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G₂-Calogero-Moser Lax operators from reduction

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ABSTRACT: We construct a Lax operator for the G_2 -Calogero-Moser model by means of a double reduction procedure. In the first reduction step we reduce the A_6 -model to a B_3 -model with the help of an embedding of the B_3 -root system into the A_6 -root system together with the specification of certain coupling constants. The G_2 -Lax operator is obtained thereafter by means of an additional reduction by exploiting the embedding of the G_2 -system into the B_3 -system. The degree of algebraically independent and non-vanishing charges is found to be equal to the degrees of the corresponding Lie algebra.

1. Introduction

The Calogero-Moser models [1, 2, 3, 4, 5, 6, 7, 8] constitute a large class of well studied interacting many particle systems. The models are very universal in the sense that they can be cast into a form in which the potential term includes a sum over all roots α of some root system Δ and the functional dependence of the potential is $V(x) \sim 1/\text{sn}^2(x)$, with sn being an elliptic function together with its various limits $1/\sinh^2(x)$, $1/\sin^2(x)$ and $1/x^2$. Often it is useful to treat the latter cases independently for their own sake. Due to their universal nature the models find a wide range of applications in physics, as for instance to characterize anyons on the lowest Landau level [9], to describe certain properties of quantum Hall droplets [10] and in various ways in conformal [11, 12, 13, 14] and boundary [15] conformal field theories.

The Hamiltonian for an n -particle Calogero-Moser system reads

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha^2 V(\alpha \cdot q) \quad g_\alpha \in i\mathbb{R}, \quad q, p \in \mathbb{R}^n, \quad (1.1)$$

with n being the dimensionality of the space in which the roots α are realized. At this point we impose only the restrictions $g_\alpha = g_{-\alpha}$ on the coupling constant, even though later on we

equate more of them for reasons to be explained. One of the most prominent feature of these models is their integrability, meaning here the existence of a sufficient number of conserved quantities (integrals of motion) I_k in involution. A standard technique to construct these charges, the so-called isospectral deformation method, goes back almost forty years [16]. It consists of formulating Lax pair operators L and M as functions of the dynamical variables q_i and p_i for $1 \leq i \leq n$, which satisfy the Lax equation $\dot{L} = [L, M]$, upon the validity of the classical equation of motion resulting from (1.1). The Lax operator is then the starting point for the construction of conserved charges of the form $I_k = \text{tr}(L^k)/k$, with $I_2 \sim \mathcal{H}$, of classical r-matrices [17, 18, 19, 20, 21, 22], spectral curves [23, 24, 25, 26] and various other important quantities. For root systems Δ , which can be associated with a Lie algebra \mathfrak{g} , i.e. crystallographic ones¹, a natural Ansatz is to expand L and M in terms of the elements H, E_α of \mathfrak{g}

$$L = p \cdot H + \sum_{\alpha \in \Delta} g_\alpha f(\alpha \cdot q) E_\alpha \quad \text{and} \quad M = m \cdot H + \sum_{\alpha \in \Delta} g_\alpha h(\alpha \cdot q) E_\alpha. \quad (1.2)$$

Alternatively, one may also expand L and M in terms of other non-commuting objects, such as Coxeter transformations, and perform a similar analysis [28, 29]. Substitution of these operators into the Lax equation yields various constraining equations, which for a given potential determine the functions $f(x)$ and $h(x)$ in L and M as defined in equation (1.2). We choose here as convention the Cartan-Weyl basis $\text{tr}(H_i H_j) = \delta_{ij}$, $\text{tr}(E_\alpha E_{-\alpha}) = 1$, which is consistent with the well-known commutation relations (e.g. [30])

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha^i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H, \quad [E_\alpha, E_\beta] = \varepsilon_{\alpha, \beta} E_{\alpha+\beta}. \quad (1.3)$$

Then by direct substitution it follows that the Lax equation holds once the functions $f_\alpha(x) = g_\alpha f(x)$ and $h_\alpha(x) = g_\alpha h(x)$ satisfy

$$g(x) = f'(x), \quad \dot{p} = \sum_{\alpha \in \Delta} \alpha f_\alpha(\alpha \cdot q) h_{-\alpha}(-\alpha \cdot q), \quad \gamma \cdot m = \sum_{\substack{\alpha, \beta \in \Delta \\ \alpha + \beta = \gamma}} \varepsilon_{\alpha, \beta} \frac{f_\alpha(\alpha \cdot q) h_\beta(\beta \cdot q)}{f_\gamma(\gamma \cdot q)}. \quad (1.4)$$

Assuming further that $I_2 = \mathcal{H}$ and the classical equation of motions resulting from (1.1), one obtains three additional equations

$$f(x)f(-x) = -V(x), \quad \dot{p} = -\frac{1}{2} \sum_{\alpha \in \Delta} \alpha g_\alpha^2 V'(\alpha \cdot q) \quad \text{and} \quad f(x) = \pm f(-x). \quad (1.5)$$

For the stated potentials it is straightforward to use the factorizing condition in (1.5) to determine $f(x)$, i.e. $1/\text{sn}(x)$, $1/\sinh(x)$, $1/\sin(x)$ and $1/x$, and therefore $g(x)$ by taking its derivative. Thus to establish the integrability of the system (1.1) reduces to the question of whether the third set of equations in (1.4) admits a solution for the vector m and therefore guarantees the existence of the operator M . As these equations are in general highly

¹For non-crystallographic root systems one may exploit the fact that they are embedded into crystallographic ones and use a reduction procedure to obtain a meaningful Lie algebraic operator [27].

overdetermined, the answer to this question depends crucially on the structure of the Lie algebra. As was observed long time ago [31], for given potentials as in (1.1) the relation

$$f(x)f'(y) - f'(x)f(y) = f(x+y)[V(x) - V(y)] \quad (1.6)$$

holds, such that the equations may be simplified further. Then the last set of equations in (1.4) reduces to

$$g_\gamma(\gamma \cdot m) = \sum_{\substack{\alpha, \beta \in \Delta \\ \alpha + \beta = \gamma}} \varepsilon_{\alpha, \beta} g_\alpha g_\beta V(\alpha \cdot q) = \sum_{\alpha \in \Delta} \varepsilon_{\alpha, \gamma} g_\alpha g_{\alpha + \gamma} V(\alpha \cdot q). \quad (1.7)$$

Clearly

$$m = \sum_{\alpha \in \Delta} \sum_{i=1}^{\ell} \varepsilon_{\alpha, \gamma_i} \frac{g_\alpha g_{\alpha + \gamma_i}}{g_{\gamma_i}} V(\alpha \cdot q) \lambda_i, \quad (1.8)$$

with λ_i being a fundamental weight, is a solution to (1.7) when γ_i is taken to be a simple root. However, for (1.8) to be a proper solution one also has to verify whether it solves (1.7) for the remaining roots. In summary, we can say that if the system (1.7) can be solved for the vector m for a particular Lie algebra \mathfrak{g} , then the system (1.1) is classically integrable. The reverse statement does not hold.

2. The G_2 -Lax operator

It turns out that only when the algebra \mathfrak{g} in (1.2) is taken to be A_ℓ one obtains directly, meaning that all quantities in (1.1) belong to A_ℓ , a solution for the Lax operator with the condition that the corresponding equation of motion holds. In all other cases one needs to devise alternative methods. The expressions for the B_ℓ , C_ℓ and D_ℓ -algebras were obtained [32, 7, 8] from $A_{2\ell}$ -theories by specific transformations of the dynamical variables and a subsequent constraint on certain coupling constants g_α . For the remaining algebras different types of techniques have been developed [32, 7, 8, 26, 28, 29]. Surprisingly for the Lie algebras $E_{6,7,8}$, F_4 and G_2 no Lax pair was known until fairly recent [26].

In particular, the latter, the G_2 -Calogero-Moser model, constitutes a standard simple example, since it can be viewed as the classical three-body problem with a two and a three-body interaction term [33]. For a specific realization of the roots (see below), the potential term in the rational case simply reads

$$V(\tilde{q}) = \frac{\tilde{g}_s^2}{2} \sum_{1 \leq i < j \leq 3} \frac{1}{(\tilde{q}_i - \tilde{q}_j)^2} + \frac{\tilde{g}_l^2}{2} \sum_{\substack{1 \leq i < j \leq 3 \\ i, j \neq k}} \frac{1}{(\tilde{q}_i + \tilde{q}_j - 2\tilde{q}_k)^2}, \quad (2.1)$$

with \tilde{g}_s, \tilde{g}_l being coupling constants. It appears to be rather surprising that despite the simplicity of this model, apart from the expressions in [26], a general simple formula for the Lax operator along the line of the original work of [32, 7, 8] may not be found in the literature. It will be the purpose of this paper to provide such a simple expression.

2.1 Direct computation

Let us commence by directly analyzing equation (1.7) for G_2 . For the explicit calculation we require first the roots of G_2 . We recall the general fact, see e.g. [34, 35, 36], that the entire root system can be generated by $h - 1$ successive actions of the Coxeter element σ on bi-coloured simple roots, i.e. $\gamma = \pm \alpha_i$, with h being the Coxeter number. It turns out to be convenient to abbreviate the roots accordingly, that is we define $\sigma^p \gamma_i =: \alpha_{i,p}$ for $1 \leq i \leq \ell = \text{rank } \mathfrak{g}$ and $0 \leq p \leq h - 1$.

For G_2 we have $\ell = 2$, $h = 6$ and the $2 \times 6 = 12$ roots are computed to

$i \setminus p$	0	1	2	3	4	5
1	$-\tilde{\alpha}_1$	$-(2\tilde{\alpha}_1 + \tilde{\alpha}_2)$	$-(\tilde{\alpha}_1 + \tilde{\alpha}_2)$	$\tilde{\alpha}_1$	$2\tilde{\alpha}_1 + \tilde{\alpha}_2$	$\tilde{\alpha}_1 + \tilde{\alpha}_2$
2	$\tilde{\alpha}_2$	$-(3\tilde{\alpha}_1 + \tilde{\alpha}_2)$	$-(3\tilde{\alpha}_1 + 2\tilde{\alpha}_2)$	$-\tilde{\alpha}_2$	$3\tilde{\alpha}_1 + \tilde{\alpha}_2$	$3\tilde{\alpha}_1 + 2\tilde{\alpha}_2$

The roots $\hat{\alpha}_{i,p}$ of the $\hat{\Delta}_{G_2}$ -root system.

In addition, we require the structure constants $\varepsilon_{\alpha,\beta}$ for the analysis of (1.7). The square of the latter can be fixed by means of the well-known formula $\varepsilon_{\alpha,\beta}^2 = \alpha^2 n(m+1)/2$, where the integers n, m are determined by the so-called α -string through β , i.e. the largest values for n, m such that $\beta + n\alpha$ and $\beta - m\alpha$ are still roots, see e.g. [37]. The overall signs are in general not fixed and are subject to convention. However, some consistency relations have to hold, resulting from the anti-symmetry of the commutator, the reality condition and the Jacobi identity when $\alpha + \beta = \gamma$

$$\varepsilon_{\alpha,\beta} = -\varepsilon_{\beta,\alpha} = \varepsilon_{\beta,-\gamma} = -\varepsilon_{-\gamma,\beta} = -\varepsilon_{\alpha,-\gamma} = \varepsilon_{-\gamma,\alpha} = \quad (2.2)$$

$$-\varepsilon_{-\alpha,-\beta} = \varepsilon_{-\beta,-\alpha} = -\varepsilon_{-\beta,\gamma} = \varepsilon_{\gamma,-\beta} = \varepsilon_{-\alpha,\gamma} = -\varepsilon_{\gamma,-\alpha}. \quad (2.3)$$

We choose here the short roots $\tilde{\alpha}_{1,p}$ to have length $\tilde{\alpha}^2 = 2$ and the long roots $\tilde{\alpha}_{2,p}$ to have length $\tilde{\alpha}^2 = 6$. As complete lists of structure constants are difficult to find in the literature, we present here a consistent choice for the $12 \times 12 = 144$ structure constants, with 60 of them non-vanishing

$\tilde{\alpha}_{i,p} \setminus \tilde{\alpha}_{j,q}$	$\tilde{\alpha}_{1,0}$	$\tilde{\alpha}_{1,1}$	$\tilde{\alpha}_{1,2}$	$\tilde{\alpha}_{1,3}$	$\tilde{\alpha}_{1,4}$	$\tilde{\alpha}_{1,5}$	$\tilde{\alpha}_{2,0}$	$\tilde{\alpha}_{2,1}$	$\tilde{\alpha}_{2,2}$	$\tilde{\alpha}_{2,3}$	$\tilde{\alpha}_{2,4}$	$\tilde{\alpha}_{2,5}$
$\tilde{\alpha}_{1,0}$	0	μ	-2	0	2	μ	0	0	0	$-\mu$	$-\mu$	0
$\tilde{\alpha}_{1,1}$	$-\mu$	0	μ	2	0	-2	0	0	0	0	μ	$-\mu$
$\tilde{\alpha}_{1,2}$	2	$-\mu$	0	μ	-2	0	$-\mu$	0	0	0	0	μ
$\tilde{\alpha}_{1,3}$	0	-2	$-\mu$	0	$-\mu$	2	μ	μ	0	0	0	0
$\tilde{\alpha}_{1,4}$	-2	0	2	μ	0	$-\mu$	0	$-\mu$	μ	0	0	0
$\tilde{\alpha}_{1,5}$	$-\mu$	2	0	-2	μ	0	0	0	$-\mu$	μ	0	0
$\tilde{\alpha}_{2,0}$	0	0	μ	$-\mu$	0	0	0	0	$-\mu$	0	μ	0
$\tilde{\alpha}_{2,1}$	0	0	0	$-\mu$	μ	0	0	0	0	μ	0	$-\mu$
$\tilde{\alpha}_{2,2}$	0	0	0	0	$-\mu$	μ	μ	0	0	0	μ	0
$\tilde{\alpha}_{2,3}$	μ	0	0	0	0	$-\mu$	0	$-\mu$	0	0	0	μ
$\tilde{\alpha}_{2,4}$	μ	$-\mu$	0	0	0	0	$-\mu$	0	μ	0	0	0
$\tilde{\alpha}_{2,5}$	0	μ	$-\mu$	0	0	0	0	μ	0	$-\mu$	0	0

The G_2 -structure constants $\varepsilon_{i,p,j,q}$ with $\mu = \sqrt{3}$.

In order to obtain the previous table we only fixed five signs by convention and determined the remaining ones by means of the relations (2.2) and (2.3).

Having assembled all necessary data, we can present a simple argument which demonstrates that it is not possible to solve the constraint (1.7) directly. Choosing all coupling constants $g_{\tilde{\alpha}}$ to be either g_s or g_l for $\tilde{\alpha}$ to be a short or long root, respectively, we may for instance add up the equation (1.7) for the three choices $\gamma = \tilde{\alpha}_{1,0}$, $\gamma = \tilde{\alpha}_{1,2}$ and $\gamma = \tilde{\alpha}_{1,4}$ and find after cancellation of g_s

$$(\tilde{\alpha}_{1,0} + \tilde{\alpha}_{1,2} + \tilde{\alpha}_{1,4}) \cdot m = 0 = 2\sqrt{3}g_l [V(\tilde{\alpha}_{1,0} \cdot q) - V(\tilde{\alpha}_{2,0} \cdot q) + V(\tilde{\alpha}_{1,4} \cdot q) - V(\tilde{\alpha}_{2,4} \cdot q)].$$

Clearly, the right hand side is not zero in general, and hence we can not solve the constraint (1.7) directly. One reaches the same conclusion by taking the expression in (1.8) and trying to verify (1.7) for γ to be a non-simple root.

However, if we switch off the two particle interaction, i.e. we take $g_s = 0$ in (2.1), we may construct a particular solution. Taking for instance $\tilde{\alpha}_1 = \varepsilon_1 - \varepsilon_2$, $\tilde{\alpha}_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_2$ as concrete realization for the simple roots of G_2 in \mathbb{R}^3 , with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$, and setting $g_s = 0$, we can solve (1.7) by

$$m_1 = 0, \tag{2.4}$$

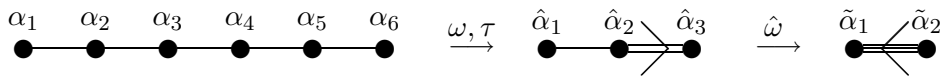
$$m_2 = \frac{g_l}{\sqrt{3}} [V(\tilde{\alpha}_{2,0} \cdot q) + V(\tilde{\alpha}_{2,4} \cdot q) - 2V(\tilde{\alpha}_{2,5} \cdot q)], \tag{2.5}$$

$$m_3 = \frac{g_l}{\sqrt{3}} [2V(\tilde{\alpha}_{2,4} \cdot q) - V(\tilde{\alpha}_{2,0} \cdot q) - V(\tilde{\alpha}_{2,5} \cdot q)]. \tag{2.6}$$

To find a general Lax operator which involves all terms of the potential one needs to device other techniques.

2.2 The G_2 -Lax operator from double reduction

The construction procedure is summarized by the following Dynkin diagrams:



In the first step we start by folding the $A_6^{(1)}$ -root system to the $A_6^{(2)}$ -root system by means of a map ω . Subsequently we constrain some of the coupling constants through a map τ , which amounts to an elimination of some particular roots of A_6 , such that we obtain two copies of a B_3 -root system. From this system we obtain the G_2 -root system, by the action of a further map $\hat{\omega}$.

The L-operator for the G_2 -Calogero-Moser model then reads

$$L(\tilde{p}, \tilde{q}) = \omega^{-1} \hat{\omega}^{-1} \tilde{p} \cdot H + \sum_{i=1}^6 \sum_{p=0}^6 \tau(g_{i,p}) f(\hat{\omega} \omega \alpha_{i,p} \cdot \tilde{q}) E_{\alpha_{i,p}} \tag{2.7}$$

$$= \omega^{-1} \hat{\omega}^{-1} \tilde{p} \cdot H + \sum_{i=1}^6 \sum_{p=0}^6 \tau(g_{i,p}) f(\alpha_{i,p} \cdot \omega^{-1} \hat{\omega}^{-1} \tilde{q}) E_{\alpha_{i,p}} \tag{2.8}$$

with $H_i, E_{\alpha_{i,p}} \in A_6$. We shall now specify the maps $\omega, \hat{\omega}, \tau$ in detail, construct the corresponding M -operator and show that the Lax equation holds upon the validity of the G_2 -equation of motion.

2.2.1 Reduction of the root systems

Let us precisely see how the root systems are embedded into each other as $\tilde{\Delta}_{G_2} \supset \hat{\Delta}_{B_3} \supset \Delta_{A_6}$. We label the 42 roots of A_6 -are as

$i \setminus p$	0	1	2	3	4
1	α_1	<u>$\alpha_2 + \alpha_3$</u>	<u>$\alpha_4 + \alpha_5$</u>	α_6	
2	$-\alpha_2$	<u>$\alpha_1 + \alpha_2 + \alpha_3$</u>	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	<u>$\alpha_4 + \alpha_5 + \alpha_6$</u>	$-\alpha_5$
3	<u>α_3</u>	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	α_4	
4	<u>$-\alpha_4$</u>	$\alpha_3 + \alpha_4 + \alpha_5$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\alpha_2 + \alpha_3 + \alpha_4$	<u>$-\alpha_3$</u>
5	α_5	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	α_2	
6	$-\alpha_6$	$\alpha_5 + \alpha_6$	$\alpha_3 + \alpha_4$	$\alpha_1 + \alpha_2$	$-\alpha_1$

The roots $\hat{\alpha}_{i,p}$ of the $\hat{\Delta}_{A_6}$ -root system.

We did not report values of $p \geq 4$, i.e. powers of the Coxeter element, for those roots which can be obtained simply by a multiplication with -1 from a root of another orbit of the Coxeter element. For instance $\alpha_{1,4} = -\alpha_{6,1}$, $\alpha_{1,5} = -\alpha_{6,2}$, etc. Let us now specify the action of the folding map ω , which acts on the simple roots of A_6

$$\alpha_i \mapsto \omega(\alpha_i) = \begin{cases} \hat{\alpha}_i & \text{for } i = 1, 2, 3 \\ \hat{\alpha}_{7-i} & \text{for } i = 4, 5, 6. \end{cases} \quad (2.9)$$

Comparing the entire root system resulting in this manner with the $(\ell = 3) \times (h = 6) = 18$ roots of $B_3^{(1)}$

$i \setminus p$	0	1	2	3	4	5
1	$\hat{\alpha}_1$	$\hat{\alpha}_2 + 2\hat{\alpha}_3$	$\hat{\alpha}_1 + \hat{\alpha}_2$	$-\hat{\alpha}_1$	$-(\hat{\alpha}_2 + 2\hat{\alpha}_3)$	$-(\hat{\alpha}_1 + \hat{\alpha}_2)$
2	$-\hat{\alpha}_2$	$\hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3$	$\hat{\alpha}_1 + 2\hat{\alpha}_2 + 2\hat{\alpha}_3$	$\hat{\alpha}_2$	$-(\hat{\alpha}_1 + \hat{\alpha}_2 + 2\hat{\alpha}_3)$	$-(\hat{\alpha}_1 + 2\hat{\alpha}_2 + 2\hat{\alpha}_3)$
3	$\hat{\alpha}_3$	$\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$	$\hat{\alpha}_2 + \hat{\alpha}_3$	$-\hat{\alpha}_3$	$-(\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3)$	$-(\hat{\alpha}_2 + \hat{\alpha}_3)$

The roots $\hat{\alpha}_{i,p}$ of the $\hat{\Delta}_{B_3}$ -root system.

it is easy to see that the map (2.9) reduces the A_6 -root system to two copies of a B_3 -root system plus 6 additional roots. We marked the 3 positive roots in the table of Δ_{A_6} , which are not mapped to $\hat{\Delta}_{B_3}$ via ω in bold and underlined the roots which are mapped to short roots in $\hat{\Delta}_{B_3}$. The unmarked roots are therefore mapped to long roots.

Having specified the map $\omega : \Delta_{A_6} \rightarrow \hat{\Delta}_{B_3}$ acting on the roots, it is important to see how this reduction is translated to the action on the dynamical variables. For this purpose we construct its “inverse” map $\omega^{-1} : \hat{\Delta}_{B_3} \rightarrow \Delta_{A_6}$, defined via the inner product relation

$$\omega(\alpha_i) \cdot \hat{\alpha}_j = \alpha_i \cdot \omega^{-1}(\hat{\alpha}_j), \quad 1 \leq i \leq 6, 1 \leq j \leq 3. \quad (2.10)$$

It is easy to verify that this is guaranteed by the map

$$\hat{\alpha}_i \mapsto \omega^{-1}(\hat{\alpha}_i) = \alpha_i + \alpha_{7-i}, \quad 1 \leq i \leq 3, \quad (2.11)$$

when taking the conventions $\alpha^2 = 2$, $\hat{\alpha}_1^2 = \hat{\alpha}_2^2 = 2$ and $\hat{\alpha}_3^2 = 1$. Now we may utilize this map to compute the reduction map when acting on the dynamical variables q, p

$$q \rightarrow \omega^{-1}(\hat{q}) = \omega^{-1} \left(\sum_{i=1}^3 \hat{y}_i \hat{\alpha}_i \right) = (y_1, y_2 - y_1, y_3 - y_2, 0, y_2 - y_3, y_1 - y_2, -y_1), \quad (2.12)$$

where we used the aforementioned Euclidean realization for the A_6 root system in \mathbb{R}^7 . To make contact with the literature, we defined a further set of variables through the relation $y_i = \sum_{k=1}^i \hat{q}_k$, such that (2.12) becomes

$$q \rightarrow (\hat{q}_1, \hat{q}_2, \hat{q}_3, 0, -\hat{q}_3, -\hat{q}_2, -\hat{q}_1). \quad (2.13)$$

This is the reduction map as employed in [32] (see also [8]).

Likewise, we reduce next the B_3 -root system to the G_2 -root system by means of the map $\hat{\omega} : \hat{\Delta}_{B_3} \rightarrow \hat{\Delta}_{G_2}$

$$\hat{\alpha}_i \mapsto \hat{\omega}(\hat{\alpha}_i) = \begin{cases} \tilde{\alpha}_1 & \text{for } i = 1, 3 \\ \tilde{\alpha}_2 & \text{for } i = 2. \end{cases} \quad (2.14)$$

The “inverse” $\hat{\omega}^{-1} : \hat{\Delta}_{G_2} \rightarrow \hat{\Delta}_{B_3}$ is obtained similarly as before, but now demanding

$$\hat{\omega}(\hat{\alpha}_i) \cdot \tilde{\alpha}_j = \hat{\alpha}_i \cdot \hat{\omega}^{-1}(\tilde{\alpha}_j), \quad 1 \leq i \leq 3, 1 \leq j \leq 2. \quad (2.15)$$

We find

$$\tilde{\alpha}_1 \mapsto \hat{\omega}^{-1}(\tilde{\alpha}_1) = \hat{\alpha}_1 + 2\hat{\alpha}_3 \quad \text{and} \quad \tilde{\alpha}_2 \mapsto \hat{\omega}^{-1}(\tilde{\alpha}_2) = 3\hat{\alpha}_2, \quad (2.16)$$

with the additional conventions $\tilde{\alpha}_1^2 = 2$ and $\tilde{\alpha}_2^2 = 6$. The reduction map, when acting on the dynamical variables \hat{q}, \hat{p} , is now evaluated as

$$\hat{q} \rightarrow \omega^{-1}(\hat{q}) = \omega^{-1} \left(\sum_{i=1}^2 \tilde{y}_i \tilde{\alpha}_i \right) = (\tilde{y}_1, 3\tilde{y}_2 - \tilde{y}_1, 2\tilde{y}_1 - 3\tilde{y}_2) = (-\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3), \quad (2.17)$$

where we realized the B_3 -roots in \mathbb{R}^3 as $\hat{\alpha}_1 = \varepsilon_1 - \varepsilon_2$, $\hat{\alpha}_2 = \varepsilon_2 - \varepsilon_3$ and $\hat{\alpha}_3 = \varepsilon_3$. The introduction of the variables \tilde{q}'_i translates into the usual G_2 constraint $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 = 0$, which corresponds to considering the three particle system in the center of mass frame. In fact, it will be convenient to introduce yet another set of variables, namely $\tilde{q}'_1 = \tilde{q}_2 - \tilde{q}_3$, $\tilde{q}'_2 = \tilde{q}_3 - \tilde{q}_1$ and $\tilde{q}'_3 = \tilde{q}_1 - \tilde{q}_2$ to make proper contact with (2.1).

Let us now see how to utilize these maps in order to reduce the corresponding potentials.

2.2.2 Reduction of the potentials

Our starting point is the A_6 -potential term in the form

$$V_{A_6}(q) = \frac{1}{2} \sum_{i=1}^6 \sum_{p=0}^6 g_{ip}^2 V(\alpha_{ip} \cdot q). \quad (2.18)$$

Below we confirm the known fact that the Lax equation dictates that all coupling constants have to be taken to be the same, i.e. $g_{ip} = g$. Then V_{A_6} is mapped into a B_3 -potential of the form

$$V_{B_3}(\hat{q}) = \frac{1}{2} \sum_{i=1}^6 \sum_{p=0}^6 \tau(g_{ip})^2 V(\omega \alpha_{ip} \cdot \hat{q}) = \frac{\hat{g}^2}{2} \sum_{p=0}^5 \left[\sum_{i=1}^2 V(\hat{\alpha}_{i,p} \cdot \hat{q}) + 2V(\hat{\alpha}_{3,p} \cdot \hat{q}) \right], \quad (2.19)$$

by means of the reduction map ω as specified above (2.9) and the map τ acting on the coupling constants as

$$g_{ip} \mapsto \tau(g_{ip}) = \begin{cases} 0 & \text{for } g_{1,5}, g_{3,5}, g_{5,5}, g_{2,2}, g_{4,2}, g_{6,2} \\ \hat{g} & \text{for } g_{1,1}, g_{1,2}, g_{2,1}, g_{2,3}, g_{3,0}, g_{3,3}, g_{4,0}, g_{4,4}, g_{5,4}, g_{5,6}, g_{6,5}, g_{6,6} \\ \frac{\hat{g}}{\sqrt{2}} & \text{otherwise.} \end{cases} \quad (2.20)$$

The map τ serves here to eliminate the aforementioned additional six roots of A_6 which have no counterpart in B_3 and at the same time it establishes a relationship between the coupling constants depending on whether the potential involves roots which are mapped to long or short roots. This relation is dictated by the Lax pair construction, and coincides with the one found by Olshanetsky and Perelomov in [32], or [8] p. 181. Note that taking merely the invariance of the Coxeter transformation as a guiding principle one could choose the coupling constants in front of the term involving long or short roots to be independent, see e.g. discussion in [27]. However, integrability demands the dependence of the coupling constants to be as stated in (2.20), such that one has only one coupling constant at ones disposal for the B_ℓ -theories.

Next we map the B_3 -potential to the G_2 -potential with the help of $\hat{\omega}$ and find

$$V_{G_2} = \frac{\tilde{g}^2}{2} \sum_{p=0}^5 \left[\sum_{i=1}^2 V(\hat{\omega} \hat{\alpha}_{i,p} \cdot \tilde{q}) + 2V(\hat{\omega} \hat{\alpha}_{3,p} \cdot \tilde{q}) \right] = \frac{\tilde{g}^2}{2} \sum_{p=0}^5 [3V(\tilde{\alpha}_{1,p} \cdot \tilde{q}) + V(\tilde{\alpha}_{3,p} \cdot \tilde{q})]. \quad (2.21)$$

In this last reduction step we did not need to specify any additional map acting on the coupling constants as the embedding is now on-to-one. For consistency we re-named, however, \hat{g} to \tilde{g} .

2.2.3 Constraints from the Lax operator

Having convinced ourselves that the potentials can be reduce properly, we still have to establish that the Lax operator exists and is indeed of the form (2.7). We commence by explicitly solving the constraint (1.7) for A_6 . In principle there are now 42×42 possible structure constants $\varepsilon_{i,p,j,q}$, with 420 of them non-vanishing. We only report here our conventions for the signs of the 35 essentials and leave it to the reader to obtain the remaining ones by means of the equations (2.2) and (2.3)

$$\varepsilon_{1,0,1,1} = \varepsilon_{1,0,2,2} = \varepsilon_{1,0,3,2} = \varepsilon_{1,1,1,2} = \varepsilon_{1,1,2,3} = \varepsilon_{1,1,3,3} = \varepsilon_{1,2,1,3} = \varepsilon_{1,2,2,4} = \varepsilon_{1,3,1,4} = \quad (2.22)$$

$$\varepsilon_{1,3,2,5} = \varepsilon_{1,3,3,5} = \varepsilon_{1,4,1,5} = \varepsilon_{1,4,2,6} = \varepsilon_{1,5,1,6} = \varepsilon_{1,5,2,0} = \varepsilon_{1,5,3,0} = \varepsilon_{1,6,1,0} = \varepsilon_{1,6,2,1} = \quad (2.23)$$

$$\varepsilon_{1,6,3,1} = \varepsilon_{2,0,1,1} = \varepsilon_{2,0,2,2} = \varepsilon_{2,1,1,2} = \varepsilon_{2,1,2,3} = \varepsilon_{2,2,1,3} = \varepsilon_{2,2,2,4} = \varepsilon_{2,3,1,4} = \varepsilon_{2,3,2,5} = \quad (2.24)$$

$$\varepsilon_{2,4,1,5} = \varepsilon_{2,4,2,6} = \varepsilon_{2,5,1,6} = \varepsilon_{2,5,2,0} = \varepsilon_{2,6,1,0} = \varepsilon_{2,6,2,1} = \varepsilon_{3,0,1,2} = \varepsilon_{3,2,1,4} = 1. \quad (2.25)$$

In fact, we verified that these choices coincide with the constants obtained directly from (1.3) when using the vector representation of A_6 . With (2.22)-(2.25) and the above mentioned realization for the simple roots, the constraint (1.7) may be solved by

$$m_i^{A_6} = g \sum_{\substack{k=1 \\ k \neq i}}^7 V_{A_6}(q_k - q_i), \quad (2.26)$$

which is known for some time [8]. Next we may solve (1.7) for the reduced systems and find

$$m_i^{B_3} = \sqrt{2}g \sum_{\substack{k=1 \\ k \neq i}}^7 \tau_{ki} V_{A_6}(\hat{\omega}^{-1}(\hat{q}_k) - \hat{\omega}^{-1}(\hat{q}_i)) \quad (2.27)$$

with $\tau_{ki} = 1$ except for $\tau_{4i} = 2$, $\tau_{(8-i)i} = 0$ and

$$m_i^{G_2} = \sqrt{2}g \sum_{\substack{k=1 \\ k \neq i}}^7 \tau_{ki} V_{A_6}(\omega^{-1}\hat{\omega}^{-1}(\tilde{q}_k) - \omega^{-1}\hat{\omega}^{-1}(\tilde{q}_i)). \quad (2.28)$$

Having presented explicit solutions to the equation (1.7), we have established the existence of the operators L and M . In particular (2.7) and the corresponding equation for M satisfy the Lax equation up to the validity of the G_2 equations of motion.

2.2.4 Conserved Charges

It is instructive to consider an explicit matrix representation for the L-operator. Using the standard vector representation of A_6 it follows directly from (2.7)

$$L = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{\lambda} \tilde{p}_{32} & f(\tilde{q}_{12}) & f(\tilde{q}_{31}) & \sqrt{2}f(\tilde{q}_{32}) & f(\tilde{q}_{13,2}) & f(\tilde{q}_{3,12}) & 0 \\ f(\tilde{q}_{21}) & \frac{\sqrt{2}}{\lambda} \tilde{p}_{31} & f(\tilde{q}_{23,1}) & \sqrt{2}f(\tilde{q}_{31}) & f(\tilde{q}_{32}) & 0 & f(\tilde{q}_{3,12}) \\ f(\tilde{q}_{13}) & f(\tilde{q}_{1,23}) & \frac{\sqrt{2}}{\lambda} \tilde{p}_{12} & \sqrt{2}f(\tilde{q}_{12}) & 0 & f(\tilde{q}_{32}) & f(\tilde{q}_{13,2}) \\ \sqrt{2}f(\tilde{q}_{23}) & \sqrt{2}f(\tilde{q}_{13}) & \sqrt{2}f(\tilde{q}_{21}) & 0 & \sqrt{2}f(\tilde{q}_{12}) & \sqrt{2}f(\tilde{q}_{31}) & \sqrt{2}f(\tilde{q}_{32}) \\ f(\tilde{q}_{2,13}) & f(\tilde{q}_{23}) & 0 & \sqrt{2}f(\tilde{q}_{21}) & \frac{\sqrt{2}}{\lambda} \tilde{p}_{21} & f(\tilde{q}_{23,1}) & f(\tilde{q}_{31}) \\ f(\tilde{q}_{12,3}) & 0 & f(\tilde{q}_{23}) & \sqrt{2}f(\tilde{q}_{13}) & f(\tilde{q}_{1,23}) & \frac{\sqrt{2}}{\lambda} \tilde{p}_{13} & f(\tilde{q}_{12}) \\ 0 & f(\tilde{q}_{12,3}) & f(\tilde{q}_{2,13}) & \sqrt{2}f(\tilde{q}_{23}) & f(\tilde{q}_{13}) & f(\tilde{q}_{21}) & \frac{\sqrt{2}}{\lambda} \tilde{p}_{23} \end{pmatrix} \quad (2.29)$$

where we abbreviated $\tilde{p}_{ij} := \tilde{p}_i - \tilde{p}_j$, $\tilde{q}_{ij} := \tilde{q}_i - \tilde{q}_j$, $\tilde{q}_{ij,k} := \tilde{q}_i + \tilde{q}_j - 2\tilde{q}_k$ and $\tilde{q}_{k,ij} := 2\tilde{q}_k - \tilde{q}_i - \tilde{q}_j$. By simple matrix multiplication we compute from this the integrals of motion of the form $I_k = \text{tr}(L^k)/k$

$$I_1 = 0, \quad I_2 = \mathcal{H}, \quad I_3 = 0, \quad I_4 = \frac{1}{4}I_2^2, \quad I_5 = 0, \quad I_6 \neq 0, \quad I_7 = 0, \quad (2.30)$$

$$I_8 = I_2I_6 - \frac{5}{96}I_2^4, \quad I_9 = 0, \quad I_{10} = \frac{3}{4}I_2^2I_6 - \frac{1}{20}I_2^5, \quad I_{11} = 0, \quad (2.31)$$

$$I_{12} = \frac{5}{12}I_2^3I_6 + \frac{1}{2}I_6^2 - \frac{19}{576}I_2^6, \quad I_{13} = 0, \dots \quad (2.32)$$

Thus we find non-vanishing and algebraically independent charges I_k only for k being a degree of G_2 , that is 2 and 6, see e.g. [34]. Computing the L -operator of the B_3 -Calogero-Moser model we verify the same property.

3. Conclusions

We have constructed a simple expression for the L -operator of the G_2 -Calogero-Moser model. We established that the constraint (1.7) may indeed be solved and therefore that the L and M -operator do exist. The operators are expanded in terms of $H, E_\alpha \in A_6$. The Lax equation constructed from these operators, with coefficients subject to the stated reduction maps, holds up to the validity of the G_2 equations of motion. To find a solution to (1.7) and thus guaranteeing the integrability of the model, we found that we are only permitted to have one coupling constant in the G_2 -theory, instead of two, what might be expected from demanding invariance under the Coxeter group. In our approach this feature is inherited from the B_3 -theory. Such a behaviour was also observed in [26]. It would be interesting to investigate if this limitation can be overcome by other techniques or to establish that this is really an intrinsic feature of the model. Furthermore, from the explicit computations of numerous integrals of motion, we found that they are only algebraically independent and non-vanishing if their degree is equal to the degree of the corresponding Lie algebra.

In our discussion we did not appeal to the explicit form of the potential and only require the relation (1.6) to be satisfied. This means that the models covered here are of the general form $V(x) \sim 1/\text{sn}^2(x)$ including a dependence of spectral parameter μ . It would be interesting to investigate the properties of the spectral curves $R(k, \mu) = \det[k\mathbb{I} - L(\mu)]$ in the spirit of [23, 24, 25, 26], resulting from the L -operator presented here.

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