Compactons versus Solitons

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We investigate whether the recently proposed $\mathcal{PT}$-symmetric extensions of generalized Korteweg-de Vries equations admit genuine soliton solutions besides compacton solitary waves. For models which admit stable compactons having a width which is independent of their amplitude and those which possess unstable compacton solutions the Painlevé test fails, such that no soliton solutions can be found. The Painlevé test is passed for models allowing for compacton solutions whose width is determined by their amplitude. Consequently these models admit soliton solutions in addition to compactons and are integrable.

I. INTRODUCTION

In a recent investigation Bender, Cooper, Khare, Mihaila and Saxena [1] have found compacton solutions, i.e. solitary wave solutions with compact support, for $\mathcal{PT}$-symmetric extensions of generalized Korteweg-de Vries (KdV) equations. The proposed models generalize various systems previously studied and are described by the Hamiltonian density

$$\mathcal{H}_{l,m,p} = -\frac{u^l}{l(l-1)} - \frac{g}{m-1} u^p (iu_x)^m. \tag{1}$$

The density $\mathcal{H}_{l,2,p}$ reduces to a modification of a Hamiltonian description [2, 3] of generalized KdV-equations [4], which are known to admit compacton solutions. For $l = 3$, $p = 0$ and $m = \varepsilon + 1$ one obtains a re-scaled version of the $\mathcal{PT}$-symmetric extension of the KdV-equation ($\varepsilon = 1$) introduced in [5]. The first $\mathcal{PT}$-symmetric extensions of the KdV-equation proposed in [6] can not be obtained from (1) as they correspond to non-Hamiltonian systems.

The virtue of $\mathcal{PT}$-symmetry, i.e. invariance under a simultaneous parity transformation $\mathcal{P} : x \to -x$ and time reversal $\mathcal{T} : t \to -t, i \to -i$, for a classical Hamiltonian is that it guarantees the reality of the energy due to its antilinear nature [5]. When quantizing $\mathcal{H}$ one also needs to ensure $\mathcal{PT}$-symmetry of the corresponding wavefunctions in order to obtain real spectra [7, 8, 9, 10]. The most natural way to implement $\mathcal{PT}$-symmetry in (1) is to keep the interpretation from the standard KdV-equation and view the field u as a velocity, such that it transforms as $u \to u$. Then $\mathcal{H}_{l,m,p}$ is $\mathcal{PT}$-symmetric for real coupling constant g and all possible real values of l, m, p. Alternatively, we could also allow a purely complex coupling constant, i.e. $g \in i\mathbb{R}$, by transforming the field as $u \to -u$, such that $\mathcal{H}_{l,m,p}$ is $\mathcal{PT}$-symmetric when l is even and $p + m$ odd. For general reviews on $\mathcal{PT}$-symmetry and non-Hermitian Hamiltonian systems see [11, 12, 13].

The equation of motion resulting from the variational principle

$$u_t = \left(\frac{\delta \int \mathcal{H} dx}{\delta u} \right)_x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{d^n}{dx^n} \frac{\partial \mathcal{H}}{\partial u_{nx}} \right)_x \tag{2}$$

for the Hamiltonian density $\mathcal{H}_{l,m,p}$ in (1) is

$$u_t + u^{l-2} u_x + g i^m u^{p-2} u_x^{m-3} \left[p(p-1) u_x^4 + 2pmu u_x^2 u_{xx} + m(m-2) u^2 u_{xx}^2 + m u^2 u_x u_{xxx} \right] = 0. \tag{3}$$

The main aim of this manuscript is to investigate whether this equation admits soliton solutions and is therefore integrable for some specific choices of the parameters l, m, p. We will also address the question of whether it is possible to find solitons and compactons in the same model or whether only one type of solutions may exist. To answer these questions one could of course construct explicitly the soliton solutions, conserved charges, Lax pairs, Dunkl operators, etc., which is usually a formidable task. Instead we will carry out the Painlevé test following a proposal originally made by Weiss, Tabor and Carnevale [14]. The test provides an indication about the existence of soliton solutions and discriminates between models, which are integrable those which are not.

II. THE PAINLEVÉ TEST

The basic assumption for the existence of a soliton solution is that it acquires the general form of a so-called Painlevé expansion [14]

$$u(x,t) = \sum_{k=0}^{\infty} \lambda_k(x,t) \phi(x,t)^{k+\alpha}. \tag{4}$$

One further demands that in the limit $\phi(x,t) \to 0$, the function $u(x,t)$ is meromorphic, such that the leading order singularity α is a negative integer and the $\lambda_k(x,t)$ are analytic functions. The general procedure of the Painlevé test consists in substituting the expansion (4) into the equation of motion, (3) for the case at hand, and determining the functions $\lambda_k(x,t)$ recursively. A partial differential equation is said to pass the Painlevé test when all $\lambda_k(x,t)$ can be computed, including enough free parameters to match the order of the differential equation. In [15] we recently applied this method to $\mathcal{PT}$-symmetric extensions of Burgers and the standard KdV-equation, where more details on the generalities and literature may be found.

A. Leading order singularities

The Painlevé test stays and falls with the possibility that the initial condition λ_0 can be determined, which is essential to commence the iterative procedure to solve the recurrence relation. We compute λ_0 by substituting the first term in the expansion (4), i.e. $u(x,t) \to \lambda_0(x,t)\phi(x,t)^\alpha$, into (3) and evaluating the values for all possible leading order singularities α. The individual terms in (3) have the following leading order behaviour: $u_t \sim \phi^{\alpha-1}$, $u^{l-2}u_x \sim \phi^{\alpha(l-1)-1}$ and all remaining terms are proportional to $\phi^{\alpha(m+p-1)-m-1}$. Therefore the leading order terms may only be cancelled if any of the following three conditions hold:

i) $\alpha - 1 = \alpha(l-1) - 1 \leq \alpha(m+p-1) - m - 1$,

which results from assuming that u_t and $u^{l-2}u_x$ constitute the leading order terms. In this case we obtain $l = 2$ and the inequality $\alpha(2-m-p) \leq -m$. Thus α remains undetermined.

ii) $\alpha - 1 = \alpha(m+p-1) - m - 1 \leq \alpha(l-1) - 1$,

which corresponds to the assumption that $u^{l-2}u_x$ is the least singular term and matching the leading orders of all the remaining ones. Then we conclude that $l \leq 2$ and α is fixed to $\alpha = m/(m+p-2)$.

iii) $\alpha(l-1) - 1 = \alpha(m+p-1) - m - 1 \leq \alpha - 1$.

which is the consequence of u_t being least singular term and the matching of the remaining ones. This means the leading order singularity of $u(x,t)$ is of the order

$$\alpha = \frac{m}{p+m-l} \in \mathbb{Z}^- \quad \text{and} \quad l \geq 2. \tag{5}$$

Cancelling the leading order terms then yields

$$\lambda_0^{(n)} = e^{2\pi i n\alpha/m}[gl(l-1)]^{-\alpha/m}(i\alpha\phi_x)^{-\alpha}, \tag{6}$$

where $1 \leq n \leq p+m-l$ indicates the different roots of the determining equation.

In principle we could also envisage a scenario in which u_t and $u^{l-2}u_x$ are the least dominant terms and the leading order singularity is cancelled by all the remaining terms. However, all these terms only differ by an overall numerical factor, such that λ_0 turns out to be zero in this case and we can therefore discard this case.

B. Resonances

A key feature of the Painlevé test is the occurrence of so-called resonances, which arise whenever the coefficient in front of a specific λ_r in the recurrence relations becomes zero. This implies that λ_r can not be determined recursively. When in this case the remaining part of the recurrence relation becomes an identity, the λ_r becomes a free parameter, otherwise the Painlevé test fails. The possible values for r can be found by substituting

$$u(x,t) \to \lambda_0(x,t)\phi(x,t)^\alpha + \lambda_r\phi(x,t)^{r+\alpha} \tag{7}$$

into (3) and computing all possible values of r for which λ_r becomes a free parameter. Considering the case *iii)* for integer values l, m, p the coefficients of the leading order $\phi^{r+\alpha(l-1)-1}$ is proportional to

$$\lambda_r g^{\alpha(2-l)/m}(r+1)(r+\alpha l)[r+\alpha(l-1)]\phi_x^{\alpha(2-l)+1}. \tag{8}$$

This means that besides the so-called fundamental resonance at $r = 1$, we also find two more resonances at $r = -\alpha l, \alpha(1-l)$. Since the differential equation (3) is of order three all these models fully pass the Painlevé test provided $\lambda_{-\alpha l}$ and $\lambda_{\alpha(1-l)}$ can indeed be chosen freely.

The standard procedure to verify this would be now to derive the recursive equation resulting from combining (4) and (3). Clearly for generic values of l, m, p this will be extremely lengthy, but even for specific choices it is fairly complicated. It suffices, however, to compute the λ_k up to $k > -\alpha l$. We will present these values for various examples for several choices of the parameters l, m, p corresponding to scenarios leading to solutions with qualitatively different kinds of behaviour.

III. GENERALIZED KDV-EQUATION

Cooper, Khare and Saxena [16] found that in the generalized KdV equation, i.e. $m = 2$, a necessary condition for compactons to be stable is to consider models with $2 < l < p+6$. This means none of the conditions *i)* or *ii)* for the leading order singularity to cancel can be satisfied. The special choice $l = p+2, 0 < p \leq 2$ guarantees that the compacton solutions have in addition a width which is independent of their amplitude [2]. For that particular case also the condition *iii)* admits no solution, such that the Painlevé test fails.

However, for models which admit stable compacton solutions having a width depending on the amplitude we can find solutions to the condition *iii)* and proceed with the Painlevé test. For instance, $m = 2, p = 1, l = 5$ is such a choice. In this case we find from (5) that $\alpha = -1$ and the leading order singularity of the corresponding differential equation is ϕ^{-5}. Computing now order by order the functions λ_k we find the two solutions

$$\begin{aligned} \lambda_0^\pm &= \pm 2i\sqrt{5g}\phi_x, \quad \lambda_1^\pm = \mp i\sqrt{5g}\frac{\phi_{xx}}{\phi_x}, \\ \lambda_2^\pm &= \mp i\frac{\sqrt{5g}}{6}\frac{\left(3\phi_{xx}^2 - 2\phi_x\phi_{xxx}\right)}{\phi_x^3}, \\ \lambda_3^\pm &= \frac{3\phi_t\phi_x^2 \mp 4i\sqrt{5g^3}\left(6\phi_{xx}^3 - 6\phi_x\phi_{3x}\phi_{xx} + \phi_x^2\phi_{4x}\right)}{48g\phi_x^5}. \end{aligned} \tag{9}$$

Crucially we observe next that $\lambda_4^\pm$ and $\lambda_5^\pm$ can be chosen arbitrarily. The remaining $\lambda_k^\pm$ for $k > 5$ can all be computed, but the expressions are all extremely cumbersome and we will therefore not report them here. Making, however, the further assumption on ϕ to be a travelling wave, i.e. $\phi(x,t) = x - \omega t$, simplifies the expressions considerably. Choosing $\lambda_4^{\pm} = \lambda_5^{\pm} = 0$ the two solutions for that scenario reduce to

$$\begin{aligned} \lambda_{3\kappa+1}^{\pm} &= \lambda_{3\kappa+2}^{\pm} = 0 \qquad \text{for } \kappa = 0,1,2,\ldots \\ \lambda_0^{\pm} &= \pm 2i\sqrt{5g},\ \lambda_3^{\pm} = -\frac{\omega}{16g}, \\ \lambda_6^{\pm} &= \mp \frac{3i\omega^2}{3584\sqrt{5}g^{5/2}},\ \lambda_9^{\pm} = \frac{\omega^3}{573440g^4}, \\ \lambda_{12}^{\pm} &= \pm \frac{33i\omega^4}{1669857280\sqrt{5}g^{11/2}}, \\ \lambda_{15}^{\pm} &= -\frac{3\omega^5}{66794291200g^7}, \ldots \end{aligned} \tag{10}$$

We conclude that the Painlevé test is passed for this choices of parameters, which means that besides stable compacton solutions, whose width depends on their amplitude, we also find genuine solitons in these models and, provided the series (4) converges, they are therefore integrable.

In the unstable compacton regime, i.e. $l \leq 2$ or $l \geq p+6$, the condition $iii)$ can not be satisfied. Consequently we do not expect to find genuine soliton solutions. We have also verified this type of behaviour for other representative examples which we do not present here.

IV. $\mathcal{PT}$-SYMMETRIC GENERALIZED KDV-EQUATION

For the $\mathcal{PT}$-symmetric extensions of the generalized KdV-equation (3) the necessary condition for compactons to be stable was extended by Bender et al [1] to $2 < l < p+3m$. Thus also for generic values of m none of the conditions $i)$ or $ii)$ for the leading order singularity to cancel can be satisfied. Furthermore, the requirement for stable compacton solutions to possess also a width which is independent of their amplitude was generalized in [1] to $l = p+m$. As for the special case $m = 2$ this value coincides with the leading order singularity resulting from the condition $iii)$ tending to infinity and therefore the Painlevé test fails.

As in the previous case, for models which have stable compacton solutions whose width is a function of their amplitude the Painlevé test has a chance to pass, as one can find a value for the leading order singularity and potentially has the correct amount of resonances. We verify this for the example $m = 3, p = 1, l = 7$, for which we obtain $\alpha = -1$ and ϕ^{-7} as the leading order singularity in (3). Since $-\alpha/m = 1/3$ in this case, we find now three non-equivalent solutions related to the different roots for the $\lambda_k^{(n)}$ with $n = 1,2,3$, of which the first terms are

$$\begin{aligned} \lambda_0^{(n)} &= -ie^{2\pi in/3}(42g)^{1/3}\phi_x, \\ \lambda_1^{(n)} &= \frac{ie^{2\pi in/3}(21g)^{1/3}\phi_{xx}}{2^{2/3}\phi_x}, \\ \lambda_2^{(n)} &= \frac{ie^{2\pi in/3}(7g)^{1/3}\left(3\phi_{xx}^2 - 2\phi_x\phi_{xxx}\right)}{2(6)^{2/3}\phi_x^3}, \\ \lambda_3^{(n)} &= \frac{ie^{2\pi in/3}(7g)^{1/3}\left(6\phi_{xx}^3 - 6\phi_x\phi_{xxx}\phi_{xx} + \phi_x^2\phi_{xxxx}\right)}{4(6)^{2/3}\phi_x^5}. \end{aligned} \tag{11}$$

From (8) we know that we should encounter resonances at the level 6 and 7, which is indeed the case as we find that $\lambda_6^{(n)}$and $\lambda_7^{(n)}$ can be chosen freely. The remaining $\lambda_k^{(n)}$ for $k > 7$ can all be computed iteratively and the Painlevé test is passed for this example.

For a travelling wave ansatz $\phi(x,t) = x - \omega t$ with the choice $\lambda_6^{(n)} = \lambda_7^{(n)} = 0$ the expressions simplify to

$$\begin{aligned} \lambda_{5\kappa+1}^{(n)} &= \lambda_{5\kappa+2}^{(n)} = \lambda_{5\kappa+3}^{(n)} = \lambda_{5\kappa+4}^{(n)} = 0, \text{ for } \kappa = 0,1,\ldots \\ \lambda_0^{(n)} &= -ie^{2\pi in/3}(42g)^{1/3},\ \lambda_5^{(n)} = \frac{e^{4\pi in/3}\omega}{36(42)^{1/3}g^{4/3}}, \\ \lambda_{10}^{(n)} &= \frac{17i\omega^2}{598752g^3}, \\ \lambda_{15}^{(n)} &= -\frac{53e^{\frac{2in\pi}{3}}\omega^3}{21555072(42)^{2/3}g^{14/3}}, \ldots \end{aligned} \tag{12}$$

Thus we observe no qualitative difference in the $\mathcal{PT}$-symmetric extensions in comparison to the case $m = 2$ and find that also in this one may have stable compacton solutions, whose width depends on their amplitude and genuine solitons at the same time.

In the unstable compacton regime, that is $l \leq 2$ or $l \geq p+3m$, the condition $iii)$ can not be satisfied and the Painlevé test fails. Once again we do not represent here other representative examples for which we obtained the same type of behaviour.

V. DEFORMATIONS OF BURGERS EQUATION

Considering $m = 1, p = 1, l = 3$ in the equation of motion (3) is a very simple example leading to a Painlevé expansion for $u(x,t)$, which can even be truncated after the second term. As this type of behaviour is reminiscent of Bäcklund transformation generating solutions found in other models [14], we present this case briefly. For this choice (3) simply reduces to

$$u_t + uu_x - 2igu_{xx} + \frac{igu}{u_x^2}\left(u_{xx}^2 - u_x u_{xxx}\right) = 0, \tag{13}$$

which can be viewed as a deformation of Burgers equation [15] corresponding to the first three terms. Proceeding as in the previous sections, we find the solution

$$u(x,t) = \frac{-6ig\phi_x}{\phi} + \frac{6ig\phi_{xx} - 3\phi_t}{2\phi_x} \tag{14}$$

provided that ϕ satisfies the equation

$$\phi_x^2\phi_{tt} + \phi_t^2\phi_{xx} = 2\phi_{tx}\phi_t\phi_x. \quad (15)$$

A travelling wave $\phi(x,t) = x - \omega t$ is for instance a solution of (15), such that we obtain the simple expression

$$u(x,t) = \frac{6ig\phi_x}{\omega t - x} + \frac{3}{2}\omega \quad (16)$$

for the solution of (13). Incidentally, the travelling wave solution for Burger's equation [14] coincides with (16).

VI. CONCLUSIONS

In previous investigations [2, 3, 16] various criteria have been found, which separate the models $\mathcal{H}_{l,m,p}$ into three distinct classes exhibiting qualitatively different types of compacton solutions, unstable compactons and stable compactons, which have either dependent or freely selectable width A and amplitude β. We have carried out the Painlevé test for various examples for each of these classes and found that all models which allow stable compactons for which the width can not be chosen independently from their amplitude pass the Painlevé test. Assuming that the Painlevé expansion (4) converges these models possess the Painlevé property [17] and allow therefore for genuine soliton solutions and are thus integrable. We found that the generalized KdV equation resulting from $\mathcal{H}_{l,2,p}$ and their $\mathcal{PT}$-symmetric extensions $\mathcal{H}_{l,m,p}$ have the same qualitative behaviour in the three different regimes. For convenience we summarize the different qualitative behaviours in the following table:

$\mathcal{H}_{l,m,p}$	compactons	solitons
$l = p + m$	stable, dependent A, β	no
$2 < l < p + 3m$	stable, independent A, β	yes
$l \leq 2$ or $l \geq p + 3m$	unstable	no

Table 1: The models $\mathcal{H}_{l,m,p}$ and their solutions.

Clearly our investigations do not constitute a full fletched mathematical proof as we based our findings on various representative examples for the different classes and it would be very interesting to settle this issue more rigorously with a generic argumentation not relying on case-by-case studies. At the same time such a treatment would probably provide a deeper understanding about the separation of the different models. Nonetheless, our findings provide enough evidence to make it worthwhile to investigate the models which pass the test with other techniques developed in the field of integrable models, whereas models which do not pass the test may be excluded from such investigations.

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