Risk capital allocation and cooperative pricing of insurance liabilities

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Abstract

The Aumann-Shapley (1974) value, originating in cooperative game theory, is used for the allocation of risk capital to portfolios of pooled liabilities, as proposed by Denault (2001). We obtain an explicit formula for the Aumann-Shapley value, when the risk measure is given by a distortion premium principle (Wang et al., 1997). The capital allocated to each instrument or (sub)portfolio is given as its expected value under a change of probability measure. Motivated by Mirman and Tauman (1982), we discuss the role of Aumann-Shapley prices in an equilibrium context and present a simple numerical example.

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1 Introduction

We discuss the problem of allocating aggregate capital requirements for a portfolio of pooled liabilities to the instruments that the portfolio consists of. Cooperative game theory (Shapley (1953), Aumann and Shapley (1974), Aubin (1981)) provides a suitable framework for cost allocation problems (e.g. Lemaire 1984). Typically, provisions are made so that the allocation of total costs does not produce disincentives for cooperation to any player (instrument) in the game or coalition of players (sub-portfolio of instruments). A closely related approach has been to determine an allocation scheme by imposing economically motivated axioms on the system of prices that it would produce (Billera and Heath (1982), Mirman and Tauman (1982)). The solution concept that emerges from the latter approaches is the celebrated Aumann-Shapley (1974) value. In the context of capital allocation, cooperation can be understood as the pooling of risky portfolios, and the cost as the 'risk capital' that a regulator decides that the holder of the portfolios should carry. This case was discussed in depth by Denault (2001), who used a cost functional based on a coherent risk measure (Artzner et al., 1999), and the Aumann-Shapley value emerged again as an appropriate solution concept. Explicit calculations of the Aumann-Shapley value were provided when the risk measure used is Expected Shortfall (this problem was solved by Tasche (2000a) in the context of performance measurement) and for the case of a risk measure used by the Securities and Exchange Commission.

In this paper we calculate an analytic formula for the Aumann-Shapley value using quantile derivatives (Tasche 2000b), for the case that the risk measure belongs to the class of distortion principles (Denneberg (1990), Wang et al. (1997)). We obtain a representation of the Aumann-Shapley value, i.e. the capital allocated to each portfolio, as the expected value of the portfolio under a change of probability measure. This representation creates a formal link between problems of allocating capital and pricing risks. We discuss this relationship through the example of a pool, which covers specific liabilities carried by a number of insurers, who in turn make cash contributions to the pool, that can be interpreted as risk premia.

It was shown by Mirman and Tauman (1982) that the Aumann-Shapley value yields equilibrium prices in a monopolistic production economy. Motivated by this work, we generalise the example to a case where the different insurers choose the extent of coverage received from the pool, by expected utility maximization. This set-up is quite different from the equilibrium models usually found in the literature on risk sharing, for example Borch (1962), Bühlmann (1980), Taylor (1995), Aase (2002). In these papers, market prices are obtained via a clearing condition, which is not applicable to the problem that we discuss. Finally, we provide a simple numerical example, where the pool offers stop-loss protection to the participating insurers.

2 Coherent risk measures and distortion principles

A coherent risk measure is defined by Artzner et al. (1999) as a functional $\rho(X)$ on a collection of random cashflows (in our case X will be a non-negative random variable representing liabilities) that satisfies the following properties:

Monotonicity: If $X \leq Y$ a.s. then $\rho(X) \leq \rho(Y)$

Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

Positive Homogeneity: If $a \in \mathbb{R}_+$ then $\rho(aX) = a\rho(X)$

Translation Invariance: If $a \in \mathbb{R}$ then $\rho(X + a) = \rho(X) + a$

$\rho(X)$ is interpreted as "the minimum extra cash that the agent has to add to the risky position X, and to invest 'prudently', to be allowed to proceed with his plans" (Artzner et al., 1999). 'Invest prudently', in this paper, means with zero interest.

All functionals satisfying the above properties allow a representation:

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}[X] \tag{1}$$

where $\mathcal{P}$ is a collection of probability measures.

Two random variables X, Y are called comonotonic if there is a random variable U and non-decreasing functions g, h such that $X = g(U), Y = h(U)$ (Denneberg, 1994a). Comonotonicity corresponds to the strongest form of positive dependence between random variables. An additional desirable property of risk measures is additivity for comonotonic risks:

Comonotonic Additivity: If X, Y comonotonic then $\rho(X+Y) = \rho(X)+\rho(Y)$

It can be shown that, if and only if $\rho(X)$ is a coherent risk measure satisfying comonotonic additivity, it has a representation as a Choquet integral with respect to a submodular set function or capacity, v (Choquet (1953), Denneberg (1994a), Delbaen (2000)):

$$\rho(X) = \int X dv = \int_0^\infty v(X > t)dt, \tag{2}$$

where $v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$.

Let $\mathbb{P}_0$ be the physical (actuarial) probability measure. If $g : [0, 1] \mapsto [0, 1]$ is a continuous, increasing and concave function, with $g(0) = 0$ and

$g(1) = 1$, then $v(A) = g(\mathbb{P}_0(A))$ is a submodular set function (Denneberg, 1994a). Thus the following integral is a comonotonic additive coherent risk measure:

$$\rho(X) = \int_0^\infty g(\mathbb{P}_0(X > t))dt, \tag{3}$$

The risk measures used in this paper will be of the above form. This type of risk measures have been axiomatically defined in the context of insurance pricing by Denneberg (1990) and Wang et al. (1997). They are termed distortion principles, since the non-linear function g 'distorts' the physical probability measure $\mathbb{P}_0$. Choquet integrals have also found application as non-linear pricing functionals in financial markets with frictions (Chateauneuf et al., 1996).

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, the set $\mathcal{P}$ of probability measures in the representation (1) corresponds to the core of the set function $v = g(\mathbb{P}_0)$ (Denneberg (1994a), Delbaen (2000)):

$$\rho(X) = \int_0^\infty g(\mathbb{P}_0(X > t))dt = \sup\{E_{\mathbb{P}}[X] : \mathbb{P}(A) \leq g(\mathbb{P}_0(A)) \ \forall A \in \mathcal{F}\} \tag{4}$$

Distortion principles have an economic justification in terms of Yaari's (1987) dual theory of choice under risk. By modifying the von Neumann-Morgenstern independence axiom, Yaari obtains an operator H dual to expected utility, given by the superadditive Choquet integral:

$$H(X) = \int_0^\infty h(\mathbb{P}_0(X > t))dx, \tag{5}$$

where $h : [0, 1] \mapsto [0, 1]$ is a continuous, increasing and convex function, which $h(0) = 0$ and $h(1) = 1$. While in expected utility theory preferences (risk aversion) are modelled by a nonlinear transformation of a random cash-flow, in Yaari's framework preferences ('uncertainty aversion') are expressed by a nonlinear distortion of the cashflow's probability distribution. The distortion principle can then be obtained by an indifference argument.

3 Cost allocation and cooperative games

3.1 The cost functional

In our application, costs corresponding to the capital that the holder of a risky portfolio (e.g. of insurance liabilities) is obliged to hold will be allocated to the different instruments (or sub-portfolios) it consists of. The cost functional used will be derived from the distortion principle defined in the previous section.

Consider a portfolio Z^u composed of n liabilities X_j, with portfolio weights $u \in [0,1]^n$:

$$Z^u = \sum_j u_j X_j \tag{6}$$

Then, as in Denault (2001), we define the cost functional $c : [0,1]^n \mapsto \mathbb{R}$:

$$c(u) = \rho(Z^u) = \rho\left(\sum_j u_j X_j\right) \tag{7}$$

This cost functional gives the capital that the holder of portfolio Z^u must hold, as a function of the vector u of portfolio weights.

3.2 Cooperative games and the fuzzy core

In this section we give a brief and informal discussion of the application of cooperative game theory to cost allocation problems. Consider n investors, each holding a liability (i.e. a positive random variable) X_i and therefore each obliged to hold (regulatory) capital $\rho(X_i)$; in aggregate they have to hold $\sum_j \rho(X_j)$. Now suppose that the investors decide to pool their liabilities, i.e. to form a portfolio $\sum_j X_j$. The aggregate capital requirement then becomes $\rho(\sum_j X_j)$. Because of the subadditivity property of coherent risk measures, it will then be $\rho(\sum_j X_j) \leq \sum_j \rho(X_j)$. This means that the level of aggregate capital required for the pooled liabilities $Z = \sum_j X_j$ will be lower than 'the sum of its parts'.

Thus, cooperation, in the form of the pooling of liabilities, produces in the aggregate capital savings for the investors of size $\sum_j \rho(X_j) - \rho(\sum_j X_j)$. Furthermore, maximal savings are produced when all investors contribute the total of their liabilities to the pool. The question that emerges now is how to allocate the costs (or equivalently the savings) to the different investors, in a fair (in some sense) and efficient way. The problem of cost allocation is thus to determine an appropriate vector $d \in \mathbb{R}^n$ such that:

$$\sum_j d_j = c(\mathbf{1}) = \rho(Z) = \rho(\sum_j X_j) \tag{8}$$

Cooperative game theory provides a conceptual framework for addressing such problems. If N is the set formed by n players (e.g. portfolios), we define an n-person cooperative game (Shapley, 1953) as an ordered pair (c, N), where c is a real-valued function $c(u_S) : \mathbb{R}^n \mapsto \mathbb{R}$. The set $S \in 2^N$ represents a coalition of players and the argument u_S is a vector in $\mathbb{R}^n$ with 1 in the ith component if the ith player belongs to S and 0 otherwise. The function $c(u_S)$ is the cost that this coalition would incur, if the players in S would cooperate and ignore the rest of the players. In our application, $c(u_S) = \rho\left(\sum_{j\in S} X_j\right)$ would be the amount of total capital required from a pool containing all liabilities carried by members of S, e.g. all instruments $X_j,\ j \in S$.

Consider all allocations $d \in \mathbb{R}^n$ of costs to the players that are efficient, that is, they add up to the aggregate cost (8). A further requirement on the allocation scheme is that there be no coalition S with 'blocking' power, that is, no coalition that would reduce its costs if it left the grand coalition N. These considerations are formalised by the concept of the core, $\mathcal{C}$, of the game:

$$\mathcal{C} = \{d \in \mathbb{R}^n | \sum_{j \in N} d_j = c(u_N) \quad \text{and} \quad c(u_S) \geq \sum_{i \in S} d_i \quad \forall S \in 2^N\} \tag{9}$$

The above setting may prove inappropriate for some applications. In the case that the players represent portfolios, it is conceivable that a player could participate in a coalition with only part of his investment (Denault, 2001). A suitable framework for such problems is provided by the theory of fuzzy cooperative games introduced by Aubin (1981). In a fuzzy game, each player participates in a coalition to a certain degree; thus the coalitions are interpreted as fuzzy sets. (This approach is closely related to the theory of non-atomic games of Aumann and Shapley (1974), where a continuum of players is considered and measureable sets stand for coalitions.)

An n-person fuzzy cooperative game is defined by a set of n players N and a function: $c(u) : [0,1]^n \mapsto \mathbb{R}$. The vector u represents a fuzzy coalition, by giving the 'level of involvement' of each player to that coalition. E.g. $u_i = 0$ means that player i does not contribute anything to coalition u, $u_i = 1$ means that he fully participates to u, and $u_i = 0.5$ means that he commits half of his 'involvement in the game' (e.g. $0.5X_i$) to coalition u. Note that the coalition of all players is now represented by the n-vector of ones, $\mathbf{1}$. The core of an n-person fuzzy cooperative game (N, c) is then defined as:

$$\mathcal{C} = \{d \in \mathbb{R}^n | c(\mathbf{1}) = \sum_{i \in N} d_i \quad \text{and} \quad c(u) \geq \sum_{i \in N} u_i d_i \quad \forall u \in [0,1]^n\} \tag{10}$$

It is easily understood that in the fuzzy setting, the number of coalitions with potential blocking power increases and, accordingly, the core shrinks. Consider the case that the cost function c is subadditive and positively homogenous:

$$c(\phi + \psi) \leq c(\phi) + c(\psi)$$

$$c(\gamma u) = \gamma c(u) \quad \text{for} \quad \gamma \geq 0$$

Then the core of the fuzzy game is convex, compact and non empty. Furthermore, if $c(u)$ is differentiable at $u = \mathbf{1}$ then the core consists only of the gradient vector of $c(u)$ at $u = \mathbf{1}$ (Aubin, 1981):

$$d_i^{\mathcal{C}} = \frac{\partial c(u)}{\partial u_i}|_{u_j=1\forall j} \tag{11}$$

This is the allocation scheme that we are going to use subsequently in the paper. We note that Tasche (2000a) obtained a similar formula, in the context of performance measurement.

The Aumann-Shapley value is the unique linear operator satisfying a set of game theoretically motivated axioms (Aumann and Shapley (1974), Aubin (1981)). The Aumann-Shapley value produces an allocation of costs:

$$d_i^{AS} = \frac{\partial}{\partial u_i} \int_0^1 c(\gamma u) d\gamma|_{u_j=1\forall j} \tag{12}$$

Besides its game theoretical derivation, the Aumann-Shapley value has been arrived at by several authors (e.g. Billera and Heath (1982), Mirman and Tauman (1982)) as the unique allocation scheme satisfying a set of economically motivated axioms. In the case that the cost function is subadditive, positively homogenenous and differentiable at $u = \mathbf{1}$, it is obvious that the Aumann-Shapley value coincides with the core of the fuzzy game (c, N):

$$d_i^{AS} = d_i^{\mathcal{C}} = \frac{\partial c(u)}{\partial u_i}|_{u_j=1\forall j} \tag{13}$$

4 The Aumann-Shapley value for distortion principles

Explicit calculations of the Aumann-Shapley value have been given in the bibliography for the cases that the cost functional is derived from the risk measure Expected Shortfall (Tasche, 2000a) and a risk measure used by the Securities and Exchange Commission (Denault, 2001). Here we provide an explicit formula for the Aumann-Shapley value in the case that the cost functional corresponds to a distortion principle, as defined in section 2.

To derive Aumann-Shapley prices, we will need to calculate derivatives of the type $\frac{\partial c(u)}{\partial u_i}$, where c is the cost function derived from the Choquet integral (3) as in (7). The calculation of Aumann-Shapley prices is performed by using the quantile representation of the Choquet integral and the analytical tool of 'quantile derivatives'.

The cost functional $c(u)$ equals the risk measure of the portfolio $Z^u = \sum_j u_j X_j$, thus it is given by the integral:

$$c(u) = \rho(Z^u) = \int Z^u dv, \tag{14}$$

where v is the set function defined by $v = g(\mathbb{P}_0)$, $g : [0,1] \mapsto [0,1]$ is a continuous, twice differentiable, strictly increasing and concave function, with $g(0) = 0$ and $g(1) = 1$.

Let S_{Z^u} be the decumulative distribution function of Z^u, $S_{Z^u}(z) = \mathbb{P}_0(Z^u > z)$. We define the distribution function $G_{Z^u} : \mathbb{R} \mapsto [0,1]$ of Z^u with respect to the monotone set function v as:

$$G_{Z^u}(z) = v(Z^u > z) = g(\mathbb{P}_0(Z^u > z)) = g \circ S_{Z^u}(z) \tag{15}$$

Assume, for simplicity, that $S_{Z^u}(z)$ is a continuous one-to-one map. Its inverse, $S^{-1}_{Z^u}(t)$ then exists and is also one-to-one. The inverse of the distribution function $G_{Z^u}(z)$ is defined in the obvious way:

$$G^{-1}_{Z^u}(t) = S^{-1}_{Z^u} \circ g^{-1}(t) \tag{16}$$

The Choquet integral admits a quantile representation (Denneberg, 1994a), which yields:

$$c(u) = \int Z^u dv = \int_0^1 G^{-1}_{Z^u}(t)dt = \int_0^1 S^{-1}_{Z^u} \circ g^{-1}(t)dt = \int_0^1 S^{-1}_{Z^u}(t)dg(t) \tag{17}$$

In the ensuing calculations we will need to calculate $\frac{\partial S^{-1}_{Z^u}(t)}{\partial u_i}$. Tasche (2000b) showed that, under some technical assumptions relating to the continuity of conditional densities (see Appendix), that we will assume to be satisfied, such derivatives exist and are given by:

$$\frac{\partial S^{-1}_{Z^u}(t)}{\partial u_i} = E[X_i | Z^u = S^{-1}_{Z^u}(t)] \tag{18}$$

Let the probability density function of Z^u be $f_{Z^u}(z)$ and the joint density of X_i and Z^u, $f_{X_i,Z^u}(x,z)$. Then we can write:

$$E[X_i | Z^u = z] = \int_0^\infty x \frac{f_{X_i,Z^u}(x,z)}{f_{Z^u}(z)} dx \tag{19}$$

We now proceed with a direct calculation of the Aumann-Shapley value. The i^{th} element of the Aumann-Shapley allocation $d^{AS} \in \mathbb{R}^n$, i.e. the amount of capital allocated to instrument X_i, is:

$$d_i^{AS} = \frac{\partial c(u)}{\partial u_i}|_{u_j=1\forall j}$$

We have:

$$\begin{aligned}
\frac{\partial c(u)}{\partial u_i} &\overset{(17)}{=} \frac{\partial}{\partial u_i}\int_0^1 S^{-1}_{Z^u}(t)dg(t) \\
&= \int_0^1 \frac{\partial}{\partial u_i} S^{-1}_{Z^u}(t)dg(t)
\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(18)}{=} \int_0^1 E[X_i|Z^u = S_{Z^u}^{-1}(t)]dg(t) \\
&= -\int_0^\infty E[X_i|Z^u = z]dg(S_{Z^u}(z)) \\
&\stackrel{(19)}{=} -\int_0^\infty \int_0^\infty x \frac{f_{X_i,Z^u}(x,z)}{f_{Z^u}(z)} g'(S_{Z^u}(z))dxdS_{Z^u}(z) \\
&= \int_0^\infty \int_0^\infty x f_{X_i,Z^u}(x,z) g'(S_{Z^u}(z))dxdz \\
&= E[X_i g'(S_{Z^u}(Z))]
\end{aligned}
$$

Let now $Z = \sum_j X_j$. The Aumann-Shapley allocation will be given by:

$$
d_i^{AS} = E[X_i g'(S_{Z^u}(Z^u))]|_{u_j=1\forall j} = E[X_i g'(S_Z(Z))] \tag{20}
$$

Thus, the capital allocated to the liability X_i, is calculated as its expected value under a change of probability measure:

$$
d_i^{AS} = E_{\mathbb{Q}}[X_i], \quad \frac{\partial \mathbb{Q}}{\partial \mathbb{P}_0} = g'(S_Z(Z)) \tag{21}
$$

The corresponding Radon-Nicodym derivative (or 'price density') thus is $D = g'(S_Z(Z))$. It is trivial that $E[D] = 1$.

Example: Consider the distortion function (e.g. Delbaen, 2000):

$$
g(t) = \frac{1 - e^{-ht}}{1 - e^{-h}} \tag{22}
$$

Then

$$
g'(t) = \frac{he^{-ht}}{1 - e^{-h}},
$$

and

$$
g'(S_Z(Z)) = \frac{he^{-he^{S_Z(Z)}}}{1 - e^{-h}} = \frac{e^{hF_Z(Z)}}{E[e^{hF_Z(Z)}]}
$$

Thus, the Aumann-Shapley allocation using this distortion function would be:

$$
d_i^{AS} = \frac{E[X_i e^{hF_Z(Z)}]}{E[e^{hF_Z(Z)}]} \tag{23}
$$

We note the formal similarity of the above equation to the Esscher premium principle obtained by Bühlmann (1980), using exponential utility functions in a competitive equilibrium framework. Our pricing (risk capital allocation) formula (23) is different to that of Bühlmann, in that the association of individual premium to collective risk is not induced by the aggregate risk itself, Z, but by its order statistics, as captured by the term $F_Z(Z)$. The fact that in our model individual premium depends only on the rank and not the scale of aggregate losses is a direct consequence of the scale invariance of preferences under Yaari's (1987) dual theory of choice, whence the risk measure used originates. In that sense our approach is complementary to the one of Bühlmann, as he considers competitive equilibrium pricing under (exponential) expected utility preferences, while we obtain cooperative cost sharing formulas under preferences expressed through (exponential) distortion functions.

Remark: Consider the Choquet integral in equation (2). According to a result due to Schmeidler (1986) (Proposition 10.1 in Denneberg (1994a)), for every Z such that $\rho(Z) < \infty$, there will exist a probability measure $\mathbb{Q}$ such that:

$$\mathbb{Q} \leq v \quad \text{and} \quad \int Z dv = \int Z d\mathbb{Q} \tag{24}$$

From the definition of the Choquet integral it immediately follows that:

$$\mathbb{Q} \leq v \Rightarrow E_{\mathbb{Q}}[X] = \int X d\mathbb{Q} \leq \int X dv = \rho(X), \ \forall X \tag{25}$$

e.g. the core property is ensured if the expectation under $\mathbb{Q}$ is used as an allocation mechanism. Note that this characterisation of the measure $\mathbb{Q}$ does not rely on any assumptions on the continuity of distributions. On the other hand, $\mathbb{Q}$ is not necessarily unique in this, more general, setting.

5 Application: valuation of pooled liabilities

5.1 Simple model of a pool

It was shown that the Aumann-Shapley value produces a cost allocation mechanism that takes the form of an expectation under a change of probability measure. This representation provides a formal link between cost allocation and pricing problems. Here we make this link explicit by an example. Consider n (re)insurers, exposed to some specific very high risks. In order to protect themselves against these risks, they form a pool. The pool takes on the individual insurers' liabilities, that is, it provides reinsurance for their individual risks. Each insurer makes cash contributions to the pool according to his risk profile, while any claims arising from the corresponding liabilities are paid by the pool. Similar pooling arrangements have emerged in the case of high risks that the insurance industry faces (for example terrorism risk), and are often set up by national governments. Another interpretation of this pooling arrangement is in the context of an insurance market with policyholder protection. Here the pooled risk would represent the insurers' liabilities in excess of the capital that each insurer holds. The pool would then pay the insurer's liabilities in case of insolvency. (We will not discuss the legal implications of such arrangements here, neither will we be concerned with a scenario where the pool itself defaults.) A third interpretation of the pooling arrangement is the case of a reinsurer determining the premiums for the different treaties that he writes. Holding a liability is less costly for a reinsurer with a widely diversified portfolio, than it is for the cedent. The savings that the reinsurer derives from his diversified portfolio can then be passed on to the buyers of the treaties in setting the premiums that he charges.

We assume that the amount of capital held by the pool is determined by a risk measure such as the one used so far in the paper. Our problem now is to calculate the contributions that the insurers have to make to the pool. On the one hand this is cost allocation problem that can be discussed in the terms of a (fuzzy) cooperative game. On the other, if we view the pool as a reinsurer, the cash contributions correspond to premia that the insurers have to pay for the liabilities ceded to the pool. The contributions represent the price that an individual insurer pays to the collective for the protection that he receives.

To make the example more specific, let $X_1, ..., X_n$ be positive random variables, corresponding to the liabilities that each insurer cedes to the pool. The pool is then exposed to risk $Z = \sum_j X_j$ and must hold capital:

$$\rho(Z) = \int_0^\infty g(S_Z(z))dz$$

As discussed in 2, this amount can be interpreted either as the minimum amount of economic capital required by a regulator, or as the pool's certainty equivalent for the risk that it takes on, assuming that its preferences are consistent with Yaari's (1987) theory of choice under risk. In a straightforward application of the result in section 4, the individual insurers have to contribute to the pool the amounts:

$$d_i = E_{\mathbb{Q}}[X_i] = E[X_i g'(S_Z(Z))]$$

5.2 Aumann-Shapley prices and equilibrium

A fundamental application of cooperative game theory has been to the study of transferable utility competitive equilibria in exchange economies. In such applications, the characteristic ('worth') function of the game represents the maximum utility that a coalition can achieve in the exchange and the (Aumann-Shapley) value is the vector of utilities that the market participants achieve at equilibrium. In the cases of a non-atomic space of players (Aumann-Shapley, 1974) and of fuzzy coalitions (Aubin, 1981) it has been shown that the core of the game coincides with the set of Walras equilibria of the economy.

Risk sharing has been examined by several authors (e.g. Borch (1962), Bühlmann (1980), Aase (2002)), in the context of competitive equilibrium in exchange economies. These models are concerned with economies where commodities corresponding to random cashflows are traded among a number of insurers and/or financial institutions. The cashflows can be insurance contracts of financial securities. Market prices and consumption levels are obtained by the maximisation of agents' expected utilities of terminal wealth and the imposition of a market-clearing condition. A more elaborate equilibrium model has been proposed by Taylor (1995), who, besides stochastic insurance liabilities and stock prices, considers consumer preferences, the presence of real assets and the relationship between insurance pricing and capitalisation.

A different approach was proposed by Mirman and Tauman (1982), who proved the existence of equilibrium in a production economy with one producer, m outputs and n consumers, when the price vector is determined by the Aumann-Shapley value, and the consumers have quasi-concave utility functions. Prices are dependent on demand, but no clearing condition applies. We will not use their result explicitly in this paper, but it will provide some motivation for generalising the example of the insurance pool discussed earlier.

If we consider a risk market, a direct analogue of Mirman and Tauman's (1982) economy would be a reinsurance monopoly, with premium charged according to Aumann-Shapley prices. Each output would correspond to a state of the world, each consumption bundle to a random variable, and the input to cash, used by the reinsurer in order to 'produce' (i.e. hold enough capital to satisfy regulators) the random cashflows sold to the primary insurers.

Perhaps more realistically, we consider a generalization of the example of the pool mentioned previously. Let again $X_1, \cdots, X_n$ be the liabilities of n primary insurance companies. However, the i^{th}, say, insurer does not necessarily purchase the precise random cashflow X_i, but buys another cashflow Y_i, again a positive random variable. The Y_i's are determined by the insurers' preferences and by the price system produced by the Aumann-Shapley value.

Let the i^{th} primary insurer's preferences be expressed by an increasing and concave utility function $u_i : \mathbb{R} \mapsto \mathbb{R}$. Each insurer carries initial capital b_i. The price that the insurer pays for the cashflow Y_i is $E[Y_i D]$, where D is the Radon-Nikodym derivative $D = g'(S_Z(Z))$, $S_Z(z) = \mathbb{P}_0(Z > z)$, $Z = \sum_j Y_j$. We see that the prices depend on the demand for reinsurance. The utility of the i^{th} insurer at the end of the time period under consideration is:

$$u_i\left(b_i - X_i + Y_i - E\left[DY_i\right]\right) \tag{26}$$

Equilibrium in this market will be reached when all insurers maximise their expected utility:

$$\max_{Y_i} Eu_i\left(b_i - X_i + Y_i - E\left[DY_i\right]\right) \quad \forall i \tag{27}$$

subject to budget constraints:

$$b_i \geq E[DY_i] \quad \forall i$$

Note that, due to a result by Bühlmann (1980), the problem (27) is equivalent to:

$$\frac{u_i'\left(b_i - X_i + Y_i - E\left[DY_i\right]\right)}{Eu_i'\left(b_i - X_i + Y_i - E\left[DY_i\right]\right)} = D \quad \text{and} \quad b_i \geq E[DY_i] \qquad \forall i \tag{28}$$

Suppose now that all utility functions are of exponential type, $u_i(t) = \frac{1}{a_i}\left(1 - e^{a_i t}\right)$. Then (28) simplifies to:

$$\frac{e^{a_i(X_i - Y_i)}}{Ee^{a_i(X_i - Y_i)}} = D \quad \text{and} \quad b_i \geq E[DY_i] \qquad \forall i \tag{29}$$

The above price density is the one underlying the Esscher premium principle, derived by Bühlmann (1980). In Bühlmann's model this price density, in conjunction with a market clearing condition, yields competitive equilibrium prices (an overview of such equilibrium models and their relationship to the Esscher principle is provided by Gerber and Pafumi (1998)). In our (cooperative) model no market clearing applies and the additional condition on market prices is their consistency with the Aumann-Shapley value, i.e. the requirement that $D = g'\left(S_Z\left(Z\right)\right)$.

If the probability space in the problem was discrete, then the existence of such an equilibrium would be guaranteed by Mirman and Tauman (1982). However, since we are concerned with a continuous probability space, this result does not necessarily hold. We will not attempt to solve this equilibrium problem analytically. Instead we turn our attention to a much more simple as well as more realistic version of the problem. Instead of allowing Y_i to be any random variable with price in the ith insurer's budget set, we postulate that it be an increasing function of the insurer's original risk X_i.

Furthermore, Y_i has a specific form corresponding to a stop-loss cover with retention K_i:[2]

$$Y_i = (X_i - K_i)_+ \tag{30}$$

The problem of calculating an equilibrium now becomes a question of determining the n-vector of K_i's by:

$$\begin{aligned} &\max_{K_i} Eu_i(b_i - \min\{X_i, K_i\} - E[(X_i - K_i)_+ g'(S_Z(Z))]), \qquad (31) \\ &\text{such that: } b_i \geq E[(X_i - K_i)_+ g'(S_Z(Z))] \ \forall i \end{aligned}$$

Note that the aggregate risk to the pool, Z, now is:

$$Z = \sum_j (X_j - K_j)_+ \tag{32}$$

There is of course no guarantee that an equilibrium exists in this modified model (or that if it exists it is unique). We further discuss the model in the next section through a numerical example.

Remark 1: The capital saving W_i that the ith insurer makes by pooling his liabilities that are in excess of K^i, equals the cost of holding this liability in the absence of cooperation (calculated by the distortion principle), minus the price that the insurer has to pay to the pool for the stop-loss cover:

$$W_i = \rho\left((X_i - K_i)_+\right) - E\left[D(X_i - K_i)_+\right] \tag{33}$$

[2] The form of this stop-loss loss cover has an interpretation in the case that we consider an insurance market with policyholder protection. We can interpret X_i as the ith insurer's liabilities, K_i as the capital that he holds and Y_i as the protection he receives from the pool, e.g. the amount that the pool pays for outstanding liabilities if i becomes insolvent, $X_i > K_i$. Then the insurer i can choose the level of K_i, subject to a regulatory minimum. Note that it is not straightforward that he would choose the lowest K_i acceptable to the regulator, as such a choice would yield a higher premium to the pool.

Remark 2: In the preceding discussion, we did not take into account the regulatory capital that the insurers have to hold for their retained liabilities. However, the inclusion of such considerations would not make a difference in the example, apart from leading to a more stringent budget constraint. If the ith insurer did not receive any cover from the pool, he would have to hold regulatory capital $\rho(X_i)$. When he receives a stop-loss cover with retention K_i, he has to hold $\rho\left(\min\{X_i, K_i\}\right)$. In this case the savings from cooperation are:

$$W_i = \rho(X_i) - \left(\rho\left(\min\{X_i, K_i\}\right) + E\left[D(X_i - K_i)_+\right]\right)$$

But since the random variables $\min\{X_i, K_i\}$ and $(X_i - K_i)_+$ are comonotonic, we have:

$$\rho\left(\min\{X_i, K_i\}\right) + \rho\left((X_i - K_i)_+\right) = \rho(X_i)$$

and thus:

$$W_i = \rho\left((X_i - K_i)_+\right) - E\left[D(X_i - K_i)_+\right],$$

which is the same as (33) above. The budget constraint would however change to:

$$b_i \geq E[(X_i - K_i)_+ D] + \rho\left(\min\{X_i, K_i\}\right).$$

In the sequel we will assume that b_i is net of capital requirements corresponding to the retained risk. Since the exponential utility function's risk aversion is invariant in absolute wealth, such a change will have no affect on the outcome.

5.3 Numerical example

We provide a simple numerical example for the model presented in 5.2. We consider 3 insurers, each holding a lognormally distributed liability, X_i,

$\ln(X) \sim N(\mu, \Sigma)$, $\mu = [0.9\ 0.9\ 0.9]^T$,

$$\Sigma = 0.3^2 \cdot \begin{pmatrix} 1 & 0.30 & 0.60 \\ 0.30 & 1 & 0.80 \\ 0.60 & 0.80 & 1 \end{pmatrix}$$

Each insurer also holds cash $b = 1$.

The insurers' preferences are, as previously, characterised by exponential utility functions, and the risk aversion parameter is $a = 3.5$ for all insurers. The risk measure used is a distortion principle with distortion function (22) and parameter $h = 10$ (fig. 1). As discussed earlier, the economy will be at equilibrium when the K_i's are such that conditions (31) hold. We assume now a market mechanism by which the retentions K_i are determined. The K_i's are given some reasonable initial values, e.g $K_i = 3,\ i = 1, 2, 3$ (these initial values do, as expected, affect the final outcome, though not radically so). Every insurer chooses his retention K_i by solving the following optimisation problem, while assuming that the other insurers' retentions $K_j,\ j \neq i$ will remain constant:

$$\begin{array}{c} \max_{K_i} Eu_i\left(b_i - \min\{X_i, K_i\} - E\left[(X_i - K_i)_+ g'\left(S_Z\left(\sum_j (X_j - K_j)_+\right)\right)\right]\right) \\ \text{such that } E\left[(X_i - K_i)_+ g'\left(S_Z\left(\sum_j (X_j - K_j)_+\right)\right)\right] \leq b_i \end{array} \tag{34}$$

When all insurers solve their optimisation problems, they announce their preferred values of K_i, the valuation measure is thus updated, and the process is repeated. Equilibrium is reached if and when the values of the K_i's stabilise. Of all vectors of K_i's that would correspond to an equilibrium we are interested only in those that could actually emerge through the above negotiation process.

We simulate the process by solving the problems (34) numerically, with the parameters given above. Optima are calculated via a simple search method and objective function evaluations take place on a simulated sample of the random vector $[X_1\ X_2\ X_3]^T$. We observe that equilibrium is indeed reached, after just few iterations (fig. 2). It can also be seen that although the different insurers share the same risk profile and preferences, their final choices of K_i's are different:

$$\begin{aligned} K_1 &= 3.34 \\ K_2 &= 3.47 \\ K_3 &= 3.50 \end{aligned} \tag{35}$$

The differences can be explained by the different degrees to which individual insurers' ceded liabilities are correlated to the aggregate, leading to different prices for their risks. Consider the pairwise correlations:

$$\begin{aligned} r\left((X_1-K_1)_+, \textstyle\sum_j (X_j-K_j)_+\right) &= 0.71 \\ r\left((X_2-K_2)_+, \textstyle\sum_j (X_j-K_j)_+\right) &= 0.76 \\ r\left((X_3-K_3)_+, \textstyle\sum_j (X_j-K_j)_+\right) &= 0.88 \end{aligned} \tag{36}$$

High dependence between individual and collective risk would result into a dearer price for the risk, thus suppressing demand for reinsurance. This explains the high retention (low coverage) that the 3rd insurer chooses, as the correlation of his liabilities to the aggregate is highest.

Under this final choice of K_i, the individual risk of each insurer's liability ceded to the pool (calculated by the distortion principle) is:

$$\begin{aligned} \rho((X_1-K_1)_+) &= 0.67 \\ \rho((X_2-K_2)_+) &= 0.57 \\ \rho((X_3-K_3)_+) &= 0.56 \end{aligned} \tag{37}$$

The above quantities correspond to the cost of holding the liabilities $(X_i - K_i)_+$, in the absence of cooperation. The premium allocated to each insurer is:

$$
\begin{aligned}
d_1 &= E[(X_1 - K_1)_+ g'(S_Z(Z)] = 0.55 \\
d_2 &= E[(X_2 - K_2)_+ g'(S_Z(Z)] = 0.47 \\
d_3 &= E[(X_3 - K_3)_+ g'(S_Z(Z)] = 0.52 \\
\textstyle\sum_j d_j &= E[Zg'(S_Z(Z)] = \rho(Z) = 1.54
\end{aligned}
\tag{38}
$$

The development of individual risks and allocated capital for the different insurers up to equilibrium is shown in figure 3. The difference between the two lines corresponds to the savings that each insurer makes by cooperating with the others. It is seen that the highest saving is made by the first insurer, whose liability is less correlated to the aggregate and whose participation in the pool is highest, while the lowest saving is made by the third insurer, whose liability has the highest correlation to the aggregate and whose participation in the pool is lowest.

6 Summary and future research

We discuss the problem of allocating capital requirements for a portfolio of stochastic liabilities to the different instruments (and sub-portfolios) that the portfolio consists of. The allocation functional used is the Aumann-Shapley value from cooperative game theory, which has been proposed as a cost allocation mechanism by several authors (Mirman and Tauman (1982), Billera and Heath (1982), Denault (2001)). In our application, we used as a risk measure the distortion principle, which is an actuarial premium calculation principle satisfying the coherence axioms of Artzner et al. (1999). The resulting allocation of costs is the unique element of the core of the associated cooperative game, in the sense that it is the unique allocation that does not produce a disincentive for cooperation to (the holder of) any instrument or sub-portfolio.

For our choice of risk measure we obtained an explicit formula for the Aumann-Shapley value. Capital requirements are calculated for each sub-portfolio as its expected value under a change of probability measure. From the resulting formula it can be seen that the amount of capital allocated to each sub-portfolio is closely related to the dependence structure between itself and the aggregate portfolio. Tail dependence between individual and aggregate risk tends to increase capital requirements, while in the extreme case of comonotonicity any savings from pooling vanish.

In the final section we utilise the representation of allocated capital as an expectation under an adjusted probability measure, in order to draw a parallel between capital allocation and insurance pricing problems. We consider the case of a pool, which reinsures the excess liabilities of a number of insurers, and charges risk premium according to Aumann-Shapley prices. Such a pooling arrangement is conceivable for protection against high industry-wide risks (e.g. terrorism) or in the case of an insurance market with policyholder protection. Motivated by an equilibrium model of Mirman and Tauman (1982) we generalise the model of the pool by letting the insurers decide themselves about the level of coverage that they receive from the pool, via expected utility maximisation. We finally present a numerical example, which illustrates the ideas previously discussed.

The main analytical result obtained in the paper, the formula for the Aumann-Shapley value, was derived under the assumption that conditional probability densities are continuous. Further research is required for the discontinuous case, in which the cost function will not be differentiable with respect to portfolio weights. The core of the related cooperative game will be the subdifferential of the cost function (Aubin, 1981) and will contain multiple allocations. It would be interesting to know whether the solution that we obtained under the continuity assumption will be one of them.

Another topic for future research is the generalisation of the methodology developed here to a dynamic setting. Prerequisite for an intertemporal allocation method is the dynamic generalisation of the risk measure used. Updating rules for submodular set functions and distorted probabilities have been studied by Denneberg (1994b), Young (1998), Wang and Young (1998), while attempts to generalise conditional expectation to the non-additive case can be found in Denneberg (2000) and Lehrer (1996).

A Quantile derivatives

This section follows Tasche (2000b). Let X be a real valued random variable. For $a \in (0,1)$ the a-quantile of X, $Q_a(X)$ is defined as:

$$Q_a(X) = \inf\{x \in \mathbb{R} | \mathbb{P}_0(X \leq x) \geq a\}$$

Now let

$$Z^u = \sum_j u_j X_j$$

be a portfolio consisting of random liabilities X_j, $j = 1, 2, ..., n$. We are interested in derivatives of the a-quantile of Z^u, with respect to the portfolio weights u_j, i.e. in expressions of the form:

$$\frac{\partial Q_a(Z^u)}{\partial u_i}$$

Such 'quantile derivatives' exist, subject to a set of technical assumptions. Let $n \geq 2$ and $(X_1, ..., X_n)$ be an $\mathbb{R}^n$-valued vector with a conditional density ϕ of X_1 given $(X_2, ..., X_n)$. ϕ satisfies the assumptions in an open set $U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$ if:

(i) For fixed $x_2, ..., x_n$, the function $t \mapsto \phi(t, x_2, ..., x_n)$ is continuous in t.

(ii) The mapping

$$(t, u) \mapsto E[\phi(u_1^{-1}(t - \sum_{j=2}^{n} u_j X_j), X_2, ..., X_n)],$$

$$\mathbb{R} \times U \mapsto [0, \infty)$$

is finite-valued and continuous.

(iii) For each $i = 2, ..., n$ the mapping

$$(t, u) \mapsto E[X_i \phi(u_1^{-1}(t - \sum_{j=2}^{n} u_j X_j), X_2, ..., X_n)],$$

$$\mathbb{R} \times U \mapsto \mathbb{R}$$

is finite-valued and continuous.

If ϕ satisfies the above assumptions in some open set $U \subset \mathbb{R}$ $\{0\} \times \mathbb{R}^{d-1}$, the quantile derivative exists and is given by:

$$\frac{\partial Q_a(Z^u)}{\partial u_i} = E[X_i | \sum_j u_j X_j = Q_a(Z^u)]$$

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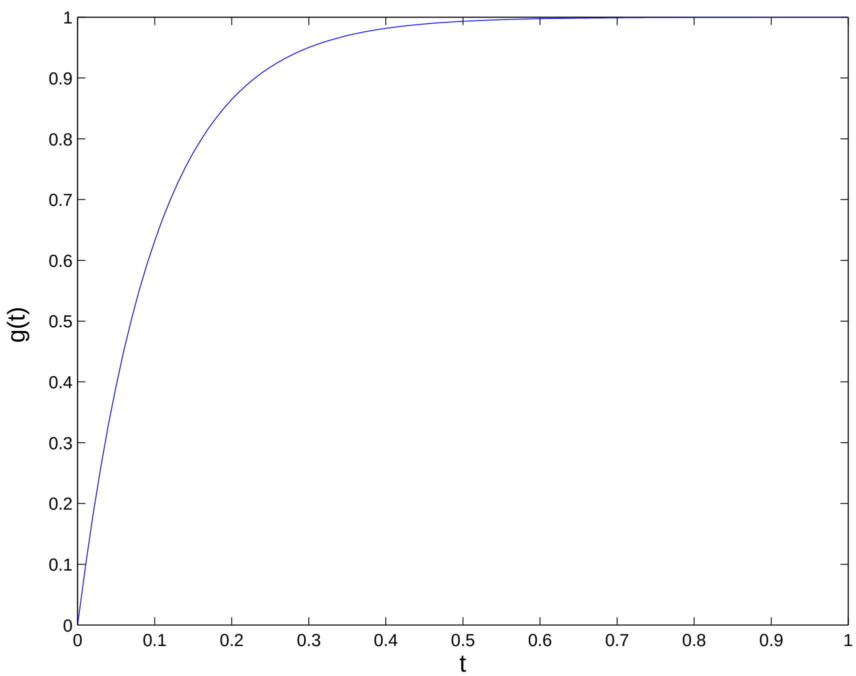

Figure 1: Distortion function $g(t) = \frac{1-e^{-ht}}{1-e^{-h}}$, h=10.

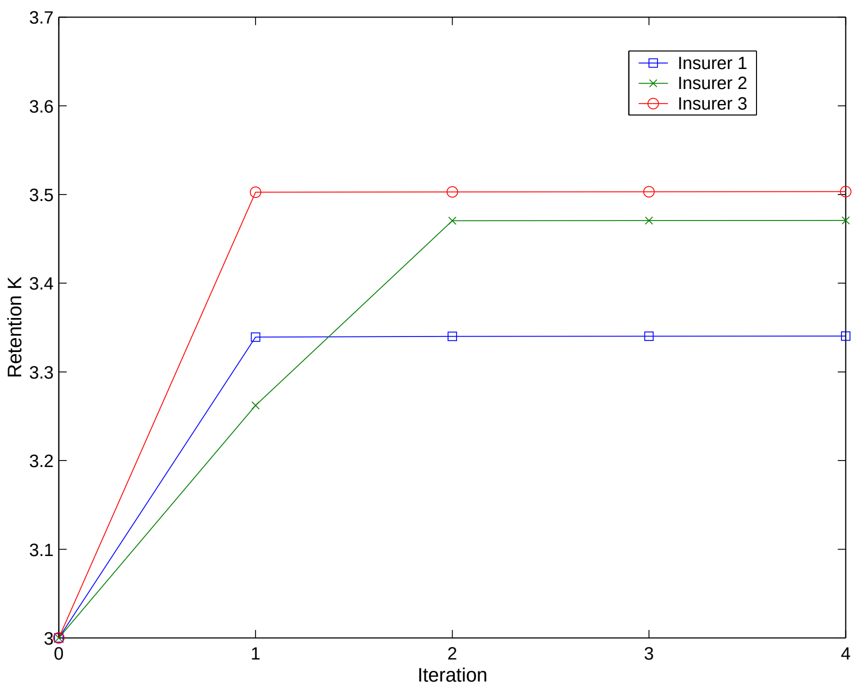

Figure 2: Development of retentions K until equilibrium is reached.

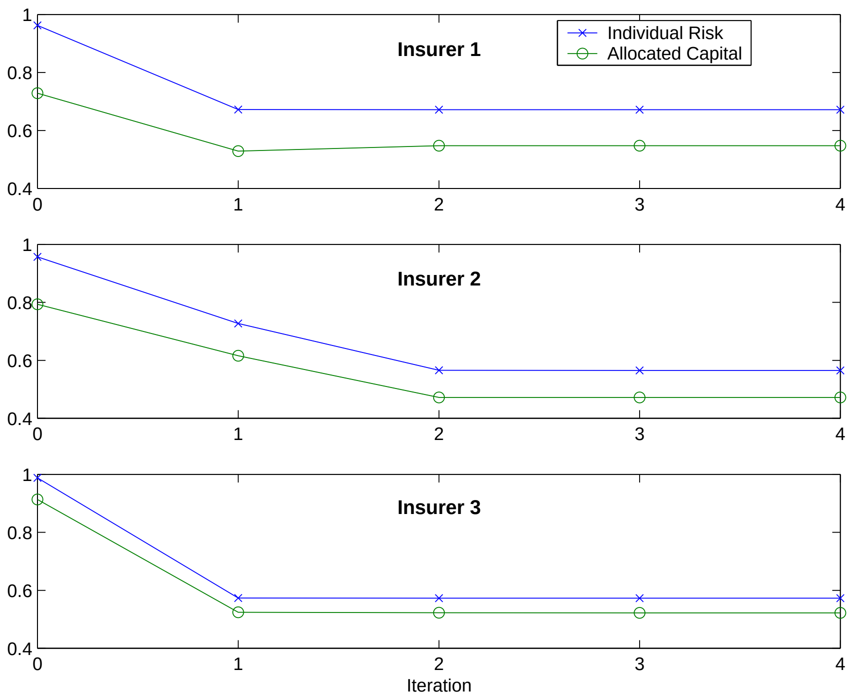

Figure 3: Risk of individual insurers' ceded liabilities and allocated capital. The difference between the lines represents savings from cooperation.