Risk measurement in the presence of background risk

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Abstract

A distortion-type risk measure is constructed, which evaluates the risk of any uncertain position in the context of a portfolio that contains that position and a fixed background risk. The risk measure can also be used to assess the performance of individual risks within a portfolio, allowing for the portfolio's re-balancing, an area where standard capital allocation methods fail. It is shown that the properties of the risk measure depart from those of coherent distortion measures. In particular, it is shown that the presence of background risk makes risk measurement sensitive to the scale and aggregation of risk. The case of risks following elliptical distributions is examined in more detail and precise characterisations of the risk measure's aggregation properties are obtained.

Keywords: Risk measures, Background risk, Capital allocation, Portfolio management, Elliptical distributions.

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1 Introduction

Risk measures are used in order to determine the *risk capital* that the holder of a portfolio of assets and liabilities has to hold, that is, to invest with low risk. Risk measures are closely related to insurance premium calculation principles, which are studied extensively by Goovaerts et al. (1984). Recent approaches to risk measures include the one by Artzner et al. (1999) who introduced a set of axioms on risk measures and introduced the term *coherent measures of risk* for risk measures that satisfy those axioms. Distortion risk measures which satisfy these axioms have been introduced in an insurance pricing context by Denneberg (1990) and Wang (1996), and in a finance setting by Acerbi (2002). These are often called *distortion risk measures* and are calculated as the expected loss of a portfolio, under an non-linear transformation of its cumulative probability distribution.

Given the aggregate risk capital of a portfolio, a separate problem that emerges is the allocation of the total amount of risk capital to the instruments (or sub-portfolios) that the portfolio consists of. The term capital allocation does generally not imply that actual amounts of capital are moved between portfolios; it rather signifies a notional exercise used primarily to evaluate the performance of risks within a portfolio. Capital allocation methods drawing on cooperative game theory were proposed by Denault (2001), based on the premise that no disincentives for the pooling of portfolios should be created by the allocation.

When a distortion risk measure is used for determining the aggregate capital corresponding to the portfolio, explicit capital allocation formulas have been obtained by Tsanakas and Barnett (2003) and Tsanakas (2004). The capital allocation mechanism obtained in the above papers can be viewed as an internal system of prices defined on the set of risks that form the aggregate portfolio. The capital allocated to a particular risk, or its price, is influenced on the dependency between that risk and the aggregate

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portfolio.

If the aggregate portfolio is fixed this approach presents a consistent method for the allocation of risk capital. However, if the portfolio is not fixed, and capital allocation is viewed as a decision tool for its re-balancing, then an inconsistency arises. Allocated capital reflects the dependence between a risk and the portfolio which contains that risk. Hence when the weight of that risk in the portfolio changes, the portfolio as a whole is affected. Therefore, the system of internal prices derived for the static portfolio changes and is no more valid for assessing the performance of each individual risk in the re-balanced portfolio.

A related situation arises when the portfolio holder is also exposed to an element of *background risk*. This means that besides the specific portfolio obtained in the financial and insurance markets, the holder is also exposed to a risk that he cannot (or will not) trade, control or mitigate. Examples of background risk are the risk to human capital for a firm, unhedgeable portfolios in an incomplete financial market and insurance risk for which no reinsurance is typically available e.g. infinite layers. Hence, the risk measurement of any other risk in the portfolio has always to be evaluated with reference to the background risk that the holder is exposed to.

While the effect of background risk on risk measurement is not well developed in the literature, its effect on risk aversion and asset allocation has been extensively studied in the economics literature. For example, Heaton and Lucas (2000) deal with portfolio choice in the presence of background risk, while Gollier and Pratt (1996) seek to characterise utility functions such that the exposure to an independent background risk increases risk aversion with respect to any other independent risk. In this paper a different approach is taken. It is proposed that the risk corresponding to a particular position or instrument is quantified via its risk contribution to a portfolio that contains itself and the background risk. That risk contribution can be determined using standard capital allocation methods. The risk contribution's sensitivity to the exposure to background risk can then be used to characterise the change in preferences that background risk induces. No assumptions on independence of risks are made.

Following that reasoning, a new distortion risk measure is defined, which provides a measure of risk for any position X, with reference to a fixed risk Y that is in the same portfolio as X. This can be interpreted either as capturing the dynamics of capital allocation under changing portfolio weights or as quantifying risk X in the presence of a background risk Y. This is akin to the portfolio holder's having a system of reference were the origin has been moved from 0 to Y. This 'new origin' is now of course a random quantity and the risk measure of X relative to Y proposed here reflects this change.

By studying the properties of that risk measure, it is possible to capture the impact of background risk on preferences, given that these preferences are captured by a distortion risk measure. It is shown that background risk increases agents' sensitivity to the aggregation of risk; while a coherent distortion risk measure is positive homogenous and subadditive, background risk potentially induces superlinearity and superadditivity. This is a notable departure from the properties of coherent risk measures, which have been criticised by some authors (e.g. Dhaene et al., 2003) for their insensitivity to the aggregation of risk. Increased sensitivity to risk aggregation in the context of the paper makes practical and intuitive sense; in an illiquid environment, such as an insurance market, a risk taker will generally require a higher price at the margin for risk that he is already highly exposed to.

In the case of general probability distributions and dependence structures, the aggregation properties of risk measurement in the presence of background risk are difficult to characterise exactly. Therefore, the special case of joint-elliptically distributed risks is examined. Elliptical distributions have become an important tool in financial risk management, because they combine heavy tails and tail-dependence with tractable aggregation properties (Embrechts et al, 2002). It is shown that in the presence of background risk, the risk measure introduced here satisfies a super-linearity property which manifests sensitivity to the exposure to 'large' risks. Moreover, it is shown that the risk measure is superadditive for comonotonic risks, which means that the pooling of perfectly correlated positions is penalised. Finally it is shown that, under some conditions (ensuring that the background risk does not operate as a hedge) the risk measure penalises increases in the variability of liabilities, as well as increases in the correlation between instruments within a portfolio.

The structure of the paper is as follows. In Section 2 capital allocation with distortion risk measures is introduced. In Section 3 distortion risk measures with reference to a background risk are introduced and their basic properties discussed. Section 4 discusses elliptical distributions and proceeds with the characterisation of the risk measure's aggregation properties in an elliptical environment. Conclusions are summarised in Section 5.

2 Capital allocation with distortion risk measures

Consider a set of random variables $\mathcal{X}$ representing insurance and financial risks, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $X \in \mathcal{X}$ assumes a positive value it is considered a loss. A *risk measure* ρ is defined as a real-valued functional on the set of risks, $\rho : \mathcal{X} \mapsto \mathbb{R}$. The quantity $\rho(X)$ represents the amount of capital that the holder of risk X has to safely invest, in order to satisfy a regulator. Risk measures have been extensively studied in actuarial science in the guise of insurance pricing functionals, see Goovaerts et al. (1984). A more recent influential approach in the financial field has been the axiomatic definition of 'coherent measures of risk' by Artzner et al. (1999).

The following *distortion risk measure* has been proposed by Denneberg (1990) and Wang (1996):

$$\rho(X) = \int_{-\infty}^{0} \left(g\left(\bar{F}_X(x)\right) - 1\right) dx + \int_{0}^{\infty} g\left(\bar{F}_X(x)\right) dx, \tag{1}$$

where $\bar{F}_X(x) = \mathbb{P}(X > x)$ is the decumulative probability distribution of X under the real-world probability measure $\mathbb{P}$, while $g : [0,1] \mapsto [0,1]$ is an increasing and concave *distortion function* such that $g(0) = 0$ and $g(1) = 1$. The distortion risk measure can be interpreted as an expected loss under a non-linear transformation of the probability distribution; observe that if $g(t) = t$, then $\rho(X) = E[X]$. The distortion risk measure (1) satisfies the coherence axioms of Artzner et al. (1999), while allowing an economic interpretation as a certainty equivalent under Yaari's (1987) dual theory of choice under risk. It is noted that distortion risk measures are essentially the same the 'spectral risk measures' of Acerbi (2002).

Consider the case where g is differentiable on $[0,1]$ with bounded first derivative, an assumption that is made throughout the paper. Moreover we assume that $\bar{F}_X$ is continuous. Then integration by parts yields the following re-writing of the distortion risk measure

$$\rho(X) = E\left[X g'\left(\bar{F}_X(X)\right)\right]. \tag{2}$$

It is noted that (2) can be easily generalised for the case of a finite number of discontinuities of $\bar{F}_X$, but this is not pursued here. The factor $g'\left(\bar{F}_X(X)\right)$ in the expectation effects a re-weighting of the probability distribution of X, placing a higher weight on adverse outcomes of X (adverse in the sense of having a high rank in the set of possible outcomes).

Suppose an economic agent (insurer, bank) holds a portfolio $Z \in \mathcal{X}$, consisting of a number of instruments $X_1, X_2, \dots, X_n \in \mathcal{X}$ such that

$$Z = X_1 + X_2 + \dots + X_n. \tag{3}$$

Then the aggregate capital that the agent has to hold is $\rho(Z) = E[Z \cdot$

$g'(\bar{F}_Z(Z))]$. The capital allocation problem consists or determining capital amounts $K_i,\ i = 1, 2, \ldots, n$, corresponding to each instrument $X_i,\ i = 1, 2, \ldots, n$ respectively, such that the allocated amounts add up to the aggregate required capital

$$\sum_j K_j = \rho(Z). \tag{4}$$

To determine the capital allocation additional criteria to (4) are of course needed. A criterion originating in cooperative game theory (Aumann and Shapley (1974), Denault (2002)) is that the allocation should produce no incentive for any instrument to leave the portfolio. This is formalised by requiring that the capital allocated to each instrument is lower than its risk measure

$$K_i \leq \rho(X_i),\ i = 1, 2, \ldots, n \tag{5}$$

We note that this requirement is consistent with the subadditivity property of coherent measures of risk, which postulates that $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$, that is, there is always a benefit in pooling portfolios. If subadditivity is accepted as a valid premise for risk management, then ensuring by (5) that portfolios are not fragmented is also desirable.

A capital allocation method for distortion risk measures, which satisfies (4) and (5) has been developed by Tsanakas and Barnett (2003) and Tsanakas (2004), who obtained the following allocation formula

$$K_i = E\left[X_i g'\left(\bar{F}_Z(Z)\right)\right] \tag{6}$$

If we consider the linear functional

$$\pi(X) = E\left[X g'\left(\bar{F}_Z(Z)\right)\right],\ X \in \mathcal{X}, \tag{7}$$

it obviously is $K_i = \pi(X_i)$. The allocated capital calculated by (6) equals the expected loss of the instrument X_i, subject to a re-weighting of its probability distribution (change of probability measure) by $g'\left(\bar{F}_Z(Z)\right)$. Thus outcomes of X_i which occur when the aggregate portfolio Z performs badly are given more weight. In this way the dependence between X_i and Z is reflected in the capital allocation, as discussed by Tsanakas (2004).

3 Risk measurement and background risk

3.1 Definition

This representation (7) gives a means of evaluating the risk of every instrument X in relation to the portfolio Z. By viewing π as a linear functional one implicitly assumes that the random variable Z, that is, the composition of the aggregate portfolio is fixed. However, this assumption does not take into account the fact that when we seek to evaluate the risk of an instrument X relative to a portfolio Z, this instrument actually forms part of the portfolio itself. In this context, when one modifies X, Z is being modified as well. This is not a problem if the issue at hand is to measure the performance of the constituent parts of a fixed portfolio. However, if the capital allocation method is used to evaluate the performance of different instruments relative to a portfolio, in order to take investment decisions, then the effect on the total portfolio of changing investment in any instrument X has to be taken explicitly into account.

Alternatively, consider the situation that Y is a risk that its holder cannot (or will not) insure or replicate in the insurance and financial markets. Thus the holder of Y cannot receive protection against this risk – so to speak, he is stuck with it. We then call Y a *background risk* for the particular agent holding it. From the point of view of an agent exposed to background risk Y, each risk X must be evaluated in relation to Y as it can only be contemplated in a portfolio that already contains Y. Hence, a risk measure would have to be defined which depends on a fixed Y but varies in X. In the context of the previous discussion, this means that the aggregate portfolio $Z = X + Y$, with respect to which the risk of X is evaluated, will again not be fixed.

The above arguments motivate the following definition:

Definition 1. *The distortion measure $\rho(X;Y)$ of risk X with respect to background risk Y is defined by:*

$$\rho(X;Y) = E\left[Xg'\left(\bar{F}_{X+Y}(X+Y)\right)\right], \tag{8}$$

where $g : [0,1] \mapsto [0,1]$ with $g(0) = 0$ and $g(1) = 1$ is increasing, concave, and differentiable with bounded first derivative.

In the above definition the probability weighting $g'\left(\bar{F}_{X+Y}(X+Y)\right)$ explicitly depends on X and therefore $\rho(X;Y)$, unlike $\pi(X)$, is not a linear functional in X. We note that the risk measure $\rho(X;Y)$ can also be expressed as the sensitivity of portfolio $X+Y$ to changes in X. Let $r(a) = \rho(aX+Y)$. Then

$$\rho(X;Y) = \left.\frac{dr(a)}{da}\right|_{a=1} \tag{9}$$

Remark: The argumentation presented here also applies to other risk measures that produce capital allocation mechanisms which are represented as linear functionals. For example, the *exponential tilting* capital allocation method proposed by Wang (2002) is defined by

$$K_i = \frac{E[X_i \exp(aZ)]}{E[\exp(aZ)]}, \quad a > 0. \tag{10}$$

Similar arguments as above could be used to define a risk measure by

$$\rho(X;Y) = \frac{E[X \exp(a(X+Y))]}{E[\exp(a(X+Y))]}, \quad a > 0. \tag{11}$$

The study of such risk measures is outside the scope of this investigation.

3.2 Elements of stochastic orders

Before proceeding to the discussion of the properties of the risk measure $\rho(X;Y)$, some results on the stochastic ordering of two-dimensional random vectors are presented here, drawing from Dhaene and Goovaerts (1996) and Müller and Stoyan (2002).

The *stop-loss order* on elements of the set of risks $\mathcal{X}$ is a natural way of comparing the riskiness of probability distributions, by comparing the stop-loss premiums of risks for any given retention:

Definition 2. *For random variables $X, Y \in \mathcal{X}$, we say that X is smaller than Y in stop-loss order and write $X \leq_{sl} Y$, whenever*

$$E[(X-d)_+] \leq E[(Y-d)_+], \ \forall d \in (-\infty, \infty) \tag{12}$$

Let $\stackrel{d}{=}$ signify equality in distribution. Consider risks $X_1 \stackrel{d}{=} Y_1$ and $X_2 \stackrel{d}{=} Y_2$ with probability distributions F_1, F_2 respectively. The random vectors (X_1, X_2) and (Y_1, Y_2) are then different only in the way that their elements depend on each other. The *Frechet Space* $\mathcal{R}_2(F_1, F_2)$ is defined as the space of two-dimensional random vectors with fixed marginals F_1 and F_2. Elements of $\mathcal{R}_2(F_1, F_2)$ can be compared in terms of their dependence structure via the partial *concordance order*:

Definition 3. *Consider the random vectors $(X_1, X_2), (Y_1, Y_2) \in \mathcal{R}_2(F_1, F_2)$, with joint cumulative distribution functions $F_{\mathbf{X}}, F_{\mathbf{Y}}$ respectively. We say that (X_1, X_2) is smaller than (Y_1, Y_2) in concordance order and write $(X_1, X_2) \leq_c (Y_1, Y_2)$, whenever either of the following equivalent conditions holds:*

i) $F_{\mathbf{X}}(x_1, x_2) \leq F_{\mathbf{Y}}(x_1, x_2)$ *for all* x_1, x_2.

ii) $Cov(h_1(X_1), h_2(X_2)) \leq Cov(h_1(Y_1), h_2(Y_2))$ *for all increasing functions* h_1, h_2 *such that the covariances exist.*

3.3 Properties of the risk measure $\rho(\cdot; Y)$

It is apparent that the properties of the functional $\rho(\cdot; Y)$ are going to be different to the ones of the distortion risk measure $\rho(\cdot)$. One could actually regard $\rho(\cdot; Y)$ as a separate class of risk measures 'parameterised' by the random variable Y. Then, it is of interest to study the properties of the risk measure $\rho(\cdot;Y)$ and the ways in which they differ from those of the standard distortion risk measure. The change in the risk measures' properties will represent the impact of background risk on risk preferences.

Some elementary properties are given by the following result.

Proposition 1. *The distortion risk measure in the presence of background risk defined by (8), where g is differentiable with bounded first derivative and the random variables referred to have continuous cumulative distributions function, satisfies the following properties:*

i)

$$\rho(X;Y) + \rho(Y;X) = \rho(X+Y) \tag{13}$$

ii) If X and Y are comonotonic

$$\rho(X;Y) = \rho(X) \tag{14}$$

iii) If X_1, X_2 and Y are comonotonic

$$\rho(X_1 + X_2;Y) = \rho(X_1;Y) + \rho(X_2;Y) \tag{15}$$

iv)

$$\rho(X;Y) \geq 0 \iff Cov\left(X, g'\left(\bar{F}_{X+Y}(X+Y)\right)\right) \geq -E[X] \tag{16}$$

v)

$$\rho(X;Y) \leq \rho(X) \tag{17}$$

vi) If $X_1 \overset{d}{=} X_2$ and $(X_1,Y) \leq_c (X_2,Y)$, then

$$\rho(X_1;Y) \leq \rho(X_2;Y) \tag{18}$$

Proof. Parts i) and iv) are straightforward. For ii) note that for a comonotonic pair (X,Y), it is known that the sum $X+Y$ is also comonotonic to

X, which implies that we have that $\bar{F}_X(X) = \bar{F}_{X+Y}(X+Y)$(Dheane et al, 2002). This yields

$$\rho(X;Y) = E[Xg'(\bar{F}_{X+Y}(X+Y))] = E[Xg'(\bar{F}_X(X))] = \rho(X). \tag{19}$$

Part iii) is a direct result of ii). Proof of v) and vi) follows directly from Proposition 5 in Tsanakas (2004). $\square$

Part i) of the result above just states that the risk of portfolio $X+Y$ can be broken down to the risk assessments of X relative to Y and of Y relative to X. Part ii) shows that the risk of an instrument X relative to a background risk Y that is comonotonic to, is equal to the risk of X with no background risk. This is because pooling comonotonic portfolios produces no diversification. Part iii) is a special case of ii) concerning the sum of risks comonotonic to the portfolio Y.

Part iv) gives a condition for the positivity of the risk measure. Roughly speaking, $\rho(X;Y)$ is positive given that X has a high enough positive dependence on (an increasing function of) the aggregate $X+Y$. This means that $\rho(X;Y)$ can be negative, even if $X \geq 0$. This could happen if X is highly negatively correlated to $X+Y$, because then the presence of X actually reduces the risk, as it reduces the variability of $X+Y$. This situation can occur in practice, especially in the trading of derivative portfolios, where often both increasing and decreasing contracts on underlying securities are sold.

Part v) reflects the origin of the risk measure in cooperative game theory, as the risk of X relative to Y is always dominated by the stand-alone risk of X. This means that the background risk Y always produces sufficient diversification benefit so as to reduce the risk of X. This is consistent with the subadditivity property of distortion risk measures; hence potential criticisms of subadditive risk measures, such as the ones expressed in Dhaene et al. (2003) also extend to $\rho(\cdot;Y)$. Finally, vi) shows that if the pair (X_2,Y)

is more concordant than the pair (X_1, Y), then the risk of X_2 relative to Y is higher that the one of X_1. The reason is that a high correlation to the portfolio Y implies less diversification.

Proposition 1 does not address the aggregation properties of the portfolio-relative risk measures. For coherent distortion risk measures, one has that the risk measure is insensitive to the size of portfolios, that is the positive homogeneity property $\rho(aX) = a\rho(X), a \in \mathbb{R}_+$. Furthermore, by the subadditivity property $\rho(X+Y) \leq \rho(X) + \rho(Y)$ it is implied that the pooling of portfolios, regardless of their dependence structure always produces diversification benefits. These properties do not necessarily carry over to the case of portfolio-relative distortion risk measures.

This divergence in aggregation behaviour can be observed heuristically. Observe that for a positive homogenous risk measure the quantity $\frac{\rho(aX)}{a}$ is constant. For the portfolio-relative risk measures, it is:

$$\frac{\rho(aX;Y)}{a} = E[X] + Cov\left(X, g'\left(\bar{F}_{aX+Y}(aX+Y)\right)\right) \tag{20}$$

The above expression obviously varies in a, therefore $\rho(aX;Y)$ is not positively homogenous. Furthermore, as the weight a becomes higher, portfolio $aX+Y$ tends to aX. On the other hand varying a does not change the marginal distribution of $g'(\bar{F}_{aX+Y}(aX+Y))$. Hence one would expect the covariance between X and $g'(\bar{F}_{aX+Y}(aX+Y))$, which is an increasing function of $aX+Y$, to grow. This means that distortion risk measures with respect to background risk will generally be super-linear, in the sense that increases in the scale of X will tend to also increase $\frac{\rho(aX;Y)}{a}$.

Regarding the issue of pooling portfolios, we can write

$$\begin{aligned} \rho(X_1+X_2;Y) = \quad & E\left[X_1 g'\left(\bar{F}_{X_1+X_2+Y}(X_1+X_2+Y)\right)\right] \\ & +E\left[X_2 g'\left(\bar{F}_{X_1+X_2+Y}(X_1+X_2+Y)\right)\right] \end{aligned} \tag{21}$$

Now $\rho(X_1+X_2;Y)$ would be subadditive if it held true that

$$E\left[X_1 g'\left(\bar{F}_{X_1+X_2+Y}(X_1+X_2+Y)\right)\right] \leq E\left[X_1 g'\left(\bar{F}_{X_1+Y}(X_1+Y)\right)\right]$$

and

$$E\left[X_2 g'\left(\bar{F}_{X_1+X_2+Y}(X_1+X_2+Y)\right)\right] \leq E\left[X_2 g'\left(\bar{F}_{X_2+Y}(X_2+Y)\right)\right].$$

However the above do not necessarily hold true. Specifically, if X_1, X_2 are highly correlated it is conceivable that the pair $(X_1, g' \circ \bar{F}_{X_1+X_2+Y}(X_1+X_2+Y))$ is more concordant than the pair $(X_1, g' \circ \bar{F}_{X_1+Y}(X_1+Y))$. Therefore, the possibility arises that the portfolio-relative risk measure becomes superadditive for certain dependence structures between risks.

For risks with general joint distributions, it is difficult to cast heuristic arguments, such as the above, in a formal setting. However, for the case of elliptical probability distributions, which have well known aggregation properties, a more precise characterisation of the risk measure's $\rho(X;Y)$ aggregation properties is possible. Such an investigation is undertaken in Section 4. First, however, the aggregation and diversification properties of $\rho(X;Y)$ are demonstrated via a simple numerical example.

Consider independent standard normal variables X and Y. To illustrate sensitivity to scale, we can evaluate the quantity $\frac{\rho(aX;Y)}{a}$ for different values of the portfolio weight a. An exponential distortion function $g(s) = \frac{1-exp(-hs)}{1-exp(-h)}$ with parameter $h = 10$ is used. As seen in figure 1, the quantity $\frac{\rho(aX;Y)}{a}$ is increasing, which implies a superlinearity that renders $\rho(X;Y)$ sensitive to the scale of X. Moreover, as a gets large this effect is reduced, since the effect of the background risk Y becomes smaller and $\rho(X;Y)$ becomes close to the positive homogenous $\rho(X)$.

To examine the sensitivity to risk aggregation of $\rho(X_1+X_2;Y)$, consider a multivariate standard normal risk profile.

$$\begin{pmatrix} X_1 \\ X_2 \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & r & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

The reduction in risk from pooling X_1 and X_2, with respect to background risk Y, is given by the diversification benefit $1 - \frac{\rho(X_1+X_2;Y)}{\rho(X_1;Y)+\rho(X_2;Y)}$. In figure 2,

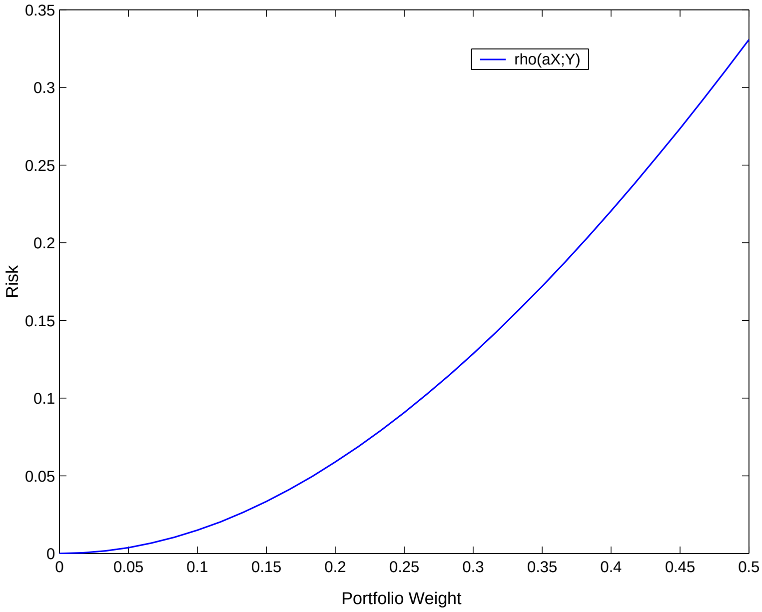

Figure 1: Sensitivity of $\rho(X;Y)$ to scale of risk X.

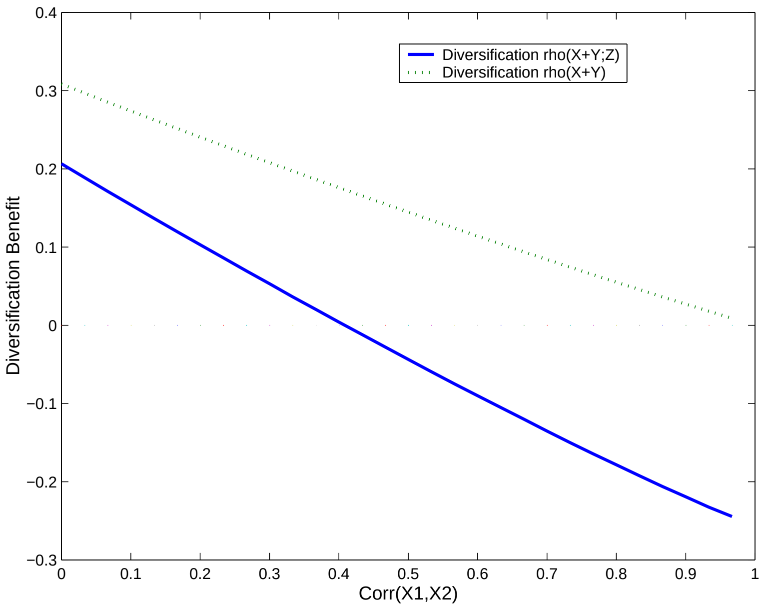

Figure 2: Diversification benefit between X_1, X_2 with respect to Y, against correlation $r(X_1, X_2)$

the diversification benefit is plotted as a function of the correlation r between X_1, X_2. It is seen that the diversification benefit reduces in the correlation. Moreover, for high enough correlation it actually becomes negative. This implies that for high levels of correlation between X_1, X_2, the background risk Y causes the risk measure $\rho(X_1 + X_2; Y)$ to become superadditive, that is, to display a sensitivity to the aggregation of risks. This is again a departure from the properties of the coherent distortion risk measure $\rho(X)$.

4 The case of elliptically distributed risks

4.1 Definition and properties of elliptical distributions

Here we briefly introduce the class of elliptical distributions (Kelker (1970), Fang et al. (1987)). Elliptical distributions can be viewed as generalisations of the multivariate normal family, which allow for heavier tails and asymptotic tail dependence. Furthermore, elliptically distributed vectors are characterised by a covariance structure similar to the one of the multivariate normal, thus creating a convenient tool for modelling symmetric risks and their aggregation properties. The importance of elliptical distributions in risk management has been highlighted by Embrechts et al. (2002) and Landsman and Valdez (2003), where an extensive list of elliptical families is also given.

Let Ψ_n be a class of functions $\psi(t) : [0, \infty) \mapsto \mathbb{R}$ such that the function $\psi(\sum_{i=1}^{n} t_i^2)$ is an $n-$dimensional characteristic function (Fang et al., 1987). It then follows that $\Psi_n \subset \Psi_{n-1} \cdots \subset \Psi_1$.

Definition 4. *Consider an $n-$dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$. The random vector $\mathbf{X}$ has a multivariate elliptical distribution, denoted by $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, \psi)$, if its characteristic function can be expressed as*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(i\mathbf{t}^T \mu)\psi\left(\frac{1}{2}\mathbf{t}^T \mathbf{\Sigma} \mathbf{t}\right) \tag{22}$$

for some column-vector μ, $n \times n$ positive-definite matrix $\mathbf{\Sigma}$, and for some function $\psi(t) \in \Psi_n$, which is called the characteristic generator.

Besides the multivariate normal family, obtained by $\psi(t) = e^{-t}$, examples of elliptical distributions are the multivariate t, logistic, symmetric stable, and exponential power families.

A vector $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, \psi)$ does not necessarily have a density. If the density exists (Fang et al., 1987), it takes form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\mathbf{\Sigma}|}} d_n \left[\frac{1}{2} (\mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \mu) \right], \tag{23}$$

where $d_n(\cdot)$ is a function called the density generator and c_n is a constant. Then the notation $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, d_n)$ can also be used.

The mean vector and covariance matrix of $\mathbf{X}$ do not necessarily exist. It is can be shown (Landsman and Valdez, 2003) that the conditions

$$\int_0^\infty d_1(x) dx < \infty \quad \text{and} \quad |\psi'(0)| \leq \infty \tag{24}$$

(where $d_1(x)$ is the density generator of the univariate marginal) are sufficient for their respective existence. If the mean and covariance exist, then they are given by

$$E(\mathbf{X}) = \mu, \qquad Cov(\mathbf{X}) = \mathbf{\Sigma}, \tag{25}$$

subject to suitable normalisation.

An important property of elliptical distributions is that linear transformations of elliptically distributed vectors are also elliptical, with the same characteristic generator. Specifically, from (22) it follows that if $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, \psi)$, A is a $m \times n$ dimensional matrix of rank $m \leq n$, and b is an m dimensional vector, then

$$A\mathbf{X} + b \sim E_m \left(A\mu + b, A\mathbf{\Sigma}A^T, \psi \right). \tag{26}$$

A direct consequence of (26) is that any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator, that is, if the diagonal elements of Σ are $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$, then for $k = 1, 2, ..., n$ it is $X_k \sim \mathbf{E}_1 \left(\mu_k, \sigma_k^2, \psi \right)$.

4.2 Distortion risk measures for elliptically distributed risks

In this section we present some result that will prove useful in the sequel, relating to representations of distortion risk measures and related functionals in the case of elliptically distributed risks. Consider $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, d_n)$ and function

$$D(x) = c_1 \int_0^x d_1(u)du. \tag{27}$$

Assuming that the mean vector and covariance matrix of $\mathbf{X}$ are finite, denote $\bar{D}(x) = D(\infty) - D(x)$.Consider now the function

$$f_{Z^*}(z) = \bar{D}\left(\frac{1}{2}z^2\right). \tag{28}$$

It is then shown in Landsman and Valdez (2003) that $f_{Z^*}(z)$ is the density of some standardised (i.e. with zero mean and unit variance) elliptical random variable, which does not necessarily come from the same family as $\mathbf{X}$. Furthermore denote $X_i^* = \mu_i + \sigma_i Z^*,\ \sigma_i = \sqrt{\sigma_{ii}},\ i = 1, \ldots, n$. Then the following result is proved by Landsman (2006).

Theorem 1. *Consider $\mathbf{X} \sim \mathbf{E}_n(\mu, \mathbf{\Sigma}, d_n)$ with finite mean vector and covariance matrix and an increasing, differentiable almost everywhere function $h(x)$ such that $E[h'(X_i^*)] < \infty,\ i = 1, \ldots, n$. Then*

$$E[(X_i - \mu_i)h(X_j)] = Cov(X_i, X_j)E[h'(X_j^*)]. \tag{29}$$

The theorem allows the explicit calculation of the expectation of an elliptical random variable, weighted by an increasing function of another elliptical variable in the same family. This makes the result particularly useful for the explicit calculation of risk measures and capital allocations which allow such representations, as shown in the result below.

For a more general version of Theorem 1 not requiring finiteness of covariances we refer to Landsman (2006). We note that the results obtained in this and the next section do not necessarily require that the covariance matrix is finite; the assumption is made to simplify the exposition.

Theorem 2. *Consider* $\mathbf{X} \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, d_2)$ *with finite mean vector and covariance matrix and function* $g : [0,1] \mapsto [0,1]$*, which increasing, concave and differentiable on* $[0,1]$ *with bounded first derivative. Then*

$$E[X_1 g'(\bar{F}_{X_2}(X_2))] = E[X_1] + \frac{Cov(X_1, X_2)}{\sqrt{Var(X_2)}} \cdot \lambda_g = \mu_1 + \frac{\sigma_{12}}{\sigma_2} \cdot \lambda_d, \tag{30}$$

where λ_g *is a positive constant specific to the elliptical family with density generator* d *and the function* g.

Proof. From Theorem 1 we have:

$$\begin{aligned} E[X_1 g'(\bar{F}_{X_2}(X_2))] &= E[X_1] + E[(X_1 - E[X_1]) g'(\bar{F}_{X_2}(X_2))] \\ &= Cov(X_1, X_2) E[h'(X_2^*)], \end{aligned} \tag{31}$$

where $h(x) = g'(\bar{F}_{X_2}(x))$ is an increasing function since

$$h'(x) = -g''(\bar{F}_{X_2}(x)) f_{X_2}(x) \geq 0 \tag{32}$$

and $X_2^* = \mu_2 + \sigma_2 Z^*$ as in Theorem 1. Now

$$\begin{aligned} E[h'(X_2^*)] &= E[h'(\mu_2 + \sigma_2 Z^*)] \\ &= \textstyle\int_{-\infty}^{\infty} -g''(\bar{F}_2(\mu_2 + \sigma_2 z)) f_{X_2}(\mu_2 + \sigma_2 z) f_{Z^*}(z) dz. \end{aligned} \tag{33}$$

Denote by F_Z and f_Z the standardised cumulative distribution and density of X_2, i.e. the distributions of random variable $Z \stackrel{D}{=} \frac{X_1 - \mu_1}{\sigma_1} \stackrel{D}{=} \frac{X_2 - \mu_2}{\sigma_2}$. Let $\bar{F}_Z(x) = 1 - F_Z(x)$. It then is

$$\bar{F}_{X_2}(\mu_2 + \sigma_2 z) = \bar{F}_Z(z), \quad f_{X_2}(\mu_2 + \sigma_2 z) = \frac{1}{\sigma_2} f_Z(z). \tag{34}$$

So

$$\begin{aligned} E[h'(X_2^*)] &= \textstyle\int_{-\infty}^{\infty} -g''(\bar{F}_Z(z)) \frac{1}{\sigma_2} f_Z(z) f_{Z^*}(z) dz \\ &= \tfrac{1}{\sigma_2} \cdot \lambda_g = \tfrac{1}{\sqrt{Var(X_2)}} \cdot \lambda_g, \end{aligned} \tag{35}$$

where $\lambda_g = \int_{-\infty}^{\infty} -g''(\bar{F}_Z(z)) \frac{1}{\sigma_2} f_Z(z) f_{Z^*}(z) dz > 0$, which does not depend on $\mu, \mathbf{\Sigma}$. $\square$

The following corollaries directly follow from Theorem 2.

Corollary 1. *For $X \sim E_1(\mu, \sigma, d_1)$ the distortion risk measure of equation (2) becomes*

$$\rho(X) = \mu + \lambda_g \sigma = E[X] + \lambda_g \sqrt{Var(X)} \tag{36}$$

Corollary 2. *For $(X_i, Z) \sim E_2(\mu, \mathbf{\Sigma}, d_2)$ the capital allocation mechanism of equation (6) becomes*

$$K_i = \mu_i + \lambda_g \frac{\sigma_{X_i,Z}}{\sigma_Z} = E[X_i] + \lambda_g \frac{Cov(X_i, Z)}{\sqrt{Var(Z)}} \tag{37}$$

Corollary 3. *For $(X, Y) \sim E_2(\mu, \mathbf{\Sigma}, d_2)$ the risk measure with respect to background risk of equation (8)*

$$\rho(X; Y) = E[X] + \lambda_g \frac{Cov(X, X + Y)}{\sqrt{Var(X + Y)}} \tag{38}$$

It was observed by Embrechts et al.(2002) that for elliptically distributed risks, Markowitz-style portfolio optimisation with a positive homogenous risk measure reduces to classic mean-variance optimisation. The corollaries above go slightly further for the specific case of distortion risk measures, as they show that the risk measure itself reduces to a 'standard deviation principle' (e.g. Goovaerts et al., 1984) and the marginal capital allocation to a simple covariance based one.

4.3 Properties of $\rho(X; Y)$ for elliptical distributions

The aggregation properties of portfolio-relative distortion risk measures were discussed in Section 3.3 in a rather heuristic manner. If however risks follow elliptical distributions, based on the results of the previous section, the aggregation properties of the risk measure can be characterised in a more precise way.

Proposition 2. *Let $(X, Y) \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$. The quantity*

$$\frac{1}{a}\rho(aX; Y), \quad a \neq 0 \tag{39}$$

is increasing in a.

Proof. By Corollary 3, we have that

$$\eta(a) = \frac{1}{a}\rho(aX;Y) = E[X] + \lambda_g \frac{Cov(X, aX+Y)}{\sqrt{Var(aX+Y)}} \tag{40}$$

Denote by $r(X,Y)$ the correlation coefficient between X and Y. By differentiating the above expression, it is shown that

$$\eta'(a) \geq 0 \Leftrightarrow 1 - r(X,Y)^2 \geq 0, \tag{41}$$

which always holds true. □

Recall that for positive homogenous risk measures the quantity $\frac{1}{a}\rho(aX)$ is constant. Thus, the above proposition reveals a form of super-linearity for $\rho(X;Y)$ in that $\frac{1}{a}\rho(aX;Y)$ is actually increasing. This shows that in the presence of background risk, aggregation of risk is penalised. In the next result the sensitivity of $\rho(X;Y)$ to aggregation risk is further characterised, by proving that it is super-additive for comonotonic risks. This means that the aggregation of highly dependent risks is being discouraged.

Proposition 3. *Let* $(X,Y,Z) \sim \mathbf{E}_3(\mu, \mathbf{\Sigma}, \psi)$ *such that* X, Y *are comonotonic. Then*

$$\rho(X+Y;Z) \geq \rho(X;Z) + \rho(Y;Z) \tag{42}$$

Proof. Since (X,Y) are comonotonic elliptical variables, they are perfectly correlated. Therefore we can write $Y = aX$ for $a > 0$ and $X = bY$ for $b = 1/a > 0$. It then is

$$\begin{gathered} \rho(X+Y;Z) \\ = E\left[Xg'\left(\bar{F}_{X+aX+Z}(X+aX+Z)\right)\right] + E\left[Yg'\left(\bar{F}_{bY+X+Z}(bY+Y+Z)\right)\right] \\ = \tfrac{1}{1+a}\rho((1+a)X;Z) + \tfrac{1}{1+b}\rho((1+b)Y;Z). \end{gathered} \tag{43}$$

Because of Proposition 2 it is

$$\frac{1}{1+a}\rho((1+a)X;Z) \geq \rho(X;Z) \quad \text{and} \quad \frac{1}{1+b}\rho((1+b)Y;Z) \geq \rho(Y;Z), \tag{44}$$

which yields the required result. □

The following corollary, characterising the sensitivity of the risk measure $\rho(X;Y)$ to increases in the direction of its argument X, is an immediate consequence of Proposition 3.

Corollary 4. *Let $(X,Y) \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$ and $\delta > 0$. Then*

$$\rho((1+\delta)X;Y) \geq \rho(X;Y) + \rho(\delta X;Y) \tag{45}$$

The sensitivity of $\rho(X;Y)$ to the variability of X is identified by the next result.

Proposition 4. *Let $(X,Y) \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$, such that $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$ and $Cov(X,Y) = \sigma_X \sigma_Y r$. Then $\rho(X;Y)$ is increasing in σ_X, if $Cov(X,Y) \geq -Var(X)$.*

Proof. It is

$$\begin{aligned}
\tfrac{\partial}{\partial \sigma_X}\rho(X;Y) &= \tfrac{\partial}{\partial \sigma_X}\left\{ E[X] + \lambda_g \tfrac{Cov(X,X+Y)}{\sqrt{Var(X+Y)}} \right\} \\
&= \lambda_g \tfrac{\partial}{\partial \sigma_X}\left\{ \sqrt{Var(X)} Cov\left(\tfrac{X}{\sqrt{Var(X)}}, \tfrac{X+Y}{\sqrt{Var(X+Y)}} \right) \right\} \\
&= \lambda_g Cov\left(\tfrac{X}{\sqrt{Var(X)}}, \tfrac{X+Y}{\sqrt{Var(X+Y)}} \right) + \\
&\quad \lambda_g \sqrt{Var(X)} \tfrac{\partial}{\partial \sigma_X} Cov\left(\tfrac{X}{\sqrt{Var(X)}}, \tfrac{X+Y}{\sqrt{Var(X+Y)}} \right)
\end{aligned} \tag{46}$$

It is thus clear that

$$Cov\left(\frac{X}{\sqrt{Var(X)}}, \frac{X+Y}{\sqrt{Var(X+Y)}} \right) \Leftrightarrow Cov(X,Y) \geq -Var(X) \geq 0 \tag{47}$$

and

$$\frac{\partial}{\partial \sigma_X} Cov\left(\frac{X}{\sqrt{Var(X)}}, \frac{X+Y}{\sqrt{Var(X+Y)}} \right) \geq 0 \tag{48}$$

are sufficient conditions for $\rho(X;Y)$ to be increasing in σ_X. Direct calculation of the derivative in (48) shows that it is always greater than zero. □

Thus, the risk measure $\rho(X;Y)$ is increasing in the variance of X, if $Cov(X,Y) \geq -Var(X) \iff Cov(X, X+Y) \geq 0$. It is intuitively appealing that an increase in variability should yield an increase in risk. The condition shows that this property is only guaranteed to hold when the portfolio X is positively correlated to the aggregate $X+Y$. The reason that the property may not hold for some negative values of the correlation between X and Y is that, under such a scenario, X operates as a hedge for Y and its variability might be useful for compensating for that of Y.

It is shown in Landsman and Tsanakas (2006) that for elliptical distributions an ordering of the diagonal elements of $\mathbf{\Sigma}$ (i.e. variances if they exist) implies stop-loss ordering of the corresponding random variables. This has the consequence:

Corollary 5. *Let* $(X,Y) \sim \mathbf{E}_2(\mu, \mathbf{\Sigma}, \psi)$*, such that* $Cov(X,Y) \geq -Var(X)$*. Then* $\rho(X;Y)$ *preserves stop-loss order.*

Dhaene and Goovaerts (1996) showed that portfolios of random variables that are highly dependent in the sense of the concordance order, yield higher stop-loss premia. As in an elliptical environment concordance order is equivalent to just ordering the correlation coefficients (e.g. Das Gupta et al. (1972), Abdous et al (2005), Landsman and Tsanakas (2006)), it follows that a high correlation between elliptical instruments results in a higher risk for the portfolio. The way this extends in the presence of background risk is reflected in the following result. The conditions again ensure that there are no high negative correlations which introduce hedging effects.

Proposition 5. *Let* $(X,Y,Z) \sim \mathbf{E}_3(\mu, \mathbf{\Sigma}, \psi)$ *and* r *be the correlation coefficient of* (X,Y)*. Then* $\rho(X+Y;Z)$ *is increasing in* r*, if* $Cov(X, Y+Z) \geq -Var(Y+Z)$ *and* $Cov(Y, X+Z) \geq -Var(X+Z)$*.*

Proof. It is:

$$\frac{\partial}{\partial r}\rho(X+Y;Z) = \lambda_g \frac{\partial}{\partial r}\frac{Cov(X+Y, X+Y+Z)}{\sqrt{Var(X+Y+Z)}} \tag{49}$$

Sufficient conditions for the above expression to be positive are

$$\begin{aligned} \frac{\partial}{\partial r} Cov\left(X, \frac{X+Y+Z}{\sqrt{Var(X+Y+Z)}}\right) \geq 0 \text{ and} \\ \frac{\partial}{\partial r} Cov\left(Y, \frac{X+Y+Z}{\sqrt{Var(X+Y+Z)}}\right) \geq 0 \end{aligned} \tag{50}$$

Direct differentiation shows that these conditions are equivalent to

$$Cov(X, Y+Z) \geq -Var(Y+Z) \text{ and } Cov(Y, X+Z) \geq -Var(X+Z) \tag{51}$$

□

5 Conclusion

A distortion-type risk measure was introduced that addresses two distinct but related issues:

- The effect of background risk on the measurement of the risk of individual positions.
- The dynamics of capital allocation, when re-balancing of the aggregate portfolio is considered.

It was shown that this new risk measure is more sensitive to the scale and aggregation of losses than the usual (coherent) distortion risk measures. This sensitivity was characterised in detail for the case of elliptical distributions.

Penalising large losses and the aggregation of highly dependent positions within a portfolio corresponds to practical concerns. For example, in an illiquid environment, such as an insurance market, a risk taker will be sensitive to aggregation by generally requiring a higher price 'at the margin' for a type risk that he is already heavily exposed to. Moreover, it is often observed that heterogeneity in the scale of risks within a portfolio reduces diversification benefits, with larger risks producing what one could call an 'elephant in the boat effect'.

Future work could include studying the effect of background risk on non-coherent risk measures, as well as a re-examination of asset allocation and pricing in the presence of background risk.

References

Abdous, B., Genest, C. and Rmillard, B. (2005), 'Dependence properties of meta-elliptical distributions,' in *Statistical Modelling and Analysis for Complex Data Problems*, eds. P. Duchesne and B. Rmillard, New York: Springer, pp. 1-15.

Acerbi, C. (2002), 'Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion', *Journal of Banking and Finance*, 26 (7), 1505-1518.

Artzner, P., F. Delbaen, J-M. Eber, and D. Heath (1999), 'Coherent Measures of Risk,' *Mathematical Finance*, 9 (3), 203-228.

Aumann, R. J., and L. S. Shapley (1974), *Values of Non-Atomic Games.* Princeton: Princeton University Press.

Das Gupta, S., M. L. Eaton, I. Olkin, M. Perlman, L. J. Savage, and M. Sobel (1972), 'Inequalities on the probability content of convex regions for elliptically contoured distributions,' in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II: Probability theory*, Berkeley: University of California Press, pp. 241-265.

Denault, M. (2001), 'Coherent Allocation of Risk Capital,' *Journal of Risk*, 4 (1), 1-34.

Denneberg, D. (1990), 'Distorted Probabilities and Insurance Premiums,' *Methods of Operations Research*, 63, 3-5.

Dhaene, J., M. Denuit, M. J. Goovaerts, R. Kaas, and D. Vyncke (2002), 'The Concept of Comonotonicity in Actuarial Science and Finance: Theory,' *Insurance: Mathematics and Economics*, 31 (1), 3-33.

Dhaene, J. and M. Goovaerts (1996), 'Dependency of Risks and Stop-Loss Order,' *ASTIN Bulletin*, 26 (2), 201-212.

Dhaene, J., M. Goovaerts, and R. Kaas (2003), 'Economic Capital Allocation Derived from Risk Measures', *North American Actuarial Journal*, 7 (2).

Embrechts, P., A. McNeil, and D. Straumann (2002), 'Correlation and Dependence in Risk Management: Properties and Pitfalls' in *Risk Management: Value at Risk and Beyond*, ed. M.A.H. Dempster. Cambridge: Cambridge University Press, pp. 176-223.

Fang, K.T., S. Kotz, and K.W. Ng (1987), *Symmetric Multivariate and Related Distributions*. London: Chapman & Hall.

Gollier, C. and J. W. Pratt (1996), 'Risk vulnerability and the tempering effect of background risk,' *Econometrica* 64 (5), 1109-1123.

Goovaerts, M.J., F. de Vylder, and J. Haezendonck (1984), *Insurance Premiums: Theory and Applications*. Amsterdam: North-Holland.

Heaton, J. and D. Lucas (2000), 'Portfolio choice in the presence of background risk,' *The Economic Journal* 110, 1-26.

Kelker, D. (1970) 'Distribution Theory of Spherical Distributions and Location-Scale Parameter Generalization,' *Sankhya* 32, 419-430.

Landsman, Z. and E. A. Valdez (2003), 'Tail Conditional Expectations for Elliptical Distributions,' *North American Actuarial Journal* 7 (4).

Landsman, Z. and A. Tsanakas (2006), 'Stochastic ordering of bivariate elliptical distributions,' *Statistics and Probability Letters* 76 (5), 488-494.

Landsman, Z. (2006), 'On the generalization of Stein's Lemma for elliptical class of distributions,' *Statistics & Probability Letters*, 76 (10), 1012-1016.

Müller, A. and D. Stoyan (2002), *Comparison Methods for Stochastic Models and Risks*, Wiley.

Tsanakas, A. (2004), 'Dynamic capital allocation with distortion risk measures,' *Insurance: Mathematics and Economics* 35, 223-243.

Tsanakas, A. and C. Barnett (2003), 'Risk capital allocation and cooperative pricing of insurance liabilities,' *Insurance: Mathematics and Economics*, 33 (2), 239-254.

Wang, S. S. (1996), 'Premium Calculation by Transforming the Premium Layer Density,' *ASTIN Bulletin*, 26 (1), 71-92.

Wang, S. (2002), 'A Set of New Methods and Tools for Enterprize Risk Capital Management and Portfolio Optimization,' 2002 *CAS Summer Forum.*

Yaari, M. (1987), 'The Dual Theory of Choice under Risk,' *Econometrica*, 55 (1), 95-115.