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$\mathcal{PT}$ invariant complex E_8 root spaces

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ABSTRACT: We provide a construction procedure for complex root spaces invariant under antilinear transformations, which may be applied to any Coxeter group. The procedure is based on the factorisation of a chosen element of the Coxeter group into two factors. Each of the factors constitutes an involution and may therefore be deformed in an antilinear fashion. Having the importance of the E_8-Coxeter group in mind, such as underlying a particular perturbation of the Ising model and the fact that for it no solution could be found previously, we exemplify the procedure for this particular case. As a concrete application of this construction we propose new generalisations of Calogero-Moser-Sutherland models and affine Toda field theories based on the invariant complex root spaces and deformed complex simple roots, respectively.

1. Introduction

It is known for more than twenty years that symmetries based on the E_8-Lie group or E_8-Coxeter (Weyl) group are known to be important in the context of 1+1 dimensional integrable models. In a field theoretical context A.B. Zamolodchikov [1] found in 1989 that the conformal field theory with central charge $c = 1/2$ perturbed by the primary field $\phi_{(1,2)}$ of conformal weight $\Delta = 1/6$ gives rise to an affine Toda field theory with an E_8-mass spectrum. On the lattice side this field theory was identified to correspond to the Ising model in a magnetic field. Remarkably, the first experimental evidence supporting these theoretical findings were reported only very recently in [2].

Furthermore, it is known that the Ising model may be perturbed by a complex field [3, 4] and still describe a meaningful physical system, despite of being related to a non-Hermitian Hamiltonian. This is related to the fact that the non-Hermitian Hamiltonian possess the property of being $\mathcal{PT}$-symmetric in a wider sense, meaning that it remains invariant under a simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$. Strictly speaking the Hamiltonian remains invariant under the more general transformation of an antilinear involutory map of which $\mathcal{PT}$-symmetry is only one example. Then by an observation of Wigner [5], made already fifty years ago, the eigenvalues of the Hamiltonian,

or any other operator with that symmetry property, are guaranteed to be real when in addition also their eigenfunctions possess this symmetry. More recently many new physically meaningful models have been constructed and properties of older models could be explained consistently exploiting this feature, for recent reviews see e.g. [6, 7, 8]. While this type of representation is usually very simple to verify for single particle Hamiltonians it is less obviously identified in multi-particle systems or field theories. Often the symmetry is only evident after a suitable change of variables or even a full separation of variables [9, 10]. Since many of such type of models are formulated generically in terms of root systems, as for instance Calogero-Moser-Sutherland models [11] or Toda field theories [12, 13], with the dynamical variables or fields lying in the dual space, the possibility to deform directly these roots was explored recently [10, 14]. This approach allows to deal with a huge class of models in a very systematic manner as it provides a well defined scheme when based on the roots rather than on a deformation of the canonical variables or fields.

The general logic followed was to identify first an element w in the Weyl (Coxeter) group $w \in \mathcal{W}$ with the involutory property $w^2 = \mathbb{I}$, view it as the analogue of the $\mathcal{P}$-operator and subsequently deform it in an antilinear fashion. The most obvious candidates to take are simple Weyl reflections. However, it was shown in [14] that root spaces with the desired properties based on this identification can only be constructed for groups of rank 2. The explicit solutions for the groups A_2, G_2 and B_2 can be found in [10] and [15], respectively. In [14] we identified the analogue of the $\mathcal{P}$-operator with either of the two factors σ_+ or σ_- of the Coxeter element σ in the form $\sigma = \sigma_- \sigma_+$ or the longest element w_0 of the Weyl group. In both cases we were able to construct explicitly the invariant complex root spaces for a large number of groups. However, we could also show that in many cases an explicit solution does either not exist based on the identifications used or leads only to trivial solutions.

In particular, no non-trivial deformation of the E_8-root system which remain invariant under antilinear transformations was found. Motivated in addition by the above mentioned importance the E_8-root systems play, the main purpose of this manuscript is to provide such a deformation. However, our procedure is very general and may in principle be applied to any element in any group.

In comparison with previous approaches we select here factorisations of an element in the Coxeter group, say $\tilde{\sigma}$, of order $\tilde{h}$ less than the Coxeter number h, i.e. $\tilde{\sigma}^{\tilde{h}} = \mathbb{I}$. We factorise them similarly as the usual Coxeter element based on the bi-colouration of the Dynkin diagram and by construction each of the two factors are then involutory maps, since all subfactors are commuting Weyl reflections being involutions themselves. We identify them as the analogue of the parity transformation and deform them to build up an antilinear involution. A reduced complex root space is then constructed from the orbits of these elements containing $\ell =$ rank $\mathcal{W} \times \tilde{h}$ roots instead of the $\ell \times h$ roots, which result when generated from the usual Coxeter element. By construction these root systems remain invariant under the action of each of the deformed factors of the chosen element when certain conditions hold.

With the above mentioned motivation in mind, one may then employ the deformed simple roots to define complex versions of E_8-affine Toda field theories or the entire deformed

root space to formulate new complex versions of Calogero-Moser-Sutherland models. We report the properties of these models elsewhere [16].

2. From factorised Weyl group elements to invariant complex rootspaces

Let us now briefly recall the main aim of the method of construction proposed so far. We use the notation of [14] and refer to it and references therein for parts of the definitions used. The aim is to construct complex extended root systems $\tilde{\Delta}(\varepsilon)$ which remain invariant under a newly defined antilinear involutory map. The standard real roots $\alpha_i \in \Delta \subset \mathbb{R}^n$ are sought to be represented in a complex space depending on some deformation parameter $\varepsilon \in \mathbb{R}$ as $\tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n$. For this purpose we define a linear deformation map

$$\delta : \ \Delta \to \tilde{\Delta}(\varepsilon), \qquad \alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha, \tag{2.1}$$

relating simple roots α and deformed simple roots $\tilde{\alpha}$ in a linear fashion via the constant deformation matrix θ_ε. Subsequently we seek an antilinear involutory map ω which leaves this root space invariant

$$\omega : \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon), \qquad \tilde{\alpha} \mapsto \omega\tilde{\alpha}, \tag{2.2}$$

this means the map satisfies $\omega : \tilde{\alpha} = \mu_1 \alpha_1 + \mu_2 \alpha_2 \mapsto \mu_1^* \omega \alpha_1 + \mu_2^* \omega \alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$ and $\omega^2 = \mathbb{I}$. Clearly there are many possibilities to achieve this.

As already mentioned, what has been investigated this far is to take simple Weyl reflections as candidates for ω, which works successfully for rank 2 groups, the two factors $\sigma_\pm$ of the Coxeter element or the longest element w_0 of the Weyl group. What has not been explored this far is to take different types of elements in $\mathcal{W}$ as starting points. Here we will indicate the general procedure and work out explicitly the concrete E_8-example. A more systematic solution procedure for other cases will be provided elsewhere [16].

We will start with an arbitrary element of the Weyl group $\tilde{\sigma} \in \mathcal{W}$. This means the element can by definition always be expressed as a product over simple Weyl reflections $\tilde{\sigma} = \prod \sigma_i$. Due to the fact that Weyl reflections do not commute there are various ways to represent elements in the same similarity class. We will therefore convert this element always into a factorised form of the following type

$$\tilde{\sigma} = \tilde{\sigma}_-\tilde{\sigma}_+ \qquad \text{with} \quad \tilde{\sigma}_\pm := \prod_{i \in \tilde{V}_\pm} \sigma_i, \tag{2.3}$$

in close analogy to the factorisation of the Coxeter element $\sigma = \sigma_-\sigma_+$ as explained in [14] and references therein. The sets $V_\pm$ are defined via the bi-colouration, meaning that the roots are separated into two sets of disjoint roots on the Dynkin diagram. However, the products do not extend over all possible elements, i.e. $\tilde{V}_\pm \subset V_\pm$ and therefore we may think of these elements as

$$\tilde{\sigma}_\pm := \sigma_\pm \prod_{i \in \tilde{V}'_\pm} \sigma_i \tag{2.4}$$

for some values j, with $\tilde{V}'_\pm \cup \tilde{V}_\pm = V_\pm$, by recalling $[\sigma_i, \sigma_j] = 0$ for $i, j \in V_+$ or $i, j \in V_-$ and $\sigma_i^2 = \mathbb{I}$. This ensures that we maintain the crucial property $\tilde{\sigma}_\pm^2 = \mathbb{I}$ and thus we select $\tilde{\sigma}_-$

or $\tilde{\sigma}_+$ as a potential candidates for the analogue of the $\mathcal{P}$-operator which we seek to deform in an antilinear fashion to construct the map ω introduced in (2.2). This is achieved by defining the antilinear deformations of the factors of the modified Coxeter element as

$$\tilde{\sigma}_\pm^\varepsilon := \theta_\varepsilon \tilde{\sigma}_\pm \theta_\varepsilon^{-1} = \tau \tilde{\sigma}_\pm \tag{2.5}$$

with τ acting as a complex conjugation and θ_ε being the deformation matrix intoduced in (2.2). By similar reasoning as in [14] we find that the properties to be satisfied by θ_ε are

$$\theta_\varepsilon^* \tilde{\sigma}_\pm = \tilde{\sigma}_\pm \theta_\varepsilon, \quad [\tilde{\sigma}, \theta_\varepsilon] = 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}. \tag{2.6}$$

These equations will be enough to determine the deformed simple roots $\tilde{\alpha}$. Before defining the entire root space associated to $\tilde{\sigma}^\varepsilon = \tilde{\sigma}_-^\varepsilon \tilde{\sigma}_+^\varepsilon$ we introduce the root space $\tilde{\Delta}$ associated to $\tilde{\sigma}$. We require for this the values $c_i = \pm 1$ assigned to the vertices of the Coxeter graphs, in such a way that no two vertices with the same values are linked together. Using then the quantity $\gamma_i = c_i \alpha_i$ for a representant similarly as in the undeformed case, we define a "reduced" Coxeter orbit as

$$\tilde{\Omega}_i := \left\{ \gamma_i, \tilde{\sigma}\gamma_i, \tilde{\sigma}^2\gamma_i, \ldots, \tilde{\sigma}^{\tilde{h}-1}\gamma_i \right\}, \tag{2.7}$$

and the entire reduced root space as

$$\tilde{\Delta} := \bigcup_{i=1}^\ell \tilde{\Omega}_i \subset \Delta_{\mathcal{W}}. \tag{2.8}$$

The length of the orbits $\tilde{\Omega}_i$ will naturally be reduced because the order of the element $\tilde{\sigma}$ will be smaller than the Coxeter number h

$$\tilde{\sigma}^{\tilde{h}} = \mathbb{I}, \qquad \text{with } \tilde{h} \leq h. \tag{2.9}$$

This means the total number of roots in $\tilde{\Delta}$ is $\ell\tilde{h}$ rather than ℓh as in the case of $\Delta_{\mathcal{W}}$. Since $[\tilde{\sigma}, \theta_\varepsilon] = 0$, the deformed orbits and root spaces are isomorphic to the undeformed ones and we may define deformed reduced orbits and a deformed rootspace as

$$\tilde{\Omega}_i^\varepsilon := \theta_\varepsilon \tilde{\Omega}_i \qquad \text{and} \qquad \tilde{\Delta}(\varepsilon) := \theta_\varepsilon \tilde{\Delta}, \tag{2.10}$$

respectively. Crucial to our construction is that the deformed root space $\tilde{\Delta}(\varepsilon)$ remains invariant under the antilinear involutory transformation $\tilde{\sigma}_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon)$. This follows from the argument

$$\tilde{\sigma}_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon \tilde{\sigma}_\pm \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon \tilde{\sigma}_\pm \tilde{\Delta} = \theta_\varepsilon \tilde{\Delta} = \tilde{\Delta}(\varepsilon) \tag{2.11}$$

if and only if $\tilde{\sigma}_\pm \tilde{\Delta} = \tilde{\Delta}$. As the root space is now reduced this might not be the case as $\tilde{\sigma}_\pm$ could map a root into the complement of $\tilde{\Delta}$. However, we may verify this explicity on a case-by-case basis.

3. Invariant complex E_8-root spaces

Our conventions for the labelling of the E_8-roots are depicted in the Dynkin diagram below. They differ slightly from the one previously used [14], but have the advantage that neither two roots labelled by odd or even numbers are connected, which allows for compact notation.
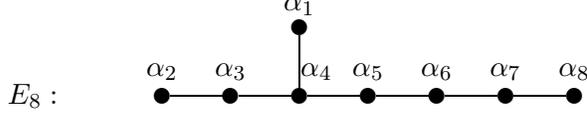

E_8 :

Figure 1: Dynkin diagram indicating the conventions of the labelling for the simple E_8-roots.

Let us now illustrate the procedure described above by selecting first of all an element of the Weyl group, for instance

$$\tilde{\sigma} = \sigma_1\sigma_-\sigma_+ = \sigma_3\sigma_5\sigma_7\sigma_2\sigma_4\sigma_6\sigma_8. \tag{3.1}$$

Acting on the vector $\vec{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ we can represent this element as

$$\tilde{\sigma} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{3.2}$$

Simply by matrix multiplication we then find that the order of this element is $\tilde{h} = 8$. Using an Ansatz for the deformation matrix similar to the one in [14]

$$\theta_\varepsilon = c_0 \mathbb{I} + (1 - c_0)\tilde{\sigma}^4 + i\sqrt{c_0^2 - c_0}(\tilde{\sigma}^2 - \tilde{\sigma}^{-2}), \qquad c_0 \in \mathbb{R}, \tag{3.3}$$

we find that the first four constraints in (2.6) are satisfied. The ususal choice $c_0 = \cosh\varepsilon$ ensures that we recover the undeformed case in the limit $\varepsilon \to 0$, that is the last requirement in (2.6). Explicitly the deformation matrix resulting from (3.2) and (3.3) reads

$$\theta_\varepsilon = \begin{pmatrix} 1 & \lambda_0 & 2\lambda_0 - i\kappa_0 & 3 - 3c_0 & 3\lambda_0 - i\kappa_0 & 3 - 3c_0 & 2\lambda_0 - i\kappa_0 & \lambda_0 \\ 0 & c_0 & 0 & i\kappa_0 & 2i\kappa_0 & i\kappa_0 & 0 & -\lambda_0 \\ 0 & 0 & c_0 - i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 & -i\kappa_0 - \lambda_0 & 0 \\ 0 & i\kappa_0 & 2i\kappa_0 & c_0 + 2i\kappa_0 & 2i\kappa_0 & 2i\kappa_0 - \lambda_0 & 2i\kappa_0 & i\kappa_0 \\ 0 & -2i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 & 2(c_0 - i\kappa_0) - 1 & -2i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 \\ 0 & i\kappa_0 & 2i\kappa_0 & 2i\kappa_0 - \lambda_0 & 2i\kappa_0 & c_0 + 2i\kappa_0 & 2i\kappa_0 & i\kappa_0 \\ 0 & 0 & -\lambda_0 - i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 & -2i\kappa_0 & c_0 - i\kappa_0 & 0 \\ 0 & -\lambda_0 & 0 & i\kappa_0 & 2i\kappa_0 & i\kappa_0 & 0 & c_0 \end{pmatrix}. \tag{3.4}$$

In order to achieve a more compact notation we introduced here the abbreviations $\kappa_0 = \sqrt{c_0^2 - c_0}$ and $\lambda_0 = 1 - c_0$. Therefore the deformed simple roots resulting from (2.1) are

$$\tilde{\alpha}_1 = \alpha_1 + \lambda_0 \left(\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8\right) - i\kappa_0 \left(\alpha_3 + \alpha_5 + \alpha_7\right), \quad (3.5)$$

$$\tilde{\alpha}_2 = c_0 \left(\alpha_2 + \alpha_8\right) - \alpha_8 + i\kappa_0 \left(\alpha_4 + 2\alpha_5 + \alpha_6\right), \quad (3.6)$$

$$\tilde{\alpha}_3 = c_0 \left(\alpha_3 + \alpha_7\right) - \alpha_7 - i\kappa_0 \left[\alpha_3 + 2\left(\alpha_4 + \alpha_5 + \alpha_6\right) + \alpha_7\right], \quad (3.7)$$

$$\tilde{\alpha}_4 = c_0 \left(\alpha_4 + \alpha_6\right) - \alpha_6 + i\kappa_0 \left[\alpha_2 + 2\left(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7\right) + \alpha_8\right], \quad (3.8)$$

$$\tilde{\alpha}_5 = \left(2c_0 - 1\right)\alpha_5 - 2i\kappa_0 \left(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\right), \quad (3.9)$$

$$\tilde{\alpha}_6 = c_0\alpha_6 + \alpha_4 \left(c_0 + 2i\kappa_0 - 1\right) + i\kappa_0 \left[\alpha_2 + 2\left(\alpha_3 + \alpha_5 + \alpha_6 + \alpha_7\right) + \alpha_8\right], \quad (3.10)$$

$$\tilde{\alpha}_7 = c_0\alpha_7 + \left(c_0 - 1\right)\alpha_3 - i\kappa_0 \left[\alpha_3 + 2\left(\alpha_4 + \alpha_5 + \alpha_6\right) + \alpha_7\right], \quad (3.11)$$

$$\tilde{\alpha}_8 = c_0\alpha_8 - \lambda_0\alpha_2 + i\kappa_0 \left(\alpha_4 + 2\alpha_5 + \alpha_6\right). \quad (3.12)$$

To construct the invariant root space we compute the undeformed reduced root space $\tilde{\Delta}$ from the orbits $\tilde{\Omega}_i$ for $1 \leq i \leq 8$ and subsequently replace the undeformed roots in a one-to-one fashion by their deformed counterparts. We evaluate

$\tilde{\Delta}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}$	1;4	3;4	-2;3;4	2;3;4;5;6	-4;5;6	4;5;6;7;8	-6;7;8	6;7
$\tilde{\sigma}^2$	$1;2;3;4^2;5;6$	5;6	-3;4;5;6	3;4;5;6;7;8	-2;3;4;5;6;7;8	2;3;4;5;6;7	-4;5;6;7	4;5
$\tilde{\sigma}^3$	$1;2;3^2;4^3;5^2;6^2;7;8$	7;8	-5;6;7;8	5;6;7	-3;4;5;6;7	3;4;5	-2;3;4;5	2;3
$\tilde{\sigma}^4$	$1;2;3^2;4^3;5^3;6^3;7^2;8$	-8	-7	-6	-5	-4	-3	-2
$\tilde{\sigma}^5$	$1;2;3^2;4^3;5^3;6^2;7^2;8$	-6;7	6;7;8	-4;5;6;7;8	4;5;6	-2;3;4;5;6	2;3;4	-3;4
$\tilde{\sigma}^6$	$1;2;3^2;4^2;5^2;6;7$	-4;5	4;5;6;7	-2;3;4;5;6;7	2;3;4;5;6;7;8	-3;4;5;6;7;8	3;4;5;6	-5;6
$\tilde{\sigma}^7$	1;3;4;5	-2;3	2;3;4;5	-3;4;5	3;4;5;6;7	-5;6;7	5;6;7;8	-7;8

We report in this table the roots which emerge as a result of computing $\tilde{\sigma}^n(\alpha_i)$ with $1 \leq n \leq 7$ and $1 \leq i \leq 8$, where we indicate multiple occurrences by a power. Since these root are either positive or negative it suffices to report the overall sign. For instance we read off from the table that $\tilde{\sigma}^3(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$ or $\tilde{\sigma}^2(\alpha_3) = -\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$.

Next we compute the action of $\tilde{\sigma}_\pm$ on the simple roots. We find

$$\begin{aligned}
&\tilde{\sigma}_-\alpha_1 = \alpha_1, &&\tilde{\sigma}_-\alpha_2 = \alpha_2 + \alpha_3 = \tilde{\sigma}^3\alpha_8, \\
&\tilde{\sigma}_-\alpha_3 = -\alpha_3, &&\tilde{\sigma}_-\alpha_4 = \alpha_3 + \alpha_4 + \alpha_5 = \tilde{\sigma}^3\alpha_6, \\
&\tilde{\sigma}_-\alpha_5 = -\alpha_5, &&\tilde{\sigma}_-\alpha_6 = \alpha_5 + \alpha_6 + \alpha_7 = \tilde{\sigma}^3\alpha_4, \\
&\tilde{\sigma}_-\alpha_7 = -\alpha_7, &&\tilde{\sigma}_-\alpha_8 = \alpha_7 + \alpha_8 = \tilde{\sigma}^3\alpha_2,
\end{aligned} \quad (3.13)$$

and

$$\begin{aligned}
&\tilde{\sigma}_+\alpha_1 = \alpha_1 + \alpha_4 = \tilde{\sigma}\alpha_1, &&\tilde{\sigma}_+\alpha_2 = -\alpha_2, \\
&\tilde{\sigma}_+\alpha_3 = \alpha_2 + \alpha_3 + \alpha_4 = \tilde{\sigma}^5\alpha_7, &&\tilde{\sigma}_+\alpha_4 = -\alpha_4, \\
&\tilde{\sigma}_+\alpha_5 = \alpha_4 + \alpha_5 + \alpha_6 = \tilde{\sigma}^5\alpha_5, &&\tilde{\sigma}_+\alpha_6 = -\alpha_6, \\
&\tilde{\sigma}_+\alpha_7 = \alpha_6 + \alpha_7 + \alpha_8 = \tilde{\sigma}^5\alpha_3, &&\tilde{\sigma}_+\alpha_8 = -\alpha_8.
\end{aligned} \quad (3.14)$$

From (3.13), (3.14) and the above table we observed that $\tilde{\sigma}_\pm\alpha_i \in \tilde{\Delta}$ for $1 \leq i \leq 8$, such that $\tilde{\sigma}_\pm\tilde{\Delta} = \tilde{\Delta}$. Therefore replacing in the above table simple roots by deformed simple

roots, $\alpha_i \mapsto \tilde{\alpha}_i$ for $1 \leq i \leq 8$, consitutes a complex root space $\tilde{\Delta}(\varepsilon)$ consisting of 64 different complex roots which remains invariant under the antilinear involutory transformations $\tilde{\sigma}_\pm^\varepsilon$. Note that in this particular case the introduction of the representative γ_i, which is often needed to avoid overcounting, is not essential.

In order to obtain non-trivial invariant root spaces with different amounts of roots we may select elements $\tilde{\sigma} \in \mathcal{W}$ of other order $\tilde{h}$. For instance we compute

$$
\begin{aligned}
\tilde{\sigma} &= \sigma_1\sigma_3\sigma_7\sigma_4\sigma_6\sigma_8, & \text{with } \tilde{h} = 4, \\
\tilde{\sigma} &= \sigma_1\sigma_3\sigma_5\sigma_7\sigma_4\sigma_6\sigma_8, & \text{with } \tilde{h} = 12, \\
\tilde{\sigma} &= \sigma_1\sigma_3\sigma_7\sigma_2\sigma_4\sigma_6\sigma_8, & \text{with } \tilde{h} = 20, \\
\tilde{\sigma} &= \sigma_1\sigma_3\sigma_5\sigma_7\sigma_2\sigma_4\sigma_8, & \text{with } \tilde{h} = 24.
\end{aligned}
\tag{3.15}
$$

Generalizing then the Ansatz (3.3) to

$$
\theta_\varepsilon = c_0\mathbb{I} + (1 - c_0)\tilde{\sigma}^{\tilde{h}/4} + i\sqrt{c_0^2 - c_0}(\tilde{\sigma}^{\tilde{h}/2} - \tilde{\sigma}^{-\tilde{h}/2}),
\tag{3.16}
$$

yields non-trivial deformation matrices satisfying the constraint (2.6). In general we may try any element of the form

$$
\tilde{\sigma} = \prod_{j \in \tilde{V}_-} \sigma_i \prod_{j \in \tilde{V}_+} \sigma_i =: \tilde{\sigma}_-\tilde{\sigma}_+
\tag{3.17}
$$

where the product might not extend over all four odd and four even roots in V_- or V_+, respectively. It is clear that this creates a large amount of possibilities. In [16] we report on how one may organise these options systematically.

4. Conclusions

We have provided a general construction procedure for invariant root spaces under antilinear involutory transformations. The starting point can be any element in the Weyl group. Since by definition such elements consist of products of Weyl reflections we may always bring it into a factorised form (3.17), such that each of the factors is comprised of elements related to simple roots which are not connected on the Dynkin diagram. Then each of the factors $\tilde{\sigma}_\pm$ will be an involution, which we can identify as an analogue to the $\mathcal{P}$-operator and subsequently we deform it to introduce the antilinear maps $\tilde{\sigma}_\pm^\varepsilon$. Solving the constraints (2.6) we construct simple deformed roots. The entire root space $\tilde{\Delta}(\varepsilon)$ may then be constructed from the union of the reduced orbits $\tilde{\Omega}_i(\varepsilon)$. By construction it remains invariant under the action of $\tilde{\sigma}_\pm^\varepsilon$ if and only if $\tilde{\sigma}_\pm\tilde{\Delta} = \tilde{\Delta}$.

We may then employ these spaces to investigate new types of non-Hermitian generalisation of Calogero models

$$
\mathcal{H}(p, q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\tilde{\alpha} \in \tilde{\Delta}(\varepsilon)} (\tilde{\alpha} \cdot q)^2 + \sum_{\tilde{\alpha} \in \tilde{\Delta}(\varepsilon)} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha} \cdot q)^2},
\tag{4.1}
$$

or the analogues of Calogero-Moser-Sutherland models when replacing the rational potential by a trigonometric or elliptic one. We may also employ only the deformed simple roots

and investigate properties of generalised versions of affine Toda field theories decribed by Lagrangians of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{i=0}^{\ell} n_0 e^{\beta \tilde{\alpha}_i \cdot \phi}. \tag{4.2}$$

For the case of E_8 we have provided the explicit construction for the rootspaces. Based on the conjecture that E_8 plays a crucial role in the understanding and realisation of perturbations of the Ising model it is conceivable that the complex deformed root systems serve to faciliate a systematic study of non-Hermitian perturbations of the Ising model. Generalisions to other types of spin chain and their scaled versions might be based on generalisations to other Coxeter groups.

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