Pricing and capital requirements for with profit contracts: modelling considerations

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Abstract

The aim of this paper is to provide an assessment of alternative frameworks for the fair valuation of life insurance contracts with a predominant financial component, in terms of impact on the market consistent price of the contracts, the embedded options, and the capital requirements for the insurer. In particular, we model the dynamics of the log-returns of the reference fund using the so-called Merton process (Merton, 1976), which is given by the sum of an arithmetic Brownian motion and a compound Poisson process, and the Variance Gamma (VG) process introduced by Madan and Seneta (1990), and further refined by Madan and Milne (1991) and Madan et al. (1998). We conclude that, although the choice of the market model does not affect significantly the market consistent price of the overall benefit due at maturity, the consequences of a model misspecification on the capital requirements are noticeable.

Keywords: fair valuation, incomplete markets, Lévy processes, Monte Carlo methods, participating contracts, solvency requirements.

1 Introduction

Our problem is motivated by the recent move towards market consistent valuation of insurance companies' assets and liabilities for accounting and solvency purposes. Although asset prices can be observed directly in the financial market, in general insurance liabilities are not fully traded, which implies the lack of proper market prices. Consequently, according to the regulators' directives, insurance companies need to develop suitable (internal) models which incorporate both market risk and insurance risk, and are market consistent, i.e. are based on the up-to-date information available at the time of valuation. These models will be used to generate market consistent distributions for the future cash flows originated by the relevant liabilities, from which a proxy for the market price can be extracted. In terms of how this is implemented in practice, we note that two approaches are currently being debated (see, for example, FSA, 2006): on the one hand, the market value of the liability can be calculated on the basis of the “fair value principle”, which, using the terminology of contingent claim pricing theory, is equivalent to risk neutral valuation. This approach should be adopted when hedges are readily available, like in the case of financial risks. In the cases in which risks are not hedgeable (for example, in the case of some insurance risk), the market value can be calculated as the sum of the expected present value of the liability itself (the so-called best estimate), and an arbitrary but quantified risk margin. Regulators agree on the use of the risk free rate of interest as discount factor to reflect the time value of money (CEIOPS, 2006).

In terms of solvency, instead, the market values of assets and liabilities need to be used to calculate the target capital or Solvency Capital Requirement (SCR); thus, the SCR should reflect the amount of capital required to meet all obligations over a specified time horizon to a defined confidence level. Hence, the calculation of the target capital should be based on suitable risk measures, such as VaR and TVaR, over a 1-year time horizon (CEIOPS, 2006).

Given the regulatory framework described above, one of the key factors the insurance companies need to deal with carefully is the development of a suitable valuation model incorporating both the market risk and the insurance risk. The features and the complexity of this model will depend on the nature of the liability to be priced; for example, very common policy types in the insurers' portfolio of products are the so-called participating contracts with minimum guarantee, which are essentially path-dependent contingent claims and, consequently, particularly sensitive to the underlying dynamics of the asset returns.

In the light of the discussion above, the aim of this paper is to analyze the impact on the market consistent price and the target capital of financially sound models for the market risk; to this purpose, we consider the example of a participating contract with minimum guarantee. In recent years, a series of studies have applied classical contingent claim theory to different types of participating contracts, building on the pioneering work of Brennan and Schwartz (1976) on unit-linked policies; thus, amongst some of the most recent works, we would cite Bacinello (2001, 2003), Ballotta et al. (2006.a,.b), and Grosen and Jørgensen (2000, 2002). All these contributions use either a standard Black-Scholes (1973) framework or its extensions to incorporate stochastic interest rates; in any case, the main assumption is that market returns (and therefore the returns on the asset fund backing the insurance policy) move continuously because driven by a diffusion process. Following this observation, Ballotta (2005), and Kassberger et al. (2008) extend the pricing framework to the case of a market specification based on different Lévy processes in order to allow for discontinuities in the returns, and therefore a more realistic description of the market.

In this respect, this contribution aims at extending these more recent works in two directions. Firstly, we want to assess the relevance of the (financial) model error by calculating the impact on the contract market consistent value of neglecting or not correctly capturing market shocks. Hence, we compare the performance of three different assumptions for the dynamic of the log-returns of the reference portfolio backing the insurance policy; specifically, we use the traditional Brownian motion, which provides the “standard” model, and two Lévy processes which allow us to depart from the assumption of normal distributed log-returns, and incorporate market shocks. The first alternative is the so-called Merton process (Merton, 1976), given by the sum of an arithmetic Brownian motion and a compound Poisson process; the second alternative is the Variance Gamma (VG) process introduced by

Madan and Seneta (1990) and further refined by Madan and Milne (1991) and Madan et al. (1998). Secondly, we assess the mispricing generated by the above mentioned models not only with respect to the arbitrage-free value of the insurance policy, but also in terms of the target capital. We note here that in the case of life insurance contracts, the time horizon involved is always quite long (20 years on average); over such a long period of time, one would expect the inclusion of jumps not to have a significant impact. In fact, the skewness of a Lévy process, like the ones considered in this paper, decreases with a factor $\sqrt{t}$ (where t is the length of the holding period), and the kurtosis decreases with a factor t; these features could imply that the Brownian motion provides an accurate representation of the stock price dynamic over longer time scales, as if the jumps tended to be "averaged out". However, the numerical experiments carried out show that, although the choice of the driving process does not affect significantly the market consistent value of the contract, the impact of the model misspecification becomes relevant when the target capital is involved. This is due to the higher flexibility of the Lévy framework in capturing the excess of kurtosis in the distribution of the reference portfolio; the feature of leptokurtosis, in fact, affects the value of the embedded options and therefore the capital required to maintain the solvability of these contracts.

The paper is organized as follows. In the next section, we present the features of the insurance contract considered for this analysis, and we also introduce the framework for the market consistent valuation and the calculation of the target capital. We then provide in section 3 the market setup and the resulting market consistent price. In section 4, we describe and test a number of numerical algorithms available to perform the required computations, the results of which are discussed in section 5. The last section presents our concluding remarks on few issues related to the pricing procedure and the model setup.

2 The participating contract: fair valuation and capital requirements

In order to assess the impact of the choice for the market model on the balance sheet of the insurance company and the corresponding capital requirements, we make use of an example based on a participating contract with minimum guarantee. More specifically, for ease of exposition, we adopt the same contract considered in Ballotta (2005); however, we consider the full specification of the policy, allowing for both leverage and terminal bonus rate (like in Ballotta et al., 2006.b). This policy is representative of a typical UK accumulating (unitized) with profit contract; however, the analysis proposed in this paper can be easily extended to any other type of participating contract, like the equity-linked policies which are so common in the North-American countries. Finally, since our focus is on the importance of the market model, in this analysis we ignore lapses and mortality.

The features of the participating contract design under analysis will be described in the next section. Based on these features, we proceed to identify the options embedded in the insurance policy, for which we develop a general framework for the calculation of the fair value, and a possible approach for the calculation of the target capital.

2.1 Contract design

The policy is initiated at time $t = 0$ by the payment of a single premium, P_0, from the policyholder to the insurance company. The premium is invested in the company's assets, A, together with the contribution from the shareholders, E_0; hence, $P_0 = \vartheta A_0$ and $E_0 = (1-\vartheta) A_0$, where $\vartheta \in (0,1]$ represents the policyholder contribution or leverage coefficient, and A_0 is the value of the insurer's assets at time $t = 0$. The contract entitles the policyholder to receive at maturity, T, an overall benefit given by the guaranteed component, $P(T)$, which includes a minimum guarantee and a scheme for the distribution of the annual returns generated by the reference fund A, and a discretionary component representing the terminal bonus

$$\gamma R(T) = \gamma \left(\vartheta A(T) - P(T)\right)^{+}, \tag{1}$$

where $\gamma \in [0,1]$ is the terminal bonus rate. Hence, the terminal bonus redistributes part of the final surplus generated by the policyholder share in the insurance company.

As to the accumulation scheme governing the guaranteed benefit, $P(T)$, we adopt the smoothed asset share scheme, so that every year after inception the guaranteed benefit is calculated as a weighted average of the unsmoothed value of the benefit at time t, and the level of the smoothed benefit at time $t-1$, i.e.

$$\begin{aligned} P(t) &= \alpha P^{1}(t) + (1-\alpha) P(t-1), \qquad \alpha \in (0,1), \quad t > 0, \\ P(0) &= P_{0,} \end{aligned}$$

where $P^{1}(t)$ is the unsmoothed asset share defined by

$$\begin{aligned} P^{1}(0) &= P_0, \\ P^{1}(t) &= P^{1}(t-1)\left(1 + r_P(t)\right), \\ r_P(t) &= \max\left\{ r_G, \beta \frac{A(t) - A(t-1)}{A(t-1)} \right\}, \end{aligned} \tag{2}$$

and $r_G \in \mathbb{R}^{++}$ and $\beta \in (0,1)$ are the guaranteed rate and the participation rate respectively. Therefore, at maturity, T, the value of the guaranteed benefit is

$$P(T) = P_0 \left[\alpha \sum_{k=0}^{T-1} (1-\alpha)^{k} \prod_{t=1}^{T-k} \left(1 + r_P(t)\right) + (1-\alpha)^{T} \right]. \tag{3}$$

Hence, the benefit follows the fluctuations of the financial market, although the presence of the guarantee imposes a floor on the downside trend of the policyholders' cashflows.

If, at the claim date, the insurance company is not capable of paying the liability due, then the policyholder receives the available assets, whilst the shareholders "walk away" empty handed. This implies that the payoff at expiration of the participating contract is

$$\Pi(T) = P(T) + \gamma R(T) - D(T), \tag{4}$$

where $D(T) = \left(P(T) - A(T)\right)^{+}$ is the payoff of the so-called default option.

2.2 Fair valuation

If the insurance company aims at setting an initial premium, P_0, which is fair in the sense that it does not originate arbitrage opportunities (and therefore is market consistent), then

$$P_0 = \hat{\mathbb{E}}\left[\tilde{\Pi}\left(T\right)\right],$$

where $\hat{\mathbb{E}}$ denotes the expectation taken under a risk neutral probability measure $\hat{\mathbb{P}}$, and $\tilde{\Pi}$ represents the contract payoff at maturity discounted at the current risk free rate of interest. Using a similar notation for the corresponding discounted cash-flows, let us define

$$V^P\left(0\right) := \hat{\mathbb{E}}\left[\tilde{P}\left(T\right)\right]; \quad V^R\left(0\right) := \hat{\mathbb{E}}\left[\tilde{R}\left(T\right)\right]; \quad V^D\left(0\right) := \hat{\mathbb{E}}\left[\tilde{D}\left(T\right)\right];$$

then, it follows from equation (4) that the "fair value" condition returns

$$P_0 + V^D\left(0\right) = V^P\left(0\right) + \gamma V^R\left(0\right). \tag{5}$$

Equation (5) shows that the price of the default option represents an additional premium that the policyholder has to pay in order to gain an "insurance" against a possible default of the company. In this sense, the default option premium can be interpreted as a safety loading, i.e. an additional source of capital aimed at hedging the risk of ruin of the insurance business originated by this type of policy (see Ballotta et al., 2006.a, .b, and Bernard et al., 2006, for a more detailed discussion of this point).

Further, equation (5) also implies that the fair terminal bonus rate is given by

$$\gamma = \frac{P_0 + V^D\left(0\right) - V^P\left(0\right)}{V^R\left(0\right)}.$$

Hence, if the policyholder's contribution is 100% of the reference fund (i.e. $\vartheta = 1$), then $\gamma = 1$. This is consistent with intuition, since in this case the policyholders would be the only group contributing to the financing of the reference portfolio, and as such they would have the right to receive the entire surplus of the company. Consequently they would fix the terminal bonus rate at its maximum value.

2.3 Target capital

As previously mentioned, the market consistent values of assets and liabilities related to insurance contracts are the key elements not only for the preparation of the company's balance sheet, but also for the calculation of the capital requirements.

For ease of exposition, in this paper the approach for the calculation of the target capital is based on the comparison between the so-called Risk Bearing Capital (RBC) and the target capital (FOPI, 2004). The RBC is defined as the difference between the total value of the assets and the market consistent price of the liabilities. Thus, we notice that, according to equation (5), the total value of the assets of the insurance company is given by the reference portfolio and the safety loading, i.e. the default option premium, which, in the following, we assume to be invested in the same fund backing the participating contract (in this respect, in this study we assume that the insurance company is passive in terms of risk management). Therefore, the RBC at time $t \in [0, T]$ is given by

$$RBC(t) = A_{tot}(t) - V^P(t) - \gamma V^R(t),$$

where A_{tot} is the total value of the insurer's assets, such that $A_{tot}(0) = A_0 + V^D(0)$. The fair value condition (5) implies that $RBC(0) = A_0 (1 - \vartheta)$. We note that, based on our model, the RBC is a stochastic process evolving under the real probability measure, which depends on the market consistent value process of the liabilities defined as a conditional expectation under the risk neutral martingale measure.

The target capital is, instead, based on the calculation of a downside risk measure relative to the change in the RBC over a 1 year time horizon. In order to take into account the time value of money, discounting at the risk free rate is applied. This reflects the implicit assumption that the target capital should represent the amount that, once invested in the money market account, guarantees enough capital strength to maintain appropriate policyholder protection and market stability with a certain confidence level. Therefore, using similar notation for the discounted cashflows as above, the target capital at year t is based on the variation

$$\widetilde{RBC}\,(t+1) - RBC(t).$$

For ease of exposition of the results, we prefer to construct a solvency index expressing the change in the RBC as a percentage of the value of the total assets of the insurance company at the valuation time, i.e.

$$s_t = \frac{\widetilde{RBC}\,(t+1) - RBC(t)}{A_{tot}(t)}.$$

In this study, we focus our attention on the TVaR (or Tail Conditional Expectation) with confidence level $1 - x$, i.e.

$$TVaR\,(x; t, t+1) := -\mathbb{E}\left(s_t \middle|\, s_t \leq c_{s_t}\,(x; t, t+1)\right),$$

where c_{s_t} is the VaR of the solvency index s_t with confidence level $1 - x$.

In order to proceed to the actual calculation of the contract market consistent value, and the related distributions of the assets and the liability which are needed to obtain the target capital, we need to specify the relevant market model and the stochastic process driving the reference portfolio. This is covered in the next section, in which we also derive the valuation framework.

3 Market consistent pricing of the embedded options

In order to price the components of the participating contract shown in section 2, we need to define a possible dynamic for the evolution of the price of the fund A. We note at this point that there is no specific recommendation from the regulators as to which model should be adopted; however, a common benchmark seems to be the RiskMetrics model (Mina and Xiao,

2001) with a given number of factors capturing the several sources of risk in the market. The Swiss Solvency Test (FOPI, 2004), for example, recommends a RiskMetrics-based standard asset model with 75 risk factors, including interest rates, FX rates, implied volatilities, credit spreads and hedge funds amongst the others. However, such a complex model might create a significant challenge in terms of intuition and understanding; further, since its main assumption is that prices move in a continuous way, this type of model neglects the abrupt movements in which most of the risk is concentrated. For those reasons, we prefer to adopt a simpler, parsimonious approach to the modelling of the reference portfolio evolution, which also reproduces the realistic properties of market price behaviour in a generic manner, i.e. without the need to fine-tune parameters to unrealistic values. In fact, while large sudden moves are generic properties of market prices, they are only obtainable in continuous processes at the price of setting parameters to extreme values. More specifically, we rely on the recent advances in the area of financial mathematics, and choose as a standard model the traditional Black-Scholes paradigm. Consequently, assets' log-returns follow a normal distribution which, in spirit, is very similar to the main assumption of the RiskMetrics model. However, since this assumption has been proven not to hold in real markets, we also consider two alternative asset models in order to incorporate market shocks. These two alternatives make use of the Merton process (Merton, 1976) and the Variance Gamma (VG) model (Madan et al., 1990, 1991, 1998).

The idea is to assess the impact of the model error when shocks are either neglected, or not correctly captured by the driving process. In the following sections, we introduce the three asset models and analyze their most relevant features in terms of skewness (i.e. the measure of the asymmetry of the probability distribution) and kurtosis (i.e. the part of the distribution's variance due to infrequent extreme deviations); we then proceed to discuss the issue of market incompleteness originated by the inclusion of shocks in the model, and hence we show how the participating contract introduced in section 2 can be evaluated.

3.1 Market modelling

Given a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ under the real probability measure $\mathbb{P}$, assume a frictionless market with continuous trading, in which a risk free security $B(t) = e^{rt}, r \in \mathbb{R}^{++}$, is traded. The insurance company's reference portfolio is then assumed to be given by

$$\begin{aligned} A(t) &= A(0)\, e^{L(t)}, \\ A(0) &= A_0, \end{aligned}$$

where L is the process governing the log-returns.

The standard model As mentioned above, the standard asset model proposed in this note agrees with the Black-Scholes paradigm, so that

$$L(t) = \mu t + \sigma_A W(t),$$

where W is a one-dimensional standard Brownian motion under the real probability measure $\mathbb{P}$, $\mu \in \mathbb{R}$ is the mean log-return and $\sigma_A \in \mathbb{R}^{++}$ is the instantaneous volatility. It follows that the expected rate of growth on the fund is $\mu + \sigma_A^2/2$. As the Brownian motion is a Gaussian process, its distribution is symmetric and mesokurtic, so that the indices of skewness and kurtosis are both equal to zero.

It is, however, a well known fact that asset log-returns exhibit fatter tails than those accommodated by the normal distribution, implying an underestimation of the likelihood of extreme events.

The Merton process-based model In order to take into account the occurrence of market shocks, the first alternative we propose is based on the so-called Merton process (Merton, 1976), which is given by the sum of a Brownian motion with drift and an independent compound Poisson process. Thus

$$L(t) = (n - \lambda\mu_X)\, t + \sigma W(t) + \sum_{k=1}^{N(t)} X(k), \quad n, \mu_X \in \mathbb{R}, \sigma \in \mathbb{R}^{++}, \tag{6}$$

where W is a one-dimensional standard $\mathbb{P}$-Brownian motion capturing the "marginal" price changes; $X \sim N(\mu_X, \sigma_X^2)$ models the size of the jumps, i.e. the "abnormal" changes in the prices due to the arrival of important new information, whose flow is regulated by a Poisson process, N, of rate $\lambda \in \mathbb{R}^{++}$. Note that W, N and X are assumed to be independent one of the other, which implies that L is a Lévy process. In particular, the characteristic function of the Merton process is

$$\begin{aligned} \phi_L(u;t) &= e^{t\left(iu(n-\lambda\mu_X) - u^2\frac{\sigma^2}{2} + \lambda(\phi_X(u)-1)\right)}, \\ \phi_X(u) &= e^{iu\mu_X - u^2\frac{\sigma_X^2}{2}}; \end{aligned} \tag{7}$$

consequently, the Lévy measure of the process L is given by

$$\upsilon_M(dx) = \frac{\lambda}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} dx.$$

Hence, the Merton process is a finite activity process. It follows that the mean log-return is $n \in \mathbb{R}$, whilst the instantaneous variance is $\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2)$ and the expected rate of growth on the fund A is $(n - \lambda\mu_X) + \sigma^2/2 + \lambda(\phi_X(1) - 1)$. The Merton process exhibits skewness and kurtosis as described by the Pearson index of asymmetry

$$\gamma_1(t) = \frac{\lambda\mu_X(\mu_X^2 + 3\sigma_X^2)}{(\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2))^{3/2}\sqrt{t}},$$

and the excess of kurtosis index

$$\gamma_2(t) = \frac{\lambda(\mu_X^4 + 6\mu_X^2\sigma_X^2 + 3\sigma_X^4)}{(\sigma^2 + \lambda(\mu_X^2 + \sigma_X^2))^2\, t}.$$

Finally, we observe that

$$\begin{aligned} sign(\gamma_1(t)) &= sign(\mu_X); \\ \gamma_2(t) &> 0. \end{aligned}$$

Therefore, the distribution of the Merton process is positively or negatively skewed according to the sign of the expected jump' size; further, it is leptokurtic.

The Variance Gamma process-based model A recent analysis offered by Carr et al. (2002) shows that, in general, market prices lack of a diffusion component, as if it was diversified away; consequently, they conclude that there is an argument for using pure jump processes, particularly of infinite activity and finite variation given their ability to capture both frequent small changes and rare large jumps. A process of this kind used in finance due to its analytical and numerical tractability is the Variance Gamma (VG) process, which is a normal tempered stable process obtained by time changing an arithmetic Brownian motion by a gamma subordinator. We follow this approach and define the second alternative asset model by

$$L(t) = (m-\theta)\,t + Z(t), \quad m \in \mathbb{R}, \tag{8}$$

where

$$Z(t) = \theta\tau_t + \xi W(\tau_t), \qquad \theta \in \mathbb{R}, \xi \in \mathbb{R}^{++},$$

is the VG process, $W = (W_t : t \geq 0)$ is a standard Brownian motion and $\tau = (\tau_t : t \geq 0)$ is a gamma process, with parameters a, $b > 0$, and independent of W. The parameter a represents the time scale of the process, i.e. it alters the intensity of the jumps of all sizes simultaneously, whilst the parameter b captures the decay rate of big jumps. It is easy to show that the characteristic function of the process L is

$$\phi_L(u,t) = e^{iu(m-\theta)t}\left(\frac{b}{b - iu\theta + u^2\frac{\xi^2}{2}}\right)^{at}. \tag{9}$$

The VG process has been introduced by Madan and Seneta (1990), and has been further refined by Madan and Milne (1991) and Madan et al. (1998). In particular, these authors consider as subordinator a gamma process with unit mean rate, i.e. with parameters $a = b = 1/k$, where $k \in \mathbb{R}^{++}$ is the variance rate, so that the random time is an unbiased reflection of calendar time. Unless otherwise stated, in the remaining of this paper we use this parametrization, and we say that the process Z is $VG(\theta, \xi, 1/k, 1/k)$; therefore

$$\phi_L(z,t) = e^{z(m-\theta)t}\left(1 - z\theta k - z^2\frac{\xi^2}{2}k\right)^{-\frac{t}{k}}. \tag{10}$$

Note that equation (10) implies that the characteristic function of the VG process exists for $z \in \mathbb{C}$ with

$$\frac{-\theta - \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2} < \Re(z) < \frac{-\theta + \sqrt{\theta^2 + \frac{2\xi^2}{k}}}{\xi^2}.$$

The VG process Z can also be represented as the difference between two independent gamma processes, which follows from the fact that

$$\phi_Z(u,t) = \left(1 - iu\theta k + u^2\frac{\xi^2}{2}k\right)^{-\frac{t}{k}} = \left(\frac{b_+}{b_+ - iu}\right)^{\frac{t}{k}}\left(\frac{b_-}{b_- + iu}\right)^{\frac{t}{k}}, \tag{11}$$

where

$$
\begin{aligned}
b_{+} &= \frac{2}{k\left(\sqrt{\theta^2+\frac{2\xi^2}{k}}+\theta\right)}, \\
b_{-} &= \frac{2}{k\left(\sqrt{\theta^2+\frac{2\xi^2}{k}}-\theta\right)}.
\end{aligned}
$$

It follows from this decomposition that the Lévy measure of the process $Z(t)$ (and therefore of the process L) is given by

$$
\upsilon_Z(dx) = \frac{1}{k}|x|^{-1}\left(e^{-b_+ x}1_{(x>0)} + e^{b_- x}1_{(x<0)}\right), \tag{12}
$$

which shows that the VG process has infinite activity with finite variation.

In this model, the mean log-return is given by $m \in \mathbb{R}$; the instantaneous variance is $(\xi^2+\theta^2 k)$; the expected rate of growth on A is $m-\theta-\frac{1}{k}\ln\left(1-\theta k-\frac{\xi^2}{2}k\right)$. As far as skewness and kurtosis are concerned, the Pearson index of asymmetry and the excess of kurtosis index are respectively

$$
\gamma_1(t) = \frac{(3\xi^2\theta k + 2\theta^3 k^2)}{(\xi^2+\theta^2 k)^{3/2}\sqrt{t}}, \quad \gamma_2(t) = \frac{(3\xi^4 k + 12\xi^2\theta^2 k^2 + 6\theta^4 k^3)}{(\xi^2+\theta^2 k)^2 t}.
$$

Therefore, the VG distribution is positively or negatively skewed according to whether $\theta > 0$ or $\theta < 0$, since $sign(\gamma_1(t)) = sign(\theta)$; further, we observe that $\gamma_2(t) > 0$ and, consequently, the distribution is leptokurtic as well.

3.2 Pricing the embedded options

The models proposed above have as a common feature the fact that the driving process is a Lévy process, i.e. a process with independent and stationary increments. This actually allows us to reduce the problem of obtaining the price of the guaranteed benefit $P(T)$ to the pricing of a European call option.

The payoff equation (3), in fact, implies that

$$
P(T) = \alpha \sum_{k=0}^{T-t-1}(1-\alpha)^k P^1(t)\prod_{i=1}^{T-t-k}(1+r_P(t+i)) + (1-\alpha)^{T-t}P(t);
$$

therefore

$$
\begin{aligned}
V^P(t) &= \hat{\mathbb{E}}\left[e^{-r(T-t)}P(T)\middle|\mathbb{F}_t\right] \\
&= \alpha P^1(t)\sum_{k=0}^{T-t-1} e^{-rk}(1-\alpha)^k \prod_{i=1}^{T-t-k}\hat{\mathbb{E}}\left[e^{-r}(1+r_P(t+i))\middle|\mathbb{F}_t\right] \\
&\quad + e^{-r(T-t)}(1-\alpha)^{T-t}P(t).
\end{aligned} \tag{13}
$$

Because of equation (2), it follows that

$$\hat{\mathbb{E}}\left[e^{-r}\left(1+r_P\left(t+i\right)\right)\middle|\,\mathbb{F}_t\right]=e^{-r}\left(1+r_G\right)+\hat{\mathbb{E}}\left[e^{-r}\left(\beta e^{L'(1)}-\left(\beta+r_G\right)\right)^{+}\right], \tag{14}$$

where L' denotes an independent copy of the Lévy process L. Analytical formulae are hence available for this part of the participating contract; we note, though, that it is not possible to derive analytical formulae for both the terminal bonus and the default option given the complex recursive nature of $P(T)$, and their high dependency on the path of the reference fund A; hence we resort to Monte Carlo simulation in order to approximate the price of these two components of the insurance contract. We need however to specify the risk neutral martingale measure $\hat{\mathbb{P}}$: except for the Brownian motion case, in fact, the market is incomplete due to the presence of shocks and, therefore, there are infinitely many pricing measures.

A general approach for selecting a suitable risk neutral martingale measure is based on the idea of extracting the market's pricing measure from the prices of traded options via calibration. However, the portfolio backing insurance policies is in general a mixture of securities such as equities, bonds and properties, for which derivatives that use it as underlying are difficult (if not impossible) to find. Consequently, the "market-implied martingale measure" cannot be extracted due to the lack of suitable option prices.

Alternative approaches rely on the construction of a Radon-Nikodým derivative linking the "rescaling" functions of the diffusion part and the jump part of the process, and the risk neutral martingale condition. A desirable feature of this Radon-Nikodým derivative (or density process) is to preserve the distribution properties of the underlying process, so that the distribution is the same before and after the change of measure. A possible density process satisfying this requirement is the so-called cumulant process, which, given our setting, takes the form of the so-called Esscher transform. The resulting probability measure has the appealing mathematical property that the unknown parameters ensuring that the martingale condition for the discounted securities price process is satisfied, can be obtained as pointwise solutions of equations depending only on the characteristic triplet of the process itself. Further, as shown by Gerber and Shiu (1994), this risk neutral martingale measure corresponds to the case of a representative agent with a power utility function. The Esscher parameter characterizing the resulting risk neutral martingale measure, in fact, is the Arrow-Pratt index of relative risk aversion (with signs changed), which for this class of utility functions is constant. As power utility functions display decreasing absolute risk aversion, investments in a risky asset increase with wealth; such an assumption is fairly plausible in (micro)economics terms, as it qualifies the risky asset under consideration as a normal good.

Due to this set of properties, in the remaining of this paper we adopt the Esscher transform method; we note that this approach has been applied in mathematical finance by Gerber and Shiu (1994) and Madan and Milne (1991) amongst the others. Conditions for the existence and integrability of such a process have been studied by Bühlmann et al. (1996), Kallsen and Shiryaev (2002) and Hubalek and Sgarra (2006). In the remaining of this section, we show how the risk neutral Esscher measure can be determined for the considered models, and how it is used to price the guaranteed benefit $P(T)$.

3.2.1 The Esscher transform and the risk neutral martingale Esscher measure

Let $L(t)$ be a Lévy motion; then the process

$$\eta(t) = \left\{ e^{hL(t)} \phi_L \left(\frac{h}{i}, t \right)^{-1} : t \geq 0 \right\}, \tag{15}$$

is a positive $\mathbb{P}$-martingale that can be used to define a change of probability measure, i.e. the Radon-Nikodým derivative of a new equivalent probability measure $\hat{\mathbb{P}}^h$, called the Esscher measure of parameter h. The process $\eta(t)$ is called the Esscher transform of parameter h. If we use the Esscher transform to determine a risk neutral martingale measure, i.e. a measure under which discounted asset prices behave like martingales, the Esscher parameter h needs to satisfy the following condition (see, for example, Gerber and Shiu, 1994):

$$r = \ln \phi_L \left(\frac{h+1}{i}, 1 \right) - \ln \phi_L \left(\frac{h}{i}, 1 \right). \tag{16}$$

In virtue of equation (7), the $\hat{\mathbb{P}}^h$-characteristic function of the log-returns driven by the Merton process is

$$\begin{aligned}
\hat{\phi}_L^h (u,t) &= \frac{\phi_L \left(\frac{iu+h}{i}, t \right)}{\phi_L \left(\frac{h}{i}, t \right)} = e^{t\left(iu\left(n+h\sigma^2-\lambda\mu_X\right) - u^2\frac{\sigma^2}{2} + \lambda^h\left(\hat{\phi}_X^h(u)-1\right)\right)}, \\
\lambda^h &= \lambda e^{h\mu_X + h^2\frac{\sigma_X^2}{2}}, \\
\hat{\phi}_X^h (u) &= e^{iu\left(\mu_X + h\sigma_X^2\right) - u^2\frac{\sigma_X^2}{2}}.
\end{aligned}$$

From the risk neutral condition (16) it follows that the Esscher parameter h solves

$$r = n + h\sigma^2 - \lambda\mu_X + \frac{\sigma^2}{2} + \lambda^h \left(\hat{\phi}_X^h (1) - 1 \right);$$

therefore, the reference fund under $\hat{\mathbb{P}}^h$ is given by

$$A(t) = A(0)\, e^{\left(r - \frac{\sigma^2}{2} - \lambda^h \left(\hat{\phi}_X^h \left(\frac{1}{i}\right) - 1 \right) \right) t + \sigma \hat{W}^h(t) + \sum_{k=1}^{\hat{N}^h(t)} \hat{X}^h(k)}, \tag{17}$$

where $\hat{W}^h$ is a $\hat{\mathbb{P}}^h$-Brownian motion, $\hat{N}^h$ is a $\hat{\mathbb{P}}^h$-Poisson process with rate λ^h, and $\hat{X}^h \sim N\left(\mu_X + h\sigma_X^2, \sigma_X^2\right)$ under $\hat{\mathbb{P}}^h$.

Similarly, using equation (10), it follows that the characteristic function of the VG-log-returns under $\hat{\mathbb{P}}^h$ is

$$\hat{\phi}_L^h (u,t) = e^{iu(m-\theta)t} \left(1 - iu\theta^h k^h + u^2 \frac{\xi^2}{2} k^h \right)^{-\frac{t}{k}}, \tag{18}$$

where

$$\begin{aligned}
\theta^h &= \theta + h\xi^2, \\
k^h &= \frac{k}{1 - h\theta k - h^2\frac{\xi^2}{2}k}.
\end{aligned}$$

Moreover, the Esscher measure exists if and only if $\phi_Z\left(\frac{h}{i},t\right)$ exists (similar results for the case $\xi=1$ have been obtained by Hubalek and Sgarra, 2006), i.e. if and only if

$$\frac{-\theta-\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2}<h<\frac{-\theta+\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2}.$$

The Esscher parameter h is the solution to

$$f(h)=r-(m-\theta)-\frac{1}{k}\ln\frac{1-h\theta k-h^2\frac{\xi^2}{2}k}{1-(h+1)\theta k-(h+1)^2\frac{\xi^2}{2}k}=0. \tag{19}$$

Note that the function $f(h)$ exists only for $h\in\left(\frac{-\theta-\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2},\frac{-\theta+\sqrt{\theta^2+\frac{2\xi^2}{k}}}{\xi^2}-1\right)$, provided that $\theta^2+\frac{2\xi^2}{k}>\frac{\xi^4}{4}$. Under these restrictions, it is possible to solve equation (19) directly and obtain

$$\begin{aligned} h &= -\frac{\theta}{\xi^2}-\frac{1}{\varepsilon}+\frac{1}{\xi^2\varepsilon}\sqrt{\xi^4+\theta^2\varepsilon^2-\xi^4\varepsilon+\frac{2\xi^2}{k}\varepsilon^2} \\ \varepsilon &= 1-e^{k(m-\theta-r)}. \end{aligned}$$

This solution to equation (19) together with equation (18) fully characterizes the risk neutral dynamic of the stock price process under $\hat{\mathbb{P}}^h$, which, in virtue of the Girsanov theorem, is given by

$$A(t)=A(0)\,e^{\left(r-\ln\hat{\phi}_Z^h\left(\frac{1}{i},1\right)\right)t+\hat{Z}^h(t)}, \tag{20}$$

and $\hat{Z}^h$ is $VG\left(\theta^h,\xi,1/k,1/k^h\right)$.

3.2.2 Pricing of the guaranteed benefit P

Based on the results presented in the previous section, it is possible to solve analytically equation (14), and therefore determine closed formulae for the price of the guaranteed benefit, expressed in equation (13), under the three market paradigms introduced in section 3.1. The result is summarized in the following.

Proposition 1 *The market consistent value of the guaranteed benefit $P(T)$, when the reference fund is driven by a Lévy process, is (for the risk neutral Esscher measure $\hat{\mathbb{P}}^h$)*

$$\begin{aligned} V^P(t) &= \alpha P^1(t)\sum_{k=0}^{T-t-1}e^{-rk}(1-\alpha)^k\left[e^{-r}(1+r_G)+C(\beta,\beta+r_G,1)\right]^{T-t-k} \\ &\quad +e^{-r(T-t)}(1-\alpha)^{T-t}P(t), \end{aligned} \tag{21}$$

with $C(\beta,\beta+r_G,1)$ denoting the price of the embedded 1-year European call option with strike $\beta+r_G$ and written on a underlying with spot value β. In particular, let Φ denote the distribution of the standard normal random variable, and h be the Esscher parameter which solves the martingale condition (16). Then *i) under the standard model,*

$$C(\beta, \beta + r_G, 1) = \beta\Phi(d_1) - e^{-r}(\beta + r_G)\Phi(d_2), \tag{22}$$

where

$$d_1 = \frac{\ln\frac{\beta}{\beta + r_G} + \left(r + \frac{\sigma_A^2}{2}\right)}{\sigma_A}; \quad d_2 = d_1 - \sigma_A.$$

ii) Under the Merton model,

$$\begin{aligned} C(\beta, \beta + r_G, 1) &= \beta \sum_{n=0}^{\infty} \frac{e^{-\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)}\left(\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)\right)^n}{n!}\Phi(d_{n;h}) \\ &\quad - e^{-r}(\beta + r_G)\sum_{n=0}^{\infty}\frac{e^{-\lambda^h}\left(\lambda^h\right)^n}{n!}\Phi\left(d'_{n;h}\right), \end{aligned} \tag{23}$$

where

$$d_{n;h} = \frac{\ln\frac{\beta}{\beta + r_G} + \left(r_{n;h} + \frac{v_n^2}{2}\right)}{v_n}, \quad d'_{n;h} = d_{n;h} - v_n,$$

and

$$\begin{aligned} r_{n;h} &= r - \lambda^h\left(\hat{\phi}_X^h\left(\frac{1}{i}\right) - 1\right) + n\ln\hat{\phi}_X^h\left(\frac{1}{i}\right), \\ v_n^2 &= \sigma^2 + n\sigma_X^2. \end{aligned}$$

iii) Under the VG model,

$$C(\beta, \beta + r_G, 1) = \beta\Psi\left(d\sqrt{1-s}, \frac{\theta^h + \xi^2}{\xi\sqrt{1-s}}, \frac{1}{k}\right) - e^{-r}(\beta + r_G)\Psi\left(d, \frac{\theta^h}{\xi}, \frac{1}{k}\right), \tag{24}$$

where

$$\begin{aligned} \Psi(a, b, c) &= \int_0^{\infty}\Phi\left(\frac{a}{\sqrt{\tau}} + b\sqrt{\tau}\right)\frac{\tau^{c-1}e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^c\Gamma(c)}d\tau; \\ d &= \frac{\ln\frac{\beta}{\beta + r_G} + r - \ln\hat{\phi}_Z^h\left(\frac{1}{i}, 1\right)}{\xi}, \quad s = k^h\left(\theta^h + \frac{\xi^2}{2}\right). \end{aligned}$$

Proof. Let $C(\beta, \beta + r_G, 1)$ denote the expectation in equation (14), then

$$C(\beta, \beta + r_G, 1) = \beta\hat{\mathbb{E}}^h\left(e^{-r}e^{L'(1)}1_{\left(L'(1) > \ln\frac{\beta + r_G}{\beta}\right)}\right) - e^{-r}(\beta + r_G)\hat{\mathbb{P}}^h\left(L'(1) > \ln\frac{\beta + r_G}{\beta}\right).$$

Therefore.

i) The result follows from the application of the Black-Scholes formula.

ii) Conditioning on the number of jumps occurring in 1 year, the process $L'(1)$ follows a normal distribution with mean $r_{n;h} - \frac{v_n^2}{2}$ and variance v_n^2. Hence,

$$\hat{\mathbb{E}}^h \left(e^{L'(1)} 1_{\left(L'(1) > \ln \frac{\beta + r_G}{\beta}\right)} \middle| \hat{N}^h(1) = n \right) = e^{r_{n;h}} \Phi(d_{n;h});$$

consequently

$$\hat{\mathbb{E}}^h \left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \ln \frac{\beta + r_G}{\beta}\right)} \right) = \sum_{n=0}^{\infty} \frac{e^{-\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)} \left(\lambda^h \hat{\phi}_X^h\left(\frac{1}{i}\right)\right)^n}{n!} \Phi(d_{n;h}).$$

Moreover,

$$\begin{aligned} \hat{\mathbb{P}}^h \left(L'(1) > \ln \frac{\beta + r_G}{\beta} \right) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda^h} \left(\lambda^h\right)^n}{n!} \hat{\mathbb{P}}^h \left(L'(1) > \ln \frac{\beta + r_G}{\beta} \middle| \hat{N}^h(1) = n \right) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda^h} \left(\lambda^h\right)^n}{n!} \Phi\left(d'_{n;h}\right), \end{aligned}$$

as required.

iii) Conditioning on the random time $\hat{\tau}^h(1)$, the process $L'(1)$ follows a normal distribution with mean $\left(r_{\tau;h} - \frac{\xi^2}{2}\right)\tau$ and variance $\xi^2 \tau$, where $r_{\tau;h} = \frac{r - \ln \hat{\phi}_Z^h\left(\frac{1}{i},1\right)}{\tau} + \theta^h + \frac{\xi^2}{2}$. Therefore,

$$\hat{\mathbb{E}}^h \left(e^{L'(1)} 1_{\left(L'(1) > \ln \frac{\beta + r_G}{\beta}\right)} \middle| \hat{\tau}^h(1) = \tau \right) = e^{r_{\tau;h}\tau} \Phi(d_{\tau;h})$$

with

$$d_{\tau;h} = \frac{\ln \frac{\beta}{\beta + r_G} + \left(r_{\tau;h} + \frac{\xi^2}{2}\right)\tau}{\xi\sqrt{\tau}}.$$

Consequently

$$\hat{\mathbb{E}}^h \left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \ln \frac{\beta + r_G}{\beta}\right)} \right) = \int_0^{\infty} \hat{\phi}_Z^h\left(\frac{1}{i}, 1\right)^{-1} e^{\left(\theta^h + \frac{\xi^2}{2}\right)\tau} \Phi(d_{\tau;h}) \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} d\tau.$$

Further, set $s = k^h\left(\theta^h + \frac{\xi^2}{2}\right)$ and $u = (1-s)\tau$. Then

$$\hat{\mathbb{E}}^h \left(e^{-r} e^{L'(1)} 1_{\left(L'(1) > \ln \frac{\beta + r_G}{\beta}\right)} \right) = \int_0^{\infty} \Phi(d_{u;h}) \frac{u^{\frac{1}{k}-1} e^{-\frac{u}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} du$$

where

$$d_{u;h} = \frac{d\sqrt{1-s}}{\sqrt{u}} + \frac{\theta^h + \xi^2}{\xi\sqrt{1-s}}\sqrt{u}.$$

Moreover,

$$\begin{aligned}
\hat{\mathbb{P}}^h\left(L'(1) > \ln\frac{\beta + r_G}{\beta}\right) &= \int_0^\infty \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} \hat{\mathbb{P}}^h\left(L'(1) > \ln\frac{\beta + r_G}{\beta}\middle| \hat{\tau}^h(1) = \tau\right) d\tau \\
&= \int_0^\infty \frac{\tau^{\frac{1}{k}-1} e^{-\frac{\tau}{k^h}}}{\left(k^h\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)} \Phi\left(d'_{\tau;h}\right) d\tau, \\
d'_{\tau;h} &= \frac{d}{\sqrt{\tau}} + \frac{\theta^h}{\xi}\sqrt{\tau},
\end{aligned}$$

which concludes the proof. ■

In order to obtain the fair value of the participating contract we also need to evaluate the premium of the terminal bonus option and the default option, which can only be done using a Monte Carlo procedure as already discussed. Proposition 1 can be used to benchmark the Monte Carlo schemes, since the price $V^P(t)$ can be easily obtained as a by-product of the computations for $V^R(t)$ and $V^D(t)$.

4 Numerical algorithms

The development of numerical algorithms for the pricing of participating contracts, and for the calculation of the corresponding risk margin and capital requirements represents in general a critical issue for insurance companies who need to develop suitable software architectures. Therefore, in this section we review the available alternative algorithms, and test their efficiency for the case of the contract considered in this note.

In order to price the guaranteed benefit, we use the closed analytical formulae developed in Proposition 1. Equations (23) and (24), though, involve the computation of an infinite sum and an improper integral respectively. Hence, for the case of the infinite series determining the Poisson distribution in equation (23), we note that the terms in the series converge to zero rapidly at infinity due to the presence of the factorial term, $n!$, in the denominator. This property allows us to compute the infinite sum with a prespecified relative tolerance ϵ, meaning that we neglect all terms smaller than ϵ times the current sum. For this numerical example, we fix $\epsilon = 10^{-15}$. For the case of the improper integral defining the function $\Psi(a, b, c)$ in equation (24), we note that a closed formula has been developed by Madan et al. (1998) (Theorem 2, equation $(A.11)$). However, due to the presence in this formula of a degenerate hypergeometric function of two variables, we prefer here to adopt a numerical quadrature scheme based on the midpoint rule, extended to accommodate a change of variable. This is implemented to map the infinite range of integration to a finite one, and it exploits the fact that the integrand functions decrease exponentially rapidly at infinity.

The price of both the terminal bonus option and the default option are obtained by Monte Carlo method, irrespective of the model, due to their path-dependent payoff design. Variance reduction techniques are used in order to speed up the convergence and improve the accuracy of the estimates. Specifically, for all models considered in this paper, we develop both sequential and stratified algorithms. The sequential algorithms use, for variance reduction purposes, the antithetic variate method (see, for example, Boyle et al., 1997); the stratified algorithms make use of the Brownian bridge and the Brownian-Gamma bridge (Ribeiro and Webber, 2004) to generate respectively the paths of the Brownian motion and the Variance Gamma process. As far as the Merton process is concerned, we use the standard property that, given the (stratified) total number of jumps in $[0, T]$, $N(T) = k$, the arrival times of the jumps have the joint distribution of the order statistics of k independent random variables uniformly distributed over $[0, T]$ (see, for example, Glasserman, 2004). We note that stratification is implemented only at maturity and is achieved with low-discrepancy sampling. The remaining values in the trajectory of the driving processes are generated by ordinary sampling since for both the cases of the Merton process and the VG process the total dimension of the low-discrepancy sequence cannot be determined a priori, therefore affecting the initialization of the sequence itself. This is due to the fact that the sequence's dimension depends on the total number of the jumps occurred by T in the case of the Merton process, and on the generator chosen for some of the random deviates required for the construction of the Brownian-Gamma bridge in the VG process case. This aspect is discussed in more details later in this section.

For low-discrepancy sampling we use a Sobol' sequence based on a routine developed by Burkardt (2006); the points in the Sobol' sequence are then randomized using the random shift rule (see, for example, Glasserman, 2004, for fuller details). Since the randomization implies that the points in the Sobol' sequence are no longer independent, the calculation of the simulation standard error is based on a "batching" procedure (see Boyle et al., 1997). Hence, given a budget of 1,000,000 replications, we run 100 batches of independent stratified samples each of size 10,000 replications. The sample standard error is then estimated from the batch means. This batching procedure is extended to the sequential algorithms as well, in order to ensure consistency in the comparison of the two methodologies.

Finally, we also make use of the control-variate technique to reduce the variance of the estimates even further; in particular, we use the results of Proposition 1 and choose the price of the policy reserve, V^P, as a control. In order to avoid the introduction of bias in the estimation, we use a few pilot runs to estimate the control variate parameter and then we use this estimate in the main simulation run (for more details, we refer to, for example, Glasserman, 2004). The control variate procedure is applied to the sample means of the 100 batches. The standard error is calculated using the same technique.

Both the antithetic variate technique and stratified Monte Carlo require the inversion of the distribution function for the generation of random deviates; therefore all Monte Carlo codes developed for this paper use this procedure, except for the case of beta deviates (see below). In particular, the inversion of the gamma distribution function is computed using an algorithm due to Moshier (2000). Finally, the beta deviates from a Beta distribution, $B(a, b)$, required for the gamma bridge are generated using the algorithms of Atkinson and Whittaker for the cases in which $a, b < 1$ with $a + b \geq 1$, and $a \leq 1, b \geq 1$; the Johnk's algorithm is used instead for the case in which $a, b < 1$ with $a + b < 1$ (see Devroye, 1986). As these algorithms are not based on the inversion principle, the beta deviates are generated using ordinary sampling. All the algorithms described above have been implemented in C/C++ environment.

For benchmarking purposes, we use the Monte Carlo procedures developed above to

	Sequential Monte Carlo (with variance reduction)			Stratified Monte Carlo (RQMC)		
	GBM	**Merton**	**VG**	**GBM**	**Merton**	**VG**
$\mathbf{V}^P(0)$	190.7750 (0.0027%)	191.8085 (0.0029%)	190.8175 (0.0012%)	190.7742 (0.0015%)	191.8121 (0.0023%)	190.8257 (0.0015%)
$\mathbf{V}^R(0)$	8.5784 (0.0634%)	8.9806 (0.0686%)	9.5199 (0.0694%)	8.6712 (0.0181%)	9.0254 (0.0357%)	9.4741 (0.0167%)
$\mathbf{V}^D(0)$	99.5536 (0.0039%)	100.8346 (0.0035%)	100.8730 (0.0048%)	99.5524 (0.0018%)	100.7681 (0.0027%)	100.8053 (0.0018%)
$\mathbf{V}^P(0)$ (closed analytical formula)						
	GBM		**Merton**		**VG**	
	190.7739		191.8112		190.8260	

Table 1: Fair values of the embedded option for the benchmark set of parameters specified in Section 5, and $\vartheta = 1$. Values based on a budget of 1,000,000 Monte Carlo replications. The numbers in parenthesis represent the error of the estimate expressed as percentage of the corresponding option price. All the estimates use the Control Variate technique.

compute V^P as well. Results are shown in Table 1. In particular, we observe that the difference between the prices calculated using the closed analytical formulae and the ones computed by Monte Carlo is less than 0.005%.

In order to test the efficiency of the algorithms developed, we consider the efficiency gain index of method A with standard error σ_A and execution time t_A versus method B with standard error σ_B and execution time t_B, which is defined as $E_{AB} = \sigma_B^2 t_B / (\sigma_A^2 t_A)$ (see, for example, Boyle et al., 1997). Results are shown in Table 2, where we test the efficiency of stratified Monte Carlo compared to sequential Monte Carlo, and the efficiency of including the control variate for further improvement of the estimates'quality. The results reported in Tables 1 and 2 show that stratified Monte Carlo is the most competitive method especially in the case of the VG economy, likely due to the need to generate two random deviates per each step of each trajectory. Further efficiency is gained by means of the control variate procedure, which is made possible by the closed analytical formulae developed for the guaranteed benefit.

5 Results

In the following, we use the results presented in section 3 and the numerical algorithms discussed in section 4 to analyze the impact of a model misspecification on the market consistent price of the participating contract, and the corresponding target capital.

For the analysis to be consistent, we need to guarantee that the adopted models represent the same underlying asset; for this purpose, we choose the distribution properties of the log-returns of the reference fund as a relevant benchmark for comparison. Hence, the parameters are chosen so that the first four cumulants of the underlying distributions of the asset log-returns are matched as closely as possible under the real probability measure. Specifically,

	GBM model	Merton model	VG model
Stratified vs Sequential (with Control Variate)			
$\mathbf{V}^R(0)$	2.0824	2.9446	75.6643
$\mathbf{V}^D(0)$	0.9832	1.3056	17.7493
Sequential with Control Variate vs Sequential without Control Variate			
$\mathbf{V}^R(0)$	87.406	114.7938	104.3374
$\mathbf{V}^D(0)$	76.5449	66.7184	87.9055
Stratified with Control Variate vs Stratified without Control Variate			
$\mathbf{V}^R(0)$	28.9972	67.4427	47.2839
$\mathbf{V}^D(0)$	62.3766	66.6516	54.6871

Table 2: Efficiency of the numerical schemes described in Section 4. The table reports the efficiency gain, E_{AB} of method A to method B. If $E_{AB} > 1$, then method A is more efficient than method B. In the top panel, method A is stratified Monte Carlo with control variate, method B is sequential Monte Carlo with control variate. In the second panel, method A is sequential Monte Carlo with control variate, method B is sequential Monte Carlo without control variate. In the bottom panel, method A is stratified Monte Carlo with control variate, method B is stratified Monte Carlo without control variate. Values based on a budget of 1,000,000 replications.

Market models parameters	
Standard model (GBM)	$\mu = 10\%$ p.a.; $\sigma_A = 20\%$ p.a.
Merton model	$n = 10\%$ p.a.; $\sigma = 18.82\%$ p.a.
	$\lambda = 0.59$; $\mu_X = -5.37\%$ p.a.; $\sigma_X = 7\%$ p.a.
VG model	$m = 10\%$ p.a.; $\theta = -3.04\%$ p.a.; $k = 0.15$; $\xi = 19.96\%$ p.a.
	$r = 3.5\%$ p.a.
Contract parameters	
$A_0 = 100$; $T = 20$ years; $\alpha = 60\%$; $\beta = 50\%$; $\vartheta = 90\%$; $r_G = 4\%$ p.a.	

Table 3: Base parameter set. The parameters are taken from Ballotta (2005).

	GBM model	Merton model	VG model
Expected rate of growth	0.12	0.1199	0.1199
$\mathbb{E}(L_1)$	0.1	0.1	0.1
	(0.1)	(0.1)	(0.1)
$\mathbb{V}ar(L_1)$	0.04	0.04	0.04
	(0.04)	(0.04)	(0.04)
γ_1	0	-0.06964	-0.06836
	(0)	(-0.0693)	(-0.0675)
γ_2	0	0.0609	0.45312
	(0)	(0.0679)	(0.459)

Table 4: Moments of the asset log-returns at time $t = 1$, based on the models considered in Section 3 and the base set of parameters given in Table 3. The numbers in parenthesis represent the estimated moments based on 1,000,000 Monte Carlo runs.

	GBM vs Merton	**GBM vs VG**	**VG vs Merton**
$\mathbf{V}^P(0)$	-0.54%	-0.03%	-0.51%
$\mathbf{V}^R(0)$	-3.92%	-8.47%	4.97%

Table 5: Model error: impact on the fair value of the guaranteed benefit and the terminal bonus for the benchmark set of parameters (unlevered contract). Mispricing calculated using the prices reported in Table 1 for the stratified Monte Carlo case.

ϑ	**GBM**	**Merton**	**VG**
0.3	6.7038	7.1569	7.4603
0.5	25.0377	25.7831	26.1078
0.7	51.4867	52.5789	52.6699
0.9	82.8095	84.1079	84.8437

Table 6: Fair value of the default option for different levels of the leverage coefficient ϑ, and the benchmark set of parameters. Values based on stratified Monte Carlo and a budget of 1,000,000 replications.

ϑ	**GBM vs Merton**	**GBM vs VG**	**VG vs Merton**
0.3	-10.14%	-6.33%	4.24%
0.5	-4.10%	-2.89%	1.26%
0.7	-2.25%	-2.08%	0.17%
0.9	-2.40%	-1.54%	0.87%

Table 7: Model error: impact on the default option price for different levels of the leverage coefficient ϑ. Mispricing calculated using the prices reported in Table 6.

$x\%$	**GBM vs Merton**	**GBM vs VG**	**VG vs Merton**
AAA	-5.69%	-15.35%	11.41%
AA	-6.98%	-15.21%	9.71%
A	-5.80%	-14.06%	9.60%
BBB	-3.54%	-10.62%	7.93%
BB	-3.32%	-8.66%	5.85%
B	-6.88%	-7.35%	0.51%

Table 8: Model error: impact on the RBC for the Standard & Poor's confidence levels considered in Figure 1. The mispricing is evaluated in correspondence of $\vartheta = 90\%$.

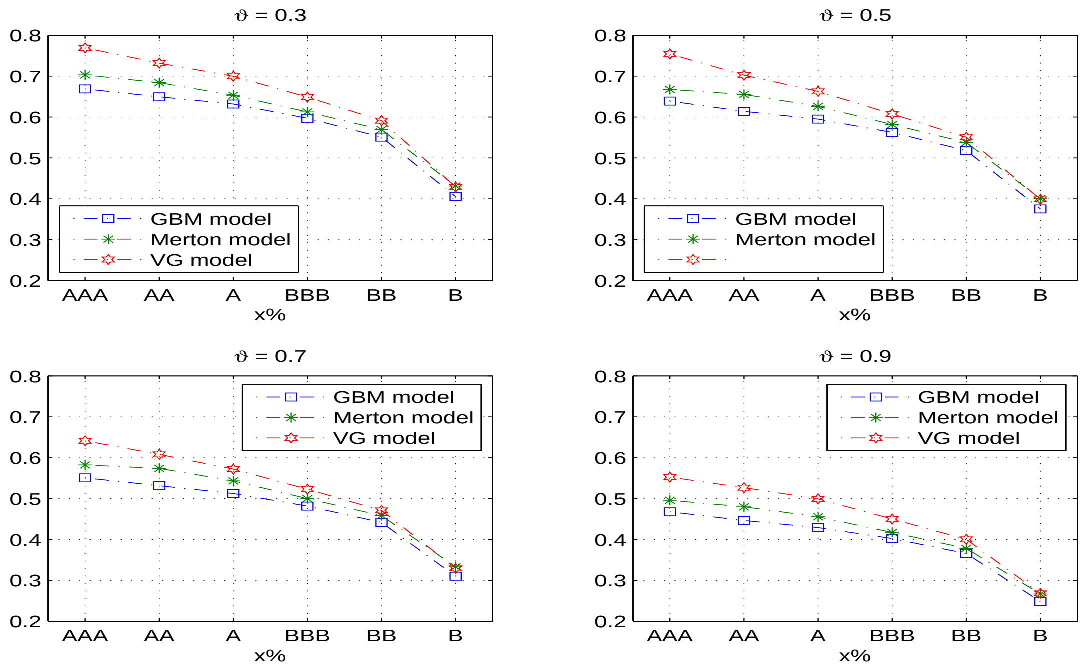

Figure 1: Risk Bearing Capital: $TVaR(s_1)$ for different levels of the leverage coefficient ϑ. For the definition of the confidence levels, we consider the percentiles provided by the Standard & Poor's classification (AAA = 99.99%; AA = 99.97%; A = 99.93%; BBB = 99.77%; BB = 99.31%; B = 93.46%). Estimation based on 100,000 simulations (from sequential Monte Carlo) for the benchmark set of parameters.

we choose an underlying asset with expected annual (log-)returns of 10% and volatility $\sigma_A =$ 20% p.a. The parameters of the compound Poisson process part in the Merton model are taken from Ballotta (2005); the diffusion coefficient σ is chosen so that the total instantaneous volatility of the process is equal to σ_A. The parameters of the VG process are then chosen accordingly. The base set of parameters is presented in Table 3; the resulting moments are instead reported in Table 4, together with the moments estimated via the Monte Carlo routines described in the previous section. In particular, we observe that the distributions of the asset log-returns generated by the two Lévy processes are both negatively skewed and leptokurtic, consistently with what discussed in section 3.1. Further, we also note that the VG process originates log-returns with the highest excess of kurtosis, despite the calibration operated on the parameters. In any case, the fact that the Lévy processes assign more probability mass to the tails of the log-returns distribution than the Brownian motion, implies that the prices for the embedded options calculated using these alternative models will be higher, especially if they are in-the-money or out-of-the-money. By the same argument, we expect similar implications on the capital requirements as well, since these quantities are directly involved with the tails of the log-returns distribution.

The mispricing of the benefits' prices are analysed in Table 5 for $\vartheta = 1$ only, since the leverage coefficient ϑ is a mere rescaling factor of the value of the benefits; equations (1) and

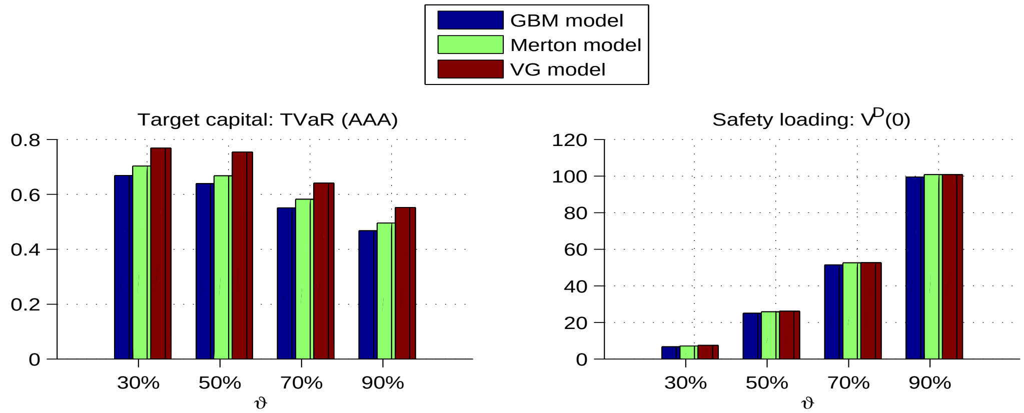

Figure 2: Contribution to the capital from the company's policyholders and shareholders for different levels of the leverage coefficient, ϑ. As ϑ increases, the policyholders become the major stakeholders of the company, and provide the largest share of the equity capital which is paid in form of the safety loading.

(3) in fact imply

$$\begin{aligned} P(T) &= \vartheta A(0)\left[\alpha \sum_{k=0}^{T-1}(1-\alpha)^k \prod_{t=1}^{T-k}\left(1+r_P(t)\right)+(1-\alpha)^T\right]=\vartheta P^U(T), \\ R(T) &= \vartheta\left(A(T)-P^U(T)\right)=\vartheta R^U(T). \end{aligned}$$

Therefore

$$V^P(0)=\vartheta V_U^P(0); \quad V^R(0)=\vartheta V_U^R(0);$$

where $V_U^{\cdot}$ denotes the "unlevered" option's value. The same argument though does not apply to the default option, as its value depends on the leverage coefficient as shown in Table 6; the mispricing generated by the model error is reported in Table 7.

Consistently with the previous considerations on the excess of kurtosis of the log-returns distribution, the results show that the standard asset model underprices each single component of the insurance contract, although the mispricing is particularly significant for the case of the terminal bonus (call) option and the default (put) option, in particular when the default option is deep out-of-the-money, i.e. for low values of ϑ. The mispricing is less severe when shocks are somehow incorporated in the model: although the VG process overprices out-of-the-money default options when compared to the Merton model, the price difference reduces sensibly as ϑ approaches 1. The observed mispricing is also reflected in the terminal bonus rate γ; for the benchmark set of parameters we obtain $\gamma = 14.26\%$ under the standard asset model, $\gamma = 18.19\%$ under the Merton process model, and $\gamma = 36.36\%$ under the VG model.

The RBC, as measured by the TVaR of the solvency index s_t defined in section 2.3, is presented in Figure 1 for different values of the leverage coefficient ϑ. The corresponding model error is reported in Table 8 for $\vartheta = 90\%$ (similar results are obtained for the other values of the parameter ϑ, and are available from the author). In particular, for this numerical

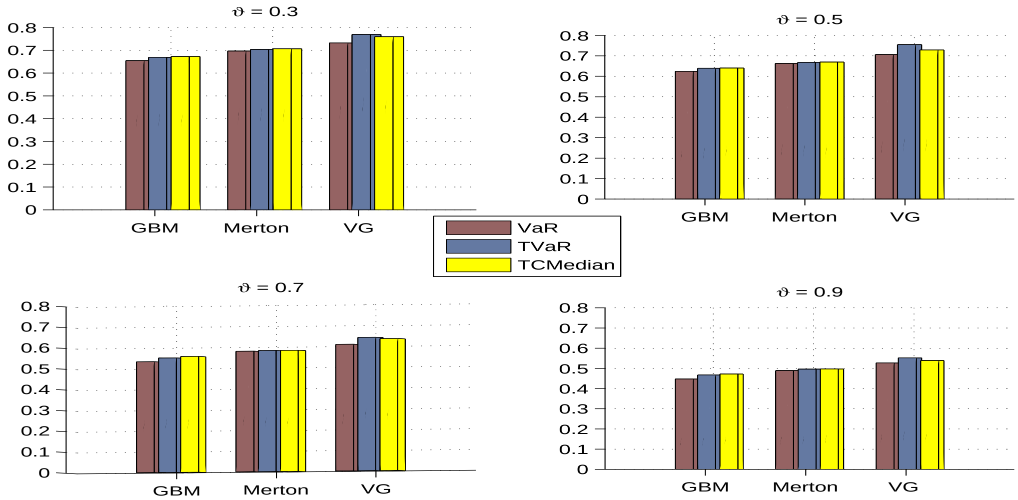

Figure 3: Comparison of solvency requirements based on different risk measures in correspondence of the AAA confidence level.

example, we consider the change in the RBC over 1 year after the inception of the contract, i.e.

$$TVaR\,(x;0,1) = -\mathbb{E}\left(s_1 \middle|\, s_1 \leq c_{s_1}(x;0,1)\right),$$

with

$$s_1 = \frac{\widetilde{RBC}\,(1) - A_0\,(1-\vartheta)}{A_0 + V^D(0)}.$$

The analysis can be easily extended to any two points in time over the lifetime of the contract (although the results would depend on the trajectory of the underlying fund).

Table 8 shows that, consistently with the findings related to the market consistent price, also in this case the standard model underestimates quite significantly the capital requirements; the inclusion of shocks in the model though reduces the magnitude of the error. Further, the mispricing becomes in general more significant in correspondence of higher confidence levels, x, which is a reflection of the different probability mass assigned by the three models to the tails of the distribution. Moreover, in Figure 1 we observe that, like the default option, the capital requirements change with the policyholder's contribution level to the reference fund of the insurance company. However, unlike the default option, the RBC decreases due to the "shift" in the ownership of the company occurring as ϑ increases. As the policyholders become the predominant stakeholders, they need to take over more of the burden of providing the capital required to offset the increased probability of default; as discussed in the previous sections, though, such a contribution comes in the form of the safety loading $V^D(0)$, rather than shareholders' equity. Figure 2 illustrates this shift for the numerical example implemented. We note that identical conclusions can be reached by using other risk measures like, for example, the VaR or the Tail Conditional Median, as shown in Figure 3.

6 Conclusions

We have developed a general framework for the market consistent pricing of insurance liabilities based on the fair value principle and the calculation of the corresponding capital requirements. In particular, we have used this framework to analyze the impact on these quantities of the inclusion in the model of market shocks. We conclude from the numerical results presented that the standard Black-Scholes economy underestimates the amount of the total liabilities and, more importantly, the value of the default option. This fact has therefore repercussions on the safety loadings and consequently on the size of the target capital. However, the mispricing is less severe when shocks are incorporated in the model, even if this might be done in a sub-optimal way.

It has to be noted that the analysis presented in this paper concerns the case in which default can only occur at maturity. Although this is a general situation in the UK, in a number of countries the supervisory authorities might impose more restrictive conditions and monitor the position of the insurance company regularly during the lifetime of the contract. In this case, if the market value of the assets is critically low during the lifetime of the contract, the regulatory authorities might decide to close down the company immediately and distribute the recovered wealth to stakeholders. A possible model based on diffusion processes for this situation of early default is provided for example by Grosen and Jørgensen (2002); the evidence presented in section 5 suggests that the inclusion of jumps in such a model would affect significantly the market value of the contract and therefore the corresponding capital requirements.

Further, the analyses are based on the assumption of constant interest rates; the techniques presented in this paper can be extended to incorporate a stochastic term structure, although this would prevent the derivation of closed form solutions for the price of the guaranteed benefit. This consideration highlights the importance of computationally efficient and accurate algorithms for the pricing of these contracts. In this note, we compare sequential and stratified Monte Carlo methods; the numerical procedures described in section 4 can be further improved by extending the use of low-discrepancy sequences to the intermediate points of the trajectory of the driving processes, although this implementation might be limited, for example, by the actual algorithm chosen to generate the beta random deviates.

An important open issue related to the implementation of fair valuation schemes in incomplete markets is the selection of the pricing measure. The analyses presented rely on the risk neutral Esscher measure; however, this might prove a quite restrictive approach since such a probability measure imposes a specific form of the investors' preferences. The lack of a market of derivative securities written on the contract reference portfolio, though, prevents the adoption of a more suitable market measure. Furthermore, the incompleteness of the market also means that the valuation framework has to take into account some non hedgeable financial risk. In this sense, perhaps it would be more appropriate to use the "best estimate plus risk margin" approach for the market consistent valuation of the liabilities; however, the full definition of the best estimate is still under discussion at regulatory level. Based on financial theory, a possible suggestion which we think might be appropriate to the task, could be interpreting the best estimate as the market price of a hedging strategy, whilst the risk margin would represent a protection against the inevitable hedging error. Finally, in this paper we assume a passive insurance company in terms of risk management of the policy; however, the capital requirements could result less onerous following the implementation of a suitable investment strategy. In this respect, we note the relevance of the model error as it could affect, as noted above, the magnitude of the hedging error. Investigation of these approaches is left for future research.

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