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A new non-Hermitian E2-quasi-exactly solvable model

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ABSTRACT: We construct a previously unknown E_2-quasi-exactly solvable non-Hermitian model whose eigenfunctions involve weakly orthogonal polynomials obeying three-term recurrence relations that factorize beyond the quantization level. The model becomes Hermitian when one of its two parameters is fixed to a specific value. We analyze the double scaling limit of this model leading to the complex Mathieu equation. The norms, Stieltjes measures and moment functionals are evaluated for some concrete values of one of the two parameters.

In [1] we introduced E_2-quasi-exactly solvable models in analogy to the notion of $sl_2(\mathbb{C})$-quasi-exactly solvability originally proposed by Turbiner [2, 3]. The different setting is motivated mathematically by the fact that solutions for E_2-quasi-exactly solvable models do not belong to the general class of hypergeometric functions which emerge as solutions from an $sl_2(\mathbb{C})$-setting. The physical motivation results from the current interest in extending the study of solvable models [4, 5, 6, 7, 1] to non-Hermitian quantum mechanical systems [8, 9, 10, 11]. The E_2-quasi-exactly solvable models are especially interesting in optical settings [12, 13, 14, 15, 16, 17, 18, 19, 20] where the fact is exploited that the Helmholtz equation results as a reduction from the Schrödinger equation. Solvable models are rare exceptions in the study of quantum mechanical systems and the model presented here should be added to that list.

The starting point for the construction of E_2-quasi-exactly systems consists of expressing the Hamiltonian operator $\mathcal{H}$ of the model in terms of the E_2-basis operators u, v and J obeying the commutation relations

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad [u, v] = 0. \tag{1}$$

instead of the standard $sl_2(\mathbb{C})$-generators. We now use the particular realization [21]

$$J := -i\partial_\theta, \quad u := \sin\theta, \quad v := \cos\theta, \tag{2}$$

and demand a specific anti-linear symmetry [22], $\mathcal{PT}_3 : J \to J, u \to v, v \to u, i \to -i$, as defined in [18], which for (2) becomes $\mathcal{PT}_3 : \theta \to \pi/2 - \theta, i \to -i$. The operators in (2)

act on the $\mathcal{PT}_3$-invariant vector spaces over $\mathbb{R}$

$$V_n^s(\phi_0) := \operatorname{span}\left\{\phi_0\left[\sin(2\theta), i\sin(4\theta), \ldots, i^{n+1}\sin(2n\theta)\right]\big|\,\theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L\right\}, \quad (3)$$

$$V_n^c(\phi_0) := \operatorname{span}\left\{\phi_0\left[1, i\cos(2\theta), \ldots, i^n\cos(2n\theta)\right]\big|\,\theta \in \mathbb{R}, \mathcal{PT}_3(\phi_0) = \phi_0 \in L\right\}. \quad (4)$$

Taking the groundstate eigenfunction to be $\phi_0^c = e^{i\kappa\cos 2\theta}$ with $\kappa \in \mathbb{R}$ we identified in [1] the following actions of combinations of the basis operators $J : V_n^{s,c}(\phi_0^c) \mapsto V_{n+1}^{c,s}(\phi_0^c)$, $uv : V_n^{s,c}(\phi_0^c) \mapsto V_{n+1}^{c,s}(\phi_0^c)$ and $i(u^2 - v^2) : V_n^{s,c}(\phi_0^c) \mapsto V_{n+1}^{s,c}(\phi_0^c)$. Quasi-exact solvability is achieved if we are able to impose suitable constraints such that $\mathcal{H}_N : V_{\tilde{n}}^{s,c}(\phi_0^c) \mapsto V_{\tilde{n}}^{s,c}(\phi_0^c)$ for specific values $\tilde{n}$. It remains a challenge to construct new models of E_2-type that satisfy the latter condition.

We introduce here the $\mathcal{PT}_3$-symmetric Hamiltonian

$$\mathcal{H}_N = J^2 + \zeta uvJ + 2i\zeta N(u^2 - v^2), \qquad \zeta, N \in \mathbb{R}, \quad (5)$$

and demonstrate explicitly that it is E_2-quasi-exactly solvable. We notice from the above that $\mathcal{H}_N : V_n^{s,c}(\phi_0^c) \mapsto V_{n+2}^{s,c}(\phi_0^c) \oplus \zeta V_{n+2}^{s,c}(\phi_0^c) \oplus V_{n+1}^{s,c}(\phi_0^c)$, such that it appears to be possible to reduce the order of the target space by imposing two additional constraints. In general, the Hamiltonian $\mathcal{H}_N$ is non-Hermitian except for $N = 1/4$, with free $\zeta \in \mathbb{R}$, which we deduce from the fact that $\mathcal{H}_N^\dagger = \mathcal{H}_{1/2-N}$. A further interesting feature of this Hamiltonian is that it reduces to the complex Mathieu Hamiltonian in the double scaling limit $\lim_{N\to\infty, \zeta\to 0}\mathcal{H}_N = \mathcal{H}_{\text{Mat}} = J^2 + 2ig(u^2 - v^2)$ for $g := N\zeta < \infty$ similarly as the Hamiltonian discussed in [1, 7].

According to (3) and (4) we make the Ansatz

$$\psi_N^c(\theta) = e^{\frac{i}{4}\zeta\cos(2\theta)}\sum_{n=0}^{\infty} i^n \frac{P_n}{\zeta^n N(1+2N)_{n-1}}\cos(2n\theta), \quad (6)$$

$$\psi_N^s(\theta) = ie^{\frac{i}{4}\zeta\cos(2\theta)}\sum_{n=1}^{\infty} i^n \frac{Q_n}{\zeta^n N(1+2N)_{n-1}}\sin(2n\theta), \quad (7)$$

for our eigenfunctions with P_n, Q_n being polynomials to be determined and $(a)_n := \Gamma(a+n)/\Gamma(a)$ denoting the Pochhammer symbol. The denominators have been extracted in such a way that P_n and Q_n become n-th order polynomials in E when ψ_N is substituted into Schrödinger equation $\mathcal{H}_N\psi_N = E\psi_N$. In this way we obtain the three-term recurrence relations

$$P_1 = EP_0, \quad (8)$$

$$P_{n+1} = 2(E - 4n^2)P_n + \zeta^2\left[4N^2 - 2N - n(n-1)\right]P_{n-1}, \quad \text{for } n = 1, 2, 3, \ldots \quad (9)$$

$$Q_2 = 2(E - 4)Q_1, \quad (10)$$

$$Q_{n+1} = 2(E - 4n^2)Q_n + \zeta^2\left[4N^2 - 2N - n(n-1)\right]Q_{n-1}, \quad \text{for } n = 2, 3, 4, \ldots \quad (11)$$

These equations may be solved in general as outlined in [1]. Taking $P_0 = 1$ we obtain for

the lowest orders

$$P_1 = E, \tag{12}$$
$$P_2 = 2E^2 - 8E + 2\zeta^2 N(2N-1),$$
$$P_3 = 4E^3 - 80E^2 + E\left(2\zeta^2\left(6N^2 - 3N - 1\right) + 256\right) + 64\zeta^2(1-2N)N,$$
$$P_4 = 8E^4 - 448E^3 + E^2\left[16\zeta^2(N-1)(2N+1) + 6272\right] - 192E\left[\zeta^2\left(6N^2 - 3N - 1\right) + 96\right]$$
$$+ 4\zeta^2 N(2N-1)\left[\zeta^2(N+1)(2N-3) + 1152\right].$$

Likewise with $Q_1 = 1$ we compute

$$Q_2 = 2E - 8, \tag{13}$$
$$Q_3 = 4E^2 - 80E + 2\zeta^2(N-1)(2N+1) + 256,$$
$$Q_4 = 8E^3 - 448E^2 + 8E\left[\zeta^2(2N^2 - N - 2) + 784\right] - 32\left[\zeta^2(10N^2 - 5N - 6) + 576\right],$$
$$Q_5 = 16E^4 - 1920E^3 + 8E^2\left[\zeta^2\left(6N^2 - 3N - 10\right) + 8736\right]$$
$$- 32E\left[\zeta^2(94N^2 - 47N - 106) + 26240\right]$$
$$+ 4\left[\zeta^4(4N^4 - 4N^3 - 13N^2 + 7N + 6) + 128\zeta^2(82N^2 - 41N - 54) + 589824\right].$$

We observe the typical feature for quasi-exactly solvable systems that the three term relation can be reset to a two-term relation at a certain level. This is due to the fact that in (9) and (11) the last term vanishes when $n = 2N$. Thus when taking N to be a half-integer, $N \in \mathbb{N}/2$, we find the typical factorization

$$P_{2N+n} = R_n P_{2N} \qquad \text{and} \qquad Q_{2N+n} = R_n Q_{2N}. \tag{14}$$

The first solutions for the factor R_n are

$$R_1 = 2E - 32N^2, \tag{15}$$
$$R_2 = 4E^2 - 16E\left(8N^2 + 4N + 1\right) + 4N\left[64N(2N+1)^2 - \zeta^2\right]. \tag{16}$$

Thus our polynomials P_n and Q_n possess the standard properties of Bender-Dunne polynomials [23].

Let us now determine the energy eigenvalues E_{2N} from the conditions $P_{2N}(E) = 0$ and $Q_{2N}(E) = 0$ for the lowest values of N. For the solutions related to (6) we compute

$$E_1^c = 0, \tag{17}$$
$$E_2^{c,\pm} = 2 \pm \sqrt{4 - \zeta^2}, \tag{18}$$
$$E_3^{c,\ell} = \frac{20}{3} + \frac{2\hat{\Omega}}{3}e^{\frac{i\pi\ell}{3}} - \frac{2}{3}\left(3\zeta^2 - 52\right)e^{-\frac{i\pi\ell}{3}}\hat{\Omega}^{-1}, \tag{19}$$

with $\ell = 0, \pm 2$ and $\hat{\Omega} := \left[280 + 36\zeta^2 + 3^{3/2}\sqrt{\zeta^6 - 4\zeta^4 + 1648\zeta^2 - 2304}\right]^{1/3}$ etc. and for the solutions related to (7) we obtain

$$E_2^s = 4, \tag{20}$$
$$E_3^{s,\pm} = 10 \pm \sqrt{36 - \zeta^2}, \tag{21}$$
$$E_4^{s,\ell} = \frac{56}{3} + \frac{2}{3}e^{\frac{i\pi\ell}{3}}\Omega - \frac{2}{3}e^{-\frac{i\pi\ell}{3}}\Omega^{-1}\left(3\zeta^2 - 196\right), \tag{22}$$

with $\ell = 0, \pm 2$ and $\Omega := \left[1144 + 36\zeta^2 + 3^{3/2}\sqrt{\zeta^6 - 148\zeta^4 + 15856\zeta^2 - 230400}\right]^{1/3}$ etc.

The exceptional points are computed from the real zeros of the discriminants Δ_{2N}^c and Δ_{2N}^s for the polynomials $P_{2N}(E)$ and $Q_{2N}(E)$, respectively,

$$\Delta_2^c = \zeta^2 - 4, \tag{23}$$
$$\Delta_3^s = \zeta^2 - 36,$$
$$\Delta_3^c = \zeta^6 - 4\zeta^4 + 1648\zeta^2 - 2304,$$
$$\Delta_4^s = \zeta^6 - 148\zeta^4 + 15856\zeta^2 - 230400,$$
$$\Delta_4^c = \zeta^{12} + 8\zeta^{10} + 6160\zeta^8 - 2119680\zeta^6 + 4128768\zeta^4 - 749850624\zeta^2 + 530841600,$$
$$\Delta_5^s = \zeta^{12} - 376\zeta^{10} + 2^47041\zeta^8 - 2^{11}11925\zeta^6 + 2^{13}207675\zeta^4 - 2^{12}19579725\zeta^2 + 2^{18}2480625.$$

We have suppressed here overall constant factors that do not contribute to the values of the zeros. Our numerical for the solutions of these equations multiplied by N are presented in table 1.

N	$\zeta_0^c N$	$\zeta_0^s N$	$\zeta_0^c N$	$\zeta_0^s N$	$\zeta_0^c N$
1	2.00000				
3/2	1.77556	9.00000			
2	1.68457	8.21937	21.0567		
5/2	1.63564	7.8691	19.4554	38.2224	
3	1.6047	7.6688	18.6864	35.5683	60.535
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	1.46877	6.92895	16.4711	30.0967	47.806

Table 1: Values of $\zeta_0 N$ computed from the positive real zeros ζ_0 of the discriminant polynomials $\Delta_{2N}^{c,s}$ approaching the critical values of the complex Mathieu equation.

We observe that in the double scaling limit the critical values for the Mathieu equation seem to be approached from above, albeit from further away than in [1]. As also noted in [1] a much better convergence can be obtained when instead of computing successively the exceptional points for each level one takes the limit directly for the three-term recurrence relation. Thus carrying out the limit $N \to \infty$, $\zeta \to 0$ with $g := N\zeta < \infty$ on (8)-(11) with the additional assumption that the coefficient functions remain finite, i.e. $\lim_{N\to\infty,\zeta\to 0} P_n =: P_n^M$ and $\lim_{N\to\infty,\zeta\to 0} Q_n =: Q_n^M$ we obtain

$$P_1^M = EP_0^M \tag{24}$$

$$-2gP_{n-1}^M + 4n^2P_n^M + \frac{1}{2}P_{n+1}^M = EP_n^M, \tag{25}$$

$$-2gQ_{n-1}^M + 4n^2Q_n^M + \frac{1}{2}P_{n+1}^M = EQ_n^M. \tag{26}$$

The recurrence relations may be viewed as two eigenvalue equations for the infinite matrices

Ξ and Θ with entries

$$\Xi_{i,j} = 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1}, \qquad \text{for } i,j \in \mathbb{N}, \tag{27}$$

$$\Theta_{i,j} = 4i^2\delta_{i,j} + \frac{1}{2}\delta_{j,i+1} - 2g^2\delta_{i,j+1} + \frac{1}{2}\delta_{i,0}\delta_{j,1}, \quad \text{for } i,j \in \mathbb{N}_0, \tag{28}$$

acting on the vectors $(Q_1^M, Q_2^M, Q_3^M, \ldots)$ and $(P_0^M, P_1^M, P_1^M, \ldots)$, respectively. The real zeros g_0 of the discriminants $\Delta^\Xi(g)$ and $\Delta^\Theta(g)$ of the characteristic polynomials $\det(\Xi^\ell - E\mathbb{I})$ and $\det(\Theta^\ell - E\mathbb{I})$ for the truncated matrices correspond to the exceptional points. The matrices differ from those reported in [1], denoted here $\Xi^{[1]}$ and $\Theta^{[1]}$, obtained from the double scaling limit analyzed therein. However, as the difference is simply $2g\Xi_{i,i+1} = \Xi_{i,i+1}^{[1]}$, $\Xi_{i+1,i} = \Xi_{i+1,i}^{[1]}2g$ the products $\Xi_{i,i+1}\Xi_{i+1,i} = \Xi_{i,i+1}^{[1]}\Xi_{i+1,i}^{[1]}$ remain invariant and thus the matrices posses the same eigenvalues. We may argue in a similar way for Θ. Thus the critical values are identical to those reported in tables 2 and 3 of [1].

Using the linear functional $\mathcal{L}$ [24, 25] acting on arbitrary polynomials p as

$$\mathcal{L}(p) = \int_{-\infty}^{\infty} p(E)\omega(E)dE, \qquad \mathcal{L}(1) = 1, \tag{29}$$

the standard norm N_n^Φ for the orthogonal polynomials $\Phi_n(E)$ is defined via

$$\mathcal{L}(\Phi_n\Phi_m) = N_n^\Phi \delta_{nm}. \tag{30}$$

The normalization in (29) implies $N_0^P = 1$ and $N_1^Q = 1$. The three-term recurrence relations (8)-(11) together with (30) lead to

$$N_n^P = \mathcal{L}(P_n^2) = \mathcal{L}(EP_{n-1}P_n) = \prod_{k=1}^{n} b_k = \frac{\zeta^{2n}}{2}(1-2N)_n(2N)_n, \qquad n = 2, 3, \ldots \tag{31}$$

$$N_n^Q = \mathcal{L}(Q_n^2) = \mathcal{L}(EQ_{n-1}Q_n) = \prod_{k=2}^{n} b_k = \frac{\zeta^{2n-2}}{2N(1-2N)}(1-2N)_n(2N)_n, \quad n = 2, 3, \ldots \tag{32}$$

with $b_1 = (N - 2N^2)\zeta^2$ and $b_n = [n(n-1) + 2N - 4N^2]\zeta^2$ for $n = 2, 3, \ldots$ Due to the fact that $\mathcal{H}_N$ is non-Hermitian when $N \neq 1/4$ these norms are not positive definite even for non-vanishing polynomials. However, for $N = 1/4$ the expressions reduce to the positive definite norms

$$N_n^P = \frac{\zeta^{2n}}{2\pi}\Gamma^2\left(\frac{1}{2}+n\right), \qquad \text{and} \qquad N_n^Q = 4\frac{\zeta^{2n-2}}{\pi}\Gamma^2\left(\frac{1}{2}+n\right). \tag{33}$$

So far we did not require the explicit expressions for the measure, but as argued in [26] the concrete formulae for $\omega(E)$ may be computed from

$$\omega(E) = \sum_{k=1}^{L}\omega_k\delta(E - E_k), \tag{34}$$

where the energies E_k are the L roots of the polynomial Φ and the L constants ω_k are determined by the L equations

$$\sum_{k=1}^{L}\omega_k\Phi_n(E_k) = \delta_{n0}, \qquad \text{for } n \in \mathbb{N}_0. \tag{35}$$

When $\Phi = P$ we have $L = 2N$ and for $\Phi = Q$ the upper limit is $L = 2N - 1$.

As examples, we solve these equations for the even solutions with $N = 1$ to

$$\omega_\pm^c = \frac{1}{2} \pm \frac{1}{\sqrt{4 - \zeta^2}}, \tag{36}$$

such that

$$N_1^P = \mathcal{L}(P_1^2) = \omega_+^c \left(E_2^{c,-} \right)^2 + \omega_-^c \left(E_2^{c,+} \right)^2 = b_1 = -\zeta^2. \tag{37}$$

Similarly we find for the odd solutions with $N = 3/2$

$$\omega_\pm^s = \frac{1}{2} \pm \frac{3}{\sqrt{36 - \zeta^2}}, \tag{38}$$

such that

$$N_2^Q = \mathcal{L}(Q_2^2) = \omega_+^s \left(E_3^{s,-} \right)^2 + \omega_-^s \left(E_3^{s,+} \right)^2 = b_2 = -4\zeta^2. \tag{39}$$

We also compute the moment functionals defined in [24, 25] as

$$\mu_n := \mathcal{L}(E^n) = \sum_{k=1}^{L} \omega_k E_k^n = \sum_{k=0}^{n-1} \nu_k^{(n)} \mu_k, \tag{40}$$

Once again also these quantities can be obtained in two alternative ways, that is either from the computation of the integrals or directly from the original polynomials P_n and Q_n without the knowledge of the constants ω_k. In the last equation the coefficients $\nu_k^{(n)}$ are defined through the expansion $P_n(E) = 2^{n-1}E^n - \sum_{k=0}^{n-1} \nu_k^{(n)} E^k$ and $Q_n(E) = 2^{n-1}E^{n-1} - \sum_{k=0}^{n-2} \nu_k^{(n)} E^k$ for the even and odd solutions, respectively. For the above even solutions with $N = 1$ we obtain

$$\mu_0^P = 1, \tag{41}$$

$$\mu_1^P = 0 = \nu_0^{(1)} \mu_0^P, \tag{42}$$

$$\mu_2^P = -\zeta^2 = \frac{1}{2} \left(\nu_0^{(2)} \mu_0^P + \nu_1^{(2)} \mu_1^P \right) = -\frac{1}{2} 2\zeta^2, \tag{43}$$

$$\mu_3^P = -4\zeta^2 = \frac{1}{4} \left(64\zeta^2 - 80\zeta^2 \right), \tag{44}$$

$$\mu_4^P = -16\zeta^2 + \zeta^4 = \frac{1}{8} \left(-4608\zeta^2 + 8\zeta^4 + 6272\zeta^2 - 448 \times 4\zeta^2 \right), \tag{45}$$

$$\mu_5^P = -64\zeta^2 + 8\zeta^4, \tag{46}$$

and similarly for the odd solutions with $N = 3/2$ we compute

$$\mu_0^Q = 1, \tag{47}$$

$$\mu_1^Q = 4, \tag{48}$$

$$\mu_2^Q = 16 - \zeta^2, \tag{49}$$

$$\mu_3^Q = 64 - 24\zeta^2, \tag{50}$$

$$\mu_4^Q = 256 - 432\zeta^2 + \zeta^4, \tag{51}$$

$$\mu_5^Q = 1024 - 7168\zeta^2 + 44\zeta^4. \tag{52}$$

Thus we have demonstrated here that the model $\mathcal{H}_N$, as defined in (5), does indeed constitute a quasi-exactly solvable model of E_2-type.

References

[1] A. Fring, E2-quasi-exact solvability for non-Hermitian models, arXiv:1411.4300.

[2] A. V. Turbiner, Quasi-Exactly-Solvable problems and sl(2) Algebra, Commun. Math. Phys. **118**, 467–474 (1988).

[3] A. Turbiner, Lie algebras and linear operators with invariant subspaces, Lie Algebras, Cohomologies and New Findings in Quantum Mechanics, Contemp. Math. AMS, (eds N. Kamran and P.J. Olver) **160**, 263–310 (1994).

[4] A. Khare and B. P. Mandal, A PT-invariant potential with complex QES eigenvalues, Phys. Lett. A **272**, 53–56 (2000).

[5] B. Bagchi, S. Mallik, C. Quesne, and R. Roychoudhury, A PT-symmetric QES partner to the Khare–Mandal potential with real eigenvalues, Phys. Lett. A **289**, 34–38 (2001).

[6] C. M. Bender and M. Monou, New quasi-exactly solvable sextic polynomial potentials, J. Phys. A **38**, 2179–2187 (2005).

[7] B. Bagchi, C. Quesne, and R. Roychoudhury, A complex periodic QES potential and exceptional points, J. Phys. A **41**, 022001 (2008).

[8] F. G. Scholtz, H. B. Geyer, and F. Hahne, Quasi-Hermitian Operators in Quantum Mechanics and the Variational Principle, Ann. Phys. **213**, 74–101 (1992).

[9] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. **80**, 5243–5246 (1998).

[10] C. M. Bender, Making sense of non-Hermitian Hamiltonians, Rept. Prog. Phys. **70**, 947–1018 (2007).

[11] A. Mostafazadeh, Pseudo-Hermitian Representation of Quantum Mechanics, Int. J. Geom. Meth. Mod. Phys. **7**, 1191–1306 (2010).

[12] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Optical Solitons in PT Periodic Potentials, Phys. Rev. Lett. **100**, 030402 (2008).

[13] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, PT-symmetric optical lattices, Phys. Rev. **A81**, 063807(10) (2010).

[14] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. Christodoulides, Observation of PT-Symmetry Breaking in Complex Optical Potentials, Phys. Rev. Lett. **103**, 093902(4) (2009).

[15] B. Midya, B. Roy, and R. Roychoudhury, A note on the PT invariant potential $4\cos^2 x + 4iV_0 \sin 2x$, Phys. Lett. **A374**, 2605–2607 (2010).

[16] H. Jones, Use of equivalent Hermitian Hamiltonian for PT-symmetric sinusoidal optical lattices, J. Phys. **A44**, 345302 (2011).

[17] E. Graefe and H. Jones, PT-symmetric sinusoidal optical lattices at the symmetry-breaking threshold, Phys. Rev. **A84**, 013818(8) (2011).

[18] S. Dey, A. Fring, and T. Mathanaranjan, Non-Hermitian systems of Euclidean Lie algebraic type with real eigenvalue spectra, Annals of Physics **346**, 28–41 (2014).

[19] S. Dey, A. Fring, and T. Mathanaranjan, Spontaneous PT-symmetry breaking for systems of noncommutative Euclidean Lie algebraic type, arXiv:1407.8097, to appear in Int. J. Theor. Phys.

[20] S. Longhi and G. Della Valle, Invisible defects in complex crystals, Annals of Physics **334**, 35–46 (2013).

[21] C. J. Isham and N. Linden, Group theoretic quantisation of strings on tori, Classical and Quantum Gravity **5**, 71–93 (1988).

[22] E. Wigner, Normal form of antiunitary operators, J. Math. Phys. **1**, 409–413 (1960).

[23] C. M. Bender and G. V. Dunne, Quasiexactly solvable systems and orthogonal polynomials, J.Math.Phys. **37**, 6–11 (1996).

[24] J. Favard, Sur les polynomes de Tchebicheff., C. R. Acad. Sci., Paris **200**, 2052–2053 (1935).

[25] F. Finkel, A. Gonzalez-Lopez, and M. A. Rodriguez, Quasiexactly solvable potentials on the line and orthogonal polynomials, J. Math. Phys. **37**, 3954–3972 (1996).

[26] A. Krajewska, A. Ushveridze, and Z. Walczak, Bender–Dunne Orthogonal Polynomials General Theory, Mod. Phys. Lett. A **12**, 1131–1144 (1997).