}$ denotes the expectation under some risk neutral probability measure $\hat{\mathbb{P}}$, w is the largest survival age, T_j, $j = 1, ..., w-(T+x_0)$, are the times of the annuity payments, $\mathcal{F}_T$ is the information flow at maturity and τ_y is a random variable representing the remaining lifetime of a policyholder aged y. Risk neutral valuation also implies that the value at time $0 \leq t < T$ of the GAO contract entered at time 0 by a policyholder then aged x_0 is

$$V_{x_0}\left(x^{(t)}, t, T-t = N - x_0 - t\right) = \hat{\mathbb{E}}\left[\left. e^{-\int_t^T r_u du} C_T 1_{\left(\tau_{x^{(t)}} > T-t\right)}\right| \mathcal{F}_t\right], \tag{3}$$

where $x^{(t)} = x_0 + t$ is the policyholder age at the valuation date t.

Equations (1)-(3) show that the GAO contract is affected by two sources of risk: the financial risk, in the form of the uncertainty related to future movements in both the equity fund and the market interest rate, and the mortality risk, captured by the random remaining lifetime of the policyholder. Ballotta and Haberman (2003) provide a closed analytical formula for equation (3) modelling the financial risk within the classical Black-Scholes economy and making use of a single-factor Heath, Jarrow and Morton (1992) framework for the term structure of interest rates. Mortality effects are taken into consideration through the survival probabilities which are calculated using the standard mortality tables adopted by UK Life Insurance companies (for pricing and reserving calculations). In sections 3 and 4 of this paper, we evaluate equation (3) still making use of the same Black-Scholes/HJM framework to model the financial risk; however, we also introduce a stochastic model for the mortality risk which is based on the reduction factor approach for projecting mortality rates.

3 A stochastic approach to mortality risk: the basic model and its extensions

In the previous section, we defined τ_x to be a random variable which represents the remaining lifetime of the policyholder and which depends on the age, x, of the policyholder at time t. The survival function of the random variable τ_x is given by

$${}_s p_x = \mathbb{P}\left(\tau_x > s \middle| \mathcal{F}_t\right),$$

where $\mathbb{P}$ is the objective probability measure. If we explicitly allow for the time dependence of the hazard rate, and we define $\mu\left(x+z, t+z\right)$ to be the hazard rate for an individual at time $t+z$ then aged $x+z$, it follows that

$${}_s p_x = \mathbb{E}\left[\left. e^{-\int_0^s \mu(x+z, t+z) dz}\right| \mathcal{F}_t\right]. \tag{4}$$

A widely used actuarial model for projecting mortality rates is the reduction factor model (Renshaw and Haberman, 2003); this has been used in the UK and US for pensioner and annuitant populations for many years, see, for example, standard tables produced by the Continuous Mortality Investigation Bureau in the UK since 1967-70 (e.g. CMI Bureau, 1999) and the General Annuity Valuation Tables in the US (Group Annuity Valuation Table Task Force, 1995).

This model has traditionally been formulated with respect to the conditional probability of dying in a year:

$$q(y,u) = q(y,0)\, RF(y,u),$$

where $q(y,0)$ represents the probability that a person aged y exact will die in the next year, based on the mortality experience for the base year (i.e. year 0), and correspondingly $q(y,u)$ relates to future year u.

Given the form of (4), we propose a simple model for the trajectory of the hazard rate over time. As in Sithole et al. (2000), we consider a reduction factor approach based on hazard rates, so that

$$\mu(y,u) = \mu(y,0)\, RF(y,u)$$

where $\mu(y,0)$ is the hazard rate for a person aged y in the base year (i.e. year 0) or period, and $\mu(y,u)$ is the hazard rate for a person attaining age y in future year u (i.e. as measured from the base year or period), and the reduction factor $RF(y,u)$ is the ratio of the hazard rates. Using a modelling approach based on generalized linear models, as proposed by Renshaw et al. (1996), Sithole et al. (2000) derive a series of models appropriate to annuitant and pensioner populations in the UK which may be reduced to the following structure:

$$RF(y,u) = e^{(\alpha+\beta y)u}. \tag{5}$$

The parameter α represents the rate of change in the hazard rate on the logarithmic scale over time and we would expect estimates that are negative. The parameter β represents an offset term that reflects a rate of change that could differ with age y.

We follow this line of reasoning and, with $y = x + z$, and $u = t + z$, use the following model for the time evolution of the hazard rate:

$$\mu(x+z, t+z) = \mu(x+z, 0)\, e^{(\alpha+\beta(x+t))(t+z)+\sigma_h Y_{t+z}}, \tag{6}$$

where $(Y_t : t \geq 0)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, which is introduced to model random variations in the forecast trend, and

$$\mu(x+z,0) = a_1 + a_2 R + e^{b_1+b_2 R+b_3\left(2R^2-1\right)}, \tag{7}$$

$$R = \frac{(x+z)-70}{50}, \qquad x \geq 50. \tag{8}$$

This model for the hazard rate for the base year corresponds to the structure for the UK standard tables for annuitant and pensioner populations for the period 1991-94, as proposed by the CMI Bureau (1999). For numerical illustrations, we use the parametrization for the standard table for female annuitants (1991-1994), and the corresponding values for α and β estimated by Sithole et al (2000): see Table 2 later[2].

[2]This choice was made because the female data set includes more observations than the corresponding

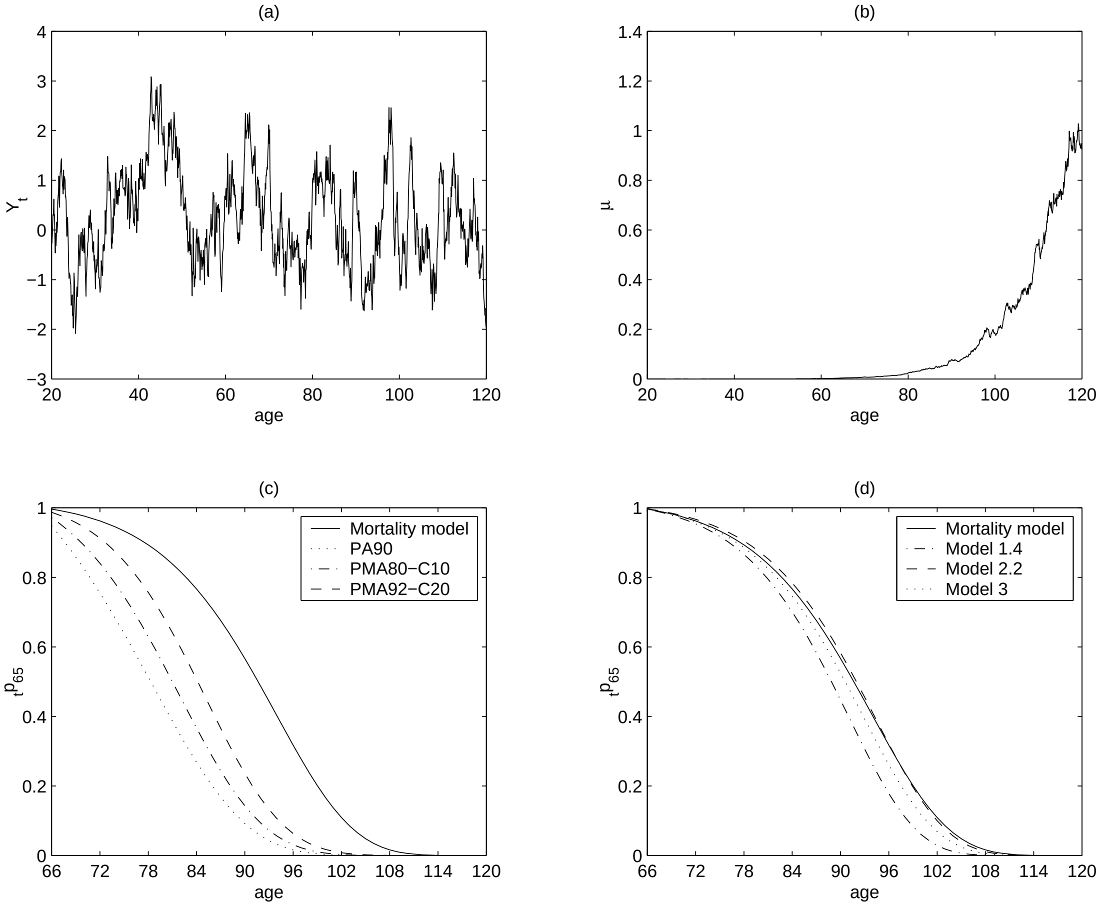

Figure 1: (a): a sample trajectory of the process Y; (b): the corresponding trajectory of the hazard rate; the parameter set is as given in Table 2; the time period covers a 100 year span: it relates in fact to the mortality evolution of an individual from age 20 to the latest survival age considered, i.e. $w = 120$; (c): post-retirement survival probabilities generated by the UK mortality tables and by the stochastic mortality model introduced in section 3; (d): post-retirement survival probabilities with longevity effects.

As far as the stochastic component of equation (6) is concerned, we assume that Y is governed by an Ornstein-Uhlenbeck process; in other words, the process Y satisfies the following stochastic differential equation,

$$\begin{cases} dY_t = -aY_t dt + dX_t \\ Y_0 = 0, \end{cases} \tag{9}$$

where $(X_t : t \geq 0)$ is a standard one-dimensional $\mathbb{P}$-Brownian motion, independent of the sources of randomness existing in the financial market. This is similar to the model proposed by Milevsky and Promislow (2001), and has the desirable property of mean reversion, with the parameter a measuring the speed of mean reversion to the long-run

male data set for immediate annuitants and, relative to other data sets, the parameters can be estimated from an extensive time series of observations: 1958-1994.

mean which is here set equal to zero. The feature of mean reversion, and in particular reversion to a long run value equal to zero, ensures the convergence to zero of survival probabilities up to very old ages. For this reason, a mean reverting process is more appealing than, say, a random walk with a tendency to move to extreme values. However, this is a strong assumption. As Cairns et al (2004) note, mean reversion implies that, if mortality improvements exceed expectations at some point, then afterwards the potential for further improvements will be reduced. Similarly, if improvements fall behind expectations, then the potential for subsequent improvements are enhanced. Hence this choice is very much a starting assumption. A sample trajectory of the Ornstein-Uhlenbeck process, Y, as specified by equation (9) , is represented alongside with the corresponding trajectory of the hazard rate, μ, in the top panels of Figure 1.

The model represented by equation (6) allows for a future mortality evolution that is random. However, it is recognised in the literature that systematic deviations from the forecasted mortality rates may take place so that a parameter risk and a model risk are present. When this is applied to the trend at the older ages, the risk is usually referred to as "longevity risk": see Marocco and Pitacco (1998), Olivieri (2001), Olivieri and Pitacco (2002), for example.

We enhance the model described above in order to incorporate longevity risk by following the approach of Olivieri and Pitacco (2002), inter alia. In order to express the range of possible evolutions of mortality, we consider a family of projected hazard rates, for a given age at entry x. Thus, we consider

$$[\mu(x+z,t+z;H(x));H(x)\in\mathcal{H}(x)],$$

where $H(x)$ is a particular hypothesis concerning the trend of mortality for individuals entering an insured group at age x, and $\mathcal{H}(x)$ represents a given set of such hypotheses. In particular, if we focus on equation (6) where the mortality trend is expressed by a set of parameters, then we could consider

$$[\mu(x+z,t+z;\theta(x));\theta(x)\in\Theta(x)], \tag{10}$$

where $\theta(x)$ denotes a vector parameter and $\Theta(x)$ denotes the corresponding multi-dimensional parameter space. To illustrate the methodology, we take $\theta(x)\equiv\alpha$ in equation (6) and consider the possible set of values for $\alpha:\Theta(x)\equiv\{-0.05,-0.03,-0.01\}$, which is shown in Table 1. We deal with the range of values for α by assuming that the parameter $\tilde{\alpha}$ is a discrete random variable and has the alternative probability functions shown in Table 1.

Panel (c) of Figure 1 compares the post retirement survival probabilities generated by three recent UK standard mortality tables (constructed by the Continuous Mortality Investigation Bureau from data collected from insurance companies) with the probabilities generated by the stochastic mortality model which we have introduced in this section, with the parametrization set out in Table 2. In particular, the PA90 table is based on data for the period 1967-70, extrapolated forward to 1990. The PMA80-C10 table is based on data for the period 1979-82, extrapolated to 2010, and the PMA92-C20 table is based on data for the period 1990-94, extrapolated to 2020. Each of these life tables is a "single

	$\tilde{\alpha} =$ with prob.	-0.05	-0.03	-0.01
Model	**1.1**	0.3	0.4	0.3
	1.2	0.2	0.6	0.2
	1.3	0.1	0.8	0.1
	1.4	1/3	1/3	1/3
(symmetric distribution)				
Model	**2.1**	0.4	0.3	0.3
	2.2	0.6	0.2	0.2
	2.3	0.8	0.1	0.1
(asymmetric distribution)				
Model	**3**	$\tilde{\alpha} \sim U(-0.05, -0.01)$		

Table 1: Parameter Risk: range of values for $\tilde{\alpha}$ with corresponding probability functions.

entry" table with extrapolation up to a single future time point and hence they differ fundamentally in character from the stochastic model represented in equation (6) which allows for explicit projection of the hazard rate on a cohort basis up to each potential survival time, t. The three standard life tables show the expected feature that survival prospects (and hence life expectancy) have improved over time. The stochastic model shows higher survival probabilities up to each age, which is a reflection of two features:

- as noted above, the standard tables are "single entry" in character, while equation (6) allows for the cohort dynamics over time;

- the standard tables relate to male pensioners in pension schemes administered by insurance companies (effectively, a case of compulsory purchase of the annuity) while the parametrization of (6) is based on the experience of female annuitants (effectively, a case of voluntary purchase of the annuity). Adverse selection effects tend to be stronger in annuitant populations than in pension populations (CMIB, 1999, Sithole et al., 2000).

Panel (d) of Figure 1 compares the post retirement survival probabilities generated by the model with allowance for longevity effects. We compare the base model with $\alpha = -0.028$ with Models 1.4, 2.2 and 3, the latter three cases incorporating heterogeneity in the value of α by way of the probability functions listed in Table 1. Comparing the survival curves for Model 1.4 and 3, we note that they have the same value of $\mathbb{E}(\tilde{\alpha}) = -0.03$, with model 1.4 having the higher variance. The effect of higher variance (corresponding to greater heterogeneity) is to reduce the survival probabilities (shift to the left). The curves for Model 3 and the base model show that, despite the former having $\mathbb{E}(\tilde{\alpha}) < -0.028$, it leads to lower survival probabilities because of the presence of heterogeneity. The survival curves for the base model and Model 2.2 are almost coincident, but the latter has higher survival probabilities at the younger ages and lower survival probabilities at the older ages, so that there is a cross-over. This again demonstrates the trade-off between heterogeneity and size of α: the characteristics of model 2.2 and the base model may be summarized as follows.

Model 2.2	$\mathbb{E}(\tilde{\alpha}) = -0.038$	positive variance
Base model	$\alpha = -0.028$	no variance.

Thus, model 2.2 has, on average, an extra 1% decline in the secular mortality trend and, in terms of survival probabilities post-retirement, this is roughly balanced by the extra presence of heterogeneity compared to the fixed α case of the base model.

4 A model for the financial risk and the GAO valuation formula

In the frictionless market introduced in section 2, assume that the insurer invests the single premium paid by each policyholder at the start of the contract into an equity fund, whose risk neutral dynamic is described by the following stochastic differential equation.

$$\begin{cases} dS_t = r_t S_t dt + \sigma_S S_t d\hat{Z}_t, \\ S_0 \in \mathbb{R}^+, \end{cases} \tag{11}$$

where $\sigma_S \in \mathbb{R}^+$ and $\left(\hat{Z}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion. Thus, S_0 is the single premium. As mentioned above, we assume that the evolution of the term structure of interest rates is given by a single-factor HJM framework and we consider the specific case in which the forward rate volatility has an exponentially decaying structure[3]. In other words, the risk-neutral dynamic of the forward rate is given by

$$df(t,T) = \left(\sigma^2 e^{-\lambda(T-t)} \int_t^T e^{-\lambda(u-t)} du\right) dt + \sigma e^{-\lambda(T-t)} d\hat{W}_t, \tag{12}$$

where $\left(\hat{W}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion correlated with $\hat{Z}$, so that

$$d\hat{W}_t d\hat{Z}_t = \rho dt$$

for any $\rho \neq 0$. This implies

$$\hat{Z}_t = \rho \hat{W}_t + \sqrt{1-\rho^2}\hat{W}'_t,$$

where $\left(\hat{W}'_t : t \geq 0\right)$ is a $\hat{\mathbb{P}}$-Brownian motion independent of $\hat{W}$. Hence, ρ represents the correlation coefficient between equity fund values and interest rates. In this setup

$$r_t = \lim_{T \to t} f(t,T)$$

[3] It can be shown that the exponentially decaying structure of the forward rate volatility leads to a mean-reverting form of the short rate that closely resembles an extended version of the Vasicek (1977) model (see, for example, Chiarella and Kwon, 2001).

and

$$P_t(T) = e^{-\int_t^T f(t,u)du}$$

is the price at time t of a unit face value zero coupon bond, with redemption date T.

According to the model for the hazard rate introduced in section 3, equation (2) can be rewritten as follows:

$$a_{x_0+T} = \hat{\mathbb{E}}\left[\sum_{j=0}^{w-(T+x_0)} e^{-\int_0^{T_j-T} \mu\left(x^{(T)}+z,T+z\right)dz} e^{-\int_T^{T_j} r_u du} \middle| \mathcal{F}_T \right],$$

where $x^{(T)} = x_0 + T$. Assume that the mortality risk is independent of any source of risk existing in the financial market, i.e. that the Brownian motion X driving the dynamic of the mortality risk, as introduced in section 3, is independent of the "financial" Wiener processes $\hat{W}$ and $\hat{Z}$. Then

$$\begin{aligned} a_{x_0+T} &= \sum_{j=0}^{w-(T+x_0)} \hat{\mathbb{E}}\left[e^{-\int_0^{T_j-T} \mu\left(x^{(T)}+z,T+z\right)dz} \middle| \mathcal{F}_T \right] \hat{\mathbb{E}}\left[e^{-\int_T^{T_j} r_u du} \middle| \mathcal{F}_T \right] \\ &= \sum_{j=0}^{w-(T+x_0)} \hat{\mathbb{E}}\left[e^{-\int_0^{T_j-T} \mu\left(x^{(T)}+z,T+z\right)dz} \middle| \mathcal{F}_T \right] P_T(T_j), \end{aligned} \tag{13}$$

where $P_T(T_j)$ is the price a time T of a zero coupon bond with unit face value and redemption date T_j. The term

$$\hat{\mathbb{E}}\left[e^{-\int_0^{T_j-T} \mu\left(x^{(T)}+z,T+z\right)dz} \middle| \mathcal{F}_T \right]$$

contained in equation (13) can be interpreted, by analogy with equation (4), as the "risk neutral survival probability" ${}_{T_j-T}\hat{p}_{x^{(T)}}$.

We note that the setting proposed in this paper defines an incomplete market: since mortality-dependent securities (like, for example, annuities) are not fully tradeable[4] in the market, the martingale condition defining $\hat{\mathbb{P}}$ does not provide any further indications as to how the change of measures for mortality should be operated. This leaves us with the issue of determining a suitable probability measure for the demographic part of equation (13). One possible approach is to let the parameter of the Girsanov density related to the mortality process be equal to zero. This implies that the survival probabilities are calculated under the real probability measure $\mathbb{P}$, as if the market were neutral with respect to systematic and unsystematic mortality risk (Dahl, 2003). Hence

$$a_{x_0+T} = \sum_{j=0}^{w-(T+x_0)} {}_{T_j-T}p_{x^{(T)}} P_T(T_j). \tag{14}$$

[4]By "not fully tradable" we mean that there is not a secondary market for these types of contract.

Consequently, from equation (3) the value of the GAO at time t is

$$
\begin{aligned}
& V_{x_0}\left(x^{(t)}, t, T-t\right) \\
= \; & \hat{\mathbb{E}}\left[e^{-\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz} e^{-\int_t^T r_u du} C_T \middle| \mathcal{F}_t \right] \\
= \; & g\hat{\mathbb{E}}\left[e^{-\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz} e^{-\int_t^T r_u du} S_T \left(\sum_{j=0}^{w-(T+x_0)} {}_{T_j-T}p_{x^{(T)}} P_T\left(T_j\right) - K \right)^{+} \middle| \mathcal{F}_t \right].
\end{aligned}
\tag{15}
$$

If $\tilde{\mathbb{P}}$ is a martingale probability measure equivalent to $\hat{\mathbb{P}}$ and defined by the density process (Geman, El Karoui and Rochet, 1995)

$$
\eta_T := \left. \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}} \right|_{\mathcal{F}_T} = e^{-\int_0^T r_u du} \frac{S_T}{S_0}, \tag{16}
$$

then equation (15) can be reduced to

$$
\begin{aligned}
& V_{x_0}\left(x^{(t)}, t, T-t\right) \\
= \; & g S_t \tilde{\mathbb{E}}\left[e^{-\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz} \left(\sum_{j=0}^{w-(T+x_0)} {}_{T_j-T}p_{x^{(T)}} P_T\left(T_j\right) - K \right)^{+} \middle| \mathcal{F}_t \right],
\end{aligned}
\tag{17}
$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the stock-risk-adjusted probability measure $\tilde{\mathbb{P}}$. Note that in the previous equation the terminal GAO payoff is "discounted", under the $\tilde{\mathbb{P}}$-measure, by the mortality term

$$
e^{-\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz}.
$$

However, as we discussed earlier, we assume that that the market is completely neutral with respect to the mortality risk. Therefore, the $\tilde{\mathbb{P}}$-dynamic of the hazard rate process μ equals its dynamic under the real probability measure $\mathbb{P}$, and the post-retirement survival probabilities, ${}_{T_j-T}p_{x^{(T)}}$, are calculated in the same spirit as equation (14).

5 Numerical calculations and sensitivity analysis

As explained in the previous sections, we implement a numerical procedure to estimate the value of the guaranteed annuity option contract based on the valuation formula contained in equation (17) and here repeated for convenience:

$$
\begin{aligned}
& V_{x_0}\left(x^{(t)}, t, T-t\right) \\
= \; & g S_t \tilde{\mathbb{E}}\left[e^{-\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz} \left(\sum_{j=0}^{w-(T+x_0)} {}_{T_j-T}p_{x^{(T)}} P_T\left(T_j\right) - K \right)^{+} \middle| \mathcal{F}_t \right],
\end{aligned}
$$

where $\tilde{\mathbb{E}}$ is the expectation taken under the stock-risk-adjusted probability measure $\tilde{\mathbb{P}}$. In particular, we recall that the pre-retirement mortality factor

$$e^{-\int_0^{T-t}\mu\left(x^{(t)}+z,t+z\right)dz}$$

and the post-retirement survival probabilities

$$_{T_j-T}p_{x^{(T)}}=\mathbb{E}\left[e^{-\int_0^{T_j-T}\mu\left(x^{(T)}+z,T+z\right)dz}\middle|\mathcal{F}_T\right], \tag{18}$$

depend on the dynamic (under the real probability measure $\mathbb{P}$) of the hazard rate process described in equations (6)-(9):

$$\begin{aligned}
\mu\left(x+z,t+z\right) &= \mu\left(x+z,0\right)e^{(\alpha+\beta(x+t))(t+z)+\sigma_h Y_{t+z}}, \\
\mu\left(x+z,0\right) &= a_1+a_2R+e^{b_1+b_2R+b_3\left(2R^2-1\right)}, \\
R &= \frac{(x+z)-70}{50}, \qquad x\geq 50; \\
& \left\{\begin{array}{l} dY_t=-aY_tdt+dX_t \\ Y_0=0. \end{array}\right.
\end{aligned}$$

Moreover, the valuation formula also requires the knowledge of the zero coupon bond prices $P_T\left(T_j\right)$ or, equivalently, the interest rate distribution under the risk-adjusted probability measure $\tilde{\mathbb{P}}$ defined in (16) by

$$\begin{aligned}
\eta_T &= e^{-\int_0^T r_u du}\frac{S_T}{S_0}=e^{-\frac{\sigma_S^2}{2}T+\sigma_S\hat{Z}_T} \\
&= e^{-\rho^2\frac{\sigma_S^2}{2}T-\left(1-\rho^2\right)\frac{\sigma_S^2}{2}T+\sigma_S\rho\hat{W}_T+\sigma_S\sqrt{1-\rho^2}\hat{W}_t'}.
\end{aligned}$$

Since the Girsanov theorem implies that

$$\begin{aligned}
\tilde{W}_t &:= \hat{W}_t-\rho\sigma_S t, \\
\tilde{W}_t' &:= \hat{W}_t'-\sigma_S\sqrt{1-\rho^2}t
\end{aligned}$$

are $\tilde{\mathbb{P}}$-standard Brownian motions, the $\tilde{\mathbb{P}}$-dynamic of the forward rate is then

$$df\left(t,T\right)=\left(\sigma e^{-\lambda(T-t)}\left(\sigma\int_t^T e^{-\lambda(u-t)}du+\rho\sigma_S\right)\right)dt+\sigma e^{-\lambda(T-t)}d\tilde{W}_t,$$

which implies that the short rate is given by

$$r_t=f\left(0,t\right)+\left(1-e^{-\lambda t}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda t}\right)+\frac{\rho\sigma\sigma_S}{\lambda}\right]+\sigma\int_0^t e^{-\lambda(t-v)}d\tilde{W}_v. \tag{19}$$

The corresponding price at time T of a zero coupon bond maturing at T_j, $j=1,2,...,w-(T+x_0)$, is[5]

$$P_T\left(T_j\right)=\frac{P_0\left(T_j\right)}{P_0\left(T\right)}e^{-\frac{1}{2}\gamma^2(T,T_j)\sigma_r^2(T)-\gamma(T,T_j)(r_T-f(0,T))}, \tag{20}$$

[5] For full details about the calculations, we refer the reader to Ballotta and Haberman (2003).

where

$$\gamma\left(T,T_j\right)=\left(\frac{1-e^{-\lambda(T_j-T)}}{\lambda}\right),$$

and

$$\sigma_r^2\left(T\right)=\sigma^2\left(\frac{1-e^{-2\lambda T}}{2\lambda}\right). \tag{21}$$

Given the path-dependent nature of the hazard rate process and the lack of knowledge about the distribution properties of the mortality factor

$$e^{-\int_0^s \mu(x+z,t+z)dz},$$

we adopt Monte Carlo techniques for the computation of the value of the contract.

The financial component of the GAO price, i.e. the prices of the zero coupon bonds, is estimated using the fact that, as shown in equation (19),

$$r_T-f\left(0,T\right)\sim N\left(m_r\left(T\right),\sigma_r^2\left(T\right)\right),$$

with

$$m_r\left(T\right)=\left(1-e^{-\lambda T}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda T}\right)+\frac{\rho\sigma\sigma_S}{\lambda}\right];\qquad \sigma_r^2\left(T\right)=\sigma^2\left(\frac{1-e^{-2\lambda T}}{2\lambda}\right).$$

Therefore, the bond prices $P_T\left(T_j\right)$ are simulated using equation (20) and by generating sample deviates from a $N\left(m_r\left(T\right),\sigma_r^2\left(T\right)\right)$ distribution.

As far as the mortality component of the GAO price is concerned, we observe that the solution to the stochastic differential equation governing the process Y,

$$Y_t=\int_0^t e^{-a(t-s)}dX_s,$$

implies that $Y_t\sim N\left(0,\xi^2\left(t\right)\right)$, with

$$\xi^2\left(t\right)=\frac{1-e^{-2at}}{2a}.$$

Note that, for any $u>t$,

$$\begin{aligned} Y_u &= \int_0^u e^{-a(u-s)}dX_s \\ &= e^{-a(u-t)}Y_t+\int_t^u e^{-a(u-s)}dX_s. \end{aligned}$$

In particular, the stochastic integral

$$\int_t^u e^{-a(u-s)}dX_s$$

is independent of Y_t and follows a Normal distribution with zero mean and variance $\xi^2\left(u-t\right)$.

Thus, for the computation of the pre-retirement mortality factor

$$e^{-\int_0^{T-t}\mu\left(x^{(t)}+z,t+z\right)dz}, \tag{22}$$

we subdivide the observed time period $[0, T-t]$ into n equal subintervals of fixed length $\Delta t = \frac{T-t}{n}$, and we define $\tau_i = i\Delta t$, $i = 0, 1, 2, ..., n$. At each step, we generate the path of the Ornstein-Uhlenbeck process Y as

$$Y_{t+\tau_i} = e^{-a(\Delta t)} Y_{t+\tau_{i-1}} + \xi\left(\Delta t\right) z_{\tau_i}$$

where $\{z_{\tau_i}\}_{i=1,...,n}$ is a sequence of independent random samples from a standardized normal distribution. Consequently, the path of the hazard rate can be generated as

$$\begin{aligned}
\mu\left(x^{(t)}+\tau_i, t+\tau_i\right) &= \mu\left(x^{(t)}+\tau_i, 0\right) e^{\left(\alpha+\beta\left(x^{(t)}+\tau_i\right)\right)(t+\tau_i)+\sigma_h Y_{t+\tau_i}}, \\
\mu\left(x^{(t)}+\tau_i, 0\right) &= a_1 + a_2 R_i + e^{b_1+b_2 R_i + b_3\left(2R_i^2-1\right)} \\
R_i &= \frac{\left(x^{(t)}+\tau_i\right)-70}{50}.
\end{aligned}$$

The integral function in equation (22) is then approximated using the trapezoidal rule, so that

$$\begin{aligned}
&\int_0^{T-t} \mu\left(x^{(t)}+z, t+z\right) dz \\
= \ &\frac{\Delta t}{2}\left[\mu\left(x^{(t)}+\tau_0, t+\tau_0\right) + \mu\left(x^{(t)}+\tau_n, t+\tau_n\right) + 2\sum_{k=1}^{n-1}\mu\left(x^{(t)}+\tau_k, t+\tau_k\right)\right].
\end{aligned}$$

Since equation (18) does not lead to a closed form expression, principally because the sum of lognormal random variables is not lognormal, the post-retirement survival probabilities ${}_{T_j-T}p_{x^{(T)}}$, for $\forall j = 1, 2, ..., w-(T+x_0)$, are calculated as well via Monte Carlo technique. Hence, the post-retirement hazard rate

$$e^{-\int_0^{T_j-T}\mu\left(x^{(T)}+z,T+z\right)dz}$$

is generated by the same procedure described above, using at each run the values of Y_T (which forms the information set available at T, $\mathcal{F}_T$) resulting from the calculation of the pre-retirement process. Regarding this last specification of the approximation procedure, we have carried out the simulation of the post-retirement survival probabilities also in the case in which the past history of the process Y is ignored, and Y_T either is approximated by its mean value (i.e. $Y_T = 0$), or is generated from its distribution (i.e. $Y_T \sim N\left(0, \xi\left(T\right)\right)$). The numerical results show that the survival probabilities obtained in both cases are the same as the ones generated by "inheriting" the Y_T values from the pre-retirement process, although both of these procedures are not theoretically correct.

The Monte Carlo experiment for the computation of both the value of the guaranteed annuity option and the post-retirement survival probabilities is carried out by generating 10,000 paths, with each path comprising 1 observation per month of the hazard rate over

Parameter Set for numerical analysis
Design parameters:
$g = 11.1\%$; $x = 50$; $T + x = N = 65$
Financial model
$S_0 = 100$; $\sigma_S = 20\%$ p.a.; $\rho = -0.5$; $f_0 = 4\%$ p.a.; $\sigma = 0.01$; $\lambda = 0.15$
Mortality model
$\alpha = -0.028$; $\beta = 0.0002$; $\sigma_h = 10\%$ p.a.; $a = 0.5$
$a_1 = \frac{0.03}{100}$; $a_2 = 0$; $b_1 = -5.265363$; $b_2 = 6.683129$; $b_3 = -0.9$

Table 2: Set of parameters used as benchmark for the comparative statics analysis. Parameters are subdivided into 3 blocks. The first group contains the parameters that characterize the individual policy; the second group contains the parameters representing the financial market components; the last group contains the parameters related to the mortality model.

each year; the antithetic variable technique is used to reduce the variance of the obtained estimates. The standard error of the estimate, expressed as percentage of the contract value, is 0.03%

In the following sections, we use the results developed in section 4 (namely, equation (17) for the GAO price), and the numerical procedure previously introduced to carry out a full sensitivity analysis for the value of the guaranteed annuity option contract. For this practical example, we incorporate a common design feature and assume that the annuity has a 5-year guarantee period, so that the first five annual payments of the annuity scheme would be definitely payable, providing that the policyholder survives to retirement age.

We subdivide the analysis into two sections. The first one relates to the study of the behavior of the GAO when the parameters "imported" into the pricing formulae from the financial market are changed one at a time, ceteris paribus. The second set of results, instead, describes the behavior of the GAO when the mortality model parameters are changed, again individually ceteris paribus.

Unless otherwise stated, the benchmark set of parameters is as given in Table 2 [6]. The value of the guaranteed annuity contract for the set of parameters as specified in Table 2, is contained in Table 3. In the same table, we also offer a comparison with the corresponding value of the GAO resulting from the valuation formula proposed by Ballotta and Haberman (2003), based on the UK mortality tables. The stochastic model for mortality trend introduced in this paper causes an increase in the contract value of about 26%.

5.1 GAO and the "financial" parameters

In this section, we illustrate the main comparative statics results for the sensitivity of the GAO to the financial parameters. In particular, we look at the contract profile versus

[6]The values of the financial parameters are very close to the ones used by Briys and de Varenne (1997), and Miltersen and Persson (1999) for the implementation of very similar market models. As far as the value of the guaranteed conversion rate is concerned, according to Bolton et al. (1997), $g = 11.1\%$ was the common guaranteed rate used in the UK by life insurance companies.

		GAO
Mortality tables	PMA92-C20	52.3029
	PMA80-C10	34.8129
	PA90	20.1225
Stochastic mortality model		65.8228

Table 3: The value of the guaranteed annuity option contract. The value resulting from modelling mortality trends as described in section 3 of this paper, is compared with the value obtained using UK mortality tables. The pricing formula used in this case is the one proposed by Ballotta and Haberman (2003).

the volatility of the equity portfolio backing the policy and its correlation to interest rates, the volatility coefficients of the forward rate, and the market interest rate. The results are shown in Figure 2. In particular, in the top panel on the left, we represent the behavior of the guaranteed annuity option as a function of the equity portfolio volatility, for different values of the correlation coefficient ρ. As shown by the chart, the value of the GAO presents a different pattern depending on whether ρ is positive or negative. When ρ is negative, the policyholder might expect the equity market to move in the opposite direction from the interest rate. In this case, the annuity guaranteed by the pension plan becomes more and more attractive as the volatility of the reference portfolio increases. In fact, if market rates of interest drop, the policy locks in a competitive annuity amount at a competitive rate. In the case of a rise in the level of the market rates, instead, the GAO might simply expire out-of-the-money. On the other hand, when ρ is positive and σ_S increases, the annuity offered in the open market is more attractive, which reduces the value of the GAO. The same argument justifies the decreasing profile of the GAO as a function of the correlation coefficient only (for fixed σ_S), and this may be deduced from the plots in the same panel.

A similar behaviour can be observed in the panels on the top-right corner and bottom-left corner of Figure 2, where the changes in value of the guaranteed annuity option arising from changes in the parameters governing the volatility structure of the forward rate, i.e. the speed or adjustment, λ, and the diffusion coefficient, σ, are summarized.

The sensitivity of the GAO to the initial (flat) redemption yield, f_0, used to calculate the initial bond prices $P_0(T)$ and $P_0(T+t), t = 0, 1, ..., w - (T+x)$, is shown in the bottom-right panel of Figure 2. In particular, we observe a decreasing pattern due to the fact that higher current interest rates make the guaranteed annuity payments less attractive than the current rates available in the market.

The trends shown in Figure 2 are consistent with the ones described by Ballotta and Haberman (2003), to which the reader is referred for fuller details.

5.2 GAO and the "mortality" parameters

In this section, we analyze the impact on GAO values of changes in mortality trends, when survival probabilities are computed using the model described in section 3 and summarized by equations (6) $-$ (9). Figure 3 shows the impact on GAO values of the

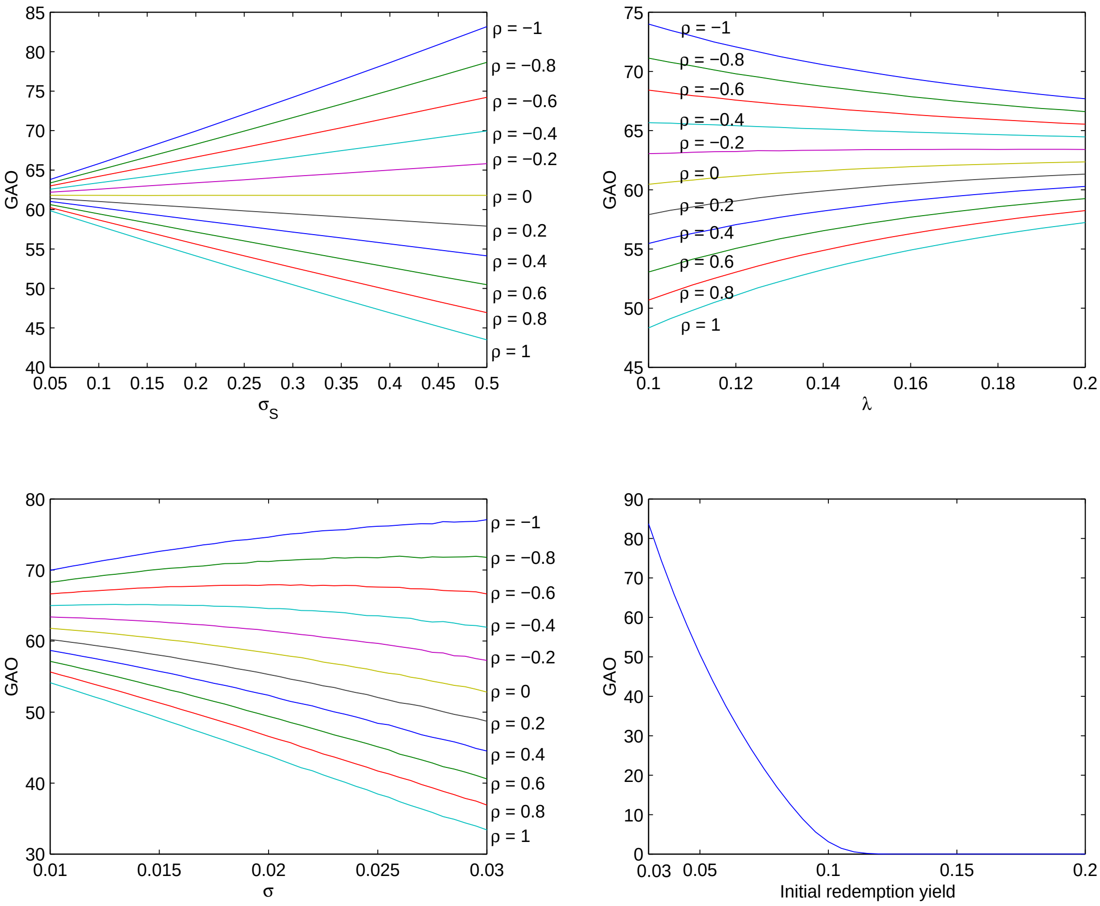

Figure 2: Sensitivity of the GAO to the key financial parameters σ_S, the portfolio volatility, λ and σ, the interest rate volatility governing parameters, and the initial redemption yield.

parameters related to the "deterministic" part of the model, i.e. the rate of change in the hazard rate over time, α, and its offset term β. As the plot shows, the value of the GAO contracts is a decreasing function of the parameter α. In fact, the more negative is α, the stronger is the downward trend in mortality rates. The final effect is then an improvement in survival probabilities and, as consequence, the GAO value contract rises in value. There is a further effect in that, as α becomes more negative, the number of annuity payments may increase because of the increase in survival probabilities - this effect also leads to an increase in value of the GAO. On the other hand, the value of the guaranteed annuity option decreases as β increases. This is consistent with the nature of this parameter, which is to mitigate with increasing age the rate of decline in mortality rates (see equation 5).

In Figure 4, we represent the behavior of the GAO value as a function of the parameters related to the stochastic component of the model for the hazard rate i.e. the amplitude σ_h of the noise process Y around its mean, and the speed of convergence, a, of Y to its long-run mean. In particular, we observe that the GAO value decreases as σ_h increases. This is due to the fact that the GAO value and the underlying probabil-

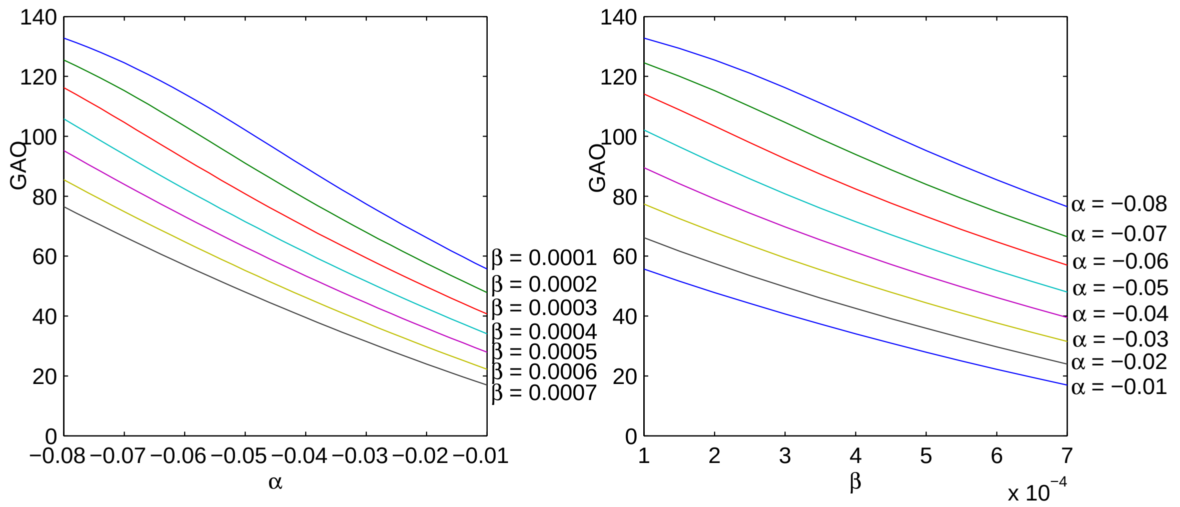

Figure 3: Effects on the guaranteed annuity option produced by the rate of change in mortality trends

ities of survival depend on the mortality path in a non-linear way (through equations (17) , (18) and (6)). As σ_h increases, the mortality trend becomes more uncertain and the chance of surviving for another year deteriorates. Thus, as the first two plots in Figure 5 show, an increase in the volatility of mortality implies smaller pre and post-retirement survival probabilities. Therefore, the value of the "coupon-bond" forming the GAO payoff (see equation 17) reduces; but we would also expect the "mortality discount factor" to reduce on average. Consequently the contract value reduces. However, the effect of σ_h becomes almost negligible as a increases. In this situation, in fact, the mean-reversion effect becomes stronger and the process Y converges to its zero long-run mean more quickly, reducing the uncertainty in the evolution of the hazard rate, as indicated by the bottom panels of Figure 5. This is shown in Figure 4 as well, by observing that the value of the contract is an increasing function of the parameter a.

Figure 6 shows the behaviour of the contract for different ages of the policyholder at inception of the contract, which corresponds to different times to maturity. In panel (a), we represent the values of the contract for different ages at entry, using the benchmark set of parameters presented in Table 2; as we can observe from the plot, the value of the GAO decreases as the time to maturity becomes shorter. This is in line with standard financial option theory: as the option contract approaches expiration, there is less and less uncertainty about the final payoff and therefore the premium for the contract decreases. Moreover, a detailed analysis of the post-retirement survival probabilities generated by the stochastic model introduced in this paper, shows that the probability to survive from age 65 onwards (calculated at age 65) is higher for an individual aged 20 at inception than for an individual who entered the contract at a later age. This implies that the payoff is higher for the former case, and so the value of the GAO is higher as well. However, as the remaining panels of Figure 6 show, the mortality component of the option contract under discussion can have a marked effect on this trend, to the extent that, for certain combinations of parameters, the trend is inverted. In particular,we observe an increasing

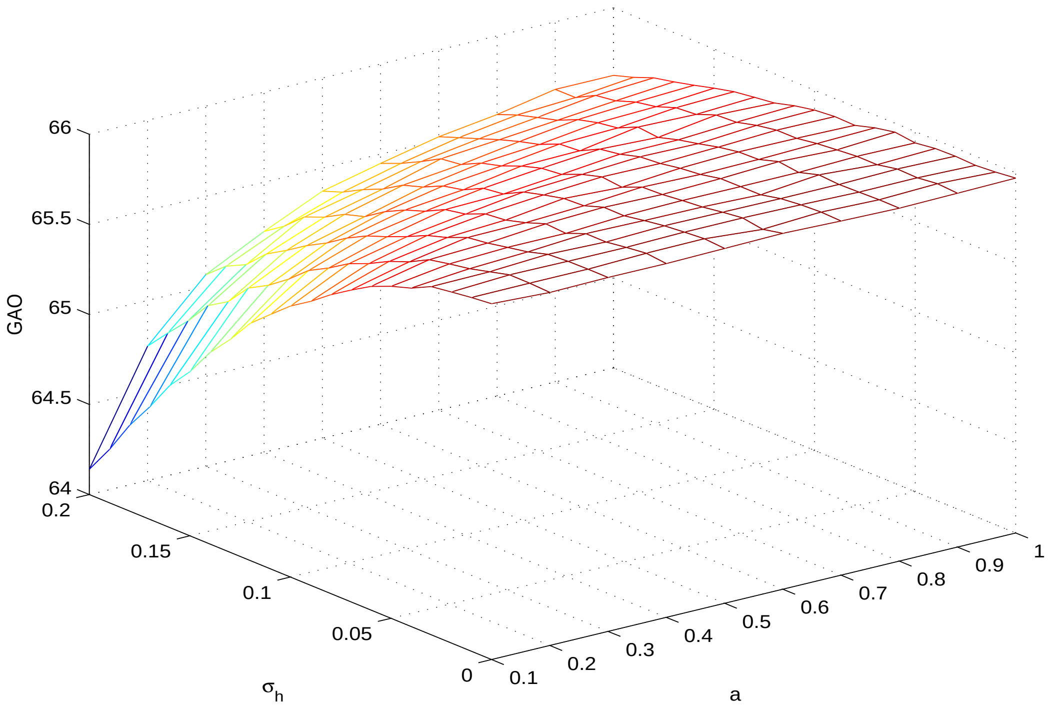

Figure 4: Sensitivity of the GAO to the parameters governing the stochastic component of the mortality model.

Model	GAO	$\mathbb{E}(\tilde{\alpha})$	$Var(\tilde{\alpha})$
Symmetric distribution case			
1.1	63.9851	-0.03	0.00024
1.2	65.1991	-0.03	0.00016
1.3	66.5339	-0.03	0.00008
1.4	63.5899	-0.03	0.00027
Asymmetric distribution case			
2.1	65.3419	-0.032	0.000276
2.2	71.4352	-0.038	0.000256
2.3	79.3129	-0.044	0.000164
Uniform distribution case			
$\tilde{\alpha} \sim U(-0.05, -0.01)$	65.6887	-0.03	0.000133
The benchmark			
$\tilde{\alpha} = -0.03$	67.9765		
$\tilde{\alpha} = -0.032$	70.1137		
$\tilde{\alpha} = -0.038$	76.8307		
$\tilde{\alpha} = -0.044$	83.7989		

Table 4: Sensitivity of the values of the GAO to the parameter error in the logarithmic rate of decline of hazard rates over time (α)

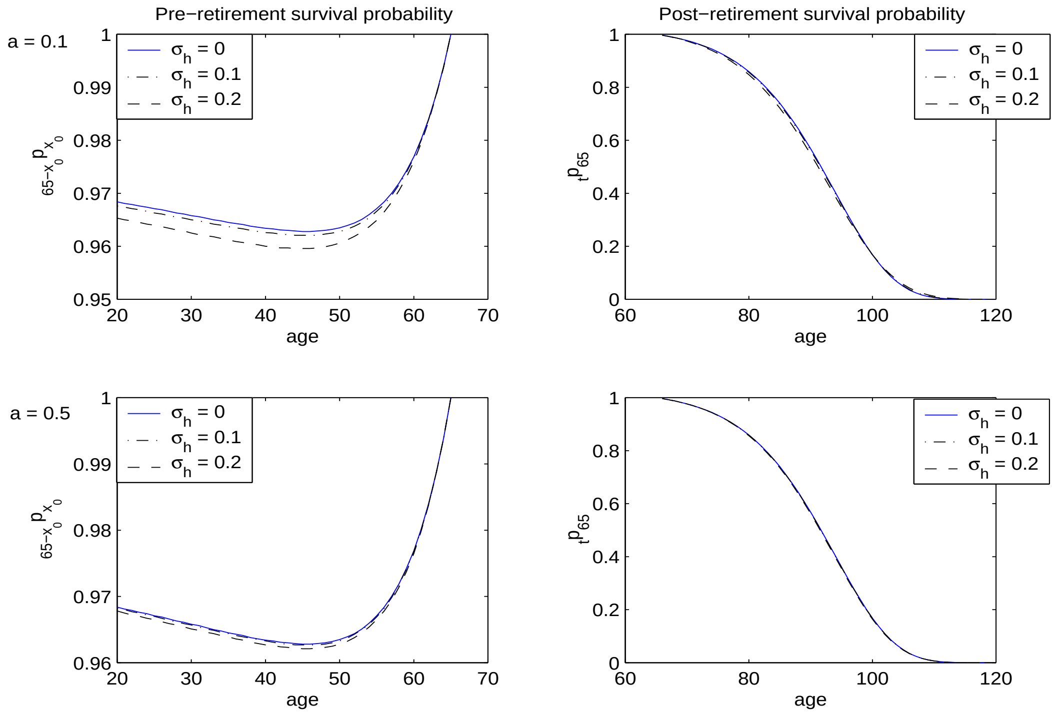

Figure 5: Pre-retirement and post-retirement survival probabilities for different values of the mortality process volatility, σ_h.

pattern for values of the parameter α less negative than -0.01 in panel (c), and less negative than -0.015 in panel (d). A detailed analysis of the post-retirement survival probabilities shows that for these particular combinations of the parameters α and β, an individual who bought the GAO at the age of 20 has a lower probability to survive from age 65 onwards than an individual who was older at inception. This is due to the features of the deterministic component of the model for the hazard rate adopted in this paper: it is clear from equation (5) that, as noted by Sithole et al. (2000), the parameters α and β cannot be chosen independently of one another if we require that the reduction factor is less than 1 for all choices of age over a given time horizon.

5.3 Longevity risk

In this section, we consider the enhanced model, equation (10), where we relax the assumption that the logarithmic rate of decline in mortality rates, α, is deterministic (and constant in particular). α is the key mortality parameter as far as trends are concerned and has been difficult to forecast in practice, as many commentators note (CMI Bureau, 1998, Sithole et al., 2000, Olivieri, 2001). Therefore, we assume that it is random and use the point distribution of Table 1 to describe its properties in order to illustrate the effect on the value of the GAO of allowing for this feature. It should be noted that we make the underlying assumption that $\tilde{\alpha}$ is independent of any other sources of randomness present in the model.

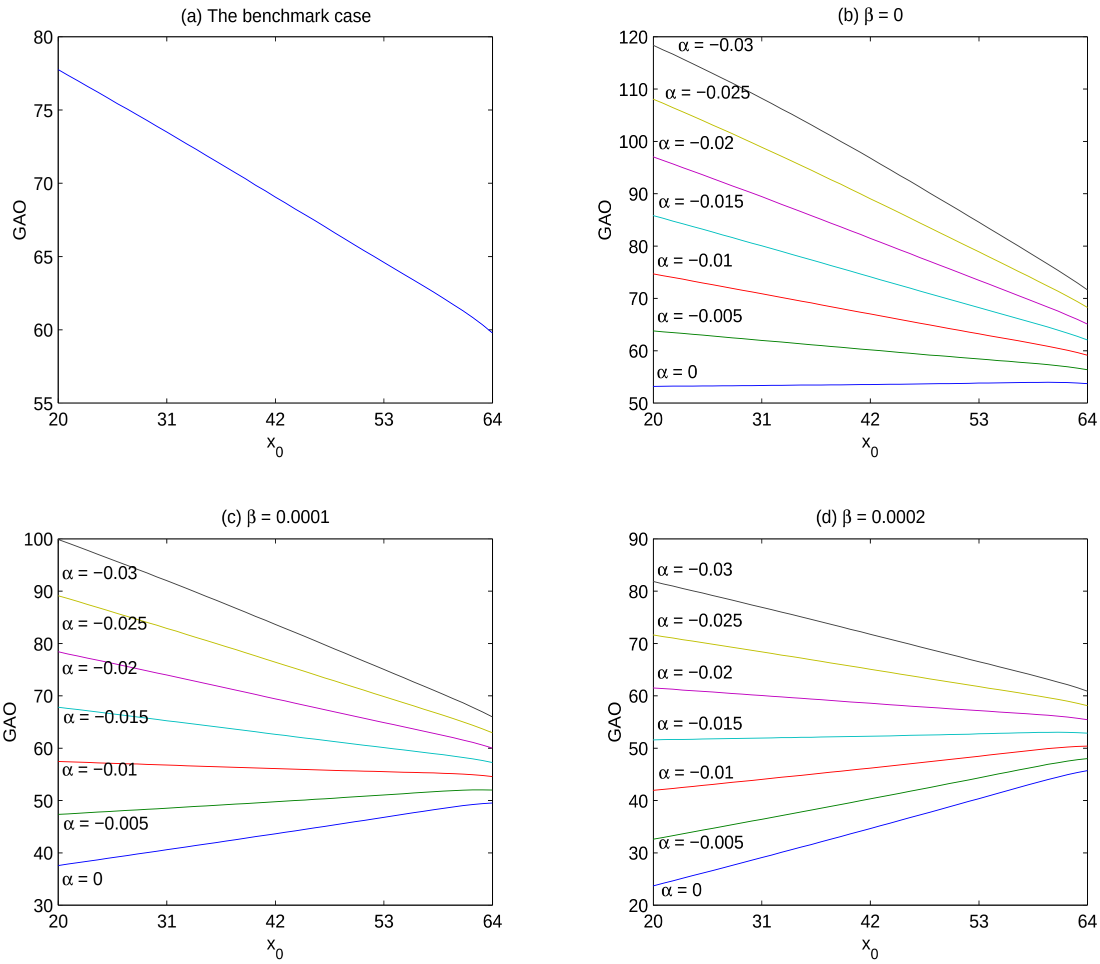

Figure 6: Values of the guaranteed annuity option for different entry ages. Panel (a): the benchmark case ($\alpha = -0.028$, $\beta = 0.0002$). Panel (b)-(c)-(d): effects of α and β on the time evolution of the GAO.

Table 4 thus shows the effect on the value of the GAO of allowing for parameter error in α. We firstly consider four symmetric distributions for $\tilde{\alpha}$ and the uniform distribution case, each with mean -0.03 (which is comparable to the value used as benchmark: see Table 2). Relative to the case with constant α, the results indicate that the presence of parameter error leads to a reduction in the GAO value and that increased uncertainty (measured by $Var(\tilde{\alpha})$) leads to larger reductions. Given the one-sided option-like nature of C_T and the non-linear dependence of the GAO value on $\tilde{\alpha}$, the presence of symmetric fluctuations in $\tilde{\alpha}$ would be expected to lead to this result, which is comparable to the sensitivities with respect to σ_h (shown in Figure 4 and discussed in section 5.2). The three asymmetric cases indicate the effect of underestimating the size of α on the value of the GAO - the comparison with the deterministic benchmark values at the foot of Table 4 show again that the presence of uncertainty leads to a marginal mediation in the value of the GAO.

5.4 Historical analysis

In this section, we address the issue of the historical problems experienced by life insurance companies in the UK as a result of issuing pension plans with GAO features attached. In particular, we use the valuation formulae previously obtained to understand how much of the GAO solvency problem is due to dramatic changes in interest rates and how much to improvements in mortality trends. The analysis carried out is an extension of a similar study performed by Ballotta and Haberman (2003). We start from a hypothetical contract issued in 1970 to a policyholder aged 20, and we follow the evolution of the value of this contract over time up to the present day. In particular, following Ballotta and Haberman (2003), we use the annual average of retail banks' base rates (Bank of England, February 2002) for the initial term structure of interest rates. In implementing the historical analysis, we make the assumption for convenience that the stochastic mortality model (introduced in section 3) applies to younger ages at entry than is justifiable by the data - this means from age 20 onwards. Results are presented in Figure 7. Related to this hypothetical contract, we also follow the evolution over time of the implied guaranteed rate, i.e. the rate of interest such that the expected present value of the guaranteed annuity equals the principal amount, or

$$gS_T a_{x+T} = S_T.$$

Results are presented in Figure 8.

Figure 8 shows that between 1970 and 1973, the implied guaranteed rate of our hypothetical contract, for the chosen set of parameters, was about 10.3% while the market interest rates were oscillating between 5% and 7%. Hence, the guaranteed annuity option was "in the money". After 1973, the market interest rate increased from a level of 9.50% in 1973 up to a maximum of 15.50% in 1980, without recording values below 9.50% till 1992. Over the same period, the implied guaranteed rate oscillated between 10.25% in 1974 and 9.90% in 1992, without being very competitive with respect to the "open market option". Thus the guaranteed annuity option contract was far "out of the money" for most of the time, recording a minimum value of 0.0001 (i.e. 0.01% of the initial premium) over the period 1979-1981. However, in 1993, market rates of interest dramatically decreased from 12% to 6%, which brought the GAO contract back "in the money". In the following years, market interest rates have been reduced further to a value of 4% in 2003, and the consequence has been the continuous rise in the value of the GAO. These results are confirmed by the features of Figure 7.

Both Figure 7 and 8 show the historical evolution of the contract for the case in which mortality is modeled as described in section 3. The resulting trends are compared to those generated by models in which the survival probabilities are computed using UK mortality tables. Two such alternative models are analyzed. As in Ballotta and Haberman (2003), we consider the case in which post-retirement survival probabilities are calculated using the PA90 mortality table only, as was the practice during the 1970s. The results are compared to those obtained for the case in which mortality improvements are incorporated by "switching" to more up-to-date mortality tables as these become available. Hence, when the valuation is performed during the period 1991-1999, the PMA80-C10 table is used, while from the year 2000 onward, the PMA92-C20 is in use.

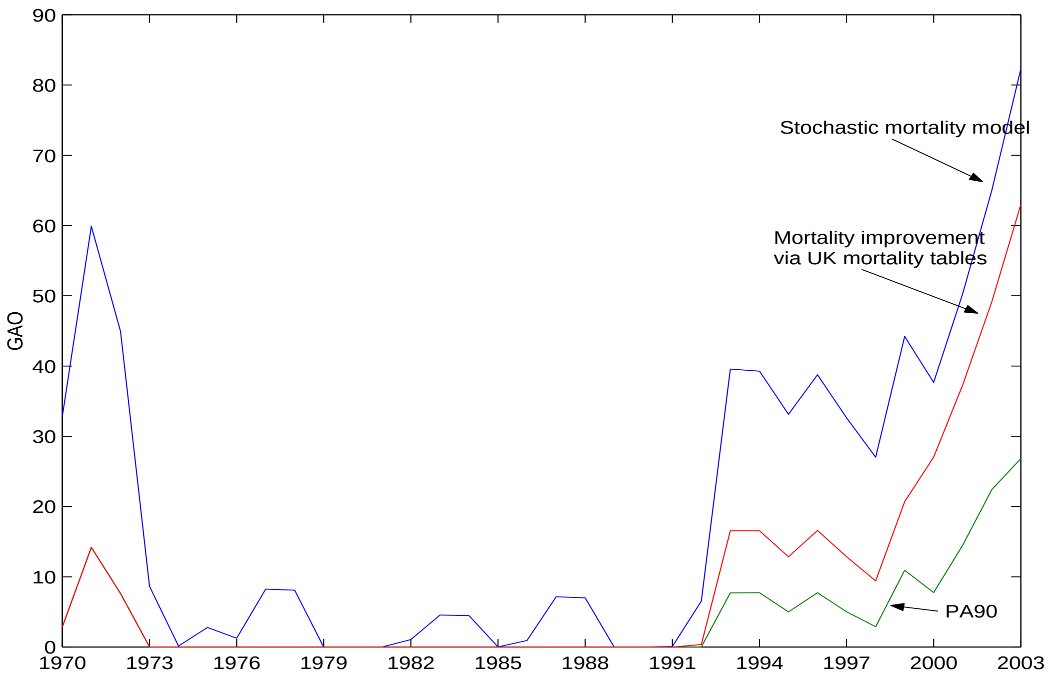

Figure 7: "Historical evolution" of a guaranteed annuity option contract issued in 1970 to a policyholder aged 20 for different specifications of the mortality model.

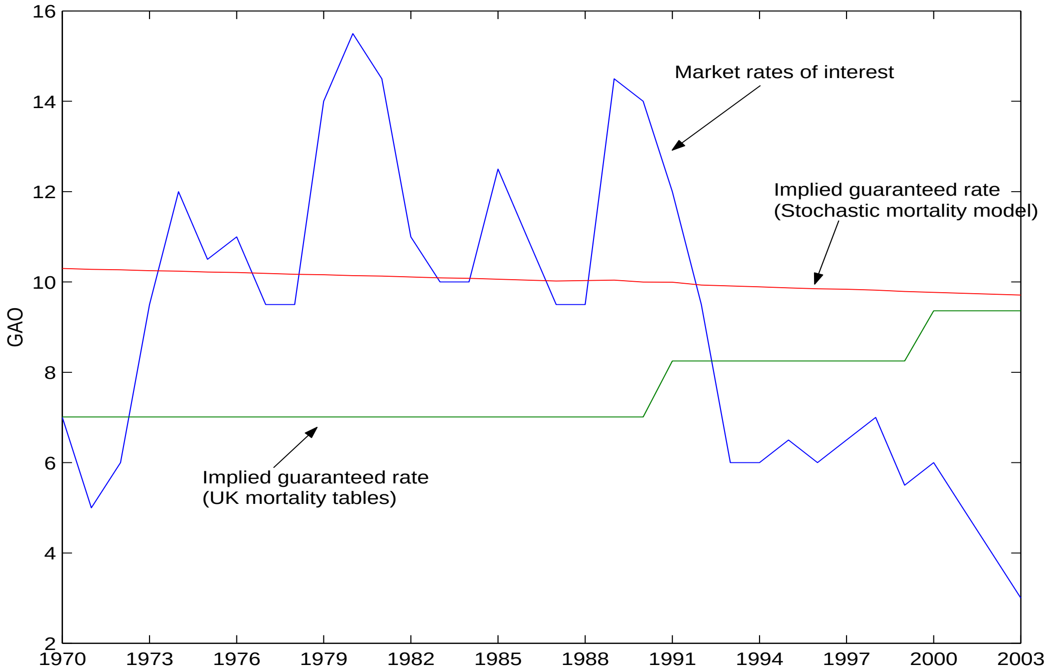

Figure 8: "Historical evolution" of the implied guaranteed rate for different specifications of the mortality model.

The model based on survival probabilities as given by the PA90 table only provides a useful benchmark to separate the effect of changing interest rates on the GAO values from the effect generated by improvements in mortality trends, as in this case no improvements are captured since only one fixed mortality table is used for valuation throughout the entire lifetime of the contract. As already observed by Ballotta and Haberman (2003), neglecting improvements in mortality rates leads to an underpricing of the guaranteed annuity option of about 55% in year 2002 with respect to the values obtained using the model based on the three mortality tables. The corresponding level of underpricing is about 65% with respect to the stochastic mortality model used.

6 Conclusions

The aim of this work has been to extend the valuation model for guaranteed annuity options proposed by Ballotta and Haberman (2003) in order to allow for stochastic uncertainty in mortality trends. The behavior of this contract with respect to changes in market conditions and mortality risk has been analyzed with numerical examples and the sensitivity analysis presented.

We have seen that the inclusion of stochastic mortality, through fluctuations around a trend (the parameter σ_h), or longevity risk, through simple distributional assumptions for the main time trend parameter $\tilde{\alpha}$, leads to a reduction in the expected value of the GAO. However, we would advise some caution in the application of this result. Our valuation formula for V_{x_0}, equation (17), relates to an expected present value obtained by the methodology of risk neutral valuation. It is possible that an insurer would also be interested, for example for reserving purposes, in the full distribution of the random present value and, in particular, in upper tail values. These percentiles are likely to depend more directly on the dimensions of stochastic mortality that we have introduced in this paper.

In the light of the analysis presented here, we identify areas where there is scope for further work. These concern limitations of the modelling framework developed so far.

One problem left open is the definition of an efficient and cost-effective risk management strategy for the guaranteed annuity option. The pricing formula expressed in equation (17) shows that these kind of products are affected by financial risk, to the extent that the value of the reference portfolio and the level of interest rates change over time, and by mortality risk. The hedging of financial risk suffers from some practical limitations, due to the very long maturities characterizing these contracts (between 45 and 60-70 years for a policyholder aged 20 at inception and who will exercise the guaranteed annuity option right at retirement). Possible approaches based on swaptions have been proposed by Bolton et al. (1997), Pelsser (2003) and Wilkie et al. (2003). Swaptions seem to be efficient when dealing with the interest rate risk incorporated in guaranteed annuity options. However, as Boyle and Hardy (2003) point out, the swaption solution is unable to deal with the equity price risk, in the sense that the number of swaptions in the replicating portfolio has to be adjusted in line with movements in the equity fund. Also, in the models mentioned above, mortality risk is incorporated by using a deterministic model for the hazard rate (i.e. the survival probabilities are calculated by means of the relevant mortality tables). This study, though, shows that the mispricing due to neglecting mortality improvements is noticeable over the long-term horizon covered by the GAO. This implies that a suitable hedging strategy for this risk as well needs to be in place. Boyle and Hardy (2003) observe that in order to hedge mortality, good estimates of the distribution of future mortality rates are required. Dahl (2004) shows that this might not be the case. He, in fact, studies possible ways of transferring mortality risk to other parties, for example by introducing mortality-linked insurance contracts, in which premiums and benefits are adapted from period to period to the development of the mortality intensity, in a manner similar to certain types of unit-linked life insurance contracts.

An important open issue related to the implementation of suitable hedging strategies, is the selection of a risk neutral probability measure to be used for the calculation of the survival probabilities, and mortality discount factors in the reserving/pricing process. As observed in section 4, the market of annuity contracts is incomplete due to the lack of fully tradeable mortality derivatives. In this work, we assume that the market is indifferent to the mortality risk and consequently calculations have been carried out under the real probability measure. Milevsky and Promislow (2001) provide a justification for this approach by arguing that both the risk that any particular policyholder is healthier than average, and the risk of overestimating the population's force of mortality can be diversified away and hedged by selling more life insurance policies via a sort of offsetting process. Biffis and Millossovich (2004) propose an alternative way of handling this problem based on the statistical measure approach (Eberlein et al., 1998). Thus, Biffis and Millossovich do not distinguish between a financial and a mortality risk neutral measure, but suggest instead the identification of the pricing risk neutral measure by calibrating the value of the insurance contract using the corresponding exchange prices. Although this approach might be closer to the spirit of the fair valuation accounting principle promoted by IASB, it has to be noted that it might prove unsatisfactory due to the lack of the required market prices. Indeed, this is actually the reason of the market incompleteness.

A limitation of the model presented in this paper is the assumption of normally distributed stock returns. In fact, empirical evidence suggests that implicit stock return distributions are skewed with a higher kurtosis than is allowable with a Black-Scholes type normal distribution (see, for example, Black, 1975, Ball and Tourus, 1985, Madan and Chang, 1996, Bakshi, Cao and Chen, 1997). Also, the valuation framework relies on a single-factor model for interest rates. There is currently an ongoing debate concerning a suitable choice of interest rate models. It seems, however, that single-factor models, although they provide tractable solutions, are insufficient to describe the risk entailed in interest rate derivatives, and for this reason they are unlikely to provide sufficient accuracy in terms of the hedging strategy.

In sections 3 and 5, we introduce and use a simple, parsimonious model for the trajectory of mortality rates over time, which is based on extensive data analyses (see Sithole et al., 2000). It is possible that this structure, in terms of a log-linear representation for the reduction factor, may be too restrictive. Hence, the effect of using other projection models could usefully be explored - for example, those based on a Lee-Carter framework (see, for example, Renshaw and Haberman, 2003).

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