The representation theory of diagram algebras

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Contents

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Abstract

In this thesis we study the modular representation theory of diagram algebras, in particular the Brauer and partition algebras, along with a brief consideration of the Temperley-Lieb algebra. The representation theory of these algebras in characteristic zero is well understood, and we show that it can be described through the action of a reflection group on the set of simple modules (a result previously known for the Temperley-Lieb and Brauer algebras). By considering the action of the corresponding affine reflection group, we give a characterisation of the (limiting) blocks of the Brauer and partition algebras in positive characteristic. In the case of the Brauer algebra, we then show that simple reflections give rise to non-zero decomposition numbers.

We then restrict our attention to a particular family of Brauer and partition algebras, and use the block result to determine the entire decomposition matrix of the algebras therein.

Introduction

Schur-Weyl duality

All of the algebras considered in this thesis appear in some generalisation of the classical Schur-Weyl duality. In 1927, Schur [Sch27] proved that the general linear group $\mathrm{GL}_m(\mathbb{C})$ and the group algebra of the symmetric group $\mathbb{C}\mathfrak{S}_n$ can be related through their action on a tensor space in the following way. Let E be a complex vector space of dimension m, then $\mathrm{GL}_m(\mathbb{C})$ acts naturally on E on the left by matrix multiplication. By extending this action diagonally we see that the n-fold tensor product $E^{\otimes n}$ is a left $\mathrm{GL}_m(\mathbb{C})$-module. We also have a right-action of $\mathfrak{S}_n$ on $E^{\otimes n}$ by place permutations. By extending both of these actions linearly to the corresponding group algebras we obtain homomorphisms

$$\begin{aligned} \alpha : \mathbb{C}\mathrm{GL}_m(\mathbb{C}) &\longrightarrow \mathrm{End}_{\mathbb{C}}(E^{\otimes n}) \\ \beta : \quad \mathbb{C}\mathfrak{S}_n \quad &\longrightarrow \mathrm{End}_{\mathbb{C}}(E^{\otimes n}) \end{aligned}$$

satisfying the following double-centraliser property.

$$\begin{aligned} \mathrm{Im}(\alpha) &= \mathrm{End}_{\mathbb{C}\mathfrak{S}_n}(E^{\otimes n}) \\ \mathrm{Im}(\beta) &= \mathrm{End}_{\mathbb{C}\mathrm{GL}_m(\mathbb{C})}(E^{\otimes n}). \end{aligned}$$

That is, the images of α and β consist precisely of those $\mathbb{C}$-endomorphisms of $E^{\otimes n}$ which commute with the symmetric group $\mathfrak{S}_n$ and the general linear group $\mathrm{GL}_m(\mathbb{C})$ respectively. Schur proved that there is an equivalence between the module categories $\mathrm{Im}(\alpha)$-**mod** and $\mathrm{Im}(\beta)$-**mod**, provided by the $(\mathbb{C}\mathrm{GL}_m(\mathbb{C}), \mathbb{C}\mathfrak{S}_n)$-bimodule $E^{\otimes n}$. This phenomenon is known as (classical) Schur-Weyl duality.

The Brauer algebra $B_n(m)$ was introduced by Brauer in [Bra37] to provide a corresponding result when replacing the general linear group by its orthogonal subgroup $\mathrm{O}_m(\mathbb{C})$. Also defined was the algebra $B_n(-m)$ for even m, obtained by replacing the general linear group by its symplectic subgroup $\mathrm{Sp}_m(\mathbb{C})$. It follows from this commuting action that the symmetric group algebra $\mathbb{C}\mathfrak{S}_n$ appears as a subalgebra of $B_n(m)$ and $B_n(-m)$.

The partition algebra was originally defined by Martin in [Mar94] over $\mathbb{C}$ as a generalisation of the Temperley-Lieb algebra (see below for a definition of this) for m-state n-site Potts models in statistical mechanics, and independently by Jones

[Jon94]. However we can also view it in the setting of Schur-Weyl duality in the following way. The subgroup of $\mathrm{O}_m(\mathbb{C})$ of permutation matrices is isomorphic to $\mathfrak{S}_m$, so we have an inherited left-action of $\mathfrak{S}_m$ on $E^{\otimes n}$. We can then ask which algebra has a right-action on $E^{\otimes n}$ which satisfies the double-centraliser property. The resulting algebra is the partition algebra $P_n(m)$. Again by the commuting action we see that the Brauer algebra $B_n(m)$ (and hence the symmetric group algebra $\mathbb{C}\mathfrak{S}_n$) appears as a subalgebra of $P_n(m)$.

The connection between these algebras via the tensor space is demonstrated in Figure 1.

$$\begin{tikzcd}
\mathrm{GL}_m(\mathbb{C}) \arrow[ddr] & & \mathbb{C}\mathfrak{S}_n \arrow[ddl] \\
\cup & & \cap \\
\mathrm{O}_m(\mathbb{C}) \arrow[r] & E^{\otimes n} & B_n(m) \arrow[l] \\
\cup & & \cap \\
\mathfrak{S}_m \arrow[uur] & & P_n(m) \arrow[uul]
\end{tikzcd}$$

Figure 1: The relations between algebras under Schur-Weyl duality.

The Temperley-Lieb algebra, introduced in [TL71] in the context of transfer matrices in statistical mechanics, also appears in a Schur-Weyl duality setting. Denote by V the 2-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, then there exists an action of the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on the $(n+1)$-fold tensor product $V^{\otimes n+1}$. This action shares the above double-centraliser property with the Temperley-Lieb algebra $TL_n(q+q^{-1})$, see [Mar92] for further details and also Figure 2.

$$U_q(\mathfrak{sl}_2(\mathbb{C})) \longrightarrow V^{\otimes n+1} \longleftarrow TL_n(q+q^{-1})$$

Figure 2: The Schur-Weyl duality setting of the Temperley-Lieb algebra.

A diagram basis

The above description of these algebras imposes certain restrictions. For instance in the case of the Brauer algebra, its action on a tensor product of m-dimensional vector spaces forces m to be a positive integer. However it is possible to define the algebras

$B_n(\delta)$, $P_n(\delta)$ and $TL_n(\delta)$ over any commutative ring R and any parameter $\delta \in R$. This definition places them in a class of algebras called *diagram algebras*, and it is in this setting that we study them here.

We again begin with the symmetric group. We can represent each permutation $\sigma \in \mathfrak{S}_n$ as a diagram in the following way: take two rows of n nodes (called northern and southern) and connect the i-th northern node to the j-th southern node by an arc if the permutation σ sends i to j. The product of two permutations is obtained by placing one diagram on top of the other, identifying central nodes, and following arcs through. For a commutative ring R, we can then form the group algebra $R\mathfrak{S}_n$ by taking linear combinations of these diagrams. See Figure 3(i) for an example.

We now form the Brauer algebra by again using diagrams consisting of two rows of n nodes, but relaxing the rule for joining them by arcs. We could previously only join northern and southern nodes, but now we allow an arc to join any two distinct nodes, so that each node is the end point of precisely one arc (see Figure 3(ii)). Multiplication is again by concatenation, except it is now possible to introduce central loops, not joined to any node. We remove these loops and multiply the resulting diagram by a power of a chosen parameter $\delta \in R$ (this is made precise in Chapter 3) . By taking linear combinations of diagrams we thus form the Brauer algebra $B_n(\delta)$. A precise defintion is given in Chapter 3.

The Temperley-Lieb algebra $TL_n(\delta)$ can now be realised as a subalgebra of the Brauer algebra $B_n(\delta)$, generated by those diagrams which are planar when arcs are drawn within the rectangle defined by the nodes. See Figure 3(iii) for an example of a planar diagram.

The partition algebra $P_n(\delta)$ is formed by again relaxing the rule for joining nodes by arcs, in particular by removing the restriction that each node is the end point of precisely one arc. This allows for multiple nodes to be connected, or indeed for a node to be disconnected from all others. Each diagram now has several equivalent forms, which we will identify. The multiplication rule is again given by concatenating diagrams, and we give a precise definition in Chapter 4. An example is given in Figure 3(iv).

It is now clear that we have the following inclusions of algebras.

$$TL_n(\delta) \subset B_n(\delta) \subset P_n(\delta).$$

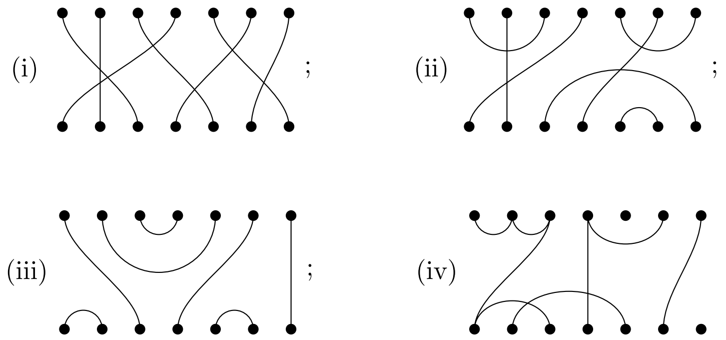


Figure 3: An example of a diagram in: (i) the symmetric group; (ii) the Brauer algebra; (iii) the Temperley-Lieb algebra; and (iv) the partition algebra.

Cellularity

Much of the study of diagram algebras exploits their property of being cellular algebras. It was proved in [GL96] that the Temperley-Lieb and Brauer algebras are cellular. König and Xi proved in [KX99] that cellular algebras can be realised as iterated inflations of smaller algebras, leading to a proof of the cellularity of the partition algebra in [Xi99]. Moreover, diagram algebras can be studied more generally through the concept of cellular stratification from [HHKP10] and in certain cases through the theory of towers of recollement from [CMPX06]. In the case of the Brauer and partition algebras, this allows us to lift results from the representation theory of the symmetric group and Brauer or partition algebras of smaller size.

A detailed history of the study of the individual algebras is given in Section 1 of the corresponding chapters.

Structure of this thesis

Chapter 1 compiles the necessary background results from representation theory, in particular the theory of cellular algebras and the modular representation theory of the symmetric group.

In Chapter 2 we study the representation theory of the Temperley-Lieb algebra. This is a very well understood topic, and we do not present any original work in this chapter. Instead, we reformulate some established results into the setting and language used in the rest of this thesis. This serves to not only provide context in the study of diagram algebras, but also to collect together in one place some results that have previously been spread across many papers. We provide a formal definition of the Temperley-Lieb algebra and recall its structure as a cellular algebra. We then discuss the representation theory in characteristic zero, and provide a geometric characterisation of this. We then consider the modular representation theory, again with a geometric view. We end this section with a brief consideration of the cases $p = 2$ and $\delta = 0$.

Chapter 3 deals with the representation theory of the Brauer algebra. We begin with the definitions and the cellular structure of the algebra, before describing the (previously known) representation theory in characteristic zero. We recall both the combinatorial and geometric description of the blocks. We then move on to the modular representation theory and extend the known partial block result to a characterisation of the limiting blocks of the Brauer algebra. Using the methods developed for this result, we show that we can determine some composition factors of cell modules in general. By restricting to a family of Brauer algebras, we then show that we can in fact describe the entire decomposition matrix. Finally, we again discuss the cases $p = 2$ and $\delta = 0$.

In Chapter 4 we move on to the partition algebra. Again, we begin with a definition of the algebra and its cellular structure. We present a combinatorial description of the representation theory in characteristic zero, and then provide a geometric reformulation of this. We then extend this geometric description to give a characterisation of the blocks of the partition algebra in positive characteristic, as well as a description of the limiting blocks using only results from characteristic zero. We next restrict our attention to a family of partition algebras and provide a complete description of the decomposition matrices in this case. Again, we end with a discussion of the cases $p = 2$ and $\delta = 0$.

Chapters 2, 3 and 4 can be read independently of each other, and as such only assume knowledge of Chapter 1. Because of this, we recycle notation chapter to chapter to not only improve legibility, but to emphasise the similarities between the algebras and their representation theory. An index of notation is provided on page 12.

Index of notation

Chapter 1

Background

1.1 Framework

We fix a prime number $p > 2$ and a p-modular system (K, R, k). That is, R is a discrete valuation ring with maximal ideal $P = (\pi)$, field of fractions $\mathrm{Frac}(R) = K$ of characteristic 0, and residue field $k = R/P$ of characteristic p. We will use $\mathbb{F}$ to denote either K or k.

We also fix a parameter $\delta \in R$ and assume that its image in k is non-zero. We will use δ to denote both the element in R and its projection in k.

For any $\mathbb{F}$-algebra A, there exists a unique decomposition of A into a direct sum of subalgebras

$$A = \bigoplus_i A_i$$

where each A_i is indecomposable as an algebra. We call the A_i the *blocks* of A. For any A-module M there exists a similar decomposition

$$M = \bigoplus_i M_i$$

where for each i,

$$A_i M_i = M_i, \quad A_j M_i = 0 \ (\forall j \neq i).$$

We say that the module M_i *lies in the block* A_i. Clearly, each simple A-module must lie in precisely one block.

Now let A be an R-algebra, free and of finite rank as an R-module. We can extend scalars to produce the K-algebra $KA = K \otimes_R A$ and the k-algebra $kA = k \otimes_R A$. Given an A-module X, we can then also consider the KA-module $KX = K \otimes_R X$ and the kA-module $kX = k \otimes_R X$.

In what follows, we will use the terms K-block and k-block to indicate that we are considering the blocks of the algebras KA and kA respectively.

Throughout this thesis we will use existing knowledge of modules over K to determine corresponding facts about modules over k. We begin by defining the reduction of a KA-module to k.

Let A be an R-algebra, and M be a finite dimensional KA-module. We can then find an integer $r \geq 0$ such that $\pi^r M$ is an A-module, and reduce this to k to obtain $k \otimes_R \pi^r M$ (see [CR81, Chapter 16] for details). Note that the choice of r is not unique, and thus we can obtain many different "reductions" of the module M. However the following proposition shows that the composition factors of the resulting modules are independent of the choice of r.

Proposition 1.1 ([CR81, Proposition 16.16])**.** *Let A be an R-algebra, M a finite dimensional KA-module, and $r_1, r_2 \in \mathbb{Z}$ such that both $\pi^{r_1} M$ and $\pi^{r_2} M$ are A-modules. Then the kA-modules $k \otimes_R \pi^{r_1} M$ and $k \otimes_R \pi^{r_2} M$ have the same composition factors.*

This is sufficient for our purposes, and we therefore make the following definition.

Definition 1.2. Let A be an R-algebra, and M be a finite dimensional KA-module. We define the *reduction* of M to k to be the module $\overline{M}$ obtained in the following way. Choose an integer $r \geq 0$, minimal such that $\pi^r M$ is an A-module (this can always be done as M is finite dimensional). Then define $\overline{M} = k \otimes_R \pi^r M$.

Not only can we reduce a module from K to k, but we can also reduce K-module homomorphisms.

Lemma 1.3. *Suppose X, Y are R-free A-modules of finite rank and let $M \subseteq KY$. If $\mathrm{Hom}_K(KX, KY/M) \neq 0$ then there is a submodule $N \subseteq kY$ such that $\mathrm{Hom}_k(kX, kY/N) \neq 0$. Moreover, N is obtained by reducing the module M to k as described above.*

Proof. Let $Q = KY/M$ be the image of the canonical quotient map $\rho : KY \to KY/M$, and let $f \in \mathrm{Hom}_K(KX, Q)$ be non-zero. As X and Y are modules of finite rank we may assume that

$$f(X) \subseteq \rho(Y) \quad \text{but} \quad f(X) \nsubseteq \pi\rho(Y) \tag{1.1}$$

for instance by considering the matrix of f and multiplying the coefficients by an appropriate power of π. Then f restricts to a homomorphism $X \to \rho(Y)$, and induces a homomorphism $\overline{f} : kX \to k\rho(Y)$. This must be non-zero since we can find $x \in X$ such that $f(x) \in \rho(Y)\backslash\pi\rho(Y)$ by (1.1).

It remains to prove that $k\rho(Y)$ can be taken to be kY/N for some $N \subset kY$, obtained by reducing the module M to k. We have the following maps:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & KY \arrow[r, two heads, "\rho"] & Q & KX \arrow[l, "f"'] \\
 & & \cup & \cup & \cup \\
0 \arrow[r] & L \arrow[r] & Y \arrow[r, two heads] \arrow[d] & \rho(Y) \arrow[d] & X \arrow[d] \\
 & & kY \arrow[r, two heads] & k\rho(Y) & kX \arrow[l, two heads, "\bar{f}"']
\end{tikzcd}$$

where $L = \mathrm{Ker}(Y \longrightarrow \rho(Y))$.

The K-module Q is torsion free. Therefore as an R-module, $\rho(Y) \subseteq Q$ must also be torsion free. Since R is a principal ideal domain (by definition of it being a discrete valuation ring), the structure theorem for modules over a principal ideal domain tells us that $\rho(Y)$ must be free. It is therefore projective, and the exact sequence

$$0 \longrightarrow L \longrightarrow Y \longrightarrow \rho(Y) \longrightarrow 0$$

is split. Then since the functors $K \otimes_R -$ and $k \otimes_R -$ preserve split exact sequences, we deduce that $M \cong KL$ and we can set $N = kL$ to complete the exact sequence

$$0 \longrightarrow N \longrightarrow kY \longrightarrow k\rho(Y) \longrightarrow 0.$$

□

1.2 Cellular algebras

The algebras we will deal with are all examples of cellular algebras. These were originally defined by Graham and Lehrer [GL96] in terms of a cellular basis. However we will present here an equivalent definition, provided by König and Xi in [KX98].

Definition 1.4 ([KX98, Definition 3.2])**.** Let A be an R-algebra where R is a commutative Noetherian integral domain. Assume there is an anti-automorphism i on A with $i^2 = \mathrm{id}_A$. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A-bimodules $\alpha : J \xrightarrow{\sim} \Delta \otimes_R i(\Delta)$ making the following diagram commutative:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta). \end{tikzcd}$$

The algebra A (with the involution i) is called *cellular* if and only if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J'_l$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The ideals $\Delta \subset J'_j$ of a cellular algebra are called *cell modules.* We label each cell module by the corresponding index of the cell ideal it lies in, and let Λ be the set of all such labels. We then give a partial order $\leq$ on Λ by defining $\mu \leq \lambda$ ($\mu, \lambda \in \Lambda$) if the cell module labelled by μ lies in a lower layer of the filtration than λ. We use the notation Δ_λ to mean the cell module indexed by $\lambda \in \Lambda$.

We now specialise to the case where A is an $\mathbb{F}$-algebra, with $\mathbb{F}$ a field. There is a subset $\Lambda^* \subset \Lambda$ such that for each $\lambda \in \Lambda^*$, the cell module Δ_λ has a simple head, denoted L_λ. Moreover, the modules L_λ with $\lambda \in \Lambda^*$ form a complete set of non-isomorphic simple A-modules.

The following proposition compiled from [GL96, Section 3] provides important results about the representation theory of cellular algebras, and will be used extensively in what follows.

Proposition 1.5. *Let A be a cellular algebra over a field $\mathbb{F}$. Then* *(i) Each cell module Δ_μ of A has a composition series with quotients isomorphic to L_λ (some set of $\lambda \in \Lambda^*$). The multiplicity of L_λ is the same in any composition series of Δ_μ and we write $d_{\lambda\mu} = [\Delta_\mu : L_\lambda]$ for this multiplicity.*

(ii) The decomposition matrix $\mathbf{D} = (d_{\lambda\mu})_{\lambda\in\Lambda^*,\mu\in\Lambda}$ is upper unitriangular, i.e. $d_{\lambda\mu} = 0$ unless $\mu \leq \lambda$ and $d_{\lambda\lambda} = 1$.

(iii) Let $\lambda, \mu \in \Lambda$. We say λ, μ are cell-linked *if $\lambda \in \Lambda^*$ and L_λ is a composition factor of Δ_μ. The classes of the equivalence relation of Λ generated by this relation are called* cell-blocks. *The intersection of a cell-block with Λ^* corresponds to an $\mathbb{F}$-block in the sense of Section 1.1.*

1.3 The symmetric group

We denote by $\mathfrak{S}_n$ the symmetric group on n letters, and by $R\mathfrak{S}_n$ the corresponding group algebra over the ring R.

1.3.1 Partitions

Given a natural number n, we define a partition $\lambda = (\lambda_1, \lambda_2, ...)$ of n to be a sequence of non-negative integers such that

(i) $\lambda_i \geq \lambda_{i+1}$ for all i

(ii) $\sum_{i>0} \lambda_i = n$.

These conditions imply that $\lambda_i = 0$ for $i \gg 0$, hence we will often truncate the sequence and write $\lambda = (\lambda_1, \dots, \lambda_l)$, where $\lambda_l \neq 0$ and $\lambda_{l+1} = 0$. We also combine repeated entries and use exponents, for instance the partition $(5, 5, 3, 2, 1, 1, 0, 0, 0, \dots)$ of 17 will be written $(5^2, 3, 2, 1^2)$.

We use the notation $\lambda \vdash n$ to mean λ is a partition of n.

We say that a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is *p-singular* if there exists t such that

$$\lambda_t = \lambda_{t+1} = \cdots = \lambda_{t+p-1} > 0,$$

i.e. some (non-zero) part of λ is repeated p or more times. Partitions that are not p-singular we call *p-regular*.

To each partition λ we may associate the Young diagram

$$[\lambda] = \{(x, y) \mid x, y \in \mathbb{Z},\ 1 \le x \le l,\ 1 \le y \le \lambda_x\}.$$

An element (x, y) of $[\lambda]$ is called a *node*. If $\lambda_{i+1} < \lambda_i$, then the node (i, λ_i) is called a *removable* node of λ. The set of removable nodes of a partition λ is denoted $\mathrm{rem}(\lambda)$. If $\lambda_{i-1} > \lambda_i$, then we say the node $(i, \lambda_i + 1)$ of $[\lambda] \cup \{(i, \lambda_i + 1)\}$ is an *addable* node of λ. The set of addable nodes of a partition λ is denoted $\mathrm{add}(\lambda)(\lambda)$. This is illustrated in Figure 1.1 below. If a partition μ is obtained from λ by removing a removable (resp. adding an addable) node $A = (x, y)$ then we write $\mu = \lambda - A$ (resp. $\mu = \lambda + A$).

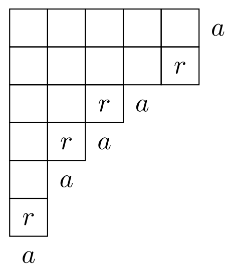


Figure 1.1: The Young diagram of $\lambda = (5^2, 3, 2, 1^2)$. Removable nodes are marked by r and addable nodes by a.

The set of partitions obtained by adding or removing a node from λ is called the *support* of λ, denoted

$$\mathrm{supp}(\lambda) = \{\lambda - A : A \in \mathrm{rem}(\lambda)\} \cup \{\lambda + A : A \in \mathrm{add}(\lambda)\}. \tag{1.2}$$

Each node (x, y) of $[\lambda]$ has an associated integer, called the *content*, given by $y - x$. We write

$$\mathrm{ct}(\lambda) = \sum_{(x,y) \in [\lambda]} (y - x). \tag{1.3}$$

By interchanging the rows and columns of the Young diagram $[\lambda]$ we obtain the diagram of a partition λ^T, which we call the *transposed* partition of λ. Using Figure 1.1 as an example, the transposed partition of $\lambda = (5^2, 3, 2, 1^2)$ is $\lambda^T = (6, 4, 3, 2^2)$.

Given two partitions λ, μ, we say that $\mu \subset \lambda$ if $[\mu] \subset [\lambda]$. In this case, we also define the *skew-partition* λ/μ to have Young diagram $[\lambda]/[\mu]$.

We will occasionally need to consider the set of all partitions, which we denote by Λ. We also let Λ^* be the set of all p-regular partitions.

There exists a partial order on the set Λ called the *dominance order with size*, denoted by $\trianglelefteq$. We say a partition λ is less than or equal to μ under this order if either $|\lambda| < |\mu|$, or $|\lambda| = |\mu|$ and $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$ for all $j \geq 1$. We write $\lambda \triangleleft \mu$ to mean $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.

1.3.2 Specht modules

The group algebra $R\mathfrak{S}_n$ is a cellular algebra, as shown in [GL96]. The cell modules are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, and are more commonly known as *Specht modules*. We denote the Specht module indexed by λ by S_R^λ, and outline their construction below. Full details can be found in [Jam78, Chapters 3,4].

A λ-*tableau* is a bijection $t : \{1, \ldots, n\} \longrightarrow [\lambda]$. We can also think of this as numbering the nodes of $[\lambda]$ with 1 to n. The group $\mathfrak{S}_n$ acts on the set of λ-tableaux by composition, $\sigma t = t \circ \sigma$, or equivalently by permuting the labelled nodes of $[\lambda]$. To each λ-tableau t there are subgroups R_t and C_t, the row stabiliser and column stabiliser respectively, of $\mathfrak{S}_n$, defined as follows:

$$R_t = \{\sigma \in \mathfrak{S}_n : \sigma \text{ fixes setwise the rows of } t\},$$

$$C_t = \{\sigma \in \mathfrak{S}_n : \sigma \text{ fixes setwise the columns of } t.\}$$

We use R_t to define an equivalence relation $\sim$ on the set of λ-tableaux:

$$t \sim t' \iff \text{there exists } \sigma \in R_t \text{ such that } t' = \sigma t.$$

The equivalence classes $\{t\}$ are called λ-*tabloids*, and $\mathfrak{S}_n$ acts on these by $\sigma\{t\} = \{\sigma t\}$. This action allows us to define the $R\mathfrak{S}_n$-module M_R^λ, spanned by λ-tabloids.

To each λ-tableau we also associate a λ-*polytabloid*, defined as the sum

$$e_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma\{t\}.$$

The cell (or Specht) module S_R^λ is then defined as the set of R-linear combinations of λ-polytabloids. It can be shown that $\sigma e_t = e_{\sigma t}$ for $\sigma \in \mathfrak{S}_n$, so this does indeed define an $R\mathfrak{S}_n$-module.

1.3.3 Representation theory of $\mathfrak{S}_n$

We now turn our attention to the structure of the Specht modules over a field. Recall that R is a discrete valuation ring with maximal ideal P, K is the field of fractions of characteristic zero, and k is the residue field R/P of characteristic p. We use $\mathbb{F}$ to denote either K or k. James then provides the following theorem:

Theorem 1.6 ([Jam78, Theorem 4.12]). *The Specht modules over K are self-dual and absolutely irreducible, and give all the ordinary irreducible representations of $\mathfrak{S}_n$.*

Recall the module $M_{\mathbb{F}}^{\lambda}$, spanned by λ-tabloids. There exists a symmetric, bilinear, $\mathfrak{S}_n$-invariant form on $M_{\mathbb{F}}^{\lambda}$, generated by the relations

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\} \\ 0 & \text{otherwise.} \end{cases}$$

We then define the module

$$D_{\mathbb{F}}^{\lambda} = S_{\mathbb{F}}^{\lambda} / (S_{\mathbb{F}}^{\lambda} \cap S_{\mathbb{F}}^{\lambda \perp}),$$

where $S_{\mathbb{F}}^{\lambda\perp}$ is the module $\{x \in M_{\mathbb{F}}^{\lambda} \mid \langle x, y \rangle = 0 \text{ for all } y \in S_{\mathbb{F}}^{\lambda}\}$. These then form the irreducible representations of $k\mathfrak{S}_n$, as shown below.

Theorem 1.7 ([Jam78, Theorem 11.5]). *As μ varies over p-regular partitions of n, D_k^{μ} varies over a complete set of inequivalent irreducible $k\mathfrak{S}_n$-modules.*

Each block of $K\mathfrak{S}_n$ contains a single Specht module, and so in characteristic 0 the decomposition matrix is trivial. In characteristic p a complete description of the decomposition matrix of $k\mathfrak{S}_n$ is yet to be found, but the blocks are known. A combinatorial approach to this is presented below.

The *hook* $h_{(x,y)}$ corresponding to the node (x, y) in the Young diagram is the subset

$$h_{(x,y)} = \{(i, y) \in [\lambda] \mid i \geq x\} \cup \{(x, j) \in [\lambda] \mid j \geq y\}$$

consisting of (x, y) and all the nodes either below or to the right of it, see Figure 1.2 for an example. A p-hook is a hook containing p nodes.

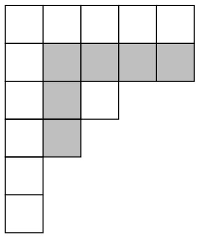


Figure 1.2: The Young diagram of $\lambda = (5^2, 3, 2, 1^2)$ with the hook $h_{(2,2)}$ shaded.

Each hook corresponds to a *rim hook*, obtained by moving each node of the hook down and to the right so that it lies on the edge of the Young diagram. An example is given in Figure 1.3.

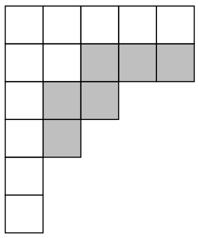


Figure 1.3: The rim hook corresponding to $h_{(2,2)}$. The node $(2, 2)$ has moved to the rim.

Given a partition λ we can successively remove rim p-hooks until we have reached a point that we can remove no more. What remains is called the *p-core* of λ, and the number of rim p-hooks removed to reach this is called the *p-weight*. It is shown in [JK81, Chapter 2.7] that the p-core is independent of the order in which we remove such hooks, and therefore both these notions are well-defined. We now state a useful theorem:

Theorem 1.8 (Nakayama's Conjecture)**.** *Two partitions λ and μ label Specht modules in the same k-block for the symmetric group algebra if and only if they have the same p-core and p-weight.*

Proof. See [JK81, Chapter 6]. $\square$

1.3.4 The abacus

Following [JK81, Chapter 2.7], we can associate to each partition an abacus diagram. This will consist of p columns, known as runners, and a configuration of beads across these. By convention we label the runners from left to right, starting with 0, and the positions on the abacus are also numbered from left to right, working down from the top row, starting with 0 (see Figure 1.4). Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n, fix a positive integer $b \geq n$ and construct the β-sequence of λ, defined to be

$$\beta(\lambda, b) = (\lambda_1 - 1 + b, \lambda_2 - 2 + b, \ldots, \lambda_l - l + b, -(l+1) + b, \ldots 2, 1, 0).$$

Then place a bead on the abacus in each position given by $\beta(\lambda, b)$, so that there are a total of b beads across the runners. Note that for a fixed value of b, the abacus is uniquely determined by λ, and any such abacus arrangement corresponds to a partition simply by reversing the above. Here is an example of such a construction:

Example 1.9. In this example we will fix the values $p = 5, n = 9, b = 10$ and represent the partition $\lambda = (5, 4)$ on the abacus. Following the above process, we first calculate the β-sequence of λ:

$$\begin{aligned}\beta(\lambda, 10) &= (5 - 1 + 10,\ 4 - 2 + 10,\ -3 + 10,\ -4 + 10, \ldots,\ -9 + 10,\ -10 + 10) \\ &= (14, 12, 7, 6, 5, 4, 3, 2, 1, 0).\end{aligned}$$

The next step is to place beads on the abacus in the corresponding positions. We also number the beads, so that bead 1 occupies position $\lambda_1 - 1 + b$, bead 2 occupies position $\lambda_2 - 2 + b$ and so on. The labelled spaces and the final abacus are shown below.

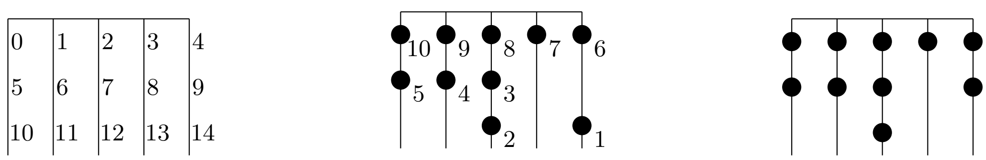


Figure 1.4: The positions on the abacus with 5 runners, the arrangement of beads (numbered) representing $\lambda = (5, 4)$, and the corresponding 5-core.

After fixing values of p and b, we will abuse notation and write λ for both the partition and the corresponding abacus with p runners and b beads. We may also then define $\Gamma(\lambda, b) = (\Gamma(\lambda, b)_0, \Gamma(\lambda, b)_1, \dots, \Gamma(\lambda, b)_{p-1})$, where

$$\Gamma(\lambda, b)_i = \left|\{j : \beta(\lambda, b)_j \equiv i \pmod{p}\}\right| \tag{1.4}$$

so that $\Gamma(\lambda, b)$ records the number of beads on each runner of the abacus of λ.

We now have alternative descriptions of certain actions we can perform on partitions. One such result given in [JK81, Chapter 2.7] is as follows: Sliding a bead down (resp. up) one space on its runner corresponds to adding (resp. removing) a rim p-hook to the Young diagram of the partition. Therefore by sliding all beads up their runners as far as they will go, we remove all rim p-hooks and arrive at the p-core (an example is shown in Figure 1.4). Therefore two partitions have the same p-core if and only if the number of beads on corresponding runners of the two abaci is the same. Therefore as in [JK81, Chapter 2.7] we may re-state Nakayama's conjecture (Theorem 1.8) as:

Theorem. *Two partitions λ, μ of n label Specht modules in the same k-block for the symmetric group algebra if and only if when represented on an abacus with p runners and b beads, the number of beads on corresponding runners is the same (i.e. $\Gamma(\lambda, b) = \Gamma(\mu, b)$).*

Chapter 2

The Temperley-Lieb algebra

2.1 Background

The Temperley-Lieb algebra $TL_n(\delta)$ appears in many different areas of mathematics. Originally introduced by Temperley and Lieb in [TL71] in the context of transfer matrices in lattice models, it appears also in the study of the Potts model and other areas of statistical mechanics. Jones [Jon83] later applied the algebra to knot theory, and defined what is now called the Jones polynomial. It is also found in the realm of representation theory as a quotient of the Hecke algebra of type A.

One of the first investigations into the representation theory of the Temperley-Lieb algebra itself was done by Martin in [Mar91], including the construction of certain primitive idempotents. Goodman and Wenzl [GW93] independently studied these idempotents, leading to a characterisation of the blocks of the Temperley-Lieb algebra $TL_n(\delta)$ in characteristic zero for $\delta \neq 0$. Westbury [Wes95] continued studying the Temperley-Lieb algebra and found a necessary and sufficient condition for there to exist a homomorphism between so called standard modules. Graham and Lehrer later formulated the notion of a cellular algebra [GL96] and proved that the Temperley-Lieb algebra is an example of this, thus showing the existence of a set of modules called cell modules. These cell modules are equal to the standard modules studied by Westbury, and therefore we obtain the decomposition matrix of the Temperley-Lieb algebra in characteristic zero. The decomposition matrix in positive characteristic was found in [CGM03] through the use of Ringel duality. In this chapter we restate these known results in a geometric form, consistent with the rest of this thesis.

2.2 Preliminaries

2.2.1 Definitions

Recall from Section 1.1 the definitions and conventions for the prime p, the p-modular system (K, R, k), the field $\mathbb{F}$ and the parameter $\delta \in R$. There is a consideration of the cases $p = 2$ and $\delta = 0$ in Section 2.5.

For a fixed $n \in \mathbb{N}$, an (n, n)-*Temperley-Lieb diagram* is a diagram formed of $2n$ nodes, arranged in two rows of n, with n arcs between them so that each node is joined to precisely one other. We also impose the condition that all arcs must be drawn inside the rectangle defined by the nodes, and futhermore within this rectangle no two arcs may cross. We call such diagrams *planar*. An example is given in Figure 2.1.

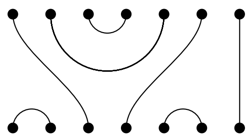


Figure 2.1: A $(7, 7)$-Temperley-Lieb diagram.

For a fixed $\delta \in R$ we then define the *Temperley-Lieb algebra* $TL_n^R(\delta)$ be the set of linear combinations of (n, n)-Temperley-Lieb diagrams. The product $x \cdot y$ of two such diagrams is found by concatenating x and y in the following way: place x on top of y and identify the central nodes. This forms a new planar diagram (since crossings cannot be formed if both diagrams are themselves planar), possibly with a number of closed loops in the centre. We remove these loops and multiply the resulting diagram by a power of δ, the multiplicity being the number of loops removed. Figure 2.2 shows an example of this multiplication.

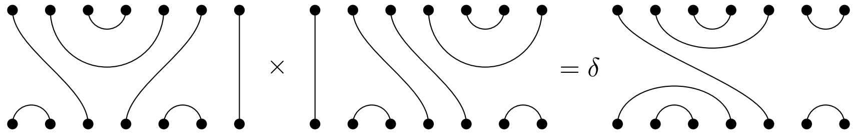
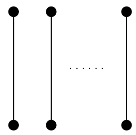
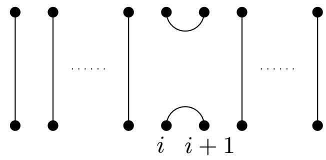


Figure 2.2: Multiplication of two diagrams in $TL_7^R(\delta)$.

Note that multiplication in $TL_n^R(\delta)$ cannot increase the number of propagating arcs (ones that join a northern and southern node). Let $J_n^{(r)}$ be the ideal generated by diagrams with at most r propagating arcs. We then have a filtration of $TL_n^R(\delta)$ by the $J_n^{(r)}$:

$$
\begin{aligned}
&0 \subset J_n^{(0)} \subset J_n^{(2)} \subset \dots J_n^{(n-2)} \subset J_n^{(n)} = TL_n^R(\delta) \qquad (n \text{ even}),\\
&0 \subset J_n^{(1)} \subset J_n^{(3)} \subset \dots J_n^{(n-2)} \subset J_n^{(n)} = TL_n^R(\delta) \qquad (n \text{ odd}).
\end{aligned} \tag{2.1}
$$

We will make note of the following elements in $TL_n^R(\delta)$ $(1 \leq i < n)$

$1 =$  , $\qquad u_i =$  .

The diagram 1 is the identity element of $TL_n^R(\delta)$, and the u_i satisfy the relations

$$
\begin{aligned}
u_i^2 &= \delta u_i,\\
u_i u_{i\pm 1} u_i &= u_i,\\
u_i u_j &= u_j u_i \;\text{ if } |i-j| > 1.
\end{aligned}
$$

It was shown in [GdHJ89, Lemma 2.8.4] that these elements generate $TL_n^R(\delta)$.

By extension of scalars we may define the algebras

$$TL_n^K(\delta) = K \otimes_R TL_n^R(\delta) \qquad \text{and} \qquad TL_n^k(\delta) = k \otimes_R TL_n^R(\delta).$$

2.2.2 Cellularity of $TL_n^{\mathbb{F}}(\delta)$

Graham and Lehrer showed in [GL96] that for any field $\mathbb{F}$, the Temperley-Lieb algebra $TL_n^{\mathbb{F}}(\delta)$ is cellular. The cell chain is given by the filtration in (2.1), and the anti-automorphism i acts by swapping corresponding northern and southern nodes in a diagram. Let Λ_{TL_n} be the set of all partitions $\lambda \vdash n$ with at most two parts, i.e. $\lambda_t = 0$ for $t > 2$, ordered by $\lhd$ (see Section 1.3.1). The cell modules $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ of $TL_n^{\mathbb{F}}(\delta)$ are then indexed by partitions $\lambda \in \Lambda_{TL_n}$, and the cellular ordering is given by the reverse of $\lhd$. Each cell module has a simple head $L_\lambda^{\mathbb{F}}(n;\delta)$. The modules $L_\lambda^{\mathbb{F}}(n;\delta)$ form a complete set of non-isomorphic simple $TL_n^{\mathbb{F}}(\delta)$-modules.

Notation. When the context is clear, we will write $\Delta_\lambda^{\mathbb{F}}(n)$ and $L_\lambda^{\mathbb{F}}(n)$ to mean $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ and $L_\lambda^{\mathbb{F}}(n;\delta)$ respectively.

We can explicitly construct an R-form of each cell module in the following way. Let $I(n,t)$, $(t = n, n-2, \ldots, 0/1)$, be the set of all (n,n)-Temperley-Lieb diagrams with precisely t propagating arcs such that the rightmost southern nodes are joined in pairs $\{n, n-1\}, \{n-2, n-3\}, \ldots, \{n-t+2, n-t+1\}$. An example is given in Figure 2.3.

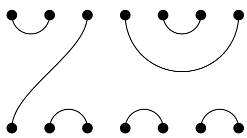
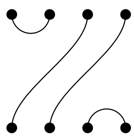
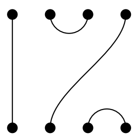
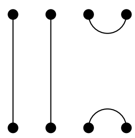


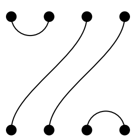
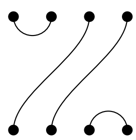
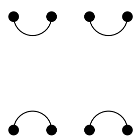
Figure 2.3: An example of a diagram in $I(7,1)$.

We then let the cell module $\Delta_\lambda^R(n)$ be the free R-module with basis $I(n, \lambda_1 - \lambda_2)$. There is a left action of $TL_n^R(\delta)$ on $\Delta_\lambda^R(n)$ by concatenating diagrams in the usual way, except we take the product to be zero if the result has fewer than $\lambda_1 - \lambda_2$ propagating lines. An example is given below.

Example 2.1. Consider the cell module $\Delta_{(3,1)}^R(4)$. This has a basis consisting of the following diagrams:

$v_1 =$  $v_2 =$  $v_3 =$ 

Recall the element u_1 from above. This acts on these basis elements on the following way:

$u_1 v_1 = \delta$  $u_1 v_2 =$  $u_1 v_3 =$  $= 0$

In a similar manner to previously, we then have

$$\Delta_\lambda^K(n) = K \otimes_R \Delta_\lambda^R(n) \quad \text{and} \quad \Delta_\lambda^k(n) = k \otimes_R \Delta_\lambda^R(n).$$

Remark. Note that we **cannot** in general provide an R-module $L_\lambda^R(n)$ such that $L_\lambda^K(n) = K \otimes_R L_\lambda^R(n)$ or $L_\lambda^k(n) = k \otimes_R L_\lambda^R(n)$.

2.3 Representation theory in characteristic zero

It was shown in [Wes95] that if q is chosen such that $0 \neq \delta = q + q^{-1}$, then $TL_n^K(\delta)$ is semisimple if q is not a root of unity. We will therefore assume in what follows that q is a root of unity, and let l be the minimal positive integer satisfying $q^{2l} = 1$. We then have the following results:

Theorem 2.2 ([Wes95, Section 9])**.** *Let $\lambda, \mu \in \Lambda_{TL_n}$. If $0 < \mu_2 - \lambda_2 < l$ and $n - \lambda_2 - \mu_2 + 1 \equiv 0$ (mod l) then there is a non-trivial homomorphism*

$$\theta : \Delta_\lambda^K(n) \longrightarrow \Delta_\mu^K(n),$$

and if $0 < \mu_2 - \lambda_2 < l$ and $n - \lambda_2 - \mu_2 + 1 \not\equiv 0$ (mod l), or $\mu_2 - \lambda_2 \geq l$, then there is no non-trivial homomorphism from $\Delta_\lambda^K(n)$ to $\Delta_\mu^K(n)$.

Theorem 2.3 ([Wes95, Section 9])**.** *The kernels and co-kernels of the homomorphisms θ above are irreducible.*

These allow us to recursively compute the composition factors of the cell modules in the following way. Suppose there is a chain of homomorphisms $\theta_i : \Delta_{\lambda^{(i)}}^K(n) \longrightarrow \Delta_{\lambda^{(i+1)}}^K(n)$, $i = 1, \dots, r$, for some set of partitions $\lambda^{(1)} \trianglerighteq \cdots \trianglerighteq \lambda^{(r)}$, satisfying the conditions of Theorem 2.2. By Proposition 1.5, the cell module $\Delta_{\lambda^{(1)}}^K(n)$ must be a simple module. Therefore by Theorem 2.3, the cell module $\Delta_{\lambda^{(2)}}^K(n)$ must have a simple maximal submodule isomorphic to $\Delta_{\lambda^{(1)}}^K(n)$ and a simple quotient. Continuing in this way we can deduce the composition factors of $\Delta_{\lambda^{(i+1)}}^K(n)$ from those of $\Delta_{\lambda^{(i)}}^K(n)$.

It is possible to reformulate Theorem 2.2 as a geometric statement, see for example [RS14, Sections 5,7]. We recall this below, changing some notation for consistency.

Let $\varepsilon_1, \varepsilon_2$ be formal symbols, and set $E_2 = \mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2$. We equip this space with an inner product by extending linearly the relations

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.

Now let W_2^l be the affine Weyl group of type A_1 generated by the reflections $s_{tl} = s_{\varepsilon_1 - \varepsilon_2, tl}$ ($t \in \mathbb{Z}$), with action on E_2 given by

$$s_{tl}(x) = x - (\langle x, \varepsilon_1 - \varepsilon_2 \rangle - tl)(\varepsilon_1 - \varepsilon_2).$$

We fix the element $\rho = (-1, -2)$ and define a shifted action of W_2^l on E_2 by

$$w \cdot x = w(x + \rho) - \rho$$

for all $w \in W_2^l$ and $x \in E_2$.

For any two-part partition $\lambda = (\lambda_1, \lambda_2)$ there is a corresponding element $\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 \in E_2$, so we can consider two-part partitions to be elements of E_2. Note also that the condition $0 < \mu_2 - \lambda_2$ is equivalent to $\lambda \trianglerighteq \mu$. Therefore we can restate Theorem 2.2 as

Theorem 2.4. *Let $\lambda, \mu \in \Lambda_{TL_n}$ with $\lambda \trianglerighteq \mu$. Then there is a non-trivial homomorphism*

$$\theta : \Delta_\lambda^K(n) \longrightarrow \Delta_\mu^K(n)$$

if and only if $\mu_2 - \lambda_2 < l$ and there is some $t \in \mathbb{Z}$ such that $\mu = s_{tl} \cdot \lambda$.

Proof. Suppose we have a non-trivial homomorphism $\Delta_\lambda^K(n) \longrightarrow \Delta_\mu^K(n)$. Then by Theorem 2.2 we have $n - \lambda_2 - \mu_2 + 1 \equiv 0 \pmod{l}$, say $n - \lambda_2 - \mu_2 + 1 = tl$ $(t \in \mathbb{Z})$. Then

$$\begin{aligned} \mu_2 &= n - \lambda_2 + 1 - tl \\ &= (\lambda_1 + \lambda_2) - \lambda_2 + 1 - tl \\ &= \lambda_1 + 1 - tl \end{aligned} \qquad \text{and } \begin{aligned} \mu_1 &= n - \mu_2 \\ &= n - (n - \lambda_2 + 1 - tl) \\ &= \lambda_2 - 1 + tl. \end{aligned}$$

But also

$$\begin{aligned} s_{tl} \cdot \lambda &= \lambda - (\langle \lambda + \rho, \varepsilon_1 - \varepsilon_2 \rangle - tl)(\varepsilon_1 - \varepsilon_2) \\ &= \lambda - (\lambda_1 - \lambda_2 + 1 - tl)(\varepsilon_1 - \varepsilon_2) \\ &= (\lambda_2 - 1 + tl)\varepsilon_1 + (\lambda_1 + 1 - tl)\varepsilon_2 \\ &= \mu. \end{aligned}$$

Now suppose that there exists a t such that $s_{tl} \cdot \lambda = \mu$. Then $\mu = (\lambda_2 - 1 + tl, \lambda_1 + 1 - tl)$, and

$$
\begin{aligned}
n - \lambda_2 - \mu_2 + 1 &= n - \lambda_2 - (\lambda_1 + 1 - tl) + 1 \\
&= n - (\lambda_1 + \lambda_2) + tl \\
&= tl \\
&\equiv 0 \pmod{l}
\end{aligned}
$$

and so by Theorem 2.2, there is a non-zero homomorphism $\Delta_\lambda^K(n) \longrightarrow \Delta_\mu^K(n)$. $\square$

Example 2.5. Consider the case $l = 5$, $n = 21$, then the reflections s_{tl} can be viewed as in Figure 2.4. The numbers correspond to the second part of the associated partition.

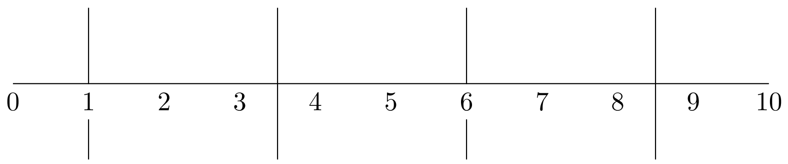


Figure 2.4: The action of W_2^5 on $\Lambda_{TL_{21}}$.

We can use this to obtain a characterisation of the blocks of the Temperley-Lieb algebra. We begin by making the following definition.

Definition 2.6. For $\lambda \in \Lambda_{TL_n}$, let $\mathcal{B}_\lambda^K(n;\delta)$ be the set of partitions μ which label cell modules $\Delta_\mu^K(n;\delta)$ in the same block as $\Delta_\lambda^K(n;\delta)$. We will also say that two partitions are in the same block if they label cell modules in the same block. Moreover, let $\mathcal{O}_\lambda^l(n)$ be the set of partitions $\mu \in \Lambda_{TL_n}$ such that $\mu \in W_2^l \cdot \lambda$.

If the context is clear, we will write $\mathcal{B}_\lambda^K(n)$ to mean $\mathcal{B}_\lambda^K(n;\delta)$.

Suppose a partition λ is fixed by one such reflection s_{tl}. Then

$$
\begin{aligned}
\lambda = s_{tl} \cdot \lambda &\implies \lambda_1 = \lambda_2 - 1 + tl \\
&\implies \lambda_1 - \lambda_2 + 1 \equiv 0 \pmod{l}.
\end{aligned}
$$

We then define $\Lambda_{TL_n}^0 = \{\lambda \in \Lambda_{TL_n} : \lambda_1 - \lambda_2 + 1 \not\equiv 0 \pmod{l}\}$ in order to prove the following.

Corollary 2.7. *Let $\lambda \in \Lambda_{TL_n}^0$. Then $\mathcal{B}_\lambda^K(n;\delta) = \mathcal{O}_\lambda^l(n)$. Furthermore, if $\lambda \in \Lambda_{TL_n} \backslash \Lambda_{TL_n}^0$, then $\mathcal{B}_\lambda^K(n) = \{\lambda\}$.*

Proof. Suppose $\lambda, \mu \in \Lambda^0_{TL_n}$ are in the same $TL_n^K(\delta)$-block. Since $TL_n^K(\delta)$ is cellular, the blocks are given by the cell-linkage classes, so it sufficient to prove the result when we have a non-zero homomorphism $\Delta_\lambda^K(n) \longrightarrow \Delta_\mu^K(n)/M$ for some module M. But this is clearly true by Theorems 2.3 and 2.4.

We now prove the converse. Suppose $\mu = w \cdot \lambda$ for some $w \in W_2^l$, and without loss of generality assume that $\lambda \trianglerighteq \mu$. Since W_2^l is a reflection group, we can write w as a product of simple reflections $w = s_{t_r l} \dots s_{t_2 l} s_{t_1 l}$, where at each stage we are reflecting across adjacent walls. We therefore have a sequence of partitions $\mu = \nu^{(0)}, \nu^{(1)}, \dots, \nu^{(r)} = \lambda$, and $|\nu_2^{(i-1)} - \nu_2^{(i)}| < l$ for each $1 \leq i \leq r$ (see Figure 2.5). Hence by Theorem 2.4, we have for each i a non-zero homomorphism

$$\Delta^K_{\nu^{(i)}}(n) \longrightarrow \Delta^K_{\nu^{(i-1)}}(n).$$

Therefore $\nu^{(i)} \in \mathcal{B}^K_{\nu^{(i-1)}}(n)$ for each i, and by the cell-linkage property we have $\mu \in \mathcal{B}^K_\lambda(n)$.

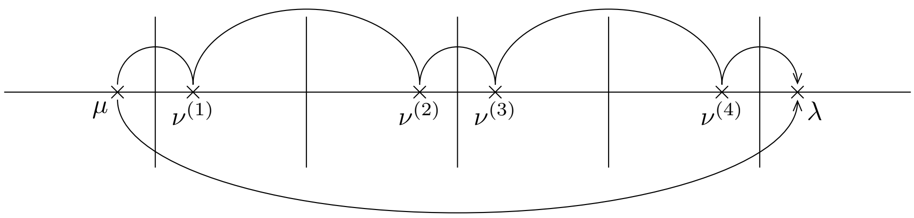


Figure 2.5: The construction of the $\nu^{(i)}$.

If $\lambda \notin \Lambda^0_{TL_n}$, then there are no partitions $\mu \neq \lambda$ such that both $\mu = s_{tl} \cdot \lambda$ and $|\mu_2 - \lambda_2| < l$. The result then follows from Theorems 2.3 and 2.4. $\square$

2.4 Modular representation theory

As in Section 2.3 we assume that $0 \neq \delta = q + q^{-1}$, and l is the lowest positive integer such that $q^{2l} = 1$. A positive characteristic version of Theorem 2.2 is then:

Theorem 2.8 ([CGM03, Theorem 5.3]). *Let $\lambda, \mu \in \Lambda_{TL_n}$. There is a non-trivial homomorphism*

$$\theta : \Delta^k_\mu(n) \longrightarrow \Delta^k_\lambda(n)$$

if and only if $n - \lambda_2 - \mu_2 + 1 \equiv 0$ (mod lp^j) for some non-negative integer j, with $0 \leq \lambda_2 - \mu_2 < lp^j$.

From [CGM03, Remark 5.4] we also see that we can compute the composition factors of cell modules recursively via a similar process to the one described after Theorem 2.3.

Again, we will rewrite this as a geometric statement. We let W_2^l be the affine Weyl group as above, and consider reflections of the form s_{tlp^j} with $t, j \in \mathbb{Z}$, $j \geq 0$. Using the same shifted action on E_2 as before, we then have

Theorem 2.9. *Let $\lambda, \mu \in \Lambda_{TL_n}$ with $\lambda \trianglerighteq \mu$ and $\mu_2 - \lambda_2 < lp^j$ for some integer $j \geq 0$. Then there is a non-trivial homomorphism*

$$\theta : \Delta_\lambda^k(n) \longrightarrow \Delta_\mu^k(n)$$

if and only if there is some $t \in \mathbb{Z}$ such that $\mu = s_{tlp^j} \cdot \lambda$.

Proof. Suppose we have a non-trivial homomorphism $\Delta_\lambda^k(n) \longrightarrow \Delta_\mu^k(n)$. Then by Theorem 2.8 we have $n - \lambda_2 - \mu_2 + 1 \equiv 0 \pmod{lp^j}$ for some j, say $n - \lambda_2 - \mu_2 + 1 = tlp^j$ ($t \in \mathbb{Z}$). Then

$$\begin{aligned} \mu_2 &= n - \lambda_2 + 1 - tlp^j \\ &= (\lambda_1 + \lambda_2) - \lambda_2 + 1 - tlp^j \\ &= \lambda_1 + 1 - tlp^j \end{aligned} \qquad \text{and } \begin{aligned} \mu_1 &= n - \mu_2 \\ &= n - (n - \lambda_2 + 1 - tlp^j) \\ &= \lambda_2 - 1 + tlp^j. \end{aligned}$$

But also

$$\begin{aligned} s_{tlp^j} \cdot \lambda &= \lambda - (\langle \lambda + \rho, \varepsilon_1 - \varepsilon_2 \rangle - tlp^j)(\varepsilon_1 - \varepsilon_2) \\ &= \lambda - (\lambda_1 - \lambda_2 + 1 - tlp^j)(\varepsilon_1 - \varepsilon_2) \\ &= (\lambda_2 - 1 + tlp^j)\varepsilon_1 + (\lambda_1 + 1 - tlp^j)\varepsilon_2 \\ &= \mu. \end{aligned}$$

Now suppose that there exists a t and j such that $s_{tlp^j} \cdot \lambda = \mu$. Then

$\mu = (\lambda_2 - 1 + tlp^j, \lambda_1 + 1 - tlp^j)$, and

$$\begin{aligned} n - \lambda_2 - \mu_2 + 1 &= n - \lambda_2 - (\lambda_1 + 1 - tlp^j) + 1 \\ &= n - (\lambda_1 + \lambda_2) + tlp^j \\ &= tlp^j \\ &\equiv 0 \pmod{lp^j} \end{aligned}$$

and so by Theorem 2.8, there is a non-zero homomorphism $\Delta_\lambda^k(n) \longrightarrow \Delta_\mu^k(n)$. $\square$

Example 2.10. We let $n = 37$, $l = 4$ and $p = 3$. Figure 2.6 shows the action of W_2^l on Λ_{TL_n}. The lines represent reflections in an lp^j-wall, and the longer the line, the larger the value of j. As before, the numbers correspond to the second part of the associated partition.

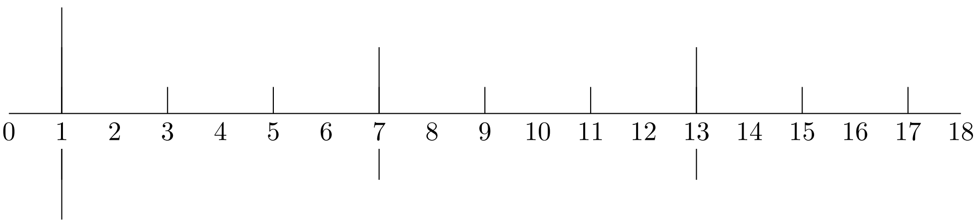


Figure 2.6: The action of W_2^4 on $\Lambda_{TL_{37}}$ in positive characteristic.

As in the previous section, we now use this to characterise the blocks of the Temperley-Lieb algebra in positive characteristic.

Definition 2.11. For $\lambda \in \Lambda_{TL_n}$, let $\mathcal{B}_\lambda^k(n;\delta)$ be the set of partitions μ which label cell modules $\Delta_\mu^k(n;\delta)$ in the same block as $\Delta_\lambda^k(n;\delta)$. Again, we will say that two partitions are in the same block if they label cell modules in the same block. Moreover, let $\mathcal{O}_\lambda^{lp^j}(n)$ be the set of partitions $\mu \in \Lambda_{TL_n}$ such that $\mu \in W_2^{lp^j} \cdot \lambda$.

If the context is clear, we will write $\mathcal{B}_\lambda^k(n)$ to mean $\mathcal{B}_\lambda^k(n;\delta)$.

Definition 2.12. Let $\Lambda_{TL_n}^0 = \{\lambda \in \Lambda_{TL_n} : \lambda_1 - \lambda_2 + 1 \not\equiv 0 \pmod{l}\}$ as before, and for $j > 0$ define

$$\Lambda_{TL_n}^j = \{\lambda \in \Lambda_{TL_n} : \lambda_1 - \lambda_2 + 1 \not\equiv 0 \pmod{lp^j} \text{ but } \lambda_1 - \lambda_2 + 1 \equiv 0 \pmod{lp^{j-1}}\}.$$

The sets $\Lambda_{TL_n}^j$ therefore consist of partitions which lie on an lp^{j-1}-wall but not an lp^j-wall.

Note that for any $\lambda \in \Lambda_{TL_n}$, there is some $m \gg 0$ such that λ does not lie on an lp^m wall. Hence there exists a unique j such that $\lambda \in \Lambda^j_{TL_n}$.

Corollary 2.13. *Let $\lambda \in \Lambda^j_{TL_n}$. Then $\mathcal{B}^k_\lambda(n;\delta) = \mathcal{O}^{lp^j}_\lambda(n)$.*

Proof. Suppose $\mu \in \mathcal{B}^k_\lambda(n)$. Since $TL^k_n(\delta)$ is cellular, the blocks are given by the cell-linkage classes, so it sufficient to consider the case when we have a non-zero homomorphism $\Delta^k_\lambda(n) \longrightarrow \Delta^k_\mu(n)/M$ for some module M. But this occurs if and only if $|\mu_2 - \lambda_2| < lp^i$ and $\mu = s_{tlp^i} \cdot \lambda$ for some i. Since $\lambda \in \Lambda^j_{TL_n}$, we must have $\mu \in \Lambda^j_{TL_n}$ and $i \geq j$. The result then follows.

Now suppose $\mu \in \mathcal{O}^{lp^j}_\lambda(n)$. Again, we must have $\mu \in \Lambda^j_{TL_n}$, and we first consider the case $\mu = s_{tlp^i} \cdot \lambda$ for some $i \geq j$. As in the proof of Corollary 2.7, we may write this as a product of reflections across adjacent walls to show that $\mu \in \mathcal{B}^k_\lambda(n)$. The rest of the proof follows as in Corollary 2.7. $\square$

Remark 2.14. The idea behind Corollary 2.13 is that we "zoom out" until the partition λ is not lying on any of the reflection walls, then the proof is exactly as in the characteristic zero case. See Figure 2.7 for an illustration of this.

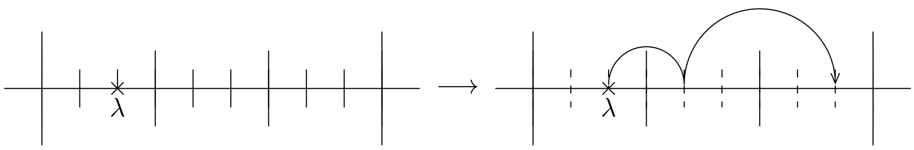


Figure 2.7: The process of the proof of Corollary 2.13. We ignore all reflections through dashed walls.

2.5 The case $\delta = 0$ or $p = 2$

In this section we briefly discuss the cases $\delta = 0$ and $p = 2$.

When $\delta = 0$ many of the results within this chapter still hold, possibly with a slight modification to the statements or proofs. We first note that the labelling set for the simple $TL^{\mathbb{F}}_n(0)$-modules is not necessarily Λ_{TL_n}. In the case $n = 2m$, the partition $\lambda = (m^2)$ no longer labels a simple module [GL96, Corollary 6.8]. In fact, we have an isomorphism $\Delta^{\mathbb{F}}_{(m^2)}(2m;0) \cong L^{\mathbb{F}}_{(m+1,m-1)}(2m;0)$. However once taking this into consideration, the representation theory of $TL^K_n(0)$ in characteristic zero remains the same. Modifications must be made to several proofs in order to show Theorem 2.2, details of which can be found in [RS14]. The geometric version of this theorem remains the same. If now we suppose that $\delta = 0$ and $p > 2$, or $p = 2$, then we cannot use the results of [CGM03]. An alternative approach would be to study the Temperley-Lieb algebra via the Hecke algebra $\mathcal{H}_n(q^2)$ of type A. The $\mathbb{F}$-algebra $\mathcal{H}_n(q^2)$ has generators $T_1, T_2, \ldots, T_{n-1}$, subject to the relations

$$\begin{aligned} T_i^2 &= q^2 + (q^2 - 1)T_i, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \\ T_i T_j &= T_j T_i \text{ if } |i - j| > 1. \end{aligned}$$

As shown in [GdHJ89, Chapter 2.11] there is an isomorphism between the Temperley-Lieb algebra $TL_n^{\mathbb{F}}(q + q^{-1})$ and a quotient of the Hecke algebra $\mathcal{H}_n(q)$, obtained from the surjection

$$\begin{aligned} \mathcal{H}_n(q^2) &\longrightarrow TL_n^{\mathbb{F}}(q + q^{-1}) \\ T_i &\longmapsto q u_i - 1 \end{aligned}$$

with kernel generated by the element $T_1 T_2 T_1 + T_1 T_2 + T_2 T_1 + T_1 + T_2 + 1$. The corresponding isomorphism shows that the representation theory of $TL_n^{\mathbb{F}}(q + q^{-1})$ appears within that of the Hecke algebra, in particular those Specht modules labelled by partitions with at most two parts. Thus we can obtain results about the Temperley-Lieb algebra in the cases above by studying the Hecke algebra in positive characteristic or when $q^2 = 1$, see for example [Fay05], [FL09] and [Fay10].

Chapter 3

The Brauer algebra

3.1 Background

The Brauer algebra first appeared in [Bra37], where it acted as the centraliser for the action of the orthogonal or symplectic group on a complex tensor space. It is possible however, to define the Brauer algebra $B_n(\delta)$ over an arbitrary commutative ring R, for any $n \in \mathbb{N}$ and $\delta \in R$. Then, rather than examining it as a way of understanding its corresponding centraliser algebra in the context of Schur-Weyl duality, we study the representation theory of $B_n(\delta)$ in its own right.

Although the orthogonal and symplectic groups are semisimple over $\mathbb{C}$, the Brauer algebra itself may not be. Brown [Bro55] showed that for generic values of δ the Brauer algebra is semisimple over $\mathbb{C}$, and furthermore gave a classification of the simple modules. Wenzl [Wen88] proved further that $B_n(\delta)$ is semisimple over $\mathbb{C}$ for all non-integer δ. Motivated by this, Rui [Rui05] provided a necessary and sufficient condition for semisimplicity, valid over an arbitrary field.

Graham and Lehrer [GL96] showed that the Brauer algebra $B_n(\delta)$ over a field $\mathbb{F}$ is a cellular algebra, with cell modules indexed by partitions of $n, n-2, n-4, \ldots, 0$ or 1 (depending on the parity of n). If we suppose $\delta \neq 0$, then in characteristic zero these partitions also label a complete set of non-isomorphic simple modules, given by the heads of the corresponding cell modules. In positive characteristic the simple modules are indexed by the subset of p-regular partitions (again under the assumption $\delta \neq 0$).

The cellularity of $B_n(\delta)$ was proved by König and Xi [KX01], where they described the Brauer algebra as an iterated inflation of symmetric group algebras. This approach allows for analogues of permutation modules and Young modules for the symmetric group to be made for the Brauer algebra. These modules have been the objects of further study, for instance by Hartmann and Paget [HP06].

Doran, Wales and Hanlon [DWH99] adapted the methods Martin [Mar96] used for the partition algebra in studying the Brauer algebra. Working from this and using the theory of towers of recollement [CMPX06], a necessary and sufficient condition for two cell modules to be in the same block in characteristic zero was given in [CDM09a]. It was shown in [CDM09b] that this is equivalent to partitions being in the same orbit under some action of a Weyl group W_n of type D_n. In the same paper, it was found that in positive characteristic, the orbits of the corresponding affine Weyl group W_n^p of type D_n on the set of partitions correspond to unions of blocks of the Brauer algebra $B_n(\delta)$.

As for the decomposition matrix itself, Martin [Mar09] showed that the decomposition numbers of cell modules over $\mathbb{C}$ are given by parabolic Kazhdan-Lusztig polynomials. In characteristic $p > 0$, Shalile [Sha13] has shown that the decomposition numbers of the Brauer algebra $B_n(\delta)$ for $n < p$ are all obtained by reducing homomorphisms from characteristic zero. In general however, the decomposition matrix of $B_n(\delta)$ in positive characteristic remains unknown.

In this chapter, we will see that in a limiting case, the orbits of an infinite affine Weyl group correspond precisely to the blocks of the Brauer algebra [Kin14a]. We will also show that the action of certain elements of the affine Weyl group give rise to non-zero decomposition numbers, and we furthermore characterise a family of Brauer algebras for which we can give a complete description of the decomposition matrix.

3.2 Preliminaries

3.2.1 Definitions

Recall from Section 1.1 the definitions and conventions for the prime p, the p-modular system (K, R, k), the field $\mathbb{F}$ and the parameter $\delta \in R$. There is a consideration of the cases $p = 2$ and $\delta = 0$ in Section 3.5.

For a fixed $n \in \mathbb{N}$, an (n, n)-*Brauer diagram* is a diagram formed of $2n$ nodes, arranged in two rows of n, with n arcs between them so that each node is joined to precisely one other. An example is given in Figure 3.1.

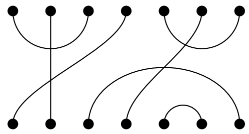


Figure 3.1: A $(7,7)$-Brauer diagram.

For a fixed $\delta \in R$ and $n \in \mathbb{N}$, the *Brauer algebra* $B_n^R(\delta)$ is defined as the set of linear combinations of (n,n)-Brauer diagrams. Multiplication of two diagrams is by concatenation in the following way: to obtain the result $x \cdot y$ given diagrams x and y, place x on top of y and identify the bottom nodes of x with those on the top of y. This new diagram may contain a number, t say, of closed loops. These we remove and multiply the final result by δ^t. An example is given in Figure 3.2 below.

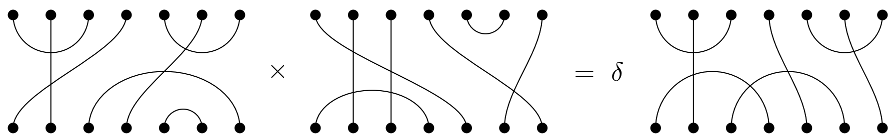


Figure 3.2: Multiplication of two diagrams in $B_7^R(\delta)$.

Note that multiplication in $B_n^R(\delta)$ cannot increase the number of propagating arcs (ones that join a northern and southern node). Let $J_n^{(r)}$ be the ideal generated by diagrams with at most r propagating arcs. We then have a filtration of $B_n^R(\delta)$ by the $J_n^{(r)}$:

$$
\begin{aligned}
&0 \subset J_n^{(0)} \subset J_n^{(2)} \subset \dots J_n^{(n-2)} \subset J_n^{(n)} = B_n^R(\delta) \qquad (n \text{ even}),\\
&0 \subset J_n^{(1)} \subset J_n^{(3)} \subset \dots J_n^{(n-2)} \subset J_n^{(n)} = B_n^R(\delta) \qquad (n \text{ odd}).
\end{aligned} \tag{3.1}
$$

We then define the algebras

$$B_n^K(\delta) = K \otimes_R B_n^R(\delta) \quad \text{and} \quad B_n^k(\delta) = k \otimes_R B_n^R(\delta).$$

Recall that the parameter δ is invertible in K. For each $n \geq 2$ we have an idempotent $e_n \in B_n^{\mathbb{F}}(\delta)$ as illustrated in Figure 3.3.

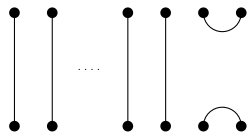
$e_n = \frac{1}{\delta}$ 

Figure 3.3: The idempotent e_n.

Using these idempotents we can define algebra isomorphisms

$$\Phi_n : B^{\mathbb{F}}_{n-2}(\delta) \longrightarrow e_n B^{\mathbb{F}}_n(\delta) e_n \tag{3.2}$$

taking a diagram in $B^{\mathbb{F}}_{n-2}(\delta)$ to the diagram in $B^{\mathbb{F}}_n(\delta)$ obtained by adding an extra northern and southern arc to the right hand end. Using this and following [Gre80, Section 6.2] we obtain the following functors as defined in [DWH99, Section 5]: an exact localisation functor

$$\begin{aligned} F_n : B^{\mathbb{F}}_n(\delta)\text{-}\mathbf{mod} &\longrightarrow B^{\mathbb{F}}_{n-2}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto e_n M; \end{aligned} \tag{3.3}$$

and a right exact globalisation functor

$$\begin{aligned} G_n : B^{\mathbb{F}}_n(\delta)\text{-}\mathbf{mod} &\longrightarrow B^{\mathbb{F}}_{n+2}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto B^{\mathbb{F}}_{n+2}(\delta) e_{n+2} \otimes_{B^{\mathbb{F}}_n(\delta)} M. \end{aligned} \tag{3.4}$$

Since $F_{n+2}G_n(M) \cong M$ for all $M \in B^{\mathbb{F}}_n(\delta)$-**mod**, G_n is a full embedding of categories. Also as

$$B^{\mathbb{F}}_n(\delta)/B^{\mathbb{F}}_n(\delta)e_n B^{\mathbb{F}}_n(\delta) \cong \mathbb{F}\mathfrak{S}_n \tag{3.5}$$

the group algebra of the symmetric group on n letters, we have from (3.2) and [Gre80, Theorem 6.2g] that the simple $B^{\mathbb{F}}_n(\delta)$-modules are indexed by the set

$$\begin{aligned} \Lambda_{B_n} &= \{\lambda : \lambda \vdash n, n-2, n-4, \dots, 0/1\} \text{ if char } \mathbb{F} = 0 \\ \text{or } \Lambda^*_{B_n} &= \{\lambda : \lambda \vdash n, n-2, n-4, \dots, 0/1,\ \lambda\ p\text{-regular}\} \text{ if char } \mathbb{F} = p > 0, \end{aligned} \tag{3.6}$$

provided $\delta \neq 0$. In characteristic zero and for a generic parameter δ, this recovers the classification of the simple modules by Brown [Bro55, Section 2]. The full classification is due to Graham and Lehrer [GL96, Section 4].

Remark 3.1. Note that we have embeddings $\Lambda_{B_n} \subset \Lambda_{B_{n+2}}$ and $\Lambda^*_{B_n} \subset \Lambda^*_{B_{n+2}}$.

For each $n \geq 1$, we can identify $B_n^{\mathbb{F}}(\delta)$ with a subalgebra of $B_{n+1}^{\mathbb{F}}(\delta)$ via an injective homomorphism that takes a diagram $x \in B_n^{\mathbb{F}}(\delta)$ to the diagram in $B_{n+1}^{\mathbb{F}}(\delta)$ obtained by adding a vertex at the right end of the northern and southern sides and joining these with an arc. Then, in addition to the localisation and globalisation functors, we have the usual restriction and induction functors:

$$\begin{aligned}\mathrm{res}_n : B_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow B_{n-1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}\\ M &\longmapsto M|_{B_{n-1}^{\mathbb{F}}(\delta)},\end{aligned}$$

$$\begin{aligned}\mathrm{ind}_n : B_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow B_{n+1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}\\ M &\longmapsto B_{n+1}^{\mathbb{F}}(\delta) \otimes_{B_n^{\mathbb{F}}(\delta)} M.\end{aligned} \tag{3.7}$$

3.2.2 Cellularity of $B_n^{\mathbb{F}}(\delta)$

Graham and Lehrer [GL96] showed that $B_n^{\mathbb{F}}(\delta)$ is a cellular algebra. The cell chain is given by refining each layer of the filtration in (3.1) by the symmetric group filtration, see [KX99] for details. The antiautomorphism i acts by swapping corresponding northern and southern nodes in a diagram. Let Λ_{B_n} be as in (3.6), and use the ordering $\lhd$ from Section 1.3.1. The cell modules $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ are then indexed by partitions $\lambda \in \Lambda_{B_n}$, and the cellular ordering is given by the reverse of $\lhd$.

When $\lambda \vdash n$, the cell module is simply a lift of the Specht module $S_{\mathbb{F}}^{\lambda}$ to the Brauer algebra using (3.5). The localisation and globalisation functors ((3.3) and (3.4)) allow us to describe the cell module $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ when $\lambda \vdash n-2t$ for some $t > 0$. In particular, Doran, Wales and Hanlon showed in [DWH99, Section 5] that the functors map cell modules to cell modules in the following way:

$$F_n(\Delta_\lambda^{\mathbb{F}}(n)) \cong \begin{cases}\Delta_\lambda^{\mathbb{F}}(n-2) & \text{if } \lambda \in \Lambda_{B_{n-2}}\\ 0 & \text{otherwise,}\end{cases}$$

$$G_n(\Delta_\lambda^{\mathbb{F}}(n)) \cong \Delta_\lambda^{\mathbb{F}}(n+2).$$

Thus when $\lambda \vdash n-2t$ for some $t > 0$, we obtain the cell module by

$$\Delta_n^{\mathbb{F}}(\lambda;\delta) \cong G_{n-2}G_{n-4}\ldots G_{n-2t}\Delta_\lambda^{\mathbb{F}}(n-2t;\delta).$$

Since we are assuming $\delta \neq 0$, over K each of the cell modules has a simple head $L_\lambda^K(n;\delta)$, and these form a complete set of non-isomorphic simple $B_n^K(\delta)$-modules.

Over k, the heads $L_\lambda^k(n;\delta)$ of cell modules labelled by p-regular partitions $\lambda \in \Lambda_{B_n}^*$ provide a complete set of non-isomorphic simple $B_n^k(\delta)$-modules.

Notation. When the context is clear, we will write $\Delta_\lambda^{\mathbb{F}}(n)$ and $L_\lambda^{\mathbb{F}}(n)$ to mean $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ and $L_\lambda^{\mathbb{F}}(n;\delta)$ respectively.

We can also define modules $\Delta_\lambda^R(n)$ over R, using the construction from [HW89]. As shown in [DWH99, Section 2] these will be cell modules for $B_n^R(\delta)$. Let $I(n,t)$, $(t = n, n-2, \ldots, 0/1)$, be the set of all (n,n)-Brauer diagrams with precisely t propagating arcs such that the rightmost southern nodes are joined in pairs $\{n, n-1\}$, $\{n-2, n-3\}, \ldots, \{n-t+2, n-t+1\}$. An example is given in Figure 3.4.

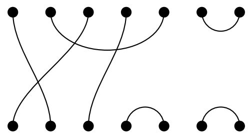


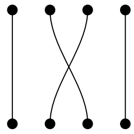
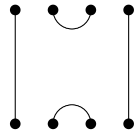
Figure 3.4: An example of diagram in $I(7,3)$.

Denote by $V(n,t)$ the free R-module with basis $I(n,t)$. There is a $(B_n^R(\delta), \mathfrak{S}_t)$-bimodule action on $V(n,t)$, where the elements of $B_n^R(\delta)$ act on the left by concatenation as normal and elements of $\mathfrak{S}_t$ act on the right by permuting the t leftmost southern nodes. Given a partition $\lambda \vdash t$, we now form the module $\Delta_\lambda^R(n) = V(n,t) \otimes_R S_R^\lambda$, where S_R^λ is the R-form of the Specht module. The action of $B_n^R(\delta)$ on this module is as follows: given a Brauer diagram $x \in B_n^R(\delta)$ and a pure tensor $v \otimes s \in \Delta_\lambda^R(n)$, we define the element

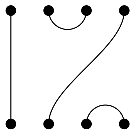
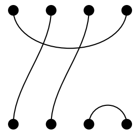
$$x(v \otimes s) = (xv) \otimes \sigma(x,v)s$$

where xv is the product of two diagrams in the usual way if the result has t propagating lines, and is zero otherwise. An example is given below.

Example 3.2. Consider the elements t and u of $B_4^R(\delta)$ as defined below.

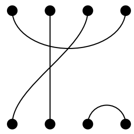
$t =$  $u =$ 

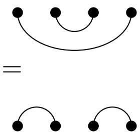
The cell module $\Delta^R_{(1^2)}(4;\delta)$ has basis $\{v \otimes s : v \in I(4,2)\}$ where s is the generator for the sign representation of $\mathfrak{S}_2$. Let x and y be the following elements of $I(4,2)$:

$x =$  $\qquad y =$ 

Then t and u act on $x \otimes s$ and $y \otimes s$ in the following way:

$$t(x \otimes s) = x \otimes s, \qquad u(x \otimes s) = \delta x \otimes s$$

$t(y \otimes s) =$  $\otimes s = y(1,2) \otimes s = y \otimes (1,2)s = -y \otimes s.$

$u(y \otimes s) =$  $= 0.$

In a similar manner to previously, we then have

$$\Delta^K_\lambda(n) = K \otimes_R \Delta^R_\lambda(n) \quad \text{and} \quad \Delta^k_\lambda(n) = k \otimes_R \Delta^R_\lambda(n).$$

Remark. Note that we **cannot** in general provide an R-module $L^R_\lambda(n)$ such that $L^K_\lambda(n) = K \otimes_R L^R_\lambda(n)$ or $L^k_\lambda(n) = k \otimes_R L^R_\lambda(n)$.

It was shown in [DWH99, Theorem 4.1, Corollary 6.4] if we apply the restriction and induction functors (3.7) to cell modules, then the result has a filtration by cell modules. The original proof for fields of characteristic zero can be made valid in arbitrary characteristic by using obvious modifications, for instance changing direct sums of modules to filtrations. We thus obtain exact sequences

$$0 \longrightarrow \biguplus_{A \in \mathrm{rem}(\lambda)} \Delta^{\mathbb{F}}_{\lambda - A}(n-1) \longrightarrow \mathrm{res}_n \Delta^{\mathbb{F}}_\lambda(n) \longrightarrow \biguplus_{A \in \mathrm{add}(\lambda)} \Delta^{\mathbb{F}}_{\lambda + A}(n-1) \longrightarrow 0$$

$$0 \longrightarrow \biguplus_{A \in \mathrm{rem}(\lambda)} \Delta^{\mathbb{F}}_{\lambda - A}(n+1) \longrightarrow \mathrm{ind}_n \Delta^{\mathbb{F}}_\lambda(n) \longrightarrow \biguplus_{A \in \mathrm{add}(\lambda)} \Delta^{\mathbb{F}}_{\lambda + A}(n+1) \longrightarrow 0, \quad (3.8)$$

where we use $\biguplus_{i=1}^{r} N_i$ to denote a module M with a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M_r = M$$

such that for each $1 \leq i \leq r$, $M_i/M_{i-1} \cong N_i$. Moreover, the successive quotients in these filtrations are ordered by dominance.

3.3 Representation theory in characteristic zero

The blocks of the Brauer algebra in characteristic zero were first determined in [CDM09a]. We briefly recount this below, but first we introduce the following notation:

Definition 3.3. For $\lambda \in \Lambda_{B_n}$, let $\mathcal{B}_\lambda^K(n;\delta)$ be the set of partitions μ which label cell modules $\Delta_\mu^K(n;\delta)$ in the same block as $\Delta_\lambda^K(n;\delta)$. If the context is clear, we will write $\mathcal{B}_\lambda^K(n)$ to mean $\mathcal{B}_\lambda^K(n;\delta)$.

We will also say that two partitions are in the same block if they label cell modules in the same block.

Definition 3.4 ([CDM09a, Definition 4.7])**.** Two partitions $\mu \subseteq \lambda$ are *δ-balanced* if: (i) there exists a pairing of the nodes in $[\lambda]\backslash[\mu]$ such that the contents of each pair sum to $1-\delta$; and (ii) if δ is even and the nodes with content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ in $[\lambda]\backslash[\mu]$ are configured as in Figure 3.5 below then the number of columns in this configuration is even.

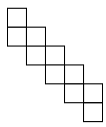


Figure 3.5: A possible configuration of the nodes of content $-\frac{\delta}{2}$ and $\frac{2-\delta}{2}$ in $[\lambda]\backslash[\mu]$.

We say two partitions $\mu \subseteq \lambda$ are *maximally δ-balanced* if μ and λ are δ-balanced and there is no partition ν such that $\mu \subsetneq \nu \subsetneq \lambda$ with ν and λ δ-balanced.

If neither $\mu \subseteq \lambda$ nor $\lambda \subseteq \mu$, then we say μ and λ are δ-balanced if there is a sequence of partitions $\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)} = \mu$ such that for each $1 \leq i < m$ the following conditions are satisfied: (i) either $\lambda^{(i)} \subseteq \lambda^{(i+1)}$ or $\lambda^{(i+1)} \subseteq \lambda^{(i)}$; and (ii) $\lambda^{(i)}, \lambda^{(i+1)}$ are δ-balanced in the previous sense.

From [CDM09a], we also have:

Theorem 3.5 ([CDM09a, Theorem 6.5])**.** *If $\mu \subseteq \lambda$ are δ-balanced, then for any maximal δ-balanced subpartition ν between μ and λ we have*

$$\mathrm{Hom}(\Delta_\lambda^K(n;\delta), \Delta_\nu^K(n;\delta)) \neq 0.$$

Corollary 3.6. *Let $\lambda, \mu \in \Lambda_{B_n}$. Then $\mu \in \mathcal{B}_\lambda^K(n;\delta)$ if and only if λ and μ are δ-balanced.*

We demonstrate this result in Example 3.7.

Example 3.7. Let $\lambda = (3,2)$, $\mu = (3)$ be partitions labelling cell modules of the algebra $B_5^K(2)$.

The module $\Delta_\lambda^K(5)$ is a Specht module, and so has a basis of λ-polytabloids (See Section 1.3.2), each generated by a standard λ-tabloid. The λ-tabloids themselves are determined by which numbers are present in the second row, and we therefore adopt the notation

$$x_{ij} = e_t, \quad \text{where } t = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline i & j \\ \cline{1-2} \end{array} \qquad (a,b,c,i,j \in \{1,2,3,4,5\})$$

and our basis of $\Delta_\lambda^K(5)$ is $x_{45}, x_{35}, x_{34}, x_{25}, x_{24}$.

From Section 3.2.2 we have $\Delta_\mu^K(5) = V(5,3) \otimes S_K^\mu$. Now S_K^μ is the trivial module and $V(5,3)$ has basis $I(5,3)$. Let v_{ij} denote the element of $I(5,3)$ with a single arc between nodes i and j, and the propagating arcs drawn without crossings. Then the cell module $\Delta_\mu^K(5)$ has basis $v_{ij} \otimes 1_{K\mathfrak{S}_3}$.

There is a homomorphism $\Delta_\lambda^K(5) \longrightarrow \Delta_\mu^K(5)$. Indeed, we map the basis elements of $\Delta_\lambda^K(5)$ as follows:

$$x_{45} \longmapsto v_{45} - v_{15} - v_{24} + v_{12},$$

$$x_{35} \longmapsto v_{35} - v_{15} - v_{23} + v_{12},$$

$$x_{34} \longmapsto v_{34} - v_{14} - v_{23} + v_{12},$$

$$x_{25} \longmapsto v_{25} - v_{15} - v_{23} + v_{13},$$

$$x_{24} \longmapsto v_{24} - v_{14} - v_{23} + v_{13}.$$

We also see that the partitions $(3,2)$ and (3) are (maximally) 2-balanced. Indeed, the skew partition $[\lambda]\backslash[\mu]$ consists of two nodes with content

$$\begin{array}{|c|c|} \hline -1 & 0 \\ \hline \end{array}$$

which sum to $1 - \delta = -1$.

This combinatorial description of the blocks was then reformulated in terms of the geometry of a reflection group in [CDM09b]. Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ be a set of formal symbols and set

$$E_n = \bigoplus_{i=1}^{n} \mathbb{R}\varepsilon_i.$$

Remark 3.8. Note that we have an embedding $E_n \subset E_{n+1}$ by adding a zero into the $(n+1)$-th position of a vector in E_n.

We have an inner product on E_n given by extending linearly the relations

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} \tag{3.9}$$

where δ_{ij} is the Kronecker delta.

Let $\Phi = \{\pm(\varepsilon_i - \varepsilon_j), \pm(\varepsilon_i + \varepsilon_j) : 1 \leq i < j \leq n\}$ be the root system of type D_n, and W_n the corresponding Weyl group, generated by the reflections s_α $(\alpha \in \Phi)$. There is an action of W_n on E_n, the generators acting by

$$s_\alpha(x) = x - \langle x, \alpha \rangle \alpha.$$

Fix the element

$$\rho(\delta) = \left(-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, \dots, -\frac{\delta}{2} - (n-1)\right).$$

We may then define a different (shifted) action of W_n on E_n given by

$$w \cdot_\delta x = w(x + \rho(\delta)) - \rho(\delta) \tag{3.10}$$

for all $w \in W_n$ and $x \in E_n$.

Note that for any partition $\lambda = (\lambda_1, \dots, \lambda_l)$ there is a corresponding element $\sum \lambda_i \varepsilon_i \in E_n$, where any λ_i not appearing in λ is taken to be zero. In this way we can view partitions as elements of E_n, and use the following notation to characterise the orbits of a partition under this action.

Definition 3.9. For $\lambda \in \Lambda_{B_n}$, let $\mathcal{O}_\lambda(n;\delta)$ be the set of partitions $\mu \in \Lambda_{B_n}$ such that $\mu \in W_n \cdot_\delta \lambda$. If the context is clear, we will write $\mathcal{O}_\lambda(n)$ to mean $\mathcal{O}_\lambda(n;\delta)$.

We then arrive at the following result.

Theorem 3.10 ([CDM09b, Theorem 4.2]). *Let $\lambda \in \Lambda_{B_n}$. Then*

$$\mathcal{B}^K_\lambda(n;\delta) = \{\mu^T : \mu \in \mathcal{O}_{\lambda^T}(n;\delta)\}.$$

Returning to Example 3.7, we can see this result explicitly.

Example 3.11. Let $\lambda = (3,2)$ and $\mu = (3)$. We saw in Example 3.7 that these partitions are in the same $B^K_5(2)$-block. Consider now the transposed partitions $\lambda^T = (2^2, 1)$ and $\mu^T = (1^3)$, then

$$\begin{aligned}
\lambda^T + \rho(2) &= \left(2 - \frac{2}{2},\ 2 - \frac{2}{2} - 1,\ 1 - \frac{2}{2} - 2,\ \ -\frac{2}{2} - 3, \dots\right) \\
&= (1, 0, -2, -4, -5, -6, \dots), \\
\mu^T + \rho(2) &= \left(1 - \frac{2}{2},\ 1 - \frac{2}{2} - 1,\ 1 - \frac{2}{2} - 2,\ \ -\frac{2}{2} - 3, \dots\right) \\
&= (0, -1, -2, -4, -5, -6, \dots).
\end{aligned}$$

We then see that $\mu^T + \rho(2) = s_{\varepsilon_1+\varepsilon_2}(\lambda^T + \rho(2))$, and hence $\mu^T = s_{\varepsilon_1+\varepsilon_2} \cdot_2 \lambda^T$.

3.4 Modular representation theory

3.4.1 Block structure

We begin by defining a positive characteristic analogue of Definition 3.3.

Definition 3.12. For $\lambda \in \Lambda_{B_n}$, let $\mathcal{B}^k_\lambda(n;\delta)$ be the set of partitions μ which label cell modules $\Delta^k_\mu(n;\delta)$ in the same block as $\Delta^k_\lambda(n;\delta)$. If the context is clear, we will again write $\mathcal{B}^k_\lambda(n)$ to mean $\mathcal{B}^k_\lambda(n;\delta)$.

As before, we will also say that two partitions are in the same block if they label cell modules in the same block.

In [CDM09b, Section 6], a partial block result was obtained in positive characteristic by adapting the geometric method from characteristic zero. If W_n is the Weyl group as in the previous section, we define W^p_n to be the corresponding affine Weyl group, generated by the reflections $s_{\alpha,rp}$ ($\alpha \in \Phi, r \in \mathbb{Z}$), with an action on E_n given by

$$s_{\alpha,rp}(x) = x - (\langle x, \alpha\rangle - rp)\alpha.$$

Using the same shifted action as in (3.10), we now give a positive characteristic version of Definition 3.9.

Definition 3.13. For $\lambda \in \Lambda_{B_n}$, let $\mathcal{O}^p_\lambda(n;\delta)$ be the set of partitions $\mu \in \Lambda_{B_n}$ such that $\mu \in W^p_n \cdot_\delta \lambda$. If the context is clear, we will write $\mathcal{O}^p_\lambda(n)$ to mean $\mathcal{O}^p_\lambda(n;\delta)$.

Remark 3.14. It is clear that we have an embedding $\mathcal{O}^p_\lambda(n;\delta) \subset \mathcal{O}^p_\lambda(n+2;\delta)$.

We then have

Theorem 3.15 ([CDM09b, Theorem 6.4])**.** *Let $\lambda \in \Lambda_{B_n}$. Then*

$$\mathcal{B}^k_\lambda(n;\delta) \subset \{\mu^T : \mu \in \mathcal{O}^p_{\lambda^T}(n;\delta)\}.$$

It is important to note that the converse of Theorem 3.15 is not true in general, i.e. two partitions may lie in the same W^p_n-orbit but their transposes are not in the same block. In fact, it is shown in [CDM09b, Section 7] that counter-examples exist for arbitrarily large values of n (see [CDM09b, Theorem 7.2]).

However we will now show that in a limiting case, the converse of Theorem 3.15 does in fact hold. The proof requires us to introduce a variation of the abacus of Section 1.3.4.

Let $\lambda \in \Lambda_{B_n}$, and construct the associated abacus as in Section 1.3.4. We saw previously that sliding beads up and down their runners corresponds to removing or adding p-hooks. Nakayama's conjecture (Theorem 1.8) then tells us that if we slide one bead up its runner and another bead the same number of spaces down, the corresponding partitions will label Specht modules in the same block for the symmetric group algebra. The following result from [HHKP10] allows us to translate this result into one we can use to relate partitions labelling cell modules for the Brauer algebra.

Theorem 3.16 ([HHKP10, Corollary 6.2])**.** *Let $\lambda, \mu \vdash n-2t$ be partitions, with $\lambda \in \Lambda^*_{B_n}$. Then*

$$[\Delta^k_\mu(n;\delta) : L^k_\lambda(n;\delta)] = [S^\mu_k : D^\lambda_k].$$

In particular, given two partitions $\lambda, \mu \vdash n-2t$, if the two Specht modules S^λ_k and S^μ_k are in the same block over the symmetric group algebra $k\mathfrak{S}_{n-2t}$, then $\mu \in \mathcal{B}^k_\lambda(n;\delta)$.

We therefore see that λ and μ, partitions of $n-2t$, are in the same $B^k_n(\delta)$-block if it is possible to reach one configuration of beads from the other by a sequence of moves that takes two beads, slides one r places down its runner and slides the other r places up. We will use the notation

$$\mu = a^r_{(i,j)}(\lambda)$$

to indicate that the abacus representing the partition μ is obtained from that representing λ by sliding bead i down by r spaces and bead j up by r spaces.

Remark 3.17. Note that this notation will only be well defined after fixing the number of runners p and beads b, which will be the case in all subsequent uses.

In [CDM09b, Section 7] it was shown that for a particular choice of b, the orbit of a partition λ under the δ-shifted action of W_n^p can be found by examination of the abacus. Recall the definition of $\Gamma(\lambda, b)$ from (1.4).

Proposition 3.18 ([CDM09b, Proposition 7.1])**.** *Choose* $b \geq n$ *with* $2b \equiv 2 - \delta$ *(mod p), and* $\lambda, \mu \in \Lambda_{B_n}$*. Then* $\mu \in \mathcal{O}_\lambda^p(b; \delta)$ *if and only if*

(i) $\Gamma(\lambda, b)_0 = \Gamma(\mu, b)_0$*, and*

(ii) $\Gamma(\lambda, b)_l + \Gamma(\lambda, b)_{p-l} = \Gamma(\mu, b)_l + \Gamma(\mu, b)_{p-l}$ *for each* $1 \leq l \leq p-1$.

This pairing of runners suggests an alternative way of viewing the abacus. Given a partition λ, we construct its abacus in the usual way. However we then link runners l and $p - l$ for each $0 < l \leq \frac{p-1}{2}$ via an arc above the diagram. We will refer to this variant as a *linked abacus*. See Figure 3.6 for an example.

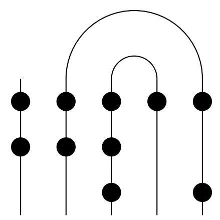


Figure 3.6: The partition $(5, 4)$ on the linked abacus.

Along with sliding beads up and down runners, we can now slide them up and over the top arc. This allows us to introduce the following move: choose two beads, and slide them both r places up their runners, over the arc and back down their respective paired runners. If one of the beads lies on runner 0, that bead simply moves up and back down the runner, visiting position zero just once.

Note that beads may both move from left to right, right to left, or one may move left and the other right. There is also no restriction on which two beads we move, provided that they move the same number of spaces and end in an unoccupied position. Figure 3.7 shows some examples of this move.

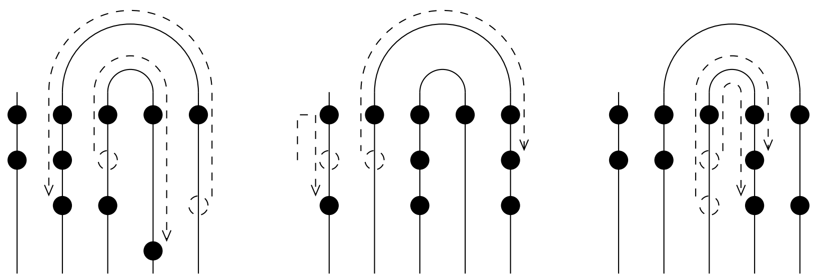


Figure 3.7: Moving beads on the linked abacus given in Figure 3.6.

We will use the notation

$$\mu = d^r_{(i,j)}(\lambda)$$

to indicate that the linked abacus representing the partition μ is obtained from that representing λ by sliding beads i and j both by r spaces up and over the arc to their paired runner (or up and down if on runner 0).

Note that the comments in Remark 3.17 also apply here.

We would like to show that $a^r_{(i,j)}$ and $d^r_{(i,j)}$ allow us to move between partitions in the same block, but this would contradict [CDM09b, Theorem 7.2]. Therefore in order to proceed we must introduce the notion of a *limiting block*.

Recall the embeddings from Remarks 3.1, 3.8 and 3.14. In addition, the functors F_n and G_n give us a full embedding of $B^k_n(\delta)$-**mod** inside $B^k_{n+2}(\delta)$-**mod**. We can then embed $\mathcal{B}^k_\lambda(n;\delta)$ inside $\mathcal{B}^k_\lambda(n+2;\delta)$, and thus consider the limits

$$\begin{aligned}
\Lambda &= \{\lambda : \lambda \vdash n \text{ for some } n \in \mathbb{N}\},\\
E_\infty &= \prod_{i=0}^{\infty} \mathbb{R}\varepsilon_i,\\
W^p_\infty &= \langle s_{\varepsilon_i-\varepsilon_j,rp}, s_{\varepsilon_i+\varepsilon_j,rp} : 1 \le i < j, r \in \mathbb{Z}\rangle,\\
\mathcal{B}^k_\lambda(\infty;\delta) &= \{\mu : \mu \in \mathcal{B}^k_\lambda(n;\delta) \text{ for some } n \in \mathbb{N}\},\\
\mathcal{O}^p_\lambda(\infty;\delta) &= \{\mu \in \Lambda : \mu \in W^p_\infty \cdot_\delta \lambda\},
\end{aligned}$$

where the element $\rho(\delta)$ is extended in the obvious way to

$$\rho(\delta) = \left(-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, \dots\right) \in E_\infty.$$

Notation. When the context is clear, we will write $\mathcal{B}^k_\lambda(\infty)$ and $\mathcal{O}^p_\lambda(\infty)$ to mean $\mathcal{B}^k_\lambda(\infty;\delta)$ and $\mathcal{O}^p_\lambda(\infty;\delta)$ respectively.

Our aim now is to use the moves $a^r_{(i,j)}$ and $d^r_{(i,j)}$ to show that in the limiting case, we have a correspondence between orbits and blocks. In other words $\mathcal{B}^k_\lambda(\infty;\delta) = \mathcal{O}^p_{\lambda^T}(\infty;\delta)$ for all $\lambda \in \Lambda$. We begin by proving the following.

Proposition 3.19. *If $\mu \in \mathcal{B}^K_\lambda(n;\delta+rp)$ for some $r \in \mathbb{Z}$, then $\mu \in \mathcal{B}^k_\lambda(n;\delta)$.*

Proof. By the cellularity of $B^K_n(\delta)$, partitions λ and μ are in the same K-block if and only if there is a sequence of partitions

$$\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t)} = \mu$$

and $B^K_n(\delta)$-modules

$$M^{(i)} \leq \Delta^K_{\lambda^{(i)}}(n;\delta+rp) \qquad (1 < i \leq t),$$

such that for each $1 \leq i < t$

$$\mathrm{Hom}(\Delta^K_{\lambda^{(i)}}(n;\delta+rp), \Delta^K_{\lambda^{(i+1)}}(n;\delta+rp)/M^{(i+1)}) \neq 0.$$

Since $\delta + rp = \delta$ in k, the application of Lemma 1.3 then shows

$$\mathrm{Hom}(\Delta^k_{\lambda^{(i)}}(n;\delta), \Delta^k_{\lambda^{(i+1)}}(n)/\overline{M^{(i+1)}}) \neq 0,$$

giving us such a sequence of partitions linking λ and μ, except now we are working with $B^k_n(\delta)$-modules. □

We have an interpretation of $d^r_{(i,j)}$ as the action of an element of the Weyl group W_b:

Lemma 3.20. *If two partitions $\lambda, \mu \in \Lambda_{B_n}$, both represented with b beads on a linked abacus with p runners, are related by the move $d^r_{(i,j)}(\lambda) = \mu$, then $\mu \in \mathcal{O}_\lambda(b;\delta')$ for $\delta' = rp - 2b + 2$.*

Proof. Suppose $\lambda = (\lambda_1, \ldots, \lambda_l)$, then

$$\beta(\lambda, b) = (\lambda_1 - 1 + b, \lambda_2 - 2 + b, \ldots, \lambda_l - l + b, -(l+1) + b, \ldots, 0).$$

The move $d^r_{(i,j)}$ can be represented in the β-sequence. Indeed, we can decompose $r = r_1 + 1 + r_2$, where r_1 is the number of positions needed to move to the top of the runner, then 1 for over the arc, and finally r_2 positions down. Then:

1. Sliding bead i up by r_1 spaces puts that bead in position $\lambda_i - i + b - r_1 p$.

2. Moving across to the paired runner places it in position $p - (\lambda_i - i + b - r_1 p)$.

3. Finally, sliding it down by r_2 spaces lands it in position

$$p - (\lambda_i - i + b - r_1 p) + r_2 p = -\lambda_i + i - b + rp.$$

A similar process leads to the same result if the bead is on runner zero: we just write $r = r_1 + r_2$ and omit Step 2.

This leaves us with the sequence

$$(\lambda_1 - 1 + b, \ldots, \underbrace{-\lambda_i + i - b + rp}_{i\text{-th place}}, \ldots, \underbrace{-\lambda_j + j - b + rp}_{j\text{-th place}}, \ldots, 0).$$

Note that this will not be the β-sequence of a partition, i.e. will not be a strictly decreasing sequence. However by re-arranging the sequence we see that setwise it must be that of $\beta(\mu, b)$, since the beads occupy the same positions in the linked abacus. Therefore, for an appropriate element $w \in \mathfrak{S}_b$ of the symmetric group on b letters, we have:

$$w^{-1}(\beta(\mu, b)) = (\lambda_1 - 1 + b, \ldots, \underbrace{-\lambda_j + j - b + rp}_{i\text{-th place}}, \ldots, \underbrace{-\lambda_i + i - b + rp}_{j\text{-th place}}, \ldots, 0)$$

and hence

$$\begin{aligned} \beta(\lambda, b) - w^{-1}(\beta(\mu, b)) &= (\lambda_i + \lambda_j - (i + j) + 2b - rp)(\varepsilon_i + \varepsilon_j) \\ &= \langle \lambda + \rho(rp - 2b + 2), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j) \\ &= \langle \lambda + \rho(\delta'), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j), \qquad (3.11) \end{aligned}$$

making the substitution $\delta' = rp - 2b + 2$ and using the inner product defined in (3.9). Now define

$$\eta = (1, 2, 3, \ldots, b), \qquad \omega = (1, 1, \ldots, 1).$$

We may then rewrite (3.11) as

$$w^{-1}(\mu - \eta + b\omega) = \lambda - \eta + b\omega - \langle \lambda + \rho(\delta'), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j),$$

and noticing that ω is invariant under the action of $\mathfrak{S}_b$:

$$\begin{aligned}
&w^{-1}(\mu - \eta) = \lambda - \eta - \langle \lambda + \rho(\delta'), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j) \\
\implies & w^{-1}\left(\mu + \rho(\delta') + \left(\frac{\delta'}{2} - 1\right)\omega\right) = \lambda + \rho(\delta') + \left(\frac{\delta'}{2} - 1\right)\omega \\
& \qquad\qquad\qquad\qquad\qquad\qquad - \langle \lambda + \rho(\delta'), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j) \\
\implies & w^{-1}(\mu + \rho(\delta')) = \lambda + \rho(\delta') - \langle \lambda + \rho(\delta'), \varepsilon_i + \varepsilon_j \rangle (\varepsilon_i + \varepsilon_j) \\
\implies & \mu = w s_{\varepsilon_i + \varepsilon_j}(\lambda + \rho(\delta')) - \rho(\delta').
\end{aligned}$$

If we write $w = (i_1\ j_1)(i_2\ j_2)\dots(i_t\ j_t)$ as a product of transpositions, then the action of this element is the same as that of $s_{\varepsilon_{i_1}-\varepsilon_{j_1}} s_{\varepsilon_{i_2}-\varepsilon_{j_2}} \dots s_{\varepsilon_{i_t}-\varepsilon_{j_t}} \in W_b$. So we may assume that $w \in W_b$. Therefore we may obtain μ by

$$\mu = w s_{\varepsilon_i + \varepsilon_j} \cdot_{\delta'} \lambda$$

where $w s_{\varepsilon_i + \varepsilon_j} \in W_b$. □

Corollary 3.21. *If two partitions $\lambda, \mu \in \Lambda_{B_n}$, both represented with b beads on a linked abacus with p runners, are related by a single move $a^r_{(i,j)}$ or $d^r_{(i,j)}$, then $\mu^T \in \mathcal{B}^k_{\lambda^T}(b)$.*

Proof. If $\mu = a^r_{(i,j)}(\lambda)$ then we apply Nakayama's Conjecture (Theorem 1.8) and Theorem 3.16 to show this result.

If $\mu = d^r_{(i,j)}(\lambda)$, then by Lemma 3.20

$$\mu \in \mathcal{O}_\lambda(b; \delta')$$

where $\delta' = rp - 2b + 2$. By Theorem 3.10, we have that $\mu^T \in \mathcal{B}^K_{\lambda^T}(n; \delta')$. Proposition 3.19 then shows that they must then be in the same $B^k_n(\delta)$-block, since our condition on b

$$2b \equiv 2 - \delta \pmod p$$

implies $\delta' \equiv \delta \pmod p$, so $\delta = \delta'$ as elements of k . □

We demonstrate this result below.

Example 3.22. Recall the linked abacus in Figure 3.6 and the leftmost one in Figure 3.7. They represent partitions $\lambda = (5, 4)$ and $\mu = (9, 4^2)$ respectively, labelling

$B_n^k(2)$-cell modules over a field k of characteristic 5. They are related by the move $\mu = d^5_{(1,3)}(\lambda)$, and we will show that $\mu^T \in \mathcal{B}^k_{\lambda^T}(17;2)$.

Indeed, if we set $\delta' = rp - 2b + 2 = 25 - 20 + 2 = 7$, then

$$\begin{aligned}
s_{\varepsilon_1+\varepsilon_3} \cdot_{\delta'} \lambda &= s_{\varepsilon_1+\varepsilon_3}(\lambda + \rho(\delta')) - \rho(\delta') \\
&= \lambda - (\lambda + \rho(\delta'), \varepsilon_1 + \varepsilon_3)(\varepsilon_1 + \varepsilon_3) \\
&= (5,4,0,0,\dots) - \left(\left(\frac{3}{2}, -\frac{1}{2}, -\frac{11}{2}, -\frac{13}{2}, \dots\right), \varepsilon_1 + \varepsilon_3\right)(\varepsilon_1 + \varepsilon_3) \\
&= (5,4,0,0,\dots) + 4(\varepsilon_1 + \varepsilon_3) \\
&= (9,4,4,0,0,\dots) = \mu.
\end{aligned}$$

Therefore the transposed partitions are in the same $B^K_{17}(7)$-block in characteristic 0. Taking the modular reduction, we have the desired result as $7 \equiv 2 \pmod 5$.

We can also deduce the following corollary.

Corollary 3.23. *If two partitions $\lambda, \mu \in \Lambda_{B_n}$, both represented with b beads on a linked abacus with p runners, are related by a sequence of moves $a^r_{(i,j)}$ and $d^r_{(i,j)}$, then $\mu^T \in \mathcal{B}^k_{\lambda^T}(\infty)$.*

Proof. Let

$$\lambda = \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)} = \mu$$

be a sequence of partitions such that

$$\lambda^{(l+1)} = \begin{cases} a^{r_l}_{(i_l,j_l)}\lambda^{(l)} & \text{or} \\ d^{r_l}_{(i_l,j_l)}\lambda^{(l)} \end{cases}$$

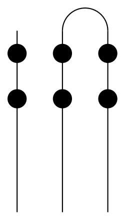
for some values $r_l \in \mathbb{Z}$ and $1 \leq i_l < j_l \leq b$ $(1 \leq l < t)$.
Then by applying Corollary 3.21 we have $\lambda^{(l+1)^T} \in \mathcal{B}^k_{\lambda^{(l)^T}}(n_l)$, where $n_l = \max(|\lambda^{(l)}|, |\lambda^{(l+1)}|)$. Therefore $\mu^T \in \mathcal{B}^k_{\lambda^T}(\infty)$. □

Remark. In Corollary 3.23, the values of n_l may exceed n, and hence the two partitions may not be in the same block when we restrict to $\mathcal{B}^k_{\lambda^T}(n)$. See Example 3.24 below for an example of this.

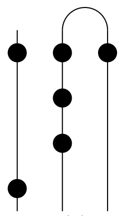
Example 3.24. Consider the Brauer algebra $B_4^k(2)$ where k is a field of characteristic 3, and the partitions $\emptyset, (2) \in \Lambda_{B_4}$. We let $b = 6$ so that the congruence $2b \equiv 2 - \delta$

(mod p) is satisfied, and construct the linked abaci of $\emptyset$ and (2) below.

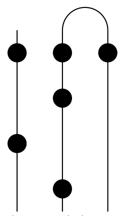
It is clear that these partitions are in the same W_n^p orbit. We can obtain the abacus of one from the other by the following steps:



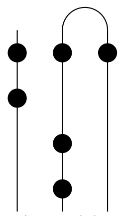
$\lambda^{(1)} = \emptyset,$



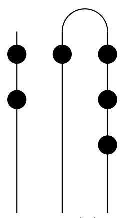
$\lambda^{(2)} = d^4_{(1,3)}(\lambda^{(1)}) = (4,3,1),$



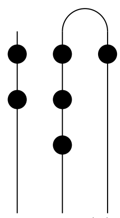
$\lambda^{(3)} = a^{-1}_{(1,2)}(\lambda^{(2)}) = (5,2,1),$



$\lambda^{(4)} = a^{-1}_{(2,3)}(\lambda^{(3)}) = (5,3),$



$\lambda^{(5)} = d^5_{(1,2)}(\lambda^{(4)}) = (3,1),$



$\lambda^{(6)} = d^4_{(1,2)}(\lambda^{(5)}) = (2).$

Note that the partitions $\lambda^{(2)}$, $\lambda^{(3)}$ and $\lambda^{(4)}$ all have size greater than four. Moreover, the partitions $\emptyset$ and $(1^2) = (2)^T$ are not in the same block, as shown by the decomposition matrix of $B_4^k(2)$ from [HP08, Section 8]:

$$\begin{array}{c} \\ (4) \\ (3,1) \\ (2^2) \\ (2,1^2) \\ (1^4) \\ (2) \\ (1^2) \\ \emptyset \end{array} \begin{array}{c} \begin{array}{ccccccc} (4) & (3,1) & (2^2) & (2,1^2) & (2) & (1^2) & \emptyset \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

We now want to show that given two partitions in the same W_n^p-orbit, their linked abaci can be related by a series of moves $a^r_{(i,j)}$ and $d^r_{(i,j)}$, and their transposes are therefore in the same $B_N^k(\delta)$-block (for some N). We do this by defining a "b-reduced linked abacus", and show that we can arrive at it from our partition using only moves $a^r_{(i,j)}$ and $d^r_{(i,j)}$.

Definition 3.25. A linked abacus with p runners is called *b-reduced* if it contains b beads, all of which are on runners 0 to $\frac{p-1}{2}$, and all beads are as high up on their runners as possible, except for the last bead on runner 0 which may be one space down.

For a fixed prime p, we say a partition is b-reduced if when represented using b beads on a linked abacus with p runners, that abacus is b-reduced.

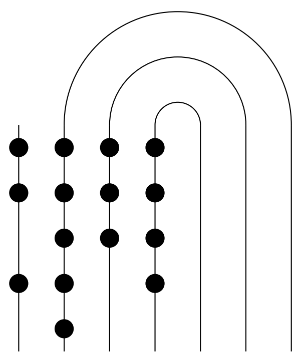


Figure 3.8: An example of a reduced linked abacus.

Given the linked abacus arrangement of a partition λ with b beads, we define the *b-reduction of* λ, written $\overline{\lambda}^b$, to be the b-reduced linked abacus satisfying:

(i) $\Gamma(\lambda, b)_0 = \Gamma(\overline{\lambda}^b, b)_0$,

(ii) $\Gamma(\lambda, b)_l + \Gamma(\lambda, b)_{p-l} = \Gamma(\overline{\lambda}^b, b)_l + \Gamma(\overline{\lambda}^b, b)_{p-l}$ for each $1 \leq l \leq p-1$, and

(iii) $|\overline{\lambda}^b| - |\lambda| \in 2\mathbb{Z}$.

Remark. This last condition determines whether or not the final bead on runner 0 is moved down a space or not, and thus ensures that the reduction is unique.

Example 3.26 below shows such a construction.

For a fixed value of b (satisfying the congruence condition of Proposition 3.18), the W_b^p-orbits on partitions represented with b beads are characterised by the number of beads on runner 0 and the total number of beads on pairs of runners l and $p-l$ for each $l > 0$. As a result, each W_b^p-orbit corresponds to a unique b-reduced partition, obtained by taking the reduction of any partition in the orbit.

We will now describe an algorithm for constructing $\overline{\lambda}^b$ from λ, using only combinations of $a^r_{(i,j)}$ and $d^r_{(i,j)}$. In order to make this process more understandable, we use these basic moves to build 4 others which makes the manipulation of the linked abacus easier:

(M1) We can slide a bead one space up a runner, provided we move another bead down a space. (See Figure 3.9)

This is simply the move $a^1_{(i,j)}$.

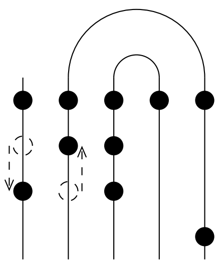


Figure 3.9: An example of move (M1).

(M2) We can move a bead and the one directly below on the same runner together over the top of the linked abacus to any (unoccupied) position on the paired runner. (See Figure 3.10)

This is the move $d^r_{(i,j)}$, where i and j are consecutive beads on the same runner.

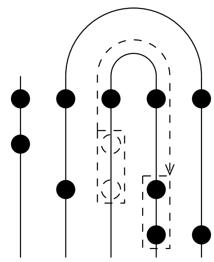


Figure 3.10: An example of move (M2).

(M3) Provided there is a runner with at least two beads on it, we can move any two beads each one space up their runner, or one bead up two spaces. (See Figures 3.11 and 3.12)

We do this by combining (M2) and (M1). Suppose we wish to move beads i and j each up by one space. We first choose the runner with at least two beads on it, and let the last two (those lowest down) be x and y, ensuring that neither are equal to i or j.

We then apply (M1) twice, specifically the moves that push bead i up and x down, then j up and y down. Next we perform $d^r_{(x,y)}$ followed by $d^{r-1}_{(x,y)}$ (for a sufficiently large value of r), so that the beads x and y finish in their original places (before the first use of (M1)). Note that we are not re-labelling beads as we move them.

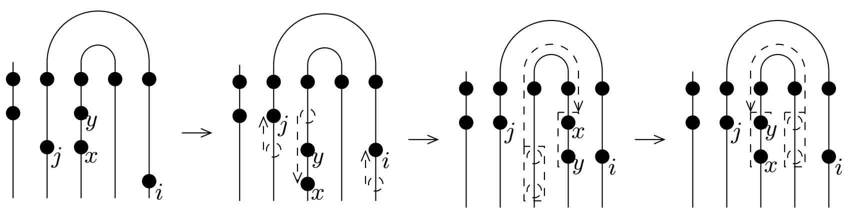


Figure 3.11: An example of move (M3).

Remark. To simplify this move, we have insisted that x and y are distinct from i and j. However we can make modifications to remove this restriction. If we follow the process as given, then after an application of (M1) there will be a bead that moves up and back to its original place, and we can then apply (M2) as usual. An example is shown in Figure 3.12. In all practical purposes however, there will usually be enough beads to allow us to use the move as originally stated.

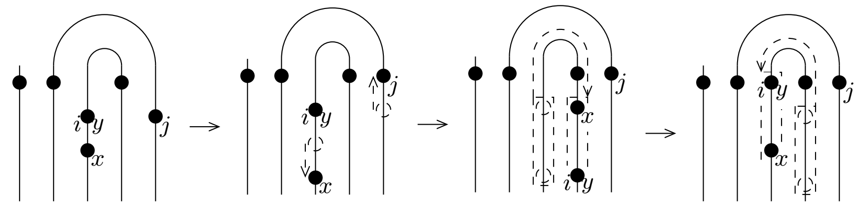


Figure 3.12: An example of move (M3) with beads i and y equal. Note that in step 2 the bead $i = y$ does not move.

(M4) Provided there is a runner with at least two beads on it, we can move any two beads each one space down their runner, or one bead down two spaces.

This is simply the reverse of the move (M3).

Now that we have detailed all the necessary moves, we describe an algorithm for obtaining the b-reduction of a given partition λ:

1. Construct the linked abacus of the partition λ in the usual way, choosing a large enough value of b so that there are at least 3 beads on runner 0, i.e. $\Gamma(\lambda, b)_0 \geq 3$.

2. Using (M3), we may move beads up in pairs so that each one is as high on the runner as it will go. Since we have chosen b so that there are always at least 3 beads on runner 0, we may use this runner when applying (M3). Now there may be a single bead with a space above it. If this is the case, and this bead is not the last bead on runner 0, then we apply (M1) and move this bead up and the last bead on runner 0 down.

3. Applying (M2) repeatedly, we then slide beads in pairs over from the right side to the left, so that there is at most one bead on the right hand runners.

4. If there are no more beads on runners $\frac{p+1}{2}$ to $p-1$, then we have the reduced abacus. So assume that there is a bead on runner l in this range. Pair the beads consecutively on runner $p-l$, starting from the one furthest down, and move

each pair down by two spaces with (M4). This ensures an empty space in the second position on runner $p - l$ (the first may or may not be filled).

5. Since there are at least 3 beads on runner 0 there must be one in the second position. Move this and the single bead on runner l by 2 spaces over the arc (this is just the move $d^2_{(i,j)}$). The bead on runner 0 returns to the same space, but there are now no beads on runner l.

6. If there is an empty space in the first position of runner $p - l$, we use (M3) to slide all the beads up in pairs to fill this gap. If the total number of beads now on the runner is odd, then the final bead cannot be moved in a pair. We then have one of the two following cases:

   (a) The last bead on runner 0 has a space before it. In this case, we can use (M3) to move this and the last bead on runner l each up by one space.

   (b) The last bead on runner 0 does not have a space before it. In this case, we use (M1) to slide the bead on runner l up and the bead on runner 0 down each by one space.

7. Repeat Steps 4-6 until there are no more beads on runners $\frac{p+1}{2}$ to $p - 1$. We then have a reduced linked abacus.

Example 3.26. Setting $p = 7$ we construct the b-reduction of $\lambda = (5^2, 4^2, 3, 2^3, 1)$ following the steps above. We describe the process below, and show this on the linked abacus in Figure 3.13.

1. Choosing $b = 22$ ensures that we satisfy the requirement that there are at least 3 beads on runner 0.

2. We can slide the last beads on runners 5 and 6 up (M3), but we must then move the last bead on runner 0 down one space and the last bead on runner 1 up one space (M1).

3. After sliding the beads over in pairs with (M2), we are left with a single bead at the top of runner 5.

4. In pairs, the beads are moved down runner 2 by two spaces each with (M4), leaving 2 empty spaces at the top.

5. The second bead on runner 0 and the bead on runner 5 are each moved by 2 spaces up and over (the move $d^2_{(i,j)}$), so that there are now no beads on runner 5.

6. In pairs, the beads are moved up runner 2 by one space each with (M3). This leaves a single bead at the bottom with an empty space before it. Therefore we use (M3) and push this and the last bead on runner 0 up one space so that we arrive at the reduced linked abacus.

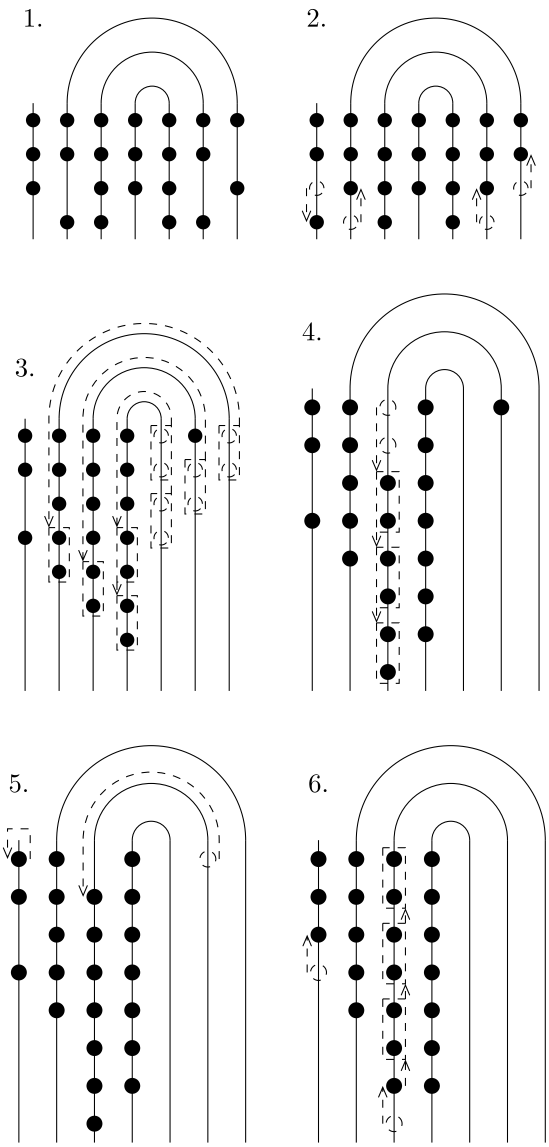


Figure 3.13: Constructing the b-reduction of $(5^2, 4^2, 3, 2^3, 1)$.

We may now state the main theorem.

Theorem 3.27. *Let $\lambda, \mu \in \Lambda$. We have $\mathcal{B}^k_{\lambda^T}(\infty;\delta) = \{\mu^T : \mu \in \mathcal{O}^p_\lambda(\infty;\delta)\}$. In other words, λ and μ are in the same W^p_∞-orbit under the δ-shifted action if and only if their transposes label cell modules in the same $B^k_n(\delta)$-block for some n (and hence all $m \geq n$).*

Proof. If $\mu^T \in \mathcal{B}^k_{\lambda^T}(\infty;\delta)$ then $\mu \in \mathcal{O}^p_\lambda(\infty;\delta)$ by Theorem 3.15.

Conversely, given $\mu \in \mathcal{O}^p_\lambda(\infty;\delta)$ choose $b \geq \max(|\lambda|,|\mu|)$ satisfying both the congruence of Proposition 3.18 and the requirement that there are at least 3 beads on runner 0 of both abaci of the partitions, i.e. $\Gamma(\lambda,b)_0 \geq 3$. Then the b-reductions $\overline{\lambda}^b$ and $\overline{\mu}^b$ of λ and μ must coincide, since they have the same number of beads on corresponding pairs of runners. Therefore it is possible to reach the linked abacus arrangement of μ from that of λ using a sequence of moves $a^r_{(i,j)}$ and $d^r_{(i,j)}$, travelling via the reduced abacus. Corollary 3.23 then gives the result. $\square$

3.4.2 Decomposition numbers

The results of the previous section give us the limiting blocks of the Brauer algebra, but do not provide any details about the structure of the blocks, in particular the composition factors of cell modules or homomorphisms between them. This section, which has been published in [Kin14a, Section 5], will give some results regarding this.

As mentioned in Section 3.3 we may view partitions as points in a Euclidean space E_n, and the (affine) Weyl group acts as a group of isometries of this. In particular, the elements $s_{\alpha,rp}$ ($\alpha \in \Phi$) correspond to reflections through hyperplanes. Let λ and μ be partitions that are related via a reflection in such a hyperplane. We will show that, possibly after swapping the roles of λ and μ, we may assume $\mu \subseteq \lambda$ or $\mu \triangleleft \lambda$ (where $\triangleleft$ denotes the dominance order with size on partitions, see Section 1.3.1 for details). Moreover, in that case there is a non-zero homomorphism from $\Delta^{\mathbb{F}}_{\lambda^T}(n)$ to $\Delta^{\mathbb{F}}_{\mu^T}(n)$. In particular, this will show that

$$[\Delta^{\mathbb{F}}_{\mu^T}(n) : L^{\mathbb{F}}_{\lambda^T}(n)] \neq 0$$

whenever the simple module $L^{\mathbb{F}}_{\lambda^T}(n)$ exists.

Proposition 3.28. *Let $\lambda, \mu \in \Lambda_{B_n}$. If there is a reflection $s_{\varepsilon_i+\varepsilon_j,rp} \in W^p_n$*

$(1 \leq i < j \leq n)$ *such that*

$$s_{\varepsilon_i+\varepsilon_j,rp} \cdot_\delta \lambda = \mu,$$

then $\mu = \lambda - (\lambda_i + \lambda_j - \delta - rp - i - j - 2)(\varepsilon_i + \varepsilon_j)$. *In particular, either* $\mu \subseteq \lambda$ *or* $\lambda \subseteq \mu$.

Proof. We have

$$\begin{aligned} s_{\varepsilon_i+\varepsilon_j,rp} \cdot_\delta \lambda &= s_{\varepsilon_i+\varepsilon_j,rp}(\lambda + \rho(\delta)) - \rho(\delta) \\ &= \lambda + \rho(\delta) - (\langle \lambda + \rho(\delta), \varepsilon_i + \varepsilon_j \rangle - rp)(\varepsilon_i + \varepsilon_j) - \rho(\delta) \\ &= \lambda - \left(\lambda_i + \lambda_j - \frac{\delta}{2} - (i-1) - \frac{\delta}{2} - (j-1) - rp\right)(\varepsilon_i + \varepsilon_j) \\ &= \lambda - (\lambda_i + \lambda_j - \delta - rp - i - j + 2)(\varepsilon_i + \varepsilon_j). \end{aligned}$$

If $(\lambda_i + \lambda_j - \delta - rp - i - j + 2) \geq 0$ then $\mu \subseteq \lambda$, whereas if $(\lambda_i + \lambda_j - \delta - rp - i - j + 2) \leq 0$ then $\lambda \subseteq \mu$. $\square$

Recall the notion of two partitions being maximally δ-balanced from Definition 3.3. We now prove the following:

Proposition 3.29. *Let* $\lambda, \mu \in \Lambda_{B_n}$. *If there is a reflection* $s_{\varepsilon_i+\varepsilon_j} \in W_n$ $(1 \leq i < j \leq n)$ *such that*

$$s_{\varepsilon_i+\varepsilon_j} \cdot_\delta \lambda = \mu$$

with $\mu \subseteq \lambda$, *then* μ^T *and* λ^T *are maximally* δ*-balanced.*

Proof. We first show that μ^T and λ^T are δ-balanced. For more details, see the proof of [CDM09b, Theorem 4.2].

From Proposition 3.28 we have $\mu = \lambda - (\lambda_i + \lambda_j - \delta - i - j + 2)(\varepsilon_i + \varepsilon_j)$. If $\mu = \lambda$ there is nothing to prove, so we will assume that $\mu \subsetneq \lambda$ and see that $[\lambda] \backslash [\mu]$ consists of two strips of nodes in rows i and j. The content of the last node in row i of $[\mu]$ is given by

$$\begin{aligned} \mu_i - i &= \lambda_i - (\lambda_i + \lambda_j - \delta - i - j + 2) - i \\ &= -\lambda_j + j + \delta - 2. \end{aligned}$$

Therefore the content of the first node in row i of $[\lambda] \backslash [\mu]$ is $\mu_i - i + 1 = -\lambda_j + j + \delta - 1$, and so the nodes in row i of $[\lambda] \backslash [\mu]$ have content

$$-\lambda_j + j + \delta - 1, -\lambda_j + j + \delta, \ldots, \lambda_i - (i-2), \lambda_i - (i-1), \lambda_i - i.$$

Similarly, the nodes in row j of $[\lambda]\backslash[\mu]$ have content

$$-\lambda_i + i + \delta - 1, -\lambda_i + i + \delta, \ldots, \lambda_j - (j-2), \lambda_j - (j-1), \lambda_j - j. \tag{3.12}$$

If we pair these two rows in reverse order, the contents of each pair sum to $\delta - 1$. If we take the transpose of the partitions, we then have a pairing of two columns of nodes such that the content of each pair sums to $1-\delta$, satisfying condition (i) of Definition 3.3 above. Moreover, since after taking the transpose we are always pairing nodes in different columns, condition (ii) is also satisfied. Hence μ^T and λ^T are δ-balanced.

Suppose now there is a partition ν such that $\mu^T \subseteq \nu^T \subsetneq \lambda^T$ with λ^T and ν^T δ-balanced. Then after transposing, $[\lambda]\backslash[\nu]$ consists of nodes from rows i and j and, since $\nu \neq \lambda$, must contain at least one of the final nodes in row i or j, say row i (the case of row j is similar).

This final node has content $\lambda_i - i$ and, since λ^T and ν^T are δ-balanced, must be paired with a node of content $\delta - 1 - (\lambda_i - i)$. But using (3.12) above, and the fact that $i < j$, we have that the only such node in row i or j of $[\lambda]\backslash[\mu]$ is the first in row j. As $[\lambda]\backslash[\nu]$ now must contain the first node in row j of $[\lambda]\backslash[\mu]$, it contains all nodes in row j of $[\lambda]\backslash[\mu]$, in particular the final node in this row.

By repeating the argument of the above paragraph, we see that $[\lambda]\backslash[\nu]$ also contains all the nodes in row i of $[\lambda]\backslash[\mu]$, hence $\nu = \mu$. Therefore $\nu^T = \mu^T$ and we deduce that μ^T and λ^T are maximally δ-balanced. □

We may now show:

Theorem 3.30. *Let $\lambda, \mu \in \Lambda_{B_n}$. If there is a reflection $s_{\varepsilon_i+\varepsilon_j,rp} \in W_n^p$ $(1 \leq i < j \leq n)$ such that*

$$s_{\varepsilon_i+\varepsilon_j,rp} \cdot_\delta \lambda = \mu$$

then we have (possibly after swapping the roles of λ and μ) $\mu \subseteq \lambda$ and $\mathrm{Hom}(\Delta^k_{\lambda^T}(n;\delta), \Delta^k_{\mu^T}(n;\delta)) \neq 0$. In particular, if $\lambda^T \in \Lambda^*_{B_n}$ is p-regular we have a non-zero decomposition number $[\Delta^k_{\mu^T}(n;\delta) : L^k_{\lambda^T}(n;\delta)] \neq 0$

Proof. If $s_{\varepsilon_i+\varepsilon_j,rp} \cdot_\delta \lambda = \mu$ then by Proposition 3.28 we have either $\mu \subseteq \lambda$ or $\lambda \subseteq \mu$. Since $s_{\varepsilon_i+\varepsilon_j,rp}$ is a reflection we may swap λ and μ if necessary and always assume the former case.

Next, notice that

$$
\begin{aligned}
s_{\varepsilon_i+\varepsilon_j,rp} \cdot_\delta \lambda &= s_{\varepsilon_i+\varepsilon_j,rp}(\lambda+\rho(\delta)) - \rho(\delta) \\
&= \lambda+\rho(\delta) - (\langle \lambda+\rho(\delta), \varepsilon_i+\varepsilon_j\rangle - rp)(\varepsilon_i+\varepsilon_j) - \rho(\delta) \\
&= \lambda+\rho(\delta) \\
&\quad - \left(\lambda_i+\lambda_j - \frac{\delta}{2} - (i-1) - \frac{\delta}{2} - (j-1) - rp\right)(\varepsilon_i+\varepsilon_j) - \rho(\delta) \\
&= \lambda+\rho(\delta+rp) \\
&\quad - \left(\lambda_i+\lambda_j - \frac{\delta+rp}{2} - (i-1) - \frac{\delta+rp}{2} - (j-1)\right)(\varepsilon_i+\varepsilon_j) \\
&\quad -\rho(\delta+rp) \\
&= \lambda+\rho(\delta+rp) - \langle \lambda+\rho(\delta+rp), \varepsilon_i+\varepsilon_j\rangle(\varepsilon_i+\varepsilon_j) - \rho(\delta+rp) \\
&= s_{\varepsilon_i+\varepsilon_j} \cdot_{\delta+rp} \lambda.
\end{aligned}
$$

Therefore we have $s_{\varepsilon_i+\varepsilon_j} \cdot_{\delta+rp} \lambda = \mu$ with $\mu \subseteq \lambda$, so by Proposition 3.29 we see that λ^T and μ^T are maximally $(\delta+rp)$-balanced. Theorem 3.5 then shows that

$$\mathrm{Hom}(\Delta^K_{\lambda^T}(n;\delta+rp), \Delta^K_{\mu^T}(n;\delta+rp)) \neq 0.$$

As the cell modules have a basis over R (see Section 3.2.2), we can consider the p-modular reductions of these and using Lemma 1.3 conclude that,

$$\mathrm{Hom}(\Delta^k_{\lambda^T}(n;\delta), \Delta^k_{\mu^T}(n;\delta)) \neq 0.$$

If now we assume that $\lambda^T \in \Lambda^*_{B_n}$, then the simple module $L^k_{\lambda^T}(n)$ exists and we have a non-zero decomposition number

$$[\Delta^k_{\mu^T}(n;\delta) : L^k_{\lambda^T}(n;\delta)] \neq 0.$$

$\square$

To prove the corresponding result for reflections of type $s_{\varepsilon_i-\varepsilon_j,rp}$, we will require the following theorem from [CP80] and an analogue of Proposition 3.28.

Theorem 3.31 ([CP80])**.** *Let $\lambda, \mu \vdash n$ and suppose the Young diagram of λ is obtained from that of μ by raising d nodes from row j to row i. Suppose $\lambda_i - \lambda_j + j - i - d$ is divisible by p^e, and also that $d < p^e$. Then* $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda_k, S^\mu_k) \neq 0$

Proposition 3.32. *Let $\lambda, \mu \in \Lambda_{B_n}$. If there is a reflection $s_{\varepsilon_i - \varepsilon_j, rp} \in W_n^p$ $(1 \leq i < j \leq n)$ such that*

$$s_{\varepsilon_i - \varepsilon_j, rp} \cdot_\delta \lambda = \mu$$

then $\mu = \lambda - (\lambda_i - \lambda_j - i + j - rp)(\varepsilon_i - \varepsilon_j)$. In particular, $|\lambda| = |\mu|$ and either $\mu \lhd \lambda$ or $\lambda \lhd \mu$, where $\lhd$ is the dominance order with size on partitions.

Proof. We have

$$\begin{aligned}
s_{\varepsilon_i - \varepsilon_j, rp} \cdot_\delta \lambda &= s_{\varepsilon_i - \varepsilon_j, rp}(\lambda + \rho(\delta)) - \rho(\delta) \\
&= \lambda + \rho(\delta) - (\langle \lambda + \rho(\delta), \varepsilon_i - \varepsilon_j \rangle - rp)(\varepsilon_i - \varepsilon_j) - \rho(\delta) \\
&= \lambda + \rho(\delta) \\
&\quad - \left(\lambda_i - \lambda_j - \frac{\delta}{2} - (i-1) + \frac{\delta}{2} + (j-1) - rp \right)(\varepsilon_i - \varepsilon_j) - \rho(\delta) \\
&= \lambda - (\lambda_i - \lambda_j - i + j - rp)(\varepsilon_i - \varepsilon_j).
\end{aligned}$$

So the effect of $s_{\varepsilon_i - \varepsilon_j, rp}$ is to remove nodes from one row of the Young diagram of λ and add the same number to another row. It is then clear that $|\lambda| = |\mu|$, and it remains to consider the following three cases:

- If $(\lambda_i - \lambda_j - i + j - rp) = 0$, then $\lambda = \mu$ and the result follows trivially.
- If $(\lambda_i - \lambda_j - i + j - rp) > 0$, then we are moving nodes in the Young diagram of λ from row i into row j. Since $i < j$, we are moving nodes into a lower row and therefore $\mu \lhd \lambda$.
- If $(\lambda_i - \lambda_j - i + j - rp) < 0$ then we move the nodes from row j to row i. Since $j > i$, we are moving nodes to a higher row and therefore $\lambda \lhd \mu$.

□

We can now prove the following:

Theorem 3.33. *Let $\lambda, \mu \in \Lambda_{B_n}$, with $\lambda^T \in \Lambda^*_{B_n}$ p-regular. If there is a reflection $s_{\varepsilon_i - \varepsilon_j, rp} \in W_n^p$ such that*

$$s_{\varepsilon_i - \varepsilon_j, rp} \cdot_\delta \lambda = \mu$$

then we have (possibly after swapping the roles of λ and μ) $\mu^T \lhd \lambda^T$ and $\mathrm{Hom}(\Delta^k_{\lambda^T}(n;\delta), \Delta^k_{\mu^T}(n;\delta)) \neq 0$. *In particular, we have a non-zero decomposition number $[\Delta^k_{\mu^T}(n;\delta) : L^k_{\lambda^T}(n;\delta)] \neq 0$.*

Proof. If $s_{\varepsilon_i-\varepsilon_j,rp}\cdot_\delta\lambda=\mu$ then by Proposition 3.32 we have either $\mu\lhd\lambda$ or $\lambda\lhd\mu$. Since $s_{\varepsilon_i-\varepsilon_j,rp}$ is a reflection we may swap λ and μ if necessary and always assume the latter case, i.e. that we are raising nodes in λ to obtain μ. We can also assume $r\neq 0$, since otherwise $s_{\varepsilon_i-\varepsilon_j,rp}\cdot_\delta\lambda$ cannot be a partition.

From Proposition 3.32 we set $d=(\lambda_i-\lambda_j-i+j-rp)$ and $e=1$ in the context of Theorem 3.31, and obtain

$$\lambda_i-\lambda_j-i+j-d=rp$$

which is divisible by p as $r\neq 0$. The condition $d<p^e$ is also satisfied, since if we were able to move p or more nodes then λ^T would not be p-regular. Therefore we may apply Theorem 3.31 to deduce

$$\mathrm{Hom}_{k\mathfrak{S}_m}(S_k^\mu,S_k^\lambda)\neq 0$$

where $m=|\lambda|$. Now recall from [JK81, Section 7] that $S_k^\lambda\otimes S_k^{(1^m)}\cong(S_k^{\lambda^T})^*$, the dual of the corresponding Specht module. Therefore

$$\mathrm{Hom}_{k\mathfrak{S}_m}((S_k^{\mu^T})^*,(S_k^{\lambda^T})^*)\neq 0$$

and consequently

$$\mathrm{Hom}_{k\mathfrak{S}_m}(S_k^{\lambda^T},S_k^{\mu^T})\neq 0.$$

Note that $\lambda\lhd\mu$ if and only if $\mu^T\lhd\lambda^T$, see for example [JK81, Lemma 1.4.11]. Thus by Theorem 3.16,

$$\mathrm{Hom}(\Delta^k_{\lambda^T}(n;\delta),\Delta^k_{\mu^T}(n;\delta))\neq 0.$$

Now as $\lambda^T\in\Lambda^*_{B_n}$, the simple module $L^k_{\lambda^T}(n)$ exists and we must then have a non-zero decomposition number

$$[\Delta^k_{\mu^T}(n;\delta):L^k_{\lambda^T}(n;\delta)]\neq 0.$$

□

3.4.3 Decomposition matrices

Recall from Definition 1.2 that we can reduce a K-module to a k-module in such a way that its composition factors are well-defined. We can therefore compute the decomposition numbers of a cell module over k by first finding the composition factors over K, then reducing these to k. Thus, we obtain the following factorisation of the decomposition matrix.

$$[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] = \sum_{\nu \in \Lambda_{B_n}} [\Delta_\mu^K(n;\delta+rp) : L_\nu^K(n;\delta+rp)][\overline{L_\nu^K(n;\delta+rp)} : L_\lambda^k(n;\delta)] \tag{3.13}$$

where $\lambda \in \Lambda_{B_n}^*$, $r \in \mathbb{Z}$ is fixed and $\delta \in R$ is identified with its image in k as usual. Note that by varying r it is possible to obtain several distinct factorisations of this form. We also obtain the following immediate result.

Lemma 3.34. *Let $\delta \in R$, $\mu \in \Lambda_{B_n}$, $\lambda \in \Lambda_{B_n}^*$ and $r \in \mathbb{Z}$. Then*

$$[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \geq [\Delta_\mu^K(n;\delta+rp) : L_\lambda^K(n;\delta+rp)].$$

Proof. This follows immediately from the fact that each part of the sum in (3.13) is non-negative. □

The factorisation (3.13) allows us to use the decomposition numbers in characteristic zero in determining those in characteristic p. However the second constituent of each summand is in general very difficult to compute. In this section we will simplify the process by placing restrictions on the values of n, p and δ. Such restrictions will be based on the semisimplicity criterion from [Rui05], which was made explicit in [RS06]. We recall the result (specialised to characteristic zero) below:

Theorem 3.35 ([RS06, Corollary 2.5])**.** *Suppose $\delta \neq 0$. Then $B_n^K(\delta)$ is semisimple if and only if $\delta \notin \mathbb{Z}(n)$, where*

$$\mathbb{Z}(n) = \{i \in \mathbb{Z} : 4-2n \leq i \leq n-2\} \backslash \{i \in \mathbb{Z} : 4-2n < i \leq 3-n, 2 \nmid i\}.$$

We henceforth assume that $4n-2 < p$. By examining the semisimplicity criterion in Theorem 3.35, note that imposing this condition ensures that for all δ, there is at most one $r \in \mathbb{Z}$ such that $B_n^K(\delta+rp)$ is non-semisimple. If there is such an r, then we assume for the rest of this section that $\delta \in \mathbb{Z}(n)$.

We now to prove the following:

Lemma 3.36. *Suppose $\lambda \in \Lambda_{B_n}$ and $4n-2 < p$. Then $\mathcal{O}_\lambda^p(n;\delta) = \mathcal{O}_\lambda(n;\delta)$.*

Proof. If $\mu \in \mathcal{O}_\lambda(n;\delta)$, then trivially we have $\mu \in \mathcal{O}_\lambda^p(n;\delta)$.

If $\mu \in \mathcal{O}^p_\lambda(n;\delta)$, then by [CDM09b, Proposition 5.3] we have a permutation $\sigma \in \mathfrak{S}_n$ such that for all $1 \leq i \leq n$, either

$$\begin{aligned} & \mu_i - i \equiv \lambda_{\sigma(i)} - \sigma(i) \pmod p \\ \text{or} \qquad & \mu_i - i + \lambda_{\sigma(i)} - \sigma(i) \equiv 2 - \delta \pmod p, \end{aligned}$$

and the latter occurs for an even number of $1 \leq i \leq n$.

Suppose that for some i, we are in the former case. The largest value that $\mu_i - i$ can take is when $\mu = (n)$ and $i = 1$, so that $\mu_1 - 1 = n - 1$. The smallest value occurs when $\mu_n = 0$, so that $\mu_n - n = -n$. Since the difference is $(n-1) + n = 2n - 1 < p$, we see that if $\mu_i - i \equiv \lambda_{\sigma(i)} - \sigma(i) \pmod p$, then in fact $\mu_i - i = \lambda_{\sigma(i)} - \sigma(i)$.

Suppose now we are in the latter case. Since $\delta \in \mathbb{Z}(n)$, Theorem 3.35 tells us that $-n + 4 \leq 2 - \delta \leq 2n - 2$, and hence for some $r \in \mathbb{Z}$

$$-n + 4 \leq \mu_i - i + \lambda_{\sigma(i)} - \sigma(i) + rp \leq 2n - 2.$$

As in the previous case, the smallest value of $\mu_i - i$ is $-n$, so we deduce that

$$\begin{aligned} -2n + rp \leq 2n - 2 &\implies rp \leq 4n - 2 \\ &\implies r \leq 0. \end{aligned}$$

Again, the maximal value of $\mu_i - i$ is $n - 1$, so we also have

$$\begin{aligned} -n + 4 \leq 2n - 2 + rp &\implies rp \geq -3n + 6 \\ &\implies r \geq 0. \end{aligned}$$

Thus $r = 0$, and we have a permutation $\sigma \in \mathfrak{S}_n$ such that for all $1 \leq i \leq n$, either

$$\begin{aligned} & \mu_i - i = \lambda_{\sigma(i)} - \sigma(i) \\ \text{or} \qquad & \mu_i - i + \lambda_{\sigma(i)} - \sigma(i) = 2 - \delta \end{aligned}$$

with the latter occurring for an even number of $1 \leq i \leq n$. By [CDM09b, Proposition 3.1], we see that $\mu \in \mathcal{O}_\lambda(n;\delta)$. $\square$

We can now strengthen the block result of Theorem 3.27 to the following.

Theorem 3.37. *Let $\lambda, \mu \in \Lambda_{B_n}$ and suppose $4n - 2 < p$. Then the following are equivalent:*

(i) $\mu^T \in \mathcal{B}^k_{\lambda^T}(n;\delta)$*;*

(ii) $\mu \in \mathcal{O}^p_\lambda(n;\delta)$*;*

(iii) $\mu \in \mathcal{O}_\lambda(n;\delta)$*;*

(iv) $\mu^T \in \mathcal{B}^K_{\lambda^T}(n;\delta)$*.*

Proof. (i) $\Longrightarrow$ (ii) by Theorem 3.15.

(ii) $\Longrightarrow$ (iii) by Lemma 3.36.

(iii) $\Longrightarrow$ (iv) by Theorem 3.10

(iv) $\Longrightarrow$ (i) by Proposition 3.19. $\square$

Now that we have a description of the blocks of this family of Brauer algebras, we can employ many more results from characteristic zero, including the notion of translation equivalence.

Definition 3.38. Let $\mathbb{F} = K$ or k, and recall the definition of $\mathrm{supp}(\lambda)$ from (1.2). Two partitions λ and λ' are $\mathbb{F}$-*translation equivalent* if for all $\mu \in \mathcal{B}^{\mathbb{F}}_\lambda(\infty)$ there is a unique element $\mu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty) \cap \mathrm{supp}(\mu)$, and

$$\mathcal{B}^{\mathbb{F}}_\lambda(\infty) \cap \mathrm{supp}(\mu') = \{\mu\}$$

and every element of $\mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty)$ arises in this way.

Example 3.39. Consider the Brauer algebra $B^K_5(2)$, and let $\lambda = (2^2, 1)$. We will show that this is translation equivalent to $\lambda' = (2, 1^2)$ (we will only consider partitions of size at most 5). Indeed , we see that $\mathrm{supp}(\lambda) = \{(2^2), (2, 1^2), (3, 2, 1), (2^3), (2^2, 1^2)\}$. Now the set $\mathcal{B}^K_{\lambda'}(4)$ is equal to $\{\mu^T : \mu \in \mathcal{O}_{\lambda'^T}(4)\}$, which is easily shown to be $\{(2, 1^2), (1^2)\}$. Thus, $\mathrm{supp}(\lambda) \cap \mathcal{B}^K_{\lambda'}(4) = \{(2, 1^2)\}$. Moreover $\mathcal{B}^K_\lambda(5) = \{(2^2, 1), (1)\}$ and $\mathrm{supp}(\lambda') = \{(2, 1), (1^3), (3, 1^2), (2^2, 1), (2, 1^3)\}$, so that $\mathrm{supp}(\lambda') \cap \mathcal{B}^K_\lambda(5) = \{(2^2, 1)\}$. It is easy to check that the following also hold

$$\mathcal{B}^K_{(1)}(5) \cap \mathrm{supp}(1^2) = \{(1)\} \qquad \text{and} \qquad \mathcal{B}^K_{(1^2)}(4) \cap \mathrm{supp}(1) = \{(1^2)\}.$$

Hence the partitions λ and λ' are translation equivalent.

Translation equivalence allows us to transfer results about the representation theory of $B^{\mathbb{F}}_{n-1}(\delta)$ to $B^{\mathbb{F}}_n(\delta)$ through the use of translation functors:

Definition 3.40. Given a $B_n^{\mathbb{F}}(\delta)$-module M, let $\mathrm{pr}_n^\lambda M$ be the projection of M onto the block $\mathcal{B}_\lambda^{\mathbb{F}}(n)$. We then define the *translation functors*

$$\mathrm{res}_n^\lambda = \mathrm{pr}_{n-1}^\lambda \mathrm{res}_n \qquad \text{and} \qquad \mathrm{ind}_n^\lambda = \mathrm{pr}_{n+1}^\lambda \mathrm{ind}_n.$$

Proposition 3.41 ([CDM11, Proposition 4.1])**.** *Suppose that $\lambda \in \Lambda_{B_n}^*$ and $\lambda' \in \Lambda_{B_{n-1}}^*$ are $\mathbb{F}$-translation equivalent. Then we have*

$$\mathrm{res}_n^{\lambda'} L_\mu^{\mathbb{F}}(n) \cong L_{\mu'}^{\mathbb{F}}(n-1) \qquad \textit{and} \qquad \mathrm{ind}_{n-1}^\lambda L_{\mu'}^{\mathbb{F}}(n-1) \cong L_\mu^{\mathbb{F}}(n)$$

for all $\mu \in \mathcal{B}_\lambda^{\mathbb{F}}(n) \cap \Lambda_{B_n}^*$ such that the corresponding μ' is an element of $\mathcal{B}_\lambda^{\mathbb{F}}(n-1) \cap \Lambda_{B_{n-1}}^*$. Furthermore, if $\nu \in \mathcal{B}_\lambda^{\mathbb{F}}(n)$ is such that ν' is in $\mathcal{B}_{\lambda'}^{\mathbb{F}}(n-1)$ then we have

$$[\Delta_\mu^{\mathbb{F}}(n) : L_\nu^{\mathbb{F}}(n)] = [\Delta_{\mu'}^{\mathbb{F}}(n-1) : L_{\nu'}^{\mathbb{F}}(n-1)].$$

Remark 3.42. Note that Theorem 3.37 implies that two partitions are k-translation equivalent if and only if they are K-translation equivalent, since both blocks are described by the W_n-orbits on E_n. Therefore in what follows we will simply call such partitions *translation equivalent*.

When viewed in the geometric situation introduced earlier in Section 3.3, translation equivalence relates partitions with the same degree of singularity, i.e. lying on precisely the same number of reflection hyperplanes in E_n. We now describe a situation that allows us to relate partitions with differing degrees of singularity. As in Remark 3.42 this definition is independent of the field, and thus is presented without reference to K or k.

Definition 3.43. We say a partition λ' *separates* λ^- and λ^+ if

(i) The partition λ' is the only element of $\mathcal{B}_{\lambda'}^{\mathbb{F}}(\infty) \cap \mathrm{supp}(\lambda^-)$,

(ii) The partition λ' is the only element of $\mathcal{B}_{\lambda'}^{\mathbb{F}}(\infty) \cap \mathrm{supp}(\lambda^+)$, and

(iii) The partitions λ^- and λ^+ are the only elements of $\mathcal{B}_{\lambda^-}^{\mathbb{F}}(\infty) \cap \mathrm{supp}(\lambda')$.

We will always assume that $\lambda^- \lhd \lambda^+$.

Proposition 3.44 ([CDM11, Theorem 4.8, Proposition 4.9])**.** *If $\lambda' \in \Lambda_{B_{n-1}}$ separates λ^- and λ^+, then*

$$\mathrm{res}_n^{\lambda'} L_{\lambda^+}^{\mathbb{F}}(n) \cong L_{\lambda'}^{\mathbb{F}}(n-1)$$

whenever $\lambda^+ \in \Lambda^*_{B_n}$ *and* $\lambda' \in \Lambda^*_{B_{n-1}}$*. Moreover, suppose that for all* $\mu', \nu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty)$ *there exist partitions* μ^+, μ^- *and* ν^+, ν^- *in* $\mathcal{B}^{\mathbb{F}}_{\lambda}(\infty)$ *such that* μ *separates* μ^+ *and* μ^-*, and* ν *separates* ν^+ *and* ν^-*. Then*

$$[\Delta^{\mathbb{F}}_{\mu^+}(n) : L^{\mathbb{F}}_{\nu^+}(n)] = [\Delta^{\mathbb{F}}_{\mu'}(n-1) : L^{\mathbb{F}}_{\nu'}(n-1)].$$

Remark 3.45. Note that the "local homomorphisms" condition from the original statement of this proposition is satisfied by Theorems 3.30 and 3.33, since the corresponding partitions are related by a simple reflection.

Our eventual aim is to show that the decomposition matrix of $B^k_n(\delta)$ is equal to that of $B^K_n(\delta)$ (recall that we have fixed $\delta \in \mathbb{Z}(n)$, if a δ exists). We begin with the following. First recall the dominance order with size $\lhd$ from Section 1.3.1, then we have

Lemma 3.46. *Suppose* $\lambda \in \Lambda_{B_n}$ *is minimal in* $\mathcal{B}^k_{\lambda}(n)$ *with respect to the order* $\lhd$*. Then for all partitions* $\mu \in \Lambda_{B_n}$

$$[\Delta^k_{\mu}(n) : L^k_{\lambda}(n)] = [\Delta^K_{\mu}(n) : L^K_{\lambda}(n)].$$

Proof. By Theorem 3.37, λ must also be minimal in $\mathcal{B}^K_{\lambda}(n)$. By the cellularity of $B^K_n(\delta)$ and $B^k_n(\delta)$, it follows that

$$[\Delta^k_{\mu}(n) : L^k_{\lambda}(n)] = [\Delta^K_{\mu}(n) : L^K_{\lambda}(n)] = \begin{cases} 1 & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda. \end{cases}$$

□

In order to show a corresponding result for non-minimal partitions, we introduce up-down diagrams from [CD11, Section 5].

Recall from Section 3.3 that we have an embedding of Λ_{B_n} in a real vector space E_n, and that by Remark 3.8 we can consider partitions to be elements of an infinite dimensional space E_∞. Then given an embedded partition $\lambda \in E_\infty$, define $x_\lambda \in E_\infty$ by

$$x_\lambda = \lambda + \rho(\delta).$$

Now to each $x_\lambda \in E_\infty$ we associate a diagram, with vertices indexed by $\mathbb{Z}_{\geq 0}$ if $x_\lambda \in \bigoplus_i \mathbb{Z}\varepsilon_i$ or by $\mathbb{Z}_{\geq 0} + \frac{1}{2}$ if $x_\lambda \in \bigoplus_i (\mathbb{Z} + \frac{1}{2})\varepsilon_i$. Each vertex is labelled with one of the symbols $\circ, \times, \vee, \wedge$, or $\diamond$. Define the sets

$$I_\wedge(x_\lambda) = \{(x_\lambda)_i : (x_\lambda)_i > 0\}, \qquad I_\vee(x_\lambda) = \{|(x_\lambda)_i| : (x_\lambda)_i < 0\}$$

$$\text{and} \qquad I_\diamond(x_\lambda) = \{(x_\lambda)_i : (x_\lambda)_i = 0\}.$$

Then each vertex r in the diagram associated with x is labelled by

$$\begin{cases} \circ & \text{if } r \notin I_\vee(x_\lambda) \cup I_\wedge(x_\lambda) \\ \times & \text{if } r \in I_\vee(x_\lambda) \cap I_\wedge(x_\lambda) \\ \vee & \text{if } r \in I_\vee(x_\lambda) \backslash I_\wedge(x_\lambda) \\ \wedge & \text{if } r \in I_\wedge(x_\lambda) \backslash I_\vee(x_\lambda) \\ \diamond & \text{if } r \in I_\diamond(x_\lambda). \end{cases}$$

Since the sequences x_λ are strictly decreasing, every such element of E_∞ is uniquely determined by its diagram. Moreover, every diagram where the vertex 0 is labelled by $\circ$ or $\diamond$ and vertices $r, r+1, r+2, \ldots$ are labelled by $\vee$ for some $r \gg 0$ corresponds to the diagram of some partition λ.

Example 3.47. Let $\lambda = (13^2, 10, 9^2, 8^3, 6^2, 3, 2, 1)$ and $\delta = 2$. Then $x_\lambda \in E_\infty$ is equal to

$$(12, 11, 7, 5, 4, 2, 1, 0, -3, -4, -8, -10, -12, -14, -15, -16, \ldots).$$

The corresponding diagram is given in Figure 3.14.

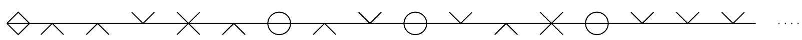


Figure 3.14: The diagram of $\lambda = (13^2, 10, 9^2, 8^3, 6, 3, 2, 1)$.

For $\lambda \in \Lambda_{B_n}$, consider the diagram of x_λ. If this diagram contains a $\diamond$, we first make a choice by replacing this wit either a $\vee$ or $\wedge$. This choice will determine the diagrams of all partitions μ in the same W_∞-orbit as λ. Indeed, we obtain such diagrams by repeatedly swapping a $\vee$ and a $\wedge$ or replacing two $\vee$s by $\wedge$s (and vice versa). Moreover, if the diagram of one partition x_μ is obtained from that of x_λ by swapping a $\vee$ with a $\wedge$ to its right, or by changing two $\wedge$s to two $\vee$s, then we have $\mu \lhd \lambda$.

Remark. In what follows, we will always assume that a choice of $\vee$ or $\wedge$ has been made for the symbol $\diamond$ in the diagram of some partition λ in each W_∞-orbit. This will then determine the symbol in position 0 of all other diagrams in this orbit.

We now adapt the proof of [CD11, Theorem 5.8] in order to show the following. First note that since we are imposing the condition $4n - 2 < p$, all partitions in Λ_{B_n} are p-regular, and thus $\Lambda^*_{B_n} = \Lambda_{B_n}$.

Lemma 3.48. *Let $\lambda \in \Lambda_{B_n}$, such that λ is not minimal in $\mathcal{B}^{\mathbb{F}}_{\lambda}(n)$ with respect to $\lhd$. Then there exists a partition $\lambda' \in \Lambda_{B_{n-1}}$ such that either:*

(i) λ and λ' are translation equivalent; or

(ii) there exists a partition λ^- such that λ' separates λ and λ^-.

Moreover, in the latter case we have for all $\mu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty)$ exactly two partitions $\mu^+, \mu^- \in \mathcal{B}^{\mathbb{F}}_{\lambda}(\infty)$ such that μ' separates μ^+ and μ^-.

Proof. Since λ is not minimal in its block, then its up-down diagram contains either a $\vee$ with a $\wedge$ somewhere to the right, or two $\wedge$s. In both cases, we may assume that between the two symbols there are only $\times$s or $\circ$s. Then the diagram must contain one of the configurations in the cases below:

(1) $\circ\wedge$,

(2) $\times\wedge$,

(3) $\vee\wedge$ or

(4) $\wedge\wedge$.

Case 1: $\circ\wedge$**.**

Consider the partition $\lambda' = \lambda - A$ (some $A \in \mathrm{rem}(\lambda)$) whose up-down diagram is obtained by swapping these symbols. We claim that λ and λ' are translation equivalent; that is for any $\mu \in \mathcal{B}^{\mathbb{F}}_{\lambda}(n)$ there is a unique $\mu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty) \cap \mathrm{supp}(\mu)$, and μ is the unique element of $\mathcal{B}^{\mathbb{F}}_{\lambda}(\infty) \cap \mathrm{supp}(\mu')$. Indeed, the up-down diagrams of any such μ and μ' must differ only in the positions corresponding to the $\circ\wedge$ as above, and the possible cases are given in Figure 3.15 below.

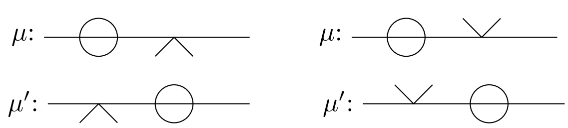


Figure 3.15: The possible configurations of the diagrams of μ and μ' in Case 1.

It is then clear that λ and λ' are translation equivalent.

Case 2: $\times\wedge$.

As in Case 1, we let the partition $\lambda' = \lambda - A$ (some $A \in \mathrm{rem}(\lambda)$) correspond to the diagram obtained by swapping these symbols. We again have a translation equivalence between these partitions, since any μ and μ' as before must have the following configurations.

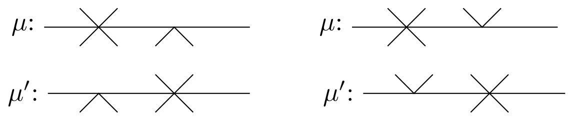


Figure 3.16: The possible configurations of the diagrams of μ and μ' in Case 2.

Case 3: $\vee\wedge$.

We let $\lambda^+ = \lambda$. Define the partition $\lambda' = \lambda^+ - A$ (some $A \in \mathrm{rem}(\lambda^+)$) to correspond to the up-down diagram of λ but with the symbols $\vee\wedge$ as above by $\times\circ$. Note that we have another partition $\lambda^- \in \mathcal{B}^{\mathbb{F}}_{\lambda^+}(\infty) \cap \mathrm{supp}(\lambda')$, as shown in Figure 3.17.

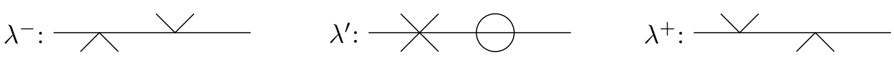


Figure 3.17: The configurations of λ^-, λ' and λ^+ in Case 3.

It is clear that λ' separates λ^- and λ^+. Moreover, for each $\mu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty)$ there are precisely two partitions $\mu^+, \mu^- \in \mathcal{B}^{\mathbb{F}}_{\lambda}(\infty) \cap \mathrm{supp}(\mu')$ corresponding to the same three configurations at the same positions as the diagrams above. Also, μ' is the unique element of $\mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty) \cap \mathrm{supp}(\mu^{\pm})$.

Case 4: $\wedge\wedge$.

We may assume that these symbols are the leftmost two, as otherwise we may proceed as in Cases 1, 2 or 3. We now have two sub-cases:

(a) Suppose first that the leftmost symbol is in position zero. First let $\lambda^+ = \lambda$, then we have a partition $\lambda' = \lambda^+ - A$ (some $A \in \mathrm{rem}(\lambda^+)$) whose diagram is as in Figure 3.18. We also have a partition $\lambda^- \in \mathcal{B}^{\mathbb{F}}_{\lambda^+}(\infty) \cap \mathrm{supp}(\lambda')$, also given in Figure 3.18. As in Case 3 we see that λ' separates λ^- and λ^+, and once again we have for all $\mu' \in \mathcal{B}^{\mathbb{F}}_{\lambda'}(\infty)$ exactly two partitions μ^+, μ^- satisfying the separation criteria.

(b) If now the leftmost symbol is in position $\frac{1}{2}$, then we have a partition $\lambda' = \lambda - A$ (some $A \in \mathrm{rem}(\lambda)$) with diagram as in Figure 3.18. As before in Cases 1 and 2, we see that these are translation equivalent.

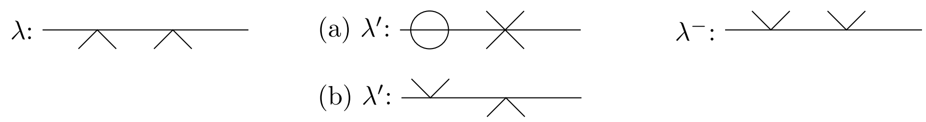


Figure 3.18: The configurations of λ^-, λ' and λ^+ in Cases 4(a) and (b).

□

We may now prove the main result of this section:

Theorem 3.49. *Suppose* $4n-2<p$. *Then for every* $\lambda, \mu \in \Lambda_{B_n}$*, we have*

$$[\Delta^k_\mu(n;\delta) : L^k_\lambda(n;\delta)] = [\Delta^K_\mu(n;\delta) : L^K_\lambda(n;\delta)].$$

Proof. If λ is minimal in its block with respect to the ordering $\triangleleft$ from Section 1.3.1, then the result follows immediately from Lemma 3.46. Otherwise, we prove this by induction on n. For $n = 0, 1$, the result is clear.

By Lemma 3.48, we can find a partition λ' such that either λ and λ' are translation equivalent, or λ' separates λ and a third partition λ^-. Note also that we obtain the same partitions whether we are working over K or k, see Remark 3.42. Therefore using Propositions 3.41 and 3.44 we see that

$$\begin{aligned}[\Delta^k_\mu(n;\delta) : L^k_\lambda(n;\delta)] &= [\Delta^k_{\mu'}(n-1;\delta) : L^k_{\lambda'}(n-1;\delta)] \\ &= [\Delta^K_{\mu'}(n-1;\delta) : L^K_{\lambda'}(n-1;\delta)] \\ &= [\Delta^K_\mu(n;\delta) : L^K_\lambda(n;\delta)],\end{aligned}$$

where the second equality follows by induction. □

We see now that the restrictions placed on n, p and δ in this section force the second matrix in the factorisation (3.13) to be the identity matrix.

Remark. The result of Theorem 3.49 is a specialisation of [Sha13, Theorem 5.3], which proves the corresponding result when $n < p$. However by restricting to $4n-2<p$, we are able to give a more concise proof.

3.5 The case $\delta = 0$ or $p = 2$

In this section we briefly discuss the cases $\delta = 0$ and $p = 2$.

When $\delta = 0$, many of the arguments within this chapter can be modified slightly in order to still hold. The first obvious change is to the idempotent e_n as defined in Figure 3.3. Since δ is no longer invertible, we must replace this with the idempotent $\widetilde{e}_n$ as defined in Figure 3.19.

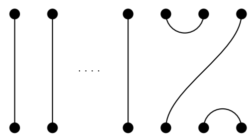
$\widetilde{e}_n =$ 

Figure 3.19: The idempotent $\widetilde{e}_n$.

We also note that the labelling set for the simple $B_n^{\mathbb{F}}(0)$-modules is no longer $\Lambda^*_{B_n}$ as defined in 3.6. As shown in [GL96, Theorem 4.17], we must exclude the empty partition. Then with slight modifications to the proofs, for instance changing e_n to $\widetilde{e}_n$, the remainder of the results in characteristic zero are still valid. Furthermore, the results in positive characteristic remain true, but care must be taken when dealing with statements involving the empty partition.

When $p = 2$, we unfortunately cannot make such modifications. The original geometric characterisation of the blocks of the Brauer algebra involves use of the element $\rho(\delta) = (-\frac{\delta}{2}, -\frac{\delta}{2} - 1, \ldots, -\frac{\delta}{2} - (n-1))$ and the value $\frac{1-\delta}{2}$, one of which will be undefined over a field of characteristic 2. Since all of the results on the modular representation theory of the Brauer algebra use this geometric construction, none of them are valid in characteristic 2.

Chapter 4

The partition algebra

4.1 Background

The partition algebra was originally defined by Martin in [Mar94] over $\mathbb{C}$ as a generalisation of the Temperley-Lieb algebra for δ-state n-site Potts models in statistical mechanics, and independently by Jones [Jon94]. Although this interpretation requires δ to be integral, it is possible to define the algebra for any δ. It was shown in [Xi99] that the partition algebra $P_n(\delta)$ over an arbitrary field $\mathbb{F}$ is a cellular algebra, with cell modules $\Delta_\lambda(n)$ indexed by partitions λ of size at most n.

If we suppose $\delta \neq 0$, then in characteristic zero each cell module has a unique simple head, and each simple module arises in this way. In positive characteristic the simple modules are given by the heads of cell modules indexed by p-regular partitions (again under the assumption $\delta \neq 0$). It is natural to then ask how the simple modules arise as composition factors of the cell modules. In the case char $\mathbb{F} = 0$ this has been entirely resolved by Martin [Mar96] and Doran and Wales [DW00], however there has previously been little investigation into the positive characteristic case.

The action of the algebra on cell modules has been studied by Halverson and Ram [HR05], who defined Jucys-Murphy type elements, and Enyang [Eny13], who constructed a seminormal basis of the modules.

Martin provides in [Mar96] a condition on λ, μ and δ for when there is a homomorphism in characteristic zero between cell modules labelled by λ and μ, provided $\delta \neq 0$. This was strengthened in [DW00] to allow for $\delta = 0$. In this chapter, we will reformulate this condition in terms of the reflection geometry of a Weyl group W_n under a δ-shifted action. Then considering the action of the corresponding affine Weyl group W_n^p, we will describe the blocks of the partition algebra in positive characteristic [BDK14]. We will also show that for a family of partition algebras we can describe the decomposition matrix entirely [Kin14b]. Independent work of Shalile [Sha14] gives the same description for a subset of these cases.

4.2 Preliminaries

4.2.1 Definitions

Recall from Section 1.1 the definitions and conventions for the prime p, the p-modular system (K, R, k), the field $\mathbb{F}$ and the parameter $\delta \in R$. There is a consideration of the cases $p = 2$ and $\delta = 0$ in Section 4.5.

For a fixed $n \in \mathbb{N}$ and $\delta \in R$, we define the partition algebra $P_n^R(\delta)$ to be the set of linear combinations of set-partitions of $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$. We call each part of a set-partition a *block*. For instance,

$$\big\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\big\}$$

is a set-partition with $n = 5$ consisting of 4 blocks. Any block with $\{i, \bar{j}\}$ as a subset for some i and j is called a *propagating block*.

We can represent each set-partition by an (n, n)-*partition diagram*, consisting of two rows of n nodes with arcs between nodes in the same block. Multiplication in the partition algebra is by concatenation of diagrams in the following way: to obtain the result $x \cdot y$ given diagrams x and y, place x on top of y and identify the bottom nodes of x with those on top of y. This new diagram may contain a number, t say, of blocks in the centre not connected to the northern or southern edges of the diagram. These we remove and multiply the final result by δ^t. An example is given in Figure 4.1 below.

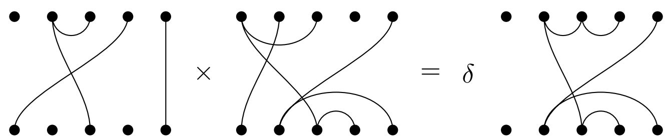


Figure 4.1: Multiplication of two diagrams in $P_5^R(\delta)$.

As shown in Example 4.1 below, there are many diagrams corresponding to the same set-partition. We will identify all such diagrams.

Example 4.1. Let $n = 5$ and consider the set-partition $\big\{\{1, 3, \bar{3}, \bar{4}\}, \{2, \bar{1}\}, \{4\}, \{5, \bar{2}, \bar{5}\}\big\}$ as above. This can be represented by the diagrams in Figure 4.2 below.

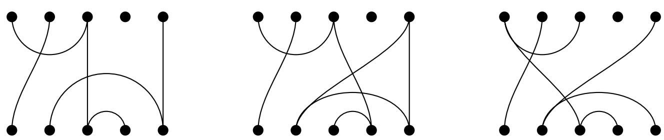
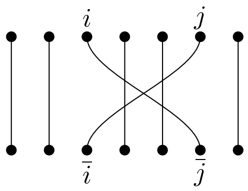
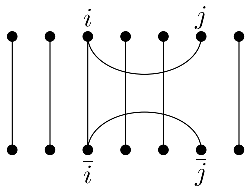


Figure 4.2: Three ways of representing the given set-partition.

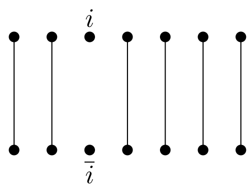
The following elements of $P_n^R(\delta)$ will be of interest:

$s_{i,j} =$ 

,

$p_{i,j} =$ 

,

$p_i =$ 

.

It was shown in [HR05] that these elements generate $P_n^R(\delta)$.

Theorem 4.2 ([HR05, Theorem 1.11(d)])**.** *The partition algebra $P_n^R(\delta)$ is generated by the elements $s_i = s_{i,i+1}$ $(i \in \mathbb{Z},\ 1 \le i < n)$, $p_{i+\frac{1}{2}} = p_{i,i+1}$ $(i \in \mathbb{Z},\ 1 \le i < n)$ and p_i $(i \in \mathbb{Z},\ 1 \le i \le n)$, with relations*

$$p_i^2 = \delta p_i, \qquad p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}, \qquad p_i p_{i\pm\frac{1}{2}} p_i = p_i, \qquad p_i p_j = p_j p_i \quad \textit{for } j \neq i,$$

$$p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}, \qquad p_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_i \quad \textit{for } |i-j| > 1,$$

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$s_i s_j = s_j s_i \quad \textit{for } |i-j| > 2,$$

$$s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}, \qquad s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}},$$

$$s_i p_i s_i = p_{i+1}, \qquad s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i = p_{i+1+\frac{1}{2}},$$

$$s_i p_j = p_j s_i \quad \textit{for } j \neq i, i+1, \qquad s_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} s_i \quad \textit{for } |i-j| > 1.$$

Notice that multiplication in $P_n^R(\delta)$ cannot increase the number of propagating blocks. We therefore have a filtration of $P_n^R(\delta)$ by the number of propagating blocks.

We will assume in what follows that $\delta \neq 0$ and work over the field $\mathbb{F} = K$ or k, so that δ is invertible. This allows us to realise the filtration by use of the idempotents e_i defined in Figure 4.3 below.

$$e_i = \frac{1}{\delta^{n-i+1}}$$

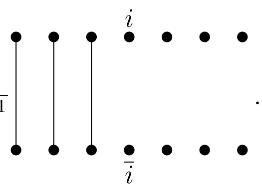


Figure 4.3: The idempotent e_i.

The filtration is then given by

$$J_n^{(0)} \subset J_n^{(1)} \subset \ldots J_n^{(n-1)} \subset J_n^{(n)} = P_n^{\mathbb{F}}(\delta) \tag{4.1}$$

where $J_n^{(r)} = P_n^{\mathbb{F}}(\delta) e_{r+1} P_n^{\mathbb{F}}(\delta)$ contains only diagrams with at most r propagating blocks. We also use e_i to construct algebra isomorphisms

$$\Phi_n : P_{n-1}^{\mathbb{F}}(\delta) \longrightarrow e_n P_n^{\mathbb{F}}(\delta) e_n \tag{4.2}$$

taking a diagram in $P_{n-1}^{\mathbb{F}}(\delta)$ and adding an extra northern and southern node to the right hand end. Using this and following [Gre80, Section 6.2] we obtain the following functors as defined in [Mar96]: an exact localisation functor

$$\begin{aligned} F_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n-1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto e_n M; \end{aligned} \tag{4.3}$$

and a right exact globalisation functor

$$\begin{aligned} G_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n+1}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} \\ M &\longmapsto P_{n+1}^{\mathbb{F}}(\delta) e_{n+1} \otimes_{P_n^{\mathbb{F}}(\delta)} M. \end{aligned} \tag{4.4}$$

Since $F_{n+1}G_n(M) \cong M$ for all $M \in P_n^{\mathbb{F}}(\delta)$-**mod**, G_n is a full embedding of categories. From the filtration (4.1) we see that

$$P_n^{\mathbb{F}}(\delta)/J_n^{(n-1)} \cong \mathbb{F}\mathfrak{S}_n \tag{4.5}$$

and so using (4.2) and following [Gre80, Theorem 6.2g], we see that the simple $P_n^{\mathbb{F}}(\delta)$-modules are indexed by the set

$$\begin{aligned} \Lambda_{P_n} &= \{\lambda : \lambda \vdash n, n-1, n-2, \dots, 1, 0\} \text{ if char } \mathbb{F} = 0 \\ \text{or } \Lambda_{P_n}^* &= \{\lambda : \lambda \vdash n, n-1, n-2, \dots, 1, 0,\ \lambda \ p\text{-regular}\} \text{ if char } \mathbb{F} = p > 0. \end{aligned} \tag{4.6}$$

In characteristic zero this is due to Martin [Mar94]. The classification of simple modules over an arbitrary field is due to Xi [Xi99].

Remark 4.3. Note that we have embeddings $\Lambda_{P_n} \subset \Lambda_{P_{n+1}}$ and $\Lambda_{P_n}^* \subset \Lambda_{P_{n+1}}^*$.

We will also need to consider the algebra $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)$, as defined by Martin [Mar00], which is the subalgebra of $P_n^{\mathbb{F}}(\delta)$ spanned by all set-partitions with n and $\bar{n}$ in the same block. As in (4.1) we have a filtration of this algebra defined by the number of propagating blocks:

$$J_{n-\frac{1}{2}}^{(1)} \subset J_{n-\frac{1}{2}}^{(2)} \subset \dots J_{n-\frac{1}{2}}^{(n-1)} \subset J_{n-\frac{1}{2}}^{(n)} = P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta) \tag{4.7}$$

where $J_{n-\frac{1}{2}}^{(r)}$ contains diagrams with at most r propagating blocks. Note that since we require the nodes n and $\bar{n}$ to be in the same block, we always have at least one propagating block. Also since n and $\bar{n}$ must always be joined, we see that $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)/J_{n-\frac{1}{2}}^{(n-1)} \cong \mathbb{F}\mathfrak{S}_{n-1}$, and so following the argument for $P_n^{\mathbb{F}}(\delta)$ above we see that the simple $P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)$-modules are indexed by $\Lambda_{P_{n-1}}$ or $\Lambda_{P_{n-1}}^*$.

Note that we have an injective map

$$\begin{aligned} P_n^R(\delta) &\longrightarrow P_{n+\frac{1}{2}}^R(\delta) \\ d &\longmapsto d \cup \big\{\{n+1, \overline{n+1}\}\big\} \end{aligned}$$

This allows us to define restriction and induction functors

$$\begin{aligned}\mathrm{res}_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}\\ M &\longmapsto M|_{P_{n-\frac{1}{2}}^{\mathbb{F}}(\delta)},\end{aligned}$$

$$\begin{aligned}\mathrm{ind}_n : P_n^{\mathbb{F}}(\delta)\text{-}\mathbf{mod} &\longrightarrow P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\mathbf{mod}\\ M &\longmapsto P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta) \otimes_{P_n^{\mathbb{F}}(\delta)} M.\end{aligned} \tag{4.8}$$

4.2.2 Cellularity of $P_n^{\mathbb{F}}(\delta)$

It was shown in [Xi99] that the partition algebra is cellular. The cell chain is given by refining each layer of the filtration in (4.1) by the symmetric group filtration, see [KX99] for details. The antiautomorphism i acts by swapping corresponding northern and southern nodes of a diagram. Let Λ_{P_n} be as in (4.6) and use the partial order $\lhd$ from Section 1.3.1. The cell modules $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ are then indexed by partitions $\lambda \in \Lambda_{P_n}$, and the cellular ordering is given by the reverse of $\lhd$.

When $\lambda \vdash n$, we obtain $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ by lifting the Specht module $S_{\mathbb{F}}^\lambda$ to the partition algebra using (4.5). The localisation and globalisation functors ((4.3) and (4.4)) allow us to describe the cell module $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ when $\lambda \vdash n-t$ for some $t>0$. In particular, Martin showed in [Mar96] that the functors map cell modules to cell modules in the following way:

$$F_n(\Delta_\lambda^{\mathbb{F}}(n)) \cong \begin{cases}\Delta_\lambda^{\mathbb{F}}(n-1) & \text{if } \lambda \in \Lambda_{P_{n-1}}\\ 0 & \text{otherwise,}\end{cases}$$

$$G_n(\Delta_\lambda^{\mathbb{F}}(n)) \cong \Delta_\lambda^{\mathbb{F}}(n+1).$$

Thus when $\lambda \vdash n-t$ for some $t>0$, we obtain the cell module by

$$\Delta_\lambda^{\mathbb{F}}(n;\delta) = G_{n-1}G_{n-2}\dots G_{n-t}\Delta_\lambda^{\mathbb{F}}(n-t;\delta).$$

Suppose $\delta \neq 0$. Over K, each of the cell modules has a simple head $L_\lambda^K(n;\delta)$, and these form a complete set of non-isomorphic simple $P_n^K(\delta)$-modules. Over k, the heads $L_\lambda^k(n;\delta)$ of cell modules labelled by p-regular partitions $\lambda \in \Lambda_{P_n}^*$ provide a complete set of non-isomorphic simple $P_n^k(\delta)$-modules.

Notation. When the context is clear, we will write $\Delta_\lambda^{\mathbb{F}}(n)$ and $L_\lambda^{\mathbb{F}}(n)$ to mean $\Delta_\lambda^{\mathbb{F}}(n;\delta)$ and $L_\lambda^{\mathbb{F}}(n;\delta)$ respectively.

We also have an explicit construction of the modules $\Delta_\lambda^R(n)$ due to Martin [Mar96]. These were shown to be equal to the cell modules for $B_n^R(\delta)$ in [DW00]. Let $I(n,t)$ be the set of (n,n)-diagrams with precisely t propagating blocks and $\overline{t+1}, \overline{t+2}, \ldots, \overline{n}$ each in singleton blocks. An example is given in 4.4 below.

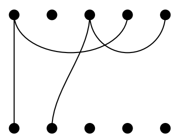


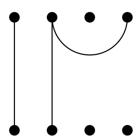
Figure 4.4: An example of a diagram in $I(5,2)$.

Then denote by $V(n,t)$ the free R-module with basis $I(n,t)$. There is a $(P_n^R(\delta), \mathfrak{S}_t)$-bimodule action on $V(n,t)$, where elements of $P_n^R(\delta)$ act on the left by concatenation as normal and elements of $\mathfrak{S}_t$ act on the right by permuting the t leftmost southern nodes. Thus for a partition $\lambda \vdash t$ we can easily show that $\Delta_\lambda^R(n) \cong V(n,t) \otimes_{\mathfrak{S}_t} S_R^\lambda$, where S_R^λ is the Specht module. The action of $P_n^R(\delta)$ on $\Delta_\lambda^R(n)$ is as follows: given a partition diagram $x \in P_n^R(\delta)$ and a pure tensor $v \otimes s \in \Delta_\lambda^R(n)$, we define the element

$$x(v \otimes s) = (xv) \otimes s$$

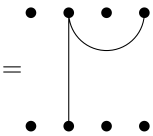
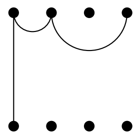
where (xv) is the product of two diagrams in the usual way if the result has t propagating blocks, and is 0 otherwise. An example is given below.

Example 4.4. Recall the elements $s_{i,j}$, $p_{i,j}$ and p_i above, and consider the cell module $\Delta_{(1^2)}^R(4;\delta)$. This has basis $\{v \otimes s : v \in I(4,2)\}$ where s is the generator for the sign representation of $\mathfrak{S}_2$. Let x be the following element of $I(4,2)$:

$x =$ 

Then we have the following action of elements of $P_4^R(\delta)$:

$$p_{2,3}(x \otimes s) = s_{2,3}(x \otimes s) = x \otimes s, \qquad p_3(x \otimes s) = \delta x \otimes s,$$

$p_1(x \otimes s) =$  $= 0, \qquad p_{1,2}(x \otimes s) =$  $= 0.$

We then have

$$\Delta_\lambda^K(n) = K \otimes_R \Delta_\lambda^R(n) \quad \text{and} \quad \Delta_\lambda^k(n) = k \otimes_R \Delta_\lambda^R(n)$$

Remark. Note that we **cannot** in general provide an R-module $L_\lambda^R(n)$ such that $L_\lambda^K(n) = K \otimes_R L_\lambda^R(n)$ or $L_\lambda^k(n) = k \otimes_R L_\lambda^R(n)$.

The algebra $P^{\mathbb{F}}_{n-\frac{1}{2}}(\delta)$ is also cellular [Mar00], with cell chain as in (4.7) and anti-automorphism i as before. We can construct the cell modules $\Delta^{\mathbb{F}}_\lambda(n - \frac{1}{2}; \delta)$ in a similar way to those of $P_n^{\mathbb{F}}(\delta)$. Let $I(n - \frac{1}{2}, t)$ be the set of (n, n)-diagrams with precisely t propagating blocks, one of which contains n and $\bar{n}$, with $\overline{t}, \overline{t+1}, \ldots, \overline{n-1}$ each in singleton blocks. Then denote by $V(n - \frac{1}{2}, t)$ the free R-module with basis $I(n - \frac{1}{2}, t)$. There is a $(P^R_{n-\frac{1}{2}}(\delta), \mathfrak{S}_{t-1})$-bimodule action on $V(n, t)$, where elements of $P^R_{n-\frac{1}{2}}(\delta)$ act on the left as normal and elements of $\mathfrak{S}_{t-1}$ act on the right by permuting the $t-1$ leftmost southern nodes. Thus for a partition $\lambda \vdash t-1$ we can define $\Delta_\lambda^R(n - \frac{1}{2}) \cong V(n - \frac{1}{2}, t) \otimes_{\mathfrak{S}_{t-1}} S_R^\lambda$, where S_R^λ is a Specht module. Note that when $\lambda \vdash n-1$, $\Delta_\lambda^R(n - \frac{1}{2}) \cong S_R^\lambda$, the Specht module. The action of $P^R_{n-\frac{1}{2}}(\delta)$ is the same as in the previous case.

We then have

$$\Delta_\lambda^K(n - \tfrac{1}{2}) = K \otimes_R \Delta_\lambda^R(n - \tfrac{1}{2}) \quad \text{and} \quad \Delta_\lambda^k(n - \tfrac{1}{2}) = k \otimes_R \Delta_\lambda^R(n - \tfrac{1}{2})$$

and as before, $\Delta_\lambda^K(n - \frac{1}{2})$ has a simple head $L_\lambda^K(n - \frac{1}{2}; \delta)$ for all λ, and $\Delta_\lambda^k(n - \frac{1}{2})$ has a simple head $L_\lambda^k(n - \frac{1}{2}; \delta)$ for each p-regular λ.

It was shown in [Mar00, Proposition 7] if we apply the restriction and induction functors (4.8) to cell modules, then the result has a filtration by cell modules. In particular, we have the following exact sequences:

$$0 \longrightarrow \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \mathrm{res}_{n+\frac{1}{2}} \Delta_\lambda^{\mathbb{F}}(n+\tfrac{1}{2}) \longrightarrow \biguplus_{A \in \mathrm{add}(\lambda)} \Delta_{\lambda+A}^{\mathbb{F}}(n) \longrightarrow 0$$

$$0 \longrightarrow \biguplus_{A \in \mathrm{rem}(\lambda)} \Delta_{\lambda-A}^{\mathbb{F}}(n-\tfrac{1}{2}) \longrightarrow \mathrm{res}_n \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \Delta_\lambda^{\mathbb{F}}(n-\tfrac{1}{2}) \longrightarrow 0$$

$$0 \longrightarrow \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \mathrm{ind}_{n-\frac{1}{2}} \Delta_\lambda^{\mathbb{F}}(n-\tfrac{1}{2}) \longrightarrow \biguplus_{A \in \mathrm{add}(\lambda)} \Delta_{\lambda+A}^{\mathbb{F}}(n) \longrightarrow 0$$

$$0 \longrightarrow \biguplus_{A \in \mathrm{rem}(\lambda)} \Delta_{\lambda-A}^{\mathbb{F}}(n+\tfrac{1}{2}) \longrightarrow \mathrm{ind}_n \Delta_\lambda^{\mathbb{F}}(n) \longrightarrow \Delta_\lambda^{\mathbb{F}}(n+\tfrac{1}{2}) \longrightarrow 0, \tag{4.9}$$

where we use $\biguplus_{i=1}^{r} N_i$ to denote a module M with a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M_r = M$$

such that for each $1 \leq i \leq r$, $M_i/M_{i-1} \cong N_i$. Moreover the successive quotients in these filtrations are ordered by dominance.

The following result from Martin [Mar00, Section 3] allows us to focus on the partition algebras $P_n^{\mathbb{F}}(\delta)$ with $n \in \mathbb{Z}$.

Proposition 4.5. *Define the idempotent*

$$\xi_{n+1} = \prod_{i=1}^{n} (1 - p_{i,n+1}) \in P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta).$$

Then we have an algebra isomorphism

$$\xi_{n+1} P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta) \xi_{n+1} \cong P_n^{\mathbb{F}}(\delta - 1)$$

which induces a Morita equivalence between the categories $P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)$-***mod*** *and* $P_n^{\mathbb{F}}(\delta-1)$-***mod***. *More precisely, using the above isomorphism the functors*

$$\begin{aligned} \Phi : P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\boldsymbol{mod} &\longrightarrow P_n^{\mathbb{F}}(\delta-1)\text{-}\boldsymbol{mod} \\ M &\longmapsto \xi_{n+1} M \\ \textit{and} \qquad \Psi : P_n^{\mathbb{F}}(\delta-1)\text{-}\boldsymbol{mod} &\longrightarrow \Phi : P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\text{-}\boldsymbol{mod} \\ N &\longmapsto P_{n+\frac{1}{2}}^{\mathbb{F}}(\delta)\xi_{n+1} \otimes_{P_n^{\mathbb{F}}(\delta-1)} N \end{aligned}$$

define an equivalence of categories. Moreover, this equivalence maps cell modules to cell modules in the following way:

$$\Phi(\Delta_\lambda^{\mathbb{F}}(n+\tfrac{1}{2})) \cong \Delta_\lambda^{\mathbb{F}}(n)$$

for all $\lambda \in \Lambda_{P_n}$.

4.3 Representation theory in characteristic zero

The blocks of the partition algebra $P_n^K(\delta)$ in characteristic 0 were described in [Mar96]. Assuming δ is an integer (otherwise the algebra is semisimple), the blocks are given by chains of partitions, each satisfying a combinatorial property determined by the previous partition in the chain. We briefly recount this below, but first we introduce some notation:

Definition 4.6. Let $\mathcal{B}_\lambda^K(n;\delta)$ be the set of partitions μ labelling cell modules in the same block as $\Delta_\lambda^K(n)$. We will also say that partitions μ and λ lie in the same block if they label cell modules in the same block. If the context is clear, we will write $\mathcal{B}_\lambda^K(n)$ to mean $\mathcal{B}_\lambda^K(n;\delta)$.

Definition 4.7. Let λ, μ be partitions, with $\mu \subset \lambda$. We say that (μ, λ) is a δ*-pair*, written $\mu \hookrightarrow_\delta \lambda$, if λ differs from μ by a strip of nodes in a single row, the last of which has content $\delta - |\mu|$.

Below is an example of this condition.

Example 4.8. We let $\delta = 7$, $\lambda = (4,3,1)$ and $\mu = (4,1,1)$. Then we see that λ and μ differ in precisely one row, and the last node in this row of λ has content 1 (see Figure 4.5). Since $\delta - |\mu| = 7 - 6 = 1$, we see that (μ, λ) is a 7-pair.

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 \\ \cline{1-1} -2 \\ \cline{1-1} \end{array} \quad \hookrightarrow_7 \quad \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1} \end{array}$$

Figure 4.5: An example of a δ-pair when $\delta = 7$.

This allows us to characterise the blocks of the partition algebra as follows.

Theorem 4.9 ([Mar96, Proposition 9]). *Each block of the partition algebra $P_n^K(\delta)$ is given by a chain of partitions*

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$$

where for each i, $(\lambda^{(i)},\lambda^{(i+1)})$ form a δ-pair, differing in the $(i+1)$-th row. Moreover there is an exact sequence of $P_n^K(\delta)$-modules

$$0 \to \Delta^K_{\lambda^{(r)}}(n) \to \Delta^K_{\lambda^{(r-1)}}(n) \to \cdots \to \Delta^K_{\lambda^{(1)}}(n) \to \Delta^K_{\lambda^{(0)}}(n) \to L^K_{\lambda^{(0)}}(n) \to 0$$

with the image of each homomorphism a simple module. In particular, each of the cell modules $\Delta^K_{\lambda^{(i)}}(n)$ for $0 \leq i < r$ has Loewy structure

$$\begin{matrix} L^K_{\lambda^{(i)}}(n) \\ L^K_{\lambda^{(i+1)}}(n) \end{matrix}$$

and $\Delta^K_{\lambda^{(r)}}(n) = L^K_{\lambda^{(r)}}(n)$.

This can be reformulated in terms of the geometry of a reflection group as follows. Let $\{\varepsilon_0, \ldots, \varepsilon_n\}$ be a set of formal symbols and set

$$E_n = \bigoplus_{i=0}^{n} \mathbb{R}\varepsilon_i.$$

Remark 4.10. Note that we have an embedding $E_n \subset E_{n+1}$ by adding a zero into the $n+1$-th position of a vector in E_n.

We have an inner product $\langle \ , \ \rangle$ on E_n given by extending linearly the relations

$$\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$$

where δ_{ij} is the Kronecker delta.
Let $\Phi = \{\varepsilon_i - \varepsilon_j : 0 \leq i, j \leq n\}$ be the root system of type A_n, and $W_n \cong \mathfrak{S}_{n+1}$ the corresponding Weyl group, generated by the reflections $s_{i,j} = s_{\varepsilon_i - \varepsilon_j}$ $(0 \leq i < j \leq n)$. There is an action of W_n on E_n, the generators acting by

$$s_{i,j}(x) = x - \langle x, \varepsilon_i - \varepsilon_j \rangle (\varepsilon_i - \varepsilon_j)$$

for all $x \in E_n$.

If we fix the element $\rho = \rho(\delta) = (\delta, -1, -2, \ldots, -n)$ we may then define a shifted action of W_n on E_n, given by

$$w \cdot_\delta x = w(x + \rho(\delta)) - \rho(\delta)$$

for all $w \in W_n$ and $x \in E_n$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, let

$$\hat{\lambda} = (-|\lambda|, \lambda_1, \dots, \lambda_n) = -|\lambda|\varepsilon_0 + \sum_{i=1}^{n} \lambda_i \varepsilon_i \in E_n.$$

Using this embedding of Λ_{P_n} into E_n we can consider the action of W_n on the set of partitions Λ_{P_n} defined by

$$w \cdot_\delta \hat{\lambda} = w(\hat{\lambda} + \rho(\delta)) - \rho(\delta)$$

where $w \in W_n$ and $\rho(\delta) = (\delta, -1, -2, \dots, -n)$. We introduce the following notation for the orbits of this action.

Definition 4.11. For $\lambda \in \Lambda_{P_n}$, let $\mathcal{O}_\lambda(n;\delta)$ be the set of partitions $\mu \in \Lambda_{P_n}$ such that $\hat{\mu} \in W_n \cdot_\delta \hat{\lambda}$. If the context is clear, we will write $\mathcal{O}_\lambda(n)$ to mean $\mathcal{O}_\lambda(n;\delta)$.

We then have the following reformulation of [Mar96]

Theorem 4.12. *For all $\lambda \in \Lambda_{P_n}$, we have $\mathcal{B}_\lambda^K(n;\delta) = \mathcal{O}_\lambda(n;\delta)$.*

Proof. We saw above that the blocks of $P_n^K(\delta)$ are given by maximal chains of partitions

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$$

where for each i, ($\lambda^{(i-1)}$,$\lambda^{(i)}$) form a δ-pair, differing in the i-th row. We claim that $\widehat{\lambda^{(i)}} = s_{0,i} \cdot_\delta \widehat{\lambda^{(i-1)}}$. Indeed,

$$\begin{aligned} s_{0,i} \cdot_\delta \widehat{\lambda^{(i-1)}} &= (\lambda_i^{(i-1)} - i, \lambda_1^{(i-1)} - 1, \dots, -|\lambda^{(i-1)}| + \delta, \dots, \lambda_n^{(i-1)} - n) - \rho(\delta) \\ &= (\lambda_i^{(i-1)} - i - \delta, \lambda_1^{(i-1)}, \lambda_2^{(i-1)}, \dots, -|\lambda^{(i-1)}| + \delta + i, \dots, \lambda_n^{(i-1)}). \end{aligned} \tag{4.10}$$

Now the partition

$$(\lambda_1^{(i-1)}, \lambda_2^{(i-1)}, \dots, -|\lambda^{(i-1)}| + \delta + i, \dots, \lambda_n^{(i-1)})$$

obtained from (4.10) differs from $\lambda^{(i-1)}$ by a strip of nodes in row i only, the last of which has content

$$(-|\lambda^{(i-1)}| + \delta + i) - i = \delta - |\lambda^{(i-1)}|$$

and so $s_{0,i} \cdot_\delta \widehat{\lambda^{(i-1)}} = \widehat{\lambda^{(i)}}$ as claimed. Therefore if $\mu \neq \nu \in \Lambda_{P_n}$ are in the same block then $\mu = \lambda^{(i)}$ and $\nu = \lambda^{(j)}$ for some $i < j$ say, and

$$\hat{\nu} = (s_{0,j} \dots s_{0,i+2} s_{0,i+1}) \cdot_\delta \hat{\mu}.$$

Conversely, suppose $\lambda, \mu \in \Lambda_{P_n}$ satisfy $\mu \in \mathcal{O}_\lambda(n;\delta)$. Since λ is a partition, the sequence $(\lambda_1-1, \lambda_2-2, \dots, \lambda_n-n)$ is strictly decreasing, and similarly for μ. Therefore if $\mu \in \mathcal{O}_\lambda(n;\delta)$, then $\hat{\mu}+\rho(\delta) = w(\hat{\lambda}+\rho(\delta))$ for some $w \in W_n$ not fixing entry 0, and we have

$$\hat{\mu}+\rho(\delta) = (\lambda_i - i, \dots)$$

for some $1 \leq i \leq n$. If $\lambda_i - i = \delta - |\lambda|$ then $\mu = \lambda$ and the result is immediate.

If now $\lambda_i - i < \delta - |\lambda|$, then

$$\hat{\mu}+\rho(\delta) = (\lambda_i - i, \dots, \lambda_j - j, \delta - |\lambda|, \lambda_{j+1}-(j+1), \dots, \lambda_{i-1}-(i-1), \lambda_{i+1}-(i+1), \dots)$$

for some j. If instead $\lambda_i - i > \delta - |\lambda|$ then

$$\hat{\mu}+\rho(\delta) =$$
$$(\lambda_i - i, \dots, \dots, \lambda_{i-1}-(i-1), \lambda_{i+1}-(i+1), \dots, \lambda_j - j, \delta - |\lambda|, \lambda_{j+1}-(j+1), \dots)$$

for some j. In either case, we have

$$\hat{\mu}+\rho(\delta) = (s_{0,i} \dots s_{0,j+2} s_{0,j+1}) \cdot_\delta (\hat{\lambda}+\rho(\delta))$$

and using the calculation in (4.10) we see that these must be elements in a chain of δ-pairs, and so are in the same block. $\square$

4.4 Modular representation theory

4.4.1 Block structure

The results of this section have been presented in [BDK14]. The contribution of this author was to define the δ-marked abacus and use it to show that two partitions in the same orbit of the affine Weyl group are in the same $P_n^k(\delta)$-block.

In characteristic zero, the blocks of the partition algebra are determined by the orbits of the Weyl group W_n of type A_n. Our aim is to show that by replacing W_n with the corresponding affine Weyl group W_n^p, we obtain a parallel result for the positive characteristic case. We begin by defining positive characteristic analogues of Definitions 4.6 and 4.7.

Definition 4.13. Let $\mathcal{B}^k_\lambda(n;\delta)$ be the set of partitions μ labelling cell modules in the same block as $\Delta^k_\lambda(n)$. We will also say that partitions μ and λ lie in the same block if they label cell modules in the same block (equivalently if $\mu \in \mathcal{B}^k_\lambda(n;\delta)$). If the context is clear, we will write $\mathcal{B}^k_\lambda(n)$ to mean $\mathcal{B}^k_\lambda(n;\delta)$.

Remark 4.14. Note that the functors F_n and G_n give us a full embedding of $P^k_n(\delta)$-**mod** inside $P^k_{n+1}(\delta)$-**mod**, and hence an embedding $\mathcal{B}^k_\lambda(n;\delta) \subset \mathcal{B}^k_\lambda(n+1;\delta)$.

Definition 4.15. Let $\lambda \vdash n$ and $\mu \vdash n-t$ for some $t \geq 0$. We say that (λ,μ) is a *(δ,p)-pair* if

$$t\delta - t|\mu| - \mathrm{ct}(\lambda) + \mathrm{ct}(\mu) - \frac{t(t-1)}{2} = 0$$

in the field k, where $\mathrm{ct}(\lambda)$ is as in (1.3).

Remark. This definition seeks to generalise the notion of δ-pairs from the characteristic zero representation theory of $P^K_n(\delta)$, and is derived using a similar proof to [DW00, Theorem 6.1]. However this no longer has a combinatorial description.

We wish to show, as in [DW00] for characteristic zero, that this provides a necessary condition for two partitions to lie in the same block in positive characteristic. However the characteristic zero proof cannot be generalised to prime characteristic, and thus we need to introduce the Jucys-Murphy elements introduced in [HR05]. They were later defined inductively in [Eny13, Section 2.3] as follows: Recall the definitions of s_i, p_i and $p_{i+\frac{1}{2}}$ from Theorem 4.2. Then

Definition 4.16. (i) Set $L_0 = 0$, $L_1 = p_1$, $\sigma_1 = 1$, $\sigma_2 = s_1$ and for $i \in \mathbb{Z}$, $i \geq 1$, define

$$L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1}$$

where for $i \geq 2$ we define

$$\begin{aligned}\sigma_{i+1} = s_{i-1}s_i\sigma_i s_i s_{i+1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\ - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i.\end{aligned}$$

(ii) Set $L_{\frac{1}{2}} = 0$, $\sigma_{\frac{1}{2}} = 1$, $\sigma_{1+\frac{1}{2}} = 1$ and for $i \in \mathbb{Z}$, $i \geq 1$, define

$$L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + (\delta - L_{i-\frac{1}{2}}) p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}}$$

where for $i \geq 2$ we define

$$\sigma_{i+\frac{1}{2}} = s_{i-1}s_i\sigma_{i-\frac{1}{2}}s_is_{i-1} + p_{i-\frac{1}{2}}L_{i-1}s_ip_{i-\frac{1}{2}}s_i + s_ip_{i-\frac{1}{2}}L_{i-\frac{1}{2}}s_ip_{i-\frac{1}{2}}$$
$$- p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_ip_{i-\frac{1}{2}} - s_ip_{i-\frac{1}{2}}p_ip_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i.$$

If we project these elements onto the quotient $P_n^k(\delta)/J_n^{(n-1)}$, where $J_n^{(n-1)}$ is defined in (4.1), then we obtain the following:

Lemma 4.17. *(i)* $\sigma_i + J_n^{(n-1)} = s_{i-1} + J_n^{(n-1)}$ *for all* $i \geq 2$,

(ii) $L_i + J_n^{(n-1)} = \sum_{j=1}^{i-1} s_{j,i} + J_n^{(n-1)}$ *for all* $i \geq 2$,

(iii) $\sigma_{i+\frac{1}{2}} + J_n^{(n-1)} = 1 + J_n^{(n-1)}$ *for all* $i \geq 0$,

(iv) $L_{i+\frac{1}{2}} + J_n^{(n-1)} = i + J_n^{(n-1)}$ *for all* $i \geq 0$,

(v) Let $Z_n = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \cdots + L_n$. *Then*

$$Z_n + J_n^{(n-1)} = \frac{n(n-1)}{2} + \sum_{1\leq i<j\leq n} s_{i,j} + J_n^{(n-1)}.$$

Proof. We prove these statements by induction on i.

(i) This is true for $i = 2$ by definition. Now let $i \geq 2$, then we have

$$\begin{aligned}\sigma_{i+1} + J_n^{(n-1)} &= s_{i-1}s_i\sigma_is_is_{i-1} + J_n^{(n-1)}\\ &= s_{i-1}s_is_{i-1}s_is_{i-1} + J_n^{(n-1)} \text{ by induction}\\ &= s_i + J_n^{(n-1)}.\end{aligned}$$

(ii) We have $L_2 + J_n^{(n-1)} = \sigma_2 + J_n^{(n-1)} = s_1 + J_n^{(n-1)}$. Now let $i \geq 2$, then we have

$$\begin{aligned}L_{i+1} + J_n^{(n-1)} &= s_iL_is_i + \sigma_{i+1} + J_n^{(n-1)}\\ &= s_i\left(\sum_{j=1}^{i-1} s_{j,i}\right)s_i + s_i + J_n^{(n-1)} \text{ by induction and using (i)}\\ &= \sum_{j=1}^{i-1} s_{j,i+1} + s_i + J_n^{(n-1)}\\ &= \sum_{j=1}^{i} s_{j,i+1} + J_n^{(n-1)}.\end{aligned}$$

(iii) We have $\sigma_{\frac{1}{2}} = 1$, and for $i \geq 1$

$$\begin{aligned}
\sigma_{i+\frac{1}{2}} + J_n^{(n-1)} &= s_{i-1}s_i\sigma_{i-\frac{1}{2}}s_is_{i-1} + J_n^{(n-1)} \\
&= s_{i-1}s_i 1 s_is_{i-1} + J_n^{(n-1)} \\
&= 1 + J_n^{(n-1)}.
\end{aligned}$$

(iv) We have $L_{\frac{1}{2}} = 0$, and for $i \geq 1$

$$\begin{aligned}
L_{i+\frac{1}{2}} + J_n^{(n-1)} &= s_iL_{i-\frac{1}{2}}s_i + \sigma_{i+\frac{1}{2}} + J_n^{(n-1)} \\
&= s_i(i-1)s_i + 1 + J_n^{(n-1)} \text{ by induction and using (iii)} \\
&= i + J_n^{(n-1)}.
\end{aligned}$$

(v) Follows immediately from (ii) and (iv).

□

We then recall the following result.

Lemma 4.18 ([HR05, Theorem 3.35], [Eny13, Lemma 3.14])**.** *Let $\mu \in \Lambda_{P_n}$ with $|\mu| = n - t$ for some $t \geq 0$. Then Z_n acts on $\Delta_\mu^k(n;\delta)$ as scalar multiplication by*

$$t\delta + \binom{|\mu|}{2} + \mathrm{ct}(\mu).$$

This allows us to provide the following necessary condition for two partitions to be in the same k-block:

Theorem 4.19. *Let $\lambda, \mu \in \Lambda_{P_n}$ be partitions. If there exists a submodule $M \subset \Delta_\mu^k(n;\delta)$ with*

$$\mathrm{Hom}(\Delta_\lambda^k(n;\delta), \Delta_\mu^k(n;\delta)/M) \neq 0$$

then (λ, μ) is a (δ, p)-pair.

Proof. By use of the localisation functor (4.3) we may assume that $\lambda \vdash n$ and $\mu \vdash n-t$. Therefore we have $\Delta_\lambda^k(n;\delta) \cong S_k^\lambda$, and the ideal $J_n^{(n-1)}$ acts as zero on $\Delta_\lambda^k(n;\delta)$.

Now suppose that $\mathrm{Hom}(\Delta_\lambda^k(n;\delta), \Delta_\mu^k(n;\delta)/M) \neq 0$. Then there exists a submodule N of $\Delta_\mu^k(n;\delta)$ with $M \subset N \subset \Delta_\mu^k(n;\delta)$ and a surjective homomorphism

$$\Delta_\lambda^k(n;\delta) \cong S_k^\lambda \longrightarrow N/M.$$

By Lemma 4.17, the element

$$Z_n - \frac{n(n-1)}{2} - \sum_{1 \le i < j \le n} s_{i,j} \tag{4.11}$$

must act as zero on N/M. We know that $\sum_{1 \le i < j \le n} s_{i,j}$ acts by the scalar $\mathrm{ct}(\lambda)$ on S_k^λ, and hence also on N/M. By Lemma 4.18, we then see that the element (4.11) acts by the scalar

$$\begin{aligned} & t\delta + \binom{|\mu|}{2} + \mathrm{ct}(\mu) - \frac{n(n-1)}{2} - \mathrm{ct}(\lambda) \\ = \ & t\delta - t|\mu| - \mathrm{ct}(\lambda) + \mathrm{ct}(\mu) - \frac{t(t-1)}{2}. \end{aligned}$$

Since this must be zero in the field k, we see that λ and μ must form a (δ, p)-pair. $\square$

As with the characteristic zero result, we now wish to reformulate this in terms of the geometry of a reflection group. Let W_n^p be the group generated by reflections $s_{i,j,rp} = s_{\varepsilon_i - \varepsilon_j, rp}$ $(0 \le i < j \le n), r \in \mathbb{Z}$, where for $x \in E_n$

$$s_{i,j,rp}(x) = x - \big(\langle x, \varepsilon_i - \varepsilon_j \rangle - rp\big)(\varepsilon_i - \varepsilon_j).$$

Using the same embedding $\hat{\lambda} \in E_n$ of a partition λ and shifted action as in Section 4.3, we define a positive characteristic analogue of Definition 4.11.

Definition 4.20. For $\lambda \in \Lambda_{P_n}$, let $\mathcal{O}_\lambda^p(n;\delta)$ be the set of partitions $\mu \in \Lambda_{P_n}$ such that $\hat{\mu} \in W_n^p \cdot_\delta \hat{\lambda}$. If the context is clear, we will write $\mathcal{O}_\lambda^p(n)$ to mean $\mathcal{O}_\lambda^p(n;\delta)$.

Remark 4.21. It is clear that we have an embedding $\mathcal{O}_\lambda^p(n;\delta) \subset \mathcal{O}_\lambda^p(n+1;\delta)$.

We then have the following characterisation of the W_n^p-orbits on Λ_{P_n}:

Lemma 4.22. *Let $\lambda, \mu \in \Lambda_{P_n}$, then we have*

$$\mu \in \mathcal{O}_\lambda^p(n;\delta) \iff \hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$$

where for $x, y \in E_n$, $x \sim_p y$ means there is a permutation $\sigma \in \mathfrak{S}_{n+1}$ such that $x_i \equiv y_{\sigma(i)}$ (mod p) for all $0 \le i \le n$.

Proof. We have $\mu \in \mathcal{O}_\lambda^p(n;\delta)$ if and only if there is some $w \in W_n$ and $\alpha \in \mathbb{Z}\Phi$ such that

$$\hat{\mu} + \rho(\delta) = w(\hat{\lambda} + \rho(\delta)) + p\alpha.$$

Conversely, we have $\hat{\mu}+\rho(\delta) \sim_p \hat{\lambda}+\rho(\delta)$ if and only if

$$\hat{\mu}+\rho(\delta) = w(\hat{\lambda}+\rho(\delta)) + px$$

for some $w \in W_n$ and $x \in \mathbb{Z}^{n+1}$. But as $\sum_{i=0}^n (\hat{\mu})_i = \sum_{i=0}^n (\hat{\lambda})_i = 0$ we see that $\sum_{i=0}^n x_i = 0$, and therefore $x \in \mathbb{Z}\Phi$. $\square$

We can now strengthen Theorem 4.19 to provide a necessary condition for the blocks of the partition algebra. We begin with the following Lemma.

Lemma 4.23. *Let $\lambda \in \Lambda^*_{P_n}$ be a p-regular partition with $|\lambda| = n > 1$. Then there exists a removable node $A \in \mathrm{rem}(\lambda)$ such that $\lambda - A$ is p-regular and there is a surjective homomorphism*

$$\mathrm{ind}_{n-\frac{1}{2}} \Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta) \longrightarrow \Delta^k_\lambda(n;\delta)$$

Proof. Since $|\lambda| > 1$, it has a removable node. Let $A \in \mathrm{rem}(\lambda)$ be such that A is in the highest row, i say, and $\lambda - A$ is p-regular. Now consider the set $S = \{\lambda - A + B : B \in \mathrm{add}(\lambda - A)\}$. By (4.9) we have a surjection

$$\mathrm{ind}_{n-\frac{1}{2}} \Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta) \longrightarrow \biguplus_{\mu \in S} \Delta^k_\mu(n;\delta) \tag{4.12}$$

where the factors on the right hand side are ordered by dominance of partitions, with the most dominant at the top. Note that for all $\mu \in S$, $|\mu| = |\lambda| - 1 + 1 = n$, and hence $\Delta^k_\mu(n;\delta) \cong S^\mu_k$ as $k\mathfrak{S}_n$-modules. Thus the module on the right hand side of (4.12) is a module for $k\mathfrak{S}_n$, inflated to $P^k_n(\delta)$, and will decompose according to the block structure of $k\mathfrak{S}_n$. We claim that λ is the most dominant partition in its block for $k\mathfrak{S}_n$. If $i = 1$, then we are done, so assume A is in row $i > 1$. By our choice of A, if we remove a node B in an earlier row, $j < i$ say, then $\lambda - B$ is not p-regular. Using this, and the fact that λ is p-regular, we deduce that λ and $\lambda - A$ must be of the form

$$\lambda = (b^a, (b-1)^{p-1}, (b-2)^{p-1}, \dots, (b-t+1)^{p-1}, (b-t)^{p-1}, b-t-1, \dots), \text{ and}$$
$$\lambda - A = (b^a, (b-1)^{p-1}, (b-2)^{p-1}, \dots, (b-t+1^{p-1}, (b-t)^{p-2}, b-t-1, \dots)$$

for some $1 < a < p$, $b > 0$ and $t \geq 0$. Now the partitions $\mu \in S$ with $\mu \rhd \lambda$ are precisely the partitions

$$\mu^{(0)} = (b+1, b^{a-1}, (b-1)^{p-1}, (b-2)^{p-1}, \dots, (b-t+1^{p-1}, (b-t)^{p-2}, b-t-1, \dots),$$
$$\mu^{(1)} = (b^{a+1}, (b-1)^{p-2}, (b-2)^{p-1}, \dots, (b-t+1^{p-1}, (b-t)^{p-2}, b-t-1, \dots),$$
$$\mu^{(2)} = (b^{a}, (b-1)^{p}, (b-2)^{p-2}, \dots, (b-t+1^{p-1}, (b-t)^{p-2}, b-t-1, \dots),$$
$$\vdots$$
$$\mu^{(t)} = (b^{a}, (b-1)^{p-1}, (b-2)^{p-1}, \dots, (b-t+1^{p}, (b-t)^{p-3}, b-t-1, \dots).$$

Now Theorem 1.8 tells us that none of these partitions are in the same $k\mathfrak{S}_n$-block as λ as required.

Therefore, $\Delta^k_\lambda(n;\delta)$ appears as a quotient in the right hand side of (4.12), and we obtain the required surjective homomorphism by composing (4.12) with the projection onto this quotient. $\square$

With this, we can now provide a geometric version of the combinatorial result of Theorem 4.19.

Theorem 4.24. *Let $\lambda \in \Lambda^*_{P_n}$, $\mu \in \Lambda_{P_n}$. If $\mathrm{Hom}(\Delta^k_\lambda(n;\delta), \Delta^k_\mu(n;\delta)/M) \neq 0$ for some $M \subseteq \Delta^k_\mu(n;\delta)$, then $\mu \in \mathcal{O}^p_\lambda(n;\delta)$.*

Proof. By use of the localisation functor (4.3) we may assume that $\lambda \vdash n$ and $\mu \vdash n-t$ for some $t \geq 0$. We prove the result by induction on n.

If $n = 0$ there is nothing to prove, so assume $n \geq 1$. If $\lambda = \emptyset$ then by cellularity we must also have $\mu = \emptyset$, and the result holds trivially.

If now $|\lambda| \geq 1$, then λ has a removable node A and by Lemma 4.23 we have a surjection

$$\mathrm{ind}_{n-\frac{1}{2}} \Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta) \longrightarrow \Delta^k_\lambda(n;\delta),$$

and so by assumption and Frobenius reciprocity

$$\begin{aligned}&\mathrm{Hom}(\mathrm{ind}_{n-\frac{1}{2}} \Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta), \Delta^k_\mu(n;\delta)/M)\\ &\qquad\cong \mathrm{Hom}(\Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta), \mathrm{res}_n(\Delta^k_\mu(n;\delta)/M)) \neq 0.\end{aligned}$$

By the restriction rule (4.9) we have either

$$\mathrm{Hom}(\Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta), \Delta^k_\mu(n-\tfrac{1}{2};\delta)/N) \neq 0$$

for some submodule $N \subset \Delta^k_\mu(n-\frac{1}{2};\delta)$, or

$$\mathrm{Hom}(\Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta), \Delta^k_{\mu-B}(n-\tfrac{1}{2};\delta)/Q) \neq 0$$

for some removable node B in row j of μ say, and some submodule $Q \subset \Delta^k_{\mu-B}(n-\frac{1}{2};\delta)$.

Applying Proposition 4.5 we have the following two cases:

Case 1 $\mathrm{Hom}(\Delta^k_{\lambda-A}(n-1;\delta-1), \Delta^k_\mu(n-1;\delta-1)/N) \neq 0$ for some submodule $N \subset \Delta^k_\mu(n-1;\delta-1)$

Case 2 $\mathrm{Hom}(\Delta^k_{\lambda-A}(n-1;\delta-1), \Delta^k_{\mu-B}(n-1;\delta-1)/Q) \neq 0$ for some removable node B in row j of μ, and some submodule $Q \subset \Delta^k_{\mu-B}(n-1;\delta-1)$.

Case 1 Applying our inductive step, we have that $\mu \in \mathcal{O}_{\lambda-A}(n-1;\delta-1)$. Using Lemma 4.22 we see that $\hat{\mu}+\rho(\delta-1) \sim_p \widehat{\lambda-A}+\rho(\delta-1)$, that is

$$(\delta-1-|\mu|, \mu_1-1, \ldots, \mu_n-n) \sim_p (\delta-|\lambda|, \lambda_1-1, \ldots, \lambda_i-i-1, \ldots, \lambda_n-n). \quad (4.13)$$

We also see from Theorem 4.19 that $(\lambda-A,\mu)$ form a $(\delta-1,p)$-pair. Since $|\lambda-A|-|\mu| = t-1$, we have

$$(t-1)(\delta-1)-(t-1)|\mu| - \mathrm{ct}(\lambda)+\mathrm{ct}(A)+\mathrm{ct}(\mu) - \frac{(t-1)(t-2)}{2} \equiv 0 \pmod p$$

from which we deduce

$$t\delta - t|\mu| - \mathrm{ct}(\lambda) + \mathrm{ct}(\mu) - \frac{t(t-1)}{2} + \mathrm{ct}(A) + |\mu| - \delta \equiv 0 \pmod p.$$

Moreover by assumption we have that (λ,μ) is a (δ,p)-pair, thus

$$\mathrm{ct}(A) = \lambda_i - i \equiv \delta - |\mu| \pmod p. \quad (4.14)$$

Combining (4.13) and (4.14), the sequences

$$\hat{\lambda}+\rho(\delta) = (\delta-|\lambda|, \lambda_1-1, \ldots, \lambda_i-i, \ldots, \lambda_n-n)$$

and

$$\hat{\mu}+\rho(\delta) = (\delta-|\mu|, \mu_1-1, \ldots, \mu_n-n)$$

are then equivalent modulo p (up to reordering). A final application of Lemma 4.22 then provides the result.

Case 2 Applying our inductive step, we have that $\mu - B \in \mathcal{O}^p_{\lambda-A}(n-1;\delta-1)$. Using Lemma 4.22 we see that $\widehat{\mu - B} + \rho(\delta-1) \sim_p \widehat{\lambda - A} + \rho(\delta-1)$, that is

$$(\delta-|\mu|, \mu_1-1, \ldots, \mu_j-j-1, \ldots, \mu_n-n) \sim_p (\delta-|\lambda|, \lambda_1-1, \ldots, \lambda_i-i-1, \ldots, \lambda_n-n). \tag{4.15}$$

We also see from Theorem 4.19 that $(\lambda - A, \mu - B)$ form a $(\delta-1, p)$-pair. Since $|\lambda - A| - |\mu - B| = t$, we have

$$t(\delta-1) - t(|\mu|-1) - \mathrm{ct}(\lambda) + \mathrm{ct}(A) + \mathrm{ct}(\mu) - \mathrm{ct}(B) - \frac{t(t-1)}{2} \equiv 0 \pmod p.$$

Moreover by assumption we have that (λ, μ) is a (δ, p)-pair, thus

$$\mathrm{ct}(A) \equiv \mathrm{ct}(B) \pmod p$$

that is,

$$\lambda_i - i \equiv \mu_j - j \pmod p. \tag{4.16}$$

Combining (4.15) and (4.16), the sequences

$$\hat{\lambda} + \rho(\delta) = (\delta - |\lambda|, \lambda_1 - 1, \ldots, \lambda_i - i, \ldots, \lambda_n - n)$$

and

$$\hat{\mu} + \rho(\delta) = (\delta - |\mu|, \mu_1 - 1, \ldots, \mu_j - j, \ldots, \mu_n - n)$$

are then equivalent modulo p (up to reordering). A final application of Lemma 4.22 then provides the result. $\square$

Since the blocks of $P^k_n(\delta)$ are generated by such homomorphisms (See Proposition 1.5), we immediately obtain the following corollary.

Corollary 4.25. *Let $\lambda \in \Lambda_{P_n}$. Then $\mathcal{B}^k_\lambda(n;\delta) \subset \mathcal{O}^p_\lambda(n;\delta)$.*

In order to prove the converse of Corollary 4.25, i.e. that two partitions in the same W^p_n-orbit are in the same $P^k_n(\delta)$-block, we need to introduce a variation of the abacus of Section 1.3.4

Recall the result of Lemma 4.22, that $\mu \in \mathcal{O}^p_\lambda(n;\delta)$ if and only if $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$. We represent this equivalence in the form of an abacus in the following way. For a fixed $\delta \in R$ and a partition $\lambda \in \Lambda_{P_n}$, choose $b \in \mathbb{N}$ satisfying

$b \geq n$. We write $\hat{\lambda}$ as a $(b+1)$-tuple by adding zeros to get a vector in E_b (see Remark 4.10), and extend $\rho(\delta)$ to the $(b+1)$-tuple

$$\rho(\delta) = (\delta, -1, -2, \ldots, -b) \in E_b.$$

We can then define the β_δ-sequence of λ to be

$$\begin{aligned} \beta_\delta(\lambda, b) &= \hat{\lambda} + \rho(\delta) + b(\underbrace{1, 1, \ldots, 1}_{b+1}) \\ &= (\delta - |\lambda| + b, \lambda_1 - 1 + b, \lambda_2 - 2 + b, \lambda_3 - 3 + b, \ldots, 2, 1, 0). \end{aligned}$$

It is clear that the equivalence in Lemma 4.22 can now also be written as $\beta_\delta(\mu, b) \sim_p \beta_\delta(\lambda, b)$. The β_δ-sequence is used to construct the *δ-marked abacus* of λ as follows:

1. Take an abacus with p runners, labelled 0 to $p-1$ from left to right. The positions of the abacus start at 0 and increase from left to right, moving down the runners.

2. Set v_λ to be the unique integer $0 \leq v_\lambda \leq p-1$ such that $\beta_\delta(\lambda, b)_0 = \delta - |\lambda| + b \equiv v_\lambda \pmod{p}$. Place a $\vee$ on top of runner v_λ.

3. For the rest of the entries of $\beta_\delta(\lambda, b)$, place a bead in the corresponding position of the abacus, so that the final abacus contains b beads.

Example 4.26 below demonstrates this construction.

Example 4.26. Let $p = 5$, $\delta = 6$, $\lambda = (2, 1)$. We choose an integer $b \geq 3$, for instance $b = 7$. Then the β_δ-sequence is

$$\begin{aligned} \beta_\delta(\lambda, 7) &= (6 - 3 + 7, 2 - 1 + 7, \ldots, 0) \\ &= (10, 8, 6, 4, 3, 2, 1, 0). \end{aligned}$$

The resulting δ-marked abacus is given in Figure 4.6.

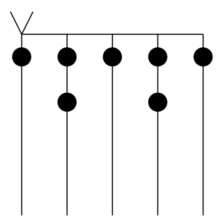


Figure 4.6: The δ-marked abacus of λ, with $\lambda = (2, 1)$ and $b = 7$.

Note that if we ignore the $\vee$ we recover James' abacus representing λ with b beads.

If the context is clear, we will use *marked abacus* to mean δ-marked abacus.

Recall the definition of $\Gamma(\lambda, b)$ from (1.4). If we now use the marked abacus, we similarly define $\Gamma_\delta(\lambda, b) = (\Gamma_\delta(\lambda, b)_0, \Gamma_\delta(\lambda, b)_1, \ldots, \Gamma_\delta(\lambda, b)_{p-1})$ by

$$\Gamma_\delta(\lambda, b)_i = \begin{cases} \Gamma(\lambda, b)_i & \text{if } i \neq v_\lambda \\ \Gamma(\lambda, b)_i + 1 & \text{if } i = v_\lambda. \end{cases}$$

This provides us with a further form of Lemma 4.22:

$$\begin{aligned} \mu \in \mathcal{O}^p_\lambda(n; \delta) &\iff \hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta) \\ &\iff \beta_\delta(\mu, b) \sim_p \beta_\delta(\lambda, b) \\ &\iff \Gamma_\delta(\mu, b) = \Gamma_\delta(\lambda, b). \end{aligned} \tag{4.17}$$

We now have a characterisation of the orbits of W^p_n in terms of the beads on the marked abacus. Before we use this to determine the blocks of the partition algebra, we demonstrate a further use of this characterisation.

The following theorem from [HHKP10] allows us to use the modular representation theory of the symmetric group in examining the partition algebra, and vice versa.

Theorem 4.27 ([HHKP10, Corollary 6.2])**.** *Let $\lambda, \mu \vdash n - t$ be partitions, with $\lambda \in \Lambda^*_{P_n}$. Then*

$$[\Delta^k_\mu(n; \delta) : L^k_\lambda(n; \delta)] = [S^\mu_k : D^\lambda_k].$$

In particular, given two partitions $\lambda, \mu \vdash n - t$, if the two Specht modules S^λ_k and S^μ_k are in the same block over the symmetric group algebra $k\mathfrak{S}_{n-t}$, then $\mu \in \mathcal{B}^k_\lambda(n; \delta)$.

We may now combine Corollary 4.25 and Theorem 4.27, and use the marked abacus to obtain a new proof of the following result:

Theorem 4.28 ([JK81, Corollary 6.1.42])**.** *Let $\lambda, \mu \vdash n$ be partitions. If the Specht modules S^λ_k and S^μ_k are in the same $k\mathfrak{S}_n$-block, then λ and μ have the same p-core.*

Proof. By Theorem 4.27, we have $\mu \in \mathcal{B}^k_\lambda(n)$. Applying Corollary 4.25 then shows that $\mu \in \mathcal{O}^p_\lambda(n)$, and so by the characterisation of the orbits in terms of the abacus (4.17):

$$\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n).$$

But since $|\lambda| = |\mu|$, we have $\delta - |\lambda| + n = \delta - |\mu| + n$, and so $v_\lambda = v_\mu$. Therefore we in fact have $\Gamma(\mu, n) = \Gamma(\lambda, n)$, and the two partitions therefore have the same p-core. $\square$

This theorem is not new, indeed it is half of Nakayama's conjecture. The striking element is that we have proved this using only the representation theory of the partition algebra, with no modular representation theory of the symmetric group.

We now use the δ-marked abacus to show that each orbit $\mathcal{O}^p_\lambda(n;\delta)$ contains a unique minimal element.

Proposition 4.29. *Let $\lambda \in \Lambda_{P_n}$, then $\mathcal{O}^p_\lambda(n)$ contains a unique minimal element $\lambda_\mathcal{O}$ (with respect to the dominance ordering with size $\lhd$). More precisely, if $\mu \in \mathcal{O}^p_\lambda(n)$ then $|\mu| \geq |\lambda_\mathcal{O}|$, with equality if and only if $\mu = \lambda_\mathcal{O}$.*

Proof. We define $\lambda_\mathcal{O}$ to be the partition such that

(i) $\Gamma_\delta(\lambda_\mathcal{O}, b) = \Gamma_\delta(\lambda, b)$,

(ii) All beads on the marked abacus of $\lambda_\mathcal{O}$ are as far up their runners as possible,

(iii) The runner $v_{\lambda_\mathcal{O}}$ is the rightmost runner i such that $\Gamma_\delta(\lambda_\mathcal{O}, b)_i$ is maximal.

The partition $\lambda_\mathcal{O}$ is well defined, i.e. it is independent of the number of beads used. To see this, note that by adding m beads to the abacus we move each existing bead m places to the right. Moreover since $v_{\lambda_\mathcal{O}} \equiv \delta - |\lambda_\mathcal{O}| + b \pmod p$, we also move the $\vee$ by m places to the right. Therefore none of the beads change their relative positions to one another and the partition $\lambda_\mathcal{O}$ remains unchanged. This is illustrated in Figure 4.7.

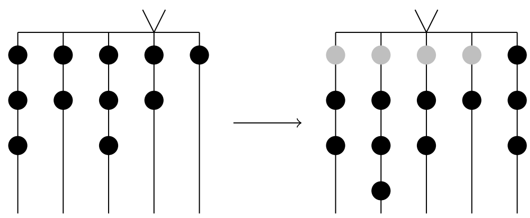


Figure 4.7: Adding 4 beads (coloured grey) to the abacus of $\lambda_\mathcal{O} = (2, 1)$. Each existing bead (coloured black) and the $\vee$ is moved 4 places to the right.

Condition (i) shows that our choice of $\lambda_{\mathcal{O}}$ is in the correct orbit (see (4.17)). Uniqueness is clear from the definition, so it remains to show that $\lambda_{\mathcal{O}}$ is minimal. Let $\mu \in \mathcal{O}^p_\lambda(n)$ be distinct from $\lambda_{\mathcal{O}}$. We will prove by induction on the size of μ that $|\mu| > |\lambda_{\mathcal{O}}|$. First we note that since $\Gamma_\delta(\mu, b) = \Gamma_\delta(\lambda, b)$, the marked abacus of μ must violate condition (ii) or (iii).

If μ does not satisfy (ii), then μ is not a p-core (see Section 1.3.4). Thus we can remove a p-hook (that is, push a bead one space up its runner on the marked abacus) and obtain a new partition $\mu' \in \mathcal{O}^p_\lambda(n)$ with size $|\mu'| = |\mu| - p < |\mu|$. Therefore by induction we have $|\lambda_{\mathcal{O}}| \leq |\mu'| < |\mu|$.

Now assume that μ does satisfy (ii), but not (iii), so that all beads are as far up their runners as possible but $v_\mu \neq v_{\lambda_{\mathcal{O}}}$. Consider the first empty space on runner v_μ, say it is in position m. Since condition (iii) does not hold we can choose $l > 0$ minimal such that a bead is occupying position $m + l$. Let μ' be the partition corresponding to the marked abacus obtained by moving this bead into position m, with runner $v_{\mu'}$ equal to the runner this bead previously occupied (see Figure 4.8). Then $\Gamma_\delta(\mu', b) = \Gamma_\delta(\lambda, b)$, and $|\mu'| = |\mu| - l < |\mu|$. So $\mu' \in \mathcal{O}^p_\lambda(n)$, and by induction we have $|\lambda_{\mathcal{O}}| \leq |\mu'| < |\mu|$ as required.

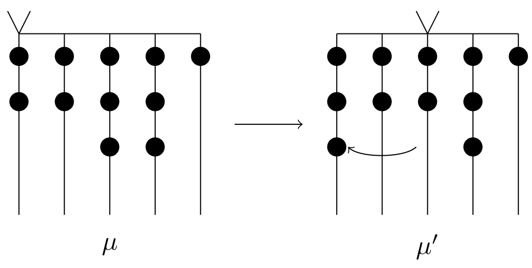


Figure 4.8: Constructing the marked abacus of μ' from μ.

$\square$

Remark. Note that since $\mathcal{O}^p_\lambda(n) \subset \mathcal{O}^p_\lambda(n+1)$ for all n (see Remark 4.21), the minimal partition in $\mathcal{O}^p_\lambda(n)$ is equal to the minimal partition in $\mathcal{O}^p_\lambda(n+1)$.

Now that we have information about the orbits of W^p_n, we will use the marked abacus to derive results about the blocks of $P^k_n(\delta)$ from our knowledge of the blocks of $P^K_n(\delta)$. Recall that we are assuming that $\delta \in R$ and are also writing δ for its image in k.

The following proposition relates blocks over a field of characteristic zero to those over a field of positive characteristic.

Proposition 4.30. *If $\mu \in \mathcal{B}^K_\lambda(n; \delta + rp)$ for some $r \in \mathbb{Z}$, then $\mu \in \mathcal{B}^k_\lambda(n; \delta)$.*

Proof. By the cellularity of $P^K_n(\delta)$, partitions λ and μ are in the same K-block if and only if there is a sequence of partitions

$$\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t)} = \mu$$

and $P^K_n(\delta)$-modules

$$M^{(i)} \leq \Delta^K_{\lambda^{(i)}}(n; \delta + rp) \qquad (1 < i \leq t)$$

such that for each $1 \leq i < t$

$$\mathrm{Hom}(\Delta^K_{\lambda^{(i)}}(n; \delta + rp), \Delta^K_{\lambda^{(i+1)}}(n; \delta + rp)/M^{(i+1)}) \neq 0.$$

Since $\delta + rp = \delta$ in k, the application of Lemma 1.3 then shows

$$\mathrm{Hom}(\Delta^k_{\lambda^{(i)}}(n; \delta), \Delta^k_{\lambda^{(i+1)}}(n; \delta)/\overline{M^{(i+1)}}) \neq 0,$$

giving us such a sequence of partitions linking λ and μ, except now we are working with $P^k_n(\delta)$-modules. □

We are now ready to begin describing the blocks of the partition algebra in positive characteristic. We begin by setting $b = n$, so that all marked abaci have n beads.

Proposition 4.31. *Let $\lambda \in \Lambda_{P_n}$. If $\lambda \neq \lambda_{\mathcal{O}}$, i.e. λ is not minimal in its orbit, then there exists a partition $\mu \in \mathcal{O}^p_\lambda(n)$ with $|\mu| < |\lambda|$ and $\mu \in \mathcal{B}^k_\lambda(n)$.*

Proof. If $\lambda \neq \lambda_{\mathcal{O}}$, then as in the proof of Proposition 4.29 either λ is not a p-core or v_λ is not the rightmost runner i such that $\Gamma(\hat{\lambda}, n)_i$ is maximal.

Suppose first that λ is not a p-core. There there is a bead, say the j-th bead, with an empty space immediately above it. We have the following three cases to consider:

Case 1 The j-th bead lies on runner v_λ.

Let μ be the partition obtained by moving the j-th bead one space up its runner, so that it now occupies position $\lambda_j - j + n - p$. Clearly $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$ since no beads are changing runners, so $\mu \in \mathcal{O}^p_\lambda(n)$. Note that $|\mu| = |\lambda| - p$, and that setwise the β_δ-sequence $\beta_\delta(\mu, n)$ must be

$$(\delta - |\lambda| + p + n, \lambda_1 - 1 + n, \ldots, \lambda_j - j - p + n, \ldots, 0) \tag{4.18}$$

since no other beads move. However since bead j lies on runner v_λ, we can find $r \in \mathbb{Z}$ such that

$$\delta - |\lambda| + n + (r+1)p = \lambda_j - j + n$$

and can therefore rewrite (4.18) as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \ldots, \delta - |\lambda| + n + rp, \ldots, 0).$$

Thus, for an appropriate element $w \in \langle s_{i,j} : 1 \leq i < j \leq n \rangle \cong \mathfrak{S}_n$ of the symmetric group on n letters, we have:

$$w^{-1}(\beta_\delta(\mu, n)) = (\lambda_j - j - rp + n, \lambda_1 - 1 + n, \ldots, \underbrace{\delta - |\lambda| + rp + n}_{j\text{-th place}}, \ldots, 0)$$

and hence

$$\begin{aligned} \beta_\delta(\lambda, n) - w^{-1}(\beta_\delta(\mu, n)) &= (\delta - |\lambda| - \lambda_j + j + rp)(\varepsilon_0 - \varepsilon_j) \\ &= \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j). \end{aligned}$$

We can rewrite this as

$$w^{-1}(\hat{\mu} + \rho(\delta) + n(1, \ldots, 1)) = \hat{\lambda} + \rho(\delta) + n(1, \ldots, 1) - \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j).$$

Since our chosen subgroup $\mathfrak{S}_n$ of $\mathfrak{S}_{n+1}$ does not act on the 0-th position, both the elements $(rp, 0, 0, \ldots, 0)$ and $n(1, 1, \ldots, 1)$ are unchanged by w^{-1}. Thus:

$$\begin{aligned} & w^{-1}(\hat{\mu} + \rho(\delta) + (rp, 0, \ldots, 0)) \\ & \qquad = \hat{\lambda} + \rho(\delta) + (rp, 0, \ldots, 0) - \langle \hat{\lambda} + \rho(\delta + rp), \varepsilon_0 - \varepsilon_j \rangle (\varepsilon_0 - \varepsilon_j) \\ \implies & w^{-1}(\hat{\mu} + \rho(\delta + rp)) = s_{0,j}(\hat{\lambda} + \rho(\delta + rp)) \\ \implies & \hat{\mu} = w s_{0,j}(\hat{\lambda} + \rho(\delta + rp)) - \rho(\delta + rp) \\ \implies & \hat{\mu} = w s_{0,j} \cdot_{\delta + rp} \hat{\lambda}. \end{aligned}$$

Therefore $\mu \in \mathcal{O}_\lambda(n; \delta + rp)$, and so by Theorem 4.12 $\mu \in \mathcal{B}^K_\lambda(n; \delta + rp)$. Proposition 4.30 then provides the final result.

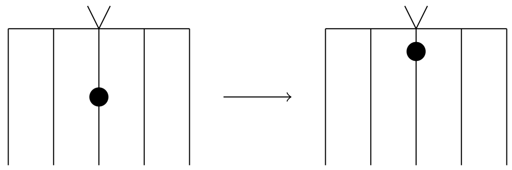


Figure 4.9: The movement of beads in Case 1.

Case 2 The j-th bead does not lie on runner v_λ and runner v_λ is empty.

Since there is a space above bead j, then by moving this bead one space up its runner and then across to runner v_λ, we obtain the abacus of a new partition μ with $|\mu| = |\lambda| - m$ for some $m > 0$. Since bead j is now on runner v_λ and occupies position $\lambda_j - j + n - m$, we see that

$$\lambda_j - j + n - m = \delta - |\lambda| + n + rp \tag{4.19}$$

for some $r \in \mathbb{Z}$. The runner v_μ is given by

$$\begin{aligned}\delta - |\mu| + n &= \delta - |\lambda| + m + n\\ &= \lambda_j - j + n - rp.\end{aligned}$$

So runner v_μ is equal to the runner previously occupied by bead j (see Figure 4.10). Therefore $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$, and so $\mu \in \mathcal{O}^p_\lambda(n)$. We also have that setwise the β_δ-sequence $\beta_\delta(\mu, n)$ is

$$(\delta - |\lambda| + m + n, \lambda_1 - 1 + n, \ldots, \lambda_j - j - m + n, \ldots, 0)$$

which, by using (4.19), we may rewrite as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \ldots, \delta - |\lambda| + n + rp, \ldots, 0)$$

and then continue as in Case 1.

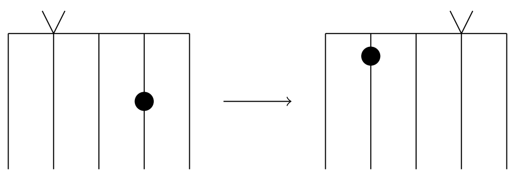


Figure 4.10: The movement of beads in Case 2.

Case 3 The j-th bead does not lie on runner v_λ and runner v_λ is not empty.

By Theorem 1.8, if we move one bead r spaces up its runner and another bead r spaces down, then the resulting partitions are in the same block for the symmetric group, and so by Theorem 4.27 are in the same block for the partition algebra. We are assuming that bead j has a space above it, and there is at least one bead on runner v_λ with a space below it (for example the final bead). Therefore we can move this bead down and bead j up by one space each and obtain a partition $\nu \in \mathcal{B}^k_\lambda(n)$ satisfying the conditions in Case 1, and so we continue as there.

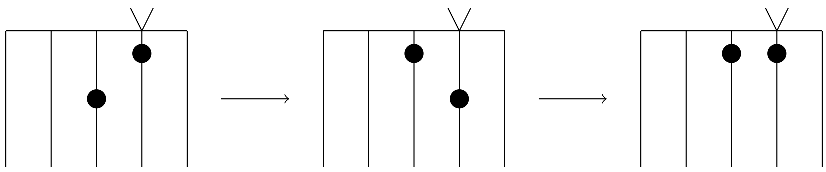


Figure 4.11: The movement of beads in Case 3.

Now suppose that λ is a p-core, but v_λ is not the rightmost runner i such that $\Gamma_\delta(\lambda, n)_i$ is maximal. Let bead j be the last bead on the rightmost runner i such that $\Gamma_\delta(\lambda, n)_i$ is maximal, so that it occupies position $\lambda_j - j + n$. Then there must exist an $m > 0$ such that position $\lambda_j - j + n - m$ is both empty and lies on runner v_λ, and let μ be the partition obtained by moving bead j to this position. Then $|\mu| = |\lambda| - m < |\lambda|$, and as in (4.19) there exists $r \in \mathbb{Z}$ such that $\lambda_j - j + n - m = \delta - |\lambda| + n + rp$, and the runner v_μ is likewise equal to the runner previously occupied by bead j. We may therefore continue as in Case 2 to obtain the final result. $\square$

We immediately deduce the following:

Corollary 4.32. *Let $\lambda \in \Lambda_{P_n}$. Then $\mathcal{O}^p_\lambda(n;\delta) \subset \mathcal{B}^k_\lambda(n;\delta)$.*

Proof. Since each orbit $\mathcal{O}^p_\lambda(n)$ contains a unique minimal element $\lambda_\mathcal{O}$, it suffices to show that $\lambda_\mathcal{O} \in \mathcal{B}^k_\lambda(n)$. We prove this by induction on $|\lambda|$.

If $|\lambda|$ is minimal, then $\lambda = \lambda_\mathcal{O}$ and there is nothing to prove. So suppose otherwise, i.e. that $\lambda \neq \lambda_\mathcal{O}$. Then by Proposition 4.31 there is a partition $\nu \in \mathcal{O}^k_\lambda(n)$ with $|\nu| < |\lambda|$ and $\nu \in \mathcal{B}^k_\lambda(n)$. By our inductive step, we then have $\lambda_\mathcal{O} \in \mathcal{B}^k_\nu(n)$. But blocks are either disjoint or coincide entirely, so $\mathcal{B}^k_\nu(n) = \mathcal{B}^k_\lambda(n)$ and $\lambda_\mathcal{O} \in \mathcal{B}^k_\lambda(n)$ as required. $\square$

We can now combine Corollaries 4.25 and 4.32 to obtain the main result of this section.

Theorem 4.33. *Let $\lambda \in \Lambda_{P_n}$, then $\mathcal{B}^k_\lambda(n;\delta) = \mathcal{O}^p_\lambda(n;\delta)$. In other words, two partitions label cell modules in the same $P^k_n(\delta)$-block if and only if their images in E_n are in the same W^p_n-orbit under the δ-shifted action.*

4.4.2 Limiting blocks

The proof of Case 3 in Proposition 4.31 requires us to use results from the modular representation theory of the symmetric group in order to continue. This is in fact a necessary step, as demonstrated in Example 4.34 below.

Example 4.34. Consider the abacus in Figure 4.12. In order to reduce the size of this partition while staying in the same orbit we must move bead j one space up its runner. However if we apply any reflection of the form $s_{0,i,rp}$ without first moving bead j up in accordance with Case 3, then we are increasing the size of the partition. The proof would then no longer be valid, as the resulting partition may be too large to reside in our indexing set, and then the cell-linkage property of Proposition 1.5 would no longer apply.

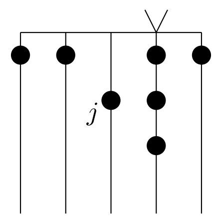


Figure 4.12: An abacus requiring moves relating to the symmetric group.

We will show that if we allow ourselves to increase the size of the intermediate partitions, then it is possible to link any partition λ to the minimal element $\lambda_{\mathcal{O}}$ of its orbit using only homomorphisms reduced from characteristic zero.

Recall from Remarks 4.3, 4.10, 4.14 and 4.21 that we can use embeddings of categories to transfer results from $P^k_n(\delta)$ to $P^k_{n+1}(\delta)$. We can repeat this process ad infinitum, and obtain the following limits

$$\begin{aligned}
\Lambda &= \{\lambda : \lambda \vdash n \text{ for some } n \in \mathbb{N}\},\\
E_\infty &= \prod_{i=0}^{\infty} \mathbb{R}\varepsilon_i,\\
W_\infty^p &= \langle s_{i,j,rp} : 0 \le i < j, r \in \mathbb{Z}\rangle,\\
\mathcal{B}_\lambda^k(\infty;\delta) &= \{\mu : \mu \in \mathcal{B}_\lambda^k(n;\delta) \text{ for some } n \in \mathbb{N}\},\\
\mathcal{O}_\lambda^p(\infty;\delta) &= \{\mu \in \Lambda : \hat{\mu} \in W_\infty^p \cdot_\delta \hat{\lambda}\},
\end{aligned}$$

where the element $\rho(\delta)$ is extended in the obvious way to

$$\rho(\delta) = (-\delta, -1, -2, -3, \dots) \in E_\infty.$$

Notation. When the context is clear we will write $\mathcal{B}_\lambda^k(\infty)$ to mean $\mathcal{B}_\lambda^k(\infty;\delta)$, and similarly for $\mathcal{O}_\lambda^p(\infty)$.

With this notion of the *limiting block* of the partition algebra, we prove the following without using any results on the modular representation theory of $\mathfrak{S}_n$.

Theorem 4.35. *Let $\lambda \in \Lambda$, then $\mathcal{B}_\lambda^k(\infty;\delta) = \mathcal{O}_\lambda^p(\infty;\delta)$. In other words, two partitions label cell modules in the same $P_n^k(\delta)$-block for some n if and only if their images in E_∞ are in the same W_∞^p-orbit under the δ-shifted action.*

Proof. Suppose $\mu \in \mathcal{B}_\lambda^k(\infty)$. Then $\mu \in \mathcal{B}_\lambda^k(n)$ for some $n \in \mathbb{N}$, and so by Corollary 4.25 we have $\mu \in \mathcal{O}_\lambda^p(n) \subset \mathcal{O}_\lambda^p(\infty)$. Note that the proof of this does not use any modular representation theory of the symmetric group.

Suppose now that $\mu \in \mathcal{O}_\lambda^p(\infty)$. Again we have $\mu \in \mathcal{O}_\lambda^p(n)$ for some $n \in \mathbb{N}$. Follow the proof of Proposition 4.31 but replace Case 3 with the following alternative:

Case 3′ The j-th bead does not lie on runner v_λ and runner v_λ is not empty.

Consider the partition μ obtained by moving bead j into the first empty space of runner v_λ. We then have $|\mu| = |\lambda| + m$ for some $m \in \mathbb{Z}$, and as in (4.19) we have

$$\lambda_j - j + n + m = \delta - |\lambda| + n + rp \tag{4.20}$$

for some $r \in \mathbb{Z}$. The runner v_μ is given by

$$\begin{aligned}
\delta - |\mu| + n &= \delta - |\lambda| - m + n\\
&= \lambda_j - j + n - rp.
\end{aligned}$$

So runner v_μ is equal to the runner previously occupied by bead j (see Figure 4.13). Therefore $\Gamma_\delta(\mu, n) = \Gamma_\delta(\lambda, n)$, but if $m > 0$ then $|\mu| > |\lambda|$ so we may not have $\mu \in \mathcal{O}^p_\lambda(n)$. However it is true that $\hat{\mu} \in W^p_\infty \cdot_\delta \hat{\lambda}$, so $\mu \in \mathcal{O}^p_\lambda(\infty)$. We also have that setwise the β_δ-sequence $\beta_\delta(\mu, n)$ is

$$(\delta - |\lambda| - m + n, \lambda_1 - 1 + n, \ldots, \lambda_j - j + m + n, \ldots, 0)$$

which, by using (4.20), we may rewrite as

$$(\lambda_j - j + n - rp, \lambda_1 - 1 + n, \ldots, \delta - |\lambda| + n + rp, \ldots, 0).$$

Arguing as in Case 1 of Proposition 4.31 we see that there is some $w \in \mathfrak{S}_n$, where again $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$ is the subgroup fixing the 0-th element, such that

$$\hat{\mu} = ws_{0,j} \cdot_{\delta - rp} \hat{\lambda}$$

and so by Theorem 4.12 and Proposition 4.30, $\mu \in \mathcal{B}^k_\lambda(\infty)$.

Now that bead j occupies the lowest position on runner v_λ, let bead j' be the bead immediately above this. Let ν be the partition obtained by moving bead j' into the space above the position previously occupied by bead j, i.e. into position $\lambda_j - j + n - p$. Then $|\nu| = |\mu| + m$ for some $m \in \mathbb{Z}$, and we repeat the argument of the previous paragraph to see that $\nu \in \mathcal{B}^k_\mu(\infty) = \mathcal{B}^k_\lambda(\infty)$.

We repeat this process for each bead that falls into Case 3 of Proposition 4.31. Then we are left only with beads in Case 1 or 2, and may therefore continue with the proof of Proposition 4.31 to arrive at the result.

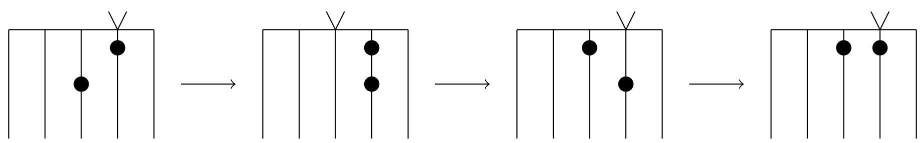


Figure 4.13: The movement of beads in Case 3′.

□

4.4.3 Decomposition matrices

Recall from Definition 1.2 that we can reduce a K-module to a k-module in such a way that its composition factors are well-defined. We can therefore compute the decomposition numbers of a cell module over k by first finding the composition factors over K, then reducing these to k. Thus, we obtain the following factorisation of the decomposition matrix:

$$[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] = \sum_{\nu \in \Lambda_{P_n}} [\Delta_\mu^K(n;\delta+rp) : L_\nu^K(n;\delta+rp)][\overline{L_\nu^K(n;\delta+rp)} : L_\lambda^k(n;\delta)], \tag{4.21}$$

where $\lambda \in \Lambda_{P_n}^*$, $r \in \mathbb{Z}$ is fixed and $\delta \in R$ is identified with its image in k. Note that by varying r it is possible to obtain several distinct factorisations of this form. We also obtain the following immediate result.

Lemma 4.36. *Let $\delta \in R$, $\mu \in \Lambda_{P_n}$, $\lambda \in \Lambda_{P_n}^*$ and $r \in \mathbb{Z}$. Then*

$$[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \geq [\Delta_\mu^K(n;\delta+rp) : L_\lambda^K(n;\delta+rp)].$$

Proof. This follows immediately from the fact that each part of the sum in (4.21) is non-negative. □

The factorisation (4.21) allows us to use the decomposition numbers in characteristic zero in determining those in characteristic p. However the second constituent of each summand is in general very difficult to compute. In this section we will simplify the process by placing restrictions on the values of n, p and δ in order to reduce the problem to cases where one of the constituents is known.

We begin by recalling some results from [DW00] which can be generalised to fields of arbitrary characteristic. Recall from Section 4.2.1 the diagram $p_{i,j}$ consisting of blocks $\{i, j, \bar{i}, \bar{j}\}$ and $\{m, \bar{m}\}$ for $m \neq i, j$, and from Section 4.2.2 the free R-module $V(n,t)$ with basis all diagrams with precisely t propagating blocks and $\overline{t+1}, \overline{t+2}, \ldots, \bar{n}$ each in singleton blocks. We now define the k-vector space

$$\Psi(n,t) = \{u \in k \otimes_R V(n,t) : p_{i,j}u = 0 \text{ for all } i \neq j\}.$$

Definition 4.37. We place a partial order $\prec$ on $I(n,t)$ by refinement of set-partitions. Let $M(n,t)$ be the set of minimal elements of $I(n,t)$ under $\prec$.

For $x, y \in I(n,t)$, we recursively define the Möbius function to be

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \preceq z \prec y} \mu(x,z) & \text{if } x \prec y \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.38. The Hasse diagram of $I(3,1)$ under $\prec$ is given below.

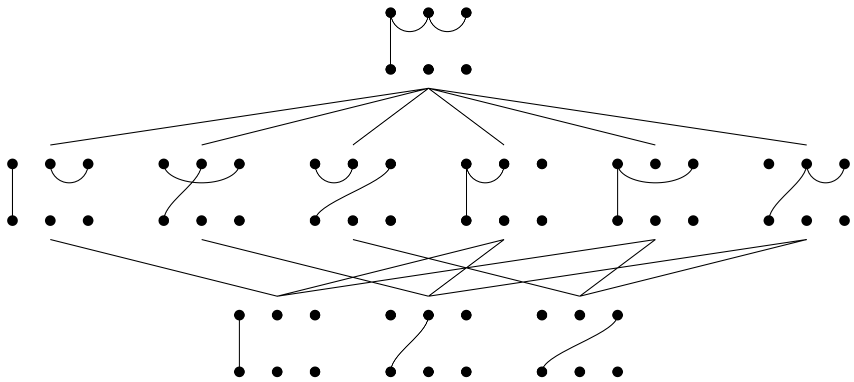


Figure 4.14: The Hasse diagram of $I(3,1)$.

The three diagrams on the bottom row are the elements of $M(3,1)$.

The proof of the following proposition is valid over a field of positive characteristic.

Proposition 4.39 ([DW00, Proposition 4.3])**.** *An R-basis for $\Psi(n,t)$ is given by the set*

$$\left\{ \sum_{x \in I(n,t)} \mu(y,x)x : y \in M(n,t) \right\}.$$

Each of these basis elements has a unique non-zero term y for some $y \in M(n,t)$. All other non-zero terms are x are strictly greater than y under $\prec$.

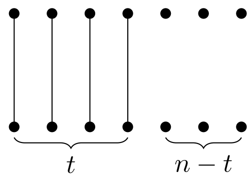
We have an action of $\mathfrak{S}_n$ on the left of $I(n,t)$ by permuting the n northern nodes, and an action of $\mathfrak{S}_t$ on the right by permuting the t leftmost southern nodes. This gives a $(\mathfrak{S}_n, \mathfrak{S}_t)$-bimodule structure on $\Psi(n,t)$. Let $\sigma \in \mathfrak{S}_n$ and $x, y \in I(n,t)$ such that $x \prec y$. Then $\sigma x \prec \sigma y$, since σy will be a refinement of the set-partition represented by σx. Therefore σ will take one basis element as given in Proposition 4.39 to another. Similarly for $\tau \in \mathfrak{S}_t$ we have $x\tau \prec y\tau$.

We then have the following decomposition of $\Psi(n,t)$ as a $(\mathfrak{S}_n, \mathfrak{S}_t)$-bimodule.

Proposition 4.40 ([DW00, Proposition 4.4]). *As a $(\mathfrak{S}_n, \mathfrak{S}_t)$-bimodule,*

$$\Psi(n,t) \cong \biguplus_{\mu \vdash t} \left(\mathrm{ind}_{\mathfrak{S}_t \times \mathfrak{S}_{n-t}}^{\mathfrak{S}_n} (S_k^\mu \boxtimes 1_{\mathfrak{S}_{n-l}}) \boxtimes S_k^\mu \right).$$

Proof. By Proposition 4.39, we can index a basis of $\Psi(n,t)$ by $M(n,t)$. Note that $\mathfrak{S}_n \times \mathfrak{S}_t$ acts transitively on this set. Let $y \in M(n,t)$ be the element below

$y =$ .

We see that $\mathfrak{S}_t \times \mathfrak{S}_{n-t}$ is a natural subgroup of $\mathfrak{S}_n$, with $\mathfrak{S}_t$ acting on the leftmost t northern nodes and $\mathfrak{S}_{n-t}$ acting on the remaining northern nodes. Then the stabiliser of y in $\mathfrak{S}_n \times \mathfrak{S}_t$ is the set of permutations

$$H = \{((\gamma, \pi), \gamma^{-1}) : \pi \in \mathfrak{S}_{n-t}, \gamma \in \mathfrak{S}_t\} \subseteq \mathfrak{S}_n \times \mathfrak{S}_t.$$

Since the action of $\mathfrak{S}_n \times \mathfrak{S}_t$ on $M(n,t)$ is transitive, we can write $\Psi(n,t) = \mathrm{ind}_H^{\mathfrak{S}_n \times \mathfrak{S}_t} 1_H$. We induce first to the subgroup $(\mathfrak{S}_t \times \mathfrak{S}_{n-t}) \times \mathfrak{S}_t$ of $\mathfrak{S}_n \times \mathfrak{S}_t$. It is clear using Frobenius reciprocity that inducing the trivial module from the subgroup $L = \{(\gamma, \gamma^{-1}) : \gamma \in \mathfrak{S}_t\}$ of $\mathfrak{S}_t \times \mathfrak{S}_t$ to $\mathfrak{S}_t \times \mathfrak{S}_t$ gives a module with filtration $\biguplus_{\mu \vdash t} (S_k^\mu \boxtimes S_k^\mu)$. Since $\mathfrak{S}_{n-t}$ has no effect here, it follows that

$$\mathrm{ind}_H^{(\mathfrak{S}_t \times \mathfrak{S}_{n-t}) \times \mathfrak{S}_t} 1_H \cong \biguplus_{\mu \vdash t} \left(S_k^\mu \boxtimes 1_{\mathfrak{S}_{n-t}}\right) \boxtimes S_k^\mu.$$

Inducing the left side of the tensor product to $\mathfrak{S}_n$ then gives the required result. This has no effect on the last factor, as seen for example by taking coset representatives in $\mathfrak{S}_n$. $\square$

Using the Littlewood-Richardson rule we obtain the following complete decomposition.

Proposition 4.41 ([DW00, Proposition 4.5]). *As a $\mathfrak{S}_n \times \mathfrak{S}_t$-module*

$$\Psi(n,t) \cong \biguplus_{\substack{\lambda \vdash n, \mu \vdash t \\ \text{with } c^\lambda_{\mu,(n-t)} = 1}} S_k^\lambda \boxtimes S_k^\mu.$$

Proof. This follows from the Littlewood-Richardson rule, generalised to an arbitrary field by James and Peel in [JP79]. The Littlewood-Richardson coefficients $c^{\lambda}_{\mu,(n-t)}$ can only be 0 or 1. Note that as a $\mathfrak{S}_n \times \mathfrak{S}_t$-module it is multiplicity free. $\square$

Proposition 4.42 ([DW00, Proposition 4.6])**.** *The elements of $\Delta^k_\mu(n)$ which are annihilated by all $p_{i,j}$ are spanned by elements of the form $u \otimes s$ for $s \in S^\mu_k$ and $u \in \Psi(n,t)$.*

Proof. First, let $u \in \Psi(n,t)$. Then for all $1 \leq i < j \leq n$ and $s \in S^\mu_k$,

$$p_{i,j}(u \otimes s) = (p_{i,j}u) \otimes s = 0 \otimes s = 0.$$

So if $w \in \Psi(n,t) \otimes S^\mu_k$, then $p_{i,j}w = 0$.

Conversely, suppose $p_{i,j}w = 0$ for some $w \in V(n,t) \otimes S^\mu_k \cong \Delta^k_\mu(n)$. We can write

$$w = \sum_{x \in I(n,t)} c_x x \otimes s_x$$

where $c_x \in k$ and $s_x \in S^\mu_k \backslash \{0\}$. We know from [DW00, Proposition 4.2] that if

$$w' = \sum_{x \in I(n,t)} \sum_{y \in M(n,t)} c_y \mu(y,x)(x \otimes s_y)$$

then $p_{i,j}w' = 0$, and therefore $p_{i,j}(w - w') = 0$. Now

$$\begin{aligned} w - w' &= \sum_{x \in I(n,t)} c_x x \otimes s_x - \sum_{x \in I(n,t)} \sum_{y \in M(n,t)} c_y \mu(y,x)(x \otimes s_y) \\ &= \sum_{x \in I(n,t)} x \otimes \left(c_x s_x - \sum_{y \in M(n,t)} c_y \mu(y,x) s_y \right). \end{aligned}$$

For $x, y \in M(n,t)$ we have $\mu(x,y) = 1$ if $x = y$ and zero otherwise. Therefore there are no terms of the form $y \otimes s_y$ ($y \in M(n,t)$) in $w - w'$, and we may write

$$w - w' = \sum_{x \in I(n,t) \backslash M(n,t)} d_x x \otimes t_x$$

where $d_x \in k$ and $t_x \in S^\mu_k \backslash \{0\}$.

If $w - w' \neq 0$, then choose $x \in I(n,t) \backslash M(n,t)$ minimal with respect to $\prec$ for which $d_x \neq 0$. As in the proof of [DW00, Theorem 4.2] there exist i, j for which $p_{i,j}x = x$, and therefore $p_{i,j}(x \otimes t_x) = x \otimes t_x$. Since $p_{i,j}$ can only join blocks of a diagram and x is minimal with respect to $\prec$, we then see that $p_{i,j}(w - w') \neq 0$, a contradiction.

Therefore $w = w'$, and by Proposition 4.39 we see that $w \in \Psi(n,t) \otimes S^\mu_k$. $\square$

The following is a very restricted case of [DW00, Proposition 4.7], but is necessary for later use. The proof of the original proposition does not generalise to fields of positive characteristic.

Proposition 4.43 ([DW00, Proposition 4.7])**.** *Let μ be a partition with $|\mu| = t < p$, and suppose $\lambda \neq \mu$ is the only partition other than μ such that $L^k_\lambda(n)$ appears as a composition factor of $\Delta^k_\mu(n)$. Then $\mu \subset \lambda$, all of the nodes in $[\lambda]/[\mu]$ are in different columns, and in fact $[\Delta^k_\mu(n) : L^k_\lambda(n)] = 1$.*

Proof. By localising we may assume that $\lambda \vdash n$. By the cellularity of $P^k_n(\delta)$ we see that $L^k_\mu(n)$ appears precisely once as a composition factor of $\Delta^k_\mu(n)$, as the head of the module. Therefore $\Delta^k_\mu(n)$ has structure

$$\begin{array}{c} L^k_\mu(n) \\ \biguplus L^k_\lambda(n). \end{array}$$

Thus there is a submodule $W \subset \Delta^k_\mu(n)$ isomorphic to $\biguplus L^k_\lambda(n)$, and therefore a sequence of modules

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{r-1} \subset W_r = W$$

such that $W_i/W_{i-1} \cong L^k_\lambda(n)$ for $1 \leq i \leq r$. Let $w_r \in W_r = W$, and consider $p_{i,j}w_r$. Since $W_r/W_{r-1} \cong L^k_\lambda(n)$ is a module for the symmetric group, it must be annihilated by $p_{i,j}$. Therefore $p_{i,j}w_r = w_{r-1}$ for some $w_{r-1} \in W_{r-1}$. By the same argument, we also see that $p_{i,j}w_{r-1} = w_{r-2}$ for some $w_{r-2} \in W_{r-2}$, and so $p^2_{i,j}w_r = w_{r-2}$. Repeating this process we arrive at $p^r_{i,j}w_r = 0$, and since $p_{i,j}$ is an idempotent we deduce that $p_{i,j}w_r = 0$ for all $w_r \in W$. By Proposition 4.42, W must then be in $\Psi(n,t) \otimes S^\mu_k$.

Consider now the module $W_1 \cong L^k_\lambda(n)$. Since $|\mu| = t < p$, as a left $k\mathfrak{S}_n$-module it is S^λ_k and we can find an idempotent e_μ such that $S^\mu_k = k\mathfrak{S}_t e_\mu$. Then for $\tau \neq \mu$, $|\tau| = |\mu|$, we have $e_\tau k\mathfrak{S}_t e_\mu = 0$ and so $e_\tau k\mathfrak{S}_t \otimes_{\mathfrak{S}_t} k\mathfrak{S}_t e_\mu = 0$. Therefore $(S^\lambda_k \boxtimes S^\tau_k) \otimes_{\mathfrak{S}_t} S^\mu_k = 0$ if $\tau \neq \mu$. This means the only terms from Proposition 4.41 we need consider in $\Psi(n,t)$ are those $\{\lambda, \mu\}$ with this given μ. By Proposition 4.41, $\mu \subset \lambda$ and the nodes of λ/μ are in different columns. Furthermore, $c^\lambda_{\mu,(n-t)} = 1$, and so there is a unique copy. $\square$

We may now present some results that allow us to use information about $P^K_n(\delta + rp)$

($r \in \mathbb{Z}$) to understand the structure of $P_n^k(\delta)$. We use the notation $\mathbf{D}(A)$ to denote the decomposition matrix of the algebra A.

We will consider separately different cases concerning the values of n and δ. The first distinction we make is due to the following Lemma:

Lemma 4.44. *Suppose there exist partitions $\lambda \vdash n$, $\mu \vdash n-t$ ($t > 0$) with $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \neq 0$. Then $\delta \in \mathbb{F}_p$, the prime subfield of k.*

Proof. Since $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \neq 0$ there exists a submodule $M \subseteq \Delta_\mu^k(n;\delta)$ such that $\mathrm{Hom}(\Delta_\lambda^k(n;\delta), \Delta_\mu^k(n;\delta)/M) \neq 0$, and hence by Theorem 4.24 we have $\mu \in \mathcal{O}_\lambda^p(n;\delta)$. Therefore by Lemma 4.22, $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$. Suppose now that $\delta \notin \mathbb{F}_p$. Entry 0 of $\hat{\mu} + \rho(\delta)$ is $\delta - |\mu|$, which cannot be an integer, and similarly for entry 0 of $\hat{\lambda} + \rho(\delta)$. However the remaining entries are all integers, so in order to have $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$ we must have $\delta - |\mu| \equiv \delta - |\lambda| \pmod p$. Thus

$$(\mu_1 - 1, \mu_2 - 2, \ldots, \mu_n - n) \sim_p (\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_n - n),$$

which is equivalent to the Specht modules S_k^μ and S_k^λ having the same p-core. In particular, this requires $|\lambda| = |\mu| + rp$ for some $r \in \mathbb{Z}$, $r \geq 0$. We will show by induction on n that $r = 0$.

If $n = 1$ there is nothing to prove as $|\lambda|, |\mu| \leq 1$.

Now suppose that $n > 1$ and assume for contradiction that $r > 0$. Following the argument of Theorem 4.24 we know that there exists $A \in \mathrm{rem}(\lambda)$ such that either:

(i) $[\Delta_\mu^k(n-1;\delta-1) : L_{\lambda-A}^k(n-1;\delta-1)] \neq 0$, or

(ii) $[\Delta_{\mu-B}^k(n-1;\delta-1) : L_{\lambda-A}^k(n-1;\delta-1)] \neq 0$ for some $B \in \mathrm{rem}(\mu)$.

Note that $\delta - 1 \notin \mathbb{F}_p$. Moreover, as $|\lambda - A| - |\mu| = rp - 1$ we see that (i) is impossible. From (ii) we obtain, by induction, that $|\lambda - A| = |\mu - B|$ and so $|\lambda| = |\mu|$ as required. □

Case 1: $\delta \notin \mathbb{F}_p$

We will show that in this case the decomposition matrix $\mathbf{D}(P_n^k(\delta))$ is equal to a block diagonal matrix, with components equal to the decomposition matrices of symmetric group algebras. A proof of this result can also be found in [HHKP10, Corollary 6.2].

Theorem 4.45 ([HHKP10, Corollary 6.2]). *Suppose $\delta \notin \mathbb{F}_p$. Then the decomposition matrix $\mathbf{D}(P_n^k(\delta))$ is equal to the block diagonal matrix*

$$\mathbf{S} = \begin{pmatrix} \boxed{\mathbf{D}(k\mathfrak{S}_n)} & & & & \\ & \boxed{\mathbf{D}(k\mathfrak{S}_{n-1})} & & & \\ & & \ddots & & \\ & & & \boxed{\mathbf{D}(k\mathfrak{S}_1)} & \\ & & & & \boxed{\mathbf{D}(k\mathfrak{S}_0)} \end{pmatrix}. \quad (4.22)$$

Proof. By the cellularity of $P_n^k(\delta)$ we immediately see that $[\Delta_\mu^k(n) : L_\lambda^k(n)] = 0$ if $|\lambda| < |\mu|$.

If $|\lambda| > |\mu|$, then by Lemma 4.44 we see that as $\delta \notin \mathbb{F}_p$, the decomposition number $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)]$ must be zero.

If now $|\lambda| = |\mu|$, then by localising we have

$$[\Delta_\mu^k(n) : L_\lambda^k(n)] = [S_k^\mu : D_k^\lambda]$$

and the result follows as these are the entries of the decomposition matrix of $k\mathfrak{S}_{|\lambda|}$. □

Case 2: $n < p$ **and** $\delta \in \mathbb{F}_p$

We will see that in this case, any non-zero decomposition numbers arise from reducing homomorphisms in the characteristic zero case of the partition algebra $P_n^K(\delta + rp)$ for some $r \in \mathbb{Z}$.

Lemma 4.46. *Let $n < p$ and $\delta \in \mathbb{F}_p$. If $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \neq 0$ then either $\lambda = \mu$ or $\mu \hookrightarrow_{\delta+rp} \lambda$ for a unique $r \in \mathbb{Z}$.*

Proof. By localising we may assume that $\lambda \vdash n$. We will prove this result by induction on n. If $n = 0$ then we have $\lambda = \mu = \emptyset$ and the result clearly holds by the cellularity of $P_0^k(\delta)$.

Since $n < p$, we must have $L_\lambda^k(n;\delta) = \Delta_\lambda^k(n;\delta) \cong S_k^\lambda$, the Specht module. If we apply the restriction functor to this module, then by the branching rule the result is non-zero:

$$\mathrm{res}_n L^k_\lambda(n;\delta) = \mathrm{res}_n \Delta^k_\lambda(n;\delta) \cong \biguplus_{A\in \mathrm{rem}(\lambda)} \Delta^k_{\lambda-A}(n-\tfrac{1}{2};\delta).$$

Since $\lambda - A \vdash n-1$ all the modules in this filtration are Specht modules, and since $n < p$ they are also simple. Thus $[\Delta^k_\mu(n;\delta) : L^k_\lambda(n;\delta)] \neq 0$ implies that we have $[\mathrm{res}_n\Delta^k_\mu(n;\delta) : L^k_{\lambda-A}(n-\frac{1}{2};\delta)] \neq 0$ for some $A \in \mathrm{rem}(\lambda)$ in row i of λ. Recall that we have an exact sequence

$$0 \longrightarrow \biguplus_{B\in\mathrm{rem}(\mu)} \Delta^k_{\mu-B}(n-\tfrac{1}{2};\delta) \longrightarrow \mathrm{res}_n\Delta^k_\mu(n) \longrightarrow \Delta^k_\mu(n-\tfrac{1}{2};\delta) \longrightarrow 0$$

and therefore a filtration of $\mathrm{res}_n\Delta^k_\mu(n;\delta)$ by modules $\Delta^k_\nu(n-\frac{1}{2};\delta)$ with $\nu = \mu$ or $\nu = \mu - B$ for some $B \in \mathrm{rem}(\mu)$.

Using the Morita equivalence from Proposition 4.5 we must therefore have $[\Delta^k_\nu(n-1;\delta-1) : L^k_{\lambda-A}(n-1;\delta-1)] \neq 0$. So by induction on n, ν and $\lambda - A$ must be a $(\delta-1+rp)$-pair for some $r \in \mathbb{Z}$.

Suppose first that $\nu = \mu$. Then since μ and $\lambda - A$ is a $(\delta-1+rp)$-pair there is some j such that

$$\lambda - A = (\mu_1, \ldots, \mu_{j-1}, \delta-1+rp-|\mu|+j, \mu_{j+1}, \ldots, \mu_m).$$

Recall that A is in row i of λ. If $j = i$ then

$$\lambda = (\mu_1, \ldots, \mu_{i-1}, \delta+rp-|\mu|+i, \mu_{i+1}, \ldots, \mu_m)$$

and so $\mu \hookrightarrow_{\delta+rp} \lambda$. If $j \neq i$, then since λ and μ are in the same k-block we must have $\hat{\mu} + \rho(\delta) \sim_p \hat{\lambda} + \rho(\delta)$. We calculate

$$\begin{aligned} |\lambda| &= |\mu| - \mu_j + (\delta - 1 + rp - |\mu| + j) + 1 \\ &= \delta + rp - \mu_j + j \end{aligned}$$

and so

$$\hat{\lambda} + \rho(\delta) = (\mu_j - j - rp, \mu_1 - 1, \mu_2 - 2, \ldots, \mu_i + 1 - i, \ldots, \delta - 1 + rp - |\mu|, \ldots, \mu_m - m).$$

By pairing equal elements from $\hat{\lambda} + \rho(\delta)$ and $\hat{\mu} + \rho(\delta)$ we are left with

$$\{\mu_j - j - rp, \mu_i + 1 - i, \delta - 1 + rp - |\mu|\} \sim_p \{\mu_j - j, \mu_i - i, \delta - |\mu|\}.$$

Clearly $\mu_j - j - rp \equiv \mu_j - j \pmod p$, so we must have $\delta - 1 + rp - |\mu| \equiv \mu_i - i \pmod p$. These are the contents of the final node in row j of λ and the penultimate node in row i respectively, and since $|\lambda| = r < p$ these cannot differ by p or more. Hence $\delta - 1 + rp - |\mu| = \mu_i - i$. But this cannot be true as these are the contents of the final nodes in different rows of $\lambda - A$.

Suppose now that $\nu = \mu - B$ for some $B \in \mathrm{rem}(\mu)$ in row k of μ. Then we have

$$\lambda - A = (\mu_1, \ldots, \mu_{j-1}, \delta + rp - |\mu| + j, \mu_{j+1}, \ldots, \mu_{k-1}, \mu_k - 1, \mu_{k+1}, \ldots, \mu_m).$$

If $k = i$, that is nodes A and B lie in the same row, then we have

$$\lambda = (\mu_1, \ldots, \mu_{j-1}, \delta + rp - |\mu| + j, \mu_{j+1}, \ldots, \mu_m)$$

and so $\mu \hookrightarrow_{\delta+rp} \lambda$.

If now we suppose $k \neq i$, $j = i$, then

$$\hat{\lambda} + \rho(\delta) = (\mu_i - i - rp, \mu_1 - 1, \ldots, \delta + rp - |\mu| + 1, \ldots, \mu_m - m),$$

Again by pairing equal elements from $\hat{\lambda} + \rho(\delta)$ and $\hat{\mu} + \rho(\delta)$ we are left with

$$\{\mu_i - i - rp, \delta + rp - |\mu| + 1, \mu_k - k - 1\} \sim_p \{\delta - |\mu|, \mu_i - i, \mu_k - k\}.$$

Therefore we have $\mu_k - k - 1 \equiv \delta + rp - |\mu| \pmod{p}$, which arguing as in the case $\nu = \mu$, $j \neq i$ is impossible. The case $k \neq i$, $j = k$ is similar.

Finally suppose k, j and i are all distinct. We have

$$\begin{aligned}
\hat{\mu} + \rho(\delta) &= (\delta - |\mu|, \ldots, \mu_i - i, \ldots, \mu_j - j, \ldots, \mu_k - k, \ldots),\\
\hat{\lambda} + \rho(\delta) &= (\mu_j - j - rp, \ldots, \mu_i - i + 1, \ldots, \delta + rp - |\mu|, \ldots, \mu_k - k - 1, \ldots),
\end{aligned}$$

and arguing as before we see that we must have $\mu_i - i + 1 = \mu_k - k$. Therefore the node A must lie directly below node B, but this cannot be true as $\lambda - A$ differs from $\mu - B$ only in row j, so adding the node A will not result in a valid partition. □

Theorem 4.47. *Let $n < p$, $\delta \in \mathbb{F}_p$ and suppose $\mu \in \Lambda_{P_n}$ is such that $\Delta^k_\mu(n;\delta) \neq L^k_\mu(n;\delta)$. Then there is a unique $r \in \mathbb{Z}$ such that*

$$[\Delta^k_\mu(n;\delta) : L^k_\lambda(n;\delta)] = [\Delta^K_\mu(n;\delta + rp) : L^K_\lambda(n;\delta + rp)]$$

for all $\lambda \in \Lambda_{P_n}$. That is, $\Delta^k_\mu(n;\delta)$ has Loewy structure

$$L_\mu^k(n;\delta)$$
$$L_\lambda^k(n;\delta)$$

for a unique λ *such that* $\mu \hookrightarrow_{\delta+rp} \lambda$.

Proof. Since $\Delta_\mu^k(n;\delta) \neq L_\mu^k(n;\delta)$ there is some $\lambda \neq \mu$ such that $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \neq 0$. By Lemma 4.46, there exists a unique $r \in \mathbb{Z}$ such that $\mu \hookrightarrow_{\delta+rp} \lambda$.

Suppose now there is another partition $\nu \neq \lambda, \mu$ such that $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] \neq 0$. Again there is a unique $r' \in \mathbb{Z}$ such that $\mu \hookrightarrow_{\delta+r'p} \nu$. We will show that this leads to a contradiction.

Consider first the case $r = r'$. Since both λ and ν are obtained from μ by adding a single row of nodes, the final node having content $\delta + rp - |\mu|$, we immediately see that we cannot be adding nodes to the same row, otherwise $\lambda = \nu$. So suppose we add nodes to row i to obtain λ and to j to obtain ν, with $i < j$. Then $\nu_j - j = \delta + rp - |\mu|$, and since ν is a partition we must have $\nu_m - m = \mu_m - m > \delta + rp - |\mu|$ for all $m < j$. In particular $\mu_i - i > \delta + rp - |\mu|$, and so we cannot add nodes to this row to obtain λ.

Suppose now that $r \neq r'$. Assume again that we are adding nodes to row i to obtain λ, and to row j to obtain ν, with $i < j$. Therefore $\lambda_i - i = \delta + rp - |\mu|$ and $\nu_j - j = \delta + r'p - |\mu|$. Notice that

$$\begin{aligned} \delta + rp - |\mu| &= \lambda_i - i \\ &> \mu_i - i \\ &= \nu_i - i \\ &> \nu_j - j \\ &= \delta + r'p - |\mu| \end{aligned}$$

and hence $r > r'$.

The hook in the Young diagram $[\lambda] \cup [\mu]$ with endpoints the last nodes of rows i and j contains $(r - r')p + 1$ nodes. Since λ and ν differ only in rows i and j, the part of this hook lying inside $[\lambda]$ contains $(r - r')p$ nodes. Therefore $|\lambda| \geq (r - r')p$, which cannot happen if $n < p$.

We have therefore shown that there cannot be two distinct partitions that appear as a composition factor of $\Delta_\mu^k(n;\delta)$ (other than μ itself). Thus we can apply Proposition 4.43 to see that $[\Delta_\mu^k(n;\delta) : L_\lambda^k(n;\delta)] = 1$, and the result follows. □

Remark 4.48. Theorem 4.47 shows us that the decomposition matrix of $P_n^k(\delta)$ when $n < p$ and $\delta \in \mathbb{F}_p$ is obtained by "putting together" all of the characteristic zero decomposition matrices for each lift of δ to K. In terms of the factorisation (4.21), this shows that for any pair of partitions (λ, μ), there is a unique $r \in \mathbb{Z}$ such that the factorisation is trivial.

Case 3: $n \geq p$ **and** $\delta \in \mathbb{F}_p$

In this case, the decomposition matrix of the partition algebra $P_n^k(\delta)$ is much more complicated. However there is still one sub-case when we can give a complete description.

Lemma 4.49. *Let $n \geq p$ and $\delta \in \mathbb{F}_p$. Then there is only one lift of $\delta \in \mathbb{F}_p$ to R such that the partition algebra $P_n^K(\delta)$ is non-semisimple if and only if $n = p$ and $\delta = p-1$.*

Proof. First notice that if $\delta < 0$ then $P_n^K(\delta)$ is always semisimple, as we can never have a δ-pair $\mu \hookrightarrow_\delta \lambda$. Combining this with [HR05, Theorem 3.27] we see that $P_r^K(\delta)$ is non-semisimple if and only if $0 \leq \delta < 2n - 1$.

Suppose that $n = p$ and $\delta = p - 1$. The lifts of δ to R are $p - 1 + rp$ with $r \in \mathbb{Z}$. Clearly there is only one value of r such that $0 \leq p - 1 + rp < 2p - 1$, namely $r = 0$, and so there is only one lift of δ producing a non-semisimple algebra.

Suppose now that $n > p$. Again, the lifts of δ to K are $\delta + rp$, only now we have at least two values of r such that $0 \leq \delta + rp < 2n - 1$. Clearly $r = 0$ satisfies this, but also since $\delta \leq p - 1$ so too does $r = 1$. Thus we have more than one lift of δ giving a non-semisimple algebra.

Finally, suppose $\delta \neq p - 1$. Once more, the lifts of δ are $\delta + rp$, and we want to satisfy the condition $0 \leq \delta + rp < 2n - 1$. We have $r = 0$ immediately, and since $\delta < p - 1 < n$ we can also choose $r = 1$, so again there is more than one lift of δ resulting in a non-semisimple algebra. $\square$

Because of this result, we will henceforth restrict our attention to the case $n = p$ and $\delta = p - 1$. We will next determine the decomposition numbers for the block $\mathcal{B}_\emptyset^k(p; p-1)$.

Lemma 4.50. *The block $\mathcal{B}_\emptyset^k(p; p-1)$ contains precisely those partitions with empty p-core.*

Proof. Using Theorem 4.33 we look instead at the orbit $\mathcal{O}^p_\emptyset(p;p-1)$, and characterise the partitions therein. This is accomplished by constructing the marked abacus of $\emptyset$ using p beads. The number of beads on each runner is given by $\Gamma(\emptyset,p)=(1,1,\ldots,1,1)$. The runner $v_\emptyset$ is given by the p-congruence class of $\delta-|\emptyset|+p\equiv p-1$, so we therefore have $\Gamma_{p-1}(\emptyset,p)=(1,1,\ldots,1,2)$. The k-block containing $\emptyset$ thus contains all partitions λ with $\Gamma_{p-1}(\lambda,p)=(1,1,\ldots,1,2)$.

Let λ be such a partition. If $v_\lambda=p-1$, then $\Gamma(\lambda)=(1,1,\ldots,1,1)$ and so λ has empty p-core. If $v_\lambda=m$ for some $0\leq m<p-1$, then

$$\Gamma(\lambda,p)=(1,\ldots,1,\underbrace{0}_{\substack{(m+1)\text{-th}\\ \text{place}}},1,\ldots,1,2).$$

Now let μ be the p-core of λ. Note that we must have $|\mu|\leq|\lambda|$. Since $\Gamma(\mu,p)=\Gamma(\lambda,p)$ and all beads are as high up their runners as possible, we can find μ explicitly. First we see that

$$\beta(\mu,p)=(2p-1,p-1,p-2,\ldots,m+1,m-1,m-2,\ldots,2,1,0)$$

and therefore

$$\begin{aligned}\mu&=\beta(\mu,p)+(1,2,\ldots,p)-(p,p,\ldots,p)\\&=(p,1,1,\ldots,1,0,0,\ldots,0).\end{aligned}$$

It is then clear that $|\mu|>p$. Since $|\mu|\leq|\lambda|$ we see that $|\lambda|>p$ and therefore λ cannot label a $P^k_p(p-1)$ cell module.

Conversely, if $\lambda\vdash t\leq p$ and has empty p-core, then $|\lambda|=p$ or 0, and

$$\Gamma_{p-1}(\lambda,p)=(1,1,\ldots,1,2)=\Gamma_{p-1}(\emptyset,p)$$

Therefore λ is in the same k-block as $\emptyset$. $\square$

Having determined which partitions lie in $\mathcal{B}^k_\emptyset(p;p-1)$, we will now determine the decomposition matrix of this block.

Lemma 4.51. *The composition series of $\Delta^k_\emptyset(p;p-1)$ is*

$$0\subset\Delta^k_{(p)}(p;p-1)\subset\Delta^k_\emptyset(p;p-1).$$

Proof. Firstly, $\emptyset \hookrightarrow_{p-1} (p)$ since the partitions differ in one row only and the final node of this row of (p) has content $p-1 = \delta - |\emptyset|$. Thus there is a non-trivial homomorphism $\Delta^K_{(p)}(p;p-1) \to \Delta^K_{\emptyset}(p;p-1)$, and so we must have $\mathrm{Hom}(\Delta^k_{(p)}(p;p-1), \Delta^k_{\emptyset}(p;p-1)) \neq 0$ by Lemma 1.3.

We must now show that there is no module N such that

$$\Delta^k_{(p)}(p;p-1) \subsetneq N \subsetneq \Delta^k_{\emptyset}(p;p-1).$$

By Lemma 4.50 and the cellularity of $P^k_p(p-1)$, any such module N would be a symmetric group module. In particular, the action of any element $p_{i,j}$ on N must be zero, and by Proposition 4.42 we see that $N \subseteq \Psi(p,0) \otimes S^{\emptyset}_k \cong \Psi(p,0)$. Now from Proposition 4.39 we have a basis for $\Psi(p,0)$ given by the set

$$\left\{ \sum_{x \in I(p,0)} \mu(y,x)x : y \in M(p,0) \right\}.$$

But $M(p,0)$ consists of only one element, namely the diagram with each node in its own block. Therefore the module $\Psi(p,0)$ is one-dimensional and is isomorphic to $\Delta^k_{(p)}(p;p-1)$. Thus there can be no module N with $\Delta^k_{(p)}(p;p-1) \subsetneq N \subsetneq \Delta^k_{\emptyset}(p;p-1)$. □

From Lemma 4.50 we have $(p) \in \mathcal{B}^k_{\emptyset}(p;p-1)$, and Lemma 4.51 shows us that in fact $[\Delta^k_{\emptyset}(p;p-1) : L^k_{(p)}(p;p-1)] = 1$. The remaining partitions in this block are all p-hook partitions, i.e. are of the form $(p-m, 1^m)$ for some $1 < m < p-1$, since these are the only partitions of p with empty p-core. Because of this, the following result from Peel allows us to complete our description of the decomposition matrix of the block $\mathcal{B}^k_{\emptyset}(p;p-1)$:

Theorem 4.52 ([Pee71, Theorem 1])**.** *Let* char $k = p > 2$. *A composition series for* $S_k^{(p-m,1^m)}$, $0 < m < p-1$, *is given by*

$$0 \subset \mathrm{Im}\ \theta^{m-1} \subset S_k^{(p-m,1^m)}$$

where $\theta^{m-1} : S_k^{(p-(m-1),1^{m-1})} \longrightarrow S_k^{(p-m,1^m)}$ *is a non-trivial* $k\mathfrak{S}_p$*-homomorphism. Further,* $S_k^{(1^p)} \cong S_k^{(2,1^{p-2})}/\mathrm{Im}\ \theta^{p-3}$.

Corollary 4.53. *Let* char $k = p > 2$. *For* $0 < m < p-1$ *we have*

$$[\Delta^k_{(p-m,1^m)}(p;p-1) : L^k_\lambda(p;p-1)] = \begin{cases} 1 & \text{if } \lambda = (p-m,1^m) \text{ or } (p-m+1, 1^{m-1}) \\ 0 & \text{otherwise.} \end{cases}$$

For $m = 0$ we have $\Delta^k_{(p)}(p; p-1) \cong L^k_{(p)}(p; p-1)$.
For $m = p-1$ we have $\Delta^k_{(1^p)}(p; p-1) \cong L^k_{(2,1^{p-2})}(p; p-1)$.

Proof. We apply Theorem 4.27 to Theorem 4.52. For $m = 0$ we use the fact that $P^k_p(p-1)$ is a cellular algebra. □

We now turn our attention to the other blocks of $P^k_p(p-1)$. The partitions here must have non-empty p-core, and since all partitions have size at most p, then they are themselves all p-cores.

Lemma 4.54. *Let $\lambda \in \Lambda_{P_p}$ be a partition with non-empty p-core. If there exists $\mu \in \mathcal{B}^k_\lambda(p)\backslash\{\lambda\}$, then $|\lambda| \neq |\mu|$.*

Proof. Choose a partition $\mu \in \mathcal{B}^k_\lambda(p)$ with $|\lambda| = |\mu|$. By the characterisation of the blocks of the partition algebra we have $\Gamma_{p-1}(\lambda, p) = \Gamma_{p-1}(\mu, p)$. As $|\lambda| = |\mu|$ we have $v_\lambda = v_\mu$, hence $\Gamma(\lambda, p) = \Gamma(\mu, p)$ and they have the same p-core. However since $|\lambda| \leq p$ and has non-empty p-core, it must in fact be that p-core. Since we also have $|\mu| \leq p$, it follows that $\mu = \lambda$. □

Theorem 4.55. *Let λ be a partition with non-empty p-core. Then the block $\mathcal{B}^k_\lambda(p)$ has the same decomposition matrix as $\mathcal{B}^K_\lambda(p)$.*

Proof. By Lemma 4.54 we can relabel

$$\mathcal{B}^k_\lambda(p) = \{\lambda^{(m)}, \lambda^{(m-1)}, \ldots, \lambda^{(1)}\}$$

where $|\lambda^{(i)}| > |\lambda^{(i-1)}|$ for $1 < i \leq m$.

Suppose $|\lambda^{(m)}| \neq p$. Then every partition in the block has size strictly less than p, and so labels a cell module for $P^k_{p-1}(p-1)$. Since the partition algebras form a tower of recollement, the decomposition matrix of the block $\mathcal{B}^k_\lambda(p; p-1)$ is the same as that of $\mathcal{B}^k_\lambda(p-1; p-1)$. We can therefore use the results of Theorem 4.47 to conclude that the decomposition numbers $[\Delta^k_{\lambda^{(i)}}(p-1) : L^k_{\lambda^{(j)}}(p-1)]$ are either 0 or 1, and the latter occurs if and only if $\lambda^{(i)} \hookrightarrow_{p-1+rp} \lambda^{(j)}$ for some $r \in \mathbb{Z}$. But since $\delta = p-1$ is the only lift of δ to K that gives a non-semisimple K-algebra, we must have $\lambda^{(i)} \hookrightarrow_{p-1} \lambda^{(j)}$.

Therefore we have

$$\begin{aligned}[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(j)}}(p)] &= [\Delta^k_{\lambda^{(i)}}(p-1) : L^k_{\lambda^{(j)}}(p-1)] \\ &= [\Delta^K_{\lambda^{(i)}}(p-1) : L^K_{\lambda^{(j)}}(p-1)] \\ &= [\Delta^K_{\lambda^{(i)}}(p) : L^K_{\lambda^{(j)}}(p)].\end{aligned}$$

Suppose now that $|\lambda^{(m)}| = p$. Then the partitions $\lambda^{(m-1)}, \lambda^{(m-2)}, \ldots, \lambda^{(1)}$ are all of size strictly less than p, and therefore label cell modules for $P^k_{p-1}(p-1)$. By the same argument as above, the decomposition matrix obtained by removing the row and column labelled by $\lambda^{(m)}$ is the same as that of $\mathcal{B}^k_{\lambda^{(1)}}(p-1)$, which is the same as in characteristic zero.

It remains to show that the decomposition numbers $[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(m)}}(p)]$ are the same as in characteristic zero. We begin by showing that $[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(m)}}(p)] = 0$ for $i < m-1$. Since $\lambda^{(m)}$ is the only partition of size p in its block, the simple module $L^k_{\lambda^{(m)}}(p)$ is a Specht module. Therefore after applying the restriction functor we have the following filtration:

$$\mathrm{res}_p L^k_{\lambda^{(m)}}(p) \cong \biguplus_{A \in \mathrm{rem}(\lambda^{(m)})} \Delta^k_{\lambda^{(m)}-A}(p-\tfrac{1}{2}).$$

Therefore we can apply the same argument as in Theorem 4.47 and see that if $[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(m)}}(p)] \neq 0$, then either $\lambda^{(i)} = \lambda^{(m)}$ or $\lambda^{(i)} \hookrightarrow_{p-1} \lambda^{(m)}$. Following the proof of Theorem 4.47 we must then have

$$[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(m)}}(p)] = \begin{cases} 1 & \text{if } i = m-1, m \\ 0 & \text{otherwise} \end{cases}$$

and hence $[\Delta^k_{\lambda^{(i)}}(p) : L^k_{\lambda^{(m)}}(p)] = [\Delta^K_{\lambda^{(i)}}(p) : L^K_{\lambda^{(m)}}(p)]$ by Theorem 4.9. □

Remark 4.56. If we denote again by $\mathbf{S}$ the block decomposition matrix of the symmetric group algebras over k (see (4.22)), then we can combine Lemma 4.51, Corollary 4.53 and Theorem 4.55 and say that the decomposition matrix $\mathbf{D}(P^k_p(p-1))$ is equal to the product $\mathbf{D}(P^K_p(p-1))\,\mathbf{S}$. In fact, we can compute $\mathbf{S}$ explicitly in this case using Corollary 4.53. A comparison of this to the factorisation (4.21) also reveals that in this case, the composition factors of $\overline{L^K_\nu(p; p-1)}$ are known.

Unfortunately without the restrictions imposed thus far, we encounter examples of partition algebras whose decomposition matrices are not obtained from the methods summarised in Remarks 4.48 or 4.56. One such is detailed below.

Example 4.57. We will show the decomposition matrix of $P_4^k(1)$ with char $k = 3$ cannot be computed as in Remarks 4.48 or 4.56. We present below the decomposition matrix of $k\mathfrak{S}_4$, which can be found in [Jam78, Appendix]:

$$\begin{array}{c} \\ S_k^{(4)} \\ S_k^{(3,1)} \\ S_k^{(2^2)} \\ S_k^{(2,1^2)} \\ S_k^{(1^4)} \end{array} \begin{array}{c} \begin{array}{cccc} D_k^{(4)} & D_k^{(3,1)} & D_k^{(2^2)} & D_k^{(2,1^2)} \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right) \end{array}.$$

We will first show that there exist non-zero decomposition numbers $[\Delta_\mu^k(4;1) : L_\lambda^k(4;1)]$ for which there is no $r \in \mathbb{Z}$ such that $\mu \hookrightarrow_{1+3r} \lambda$, thus not following Remark 4.48. Indeed, examination of the decomposition matrix of $k\mathfrak{S}_4$ combined with Theorem 4.27 shows us that $\Delta_{(2^2)}^k(4;1)$ has a submodule isomorphic to $L_{(4)}^k(4;1)$. Therefore $[\Delta_{(2^2)}^k(4;1) : L_{(4)}^k(4;1)] \neq 0$, but $(2^2) \not\subset (4)$ and so there cannot exist an integer r with $(2^2) \hookrightarrow_{\delta+rp} (4)$.

We will now show that the decomposition matrix of $P_4^k(1)$ is not equal to the product of the decomposition matrices $\mathbf{D}(P_4^K(1{+}3r))\,\mathbf{S}$, for any $r \in \mathbb{Z}$. The semisimplicity criterion of [HR05, Theorem 3.27] shows us that we must consider $r = 0, 1$.

Consider first the case $r = 0$, that is $P_4^K(1)$. We let $\lambda = (2,1^2)$ and $\mu = (2,1)$, then we have $\delta - |\mu| = -2$. Note that these partitions differ by a single node of content -2 in the third row, and therefore form a 1-pair. By Theorem 4.9 we thus have $[\Delta_{(2,1)}^K(4;1) : L_{(2,1^2)}^K(4;1)] = 1$, and so by Lemma 1.3

$$[\Delta_{(2,1)}^k(4;1) : L_{(2,1^2)}^k(4;1)] \neq 0. \tag{4.23}$$

Now consider the case $r = 1$, i.e. $P_4^K(4)$. Let $\lambda = (4)$ and $\mu = (1)$, then we have $\delta - |\mu| = 3$. These partitions differ by a strip of nodes in the first row, the last of which has content 3, and therefore form a 4-pair. By Theorem 4.9 we see that $[\Delta_{(1)}^K(4;4) : L_{(4)}^K(4;4)] = 1$, and so by Lemma 1.3

$$[\Delta_{(1)}^k(4;1) : L_{(4)}^k(4;1)] \neq 0. \tag{4.24}$$

If the decomposition matrix $\mathbf{D}(P_4^k(1))$ was equal to the product $\mathbf{D}(P_4^K(1+3r))\,\mathbf{S}$

for some $r \in \mathbb{Z}$, we would have the following expansion for every $\mu \in \Lambda_{P_4}$, $\lambda \in \Lambda^*_{P_4}$:

$$[\Delta^k_\mu(4;1) : L^k_\lambda(4;1)] = \sum_{\nu \in \Lambda_{P_4}} [\Delta^K_\mu(4;1+3r) : L^K_\nu(4;1+3r)][S^\nu_k : D^\lambda_k].$$

First let $r = 0$, $\lambda = (4)$ and $\mu = (1)$. By examining the decomposition matrix of $k\mathfrak{S}_4$, we see that the only partitions ν for which $[S^\nu_k : D^{(4)}_k] \neq 0$ are $\nu = (4)$ and $\nu = (2^2)$. The above factorisation then becomes

$$\begin{aligned}[\Delta^k_{(1)}(4;1) : L^k_{(4)}(4;1)] &= [\Delta^K_{(1)}(4;1) : L^K_{(4)}(4;1)][S^{(4)}_k : D^{(4)}_k] \\ &\quad + [\Delta^K_{(1)}(4;1) : L^K_{(2^2)}(4;1)][S^{(2^2)}_k : D^{(4)}_k] \\ &= [\Delta^K_{(1)}(4;1) : L^K_{(4)}(4;1)] + [\Delta^K_{(1)}(4;1) : L^K_{(2^2)}(4;1)].\end{aligned}$$

From Theorem 4.9 we know that all non-decomposition numbers in characteristic zero correspond to δ-pairs. However neither (4) and (1) nor (2^2) and (1) are 1-pairs, and therefore both these decomposition numbers are zero. This contradicts (4.24), and the factorisation must in fact not be valid for $r = 0$.

Now let $r = 1$, $\lambda = (2,1^2)$ and $\mu = (2,1)$. Again by examining the decomposition matrix of $k\mathfrak{S}_4$, we see that the only partition ν for which $[S^\nu_k : D^{(2,1^2)}_k] \neq 0$ is $\nu = (2,1^2)$. The factorisation then becomes

$$\begin{aligned}[\Delta^k_{(2,1)}(4;1) : L^k_{(2,1^2)}(4;1)] &= [\Delta^K_{(2,1)}(4;4) : L^K_{(2,1^2)}(4;4)][S^{(2,1^2)}_k : D^{(2,1^2)}_k] \\ &= [\Delta^K_{(2,1)}(4;4) : L^K_{(2,1^2)}(4;4)].\end{aligned}$$

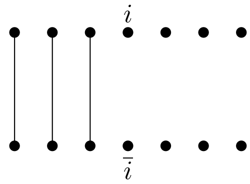
Again we see that $(2,1^2)$ and $(2,1)$ is not a 4-pair, and therefore this decomposition number is zero. This contradicts (4.23), and the factorisation is not valid for $r = 1$.

Since $\delta = 1$ and $\delta = 4$ are the only values of δ such that $P^K_4(\delta)$ is non-semisimple, we see that there is no $r \in \mathbb{Z}$ that allows us to express the decomposition matrix in characteristic p as a product as above.

4.5 The case $\delta = 0$ or $p = 2$

In this section we briefly discuss the cases $\delta = 0$ and $p = 2$.

When $\delta = 0$, most of the arguments within this chapter can be modified slightly in order to still hold. The first obvious change is to the idempotents e_i as defined in Figure 4.3. Since δ is no longer invertible, we must replace this with the quasi-idempotent $\widetilde{e}_i$ as defined in Figure 4.15.

$\widetilde{e}_i =$ 

.

Figure 4.15: The quasi-idempotent $\widetilde{e}_i$.

This will require modifications to the proofs of the statements involving the functors F_n and G_n, details of which can be found in [DW00, Section 3]. We also note that the labelling set for the simple $P_n^{\mathbb{F}}(0)$-modules is no longer $\Lambda_{P_n}^*$ as defined in 4.6. As shown in [GL96, Theorem 4.17], we must exclude the empty partition. Then, taking care with statements involving the cell module $\Delta_{\emptyset}^{\mathbb{F}}(n;\delta)$ or localisation/globalisation, the remainder of the results in characteristic zero are still valid. The same is true of the results on the modular representation theory of the partition algebra.

Now consider the case $p = 2$. All of the proofs involved in the geometric characterisation of the blocks still hold in characteristic 2. Note for instance in Definition 4.15 of the (δ, p)-pair that although we are dividing by 2, we can cancel this with the term $t(t-1)$. The results on the decomposition matrices are also valid in characteristic 2, however instead of using the result of Peel in Theorem 4.52 and Corollary 4.53, we simply calculate the decomposition matrix of $k\mathfrak{S}_2$.

Bibliography

[BDK14] C. Bowman, M. De Visscher, and O. King, *The block structure of the partition algebras in characteristic $p \geq 0$*, arXiv:1402.5157, 2014.

[Bra37] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.

[Bro55] W. P. Brown, *An algebra related to the orthogonal group*, Michigan Math. J. **3** (1955), 1–22.

[CD11] A. G. Cox and M. De Visscher, *Diagrammatic Kazhdan–Lusztig theory for the (walled) Brauer algebra*, J. Algebra **340** (2011), 151–181.

[CDM09a] A. G. Cox, M. De Visscher, and P. P. Martin, *The blocks of the Brauer algebra in characteristic zero*, Representation Theory **13** (2009), 272–308.

[CDM09b] ———, *A geometric characterisation of the blocks of the Brauer algebra*, J. London Math. Soc. **80** (2009), 471–494.

[CDM11] ———, *Alcove geometry and a translation principle for the Brauer algebra*, J. Pure Appl. Algebra **215** (2011), 335–367.

[CGM03] A. G. Cox, J. J. Graham, and P. P. Martin, *The blob algebra in positive characteristic*, J. Algebra **266** (2003), 584–635.

[CMPX06] A. G. Cox, P. P. Martin, A. E. Parker, and C. C. Xi, *Representation theory of towers of recollement: theory, notes and examples*, J. Algebra **302** (2006), 340–360.

[CP80] R. W. Carter and M. T. J. Payne, *On homomorphisms between Weyl modules and Specht modules*, Math. Proc. Camb. Phil. Soc. **87** (1980), 419–425.

[CR81] C. W. Curtis and I. Reiner, *Methods of representation theory: with applications to finite groups and orders*, vol. 1, Wiley New York, 1981.

[DW00] W. F. Doran and D. B. Wales, *The Partition Algebra Revisited*, J. Algebra **231** (2000), 265–330.

[DWH99] W. F. Doran, D. B. Wales, and P. J. Hanlon, *On the semisimplicity of the Brauer centralizer algebras*, J. Algebra **211** (1999), 647–685.

[Eny13] J. Enyang, *A seminormal form for partition algebras*, J. Comb. Theory Ser. A **120** (2013), 1737–1785.

[Fay05] M. Fayers, *Irreducible Specht modules for Hecke algebras of type a*, Adv. Math. **193** (2005), 438–452.

[Fay10] ———, *On the irreducible Specht modules for Iwahori–Hecke algebras of type a with $q = -1$*, J. Algebra **323** (2010), 1839–1844.

[FL09] M. Fayers and S. Lyle, *Some reducible Specht modules for Iwahori–Hecke algebras of type a with $q = -1$*, J. Algebra **321** (2009), 912–933.

[GdHJ89] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, vol. 14, Springer-Verlag, 1989.

[GL96] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[Gre80] J. A. Green, *Polynomial representations of GL_n*, Lecture Notes in Mathematics, vol. 830, Springer, 1980.

[GW93] F. M. Goodman and H. Wenzl, *The Temperley-Lieb algebra at roots of unity*, Pacific J. Math. **161** (1993), 307–334.

[HHKP10] R. Hartmann, A. Henke, S. König, and R. Paget, *Cohomological stratification of diagram algebras*, Math. Ann. **347** (2010), 765–804.

[HP06] R. Hartmann and R. Paget, *Young modules and filtration multiplicities for Brauer algebras*, Math. Z. **254** (2006), 333–357.

[HP08] A. Henke and R. Paget, *Brauer algebras with parameter $n = 2$ acting on tensor space*, Algebras and Representation Theory **11** (2008), 545–575.

[HR05] T. Halverson and A. Ram, *Partition algebras*, European J. Combin. **26** (2005), 869–921.

[HW89] P. J. Hanlon and D. B. Wales, *On the decomposition of Brauer's centralizer algebras*, J. Algebra **121** (1989), 409–445.

[Jam78] G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer-Verlag, 1978.

[JK81] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, 1981.

[Jon83] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[Jon94] ———, *The Potts model and the symmetric group*, Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 259–267.

[JP79] G. D. James and M. H. Peel, *Specht series for skew representations of symmetric groups*, J. Algebra **56** (1979), 343–364.

[Kin14a] O. King, *The limiting blocks of the Brauer algebra in characteristic p*, J. Algebra **397** (2014), 168–189.

[Kin14b] ———, *On the decomposition matrix of the partition algebra in positive characteristic*, arXiv:1403.6691, 2014.

[KX98] S. König and C. C. Xi, *On the structure of cellular algebras*, Algebras and modules II (Geiranger 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., 1998, pp. 365–386.

[KX99] ———, *Cellular algebra: inflations and Morita equivalences*, J. London Math. Soc. **60** (1999), 700–722.

[KX01] S. König and C. C. Xi, *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc. **353** (2001), 1489–1505.

[Mar91] P. P. Martin, *Potts models and related problems in statistical mechanics*, Advances in Statistical Mechanics, vol. 5, World Scientific, 1991.

[Mar92] ———, *On Schur-Weyl duality, A_n Hecke algebras and quantum $sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$*, Int. J. Mod. Phys. A **7** (1992), 645–673.

[Mar94] ———, *Temperley-Lieb algebras for non-planar statistical mechanics – the partition algebra construction*, J. Knot Theory Ramifications **3** (1994), 51–82.

[Mar96] ———, *The Structure of the Partition Algebras*, J. Algebra **183** (1996), 319–358.

[Mar00] ———, *The partition algebra and the Potts model transfer matrix spectrum in high dimensions*, J. Phys. A **33** (2000), no. 19, 3669–3695.

[Mar09] ———, *The decomposition matrices of the Brauer algebra over the complex field*, arXiv:0908.1500, 2009.

[Pee71] M. H. Peel, *Hook representations of the symmetric groups*, Glasgow Math. J. **12** (1971), 136–149.

[RS06] H. Rui and M. Si, *A criterion on the semisimple Brauer algebras II*, J. Comb. Theory Ser. A **113** (2006), 1199–1203.

[RS14] D. Ridout and Y. Saint-Aubin, *Standard Modules, Induction and the Temperley-Lieb Algebra*, to appear in Adv. Theor. Math. Phys. **18**, 2014.

[Rui05] H. Rui, *A criterion on the semisimple Brauer algebras*, J. Comb. Theory Ser. A **111** (2005), 78–88.

[Sch27] I. Schur, *Über die rationalen Darstellungen der allgemeinen linearen Gruppe*, Sitzungsberichte Akad. Berlin (1927), 58–75.

[Sha13] A. Shalile, *Decomposition numbers of Brauer algebras in non-dividing characteristic*, Preprint, 2013.

[Sha14] ———, *On the modular representation theory of the partition algebra*, arXiv:1403.5132, 2014.

[TL71] H. N. V. Temperley and E. H. Lieb, *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem*, Proc. Roy. Soc. London Ser. A **322** (1971), 251–280.

[Wen88] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. of Math. **128** (1988), 173–193.

[Wes95] B. W. Westbury, *The representation theory of the Temperley-Lieb algebras*, Math. Z. **219** (1995), 539–565.

[Xi99] C. Xi, *Partition algebras are cellular*, Compositio Mathematica **119** (1999), 107–118.