Minimal length in Quantum Mechanics and non-Hermitian Hamiltonian systems

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Deformations of the canonical commutation relations lead to non-Hermitian momentum and position operators and therefore almost inevitably to non-Hermitian Hamiltonians. We demonstrate that such type of deformed quantum mechanical systems may be treated in a similar framework as quasi/pseudo and/or $\mathcal{PT}$-symmetric systems, which have recently attracted much attention. For a newly proposed deformation of exponential type we compute the minimal uncertainty and minimal length, which are essential in almost all approaches to quantum gravity.

I. INTRODUCTION

The conventional space-time structure described locally in flat Minkowski space has not been confirmed experimentally up to several orders of the Planck length. This fact allows for the possibility to accommodate various modifications of the short distance structure which are needed to achieve consistency in several types of quantum theories, especially those which aim to incorporate gravity [1]. Numerous investigations in string theory [2] and alternative approaches to quantum gravity [3] have indicated the necessity to introduce a so-called minimal length, which constitutes a bound beyond which the localization of space-time events is no longer possible. Such type of limitations inevitably require some generalizations of the uncertainty relations, which usually originate from a modification of the related canonical commutation relations [4, 5, 6, 7, 8]. There exist also alternative mechanisms giving rise to a minimal length, but ultimately these approaches are all related [9]. Modifying the canonical commutation relations will almost unavoidably lead to operators, which are non-Hermitian or strictly speaking not self-adjoint and therefore they are not observable. Consequently the Hilbert space structure needs to be modified. This is a familiar scenario encountered in the context of non-Hermitian Hamiltonian systems with real eigenvalues; a field of interest initiated around ten years ago by the seminal paper [10] and which is a topic currently also explored experimentally [12], see [11] for a recent review. However, in the latter context the starting point is very different, namely a Hamiltonian rather than the set of canonical variables. The main objective here is to explore the similarities between these two scenarios and unravel their differences.

II. QUASI/PSEUDO HERMITIAN VERSUS DEFORMED QUANTUM MECHANICS

Non-Hermitian Hamiltonian systems with real eigenvalues allow for a consistent quantum mechanical description, may they be either quasi-Hermitian [13], pseudo-Hermitian [14] or/and $\mathcal{PT}$-symmetric [15]. When dealing with such type of systems one usually starts from some physical or mathematical motivation for a non-Hermitian Hamiltonian $H \neq H^\dagger$. Next one seeks a positive operator ρ, whose adjoint action corresponds to the Hermitian conjugation $\rho H \rho^{-1} = H^\dagger$. Factorizing this operator into a product of a Dyson operator η and its Hermitian conjugate in the form $\rho = \eta^\dagger \eta$ allows in a sufficient manner to compute an isospectral Hermitian Hamiltonian counterpart from $h = \eta H \eta^{-1} = h^\dagger$. A well known and important feature in this context is the fact that when given only the Hamiltonian H the subsequently constructed operator ρ, and therefore also η and h, is not unique. However, as was argued in [13] uniqueness can be achieved by one additional choice: One could fix directly the transforming operators ρ, η or one physical observable o' or $\mathcal{O}'$ either in the Hermitian or non-Hermitian system, respectively. These two choices, i.e. a definite form for the Hamiltonian H and an additional observable, will fix the metric uniquely, such that there are no ambiguities left in the interpretation of the physical observables. In the corresponding Hermitian Hamiltonian counterpart description of the system all physical observables o may be determined from $\eta \mathcal{O} \eta^{-1} = o$.

Schematically summarized the above can be described by the following sequence of steps

$$\begin{aligned} H \neq H^\dagger \overset{\rho}{\Rightarrow} \rho H \rho^{-1} = H^\dagger \overset{\rho=\eta^\dagger\eta}{\Rightarrow} \;& \eta H \eta^{-1} = h = h^\dagger \\ \overset{o',\mathcal{O}',\eta'}{\Rightarrow} \;& \eta \mathcal{O} \eta^{-1} = o. \end{aligned} \qquad (1)$$

As a consequence of the non-Hermiticity of H its eigenstates no longer form an orthonormal basis and the Hilbert space representation has to be modified. This is achieved by utilizing the operator ρ as a metric to define a new inner product $\langle \;\; | \;\; \rangle_\rho$ in terms of the standard inner product $\langle \;\; | \;\; \rangle$ as $\langle \Phi | \Psi \rangle_\rho := \langle \Phi | \rho \Psi \rangle$, for arbitrary states $\langle \Phi |$ and $| \Psi \rangle$. Observables $\mathcal{O}$ are then by construction Hermitian with regard to this metric $\langle \Phi | \mathcal{O} \Psi \rangle_\rho := \langle \mathcal{O} \Phi | \Psi \rangle_\rho$.

The treatment of deformed quantum mechanical systems is somewhat similar, although there are some crucial differences regarding the uniqueness of the construction resulting from a contrary starting point. In this latter context one commences from a modified version of the commutation relation between the dynamical variables corresponding to the position operator X and momentum operator P

$$[X(\alpha), P(\alpha)] = i\hbar f(X(\alpha), P(\alpha)), \tag{2}$$

for some arbitrary function f and α being a deformation parameter or possibly a collection of them. One recovers Heisenberg's canonical commutation relations for the standard momentum p_0 and position operator x_0 in some well defined limit

$$\lim_{\alpha\to 0}(X(\alpha), P(\alpha)) = (x_0, p_0) \quad \text{with } [x_0, p_0] = i\hbar. \tag{3}$$

In general, the operators defined in (2) are non-Hermitian, i.e. $X^\dagger \neq X$, $P^\dagger \neq P$, and one therefore has to proceed similarly as indicated in (1). First one seeks the metric operator ρ such that its adjoint action Hermitian conjugates the new variables X and P as $\rho(X,P)\rho^{-1} = (X^\dagger, P^\dagger)$. As opposed to the treatment of non-Hermitian Hamiltonian systems with real eigenvalues this metric will be unique, because one has already selected two observables from the very beginning which is sufficient according to the argumentation in [13]. Next one has to select a Hamiltonian in order to specify the physical system one wishes to describe. This is most naturally formulated in terms of the variables X and P, such that non-Hermitian Hamiltonians, i.e. $H^\dagger(X,P) \neq H(X,P)$, almost inevitably arise from deformations of the type (2), when the undeformed Hamiltonian $H(x_0,p_0) = \lim_{\alpha\to 0} H(X,P)$ was taken to be Hermitian. The deformed quantum mechanical system is now uniquely determined. The Hilbert space is constructed in the same manner as above, that is by a redefinition of the inner product by utilizing the metric operator ρ. Factorizing the metric operator into a product of η and its Hermitian conjugate allows as an alternative view to consider the entire system in the equivalent picture of its isospectral Hermitian counterpart

$$\eta(X,P,H,\ldots)\eta^{-1} = (x,p,h,\ldots) = (x^\dagger, p^\dagger, h^\dagger, \ldots). \tag{4}$$

When the undeformed Hamiltonian $H(x_0,p_0)$ is not Hermitian one also has to carry out the steps in (1) as X and P are no longer observables in that case.

The procedure works as shown below

$$\begin{aligned} &(x_0,p_0) \to (X,P) \overset{\rho}{\Rightarrow} \rho(X,P)\rho^{-1} = (X^\dagger, P^\dagger) \overset{\eta}{\Rightarrow} (x,p) \\ &= \eta(X,P)\eta^{-1} \Rightarrow \eta H(X,P)\eta^{-1} = \begin{cases} h(x,p) \\ H(x,p) \Rightarrow (1) \end{cases} \end{aligned} \tag{5}$$

We illustrate now the above general statements with some concrete examples:

There are various possibilities to deform the standard canonical commutation relations between the dynamical variables X and P. One might for instance deform Heisenberg's canonical commutation relations as $PX - qXP = i\hbar$ [6]. Here we will instead assume a q-deformation of the corresponding commutation relations between the creation operator $a^\dagger$ and the annihilation operator a in the general form

$$aa^\dagger - q^2 a^\dagger a = q^{g(N)}, \qquad \text{with } N = a^\dagger a \tag{6}$$

where g is some arbitrary function of the number operator N. The case g taken to be just 0 corresponds to the deformation studied for example in [4], whereas when $g(N) = N$ we recover the version explored for instance in [5]. In both cases the deformed Fock space can be constructed explicitly [7].

Assuming the representation for X and P to be still linear in a and $a^\dagger$ we define

$$X = \alpha a^\dagger + \beta a, \quad P = i\gamma a^\dagger - i\delta a, \qquad \alpha,\beta,\gamma,\delta \in \mathbb{R}. \tag{7}$$

Then with the help of (6) we can write $[X,P]$ as

$$\begin{aligned} [X,P] &= i\hbar q^{g(N)}(\alpha\delta + \beta\gamma) \\ &+ \frac{i\hbar(q^2-1)}{\alpha\delta+\beta\gamma}\left(\delta\gamma X^2 + \alpha\beta P^2 + i\alpha\delta XP - i\beta\gamma PX\right), \end{aligned} \tag{8}$$

together with the constraint $4\alpha\gamma = (q^2+1)$. The canonical commutation relations (3) are obtained in the simultaneous limit $(\alpha\delta + \beta\gamma) \to 1$, $q \to 1$. The relations (8) simplify by taking the limit $\beta \to \alpha$, $\delta \to \gamma$, in which case the commutator (8) reduces to a version studied for instance in [4] for the special case $g(N) = 0$

$$[X,P] = i\hbar q^{g(N)} + \frac{i\hbar}{4}(q^2-1)\left(\frac{X^2}{\alpha^2} + \frac{P^2}{\gamma^2}\right). \tag{9}$$

We may simplify (9) further by taking different limits to obtain a pure P-dependence on the right hand side.

Taking $g(N) = 0$ in (9), parameterizing the deformation parameter in the form $q = e^{2\tau\gamma^2}$ with $\tau \in \mathbb{R}^+$ and taking the limit $\gamma \to 0$ we obtain the simple deformation

$$[X,P] = i\hbar\left(1 + \tau P^2\right). \tag{10}$$

It is easy to find a representation for X and P which will reproduce (10) in terms of the standard momentum and position operators p_0 and x_0, respectively. We may for instance select $X = (1+\tau p_0^2)x_0$ and $P = p_0$. The undeformed case is obviously recovered in the limit $\tau \to 0$. As announced in the previous section we find that the operators associated to the deformed commutation relations are in general not Hermitian $X^\dagger = X + 2\tau i\hbar P$ and $P^\dagger = P$, albeit the simplified version (10) still allows one operator to remain Hermitian. According to (5), the unique metric operator ρ is constructed to $\rho = (1+\tau P^2)^{-1}$. At this point the positivity of τ becomes important, as it ensures the absence of singularities in the metric. In a momentum space representation $x_0 = i\hbar\partial_{p_0}$ this metric corresponds to the one found in [4], which in that formulation may be obtained through integration by parts.

Next we select the deformed harmonic oscillator as a specific example supplemented by the deformed commutation relations (10) together with the above mentioned representations for X and P

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)x_0(1+\tau p_0^2)x_0, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1+\tau p_0^2)x_0\right]. \end{aligned} \tag{11}$$

As the dynamical variables X and P are no longer Hermitian, the standard harmonic oscillator Hamiltonian in terms of the variables x_0 and p_0 ceases to be Hermitian as well when replacing (x_0,p_0) by (X,P)

$$H_{ho}^{\dagger}(X,P) = \rho(X,P) H_{ho}(X,P) \rho^{-1}(X,P). \tag{12}$$

By construction the Hamiltonian in (11) is pseudo and quasi Hermitian as can be checked very easily. Moreover the third version in (11) could be a standard starting point in the context of non-Hermitian $\mathcal{PT}$-symmetric quantum mechanical models, as the Hamiltonian evidently respects this symmetry, $[\mathcal{PT},H_{ho}] = 0$. The simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$ are realized as $\mathcal{P}: x_0 \to -x_0$, $p_0 \to -p_0$; $\mathcal{T}: x_0 \to x_0$, $p_0 \to -p_0$, $i \to -i$. The undeformed and deformed dynamical variables transform in the same manner under a $\mathcal{PT}$-operation, such that the $\mathcal{PT}$-symmetry is preserved in the deformation process $H(x_0,p_0) \to H(X,P)$.

Next we compute the corresponding quantities in the standard framework of the Hermitian counterpart. From the explicit form of ρ it is trivial to find the Dyson map $\eta = (1+\tau P^2)^{-1/2}$ according to (5), such that the physical variables corresponding to momentum and position in the Hermitian system result according to (4) to

$$x = (1+\tau p_0^2)^{1/2} x_0 (1+\tau p_0^2)^{1/2} \quad \text{and} \quad p = p_0. \tag{13}$$

These operators satisfy the same deformed canonical commutation relations as their counterparts in the non-Hermitian version of the theory. Consequently the Hermitian counterpart Hamiltonian becomes

$$\begin{aligned} h_{ho} &= \eta H_{ho} \eta^{-1} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2, \qquad (14)\\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)^{\frac{1}{2}} x_0 (1+\tau p_0^2) x_0 (1+\tau p_0^2)^{\frac{1}{2}}, \\ &= \left(\frac{1}{2m} - m\omega^2\hbar^2\tau^2\right) p_0^2 + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 \right. \\ &\qquad \left. + 4i\hbar\tau p_0 (1+\tau p_0^2) x_0 - \hbar^2 \tau\right]. \end{aligned}$$

Recalling that the commutation relations were the starting point in the first place, we notice that the non-Hermitian nature of the construction could have been avoided from the beginning when selecting the equivalent Hermitian representation (13) right from the start, although a priori this would be less obvious to guess.

Taking now $g(N) = N/\gamma$ and parameterizing the deformation parameter in the form $q = e^{\tau\gamma^3}$ with $\tau \in \mathbb{R}^+$, the limit $\gamma \to 0$ yields an exponential deformation of the canonical commutation relations

$$[X,P] = i\hbar e^{\tau P^2}. \tag{15}$$

Representations for X and P in terms of x_0 and p_0 are easily found. We choose $X = e^{\tau p_0^2} x_0$ and $P = p_0$, which reduces to the previous deformation to first order in τ. As we found also above, the representations are not Hermitian $X^\dagger = X + 2\tau i\hbar P e^{\tau P^2}$ and $P^\dagger = P$. The unique metric operator which conjugates X and P is then found to be $\rho = e^{-\tau P^2}$.

Let us now specify a concrete Hamiltonian depending on these variables. Of course we could also study the deformed harmonic oscillator in terms of the variables corresponding to (15) along the lines outlined above, but instead we investigate a non-Hermitian version of it, the so-called Swanson model [16]

$$H_S(X,P) = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 + i\mu\{X,P\}. \tag{16}$$

Even in the limit $\tau \to 0$ this Hamiltonian remains non-Hermitian and consequently replacing the non-Hermitian variables (X,P) by their Hermitian counterparts according to (5)

$$x = e^{\frac{\tau}{2}p_0^2} x_0 e^{\frac{\tau}{2}p_0^2} \qquad \text{and} \qquad p = p_0, \tag{17}$$

with $\eta = e^{-\frac{\tau}{2}P^2}$ will not render it Hermitian

$$H_S(x,p) = \frac{x^2}{2m} + \frac{m\omega^2}{2} x^2 + i\mu\hbar m\omega^2\{x,p\}. \tag{18}$$

Nonetheless, we may invoke the steps indicated in (1) and find an isospectral Hermitian counterpart

$$h_S(x,p) = \left(\frac{1}{2m} + 2\mu^2\hbar^2 m\omega^2\right) p^2 + \frac{m\omega^2}{2} x^2, \tag{19}$$

using the transformation $\tilde{\eta} H_S(x,p) \tilde{\eta}^{-1} = h_S(x,p)$ with $\tilde{\eta} = e^{\mu p_0^2}$. A crucial point to notice is that the Hermitian variables (x,p) are only observables in the Hermitian system $h_S(x,p)$, whereas the counterparts in the deformed non-Hermitian system are $\tilde{X} = \tilde{\eta}^{-1}\eta X \eta^{-1}\tilde{\eta}$ and $\tilde{P} = \tilde{\eta}^{-1}\eta P \eta^{-1}\tilde{\eta} = p_0$.

Similar arguments will always hold when $\tilde{\eta}$ and η commute, which for the deformations (10) and (15) is the case when $\tilde{\eta}$ only depends on p_0. A further non-trivial example for this is for instance the $-x^4$-potential, which despite being unbounded from below can be shown to posses a well defined spectrum when making a suitable variable transformation and a subsequent similarity transformation of the form $\tilde{\eta} = e^{\alpha p_0^3 + \beta p_0}$ with $\alpha, \beta \in \mathbb{R}$ [17].

III. MINIMAL LENGTH

An important physical consequence resulting from the deformation of the canonical commutation relations is the unavoidable occurrence of a minimal length. Considering the uncertainty relations for the deformed canonical observables of the system (16)

$$\Delta\tilde{X}\Delta\tilde{P} = \Delta x \Delta p \geq \frac{1}{2}\left|\left\langle [\tilde{X},\tilde{P}] \right\rangle_{\tilde{\rho}\rho^{-1}}\right| = \frac{1}{2}\left|\langle [x,p]\rangle\right| \tag{20}$$

it is clear that a minimal length must always arise once the right hand side of (20) involves higher powers of $\tilde{P}$.

One is then naturally led to a minimal uncertainty in the limit $\Delta\tilde{X} \to 0$ as then the momentum uncertainty $\Delta\tilde{P}$ becomes very large, but eventually the linear behaviour on the left hand side of the inequality will be too weak to balance the higher powers of $\tilde{P}$ on the right hand side. Consequently the limit $\Delta\tilde{X} \to 0$ can not be reached without violating the inequality (20) and a localization in space is no longer possible.

We now compute the explicit value for the minimal length for the deformed commutation relation (15) associated to the Hamiltonian (16)

$$\Delta\tilde{X}\Delta\tilde{P} \geq \frac{i\hbar}{2} e^{\tau\langle\tilde{P}^2\rangle_{\tilde{\rho}\rho^{-1}}} = \frac{i\hbar}{2} e^{\tau\left[\langle\tilde{P}\rangle^2_{\tilde{\rho}\rho^{-1}} + (\Delta\tilde{P})^2\right]}. \quad (21)$$

In order to determine the minimal value for $\Delta\tilde{X}$ we have to solve the two equations

$$\partial_{\Delta\tilde{P}} f(\Delta\tilde{X}, \Delta\tilde{P}) = 0 \qquad \text{and} \qquad f(\Delta\tilde{X}, \Delta\tilde{P}) = 0, \quad (22)$$

with $f(\Delta\tilde{X}, \Delta\tilde{P}) = \Delta\tilde{X}, \Delta\tilde{P} - \frac{i\hbar}{2} e^{\tau\left[\langle\tilde{P}\rangle^2_{\tilde{\rho}\rho^{-1}} + (\Delta\tilde{P})^2\right]}$ for $\Delta\tilde{X}$. There is no general solution to this equation, but we may find a minimal value order by order in τ, when expanding $f(\Delta\tilde{X}, \Delta\tilde{P})$ in powers of τ as $f(\Delta\tilde{X}, \Delta\tilde{P}) = a_0 + a_1\tau + a_2\tau^2 + \ldots$ To first order we find the minimal uncertainty

$$\Delta\tilde{X}^{(1)}_{\min} = \hbar\sqrt{\tau}\sqrt{1 + \tau\left\langle\tilde{P}\right\rangle^2_{\tilde{\rho}\rho^{-1}}}, \quad (23)$$

which corresponds to the value already found in [4]. This is to be expected as to that order the deformed commutation relations (10) and (15) coincide. Expanding f to second order and solving (22) thereafter yields

$$(\Delta\tilde{X}^{(2)}_{\min})^2 = \frac{\hbar^2\tau}{27}\left\{\left[7 + 4\tau\langle p\rangle^2(2 + \tau\langle p\rangle^2)\right]^{\frac{3}{2}} + (1 + \tau\langle p\rangle^2)\left[17 + 8\tau\langle p\rangle^2(2 + \tau\langle p\rangle^2)\right]\right\}, \quad (24)$$

where we used $\left\langle\tilde{P}\right\rangle^2_{\tilde{\rho}\rho^{-1}} = \langle p\rangle$. Since τ is positive the smallest values this expression can acquire, say $\Delta\tilde{X}_0^{(\kappa)}$ with κ denoting the order, occur when $\langle p\rangle = 0$ since $\langle p\rangle$ is always real. We perform this minimization order by order in units of $\hbar\sqrt{\tau}$: $\Delta\tilde{X}_0^{(1)} = 1$, $\Delta\tilde{X}_0^{(2)} = \sqrt{\frac{17+7\sqrt{7}}{27}} = 1.14698088$, $\Delta\tilde{X}_0^{(3)} = 1.16373131$, $\Delta\tilde{X}_0^{(4)} = 1.16562060$, $\Delta\tilde{X}_0^{(5)} = 1.16580546$, $\Delta\tilde{X}_0^{(6)} = 1.16582082$, $\Delta\tilde{X}_0^{(7)} = 1.16582191$, $\Delta\tilde{X}_0^{(8)} = 1.16582198$, $\Delta\tilde{X}_0^{(9)} = 1.1658219905$, $\Delta\tilde{X}_0^{(10)} = 1.1658219907$. Thus for the deformed commutation relations (15) we observe a fast convergence of the absolute minimal length to a value of $\Delta\tilde{X}_0 \approx 1.16582199\hbar\sqrt{\tau}$. This means the minimal length resulting either from the deformation (10) or (15) differ by around 16%, which allows for new opportunities in the context of theories describing quantum gravity.

IV. CONCLUSIONS

We demonstrated that non-Hermitian Hamiltonian systems and deformed quantum mechanics based on deformations of the uncertainty relations may be treated in a similar fashion. The key difference between the two scenarios is that in the former the starting point are Hamiltonians whereas in the latter one commences with a set of dynamical variables. Uniqueness of the construction is therefore only guaranteed in the latter case. The absolute minimal length is entirely governed by the choice of the observables and not by the explicit form of the underlying Hamiltonian. Representation for the generic version (8) will lead to more involved non-Hermitian Hamiltonians, different minimal uncertainties and lengths and in addition to minimal momenta.

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[1] X. Calmet, M. Graesser, and S. D. H. Hsu, Phys. Rev. Lett. **93**, 211101 (2004).
[2] D. Gross and P. Mende, Nucl. Phys. **B303**, 407 (1988); D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. **B216**, 41 (1989).
[3] C. Rovelli, Living Rev. Rel **11**, 5 (2008).
[4] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. **D52**, 1108 (1995).
[5] G. Brodimas, A. Jannussis, and R. Mignani, J. Phys. **A25**, L329 (1992).
[6] J. Schwenk and J. Wess, Phys. Lett. **B291**, 273 (1992).
[7] L. C. Biedenham, J. Phys. **A22**, L873 (1989); A. J. Macfarlane, J. Phys. **A22**, 4581 (1989); C.-P. Su and H.-C. Fu, J. Phys. **A22**, L983 (1989).
[8] C. Quesne and V. M. Tkachuk, SIGMA **3**, 016 (2007).
[9] S. Hossenfelder, Class. Quant. Grav. **23**, 1815 (2006).
[10] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[11] C. M. Bender, Rept. Prog. Phys. **70**, 947 (2007).
[12] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. **103**, 030402 (2009).
[13] J. Dieudonné, Proc. of the Int. Symp. on Linear Spaces, Pergamon, Oxford, 115 (1961); F. G. Scholtz, H. B. Geyer, and F. Hahne, Ann. Phys. **213**, 74 (1992).
[14] M. Froissart, Il Nuovo Cimento **14**, 197 (1959); E. C. G. Sudarshan, Phys. Rev. **123**, 2183 (1961); A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).
[15] C. M. Bender, D. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002); Phys. Rev. **D70**, 025001 (2004); E. Wigner, J. Math. Phys. **1**, 409 (1960).
[16] M. S. Swanson, J. Math. Phys. **45**, 585 (2004).
[17] H. F. Jones and J. Mateo, Phys. Rev. **D73**, 085002 (2006).