Chover-type laws of the k-iterated logarithm for weighted sums of strongly mixing sequences

Lorenzo Trapani[1]

Cass Business School, City University London, 106 Bunhill Row, London EC1Y 8TZ, UK.

Abstract

This note contains a Chover-type Law of the k-Iterated Logarithm for weighted sums of strong mixing sequences of random variables with a distribution in the domain of a stable law. We show that the upper part of the LIL is similar to other studies in the literature; conversely, the lower half is substantially different. In particular, we show that, due to the failure of the classical version of the second Borel-Cantelli lemma, the upper and the lower bounds are separated, with the lower bound being further and further away as the memory of the sequence increases.

Keywords: Chover Law of the Iterated Logarithm, strongly mixing sequence of random variables, slowly varying function
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1. Introduction

Let $\{X_i, 1 \leq i < \infty\}$ be a sequence of real-valued random variables defined on a probability space (Ω, F, P), and define

$$\alpha(m) = \sup_k \left\{ |P(A \cap B) - P(A) P(B)| \text{ s.t. } \forall k \in \mathbb{N}, A \in F_1^k, \ B \in F_{k+m}^{\infty} \right\}, \tag{1}$$

Email address: L.Trapani@city.ac.uk (Lorenzo Trapani)

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where $F_j^{j+k} = \sigma(X_i, j \leq i \leq j+k)$. The sequence $\{X_i,\, 1 \leq i < \infty\}$ is said to be *strongly mixing* (or *α-mixing*) if $\alpha(m) = 0$ as $m \to \infty$ - we refer to Davidson [4] *inter alia*, for a comprehensive treatment of mixing.

The notion of strong mixing plays an important role in asymptotic theory, also because it has the relative advantage of being weaker than other types of mixing (see e.g. Davidson [4, Ch. 13.6]). In the context of the Law of the Iterated Logarithm (LIL), however, strong mixing poses some problems, essentially due to the difficulty in using the second Borel-Cantelli lemma. Although the classical version of this result requires independence, it is well known that such assumption can be relaxed: e.g. Iosifescu and Theodorescu [5, Lemma 1.1.2] show the validity of the lemma for uniformly mixing sequences. However, in the case of strongly mixing sequences, the second Borel-Cantelli lemma does not follow automatically, and in order to derive it, restrictions have to be imposed on the size of the probabilities (Yoshihara [13]), or on the rate of divergence of the sum of the probabilities (Tasche [10]). Oodaira and Yoshihara [7] and Yoshihara [13] study the LIL for strongly mixing sequences, in the case of random variables with finite second moments.

In this contribution, we derive the Law of the k-Iterated Logarithm for weighted sums of strongly mixing sequences of random variables with infinite second moment. In this respect, the subject of this note belongs in the family of the so-called "Chover-type LIL" - see e.g. Chover [3], Mikosch [6], Vasudeva [12], Cai [1][2], Wu and Jiang [14][15]. One major difference with such contributions, however, is that in our case the upper and the lower halves of the LIL are separated. In particular, the upper half of the LIL is exactly the same as in other studies. On the other hand, the lower half is substantially different: a different norming sequence is required, which is slower the slower the convergence of $\alpha(m)$ to zero as $m \to \infty$. As a consequence, the LIL bifurcates into two separate results, thereby not being a sharp result any more.

Let the common, non-degenerate distribution of the X_is be denoted as F, assumed to be in the domain of attraction of a stable distribution with tail exponent $\gamma \in (0, 2)$, viz.

$$1 - F(x) = \frac{c_1(x) + o(1)}{x^\gamma} L(x) \text{ and } F(-x) = \frac{c_2(x) + o(1)}{x^\gamma} L(x), \qquad (2)$$

as $x \to \infty$. Consider the following assumptions:

Assumption 1 *[distribution]* In (2), it holds that *(i)* $c_i(x) \geq 0$, $i = 1, 2$ with $\lim_{x\to\infty} c_i(x) = c_i$ and $c_1 + c_2 > 0$; *(ii)* the function $L(x) \geq 0$ is slowly varying at infinity in the Karamata sense, i.e.

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1 \text{ for any } x > 0.$$

Assumption 2 *[dependence]* The sequence $\{X_i, 1 \leq i < \infty\}$ is strongly mixing with mixing numbers $\alpha(m) = O\left(m^{-\theta}\right)$ and $\theta > 1$.

By Theorem 1.2 in Seneta [9, p.2], Assumption 1 entails that $L(x)$ has the representation $L(x) = l_1(x) \exp\left\{\int_B^x t^{-1} l_2(t)\, dt\right\}$, for some $B > 0$, $l_1(x) \geq 0$, $\lim_{x\to\infty} l_1(x) = l_1 > 0$, and $\lim_{x\to\infty} l_2(x) = 0$.

In Assumption 2, a polynomial rate of decay is assumed for $\alpha(m)$. Alternatively, an exponential rate of decay can be assumed - this is in the next assumption, which constitutes an alternative to Assumption 2.

Assumption 2'. *[exponential mixing rate]* The sequence $\{X_i, 1 \leq i < \infty\}$ is strongly mixing with mixing numbers $\alpha(m) = O(b^m)$ for some $b \in (0, 1)$.

The remainder of the paper is organised as follows. The main results, namely the upper and lower bounds for the LIL, are presented in Section 2. In Section 4, we report all the technical lemmas that are required for the proofs of the two theorems in Section 2.

2. Main results

Let $S_n = \sum_{i=1}^n h\left(\frac{i}{n}\right) X_i$, where $h(\cdot)$ is a nontrivial, pointwise continuous function of bounded variation on $[0, 1]$. We present the upper and lower halves of the LIL as two separate theorems. Let $\ln_k x$ denote the k-iterated logarithm of x (e.g. $\ln_2 x = \ln \ln x$) truncated at zero, and define the product $\prod_{p=1}^{j}(\cdot)$ as 1 for $j = 0$.

It holds that

Theorem 2.1. *Let Assumptions 1 and 2 hold. For any $\varepsilon > 0$ and any $j \in \mathbb{N} \cup \{0\}$*

$$\limsup_{n\to\infty} \left(\frac{|S_n - a_n|}{b_n}\right)^{1/\ln_{j+2} n} \leq e^{\frac{1+2\varepsilon}{\gamma}} \text{ a.s.}$$

where $a_n = \sum_{i=1}^{n} E\left[h\left(\frac{i}{n}\right)X_i\right]$ *when* $\gamma > 1$ *and zero otherwise, and* $b_n = n^{\frac{1}{\gamma}}$ $\left[\left(\prod_{p=1}^{j}\ln_p n\right)\ln_{j+1}^{1+\varepsilon} n\right]^{\frac{1}{\gamma}} L_1(n)$, *with* $L_1(n)$ *slowly varying at infinity in the Karamata sense. Under Assumptions 1 and 2', the same result holds.*

Theorem 2.2. *Let Assumptions 1 and 2 hold, and recall that* θ *is the size of the mixing numbers* $\alpha(m)$*. For any* $\varepsilon \in \left(0, \frac{1}{2}\right)$ *and any* $j \in \mathbb{N} \cup \{0\}$

$$\limsup_{n\to\infty}\left(\frac{|S_n - a_n|}{b_n^{\theta}}\right)^{1/\ln_{j+2} n} \geq e^{\frac{1-2\varepsilon}{\gamma}} \quad a.s.$$

where a_n *is defined in Theorem 2.1 and* $b_n^{\theta} = n^{\frac{\theta}{1+\theta}\frac{1}{\gamma}}\left[\left(\prod_{p=1}^{j}\ln_p n\right)\ln_{j+1}^{1-\varepsilon} n\right]^{\frac{1}{\gamma}}$ $L_2(n)$, *with* $L_2(n)$ *slowly varying at infinity in the Karamata sense. Under Assumptions 1 and 2', the same result holds with* $b_n^{\theta} = \left(\frac{n}{\ln n}\right)^{\frac{1}{\gamma}}\left[\left(\prod_{p=1}^{j}\ln_p n\right)\ln_{j+1}^{1-\varepsilon} n\right]^{\frac{1}{\gamma}}$ $L_2(n)$.

Remark 2.1. *Theorem 2.1 contains the upper half of the Law of the k-Iterated Logarithm. As can be seen comparing the result with e.g. Corollary 2.4 in Cai [1], this is the "classical" upper bound for the LIL, which is typically found in the case of independent data. The only difference, in the proof, is the maximal inequality employed here.*

Remark 2.2. *Theorem 2.2 contains the lower bound of the LIL. The norming sequence* b_n^{θ} *is different than in Theorem 2.1, and of lower magnitude. In particular, as mentioned in the Introduction,* b_n^{θ} *depends on the size of the mixing coefficients,* θ*: the higher the dependence (and, therefore, the lower* θ*), the smaller the norming sequence, and vice versa. This is not the case under Assumption 2': the norming sequence* b_n^{θ} *does not depend on* b *in case of exponential mixing.*

As mentioned in the Introduction, the Law of the Iterated Logarithm for strongly mixing sequences has been shown by Oodaira and Yoshihara [7] and Yoshihara [13]. These studies require the existence of second moments, and are based on a different assumption on the mixing numbers - e.g. in Oodaira and Yoshihara [7, Theorem 5], it is required that $E|X_1|^{2+\delta} < \infty$ and $\sum_{m=1}^{\infty}\alpha^{\frac{\delta'}{2+\delta'}}(m) < \infty$ for $0 < \delta' < \delta$. Although "classical" studies in the field of Chover-type laws usually require the tail index γ to be constrained to be stricly smaller than 2, the method of proof employed in here can be readily generalised to the case of $\gamma = 2$.

Lemma 2.1. *Let Assumptions 1 and 2 hold, with $\gamma = 2$. Then Theorems 2.1 and 2.2 hold; the same result holds under Assumptions 1 and 2'.*

Remark 2.3. *The Lemma follows immediately from the proofs of the main results and it is therefore reported without proof. Note that, in this case, the norming sequence for the upper bound is the familiar one that is typically found in Hartman-Wintner-type Laws of the Iterated Logarithm, i.e.* $b_n = \sqrt{n \left(\prod_{p=1}^{j} \ln_p n \right) \ln_{j+1}^{1+\varepsilon} n \; L_1(n)}$*; on the other hand, given Assumptions 2 or 2', there is the usual gap between upper and lower bounds.*

Remark 2.4. *The Lemma is valid even in the case of $\gamma > 2$. In such case, the proofs of Lemmas 4.1-4.5 can (again) be readily extended. As can be expected, the norming sequence for the upper bound cannot be faster than* $b_n = \sqrt{n \left(\prod_{p=1}^{j} \ln_p n \right) \ln_{j+1}^{1+\varepsilon} n \; L_1(n)}$*, otherwise Lemma 4.3 does not hold - in particular, term II_a in equation (7) diverges.*

3. Conclusions

This note studies the Law of the k-Iterated Logarithm for strongly mixing sequences. Most of the attention is devoted to the case of random variables with common distribution in the domain of attraction of the normal distribution and tail index $\gamma \in (0, 2)$; however, proofs and results can be readily extended e.g. to the case of finite second moment, i.e. $\gamma = 2$. The Law of the Iterated Logarithm (LIL) is a delicate result, in that it is, usually, an equality. In addition to being of interest per se, the LIL finds natural applications when *exact* rates of convergence are required - an example, under investigation of the author, is the case of a randomised test for the finiteness of the k-th absolute moment of a random variable (see Trapani [11]). In general, when exact rates are required, the LILs derived in this paper can be of help, e.g. to strengthen the results given by a Law of Large Numbers or by a Central Limit Theorem. More generally, derivations, in statistics, often involve the use of inequalities - heuristically, there is a wide amount of such inequalities providing upper bounds, whereas lower bounds are relatively less explored; results like Theorem 2.2 bridge this gap.

The results in this paper have one interesting theoretical implication: when considering strongly mixing sequences, the upper and lower halves of the LIL are not the same, and therefore the LIL is no longer a equality. We show in the paper that this is due to the failure of the (standard) formulation of the second Borel-Cantelli Lemma: the upper bound of the LIL is the same as when having an independent sequence, but the lower bound, on the other hand, is not. Specifically, we find that in order for the partial sums to be bounded from below, the norming constant has to be of smaller order of magnitude (and, therefore, it is slower) than e.g. in the case of independence. Despite such negative feature, this paper provides almost sure upper and lower bounds for the (weighted) partial sums of a sequence of strongly mixing random variables.

4. Lemmas and proofs

Henceforth, we denote with $f(x) > 0$ a function such that

$$\limsup_{x \to \infty} \sup_{0 \le t \le x} \frac{f(t)}{f(x)} < \infty. \tag{3}$$

Further, we define the truncated sequence $X_i^{(n)} = X_i I\left[|X_i| \le b_n\right]$, and the corresponding partial sums $S_i^{(n)} = \sum_{j=1}^{i} \left[h\left(\frac{j}{n}\right) X_j^{(n)} - Eh\left(\frac{j}{n}\right) X_j^{(n)}\right]$. The proof of Theorem 2.1 requires some preliminary lemmas, some of which are similar to those found in other studies (e.g. Cai [1]).

Lemma 4.1. *Let Assumption 2 hold. Then, for any sequence* r_n

$$P\left[\max_{1 \le i \le n} \left|S_i^{(n)}\right| \ge x r_n\right] \le \frac{8n}{x^2 r_n^2} \left[\sup_{0 \le z \le 1} |h(z)|^2\right] E\left|X_1^{(n)}\right|^2 + \frac{32(n-1)}{x} \alpha(n). \tag{4}$$

The same result holds for Assumption 2'.

Proof. The lemma is an application of Theorem 4 and Lemma 1 in Rio [8]. Indeed, Theorem 4 in Rio [8] stipulates that

$$P\left[\max_{1 \le i \le n} \left|S_i^{(n)}\right| \ge 2x\right] \le \frac{2}{x^2} \sum_{i=1}^{n} E\left|h\left(\frac{i}{n}\right) X_i^{(n)}\right|^2 + \frac{4}{x} \sum_{i=1}^{n-1} \delta_{i,1}(n), \tag{5}$$

where $\delta_{i,1}(n) \le 2\int_0^{2\alpha(n)} Q_{i+1}(u)\, du$, where $Q_{i+1}(u)$ is the inverse of $P\left[\left|X_{i+1}^{(n)}\right| > u\right]$. This result holds for any weighted sequence $h\left(\frac{i}{n}\right) X_i$, since transformations of mixing sequences are mixing of the same size (e.g. Theorem 14.1 in Davidson [4]). Since the X_is have identical distribution, $E\left|X_i^{(n)}\right|^2 = E\left|X_1^{(n)}\right|^2$ and we write $Q_{i+1}(u) = Q_1(u)$ for each i. Also, by the boundedness of $X_i^{(n)}$, $\int_0^{2\alpha(n)} Q_1(u)\,du \leq 2\alpha(n)\,r_n$, so that $\sum_{i=1}^{n-1} \delta_{i,1}(n) \leq 4(n-1)\,\alpha(n)\,r_n$. Substituting these calculations in (5), and noting that $h(\cdot)$ is bounded, yields (4). $\square$

Lemma 4.2. *Let Assumption 1 hold, and let $f(x)$ be a function with property (3) such that $\int_1^\infty [xf(x)]^{-1}\,dx < \infty$ and define $c_n = [nf(n)]^{\frac{1}{\gamma}}$. Then, as $n \to \infty$*

$$c_n^{-1} \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Eh\left(\frac{j}{n}\right) X_j^{(n)} \right| \to 0.$$

Proof. The proof is the same as for equation (2.5) in Cai [1], where it is shown that

$$c_n^{-1} \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Eh\left(\frac{j}{n}\right) X_j^{(n)} \right| < \frac{\varepsilon}{2}, \tag{6}$$

for $\varepsilon > 0$. $\square$

Lemma 4.3. *Let Assumption 2 hold. Then, under the conditions of Lemma 4.2, for any $\varepsilon > 0$ it holds that*

$$\sum_{n=1}^{\infty} n^{-1} P\left[\max_{1\leq j\leq n} |S_j| > \varepsilon c_n\right] < \infty.$$

The same result holds under Assumption 2'.

Proof. It holds that

$$\begin{aligned} P\left[\max_{1\leq j\leq n} |S_j| > \varepsilon c_n\right] \;\leq\;& P\left[\max_{1\leq j\leq n} |X_j| > c_n\right] \\ &+ P\left[\max_{1\leq j\leq n} \left|S_j^{(n)}\right| > \varepsilon c_n - \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} Eh\left(\frac{j}{n}\right) X_j^{(n)} \right|\right]. \end{aligned}$$

Using Lemma 4.2, and specifically equation (6), the expression above can be rewritten as

$$P\left[\max_{1\leq j\leq n} |S_j| > \varepsilon c_n\right] \leq \sum_{j=1}^{n} P\left[|X_j| > c_n\right] + P\left[\max_{1\leq j\leq n} \left|S_j^{(n)}\right| > \frac{\varepsilon}{2} c_n\right].$$

Therefore, in order to prove the lemma, we need to show that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P\left[|X_j| > c_n\right] < \infty, \\ II &= \sum_{n=1}^{\infty} n^{-1} P\left[\max_{1\leq j\leq n} \left|S_j^{(n)}\right| > \frac{\varepsilon}{2} c_n\right] < \infty. \end{aligned}$$

Consider I; we have $I = \sum_{n=1}^{\infty} P\left[|X_n| > c_n\right]$; thus, by Assumption 1, $I \leq C'$ $\sum_{n=1}^{\infty} c_n^{-\gamma} L(n) \leq C \int_1^{\infty} \left[x f(x) L(x)\right]^{-1} dx$. Using the properties of regular variation (see Seneta [9]; in particular, Proposition 3 on p. 18 and Theorem 2.6 on p. 65), this integral is equivalent to $L^{-1}(1) \int_1^{\infty} \left[x f(x)\right]^{-1} dx$, which is finite by assumption, so that $I < \infty$. As far as II is concerned, using Lemma 4.1 we have

$$II \leq \frac{32}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{n}{n c_n^2} E\left|X_1^{(n)}\right|^2 + \frac{64}{\varepsilon} \sum_{n=1}^{\infty} \frac{(n-1)}{n} \alpha(n) = II_a + II_b. \tag{7}$$

Clearly, $II_b \leq C \sum_{n=1}^{\infty} \alpha(n) < \infty$ by Assumption 2; this also holds under Assumption 2', and it is the only part of the proof in which the mixing numbers play a role. The proof that $II_a < \infty$ is in equation (2.13) in Cai [1] - see also the proof of Theorem 1(ii) in Wu ad Jiang [14]. Thus, $II < \infty$, which proves the lemma. □

Lemma 4.4. *Under the conditions of Lemma 4.3 it holds that*

$$\limsup_{n\to\infty} \frac{|S_n - a_n|}{c_n} = 0 \ a.s.$$

Proof. The proof is similar to that in the proof of Corollary 2.4 in Cai [1]. In particular, note that for any n there is a $k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$; further, there exists some $t \in [0, 1)$ it holds that $n = 2^{k+t}$. Using the short-hand notation $P_k \equiv P\left[\max_{1\leq i\leq 2^{k+t}} |S_i| > \varepsilon\left(2^{k+1} f\left(2^{k+t}\right) L\left(2^{k+t}\right)\right)^{\frac{1}{\gamma}}\right]$ we have

$$\sum_{k=0}^{\infty} P_k \leq 2 \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{2^{k+1}-1} P_k \leq 2 \sum_{n=1}^{\infty} \frac{1}{n} P\left[\max_{1\leq j\leq n} |S_j| > \varepsilon c_n\right] < \infty,$$

by Lemma 4.3. Therefore, as $n \to \infty$

$$\frac{\max_{1\leq i\leq 2^{k+t}} |S_i|}{\left[2^{k+1} f\left(2^{k+t}\right) L\left(2^{k+t}\right)\right]^{\frac{1}{\gamma}}} = 0 \text{ a.s.},$$

which entails

$$\frac{|S_n|}{[nf(n)]^{\frac{1}{\gamma}}} \leq \frac{\max_{1\leq i\leq 2^{k+t}}|S_i|}{\left[2^{k+1}f\left(2^{k+t}\right)L\left(2^{k+t}\right)\right]^{\frac{1}{\gamma}}}\frac{\left[2^{k+1}f\left(2^{k+t}\right)L\left(2^{k+t}\right)\right]^{\frac{1}{\gamma}}}{[nf(n)]^{\frac{1}{\gamma}}} \leq C\frac{\max_{1\leq i\leq 2^{k+t}}|S_i|}{\left[2^{k+1}f\left(2^{k+t}\right)L\left(2^{k+t}\right)\right]^{\frac{1}{\gamma}}} = 0 \text{ a.s.},$$

so that finally

$$\limsup_{n\to\infty}\frac{|S_n|}{[nf(n)]^{\frac{1}{\gamma}}} = 0 \text{ a.s.}$$

□

Proof of Theorem 2.1. The Theorem follows immediately from Lemma 4.4, upon setting $f(x) = L_1(x)\left(\prod_{p=1}^{j}\ln_p x\right)\ln_{j+1}^{1+\varepsilon}x$, and using equation (2.11) in Wu and Jiang [14]. □

We now prove Theorem 2.2. Define

$$\bar{\alpha}(m) = \sup_k\left\{|P(A\cap B) - P(A)P(B)| \text{ s.t. } \forall k\in\mathbb{N},\ A\in F_1^k,\ B\in F_{k+m}^{k+m}\right\}. \tag{8}$$

Clearly, $\bar{\alpha}(m) \leq \alpha(m)$.

Henceforth, we use the short-hand notation "i.o." for "infinitely often".

Lemma 4.5. *Let Assumptions 1 and 2 hold, and define the sequence* $c_n^{\theta} = n^{\frac{\theta}{1+\theta}\frac{1}{\gamma}}\varphi(n)^{\frac{1}{\gamma}}$, *with* $\int_1^{\infty}[x\varphi(x)]^{-1}dx = \infty$. *Then*

$$P\left[|X_1| > c_n^{\theta} \text{ i.o.}\right] = 1. \tag{9}$$

Under Assumptions 1 and 2', (9) holds for $c_n^{\theta} = \left(\frac{n}{\ln n}\right)^{\frac{1}{\gamma}}\varphi(n)^{\frac{1}{\gamma}}$.

Proof. We report the proof in detail under Assumption 2, and, at the end, sketch the proof under Assumption 2'. Let $r = \theta - 1$; clearly, by Assumption 2, $r\in[-1,+\infty)$. The lemma is based on applying Theorem 2.2 in Tasche [10]; essentially the same results could be obtained using Theorem 1 in Yoshihara [13]. Indeed, the result in the lemma requires two sufficient conditions. Firstly, it is required that $\sum_{n=1}^{\infty}n^r\bar{\alpha}_n < \infty$; but since $\sum_{n=1}^{\infty}n^r\bar{\alpha}_n \leq \sum_{n=1}^{\infty}n^r\alpha_n < \infty$ on account of Assumption 2, this condition holds. Secondly, it is required that $\sum_{n=1}^{\infty}n^{-\frac{1}{2+r}}P\left[|X_1| > c_n^{\theta}\right] = \infty$. Indeed

$$P\left[|X_1| > c_n^{\theta}\right] = C\left(c_n^{\theta}\right)^{-\gamma}L\left(c_n^{\theta}\right) \sim Cn^{-\frac{\theta}{1+\theta}}\varphi^{-1}(n),$$

for some $C > 0$ in view of Assumption 1 and of the properties of regular variations (see Seneta [9]). Thus

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2+r}} P\left[|X_1| > c_n^{\theta}\right]$$
$$\sim \quad C \sum_{n=1}^{\infty} n^{-\frac{1}{1+\theta}} n^{-\frac{\theta}{1+\theta}} \varphi^{-1}(n) = C \sum_{n=1}^{\infty} \frac{1}{n\varphi(n)},$$

which diverges by virtue of the integral test $\int_1^{\infty} [x\varphi(x)]^{-1} dx = \infty$. This proves (9) under Assumption 2. If Assumption 2' holds, Theorem 2.2 in Tasche [10] can still be applied as long as $\sum_{n=1}^{\infty} \frac{P\left[|X_1|>c_n^{\theta}\right]}{\ln n} = \infty$. Indeed, since in this case

$$P\left[|X_1| > c_n^{\theta}\right] = C\left(c_n^{\theta}\right)^{-\gamma} L\left(c_n^{\theta}\right) \sim C\frac{\ln n}{n}\varphi^{-1}(n),$$

we have

$$\sum_{n=1}^{\infty} \frac{P\left[|X_1| > c_n^{\theta}\right]}{\ln n} \sim C\sum_{n=1}^{\infty} \frac{\ln n}{n \ln n}\varphi^{-1}(n) = C\sum_{n=1}^{\infty} \frac{1}{n\varphi(n)},$$

which gives the desired result. Henceforth, the proof is the same as for the case of Assumption 2. □

Proof of Theorem 2.2. Standard calculations yield that Lemma 4.5 holds when $\varphi(n) = \left(\prod_{p=1}^{j} \ln_p n\right) \ln_{j+1}^{1-\varepsilon} n$. Hence

$$P\left[|X_1| > n^{\frac{\theta}{1+\theta}\frac{1}{\gamma}} \varphi(n)^{\frac{1}{\gamma}} \text{ i.o.}\right] = 1; \tag{10}$$

by virtue of Davidson [4, Theorem 14.1], this result holds for any transformation of the type $X_i \longmapsto h\left(\frac{i}{n}\right) X_i$. This entails that

$$\limsup_{n\to\infty} \frac{|S_n - a_n'|}{\left[n^{\frac{\theta}{1+\theta}}\varphi(n)\right]^{\frac{1}{\gamma}}} = +\infty \text{ a.s.}, \tag{11}$$

for every sequence a_n'. This can be shown by contradiction: let $b_n^{\theta,\varepsilon} = \left[n^{\frac{\theta}{1+\theta}}\varphi(n)\right]^{\frac{1}{\gamma}}$ and suppose

$$\limsup_{n\to\infty} \frac{|S_n - a_n'|}{\left[n^{\frac{\theta}{1+\theta}}\varphi(n)\right]^{\frac{1}{\gamma}}} = d_0 \text{ a.s.}, \tag{12}$$

for some $d_0 < \infty$. We have

$$\begin{aligned}
\frac{\left|X_n - \left(a'_n - a'_{n-1}\right)\right|}{b_n^{\theta,\varepsilon}} &= \frac{\left|S_n - S_{n-1} - \left(a'_n - a'_{n-1}\right)\right|}{b_n^{\theta,\varepsilon}} \frac{1}{|h(1)|} \\
&\leq C\left[\frac{|S_n - a'_n|}{b_n^{\theta,\varepsilon}} + \frac{\left|S_{n-1} - a'_{n-1}\right|}{b_{n-1}^{\theta,\varepsilon}} \frac{b_{n-1}^{\theta,\varepsilon}}{b_n^{\theta,\varepsilon}}\right] \\
&\leq C\left[\frac{|S_n - a'_n|}{b_n^{\theta,\varepsilon}} + \frac{\left|S_{n-1} - a'_{n-1}\right|}{b_{n-1}^{\theta,\varepsilon}}\right],
\end{aligned}$$

where the last passage holds on account of $b_n^{\theta,\varepsilon}$ being a non-decreasing sequence. By (12), this entails that

$$\limsup_{n\to\infty} \frac{|X_n|}{b_n^{\theta,\varepsilon}} \leq 2d_0 + \limsup_{n\to\infty} \frac{\left|\left(a'_n - a'_{n-1}\right)\right|}{b_n^{\theta,\varepsilon}} < \infty \text{ a.s.;}$$

due to $\frac{X_n}{b_n^{\theta,\varepsilon}} \stackrel{p}{\to} 0$, we have $\limsup_{n\to\infty} \frac{\left|\left(a'_n - a'_{n-1}\right)\right|}{b_n^{\theta,\varepsilon}} = 0$ a.s., which entails $P\left[|X_1| > n^{\frac{\theta}{1+\theta}\frac{1}{\gamma}} \varphi(n)^{\frac{1}{\gamma}} \text{ i.o.}\right] = 0$. But this contradicts (10), so that (11) is the correct statement.

Now, using Lemma 3 in Wu and Jiang [14] we have that, for some arbitrary κ

$$\begin{aligned}
\frac{|S_n - a_n|}{\left[n^{\frac{\theta}{1+\theta}} \prod_{p=1}^{j} \ln_p n\right]^{\frac{1}{\gamma}}} &= \frac{|S_n - a_n|}{\left[n^{\frac{\theta}{1+\theta}} \varphi(n)\right]^{\frac{1}{\gamma}}} \frac{\left[n^{\frac{\theta}{1+\theta}} \varphi(n)\right]^{\frac{1}{\gamma}}}{\left[n^{\frac{\theta}{1+\theta}} \prod_{p=1}^{j} \ln_p n\right]^{\frac{1}{\gamma}}} \\
&\geq \left(\ln_{j+1} n\right)^{\frac{1-\varepsilon}{\gamma}} \left(\ln_{j+1} n\right)^{-(1-\varepsilon)\kappa};
\end{aligned}$$

setting $\kappa = \frac{\varepsilon}{1-\varepsilon}\gamma$, we finally have

$$\frac{|S_n - a_n|}{\left[n^{\frac{\theta}{1+\theta}} \prod_{p=1}^{j} \ln_p n\right]^{\frac{1}{\gamma}}} = \left(\ln_{j+1} n\right)^{\frac{1-2\varepsilon}{\gamma}},$$

whence the statement of the theorem. The proof under Assumption 2' follows exactly the same passages and thus it is omitted. □

The proof of Lemma 2.1 is a straightforward generalisation of the results above, and it is left as an exercise.

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