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To split or not to split: Capital allocation with convex risk measures

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Abstract

Convex risk measures were introduced by Deprez and Gerber (1985). Here the problem of allocating risk capital to subportfolios is addressed, when aggregate capital is calculated by a convex risk measure. The Aumann-Shapley value is proposed as an appropriate allocation mechanism. Distortion-exponential measures are discussed extensively and explicit capital allocation formulas are obtained for the case that the risk measure belongs to this family. Finally the implications of capital allocation with a convex risk measure for the stability of portfolios are discussed.

Keywords: Convex measures of risk, capital allocation, Aumann-Shapley value, inf-convolution.

1 Introduction

The formal study of risk measures has been an important part actuarial research since the 1970s (e.g. Bühlmann (1970), Gerber (1974), Goovaerts (1984)). In an actuarial context, risk measures were originally associated

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with the calculation of insurance premia. The emergence of risk-based regulatory regimes in banking and insurance motivated a renewed interest in risk measures, this time as functionals that give the required level of safely invested risk capital that the holder of a risky portfolio has to hold (Artzner et al, 1999).

A particular focus has often been placed on alternative sets of properties or axioms for risk measures, that are considered desirable in a particular context. For example, the properties of positive homogeneity and subadditivity have often been considered as appropriate (e.g. Wang, 1996)). These properties, partly characterising coherent measures of risk (Artzner et al., 1999), ensure that proportional increases in risk exposures only yield a proportional increase in risk capital and that the pooling of portfolios always reduces risk capital requirements due to diversification.

Given aggregate risk capital requirements, capital can be allocated down to subportfolios e.g. for performance measurement purposes (Tasche, 2004). If the risk measure is positive homogenous and subadditive, allocations based on marginal costs produce allocated capital amounts that are smaller than the respective risk capital levels corresponding to the subportfolios on a stand-alone basis (Aubin, 1981). This argument is motivated by game theory and implies that such allocations produce no incentives for the fragmentation of portfolios (Denault, 2001).

A weaker requirement on risk measures than positive homogeneity / subadditivity is convexity, proposed by Deprez and Gerber (1985), who introduce convex risk measures and study them in the context of optimal risk exchanges. While convexity still acknowledges diversification, risk capital is no more scale-independent – in fact it is increasing per unit of exposure. Moreover, it is possible that for some portfolios pooling increases aggregate risk. Convex risk measures were introduced in the mathematical finance literature by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) and spurred a lively research area including dynamic generalisations (Detlefsen and Scandolo, 2005).

While there are some studies of capital allocation using convex risk measures (see e.g. Dhaene et al. (2003) and, in a finance setting, Filipović

and Kupper (2006)), most of current research on convex risk measures is not in that context. Instead, they are often considered as utility-like decision functionals in risk exchange situations (Barrieu and El Karoui (2005), Jouini et al. (2007)) or tools for pricing in incomplete markets (Klöppel and Schweizer, 2007).

Risk capital allocation with convex risk measures poses a number of challenges which the present contribution aims to address:

a) As convex risk measures are generally not positive homogenous, it is no more possible to use marginal costs as a capital allocation mechanism. This is because additivity of marginal risk contributions to the aggregate is guaranteed by Euler's theorem for positive homogenous functions, which does not hold in the more general case.

b) The convex risk measures typically proposed in the literature have properties that are difficult to reconcile with risk management priorities. For example the popular exponential or entropic risk measure (Gerber (1974), Föllmer and Schied (2002)) becomes superadditive for positively dependent risks. This implies that savings in risk capital can only achieved when pooling negatively correlated positions, which a very strict requirement.

c) The game theoretical argument for capital allocation is more difficult to make. According to the properties of convex risk measures, it can be desirable to split a portfolio. What this implies for potential capital allocation methods is unclear.

These three issues are addressed as follows. In Section 2, a brief review of risk measures and capital allocation is given. The Aumann-Shapley value (Aumann and Shapley, 1974), originating in cooperative game theory, is proposed as an appropriate capital allocation mechanism. The Aumann-Shapley allocation can be viewed as a genaralisation of marginal costs that is applicable even when the risk measure is not homogenous.

In Section 3, a convex risk measure with a flexible set of properties is discussed. The risk measure is derived as a combination of the convex (non-homogenous) exponential (Gerber (1974), Föllmer and Schied (2002)) and

coherent distortion (Wang (1996), Acerbi (2002)) risk measures, which are obtained as special cases. This risk measure allows the introduction of some sensitivity to the scale of potential losses, without being excessively penal on risk aggregation.

Based on the work of Carlier and Dana (2003) a condition for the (Gateaux) differentiability of the risk measure is established. This allows the derivation of explicit capital allocation formulas for the Aumann-Shapley value. It is also shown that in the more general case where the risk measure is not differentiable, the use of a particular subgradient yields a capital allocation mechanism essentially identical to the Aumann-Shapley allocation, which is thus generalised.

In Section 4 conditions are examined under which the Aumann-Shapley allocation produces incentives for the splitting of portfolios. If such splitting turns out to be beneficial in the sense of savings in aggregate capital, the question arises as to how the portfolio should be optimally restructured. By transferring the results in Barrieu and El Karoui (2005) to the present context, it is demonstrated that using a (non-homogenous) convex risk measure for capital allocation produces an incentive for infinite fragmentation of portfolios. This clarifies some of the difficulties for using convex risk measures in a risk management context. It is argued that the rather uncomfortable issue of portfolio fragmentation can be addressed by posing some cost-induced constraints to the extent that portfolios can be split.

2 Capital allocation for convex risk measures

2.1 Risk measures

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a set of liabilities $\mathcal{X}$ defined thereon. Elements of $\mathcal{X}$ are bounded and are interpreted as the random losses from financial portfolios at a fixed future date. In particular an outcome of a random variable $X \in \mathcal{X}$ will be a considered a loss if $X(\omega) > 0$. It is moreover assumed that all payments are discounted by the risk free rate.

A risk measure is defined as a function $\rho : \mathcal{X} \mapsto \mathbb{R}$. $\rho(X)$ can be taken to represent the level of safely invested economic or risk capital that the owner

of X has to hold in order to make the portfolio $X - \rho(X)$ acceptable to e.g. a regulator or rating agency (Artzner et al., 1999).Several axiomatic characterisations of risk measures have been proposed in the literature, along with corresponding representation results. Risk measures satisfying a convexity property where introduced and discussed in detail by Deprez and Gerber (1985), while more recently, *convex measures of risk* have been defined axiomatically by Frittelli and Rosazza Giannin (2002), Föllmer and Schied (2002).

Definition 1. *A* convex measure of risk *is a risk measure ρ satisfying the following set of properties:*

- *Monotonicity: $X_1 \geq X_2 \implies \rho(X_1) \geq \rho(X_2)$;*

- *Translation invariance: $\rho(X + a) = \rho(X) + a, \ a \in \mathbb{R}$;*

- *Convexity: $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2), \ \lambda \in [0, 1], X_1, X_2 \in \mathcal{X}$.*

Besides characterising diversification, convexity implies that the function $f(a) = \rho(aX)/a$ is an increasing one for $a > 0$ (e.g. Deprez and Gerber, 1985). This can be interpreted as the capital requirements per unit of exposure being increasing in portfolio size, thus inducing a penalty for the aggregation of large risks.

Such sensitivity to risk aggregation vanishes in the case of *coherent measures of risk*, introduced by Artzner et al. (1999).

Definition 2. *A* coherent measure of risk *is a convex measure of risk ρ satisfying the additional property*

- *Positive homogeneity: $\rho(aX) = a\rho(X), a > 0$.*

It is noted that convexity and positive homogeneity imply the property of

- Subbaditivity: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).X_1, X_2 \in \mathcal{X}$.

Subadditivity is an altogether stronger requirement on diversification than convexity, implying that the merging of portfolios always yields a reduction in aggregate risk.

2.2 Aumann-Shapley allocations

Consider now the portfolio $X = \sum_j^n X_j$. It is sometimes of interest, e.g. for performance management purposes, to allocate the aggregate capital requirement $K = \rho(X)$ to the subportfolios $X_j, j = 1, \ldots, n$. Thus the capital allocation problem can be defined as a search for a vector of real numbers $(K_1, \ldots, K_n)$ such that $\sum_{j=1}^n K_j = K$.

We consider here a broader definition of the capital allocation problem:

Capital allocation problem: *For fixed $X \in \mathcal{X}$, and risk measure ρ find a linear functional $\psi\rho(\cdot; X) : \mathcal{X} \mapsto \mathbb{R}$, termed a* capital allocation rule *such that $\psi\rho(X; X) = \rho(X)$*

By the presumed linearity of $\psi\rho(\cdot; X)$ it directly follows that $\sum_{j=1}^n \psi\rho(X_j; X) = \rho(X)$, so that one can set $K_i = \psi\rho(X_i; X)$. Moreover, the allocated capital amounts calculated by $\psi\rho(\cdot; X)$ add up to $\rho(X)$ for *any* random variables $X_j \in \mathcal{X}$ such that $\sum_{j=1}^n X_j = X$. This makes the capital allocated to a risk independent of the way that the aggregate portfolio is partitioned into subportfolios.

In order to construct an appropriate $\psi\rho(\cdot; X)$ for capital allocation one has to also require that $\psi\rho(X_i; X)$ does in some way reflect the risk that X_i adds to the aggregate portfolio X. Marginal-cost-type allocation mechanisms address this by considering the sensitivity of aggregate risk to small changes in the exposure to particular subportfolios. This is formalised via the concept of *Gateaux derivatives*:

Definition 3. *If for risk measure ρ and fixed $X \in \mathcal{X}$ the limit*

$$D\rho(Y; X) = \lim_{t \to 0} \frac{\rho(X + tY) - \rho(X)}{t}$$

i) exists for all $Y \in \mathcal{X}$ and

ii) the mapping $Y \mapsto D\rho(Y; X)$ is a linear bounded functional,

then ρ is Gateaux differentiable *at X and $D\rho(Y; X)$ is the* Gateaux derivative *of ρ at X in the direction of Y.*

It is easily seen that for fixed X, a risk measure ρ that is Gateaux differentiable at γX, $\gamma \in [0, 1]$ and satisfies $\rho(0) = 0$ can be recovered from

its derivative by (Deprez and Gerber, 1985)

$$\rho(X) = \int_0^1 D\rho(X; \gamma X)d\gamma. \tag{1}$$

This simple fact motivates the definition of a capital allocation rule.

Definition 4. *For aggregate portfolio $X \in \mathcal{X}$ and a risk measure ρ that is Gateaux differentiable at γX, $\gamma \in [0,1]$,* the *Aumann-Shapley capital allocation rule $\psi^{AS}\rho$ is defined by*

$$\psi^{AS}\rho(Y; X) = \int_0^1 D\rho(Y; \gamma X)d\gamma. \tag{2}$$

This capital allocation mechanism takes its name from the concept of the Aumann-Shapley (1974) value, which has given rise to cost allocation mechanisms similar to equation (2) in the operational research literature, e.g. Billera and Heath (1982), Mirman and Tauman (1982). Capital allocations derived from the Aumann-Shapley value were discussed in the risk management literature by Denault (2001) who mainly deals with the special case of coherent risk measures, which is also the focus of Kalkbrenner (2005).

Remark 1: In the case of coherent risk measures, the positive homogeneity implies that

$$\rho(X) = D\rho(X; X), \quad \psi^{AS}\rho(Y; X) = D\rho(Y; X), \tag{3}$$

hence capital allocation reduces to marginal costs. This capital allocation mechanism has been derived by Tasche (2004) from the perspective of performance measurement and been known as the *Euler principle*; the term refers to Euler's theorem for homogenous functions, which guarantees the additivity of marginal costs to the aggregate).$\square$

Remark 2: Consider risk measures defined by

$$\rho(X) = E[X\zeta(X)], E[\zeta(X)] = 1 \tag{4}$$

where ζ is an increasing function (Dhaene et al. (2005), Furman and Zitikis (2007)). This construction can be used to define many well known classes

of risk measures, e.g. Esscher measures (Goovaerts et al, 1984) and spectral measures (Wang (1996), Acerbi (2002)). If the function ζ is differentiable then it is a simple exercise to show that the Aumann-Shapley allocation rule is:

$$\psi^{AS}\rho(Y;X) = E[Y\zeta(X)]. \tag{5}$$

Hence, if the risk measure is defined as an expected loss subject to a re-weighting of possible outcomes, then the capital allocated to any subportfolio Y equals the expected loss Y, subject to weighting induced by the aggregate risk. This representation is particularly helpful, when implementing capital allocations via Monte-Carlo simulation.□

3   Distortion-exponential risk measures

3.1   Preliminaries

Before discussing the risk measure that this section deals with, some preliminary material on Choquet integrals (see e.g. Denneberg (1994), Carlier and Dana (2003)) and dependence between random variables (see e.g. Müller and Stoyan, 2002) is presented.

3.1.1   Choquet integrals with respect to distorted probabilities

Consider an increasing *distortion function* $g : [0,1] \mapsto [0,1]$. Then the set function $g(\mathbb{P}) : \mathcal{F} \mapsto [0,1]$ is called a *distorted probability*. Distorted probabilities are special cases of *capacities*. Integrals with respect to distorted (rather than the usual, additive) probabilities can be defined as follows.

Definition 5. *The* Choquet integral *of $X \in \mathcal{X}$ with respect to the distorted probability $g(\mathbb{P})$ is defined by*

$$E_g(X) = -\int_{-\infty}^{0} (1 - g(\mathbb{P}(X > t)))dt + \int_{0}^{\infty} g(\mathbb{P}(X > t))dt. \tag{6}$$

Denote by F_X the cumulative distribution function of random variable X and by F_X^{-1} the quantile function:

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(p) \geq p\}. \tag{7}$$

Also denote by U_X a random variable that is uniform on $[0, 1]$ and satisfies $F_X^{-1}(U_X) = X$ (if F_X is strictly increasing it obviously is $U_X = F_X(X)$). The next result collects some important results, which can be found in Carlier and Dana (2003).

Theorem 1. *Assume the distortion function g is concave.*

  i)

$$E_g(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}(X)$$
$$\mathcal{Q} = \{\mathbb{Q} \text{ is a probability measure such that } \mathbb{Q}(A) \leq g(\mathbb{Q}(A)), \forall A \in \mathcal{F}\} \tag{8}$$

  ii) If in addition g is differentiable

$$E_g(X) = \int_0^1 F_X^{-1}(t) g'(1 - t) dt \tag{9}$$

  iii) For differentiable g, the probability measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = g'(1 - U_X)$ is a maximiser in expression (8) such that

$$E_g(X) = E_{\mathbb{Q}}(X) = E(X \cdot g'(1 - U_X)) \tag{10}$$

3.1.2 Dependence between random variables

Comprehensive treatments of the concepts in this section can be found in Müller and Stoyan (2002) and (for the case of non-negative variables) Dhaene and Goovaerts (1996).

Consider the set $\mathcal{A} := \mathcal{A}(F_1, F_2)$ of bivariate random vectors (X_1, X_2) with fixed cumulative probability distributions F_1, F_2. Thus the elements of $\mathcal{A}$ are only distinguished from each other with respect to their dependence structure. A partial order on $\mathcal{A}$ can be defined.

Definition 6. *The random vector $(Y_1, Y_2) \in \mathcal{A}$ with joint cumulative distribution F_Y is more concordant than $(X_1, X_2) \in \mathcal{A}$ with joint cumulative distribution F_X, denoted by*

$$(X_1, X_2) \preceq_c (Y_1, Y_2), \tag{11}$$

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if

$$F_X(x_1, x_2) \leq F_Y(x_1, x_2) \tag{12}$$

for all x_1, x_2 in the domains of F_1, F_2.

A further characterisation of the concordance order is via the following result.

Lemma 1.

$$(X_1, X_2) \preceq_c (Y_1, Y_2) \Leftrightarrow E(h_1(X_1)h_2(X_2)) \leq E(h_1(Y_1)h_2(Y_2)), \tag{13}$$

for all increasing functions h_1, h_2 such that the expectations exist.

The concepts of comonotonic and countermonotonic random variables are now introduced (see e.g. Dhaene et al, 2002).

Definition 7. *i) Two random variables X_1, X_2 are called* comonotonic *if there is random variable Z such that $X_1 = h_1(Z), X_2 = h_2(Z)$ for increasing functions h_1, h_2.*

ii) Two random variables X_1, X_2 are called countermonotonic *if there is random variable Z such that $X_1 = h_1(Z), X_2 = h_2(Z)$ for increasing h_1 and decreasing h_2.*

An example of comonotonic variables are X, U_X as defined before. Comonotonicity (countermonotonicity) forms the strongest form of positive (negative) dependence between two random variables with fixed marginal distributions, corresponding to the Frechet-Hoeffding upper (lower) bound of their joint distribution. This has the following implications.

Lemma 2. *Consider random vectors $(X_1, X_2) \in \mathcal{A}$, $(X_1^c, X_2^c) \in \mathcal{A}$ comonotonic, and $(X_1^{-c}, X_2^{-c}) \in \mathcal{A}$ countermonotonic. Then*

i)

$$(X_1^{-c}, X_2^{-c}) \preceq_c (X_1, X_2) \preceq_c (X_1^c, X_2^c) \tag{14}$$

ii)

$$E(h_1(X_1^{-c})h_2(X_2^{-c})) \leq E(h_1(X_1)h_2(X_2)) \leq E(h_1(X_1^c)h_2(X_2^c)) \tag{15}$$

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Finally, a weaker form of positive (negative) dependence than co(counter)-monotonicity is defined.

Definition 8. *Consider independent random vectors $(X_1^I, X_2^I) \in \mathcal{A}$. Then $(X_1, X_2) \in \mathcal{A}$ is called* positive (negative) quadrant dependent *if*

$$(X_1^I, X_2^I) \preceq_c (\succeq_c)(X_1, X_2). \tag{16}$$

3.2 Definition and properties of the risk measure

A flexible class of convex risk measures is now discussed.

Definition 9. *For $a \geq 0$ and concave, differentiable distortion function g, the* distortion-exponential risk measure $\rho_{g,a}$ *is defined by*

$$\rho_{g,a}(X) = \frac{1}{a} \ln E_g\left(e^{aX}\right), a > 0 \tag{17}$$

$$\rho_{g,a}(X) = E_g\left(X\right), a = 0. \tag{18}$$

The convexity of the risk measure $\rho_{g,a}$ is now established.

Proposition 1. *The risk measure $\rho_{g,a}$ is a convex measure of risk.*

Proof. Translation invariance: $\rho_{g,a}(X + b) = \rho_{g,a}(X) + b$ follows directly from the definition of the risk measure.

Monotonicity: Let F_i be the cdf of X_i and F_i^{-1} its quantile function. Then $X_1 \leq X_2 \implies F_1^{-1}(p) \leq F_2^{-1}(p), p \in [0,1] \implies \int_0^1 \exp(aF_1^{-1}(p))g'(1-p)dp \leq \int_0^1 \exp(aF_2^{-1}(p))g'(1-p)dp$, from which, in view of Theorem 1ii) the result follows.

Convexity: Consider $\frac{d\mathbb{Q}}{d\mathbb{P}} = g'(1 - U_Z)$ where $Z = \lambda X_1 + (1-\lambda)X_2$. Then

$$
\begin{aligned}
\rho_{g,a}(Z) \qquad &= \tfrac{1}{a} \ln E_{\mathbb{Q}}(e^{aZ}) \\
&\leq \lambda \tfrac{1}{a} \ln E_{\mathbb{Q}}(e^{aX_1}) + (1-\lambda)\tfrac{1}{a} \ln E_{\mathbb{Q}}(e^{aX_2}) \\
&= \lambda \tfrac{1}{a} \ln E(e^{aX_1} g'(1 - U_Z)) + (1-\lambda)\tfrac{1}{a} \ln E(e^{aX_2} g'(1 - U_Z)) \\
&\leq \lambda \tfrac{1}{a} \ln E(e^{aX_1} g'(1 - U_{X_1})) + (1-\lambda)\tfrac{1}{a} \ln E(e^{aX_2} g'(1 - U_{X_2})) \\
&= \lambda \rho_{g,a}(X_1) + (1-\lambda)\rho_{g,a}(X_1)
\end{aligned}
\tag{19}
$$

where the first inequality follows from the convexity of the functional $\rho(X) = \frac{1}{a}E_{\mathbb{Q}}(e^{aX})$ (proved in Deprez and Gerber (1985)) and the second from Lemma 2ii). $\qquad\square$

The risk measure $\rho_{g,a}$ has two well known special cases:

- For $a = 0$ we get the *distortion* or *spectral risk measure* (Wang (1996), Acerbi (2002))

$$\rho_g(X) = E\left(X \cdot g'(1 - U_X)\right). \tag{20}$$

  This is known to be a coherent measure of risk, which follows directly from the positive homogeneity of ρ_g (note that for $a \geq 0$, $U_{aX} = U_X$).

- For $g(t) = t$, we get the *exponential risk measure* (termed originally the *exponential premium principle* by Gerber (1974))

$$\rho_a(X) = \frac{1}{a} \ln E\left(e^{aX}\right), \tag{21}$$

  which is known to be a convex risk measure (see e.g. Föllmer and Schied (2002), where it is referred to as the *entropic risk measure*).

While the exponential risk measure is convex, its aggregation properties can be considered too extreme, since for any positive quadrant dependent X_1, X_2, it can be easily shown that $\rho_a(X_1 + X_2) \geq \rho_a(X_1) + \rho_a(X_2)$. This means there is no diversification benefit for even weakly positively dependent risks, which is a rather harsh requirement. For the risk measure $\rho_{g,a}$ this effect is moderated, as the aggregation of positively dependent risks will produce a diversification benefit, as long as a (or the risk $X_1 + X_2$ itself) is small enough, as seen by $\lim_{a \to 0} \rho_{g,a}(X_1 + X_2) = \rho_{g,0}(X_1 + X_2) \leq \rho_{g,0}(X_1) + \rho_{g,0}(X_2)$.

For extreme cases of dependence between X_1, X_2, the aggregation properties of $\rho_{g,a}$ can be easily characterised.

Lemma 3. *i) For comonotonic random variables X_1^c, X_2^c,*

$\rho_{g,a}(X_1^c + X_2^c) \geq \rho_{g,a}(X_1^c) + \rho_{g,a}(X_2^c).$

i) For countermonotonic random variables X_1^{-c}, X_2^{-c},

$\rho_{g,a}(X_1^{-c} + X_2^{-c}) \leq \rho_{g,a}(X_1^{-c}) + \rho_{g,a}(X_2^{-c}).$

Proof. First we note that the properties of comonotonicity and countermonotonicity are not affected by performing the change of probability measure $\frac{dQ}{d\mathbb{P}} = g'(1 - U_{X_1 + X_2})$. Second, the exponential risk measure ρ_a is

superadditive for comonotonic and subadditive for countermonotonic risk, as these are extreme cases of Positive and Negative Quadrant Dependence respectively.

For proof of part i) we note that for comonotonic X_1^c, X_2^c, $U_{X_1^c} = U_{X_2^c} = U_{X_1^c+X_2^c}$. Hence

$$
\begin{aligned}
\rho_{g,a}(X_1^c + X_2^c) &= \tfrac{1}{a} \ln E(e^{a(X_1^c+X_2^c)} g'(1 - U_{X_1^c+X_2^c})) \\
&\geq \tfrac{1}{a} \ln E(e^{aX_1^c} g'(1 - U_{X_1^c+X_2^c})) + \tfrac{1}{a} \ln E(e^{aX_2^c} g'(1 - U_{X_1^c+X_2^c})) \\
&= \tfrac{1}{a} \ln E(e^{aX_1^c} g'(1 - U_{X_1^c})) + \tfrac{1}{a} \ln E(e^{aX_2^c} g'(1 - U_{X_2^c})) \\
&= \rho_{g,a}(X_1^c) + \rho_{g,a}(X_2^c).
\end{aligned}
\tag{22}
$$

For proof of part ii), we consider countermonotonic X_1^{-c}, X_2^{-c}

$$
\begin{aligned}
\rho_{g,a}(X_1^{-c} + X_2^{-c}) &= \tfrac{1}{a} \ln E(e^{a(X_1^{-c}+X_2^{-c})} g'(1 - U_{X_1^{-c}+X_2^{-c}})) \\
&\leq \tfrac{1}{a} \ln E(e^{aX_1^{-c}} g'(1 - U_{X_1^{-c}+X_2^{-c}})) + \tfrac{1}{a} \ln E(e^{aX_2^{-c}} g'(1 - U_{X_1^{-c}+X_2^{-c}})) \\
&\leq \tfrac{1}{a} \ln E(e^{aX_1^{-c}} g'(1 - U_{X_1^{-c}})) + \tfrac{1}{a} \ln E(e^{aX_2^{-c}} g'(1 - U_{X_2^{-c}})) \\
&= \rho_{g,a}(X_1^{-c}) + \rho_{g,a}(X_2^{-c}).
\end{aligned}
\tag{23}
$$

$\square$

Remark 3: The risk measure $\rho_{g,a}$ was introduced in a slightly different context by Tsanakas and Desli (2003), who derived using indifference arguments in the context of rank-dependent expected utility theory (e.g. Quiggin, 1982), with an exponential utility function $u(x) = \frac{1}{a}(1 - \exp(-ax))$ and a distortion function $h(t) = 1 - g(1 - t)$, which considers the preference functionals of the form $U(W) = E_h(u(W))$. Then it can be easily shown that $\rho_{g,a}(X)$ is the solution of the equation $U(\rho_{g,a}(X) - X) = 0$. For such a perspective see Tsanakas and Desli (2003) and also the review Denuit et al. (2006). $\square$

3.3 Differentiability and capital allocation

Here we derive Aumann-Shapley allocations for the convex risk measure $\rho_{g,a}$ introduced in Section 3.2. For this, conditions under which the risk measure is Gateaux differentiable need to be stated. Differentiability of quantiles has

been studied by Tasche (2004) and of Choquet integrals by Carlier and Dana (2003) and Marinacci and Montrucchio (2004). For distortion-exponential risk measures, the following result holds.

Proposition 2. *For concave and differentiable g, the risk measure $\rho_{g,a}$ is Gateaux differentiable at $X \in \mathcal{X}$, if and only if F_X^{-1} is strictly increasing. Then the Gateaux derivative is*

$$D\rho_{g,a}(Y; X) = \frac{E\left(Ye^{aX}g'(1 - U_X)\right)}{E\left(e^{aX}g'(1 - U_X)\right)}, \tag{24}$$

where $U_X = F_X(X)$. Moroever, if $g(t) = t$, the condition on F_X^{-1} is not necessary.

Proof. By Corollaries 2 and 3 in Carlier and Dana (2003), we have that the Choquet integrals $E_g(X), E_g(e^{aX})$ are differentiable with Gateaux derivatives $E(Yg'(1 - F_X(X))), E(aYe^{aX}g'(1 - F_X(X)))$ respectively (note that due to the monotonicity of F_X^{-1} we have $U_X = F_X(X)$). This produces the required result in the case $a = 0$. For $a > 0$ the result follows simply by taking the derivative

$$\left.\frac{\partial}{\partial t}\rho_{g,a}(X + tY)\right|_{t=0} = \frac{\frac{\partial}{\partial t}E_g(e^{a(X+tY)})\big|_{t=0}}{aE_g(e^{aX})} = \frac{E(aYe^{aX}g'(1 - F_X(X)))}{aE(e^{aX}g'(1 - F_X(X)))}. \tag{25}$$

The special case $g(t) = t$ is considered by Deprez and Gerber (1985) when dealing with the exponential risk measure. $\square$

Aumann-Shapley allocations are now readily derived.

Corollary 1. *For aggregate portfolio X with strictly increasing F_X^{-1} and risk measure $\rho_{g,a}$ with concave, differentiable g, the Aumann-Shapley capital allocation rule is given by*

$$\psi^{AS}\rho_{g,a}(Y; X) = \int_0^1 \frac{E\left(Ye^{\gamma aX}g'(1 - F_X(X))\right)}{E\left(e^{\gamma aX}g'(1 - F_X(X))\right)}d\gamma. \tag{26}$$

Corollary 2. *For aggregate portfolio X with strictly increasing F_X^{-1} and risk measure ρ_g with concave, differentiable g, the Aumann-Shapley capital allocation rule is given by*

$$\psi^{AS}\rho_g(Y; X) = E\left(Yg'(1 - F_X(X))\right). \tag{27}$$

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Corollary 3. *For aggregate portfolio X and risk measure ρ_a, the Aumann-Shapley capital allocation rule is given by*

$$\psi^{AS}\rho_a(Y;X) = \int_0^1 \frac{E\left(Ye^{\gamma aX}\right)}{E\left(e^{\gamma aX}\right)}d\gamma. \tag{28}$$

It is noted that Corollary 2 was derived in Tsanakas and Barnett (2003), while Corollary 3 could be seen as exploiting the representation of the exponential risk measure as a 'mixture of Esscher measures' (Gerber and Goovaerts, 1981).

The question now arises as to how to allocate capital using risk measure $\rho_{g,a}$ in the more general case that F_X^{-1}. In particular, will a capital allocation rule similar to (26) be still meaningful in that context? We start with the observation that even when the risk measure is not Gateaux differentiable at X, the derivative in the direction of X itself (or a variable which is comonotonic to X) does exist.

Lemma 4. *For concave and differentiable g and comonotonic X, Y the function $t \mapsto \rho_{g,a}(X + tY)$ is differentiable at $t = 0$. For $a > 0$ the derivative equals*

$$\left.\frac{\partial}{\partial t}\rho(X + tY)\right|_{t=0} = \frac{E\left(Ye^{aX}g'(1 - U_X)\right)}{E\left(e^{aX}g'(1 - U_X)\right)} \tag{29}$$

and for $a = 0$

$$\left.\frac{\partial}{\partial t}\rho(X + tY)\right|_{t=0} = E\left(Yg'(1 - U_X)\right). \tag{30}$$

Proof. Observe that for comonotonic X, Y, $X, X + tY$ are comonotonic as well, implying $U_X = U_{X+tY}$. Hence

$$
\begin{aligned}
\left.\frac{\partial}{\partial t}\rho(X + tY)\right|_{t=0} &= \lim_{t \to 0} \frac{\frac{1}{a}E\left(e^{a(X+tY)}g'(1 - U_{X+tY})\right) - \frac{1}{a}E\left(e^{aX}g'(1 - U_X)\right)}{t} \\
&= \lim_{t \to 0} \frac{\frac{1}{a}E\left(e^{a(X+tY)}g'(1 - U_X)\right) - \frac{1}{a}E\left(e^{aX}g'(1 - U_X)\right)}{t} \\
&= \frac{E\left(Ye^{aX}g'(1 - U_X)\right)}{E\left(e^{aX}g'(1 - U_X)\right)}
\end{aligned}
\tag{31}
$$

The proof for $a = 0$ is similar. $\quad\square$

Furthermore it can be shown that even when the risk measure is not Gateaux differentiable, expression (24) corresponds to a subgradient of $\rho_{g,a}$.

Recall that a linear functional $Y \mapsto D^*\rho(Y; X)$ is a *subgradient* of a convex risk measure ρ at X if for all $Y \in \mathcal{X}$:

$$\rho(X + Y) - \rho(X) \geq D^*\rho(Y; X) \tag{32}$$

Proposition 3. *Expression (24) in Proposition 2 corresponds to a subgradient of $\rho_{g,a}$ at X, regardless of whether F_X^{-1} is strictly increasing.*

Proof. Consider

$$\begin{aligned}
\rho(X + Y) - \rho(X) &= \tfrac{1}{a}E\left(e^{a(X+Y)}g'(1 - U_{X+Y})\right) - \tfrac{1}{a}E\left(e^{aX}g'(1 - U_X)\right) \\
&\geq \tfrac{1}{a}E\left(e^{a(X+Y)}g'(1 - U_X)\right) - \tfrac{1}{a}E\left(e^{aX}g'(1 - U_X)\right)
\end{aligned} \tag{33}$$

by Lemma 2. Observe now that $\frac{E\left(Ye^{aX}g'(1-U_X)\right)}{E\left(e^{aX}g'(1-U_X)\right)}$ is the Gateaux derivative of the differentiable exponential risk measure at X, under the change of measure $\frac{d\mathbb{Q}}{d\mathbb{P}} = g'(1 - U_X)$. Hence, by the convexity of the exponential risk measure

$$\frac{1}{a}E\left(e^{a(X+Y)}g'(1 - U_X)\right) - \frac{1}{a}E\left(e^{aX}g'(1 - U_X)\right) \geq \frac{E\left(Ye^{aX}g'(1 - U_X)\right)}{E\left(e^{aX}g'(1 - U_X)\right)}, \tag{34}$$

which completes the proof. $\square$

Consider now the capital allocation rule

$$\psi^{AS*}\rho_{g,a}(Y; X) = \int_0^1 \frac{E\left(Ye^{\gamma aX}g'(1 - U_X)\right)}{E\left(e^{\gamma aX}g'(1 - U_X)\right)}d\gamma, \tag{35}$$

where the particular subgradient of $\rho_{g,a}$ discussed above is being used to construct the linear functional $\psi^{AS*}\rho_{g,a}(\cdot; X)$. Given that in the case of non-differentiable $\rho_{g,a}$ there will be a multitude of subgradients, the particular choice is motivated by observing that from Lemma 4 we have that

$$\rho_{a,g}(X) = \int_0^1 \frac{E\left(Xe^{\gamma aX}g'(1 - U_X)\right)}{E\left(e^{\gamma aX}g'(1 - U_X)\right)}d\gamma. \tag{36}$$

Therefore, defining a capital allocation rule by (35) will satisfy the requirement $\psi^{AS*}\rho_{g,a}(X; X) = \rho_{g,a}(X)$.

3.4 Small a

One can consider the risk measure $\rho_{g,a}$ in (17) as a modification of coherent risk measure ρ_g, with an "add-on" for scale-dependence, represented by the risk aversion parameter a. As coherent, rather than convex, risk measures are the norm in risk management, it makes sense to ask what happens when a small value is chosen for a.

Assume for simplicity that $\rho_{g,a}$ is Gateaux differentiable at a fixed X and denote $\frac{d\mathbb{Q}}{d\mathbb{P}} = g'(1 - U_X)$. Then, we can use the first terms of the cumulant expansion to write $\rho_{g,a}$ in terms of the first three central moments under $\mathbb{Q}$:

$$\rho_{g,a}(X) = \frac{1}{a}E_{\mathbb{Q}}\left(e^{aX}\right) \approx E_{\mathbb{Q}}(X) + \frac{a}{2}E_{\mathbb{Q}}\left((X - E_{\mathbb{Q}}(X))^2\right) + \frac{a^2}{6}E_{\mathbb{Q}}\left((X - E_{\mathbb{Q}}(X))^3\right).$$
(37)

It is then simple to show that the Aumann-Shapley allocation is given by a weighted sum of expectation, covariance and co-skewness under $\mathbb{Q}$:

$$\psi^{AS}(Y; X) \approx \quad E_{\mathbb{Q}}(Y) + \frac{a}{2}E_{\mathbb{Q}}\left((Y - E_{\mathbb{Q}}(Y))(X - E_{\mathbb{Q}}(X))\right)$$
$$+ \frac{a^2}{6}E_{\mathbb{Q}}\left((Y - E_{\mathbb{Q}}(Y))(X - E_{\mathbb{Q}}(X))^2\right).$$
(38)

4 Splitting portfolios

4.1 Incentives produced by allocation

Given aggregate portfolio X and risk measure ρ it is of interest to characterise situations where $\psi^{AS}\rho(Y; X) \le \rho(Y)$ for some subportfolio Y. If such a relation holds, and assuming that allocated risk capital carries a cost, the allocation does not give an incentive to split Y from the aggregate portfolio X, as since the this would yield a increase in the level of capital required to support Y. This relates to the game-theoretical concept of the *core*, for a detailed discussion of which in a risk management context see Denault (2001) and, in association with Choquet integrals, Carlier and Dana (2003).

If the risk measure is positive homogenous and subadditive, the inequality $\psi^{AS}\rho(Y; X) \le \rho(Y)$ always holds, as shown by Aubin (1981). Thus, the Aumann-Shapley allocation (reduced to marginal costs) is consistent with the requirement induced by subadditivity that it is never optimal to split a portfolio.

In the more general setting of convex risk measures, characterisation of the incentives produced by the Aumann-Shapley capital allocation rule is less straightforward. In the sequel some rather strong conditions on the risk measure and on Y that are sufficient for $\psi^{AS}\rho(Y;X) \leq \rho(Y)$ are presented. For simplicity of exposition, for the rest of this section assume that ρ is Gateaux differentiable at X.

First we observe that even if the risk measure is not generally subadditive, a form of subadditivity with respect to the particular risks X,Y is sufficient for no incentives for splitting to be induced by the allocation.

Lemma 5. *Consider convex risk measure ρ and $X,Y \in \mathcal{X}$. Assume that for each $\gamma \in [0,1]$ there exists $t_\gamma \in (0,1)$ such that*

$$\rho(\gamma X + t_\gamma Y) \leq \rho(\gamma X) + \rho(t_\gamma Y). \tag{39}$$

Then

$$\psi^{AS}\rho(Y;X) \leq \rho(Y). \tag{40}$$

Proof. We observe that

$$D\rho(Y,\gamma X) = \frac{1}{t_\gamma}D\rho(t_\gamma Y, \gamma X) \leq \frac{1}{t_\gamma}\left(\rho(\gamma X + t_\gamma Y) - \rho(\gamma X)\right) \leq \frac{1}{t_\gamma}\rho(t_\gamma Y) \leq \rho(Y) \tag{41}$$

where the first and third inequalities are due to the convexity of ρ and the second one is due to the stated subadditivity assumption. Hence it follows that

$$D\rho(Y;\gamma X) \leq \rho(Y) \implies \int_0^1 D\rho(Y;\gamma X)d\gamma \leq \int_0^1 \rho(Y)d\gamma, \tag{42}$$

which yields the required result. $\square$

Lemma 5 yields the obvious corollary:

Corollary 4. *For a coherent risk measure ρ and any $X,Y \in \mathcal{X}$,*

$$\psi^{AS}\rho(Y;X) \leq \rho(Y). \tag{43}$$

Example 1. *Consider distortion measure $\rho_g(X) = E(X \cdot g'(1 - U_X))$. Then by Corollary 4 and the subadditivity of the risk measure we have $\psi^{AS}\rho_g(Y;X) \leq \rho_g(Y)$ for all $X,Y \in \mathcal{X}$.*

A special case takes place when the risk measure is additive over sub-portfolios. In that case the aggregation and diversification effects cancel each other out in the portfolio and the holder should be indifferent as to whether he should split the portfolio or not. The capital allocation should recognise this by allocating to each subporfolio the amount equal to its stand-alone risk (note that this is a form of the *additivity over games property* in game theory, see for example Billera and Heath (1982)). The following lemma is simple to derive.

Lemma 6. *Consider convex risk measure ρ and $X_1, X_2 \in \mathcal{X}$. Assume that for all $\gamma_1, \gamma_2 \in [0,1]$ there it is*

$$\rho(\gamma_1 X_1 + \gamma_2 X_2) = \rho(\gamma_1 X_1) + \rho(\gamma_2 X_2). \tag{44}$$

Then

$$\psi^{AS}\rho(X_i; X_1 + X_2) = \rho(X_i), \ i = 1, 2. \tag{45}$$

Example 2. *Consider $\rho_a(X) = \frac{1}{a} E\left(e^{aX}\right)$ and X_1, X_2 stochastically independent. Then the condition of Lemma 6 is satisfied and we have:*

$$\psi^{AS}\rho(X_i; X_1 + X_2) = \int_0^1 \frac{E\left(X_i e^{a(X_1 + X_2)}\right)}{E\left(e^{a(X_1 + X_2)}\right)} d\gamma = \int_0^1 \frac{E\left(X_i e^{aX_i}\right)}{E\left(e^{aX_i}\right)} d\gamma = \rho_a(X_i) \tag{46}$$

Example 3. *Consider $\rho_g(X) = E\left(Xg'(1 - U_X)\right)$ and X_1, X_2 comonotonic. Then the condition of Lemma 6 is again satisfied and noting that $U_{X_1} = U_{X_2} = U_{X_1 + X_2}$ we have:*

$$\psi^{AS}\rho(X_i; X_1 + X_2) = E(X_i \cdot g'(1 - U_{X_1 + X_2})) = E(X_i \cdot g'(1 - U_{X_i})) = \rho_g(X_i) \tag{47}$$

An alternative rather strong sufficient condition for the lack of incentives to split is that the random variables X, Y be independent. If we set $X = \sum_j X_j, Y = X_i$, that is, X_i is a subportfolio of X, independence implies that X_i is in some way hedged by other instruments in X.

Lemma 7. *Consider convex risk measure ρ, such that $\rho(X) \geq E(X), \forall X \in \mathcal{X}$. If $X, Y \in \mathcal{X}$ are independent under $\mathbb{P}$, then*

$$\psi^{AS*}(Y; X) \leq \rho(Y). \tag{48}$$

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Proof. By the Riesz Representation Theorem we can write the linear functional $\psi^{AS}\rho(Y;X) = E(Y \cdot \zeta_X)$, where the density $\zeta_X, E(\zeta_X) = 1$ only depends on X. Thus independence of X, Y gives

$$\psi^{AS}\rho(Y;X) = E(Y \cdot \zeta_X) = E(Y)E(\zeta_X) = E(Y) \leq \rho(Y). \qquad (49)$$

$\square$

4.2   Optimal splitting

Section 4.1 showed that there are situations where capital allocation may give incentives for the splitting of portfolios. Arguably one should then have to examine the potential benefits of actually (and optimally) proceeding with such splitting. This is a problem which has been addressed in the different context of optimal risk exchanges, by several authors, including Deprez and Gerber (1985), Barrieu and El Karoui (2005), and Jouini et al. (2007).

Assume that the holder of portfolio X is able to split it into two parts, X_1, X_2 (which are not defined in advance in terms of line-of-business etc). By 'splitting' we mean separation so that no cross-subsidy between X_1, X_2 can take place, for example by creating distinct legal entities possibly operating in different markets. Consider now convex risk measures ρ_1, ρ_2 representing the capital requirements that portfolios X_1, X_2 attract. Then the optimal split of X is obtained by minimising the quantity $\rho_1(X_1) + \rho_2(X_2)$ such that $X_1 + X_2 = X$. Formally this corresponds to the *infimal convolution of ρ_1, ρ_2*, denoted by $\rho_1 \square \rho_2$ and defined by:

$$\rho_1 \square \rho_2(X) = \inf_{X_2 \in \mathcal{X}} \{\rho_1(X - X_2) + \rho_2(X_2)\}. \qquad (50)$$

If the holder of X is able to split the risk as described above, then $\rho_1 \square \rho_2(X)$ is the 'real' risk measure that he uses as it represents the amount of risk capital that he needs to provide, after optimising the structure of his portfolio.

Consider the case that $\rho_1 = \rho_2 = \rho_g$, that is, both risk measures are equal and coherent. Then, by subadditivity of ρ_g it is obvious that

$$\rho_g \square \rho_g(X) = \rho_g(X), \qquad (51)$$

reflecting the requirement that portfolios are not split. In the more general case of convex risk measures the following characterisation holds (Barrieu and El Karoui, 2005).

Proposition 4. *The infimal convolution of convex risk measures* ρ_1, ρ_2

$$\rho_1 \square \rho_2(X) = \inf_{X_2 \in \mathcal{X}} \{\rho_1(X - X_2) + \rho_2(X_2)\}, \tag{52}$$

is itself a convex measure or risk.

More can be said if the risk measures ρ_1, ρ_2 belong to the same *family of dilated risk measures*, that is, if they can be written as

$$\rho_i(X) = \frac{1}{a_i} \rho_0(a_i X), \ i = 1, 2, \tag{53}$$

for some convex risk measure ρ_0 and positive *risk aversion parameters* a_1, a_2. For example, the distortion-exponential risk measures studied in Section 3 are a family of dilated risk measures for a fixed distortion function g as:

$$\rho_{g,a}(X) = \frac{1}{a} \rho_{g,1}(aX). \tag{54}$$

Then the following result holds (Barrieu and El Karoui, 2005)

Proposition 5. *Consider the family of dilated convex risk measures* ρ_a, $a > 0$. *Then:*

i) *For all* $a_1, a_2 > 0$,

$$\rho_{a_1} \square \rho_{a_2}(X) = \inf_{X_2 \in \mathcal{X}} \{\rho_{a_1}(X - X_2) + \rho_{a_2}(X_2)\} = \rho_a(X), \tag{55}$$

where $a = (a_1^{-1} + a_2^{-1})^{-1}$.

ii) *The optimal portfolio split is given by*

$$X_1^* = X - X_2^* = \frac{\frac{1}{a_1}}{\frac{1}{a_1} + \frac{1}{a_2}} X, \ X_2^* = \frac{\frac{1}{a_2}}{\frac{1}{a_1} + \frac{1}{a_2}} X. \tag{56}$$

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Hence the infimal convolution is in the same family of dilated risk measures, but with a risk aversion parameter smaller than that of each of the original risk measures. Moreover the optimal split is a proportional one, with the risk retained in each portfolio inversely proportional to the corresponding risk aversion parameter.

Even after optimally splitting the aggregate portfolio into risks X_1^*, X_2^*, capital can still be allocated to any subportfolio by the Aumann-Shapley value. For example for distortion exponential risk measures $\rho_{g,a_1}, \rho_{g,a_2}$, we have by Proposition 5 that $\rho_{g,a_1} \square \rho_{g,a_2}(X) = \rho_{g,a}(X)$ and hence we obtain

$$\psi^{AS} \rho_{g,a}(Y; X) = \int_0^1 \frac{E\left(Y e^{\gamma a X} g'(1 - U_X)\right)}{E\left(e^{\gamma a X} g'(1 - U_X)\right)} d\gamma, \tag{57}$$

for $a = (a_1^{-1} + a_2^{-1})^{-1}$. In particular if we consider allocating capital to the portfolios X_1^*, X_2^*, it is simple to show that

$$\psi^{AS} \rho_{g,a}(X_i^*; X) = \rho_{g,a_i}(X_i^*), \ i = 1, 2, \tag{58}$$

meaning the capital allocation and risk allocation (splitting) processes are consistent.

Remark 4: Many of the concepts presented in Sections 4.2 and 4.3 appear in a rather different setting in Aumann and Shapley (1974). Very loosely speaking, in a market with an infinity of (infinitely small) traders, they define the maximal utility that each group of traders can achieve by trading with each other. They then proceed to show that this utility is a superadditive function, meaning that it is always optimal for traders to take part in an exchange.

4.3 When to stop splitting?

In Section 4.2 the case where it is optimal to split the aggregate portfolio in two parts was examined. The question then naturally arises as to whether this optimisation process removes any incentives for further splitting the portfolio. More generally, into how many fragments must one split a portfolio before splitting does no more produce a saving in aggregate risk capital?

For simplicity assume here that the capital for each portfolio is calculated by using the same convex measure of risk ρ (in fact considering a family of dilated risk measures changes little in the subsequent discussion). Then, optimal splitting of X into n sub-portfolios follows from the solution of

$$\square_{j=1}^{n}\rho(X) = \inf_{X_1,\dots,X_n}\{\rho(X_1) + \cdots + \rho(X_n) : \sum_{j=1}^{n}X_j = X\}. \tag{59}$$

By Proposition 5 it is

$$\square_{j=1}^{n}\rho(X) = n\rho\left(\frac{1}{n}X\right). \tag{60}$$

By the convexity of ρ it then follows that $\square_{j=1}^{n}\rho(X)$ is decreasing in $n > 0$. Hence, *regardless how much the portfolio has been split it is always optimal to split a bit further.* In fact an optimal structure, corresponding to $\square_{j=1}^{n}\rho(X)$ becoming subadditive (coherent) is achieved for $n \rightarrow \infty$ (Barrieu and El Karoui, 2005).

This particular argument demonstrates some difficulties in the use of convex risk measures in risk management. In the case of distortion-exponential measures, using the coherent measure $\rho_{g,a=0}$ creates a stable portfolio. However the introduction of even a slight dependence on the the scale of losses, by using $\rho_{g,a}$ even with an arbitrarily small a, produces an incentive for infinite fragmentation of portfolios. Note that this still happens when we start with a particular pair of sub-portfolios (e.g. business lines) X_1, X_2 such that $X_1 + X_2 = X$ and $\rho(X) \leq \rho(X_1) + \rho(X_1)$. That is, even if the initial configuration of the portfolio is such that benefits from pooling risks occur, once splitting without any constraints is allowed, fragmentation of the portfolio is inevitable.

One way to address this somewhat counterintuitive situation is to introduce some frictions that would impede the splitting of portfolios. For example, assume that holding each separate portfolio incurs a fixed cost that is equal to c. Then, assuming again the use of a single convex risk measure, the portfolio X will be split in n parts equal to X/n until the cost of maintaining n separate portfolios exceeds the saving in aggregate risk

capital. I.e. n is the largest integer such that

$$(n-1)\rho\left(\frac{X}{n-1}\right) + (n-1)c \geq n\rho\left(\frac{X}{n}\right) + nc \qquad (61)$$

Assuming a small a and using the first 2 terms of the approximation (37) yields

$$n(n-1) \leq \frac{1}{2}\frac{aVar_{\mathbb{Q}}(X)}{c}. \qquad (62)$$

Hence the maximum number of separate portfolios that X is split into is increasing in the risk aversion parameter a and the variability of X as captured by $Var_{\mathbb{Q}}(X)$, while being decreasing in the cost of setting up a new portfolio.

A different way of limiting the fragmentation of portfolios is to consider a number of sub-portfolios $X_1, \ldots, X_n, \sum_{j=1}^{n} X_j = X$ that are impossible to split into smaller parts. In this case the framework is that of *atomic* cooperative games, which is discussed in a cost allocation context by Lemaire (1984) and Denault (2001). Several solution concepts (allocations) can then be obtained, for example by attempting to maximise the savings from cooperation (pooling) that each possible combination of subportfolios makes, as formalised by the *nucleolus* and *least core* of the game (Schmeidler, 1969). Examining such models is beyond the scope of this paper.

5   Conclusion

It was demonstrated that in the case of convex risk measures, capital can be allocated by the Aumann-Shapley value, which is viewed as a generalisation of marginal costs. It was also argued that there exist convex risk measures that are flexible enough to be useful in a risk management context, such as the risk measure $\rho_{g,a}$ studied extensively in this paper.

It was shown that the use of a convex risk measure can give an incentive for the infinite fragmentation of portfolios. This demonstrates that penalising the scale of losses by dropping the homogeneity requirement of coherent risk measures has very strong (possibly undisirable) consequences. Hence the use of convex risk measures in capital allocation can be considered problematic at quite a fundamental level. A possible way to resolve this issue

is to introduce some constraints to the extent that portfolios can be split, e.g. by associating a cost with each separate portfolio. Embedding these arguments to a realistic framework remains a subject for further research.

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