BLOCKS WITH TRANSITIVE FUSION SYSTEMS

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ABSTRACT. Suppose that all nontrivial subsections of a p-block B are conjugate (where p is a prime). By using the classification of the finite simple groups, we prove that the defect groups of B are either extraspecial of order p^3 with $p \in \{3,5\}$ or elementary abelian.

1. INTRODUCTION

Let p be a prime, and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P (cf. [1] and [8]). We call $\mathcal{F}$ *transitive* if any two nontrivial elements in P are $\mathcal{F}$-conjugate. In this case, P has exponent $\exp(P) \le p$, and $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$. This paper is motivated by the following:

Conjecture 1.1. *(cf. [23]) Let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where p is a prime. Then P is either extraspecial of order p^3 or elementary abelian.*

Moreover, if P is extraspecial of order p^3 then results by Ruiz and Viruel [26] imply that $p \in \{3,5,7\}$. Note that the conjecture is trivially true for $p = 2$ since groups of exponent 2 are abelian. Thus Conjecture 1.1 is only of interest for $p > 2$. The aim of this paper is to prove the conjecture above for saturated fusion systems coming from blocks.

Theorem 1.2. *Let p be a prime, and let B be a p-block of a finite group G with defect group P. If the fusion system $\mathcal{F} = \mathcal{F}_P(B)$ of B on P is transitive then P is either extraspecial of order p^3 or elementary abelian.*

If P is extraspecial of order p^3 then the results in [26] and [20] imply that $p \in \{3,5\}$. We call a block B with defect group P and transitive fusion system $\mathcal{F}_P(B)$ *fusion-transitive*. Whenever B has full defect then the theorem is a consequence of the results in [23]. In our proof of the theorem above, we will make use of the classification of the finite simple groups.

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2. Saturated fusion systems

We begin with some results on arbitrary saturated fusion systems.

Proposition 2.1. *Let p be a prime, and let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where $|P| \geq p^4$. Suppose that P contains an abelian subgroup of index p. Then P is abelian.*

Proof. We assume the contrary. Then $p > 2$.

Suppose first that P contains two distinct abelian subgroups A, B of index p. Then $AB = P$, $A \cap B \subseteq \mathrm{Z}(P)$ and $|P : A \cap B| = p^2$. Since P is nonabelian we conclude that $|P : \mathrm{Z}(P)| = p^2$. Thus $1 \neq P' \subseteq \mathrm{Z}(P)$. Since $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$, we conclude that $P' = \mathrm{Z}(P)$. Hence there are $x, y \in P$ such that $P = \langle x, y \rangle$. Then $P' = \langle [x, y] \rangle$ (cf. III.1.11 in [17]); in particular, we have $|P'| = p$ and $|P| = p^3$, a contradiction.

It remains to consider the case where P contains a unique abelian subgroup A of index p. Let Z be a subgroup of order p in $\mathrm{Z}(P)$, and let B be an arbitrary subgroup of order p in A. By transitivity, there is an isomorphism $\phi : B \longrightarrow Z$ in $\mathcal{F}$. By definition, Z is fully $\mathcal{F}$-normalised. Thus, by Proposition 4.20 in [8], Z is also fully $\mathcal{F}$-automised and receptive. Hence ϕ extends to a morphism $\psi : N_\phi \longrightarrow P$ in $\mathcal{F}$. Since $|B| = p$ we have

$$A \subseteq \mathrm{N}_P(B) = \mathrm{C}_P(B) \subseteq N_\phi$$

(cf. p. 99 in [8]). Since $\psi(A)$ is also an abelian subgroup of index p in A we conclude that $\psi(A) = A$. Thus $\psi|A \in \mathrm{Aut}_{\mathcal{F}}(A)$, and $\psi|A$ maps B to Z. This shows that $\mathrm{Aut}_{\mathcal{F}}(A)$ acts transitively on the set of subgroups of order p in A.

In the following, we view A as a vector space over $\mathbb{F}_p$ and $G := \mathrm{Aut}_{\mathcal{F}}(A)$ as a subgroup of $\mathrm{GL}(A)$. If S denotes the group of scalar matrices in $\mathrm{GL}(A)$ then $H := GS$ is a transitive subgroup of $\mathrm{GL}(A)$. The transitive linear groups were classified by Hering (cf. [16] or Remark XII.7.5 in [18]). We are going to use the list in Theorem 15.1 of [27].

Before we do this, we observe the following. By the uniqueness of A, A is fully $\mathcal{F}$-automised, i.e. $P/A = \mathrm{N}_P(A)/\mathrm{C}_P(A) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(A))$. Thus $G = \mathrm{Aut}_{\mathcal{F}}(A)$ and $H = GS$ both have a Sylow p-subgroup of order p.

Now we write $|A| = p^n$ and go through the list in Theorem 15.1 of [27]:

(i) $H \subseteq \Gamma\mathrm{L}_1(p^n)$; in particular, $|H|$ divides $|\Gamma\mathrm{L}_1(p^n)| = n(p^n - 1)$.

In this case we can identify A with the finite field $L := \mathbb{F}_{p^n}$. Moreover, P is the semidirect product of L with $B = \langle \beta \rangle$ where β is a field automorphism of L. For $x \in L$, we have $x\beta \in P$ and

$$1 = (x\beta)^p = x\beta x\beta \dots x\beta = x\beta(x)\beta^2(x) \dots \beta^{p-1}(x) = \mathrm{N}_K^L(x)$$

where K is the fixed field of β. However, it is known that $\mathrm{N}_K^L(L) = K$, a contradiction.

(ii) $n = km$ where $k \geq 2$ and $\mathrm{SL}_k(p^m) \trianglelefteq H$.
Since the Sylow p-subgroups of H have order p, we conclude that $m = 1$ and $k = 2$. Then $n = 2$ and $|P| = p^3$, a contradiction.

(iii) $n = km$ where $k \geq 4$ is even and $\mathrm{Sp}_k(p^m)' \trianglelefteq H$.
Since $p > 2$ we have $\mathrm{Sp}_k(p^m)' = \mathrm{Sp}_k(p^m)$. Thus $\mathrm{Sp}_k(p^m)$ has a Sylow p-subgroup of order $p^{k^2/4} \geq p^4$, a contradiction.

(iv) $n = 6m$, $p = 2$ and $G_2(2^m)' \trianglelefteq H$.
This case is impossible as $p > 2$.

(v) $n = 2$ and $p \in \{5, 7, 11, 19, 23, 29, 59\}$.
Then $|P| = p^3$ which is again a contradiction.

(vi) $n = 4$, $p = 2$ and $H \cong \mathfrak{A}_7$.
This case is also impossible as $p > 2$.

(vii) $n = 4$, $p = 3$ and H is one of the groups in Table 15.1 of [27].
In this case we have $|P| = 3^5 = 243$. Then Proposition 15.12 in [27] leads to a contradiction.

(viii) $n = 6$, $p = 3$ and $H \cong \mathrm{SL}_2(13)$.
In this case we have $|P| = 3^7 = 2187$. However, one can check that P has exponent 9 in this case, a contradiction. □

Proposition 2.2. *Let P be a nonabelian p-group with a transitive fusion system. Then P is indecomposable (as a direct product).*

Proof. Let $P = N_1 \times \cdots \times N_k$ be a decomposition into indecomposable factors $N_i \neq 1$. Assume by way of contradiction that $k \geq 2$. Since P carries a transitive fusion system we have

$$\mathrm{Z}(N_1) \times \cdots \times \mathrm{Z}(N_k) = \mathrm{Z}(P) \subseteq P' = N_1' \times \cdots \times N_k'.$$

Let $1 \neq x \in \mathrm{Z}(N_1)$. By hypothesis there exists $\alpha \in \mathrm{Aut}(P)$ such that $\alpha(x) \in \mathrm{Z}(P) \setminus (\mathrm{Z}(N_1) \cup \ldots \cup \mathrm{Z}(N_k))$. By the Krull-Remak-Schmidt Theorem (see Satz I.12.5 in [17]) there is a normal automorphism β of P such that $\beta(N_i) = \alpha(N_1)$ for some $i \in \{1, \ldots, k\}$. In particular, there is $y \in \mathrm{Z}(N_i)$ such that $\beta(y) = \alpha(x)$. By Hilfssatz I.10.3 in [17], for every $g \in P$ there is a $z_g \in \mathrm{Z}(P)$ such that $\beta(g) = gz_g$. Obviously the map $P \longrightarrow \mathrm{Z}(P)$, $g \longmapsto z_g$, is a homomorphism. Since $\mathrm{Z}(N_i) \subseteq N_i'$, we obtain $z_y = 1$. This gives the contradiction $\alpha(x) = \beta(y) = y \in \mathrm{Z}(N_i)$. □

Proposition 2.3. *Let $P = \prod_{i=1}^{\infty} P_i^{a_i}$ where $P_i = C_{p^{r_i}} \wr C_p \wr \ldots \wr C_p$ (i factors in the wreath product) and $a_i \in \mathbb{N}_0$, $r_i \in \mathbb{N}$ for $i \in \mathbb{N}$. Moreover, let U be a normal subgroup of P such that P/U is cyclic, and let Z be a cyclic subgroup of $\mathrm{Z}(U)$. Suppose that $R := U/Z$ supports a transitive fusion system. Then R has order p^3 or is elementary abelian.*

Proof. We assume the contrary. Then $|R| \geq p^4$ and $p > 2$.

Suppose first that $r_j > 1$ for some $j > 1$. Since $p > 2$, P' contains a subgroup isomorphic to $C_{p^{r_j}} \times C_{p^{r_j}}$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction.

Thus $r_j = 1$ for $j > 1$, and P_j is the iterated wreath product of j copies of C_p in this case.

Suppose next that $a_j > 0$ for some $j \geq 3$. Since $p > 2$, P' contains a subgroup isomorphic to $P_{j-1} \times P_{j-1}$. By Satz III.15.3 in [17], P_{j-1} has exponent $p^{j-1} \geq p^2$. Since $P' \subseteq U$ we conclude that $\exp(R) \geq p^2$, a contradiction again.

Thus $P = P_1^{a_1} \times P_2^{a_2}$ where $P_1 = C_{p^{r_1}}$ and $P_2 = C_p \wr C_p$. If $a_2 \leq 1$ then P and R contain abelian subgroups of index p. In this case Proposition 2.2 gives a contradiction.

Hence we may assume that $a_2 \geq 2$. Let $\pi : P \longrightarrow P_2^{a_2}$ be the relevant projection. Since $\exp(P_2) = p^2$ we cannot have $\pi(U) = P_2^{a_2}$. On the other hand, P_2/P_2' is elementary abelian. Since $P_2^{a_2}/\pi(U)$ is cyclic, $\pi(U)$ is a maximal subgroup of $P_2^{a_2}$. Let $\pi_1 : P_2^{a_2} \longrightarrow P_2^{a_2-1}$ be the projection onto the direct product of the first $a_2 - 1$ copies of P_2, and let $\pi_2 : P_2^{a_2} \longrightarrow P_2^{a_2-1}$ be the projection onto the direct product of the last $a_2 - 1$ copies of P_2.

Now suppose that $a_2 \geq 3$. Then an argument similar to the one above shows that $\pi_1(\pi(U))$ is a maximal subgroup of $P_2^{a_2-1} = \pi_1(P_2^{a_2})$. Thus $\mathrm{Ker}(\pi_1) \subseteq \pi(U)$ and, similarly, $\mathrm{Ker}(\pi_2) \subseteq \pi(U)$. Thus $\pi(U)$ contains a subgroup isomorphic to P_2^2. Hence $\exp(R) \geq p^2$, a contradiction.

We are left with the case $a_2 = 2$, i.e. $P = A \times P_2 \times P_2$ where $A = P_1^{a_1} \cong C_{p^{r_1}}^{a_1}$ is abelian. Since $\pi(U)$ is a maximal subgroup of $P_2 \times P_2$, we see that $A \times \pi(U)$ is a maximal subgroup of P. Let $x \in P$ such that $P = U\langle x\rangle$. Then $U\langle x^p\rangle \subseteq A \times \pi(U)$. Since $|P : U\langle x\rangle| \leq p$ we conclude that $U\langle x^p\rangle = A \times \pi(U)$. Note that $x^p \in \mho(P) \subseteq \mathrm{Z}(P)$.

Suppose that $\exp(A) > p$, and choose an element $a \in A$ of maximal order. We write $x = x_1x_2$ with $x_1 \in A$ and $x_2 \in P_2^2$, we write $a = ux^{pi}$ with $u \in U$ and $i \in \mathbb{Z}$, and we write $u = u_1u_2$ with $u_1 \in A$ and $u_2 \in P_2^2$. Then $a^p = u^p x^{p^2 i} = u_1^p x_1^{p^2 i} u_2^p x_2^{p^2 i} = u_1^p x_1^{p^2 i} u_2^p$. We conclude that $u_2^p = 1$ and $a^p = u_1^p x_1^{p^2 i}$. Thus $p < \exp(A) = |\langle a\rangle| = |\langle u_1\rangle| = |\langle u\rangle|$, and $1 \neq u^p \in \mho(U) \cap A$.

By Aufgabe III.15.36 in [17], the elements of order 1 or p form a union of two maximal subgroups. Thus P_2^2 contains $p^{2p-2}(2p-1)^2 < p^{2p+1}$ elements of order 1 or p. Hence $\pi(U)$ contains elements of order p^2; in particular, $\mho(U)$ is noncyclic. Since $\mho(U) \subseteq Z$, this is a contradiction.

This contradiction shows that $\exp(A) \leq p$, i.e. $P = A \times P_2 \times P_2$ where A is elementary abelian. Hence P/P' is elementary abelian. Since P/U is cyclic we conclude that U is a maximal subgroup of P. Thus $U = A \times \pi(U)$ and $\mho(U) \subseteq \pi(U)$. Since $\pi(U)$ contains elements of order p^2, we have $1 \neq \mho(U) \subseteq Z$. On the other hand, Satz III.15.4 in [17] implies that $\mathrm{Z}(U)$ is elementary abelian. Thus $|Z| = p$ and $Z = \mho(U) \subseteq \pi(U)$. Since R supports a transitive fusion system we have

$$AZ/Z \subseteq \mathrm{Z}(U)/Z \subseteq \mathrm{Z}(R) \subseteq R' = U'Z/Z = \pi(U)'Z/Z \subseteq \pi(U)/Z.$$

Therefore $A = 1$, i.e. $P = P_2 \times P_2$. Recall that U is a maximal subgroup of P and that $\pi_1, \pi_2 : P \longrightarrow P_2$ denote the two projections. Without loss of generality we have $\pi_1(U) = P_2$. Since $\mho(U)$ is cyclic, $K_1 := \mathrm{Ker}(\pi_1)$ has order p^p and exponent p.

If $\pi_2(U) \neq P_2$ then $U = P_2 \times \pi_2(U)$ and $\exp(\pi_2(U)) = p$. Thus $Z = \mho(U) \subseteq P_2 \times 1$ and $R \cong P_2/Z \times \pi_2(U)$, a contradiction to Proposition 2.2.

Thus we must also have $\pi_2(U) = P_2$. Then also $K_2 := U \cap \mathrm{Ker}(\pi_2)$ has order p^p and exponent p. Moreover, we have $K_1 \times K_2 \subseteq U$.

We may choose elements $x, y \in U$ such that $\pi_1(x)$ and $\pi_2(x)$ have order p^2. Since $\langle x^p \rangle = Z = \langle y^p \rangle$ we see that $\pi_2(x)$ and $\pi_1(y)$ have order p^2. However, we may choose y such that yK_1 contains an element y' such that $\pi_2(y')$ has order p. Since $\pi_1(y) = \pi_1(y')$ still has order p^2, we have a final contradiction. □

3. Blocks

We now present the proof of Theorem 1.2.

Proof. Suppose that the result is false. Then P is nonabelian with $|P| \geq p^4$ and $p > 2$.

By [1, Proposition IV.6.3] we may assume that B is quasiprimitive. This means that, for any normal subgroup H of G, B covers a unique p-block of H.

Now let H be a normal subgroup of G, and let b be the unique p-block of H covered by B. Suppose that $P \cap H = 1$. (This is satisfied, for example, whenever H is a p'-subgroup.) Then b has defect zero. By Clifford theory, there exist a finite group G^*, a central p'-subgroup H^* of G^*, and a p-block B^* of G^* with defect group $P^* \cong P$ such that $\mathcal{F}_{P^*}(B^*)$ is equivalent to $\mathcal{F}$. Thus we may replace G by G^* and B by B^*.

Repeating the argument above we may therefore assume that every normal subgroup H of G with $P \cap H = 1$ is central. In particular, we have $\mathrm{O}_{p'}(G) \subseteq \mathrm{Z}(G)$.

It is well-known that $M := \mathrm{O}_p(G) \subseteq P$. Suppose first that $M \neq 1$. Since $\mathcal{F}$ is transitive this implies $M = P$. Then $\Phi(P)$ is a normal subgroup of G and properly contained in P. Since $\mathcal{F}$ is transitive, we must have $\Phi(P) = 1$. Thus P is elementary abelian in this case.

Hence, in the following, we may assume that $\mathrm{O}_p(G) = 1$. Then $\mathrm{F}(G) = \mathrm{O}_{p'}(G) = \mathrm{Z}(G)$. Moreover, the layer $\mathrm{E}(G)$ is nontrivial. Let b be the unique p-block of $\mathrm{E}(G)$ covered by B. Then b has defect group $P \cap \mathrm{E}(G) \neq 1$. Since B is transitive, this implies that $P \subseteq \mathrm{E}(G)$.

Let $L_1, \ldots, L_n$ denote the components of G. Then $\mathrm{E}(G) = L_1 * \cdots * L_n$ is a central product. For $i = 1, \ldots, n$, the unique p-block b_i of L_i covered by b has defect group $P_i := P \cap L_i \neq 1$. Moreover, we have $P = P_1 \times \cdots \times P_n$. Since $\mathcal{F}$ is transitive, this implies that $n = 1$. Thus $\mathrm{E}(G) = L_1 =: L$ is quasisimple, and $G/\mathrm{Z}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(L)$.

If $|P| = p^4$ then Proposition 15.14 in [27] gives a contradiction. Thus we may assume that $|P| \geq p^5$; in particular, $|L|$ is divisible by p^5. If P is a Sylow p-subgroup of G then the results of [23] imply our theorem. Hence we may assume that $|G|$ is divisible by p^6.

We now make use of the classification of the finite simple groups and discuss the various possibilities for the simple group $\mathrm{F}^*(G)/\mathrm{Z}(G) \cong L/\mathrm{Z}(L)$. Since $\mathcal{F}$ is transitive we have $\mathrm{C}_L(u) \cong \mathrm{C}_L(v)$ for any $u, v \in P \setminus \{1\}$. This will be a very useful fact.

It can be checked with GAP [13] that $L/\mathrm{Z}(L)$ cannot be a sporadic simple group. Similarly, $L/\mathrm{Z}(L)$ cannot be a simple group with an exceptional Schur multiplier.

Suppose that $L = \mathfrak{A}_n$ is an alternating group. Then P is a defect group of a p-block of $\mathfrak{A}_n$. Hence P is also a defect group of a p-block of the symmetric group $\mathfrak{S}_n$. Thus P is a direct product of (iterated) wreath products of groups of order p. Since $C_p \wr C_p$ has exponent p^2 we conclude that P is a direct product of groups of order p, and the result follows in this case.

Suppose next that $L = \hat{\mathfrak{A}}_n$ is the 2-fold cover of $\mathfrak{A}_n$. We may assume that b is a faithful block of $\hat{\mathfrak{A}}_n$. In this case the defect groups of b have a similar structure as those in $\mathfrak{A}_n$ (cf. [24, Theorem 5.8.8]), so we are done here by the same argument.

Suppose now that $L/\mathrm{Z}(L)$ is a group of Lie type in characteristic p. Then the p-block b of L has full defect, i.e. P is a Sylow p-subgroup of L. Since $\mathcal{F}$ is transitive, every nontrivial element $u \in P$ is conjugate in G to an element $v \in \mathrm{Z}(P)$. Thus $|L : \mathrm{C}_L(u)| = |L : \mathrm{C}_L(v)|$ is not divisible by p. Therefore the results in [25] imply that P is abelian.

Finally suppose that $L/\mathrm{Z}(L)$ is a group of Lie type in characteristic $r \neq p$. First we deal with the exceptional groups of Lie type. Let $S \in \mathrm{Syl}_p(L)$. By §10.1 in [14], S contains an abelian normal subgroup N such that S/N is isomorphic to a subgroup of the Weyl group of $L/\mathrm{Z}(L)$. If $|S/N| \leq p$, then Proposition 2.1 gives a contradiction. This already implies the claim for $p \geq 7$. Now let $p = 5$. Then by the same argument we may assume that $L/\mathrm{Z}(L) \cong E_8(q)$ where $q \equiv \pm 1 \pmod 5$. This case will be handled in Section 6. Now let $p = 3$. Here we need to discuss the following groups: F_4, E_6, ${}^2E_6$, E_7 and E_8. For $L/\mathrm{Z}(L) \cong F_4(q)$ we have $|P| \leq p^6$ and the result follows by Proposition 15.13 in [27]. The remaining cases will be discussed in Section 6.

We may therefore assume that $L/\mathrm{Z}(L)$ is a classical group. In this case our theorem follows from the results of the next section. □

4. Classical Groups in non-describing characteristic

We keep the notation of the previous section. We suppose in this section that $L/\mathrm{Z}(L)$ is a simple group of Lie type in characteristic r, $r \neq p$. Let q be a power of r. Suppose that $L = \mathbf{L}^F/Z$, where $\mathbf{L}$ is a simple simply connected algebraic group defined over an algebraic closure $\bar{\mathbb{F}}_q$ of a field $\mathbb{F}_q$ of q elements, $F : \mathbf{L} \to \mathbf{L}$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{L}$ and Z is a central subgroup of $\mathbf{L}^F$. Note that by the classification of finite simple groups, we may assume that if q is a power of 2, then $\mathbf{L}$ is not of type C_n. Let $\tilde{b}$ be the block of $\mathbf{L}^F$ dominating b and $\tilde{P}$ be a defect group of $\tilde{b}$ such that $\tilde{P}Z/Z = P$.

We define groups $\mathbf{H}$ as follows. If $L/\mathrm{Z}(L) = B_n(q)$, then $\mathbf{H} = \mathrm{SO}_{2n+1}(\bar{\mathbb{F}}_q)$. If $L/\mathrm{Z}(L) = C_n(q)$, then $\mathbf{H} = \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q)$. If $L/\mathrm{Z}(L) = D_n^{\pm}(q)$, then $\mathbf{H} = \mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$. Here, if q is a power of 2, and $\mathbf{L}$ is of type B_n, then by $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$ we mean the adjoint simple group of type B_n. If q is a power of 2 and if $\mathbf{L}$ is of type D_n, then by $\mathrm{SO}_{2n}(\bar{\mathbb{F}}_q)$ we mean the simple algebraic group of type D_n corresponding to the root datum $(X, \Phi, Y, \Phi^\vee)$ for which the fundamental roots are $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_{n-1} + e_n$ and $X = \{\sum_{i=1}^n a_i e_i : a_i \in \mathbb{Z}\}$ for an orthonormal basis, $e_1, e_2, \cdots, e_n$, of n-dimensional Euclidean space. We may and will assume that $\mathbf{H}$ is an F-stable quotient of $\mathbf{L}$.

Proposition 4.1. *Suppose that p is an odd prime and $L/\mathrm{Z}(L)$ is a classical group in non-describing characteristic different from triality D_4. Suppose that B is a fusion-transitive block with P of order at least p^5. Then P is abelian.*

Proof. Suppose that $L/\mathrm{Z}(L)$ is the projective special linear group $\mathrm{PSL}_n(q)$, so $\mathbf{L} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and $L = \mathrm{SL}_n(q)$. Let D be a defect group of a block of $\mathrm{GL}_n(q)$ covering $\tilde{b}$ such that $\tilde{P} = D \cap \mathrm{SL}_n(q)$. By the results of Fong and Srinivasan on blocks of finite general linear groups [12, Theorem (3C)], D is isomorphic to the Sylow p-subgroup of a direct product of general linear groups over finite extensions of $\mathbb{F}_q$. Since $\mathrm{Z}(L)$ and $D/\tilde{P}$ are cyclic, the claim follows from Proposition 2.3. The case that $L/\mathrm{Z}(L)$ is the projective special unitary group can be handled similarly.

Now consider the case that $L/\mathrm{Z}(L)$ is of type B, C or D. Then $\tilde{P}$ is a defect group of $\mathbf{L}^F$. Let $1 \neq z \in \mathrm{Z}(\tilde{P})$. Since p is odd, $\mathrm{C}_{\mathbf{L}}(z)$ is a Levi subgroup of $\mathbf{L}$. For any subset A of $\mathbf{L}$, denote by $\overline{A}$ the image of A under the isogeny from $\mathbf{L}$ onto $\mathbf{H}$ and denote by U the kernel of the isogeny. Since U is a central 2-subgroup of $\mathbf{L}$, $\overline{\mathrm{C}_{\mathbf{L}}(z)} = \mathrm{C}_{\mathbf{H}}(\bar{z})$.

The group $\mathrm{C}_{\mathbf{H}}(\bar{z})$ is a direct product

$$\mathrm{C}_{\mathbf{H}}(\bar{z}) = \mathbf{H}_0 \times \cdots \times \mathbf{H}_r,$$

where $\mathbf{H}_0$ is either the identity or a classical group and for $i \geq 1$, $\mathbf{H}_i$ is a direct product of general linear groups with F transitively permuting the factors. This follows easily from the standard description of the root datum of $\mathbf{H}$. So,

$$\mathrm{C}_{\mathbf{H}}(\bar{z})^F = \mathbf{H}_0^F \times \cdots \times \mathbf{H}_r^F,$$

where $\mathbf{H}_i^F$ is a finite general linear or unitary group for $i \geq 1$ and $\mathbf{H}_0^F$ is a finite classical group (possibly the identity).

Let $\mathbf{L}_i$ be the inverse image in $\mathrm{C}_{\mathbf{L}}(z)$ of $\mathbf{H}_i$, $0 \leq i \leq r$. Then $\mathbf{L}_i$ is a normal F-stable subgroup of $\mathrm{C}_{\mathbf{L}}(z)$, $\mathrm{C}_{\mathbf{L}}(z) = \mathbf{L}_0 \cdots \mathbf{L}_r$ and

$$[\mathbf{L}_i, \mathbf{L}_0 \cdots \mathbf{L}_{i-1}\mathbf{L}_{i+1} \cdots \mathbf{L}_r] \leq \mathbf{L}_i \cap (\mathbf{L}_0 \cdots \mathbf{L}_{i-1}\mathbf{L}_{i+1} \cdots \mathbf{L}_r) = U.$$

We claim that $\overline{\mathbf{L}_i^F}$ is a normal subgroup of $\mathbf{H}_i^F$ of 2-power index. Indeed, let M be the inverse image in $\mathbf{L}_i$ of $\mathbf{H}_i^F$. Then M is F-stable since U is F-stable. Further, $[M,F] \le U$. Since U is central in M, the map $M \to U$ defined by $x \to x^{-1}F(x)$ is a group homomorphism. The kernel of this map is $\mathbf{L}_i^F$ whence $\mathbf{L}_i^F$ is a normal subgroup of M and the index of $\mathbf{L}_i^F$ in M divides $|U|$. The claim follows since U is a 2-group.

The claim implies that $\mathbf{L}_0^F \cdots \mathbf{L}_r^F$ is a normal subgroup of 2-power index of $\mathrm{C}_{\mathbf{L}}(z)^F$. So, $\tilde{P}$ is a defect group of $\mathbf{L}_0^F \cdots \mathbf{L}_r^F$. The commutator relationship given above then implies that $\tilde{P}$ is a direct product $P_0 \cdots P_r$, where P_i is a defect group of $\mathbf{L}_i^F$, $0 \le i \le r$. By Proposition 2.2, $\tilde{P} = P_i$ for some i, $1 \le i \le r$. Since z is central in $\mathrm{C}_{\mathbf{L}}(z)$, $i \ge 1$ and $\mathbf{H}_i^F$ is a general linear or unitary group with a central p-element. Let $R = \tilde{P} \cap [\mathbf{L}_i, \mathbf{L}_i]^F$, a defect group of $[\mathbf{L}_i, \mathbf{L}_i]^F$. By suitably replacing $\tilde{P}$ by an $\mathbf{L}_i^F$-conjugate, we may assume that the relevant block of $[\mathbf{L}_i, \mathbf{L}_i]^F$ is $\tilde{P}$-stable and hence that $\tilde{P}$ is a defect group of $[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P}$.

The isogeny $\mathbf{L}_i \to \mathbf{H}_i$ restricts to an isogeny $[\mathbf{L}_i, \mathbf{L}_i] \to [\mathbf{H}_i, \mathbf{H}_i]$ with kernel $U \cap [\mathbf{L}_i, \mathbf{L}_i]$. However $[\mathbf{H}_i, \mathbf{H}_i]$ is a simply connected semisimple group, being the direct product of special linear groups. Thus, $U \cap [\mathbf{L}_i, \mathbf{L}_i] = 1$ and the restriction of the isogeny to $[\mathbf{L}_i, \mathbf{L}_i]$ is an abstract group isomorphism from $[\mathbf{L}_i, \mathbf{L}_i]$ to $[\mathbf{H}_i, \mathbf{H}_i]$ which commutes with F. Consequently, $[\mathbf{L}_i, \mathbf{L}_i]^F \cong [\mathbf{H}_i, \mathbf{H}_i]^F$. Also, $U \cap [\mathbf{L}_i, \mathbf{L}_i]\tilde{P} = 1$ and the induced map $[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P} \to \mathbf{H}_i^F$ is injective. Thus $\overline{\tilde{P}} \cong \tilde{P} \cong P$ is a defect group of $\overline{[\mathbf{L}_i, \mathbf{L}_i]^F \tilde{P}} \cong [\mathbf{H}_i, \mathbf{H}_i]^F \overline{\tilde{P}}$. Since $\mathbf{H}_i^F$ is a finite general linear or unitary group, the result now follows from [12, Theorem (3C)] and Proposition 2.3 in the same way as for the case that $L/\mathrm{Z}(L)$ is a projective special linear or unitary group. □

5. On A_{p-1}-components

Lemma 5.1. *Suppose that p is an odd prime and let G be a finite group isomorphic to one of the groups $\mathrm{SL}_p(q)$ or $\mathrm{SU}_p(q)$ for some prime power q not divisible by p. Let U be a non-abelian p-subgroup of G. Then U contains a normal abelian subgroup U_0 of index p such that any element of $U \setminus U_0$ has order p. If $|U| \ge p^{p+1}$, then U_0 contains an element of order p^2.*

Proof. First, consider the case that G is special linear or unitary. By replacing q if necessary by some power we may assume that $U \le \mathrm{SL}_p(q)$ and p divides $q-1$. Let S_0 be the Sylow p-subgroup of the group of diagonal matrices of $\mathrm{SL}_p(q)$ and let σ be a non-diagonal, monomial matrix in $\mathrm{SL}_p(q)$ of order p. Then $S := \langle S_0, \sigma \rangle$ is a Sylow p-subgroup of $\mathrm{SL}_p(q)$, S_0 is normal in S, abelian, of index p in S, rank $p-1$ and any element of S not in S_0 has order p. Let $U_0 = U \cap S_0$. Then U_0 has index at most p in U. On the other hand, since U is non-abelian and S_0 is abelian, U is not contained in U_0. Thus U_0 has index p in U, proving the first assertion. Now suppose that U has exponent p. Then U_0 is elementary abelian. On the other hand, $U_0 \le S_0$ and the p-rank of S_0 is $p-1$. Hence, $|U| = p|U_0| \le p^p$.

□

In the rest of this section, p will denote a fixed prime and $\mathbf{G}$ will denote a connected reductive group in characteristic $r \neq p$ with a Frobenius morphism F with respect to some $\mathbb{F}_{r'}$ structure for some power r' of r. In what follows, whenever we talk of a component of $\mathbf{G}$, we will mean a simple component of $[\mathbf{G}, \mathbf{G}]$.

We need a slight variation of the previous lemma.

Lemma 5.2. *Suppose that p is odd. If $[\mathbf{G}, \mathbf{G}] = \mathrm{SL}_p$, then any p-subgroup of $\mathbf{G}^F$ has an abelian subgroup of index p.*

Proof. Since $\mathbf{G} = \mathrm{Z}^\circ(\mathbf{G})[\mathbf{G}, \mathbf{G}]$ any element and hence any subgroup of $\mathbf{G}^F$ is contained in $\mathrm{Z}^\circ(\mathbf{G})^{F^d}[\mathbf{G}, \mathbf{G}]^{F^d}$ for some $d \geq 1$. This can be seen as follows. Since $\mathbf{G} = \mathrm{Z}^\circ(\mathbf{G})[\mathbf{G}, \mathbf{G}]$, any element u of $\mathbf{G}$ can be written in the form $u = xy$, where $x \in \mathrm{Z}^\circ(\mathbf{G})$ and $y \in [\mathbf{G}, \mathbf{G}]$. Let $\iota : \mathbf{G} \to \mathrm{GL}_n$ be an embedding. Then for some power, say F^t of F, some power, say s of r, and for all $g \in \mathbf{G}$, $F^t \circ \iota(g) = F_s(\iota(g))$ where F_s is the standard Frobenius morphism of GL_n raising every matrix entry to the s-th power. The claim follows since for any $h \in \mathrm{GL}_n$, $F_s^m(h) = h$ for some natural number m. Since any Sylow p-subgroup of $\mathrm{Z}^\circ(\mathbf{G})^{F^d}[\mathbf{G}, \mathbf{G}]^{F^d}$ is of the form R_1R_2, where R_1 is a Sylow p-subgroup of $\mathrm{Z}^\circ(\mathbf{G})^{F^d}$ and R_2 is a Sylow p-subgroup of $[\mathbf{G}, \mathbf{G}]^{F^d}$, the result follows from the previous Lemma and the fact that R_1 is central in R_1R_2. □

Lemma 5.3. *Suppose that p is odd. Let $\mathbf{X} = \mathrm{SL}_p$ be an F-stable component of $\mathbf{G}$ such that $\mathbf{X}^F$ has a central element of order p and let $\mathbf{Y}$ be the product of all other components of $\mathbf{G}$ and $\mathrm{Z}^\circ(\mathbf{G})$. Let P be a p-subgroup of $\mathbf{G}^F$ such that $P \cap \mathbf{X}^F$ is non-abelian of order at least p^p and P is not contained in $\mathbf{X}^F\mathbf{Y}^F$. Then there exists an element of order p^2 in P. Further, if Z is a central subgroup of $\mathbf{G}^F$ of order p such that P/Z has exponent p, then $Z \leq \mathbf{X}^F$.*

Proof. Let $\tilde{P}$ be the inverse image of P under the surjective group homomorphism $\mathbf{X} \times \mathbf{Y} \to \mathbf{G}$ induced by multiplication. The kernel of the multiplication map is isomorphic to $\mathbf{X} \cap \mathbf{Y} = \mathrm{Z}(\mathbf{X}) \cap \mathrm{Z}(\mathbf{Y})$. Since $\mathbf{X}$ is a simple group of type A_{p-1}, the kernel of the multiplication map is a group of order p and in particular, $\tilde{P}$ is a finite p-group. Let $P_1 \leq \mathbf{X}$ be the image of $\tilde{P}$ under the projection of $\mathbf{X} \times \mathbf{Y} \to \mathbf{X}$. Clearly P_1 contains $P \cap \mathbf{X}^F$. We claim that $P \cap \mathbf{X}^F$ is proper in P_1. Indeed, otherwise $\tilde{P} \leq (P \cap \mathbf{X}^F) \times \mathbf{Y}$, whence $P \leq (P \cap \mathbf{X}^F)\mathbf{Y}$. This implies that $P \leq (P \cap \mathbf{X}^F)(P \cap \mathbf{Y}^F) \leq P \cap \mathbf{X}^F\mathbf{Y}^F$, a contradiction. Since $P \cap \mathbf{X}^F$ is assumed to have order at least p^p, the claim implies that $|P_1| \geq p^{p+1}$.

Now P_1 is a finite subgroup of $\mathbf{X}$, thus of some finite special linear (or unitary) group. Hence, by Lemma 5.1, there exists an element $x \in P_1$ of order p^2. Let $y \in \mathbf{Y}$ be such that $w = xy \in P$. Since $P \cap \mathbf{X}^F$ is non-abelian again by Lemma 5.1, there exists $\sigma \in P \cap \mathbf{X}^F$ such that $x\sigma$ has order p. Then w and $w\sigma \in P$, $w^p = x^py^p$ and $(w\sigma)^p = y^p$. Then either $w^p \neq 1$ or $(w\sigma)^p \neq 1$, proving the first part of the result.

Suppose that P/Z has exponent p. Then, $w^p, (w\sigma)^p$ are in Z. Hence $x^p \in Z$. Since $1 \neq x^p$ and Z has order p the second assertion follows. $\square$

Lemma 5.4. *Let $\mathcal{X}$ be an F-stable subset of components of $\mathbf{G}$. Let $\mathbf{X}$ be the product of all elements of $\mathcal{X}$ and let $\mathbf{Y}$ be the product of $\mathrm{Z}^\circ(\mathbf{G})$ and all the components of $[\mathbf{G}, \mathbf{G}]$ not in $\mathcal{X}$.*

- (i) *Let P be a defect group of a block b of $\mathbf{G}^F$. Then $P \cap \mathbf{X}^F\mathbf{Y}^F$ is a defect group of a block of $\mathbf{X}^F\mathbf{Y}^F$ covered by b and is of the form P_1P_2, where P_1 is a defect group of a block of $\mathbf{X}^F$ covered by b and P_2 is a defect group of a block of $\mathbf{Y}^F$ covered by b. If $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ has p'-order, then $P = P_1P_2$ and the product is direct.*
- (ii) *Let c be a p-block of $\mathbf{X}^F\mathbf{Y}^F$. Then the index of the stabiliser of c in $\mathbf{G}^F$ is prime to p. Suppose further that $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ is a p-group. Then c is $\mathbf{G}^F$-stable, c is covered by a unique block of $\mathbf{G}^F$ and if P is a defect group of the block of $\mathbf{G}^F$ covering c, then $P \cap \mathbf{X}^F\mathbf{Y}^F$ is a defect group of c and $P/(P \cap \mathbf{X}^F\mathbf{Y}^F) \cong \mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$.*

Proof. The first statement of (i) follows from the theory of covering blocks as $\mathbf{X}^F\mathbf{Y}^F$ is a normal subgroup of $\mathbf{G}^F$, $\mathbf{X}^F$ and $\mathbf{Y}^F$ centralise each other and $\mathbf{X}^F \cap \mathbf{Y}^F = \mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F \subseteq \mathrm{Z}(\mathbf{G})^F$ is central in $\mathbf{X}^F\mathbf{Y}^F$. The second assertion of (i) follows from the first assertion, the fact that $|\mathbf{G}^F| = |\mathbf{X}^F||\mathbf{Y}^F|$ and $\mathbf{X}^F \cap \mathbf{Y}^F = \mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$.

We now prove (ii). Let $u \in \mathbf{G}^F$ be a p-element. Then $u = xy$, with $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ such that $x^{-1}F(x) = yF(y^{-1})$ is an element of $\mathrm{Z}(\mathbf{X}) \cap \mathrm{Z}(\mathbf{Y})$. We may assume without loss of generality that x and y are p-elements. The block c of $\mathbf{X}^F\mathbf{Y}^F$ is a product c_1c_2 of blocks c_1 of $\mathbf{X}^F$ and c_2 of $\mathbf{Y}^F$. Thus, it suffices to prove that ${}^x c_1 = c_1$ and ${}^y c_2 = c_2$.

Now consider a regular embedding $\mathbf{X} \leq \tilde{\mathbf{X}}$, where $\tilde{\mathbf{X}}$ is a connected reductive group with connected centre containing $\mathbf{X}$ as a closed subgroup, such that $[\tilde{\mathbf{X}}, \tilde{\mathbf{X}}] = [\mathbf{X}, \mathbf{X}]$ and such that F extends to a Frobenius morphism of $\tilde{\mathbf{X}}$. Since $x^{-1}F(x) \in \mathrm{Z}(\mathbf{X}) \leq \mathrm{Z}^\circ(\tilde{\mathbf{X}})$, $x = x_1z$ for some $x_1 \in \tilde{\mathbf{X}}^F$, and $z \in \mathrm{Z}^\circ(\tilde{\mathbf{X}})$. We may assume also that x_1 is a p-element. Then ${}^x c_1 = {}^{x_1} c_1$. On the other hand, c_1 contains an ordinary irreducible character χ in a Lusztig series corresponding to a semisimple element of order prime to p in the dual group of $\mathbf{X}$, hence the index in $\tilde{\mathbf{X}}^F$ of the stabiliser in $\tilde{\mathbf{X}}^F$ of χ has order prime to p (see for instance [3, Corollaire 11.13]). This proves the first assertion. If $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ is a p-group, then $|\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F| = |\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F|$ is a power of p. By the first assertion, c is $\mathbf{G}^F$-stable and by standard block theory, there is a unique block of $\mathbf{G}^F$ covering c. The second assertion of (ii) now follows from (i). $\square$

Lemma 5.5. *Suppose that p is odd. Let $\mathbf{X}$ be an F-stable component of $\mathbf{G}$ of type A_{p-1} and let $\mathbf{Y}$ be the product of all other components of $\mathbf{G}$ and $\mathrm{Z}^\circ(\mathbf{G})$. Suppose that*

$\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F \neq 1$ *and that* P *is a defect group of* $\mathbf{G}^F$ *such that* $P \cap \mathbf{X}^F$ *is abelian. Then there exists an* F*-stable torus* $\mathbf{T}$ *of* $\mathbf{X}$ *such that* P *is a defect group of* $(\mathbf{YT})^F$.

Proof. In the proof, we will identify blocks with the corresponding central primitive idempotents. Let b be a block of $\mathbf{G}^F$ with P as defect group and let $P_0 := P \cap \mathbf{X}^F\mathbf{Y}^F$. The hypothesis implies that $|\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F| = p$. So, by Lemma 5.4, b is a block of $\mathbf{X}^F\mathbf{Y}^F$, P_0 is a defect group of b as block of $\mathbf{X}^F\mathbf{Y}^F$ and P/P_0 is isomorphic to $\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$. Let $b = b_1b_2$, where b_1 is the block of $\mathbf{X}^F$ covered by b and b_2 is the block of $\mathbf{Y}^F$ covered by b.

Let $u \in P$ generate P modulo P_0 and write $u = xy$, $x \in \mathbf{X}$, $y \in \mathbf{Y}$. Since u is a p-element, we may assume that both x and y are p-elements.

Now consider an F-compatible regular embedding of $\mathbf{X}$ in $\tilde{\mathbf{X}}$ such that $\tilde{\mathbf{X}}^F$ is a finite general linear (or unitary) group. Since $\mathrm{Z}(\tilde{\mathbf{X}})$ is connected, there exists $z \in \mathrm{Z}^\circ(\tilde{\mathbf{X}})$ such that $g := xz^{-1} \in \tilde{\mathbf{X}}^F$. Further, we may choose z such that g is a p-element. Since $u = xy$ normalises P_1, x normalises P_1 and therefore g normalises P_1. Therefore $S = \langle P_1, g\rangle \leq \tilde{X}^F$ is a p-group. Since u normalises b_1 it also follows that b_1 is S-stable.

We claim that there exists a block of $\tilde{\mathbf{X}}^F$ covering b_1 with a defect group D containing S. Indeed, in order to prove the claim, it suffices to prove that $\mathrm{Br}_S(b_1) \neq 0$. Since b_1 and b_2 are both $\mathbf{G}^F$-stable,

$$0 \neq \mathrm{Br}_P(b) = \mathrm{Br}_P(b_1)\mathrm{Br}_P(b_2)$$

and consequently $\mathrm{Br}_P(b_1) \neq 0 \neq \mathrm{Br}_P(b_2)$. Hence writing $b_1 = \sum_{v\in\mathbf{X}^F} \alpha_v v$ as an element of the modular group algebra of $\mathbf{X}^F$ there exists $v \in \mathbf{X}^F$ with α_v non-zero such that v centralises P and in particular v centralises P_1 and u. Since z is central, and y centralises $\mathbf{X}$, we have that v also commutes with g. Hence v centralises S and it follows that $\mathrm{Br}_S(b_1) \neq 0$, proving the claim.

By the block theory of finite general linear (or unitary) groups (see [12]; noting that p divides $q-1$ in the linear case and that p divides $q+1$ in the unitary case) D is a Sylow p-subgroup of the centraliser of some semisimple element of $\tilde{\mathbf{X}}^F$. Since by hypothesis $P_1 = D \cap \mathbf{X}^F$ is abelian, we have that D is abelian, hence D is the Sylow p-subgroup of $\tilde{\mathbf{T}}^F$ for some F-stable maximal torus $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{X}}$. Set $\mathbf{T} = \mathbf{X} \cap \tilde{\mathbf{T}}$, an F-stable maximal torus of $\mathbf{X}$. Then $P_1 = D \cap \mathbf{X}^F$ is a Sylow p-subgroup of $\mathbf{T}^F$. Now $g = xz \in S \leq D \leq \tilde{\mathbf{T}}$, and $z \in \tilde{\mathbf{T}}$ (as z is central), hence $x = gz^{-1} \in \tilde{\mathbf{T}} \cap \mathbf{X} = \mathbf{T}$.

Set $\mathbf{G}_0 = \mathbf{TY}$. We have $u = xy \in \mathbf{G}_0^F$. Since $\mathbf{X} \cap \mathbf{Y} \leq \mathrm{Z}(\mathbf{X}) \leq \mathbf{T}$, we have that $\mathbf{G}_0^F \cap \mathbf{X}^F\mathbf{Y}^F = \mathbf{T}^F\mathbf{Y}^F$ and $\mathbf{G}_0^F/\mathbf{T}^F\mathbf{Y}^F$ is isomorphic to a subgroup of $\mathbf{G}^F/\mathbf{X}^F\mathbf{Y}^F$ and in particular has order p. Hence $\mathbf{G}_0^F = \langle \mathbf{T}^F\mathbf{Y}^F, u\rangle$. Let e be a block of $\mathbf{T}^F$ such that $eb_2 \neq 0$. Since $\mathbf{T}^F$ and $\mathbf{Y}^F$ commute, eb_2 is a block of $\mathbf{T}^F\mathbf{Y}^F$. Since $\mathbf{T}$ is central in $\mathbf{G}_0$, e is $\mathbf{G}_0^F$-stable. Further, b_2 is P-stable hence b_2 is $\mathbf{G}_0^F$-stable. So eb_2 is a $\mathbf{G}_0^F$-stable block of $\mathbf{T}^F\mathbf{Y}^F$ and therefore a block of $\mathbf{G}_0^F$. Since P_1 is the Sylow p-subgroup of $\mathbf{T}^F$ and $\mathbf{T}^F$ is abelian, P_1 is the defect group of e and P_2 is a defect group of b_2. Thus, P_1P_2 is a defect group of eb_2 as block of $\mathbf{T}^F\mathbf{Y}^F$. Since

$\mathrm{Br}_P(eb_2) = \mathrm{Br}_P(e)\mathrm{Br}_P(b_2)$ is non-zero, it follows by order considerations that P is a defect group of eb_2. □

6. The case $p = 3, 5$

In this section we handle the remaining exceptional groups of Lie type for $p \leq 5$.

Lemma 6.1. *Let G, H be finite groups, B a p-block of G and C a p-block of H such that B and C are Morita equivalent. Let P be a defect group of B, and Q a defect group of C. Suppose that P has exponent p. Then P is abelian if and only if Q is abelian. Further, P has an abelian subgroup of index p if and only if Q has an abelian subgroup of index p.*

Proof. By [21, Satz J], the exponent of defect groups is an invariant of Morita equivalence, hence Q has exponent p. In particular any abelian subgroup of P or of Q is elementary abelian. The remaining statements follow by the fact that Morita equivalence preserves the rank of the corresponding defect groups (see [2, Theorem 2.6]). □

Lemma 6.2. *Let $\mathbf{L}$ be connected reductive, with Frobenius morphism F, and let Z be a central p-subgroup of $\mathbf{L}^F$. Let b be a block of $\mathbf{L}^F$ and P a defect group of b. Suppose that P/Z is non-abelian, supports a transitive fusion system and $|P/Z| \geq p^4$. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{L}$, let c be a Bonnafé-Rouquier correspondent of b in $\mathbf{H}$ and let Q be a defect group of c. Then Q/Z has exponent p and Q/Z does not have an abelian subgroup of index p. In particular, a Sylow p-subgroup of $\mathbf{H}^F$ does not have an abelian subgroup of index p.*

Proof. Let $\bar{b}$ be the block of $\mathbf{L}^F/Z$ dominated by b and let $\bar{c}$ be the block of $\mathbf{H}^F/Z$ dominated by c. By [10, Prop. 4.1], $\bar{b}$ and $\bar{c}$ are Morita equivalent. Further, P/Z is a defect group of $\bar{b}$ and Q/Z is a defect group of $\bar{c}$. The result now follows from Lemma 2.1 and Lemma 6.1. □

Proposition 6.3. *Let $\mathbf{L}$ be connected reductive, in characteristic $r \neq p = 3$ with Frobenius morphism F, and suppose that $[\mathbf{L}, \mathbf{L}]$ is simply connected of type E_6 in characteristic $r \neq 3$. Let Z be a cyclic subgroup of $\mathrm{Z}(\mathbf{L}^F)$ of order 1 or 3 and let P be a defect group of $\mathbf{L}^F$. Suppose that P/Z supports a transitive fusion system and $|P/Z| \geq 3^7$. Suppose further that either $Z = 1$ or that $\mathbf{L}$ is simple. Then P/Z is abelian.*

Proof. Suppose that P/Z is non-abelian. Let $\mathbf{H}$ be an F-stable Levi subgroup of $\mathbf{L}$ and c a block of $\mathbf{H}^F$ such that c is quasi-isolated and b and c are Bonnafé-Rouquier correspondents. Let $s \in \mathbf{H}^*$ be a semisimple label of c (and b). Since b and c are Bonnafé-Rouquier correspondents, $\mathrm{C}_{\mathbf{L}^*}(s) = \mathrm{C}_{\mathbf{H}^*}(s)$. Let Q be a defect group of c. By Lemma 6.2, we may assume that Q/Z has exponent 3 and does not have an abelian subgroup of index 3. Note that all components of $\mathbf{L}$ and hence of $\mathbf{H}$ are simply connected.

If $\mathbf{H}^F$ has a component of type D_4 or D_5, then the only other possible components are of type A_1. We get a contradiction by Lemma 5.4(i), Lemma 6.2 and the fact that finite groups of type $D_4(q)$, $D_5(q)$, ${}^2D_4(q)$, ${}^2D_5(q)$ and ${}^3D_4(q)$ have a Sylow 3-subgroup with an abelian subgroup of index 3.

Thus, either all components of $\mathbf{H}$ are of type A or $\mathbf{H}$ has a component of type E_6. Let us first consider the case that all components of $\mathbf{H}$ are of type A. In particular, $\mathrm{C}^\circ_{\mathbf{H}^*}(s)$ is a Levi subgroup of $\mathbf{H}^*$ and since s has order prime to 3, $\mathrm{C}_{\mathbf{L}^*}(s) = \mathrm{C}_{\mathbf{H}^*}(s)$ is connected. It follows that s is central in $\mathbf{H}^*$, hence that Q is a defect group of a unipotent block of $\mathbf{H}^F$.

Suppose that $\mathbf{H}$ has a component $\mathbf{X}$ of type A_5. Then $\mathbf{X}$ is F-stable and is the only component of $\mathbf{H}$. If $\mathbf{X}^F$ does not contain a central element of order 3, then by Lemma 5.4(i), a Sylow 3-subgroup of $\mathbf{H}^F$ is a direct product of a Sylow 3-subgroup of $\mathbf{X}^F$ with the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$. Furthermore in this case a Sylow 3-subgroup of $\mathbf{X}^F$ has an abelian subgroup of index 3. If $\mathbf{X}^F$ contains a central element of order 3, then by [5, Prop. 3.3 and Theorem], the principal block is the only unipotent block of $\mathbf{X}^F$, and it follows that Q/Z has an element of order 9 since $\mathrm{PSL}_6(q)$ (respectively $\mathrm{PSU}_6(q)$) has elements of order 9 if $3 \mid q-1$ (respectively $3 \mid q+1$).

Suppose that $\mathbf{H}$ has a component of type A_4. Then the only other possible component is of type A_1 and it follows from Lemma 5.4(i) that a Sylow 3-subgroup of $\mathbf{H}^F$ has an abelian subgroup of index 3.

Suppose that $\mathbf{H}$ has a component $\mathbf{X}$ of type A_3. If all other components are of type A_1, then the above argument applies. If $\mathbf{H}$ has a component of type A_2, say $\mathbf{Y}$, then this is the only other component of $\mathbf{H}$. If the Sylow 3-subgroups of $\mathbf{X}^F$ are abelian, then Lemma 5.4(i) and Lemma 5.2 give the result. Thus, we may assume that the Sylow 3-subgroups of $\mathbf{X}^F$ are non-abelian. Thus, $\mathbf{X}^F$ is isomorphic to $\mathrm{SL}_4(q)$ (respectively $\mathrm{SU}_4(q)$) with $3 \mid q-1$ (respectively $3 \mid q+1$). Consequently, the principal block is the unique unipotent block of $\mathbf{X}^F$. In particular, Q contains a Sylow 3-subgroup of $\mathbf{X}^F$ and Q/Z has an element of order 9.

Thus, we may assume that all components of $\mathbf{H}$ are of type A_2 or A_1. By rank considerations, there can be at most two components of type A_2. By Lemma 5.4 (i) and Lemma 5.2 we may assume that there are two F-stable components $\mathbf{X}$ and $\mathbf{Y}$ of type A_2 such that both $\mathbf{X}^F$ and $\mathbf{Y}^F$ have central elements of order 3. Consequently, the principal block of $\mathbf{X}^F$ is the only unipotent block of $\mathbf{X}^F$ and similarly for $\mathbf{Y}^F$. The only other component of $\mathbf{H}$, if it exists is of type A_1, which also has a unique unipotent block. Hence Q is a Sylow 3-subgroup of $\mathbf{H}^F$.

Since $\mathbf{H}$ is a Levi subgroup of $\mathbf{L}$, there is surjective group homomorphism from $\mathrm{Z}(\mathbf{G})/\mathrm{Z}^\circ(\mathbf{G})$ to $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$ (see [3, Prop. 4.1]) and by hypothesis, $[\mathbf{L}, \mathbf{L}]$ is simple of type E_6. Hence $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$ is cyclic of order 1 or 3. Since $\mathbf{X}$ and $\mathbf{Y}$ are the only components of $\mathbf{H}$ with central elements of order 3, it follows that either $\mathrm{Z}(\mathbf{X})$ or $\mathrm{Z}(\mathbf{Y})$ covers $\mathrm{Z}(\mathbf{H})/\mathrm{Z}^\circ(\mathbf{H})$. Thus, either $\mathrm{Z}(\mathbf{X}) \leq \mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H})$ or $\mathrm{Z}(\mathbf{Y}) \leq \mathrm{Z}(\mathbf{X})\mathrm{Z}^\circ(\mathbf{H})$.

Assume that $\mathrm{Z}(\mathbf{X}) \le \mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H})$. Let $\mathbf{U}$ be the product of all components of $\mathbf{H}$ other than $\mathbf{X}$ and $\mathrm{Z}^\circ(\mathbf{H})$. Then, $\mathrm{Z}(\mathbf{X})^F \le (\mathrm{Z}(\mathbf{Y})\mathrm{Z}^\circ(\mathbf{H}))^F \le \mathbf{U}^F$ and hence $3 \mid |\mathbf{X}^F \cap \mathbf{U}^F|$. Since Q is a Sylow 3-subgroup of $\mathbf{H}^F$ and $|\mathbf{H}^F| = |\mathbf{X}^F||\mathbf{U}^F|$, Q is not contained in $\mathbf{X}^F\mathbf{U}^F$. Further, $Q \cap \mathbf{X}^F$ is a Sylow 3-subgroup of $\mathbf{X}^F$ and in particular is non-abelian of order at least 3^3. By Lemma 6.2, Q/Z has exponent 3. So, by Lemma 5.3, $1 \ne Z \le \mathrm{Z}(\mathbf{X})$ whence $Z = Z(\mathbf{X})$. Since $Z \ne 1$, $\mathbf{L}$ is simple by hypothesis. In particular, $Z = \mathrm{Z}(\mathbf{X})$ covers $\mathrm{Z}(\mathbf{G})/\mathrm{Z}^\circ(\mathbf{G})$. It follows that $\mathrm{Z}(\mathbf{Y}) \le \mathrm{Z}(\mathbf{X})\mathrm{Z}^\circ(\mathbf{H})$. By the same argument as above with $\mathbf{Y}$ replacing $\mathbf{X}$, we get that $Z = \mathrm{Z}(\mathbf{Y})$. In particular $\mathrm{Z}(\mathbf{X}) = \mathrm{Z}(\mathbf{Y})$, a contradiction since $\mathbf{X} \cap \mathbf{Y} = 1$.

Finally, consider the case that $\mathbf{H}$ has a component of type E_6. Then $\mathbf{H} = \mathbf{L}$ and $b = c$. Let b_0 be a block of $[\mathbf{L}, \mathbf{L}]^F$ covered by b and let $P_0 = P \cap [\mathbf{L}, \mathbf{L}]^F$ be a defect group of b_0. Let R be the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{L})^F$. By Lemma 5.4(i) applied with $\mathbf{X} = [\mathbf{L}, \mathbf{L}]$ and $\mathbf{Y} = \mathrm{Z}^\circ(\mathbf{L})$, $P \cap [\mathbf{L}, \mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F = P_0R$. So, P/P_0R is a subgroup of $\mathbf{L}^F/([\mathbf{L}, \mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F)$. Since $\mathbf{L}^F/([\mathbf{L}, \mathbf{L}]^F\mathrm{Z}^\circ(\mathbf{L})^F)$ is either trivial or has order 3, we have that P_0R has index at most 3 in P. If P_0 is abelian, then P and hence P/Z has an abelian subgroup of index 3. Thus, P_0 is non-abelian. We claim that $R \le P_0$. Indeed, by hypothesis, either $Z = 1$ or $[\mathbf{L}, \mathbf{L}] = \mathbf{L}$. If $\mathbf{L} = [\mathbf{L}, \mathbf{L}]$, then $R = 1$ and the claim holds trivially. If $Z = 1$, then P supports a transitive fusion system. Hence $R \le \mathrm{Z}(P) \le [P, P] \le [\mathbf{L}, \mathbf{L}]^F$ and the claim is proved. Thus, $P_0 = PR$ has index at most 3 in P.

Assume first that b_0 is unipotent. The unipotent 3-blocks of exceptional groups have been described in [11]. If b_0 is the principal block, then P/Z has exponent greater than 3. So, b_0 is non-principal and P_0 is non-abelian. By [11] (last part of the proofs for Tableau I), P_0 is the extension of a homocyclic group, say T, of rank 2 by a group of order 3. If T is not elementary abelian, then TZ/Z has exponent at least 9 and hence so does P/Z. Thus, we may assume that T is elementary abelian. So, $|P_0| = 3^3$ and $|P| \le 3^4$, a contradiction.

So, we may assume that b_0 is quasi-isolated but not unipotent. Here the blocks are described in [19, Section 4.3]. In particular, b_0 corresponds to one of lines 13, 14, or 15 of Table 4 of [19] (and the corresponding Ennola duals; see the last remark of Section 4 of [19]). If b_0 corresponds to line 15, then P_0 is abelian. If b_0 corresponds to line 14, then P_0 is the extension of a homocyclic group, say T, of rank 4 by a group of order 3. If T is not elementary abelian, then TZ/Z has exponent at least 9 and if T is elementary abelian, then $|P_0| \le 3^5$, whence $|P| \le 3^6$, a contradiction. If b_0 corresponds to line 13, then P_0 contains a subgroup isomorphic to a Sylow 3-subgroup of $\mathrm{SL}_6(q)$ with $3 \mid q - 1$. In particular, $\mho^1(P)$ is not cyclic. On the other hand, since P/Z has exponent 3, $\mho^1(P) \le Z$. This is a contradiction as Z is cyclic. □

Proposition 6.4. *Suppose that either $p = 3$ and $\mathbf{L}$ is simple and simply connected of type E_7 or E_8 in characteristic $r \ne 3$ or that $p = 5$ and $\mathbf{L}$ is simple of type E_8 in characteristic $r \ne 5$. Let F be a Frobenius morphism on $\mathbf{L}$ and let P be a defect group* *of a p-block of* $\mathbf{L}^F$*. Suppose that P supports a transitive fusion system and* $|P| \geq 3^7$ *if* $p = 3$*. Then P is abelian.*

Proof. Suppose if possible that P is not abelian. As before P has exponent p, and is indecomposable and P does not have an abelian subgroup of index p. Let $z \in \mathrm{Z}(P)$. Since $\mathbf{L}$ is simply connected, $\mathbf{H} := \mathrm{C}_{\mathbf{L}}(z)$ is a connected reductive subgroup of $\mathbf{L}$ of maximal rank and of semisimple rank at most 8 and by [24, Chapter 5, Theorem 9.6], P is a defect group of $\mathbf{H}^F$. The possible components of $\mathbf{H}$ are of type A, D, E_6 or E_7.

Let $\mathcal{X}$ be an F-stable subset of components of $\mathbf{H}$ and let $\mathbf{X}$ be the product of the elements of $\mathcal{X}$. Suppose that $\mathbf{X}^F$ does not have a central element of order p. By Lemma 5.4(i), $P = (P \cap \mathbf{X}^F) \times (P \cap \mathbf{Y}^F)$ where $\mathbf{Y}$ is the product of $\mathrm{Z}^\circ(\mathbf{H})$ and all components of $\mathbf{H}$ other than those in $\mathcal{X}$. The indecomposability of P implies that either $P \leq \mathbf{X}^F$ or $P \leq \mathbf{Y}^F$. Since z is a central p-element of $\mathbf{H}^F$, and $\mathbf{X}^F$ does not have a central element of order p, it follows that $P \leq \mathbf{Y}^F$. By replacing $\mathbf{H}$ by $\mathbf{Y}$, we may assume that the fixed points of every F-orbit of components of $\mathbf{H}$ have central elements of order p ($\mathbf{Y}$ may have rank less than $\mathbf{H}$). Thus, if $p = 5$ the only possible components are of type A_4 and if $p = 3$, then the only possible components are of type A_2, A_5, A_8 or E_6.

Suppose that $\mathbf{H}$ has an F- stable component $\mathbf{X}$ of type A_{p-1}. Let $\mathbf{Y}$ be the product of all components of $\mathbf{H}$ other than those in $\mathbf{X}$ with $\mathrm{Z}^\circ(\mathbf{H})$. By Lemma 5.4(i) and the indecomposability of P, we may assume that $\mathrm{Z}(\mathbf{X})^F \cap \mathrm{Z}(\mathbf{Y})^F$ and hence $\mathbf{H}^F/\mathbf{X}^F\mathbf{Y}^F$ has order p. So, by Lemma 5.4(ii), P is not contained in $\mathbf{X}^F\mathbf{Y}^F$. By Lemma 5.5, we may assume that $P \cap \mathbf{X}^F$ is not abelian since otherwise we can replace $\mathbf{X}$ by a torus. Since $\mathbf{X}^F$ has a central element of order p, $\mathbf{X}^F$ is a special linear (respectively unitary) group. The only non-abelian defect groups of a finite special linear (or unitary) group of degree p in non-describing characteristic are Sylow p-subgroups and $P \cap \mathbf{X}^F$ is a non-abelian defect group of $\mathbf{X}^F$. Thus, $P \cap \mathbf{X}^F$ is a Sylow p-subgroup of $\mathbf{X}^F$ and consequently has order at least p^p. Since we have shown above that P is not contained in $\mathbf{X}^F\mathbf{Y}^F$, by Lemma 5.3, P has an element of order p^2, a contradiction. Thus, we may assume that any component of $\mathbf{H}$ of type A_{p-1} lies in an F-orbit of size at least 2.

If $p = 5$, the only case left to consider is that $\mathbf{H}$ has two components of type A_4 (and these are the only ones) transitively permuted by F. In this case, by rank considerations, $\mathrm{Z}^\circ(\mathbf{H})$ is trivial, and hence $\mathbf{H}^F$ is isomorphic to a special linear or unitary group. In particular the Sylow 5-subgroups of $\mathbf{H}^F$ have an abelian subgroup of index 5, a contradiction. This completes the proof for the case that $p = 5$.

Now assume that $p = 3$. Let us first consider the case that there is a component $\mathbf{X}$ of $\mathbf{H}$ of type A_8. Then $\mathbf{H} = \mathbf{X} = \mathrm{SL}_8$ and we may argue as in the first part of the proof of Proposition 4.1.

Let us next consider the case that there is a component $\mathbf{X}$ of $\mathbf{H}$ of type A_5. If $\mathbf{X}$ also has a component of type A_2, then by rank consideration this is the unique component of type A_2 and we have ruled out this situation above. Thus $\mathbf{X}$ is the unique component of $\mathbf{H}$. Let P_0 be a defect group of a covered block of $\mathbf{X}^F$. The Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$ is contained in $\mathrm{Z}(P)$ and $\mathrm{Z}(P) \leq [P,P] \leq [\mathbf{X},\mathbf{X}] \cap \mathbf{H}^F \leq \mathbf{X}^F$, hence we have that the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$ is contained in $\mathbf{X}^F$ and in particular has order at most 3. Thus, P_0 has index at most 3 in P. In particular P_0 is non-abelian. Now $\mathbf{X} = \mathbf{M}/Z$, where $\mathbf{M}$ is a special linear group of degree 6 (with a compatible F-action) and Z is a central subgroup. Since $\mathrm{Z}(\mathbf{M})$ is cyclic of order 6 (or 3 if $r = 2$) and since $\mathbf{X}$ has a central element of order 3, Z is either trivial or of order 2, Z is F-stable and $Z^F = Z$. Further, $\mathbf{M}^F/Z$ is a normal subgroup of $\mathbf{X}^F = (\mathbf{M}/Z)^F$ of index $|Z|$. Thus P_0 is a defect group of $\mathbf{M}^F/Z$ and up to isomorphism a defect group of $\mathbf{M}^F$ and $\mathbf{M}^F = \mathrm{SL}_6(q)$ (respectively $\mathrm{SU}_6(q)$). Since $\mathbf{M}^F/Z$ has index prime to 3, $\mathbf{M}^F/Z$ contains the 3-part of the centre of $\mathbf{X}^F$, hence $\mathbf{M}^F$ has a central element of order 3. Thus, P_0 is the intersection with $\mathbf{X}^F$ of a Sylow 3-subgroup of the centraliser of a semisimple $3'$-element of $\mathrm{GL}_6(q)$ (or $\mathrm{GU}_6(q)$). Since P_0 has exponent 3 and is non-abelian, the possible structures of semisimple centralisers in $\mathrm{GL}_6(q)$ (or $\mathrm{GU}_6(q)$) force that the centraliser in $\mathrm{GL}_6(q)$ (respectively $\mathrm{GU}_6(q)$) has the form $\mathrm{GL}_3(q^2)$. Hence $|P_0| \leq p^3$ and $|P| \leq p^4$ a contradiction.

Suppose $\mathbf{H}$ has a component of type E_6. Arguing as in the previous case $\mathbf{H}$ has no components of type A_2 and hence the E_6-component is the unique component of $\mathbf{H}$. This component is of simply connected type since as explained in the beginning of the proof we may assume that the F-fixed point subgroup of every F-orbit of components of $\mathbf{H}$ has central elements of order 3 and we are done by Proposition 6.3 (note that we apply Proposition 6.3 here in the case that $Z = 1$).

The only case left to consider is that all components of $\mathbf{H}$ are of type A_2 and no component is F-stable. By rank considerations and the fact that groups of type E_8 do not have semisimple centralisers with component type A_2^4 (see the tables in [9]), we are left with two possibilities: either $\mathbf{H}$ has exactly three components, all of type A_2 and in a single F-orbit or $\mathbf{H}$ has exactly two components both of type A_2 and in a single F-orbit. In any case, $[\mathbf{H},\mathbf{H}]^F$ has a quotient or subgroup H_0 isomorphic to $\mathrm{PSL}_3(q)$ (respectively $\mathrm{PSU}_3(q)$) for some q such that $|[\mathbf{H},\mathbf{H}]^F|/|H_0|$ equals 1 or 3. Let $P_0 = P \cap [\mathbf{H},\mathbf{H}]$ and let P_0' be either the intersection of P_0 with H_0 or the image of P_0 in H_0. Then P_0' has exponent 3. Since any 3-subgroup of a finite projective special linear or unitary group of degree 3 has an abelian subgroup of index 3 and since the 3-rank of these groups is 2, it follows that $|P_0'| \leq 3^3$. Hence $|P_0| \leq 3^4$.

We claim that the index of P_0 in P is at most 3. Indeed, let R be the Sylow 3-subgroup of $\mathrm{Z}^\circ(\mathbf{H})^F$. Then $R \leq \mathrm{Z}(P) \leq [P,P] \leq [\mathbf{H},\mathbf{H}]$, that is $R \leq P_0$. On the other hand, $|P/P_0R|$ divides $|\mathrm{Z}([\mathbf{H},\mathbf{H}]^F)|_3$ and we have seen from the structure of $[\mathbf{H},\mathbf{H}]^F$ that $\mathrm{Z}([\mathbf{H},\mathbf{H}]^F)$ has order at most 3. This proves the claim. Hence $|P| \leq 3^5$, a contradiction. $\square$

7. Consequences

We note some consequences of Theorem 1.2.

Theorem 7.1. *Let B be a block of a finite group such that $k(B) - l(B) = 1$ (e. g. a block with multiplicity 1). Then B has elementary abelian defect groups.*

Proof. See proof of Theorem 3.6 in [23]. □

Corollary 7.2. *Let B be a block of a finite group such that $k(B) = 3$. Then B has elementary abelian defect groups.*

Proof. We have $l(B) \in \{1, 2\}$. In case $l(B) = 1$ it was shown by Külshammer [22] that the defect groups of B have order 3. The remaining case $l(B) = 2$ follows from Theorem 7.1. □

References

[1] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, Vol. 391, Cambridge University Press, Cambridge, 2011.

[2] C. Bessenrodt and W. Willems, *Relations between complexity and modular invariants and consequences for p-soluble groups*, J. Algebra **86** (1984), 445–456.

[3] C. Bonnafé, *Sur les caractères des groupes réductifs finis a centre non connexe: applications aux groupes spéciaux linéaires et unitaires*, Astérisque **306** (2006).

[4] C. Bonnafé, R. Rouquier, *Catégories dérivées et variétés de Deligne-Lusztig* , Publ. Math. Inst. Hautes Études Sci. **97** (2003), 1–59.

[5] M. Cabanes and M. Enguehard, *On unipotent blocks and their ordinary characters*, Invent. Math. **117** (1994), 149–164.

[6] M. Cabanes and M. Enguehard, *On blocks of finite reductive groups and twisted induction*, Advances Math. **145** (1999), 189–229.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[8] D. A. Craven, *The Theory of Fusion Systems*, Cambridge Studies in Advanced Mathematics, Vol. 131, Cambridge University Press, Cambridge, 2011.

[9] D. I. Deriziotis, *Centralisers of semisimple elements of the Chevalley groups E_7 and E_8*. Tokyo. J. Math **6** (1983), 191–216.

[10] C. Eaton, R. Kessar, B. Külshammer, B. Sambale, *2-blocks with abelian defect groups* Adv. Math. **254** (2014), 706–735.

[11] M. Enguehard, *Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais*, J. Algebra **230** (2000), 334–377.

[12] P. Fong and B. Srinivasan, *The blocks of finite general linear and unitary groups*, Inv. Math. **69** (1982), 101–153.

[13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.6.4*, 2013, `http://www.gap.system.org`.

[14] D. Gorenstein and R. Lyons, *The local structure of finite groups of characteristic 2 type*, Mem. Amer. Math. Soc. **42** (1983).

[15] D. Gorenstein, R. Lyons and R. Solomon, *The classification of finite simple groups 3*, Mathematical Surveys and Monographs **40**, American Mathematical Society (1998).

[16] C. Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order*, Geom. Dedic. **2** (1974), 425–460.
[17] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin 1967.
[18] B. Huppert and N. Blackburn, *Finite groups III*, Springer-Verlag, Berlin 1982.
[19] R. Kessar and G. Malle, *Quasi-isolated blocks and Brauer's height zero conjecture*, Ann. of Math. (2) **178** (2013) 321–384.
[20] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra **319** (2008), 806–823.
[21] B. Külshammer, *Bemerkungen über die Gruppenalgebra als symmetrische Algebra II*, J. Algebra **75** (1982), 59-69.
[22] B. Külshammer, *Symmetric local algebras and small blocks of finite groups*, J. Algebra **88** (1984), 190–195.
[23] B. Külshammer, G. Navarro, B. Sambale and P. H. Tiep, *Finite groups with two conjugacy classes of p-elements and related questions for p-blocks*, Bull. London Math. Soc. **46** (2014), 305–314
[24] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press Inc., Boston, MA, 1989.
[25] G. Navarro and P. H. Tiep, *Abelian Sylow subgroups in a finite group*, J. Algebra **398** (2014), 519-526.
[26] A. Ruiz and A. Viruel, *The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p*, Math. Z. **248** (2004), 45–65.
[27] B. Sambale, *Blocks of finite groups and their invariants*, Lecture Notes in Math. Vol. 2127, Springer-Verlag, Berlin, 2015.
[28] R. Steinberg, *Lectures on Chevalley Groups*, Notes by J. Faulkner and R. Wilson, Mimeographed notes, Yale University Mathematics Department (1968).

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