Affine Toda field theories related to Coxeter groups of non-crystallographic type

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Abstract: We propose affine Toda field theories related to the non-crystallographic Coxeter groups H_2, H_3 and H_4. The classical mass spectrum, the classical three-point couplings and the one-loop corrections to the mass renormalisation are determined. The construction is carried out by means of a reduction procedure from crystallographic to non-crystallographic Coxeter groups. The embedding structure explains for various affine Toda field theories that their particles can be organised in pairs, such that their relative masses differ by the golden ratio.

1. Introduction

The Ising model is generally considered as the prime example of integrable models. When viewed in the continuous limit as a $c = 1/2$ conformal field theory [1], it is a well known fact that it can be realized as an $E_8^{(1)} \otimes E_8^{(1)}/E_8^{(2)}$-coset model [2]. Even when the conformal symmetry is broken, by perturbing the theory with a primary field of scaling dimension $(1/16, 1/16)$ [3], the E_8 structure survives in form of a (minimal) E_8-affine Toda field theory (ATFT) [4, 5]. It will be one of the results in this paper to show that there is an even more fundamental structure than E_8 underlying this particular model, the non-crystallographic Coxeter group H_4. We draw here on the observation made first by Sherbak in 1988 [6], namely that H_4 can be embedded into E_8, see also [7, 8] for further developments of these mathematical structures. Loosely speaking, one may regard the E_8-theory as two copies of H_4-theories. We get a first glimpse of this structure from a more physical point of view when we bring the mass spectrum of minimal E_8-affine Toda field theory found originally in [3] into the form

$$\begin{array}{llll} m_1 = 1, & m_2 = 2\cos\frac{\pi}{30}, & m_3 = \sqrt{\sin\frac{11\pi}{30}/\sin\frac{\pi}{30}}, & m_4 = 2\phi\cos\frac{7\pi}{30}, \\ m_5 = \phi m_1, & m_6 = \phi m_2, & m_7 = \phi m_3, & m_8 = \phi m_4. \end{array} \tag{1.1}$$

We have set here the overall mass scale to one. Remarkably, these mass ratios are the same in the classical as well as in the quantum theory, as all masses renormalize with an overall factor, see [9, 10, 11] and references therein. We observe here that there are four "fundamental" masses present in the theory, whereas the other ones can be obtained simply by a multiplication with the golden ratio

$$\phi = \frac{1}{2}(1+\sqrt{5}) = \phi^2 - 1 \ . \tag{1.2}$$

It will turn out that each of the sets (m_1, m_2, m_3, m_4) and (m_5, m_6, m_7, m_8) can be associated with an H_4-ATFT.

The other popular integrable quantum field theory is the sine-Gordon model, see e.g. [12, 13] and references therein. It is a well known fact [14, 15] that once the coupling constant ν is taken to be $1/\nu = n$, with n being an integer, the backscattering amplitudes vanish and the theory reduces to a minimal D_{n+1}-ATFT. In particular for $n = 5$ we find a similar pattern for the mass ratios as discussed above. The D_6-ATFT mass spectrum reads up to an overall mass scale

$$\begin{array}{lll} m_1 = \phi^{-1}, & m_2 = \sqrt{1+\phi^{-2}}, & m_3 = 1, \\ m_4 = \phi m_1, & m_5 = \phi m_2, & m_6 = \phi m_3. \end{array} \tag{1.3}$$

It this case the sets (m_1, m_2, m_3) and (m_4, m_5, m_6) can be associated with an H_3-ATFT.

The above mentioned structure can be explained simply by the fact that H_4 can be embedded into E_8 and H_3 into D_6, such that the non-crystallographic structure is "visible" inside the theories related to crystallographic Coxeter groups.

Besides having a "non-crystallographic pattern" inside theories related to crystallographic Coxeter groups, it is interesting to ask the question whether it is possible to construct theories purely based on these latter groups. In particular, H_3 being a three-dimensional symmetry group of the icosahedron, a regular solid with 20 triangle faces, finds a natural application in physical [16, 17, 18], chemical [19, 20] an even biological systems [21, 22, 23, 24, 25]. In the context of integrable (solvable) models, Calogero-Sutherland models have been formulated based also on H_3 [26, 27] and it should be possible to extend these investigations to other non-crystallographic Coxeter groups. However, so far no ATFT for such type of group has been considered, the main reason being that unlike for crystallographic ones, in this case there is no Lie algebra at disposal, which is vital in that context. This deficiency can be overcome by exploiting the embedding structure and reduce the theories associated to crystallographic Coxeter groups to new types of theories related entirely to non-crystallographic ones. This is somewhat similar in spirit to the folding procedure carried out by Olive and Turok [28], who constructed ATFT for non-simply laced Lie algebras from those related to simply laced algebras, exploiting the embedding of the former into the latter. However, in comparison, there is one crucial difference. Whereas in the folding scenario [28] the reduced models are identical to a formulation purely in terms of non-simply laced algebras, the models we obtain here vitally rely in their construction on the embedding and can not be formulated directly in terms of non-crystallographic Coxeter groups on the level of the Lagrangian.

Our manuscript is organized as follows: In section 2 we review and develop the mathematics associated to the embedding of non-crystallographic into crystallographic Coxeter groups. In section 3 we apply these notions to affine Toda field theory and construct in particular their classical mass spectra and fusing structures. In addition we start the development of a quantum field theory by computing the mass renormalisations. We state our conclusions in section 4. In the appendix we present explicit computations of various orbits of coloured simple roots related to non-crystallographic and crystallographic Coxeter groups and exhibit how they can be embedded into one another.

2. Embedding of non-crystallographic into crystallographic Coxeter groups

Coxeter graphs are finite graphs, whose edges are labelled by some integers m_{ij} joining the vertices i and j [29]. To each of these graphs one can associate a finite reflection group. When the crystallographic condition is satisfied, that is m_{ij} for $i \neq j$ takes only the values $2, 3, 4$ or 6, these groups are Weyl groups. In contrast to the non-crystallographic groups, the crystallographic ones can be related to Lie algebras and Lie groups. Lie theory is exploited largely in the context of integrable models, which is one of the main reasons why crystallographic groups enjoy wider applications. Here we use the embedding of non-crystallographic groups into crystallographic ones, such that we can still exploit the Lie structure related to the larger groups.

2.1 Root systems

In order to assemble the necessary mathematical notions, we start by introducing a map ω from a root system Δ^c which is invariant under the action of a crystallographic Coxeter group $\mathcal{W}$ of rank ℓ into the union of two sets $\tilde{\Delta}^{nc}$ related to a non-crystallographic group $\tilde{\mathcal{W}}$ of rank $\tilde{\ell} = \ell/2$

$$\omega : \; \Delta^c \to \tilde{\Delta}^{nc} \cup \phi\tilde{\Delta}^{nc} \; . \tag{2.1}$$

Throughout this manuscript we adopt the notation that quantities related to crystallographic and non-crystallographic groups are specified with the same symbol, distinguished by an additional tilde, e.g. $\alpha \in \Delta^c = \Delta$ and $\tilde{\alpha} \in \tilde{\Delta}^{nc} = \tilde{\Delta}$. Introducing a special labelling for the vertices on the Coxeter graphs, or equivalently the simple roots, we can always realize this map as

$$\alpha_i \mapsto \omega(\alpha_i) = \begin{cases} \tilde{\alpha}_i & \text{for} \quad 1 \leq i \leq \tilde{\ell} = \ell/2 \\ \phi\tilde{\alpha}_{i-\tilde{\ell}} & \text{for} \quad \tilde{\ell} < i \leq \ell. \end{cases} \tag{2.2}$$

Our labelling allows for a generic treatment of the embedding and differs for instance from the one used in [6, 7, 8]. A further important property guaranteed by our conventions is $\alpha_i \cdot \alpha_{i+\tilde{\ell}} = 0$. Both types of root systems Δ and $\tilde{\Delta}$ are equipped with a symmetric bilinear form or inner product. In [7] Moody and Patera noticed the remarkable fact that the map ω is an isometric isomorphism, such that we may compute inner products in the root system Δ from inner products in $\tilde{\Delta}$

$$\alpha \cdot \beta = R(\omega(\alpha) \cdot \omega(\beta)). \tag{2.3}$$

Here the map R, called a rational form relative to ϕ, extracts from a number of the form[1] $a+\phi b$ with $a,b\in\mathbb{Q}$ the rational part a

$$R(a+\phi b)=a. \tag{2.4}$$

We normalize all our roots to have length 2, such that $\alpha^2=\tilde{\alpha}^2=2$ for $\alpha\in\Delta$ and $\tilde{\alpha}\in\tilde{\Delta}$. According to (2.3) we may therefore compute the Cartan matrix related to Δ entirely from inner products in $\tilde{\Delta}$

$$K_{ij}=\frac{2\alpha_i\cdot\alpha_j}{\alpha_j\cdot\alpha_j}=R(\omega(\alpha_i)\cdot\omega(\alpha_j))\ =R(\phi^{t_i+t_j}\tilde{\alpha}_{i-t_i\tilde{\ell}}\cdot\tilde{\alpha}_{j-t_j\tilde{\ell}}). \tag{2.5}$$

where $t_i=0$ for $1\leq i\leq\tilde{\ell}$ and $t_i=1$ for $\tilde{\ell}<i\leq\ell$. For our purposes it will be most important to achieve also the opposite, which can not be found in [7], namely to compute inner products in $\tilde{\Delta}$ from those in Δ. For this aim we introduce here the map

$$\tilde{\omega}:\ \tilde{\Delta}\to\Delta\oplus\phi\Delta, \tag{2.6}$$

which acts on the simple roots in $\tilde{\Delta}$ as

$$\tilde{\alpha}_i\mapsto\tilde{\omega}(\tilde{\alpha}_i)=(\alpha_i+\phi\alpha_{i+\tilde{\ell}})\qquad\text{for}\ \ 1\leq i\leq\tilde{\ell}\ . \tag{2.7}$$

Note that $\tilde{\omega}(\tilde{\alpha}_i)^2=2+2\phi^2$, such that $\tilde{\omega}$ is not an isometry. Instead, we find that inner products in $\tilde{\Delta}$ are related to inner products in Δ by means of

$$\omega(\alpha_i)\cdot\tilde{\alpha}_j=\alpha_i\cdot\tilde{\omega}(\tilde{\alpha}_j)\qquad\text{for}\ \ 1\leq j\leq\tilde{\ell}\ ,1\leq i\leq\ell\ . \tag{2.8}$$

Expanding (2.8) yields immediately a relation between the Cartan matrices of Δ and $\tilde{\Delta}$

$$K_{ij}+\phi K_{i(j+\tilde{\ell})}=\begin{cases}\tilde{K}_{ij} & \text{for}\ \ 1\leq i\leq\tilde{\ell}\\ \phi\tilde{K}_{(i-\tilde{\ell})j} & \text{for}\ \ \tilde{\ell}\ <i\leq\ell.\end{cases} \tag{2.9}$$

Noting that for the Coxeter groups we consider the Cartan matrices are symmetric, such that the relations (2.9) also hold with $i\leftrightarrow j$. As the right hand side involves only one inner product in $\tilde{\Delta}$, this formula achieves our objective and we may now express inner products in $\tilde{\Delta}$ in terms of those in Δ. Recalling that simple roots and fundamental weights λ_i are related as $\alpha_i=\sum_j K_{ij}\lambda_j$ and $\tilde{\alpha}_i=\sum_j\tilde{K}_{ij}\tilde{\lambda}_j$, it follows directly that (2.8) also holds when we replace simple roots by fundamental weights. As $K_{ij}^{-1}=\lambda_i\cdot\lambda_j$, this means that (2.9) also holds for the inverse matrices

$$K_{ij}^{-1}+\phi K_{i(j+\tilde{\ell})}^{-1}=\begin{cases}\tilde{K}_{ij}^{-1} & \text{for}\ \ 1\leq i\leq\tilde{\ell}\\ \phi\tilde{K}_{(i-\tilde{\ell})j}^{-1} & \text{for}\ \ \tilde{\ell}\ <i\leq\ell\end{cases} \tag{2.10}$$

[1]Note that higher powers of ϕ can be reduced to that form by a repeated use of (1.2), e.g. $\phi^2=1+\phi$, $\phi^3=1+2\phi$, $\phi^4=2+3\phi$, $\phi^5=3+5\phi$,..., $\phi^n=f_{n-1}+\phi f_n$ where f_n is the n-th Fibonacci number obeying the recursive relation $f_{n+1}=f_n+f_{n-1}$.

and (2.2), (2.7) for $\alpha_i \to \lambda_i$, $\tilde{\alpha}_i \to \tilde{\lambda}_i$. With regard to our application it will be particularly important to relate the eigensystems K and $\tilde{K}$. Abbreviating $\kappa_{ij} = K_{ij}$ and $\hat{\kappa}_{ij} = K_{i(j+\tilde{\ell})}$ for $1 \leq i, j \leq \tilde{\ell}$ we can block-decompose the Cartan matrix K further and verify that

$$U^{-1}KU = U^{-1}\begin{pmatrix} \kappa & \hat{\kappa} \\ \hat{\kappa} & \kappa+\hat{\kappa} \end{pmatrix} U = \begin{pmatrix} \tilde{K}(\phi) & 0 \\ 0 & \tilde{K}(-\phi^{-1}) \end{pmatrix} \quad \text{with } \; U = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \phi\mathbb{I} & -\mathbb{I}/\phi \end{pmatrix}, \tag{2.11}$$

where $\mathbb{I}$ is the $\tilde{\ell} \times \tilde{\ell}$ unit matrix. This means of course that the ℓ eigenvalues of K, which are known to be of the form $e_n = 4\sin^2(\pi s_n/2h)$ with h being the Coxeter number (see below for more details). The s_n label the ℓ exponents of $\mathcal{W}$ and are identical to the union of the $\tilde{\ell}$ eigenvalues of $\tilde{K}(\phi)$ and $\tilde{K}(-\phi^{-1})$

$$\mathcal{S} = \{s_1, s_2, \ldots, s_\ell\} = \tilde{\mathcal{S}}(\phi) \cup \tilde{\mathcal{S}}'(-\phi^{-1}). \tag{2.12}$$

The eigenvalues are invariant under the change $\phi \to -\phi$. Labelling the ℓ eigenvectors of K by y_n, such that $Ky_n = e_n y_n$, we can construct the eigenvectors $\tilde{y}_{\tilde{s}}(\phi)$ and $\tilde{y}_{\tilde{s}'}(-\phi^{-1})$ of $\tilde{K}(\phi)$ and $\tilde{K}(-\phi^{-1})$, respectively, from (2.11) as

$$U^{-1}y_s = \tilde{y}_{\tilde{s}}(\phi) \otimes \tilde{y}_{\tilde{s}'}(-\phi^{-1}). \tag{2.13}$$

Conversely, we can construct the eigenvectors of K from the knowledge of the eigenvectors of $\tilde{K}(\phi)$ and $\tilde{K}(-\phi^{-1})$

$$y_{\tilde{s}} = \tilde{y}_{\tilde{s}}(\phi) \otimes \phi\tilde{y}_{\tilde{s}}(\phi) \qquad \text{and} \qquad y_{\tilde{s}'} = \tilde{y}_{\tilde{s}'}(-\phi^{-1}) \otimes (-\phi^{-1})\tilde{y}_{\tilde{s}'}(-\phi^{-1}) \, . \tag{2.14}$$

The first identity follows by exploiting (2.9)

$$Ky_{\tilde{s}} = K\begin{pmatrix} \tilde{y}_{\tilde{s}}(\phi) \\ \phi\tilde{y}_{\tilde{s}}(\phi) \end{pmatrix} = \begin{pmatrix} \kappa & \hat{\kappa} \\ \hat{\kappa} & \kappa+\hat{\kappa} \end{pmatrix}\begin{pmatrix} \tilde{y}_{\tilde{s}}(\phi) \\ \phi\tilde{y}_{\tilde{s}}(\phi) \end{pmatrix} = \begin{pmatrix} \tilde{K}(\phi) & 0 \\ 0 & \tilde{K}(\phi) \end{pmatrix}\begin{pmatrix} \tilde{y}_{\tilde{s}}(\phi) \\ \phi\tilde{y}_{\tilde{s}}(\phi) \end{pmatrix}. \tag{2.15}$$

The second relation in (2.14) is obtained by the same argumentation with $\phi \to -\phi^{-1}$. These facts will not only be crucial to formulate new types of ATFT, but also to explain patterns in well studied models. The knowledge of the distribution of the exponents with respect to the embedding is important as they grade the conserved charges. The classical masses are known [10, 30, 31] to organise as components of the Perron-Frobenius vector y_1, such that (2.14) explains the aforementioned mass patterns (1.1) and (1.3).

2.2 Coxeter groups

Having related the root systems $\tilde{\Delta}$ and Δ, let us see next how to relate the corresponding Coxeter groups $\mathcal{W}$ and $\tilde{\mathcal{W}}$, which leave them invariant. We recall [32, 29] that Coxeter groups are generated by reflections on the hyperplane through the origin orthogonal to simple roots α_i

$$\sigma_i(x) = x - 2\frac{x \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i \qquad \text{for } 1 \leq i \leq \ell, \; x \in \mathbb{R}^\ell. \tag{2.16}$$

We can then think of the Coxeter group $\mathcal{W}$ as the set of all words in the generators $\{\sigma_i\}$ subject to the relations

$$(\sigma_i\sigma_j)^{m_{ij}} = 1, \qquad 1 \leq i, j \leq \ell, \tag{2.17}$$

where the m_{ij} can take the values $1, 2, 3, 4, 5$ or 6. Here we focus on the Coxeter groups for which the Cartan matrix is symmetric, in which case $m_{ij} = \pi \arccos^{-1}(-K_{ij}/2)$ are the integers labelling the edges of the Coxeter graph mentioned at the beginning of this section. $\tilde{\mathcal{W}}$ is constructed analogously when replacing K by $\tilde{K}(\phi)$. Note that when we use instead of $\tilde{K}(\phi)$ the matrix $\tilde{K}(-\phi^{-1})$, which naturally emerges through (2.11), this will lead to the same Coxeter relations.

From a group theoretical point of view we can identify $\tilde{\sigma}_i \hookrightarrow \sigma_i \sigma_{i+\tilde{\ell}}$. With the help of the map ω we can relate the reflections $\tilde{\sigma}_i$ building up $\tilde{\mathcal{W}}$ to those constituting $\mathcal{W}$ as

$$\tilde{\sigma}_i \omega = \omega \sigma_i \sigma_{i+\tilde{\ell}} \qquad \text{for} \;\; 1 \leq i \leq \tilde{\ell} \,. \tag{2.18}$$

This is seen easily by acting on a simple root in Δ

$$\begin{aligned} \omega \sigma_i \sigma_{i+\tilde{\ell}}(\alpha_j) &= \omega \left[\alpha_j - (\alpha_j \cdot \alpha_{i+\tilde{\ell}}) \alpha_{i+\tilde{\ell}} - (\alpha_j \cdot \alpha_i) \alpha_i \right] &(2.19)\\ &= \omega(\alpha_j) - \phi(\alpha_j \cdot \alpha_{i+\tilde{\ell}}) \tilde{\alpha}_i - (\alpha_j \cdot \alpha_i) \tilde{\alpha}_i, &(2.20)\end{aligned}$$

where in (2.19) we used twice (2.16) and the fact that in our labelling we always have $\alpha_i \cdot \alpha_{i+\tilde{\ell}} = 0$. Then (2.20) simply follows upon using (2.2). On the other hand using (2.16) for $\tilde{\mathcal{W}}$ and (2.8) thereafter, we obtain

$$\begin{aligned} \tilde{\sigma}_i \omega(\alpha_j) &= \omega(\alpha_j) - (\omega(\alpha_j) \cdot \tilde{\alpha}_i) \tilde{\alpha}_i &(2.21)\\ &= \omega(\alpha_j) - \phi(\alpha_j \cdot \alpha_{i+\tilde{\ell}}) \tilde{\alpha}_i - (\alpha_j \cdot \alpha_i) \tilde{\alpha}_i. &(2.22)\end{aligned}$$

In [7] a similar identification has been made for the specific case of the embedding $H_4 \hookrightarrow E_8$, which relies on the property of an inflation map which mimics the action of ω entirely inside Δ. Here we avoid the introduction of such a quantity.
Furthermore, for the second map $\tilde{\omega}$ we have the supplementary identity

$$\tilde{\omega} \tilde{\sigma}_i = \sigma_i \sigma_{i+\tilde{\ell}} \, \tilde{\omega} \qquad \text{for} \;\; 1 \leq i \leq \tilde{\ell} \,, \tag{2.23}$$

which follows from a similar argument as (2.18) upon using (2.16), (2.7) and (2.9). As we saw already in (2.8), we note here that $\tilde{\omega}$ plays the role of the "inverse" of ω. There is no analogue to the inflation map in this case. More precisely we find

$$\omega \tilde{\omega} = (1 + \phi^2) \, \mathbb{I} \qquad \text{and} \qquad \tilde{\omega} \omega = \begin{pmatrix} \mathbb{I} & \phi \mathbb{I} \\ \phi \mathbb{I} & \phi^2 \mathbb{I} \end{pmatrix}, \tag{2.24}$$

where $\mathbb{I}$ is the $\tilde{\ell} \times \tilde{\ell}$ unit matrix already encountered in (2.11).

For the application we have in mind, it is important to note that the entire root system Δ can be separated into orbits Ω_i, such that $\Delta = \bigcup_{i=1}^{\ell} \Omega_i$, each containing h roots. Here h is the order of the Coxeter element σ, i.e. the Coxeter number already introduced after (2.11), which is a product of ℓ simple reflections with the property $\sigma^h = 1$. As the reflections do not commute in general, such that Coxeter elements only form conjugacy classes, we have to specify our conventions. For this purpose we attach values $c_i = \pm 1$ to the vertices i of the Coxeter graph, in such a way that no two vertices related to the same value are linked together. The vertices then separate into two disjoint sets $V_\pm$ and the Coxeter element $\sigma = \prod_{i\in V_-} \sigma_i \prod_{i\in V_+} \sigma_i$ is uniquely defined. Introducing "coloured" simple roots as $\gamma_i = c_i \alpha_i$, each orbit $\Omega_i(\tilde{\Omega}_i)$ is then generated by $h(\tilde{h})$ successive actions of $\sigma(\tilde{\sigma})$ on $\gamma_i(\tilde{\gamma}_i)$ [33, 34]. Since we know how to relate the simple reflections of $\mathcal{W}$ and $\tilde{\mathcal{W}}$ by means of (2.18) and (2.23), it is obvious that the Coxeter elements are intertwined as

$$\tilde{\sigma}\omega = \omega\sigma \qquad \text{and} \qquad \tilde{\omega}\tilde{\sigma} = \sigma\tilde{\omega}. \tag{2.25}$$

Note further that

$$\tilde{c}_i = c_i = c_{i+\tilde{\ell}}. \tag{2.26}$$

which is important for (2.25) to work, as it guarantees that (2.18) is not an obstacle for the above mentioned separation of the product into different colours. We then find that

$$\omega(\Omega_i) = \begin{cases} \tilde{\Omega}_i & \text{for} \quad 1 \leq i \leq \tilde{\ell} = \ell/2 \\ \phi\tilde{\Omega}_i & \text{for} \quad \tilde{\ell} < i \leq \ell. \end{cases} \tag{2.27}$$

Thus we can realize the map (2.1) orbit by orbit.

After this generic preliminaries let us discuss in detail the concrete examples of the embeddings $H_2 \hookrightarrow A_4$, $H_3 \hookrightarrow D_6$ and $H_4 \hookrightarrow E_8$.

2.3 The embedding $H_2 \hookrightarrow A_4$

We start with the most simple example, that is the embedding of H_2 (also referred to as $I(5)$, see e.g. [29]) into A_4. First of all we have to fix our conventions for naming the simple roots, which we do by means of the following Coxeter graph (where we adopt the common rule [32, 29] that the label $m_{ij} = 3$ corresponds to one lace)

$$\overset{\alpha_1}{\bullet}\!\!-\!\!\!-\!\!\overset{\alpha_4}{\bullet}\!\!-\!\!\!-\!\!\overset{\alpha_3}{\bullet}\!\!-\!\!\!-\!\!\overset{\alpha_2}{\bullet} \qquad \xrightarrow{\omega} \qquad \overset{\tilde{\alpha}_1}{\bullet}\!\!-\!\!\!-\!\!\overset{\phi\tilde{\alpha}_2}{\bullet}\!\!-\!\!\!-\!\!\overset{\phi\tilde{\alpha}_1}{\bullet}\!\!-\!\!\!-\!\!\overset{\tilde{\alpha}_2}{\bullet}$$

These conventions guarantee that we can realize the map ω as defined in (2.2), which is also indicated in the above diagrams. Accordingly, the Cartan matrix of A_4, as defined in general in (2.5), reads

$$K = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} = R \begin{pmatrix} \tilde{K} & \phi\tilde{K} \\ \phi\tilde{K} & \phi^2\tilde{K} \end{pmatrix}, \tag{2.28}$$

where the Cartan matrix of H_2 is

$$\tilde{K} = \begin{pmatrix} 2 & -\phi \\ -\phi & 2 \end{pmatrix}. \tag{2.29}$$

We may read off relation (2.9) directly by comparing (2.28) with (2.29). Furthermore, we note from the second relation in (2.28) that the identity (2.5) holds by reducing the higher powers of ϕ as indicated above. The Coxeter numbers are $h = \tilde{h} = 5$ and the set of exponents separate according to (2.12) into

$$\{1,2,3,4\} = \{1,4\} \cup \{2,3\}. \tag{2.30}$$

Let us now see in detail how ω acts on the orbits Ω_i and how the map (2.27) is realized. We choose $\sigma = \sigma_1\sigma_3\sigma_2\sigma_4$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_2$ for the Coxeter element of A_4 and H_2, respectively. The corresponding orbits Ω_i and $\tilde{\Omega}_i$ are computed by successive actions of σ and $\tilde{\sigma}$, respectively, on the simple roots. One realizes that the map ω relates them indeed as specified in (2.27). Indicating in the first column the elements σ^p $(\tilde{\sigma}^p)$ for $1 \leq p \leq h = \tilde{h}$ with which we act on the roots reported in the second row, we find for instance

	Ω_1	$\omega(\Omega_1) = \tilde{\Omega}_1$	Ω_3	$\omega(\Omega_3) = \phi\tilde{\Omega}_1$
$\sigma^0, \tilde{\sigma}^0$	α_1	$\tilde{\alpha}_1$	α_3	$\phi\tilde{\alpha}_1$
$\sigma^1, \tilde{\sigma}^1$	$\alpha_3 + \alpha_4$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\phi^2(\tilde{\alpha}_1 + \tilde{\alpha}_2)$
$\sigma^2, \tilde{\sigma}^2$	α_2	$\tilde{\alpha}_2$	α_4	$\phi\tilde{\alpha}_2$
$\sigma^3, \tilde{\sigma}^3$	$-\alpha_2 - \alpha_3$	$-\phi\tilde{\alpha}_1 - \tilde{\alpha}_2$	$-\alpha_1 - \alpha_3 - \alpha_4$	$-\phi^2\tilde{\alpha}_1 - \phi\tilde{\alpha}_2$
$\sigma^4, \tilde{\sigma}^4$	$-\alpha_1 - \alpha_4$	$-\tilde{\alpha}_1 - \phi\tilde{\alpha}_2$	$-\alpha_2 - \alpha_3 - \alpha_4$	$-\phi\tilde{\alpha}_1 - \phi^2\tilde{\alpha}_2$
$\sigma^5, \tilde{\sigma}^5$	α_1	$\tilde{\alpha}_1$	α_3	$\phi\tilde{\alpha}_1$

In order to establish the last identification $\omega(\Omega_3) = \phi\tilde{\Omega}_1$ we simply need to make use of relation (1.2). Furthermore, we obtain for the remaining orbits

	Ω_2	$\omega(\Omega_2) = \tilde{\Omega}_2$	Ω_4	$\omega(\Omega_4) = \phi\tilde{\Omega}_2$
$\sigma^0, \tilde{\sigma}^0$	$-\alpha_2$	$-\tilde{\alpha}_2$	$-\alpha_4$	$-\phi\tilde{\alpha}_2$
$\sigma^1, \tilde{\sigma}^1$	$\alpha_2 + \alpha_3$	$\phi\tilde{\alpha}_1 + \tilde{\alpha}_2$	$\alpha_1 + \alpha_3 + \alpha_4$	$\phi^2\tilde{\alpha}_1 + \phi\tilde{\alpha}_2$
$\sigma^2, \tilde{\sigma}^2$	$\alpha_1 + \alpha_4$	$\tilde{\alpha}_1 + \phi\tilde{\alpha}_2$	$\alpha_2 + \alpha_3 + \alpha_4$	$\phi\tilde{\alpha}_1 + \phi^2\tilde{\alpha}_2$
$\sigma^3, \tilde{\sigma}^3$	$-\alpha_1$	$-\tilde{\alpha}_1$	$-\alpha_3$	$-\phi\tilde{\alpha}_1$
$\sigma^4, \tilde{\sigma}^4$	$-\alpha_3 - \alpha_4$	$-\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2)$	$-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$-\phi^2(\tilde{\alpha}_1 + \tilde{\alpha}_2)$
$\sigma^5, \tilde{\sigma}^5$	$-\alpha_2$	$-\tilde{\alpha}_2$	$-\alpha_4$	$-\phi\tilde{\alpha}_2$

The identification $\omega(\Omega_4) = \phi\tilde{\Omega}_2$ follows upon using (1.2). Having established how the root system Δ can be mapped into $\tilde{\Delta} \cup \phi\tilde{\Delta}$ orbit by orbit, we want to see next how $\mathcal{W}$ relates to $\tilde{\mathcal{W}}$. In principle we have to check all $|\tilde{\mathcal{W}}|$ relations in (2.17). However, it suffices to establish the identification (2.18) for the generating relations of $\tilde{\mathcal{W}}$. It is known that H_2 can be generated entirely from $\tilde{\sigma}_i^2 = 1$ for $i = 1,2,3$ together with

$$(\tilde{\sigma}_1\tilde{\sigma}_2)^5 = 1, \tag{2.31}$$

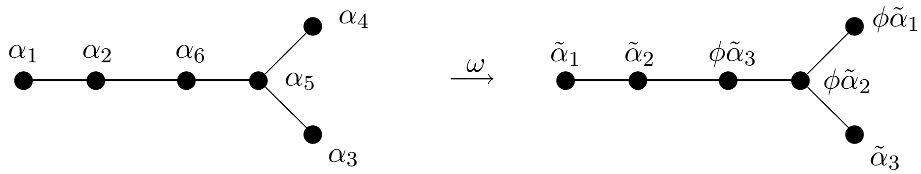
which follows directly from the previous tables, since $\sigma = \tilde{\sigma}_1\tilde{\sigma}_2$. In A_4 this corresponds to

$$(\sigma_1\sigma_3\sigma_2\sigma_4)^5 = 1 \ . \tag{2.32}$$

The remaining relations are trivially satisfied. We know that by definition $\sigma_i^2 = 1$ and therefore this also holds when squaring the product of the embedding on the right hand side of (2.18) $(\sigma_i\sigma_{i+\tilde{\ell}})^2 = \sigma_i^2\sigma_{i+\tilde{\ell}}^2 = 1$. We used here the last equality in (2.26), such that σ_i and $\sigma_{i+\tilde{\ell}}$ commute.

2.4 The embedding $H_3 \hookrightarrow D_6$

In this case we fix our conventions as



The Cartan matrix (2.5) of D_6 then reads

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & -1 & 2 \end{pmatrix} = R \begin{pmatrix} \tilde{K} & \phi\tilde{K} \\ \phi\tilde{K} & \phi^2\tilde{K} \end{pmatrix}, \tag{2.33}$$

where we also exhibit relation (2.5). Noting further that the Cartan matrix of H_3 is

$$\tilde{K} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\phi \\ 0 & -\phi & 2 \end{pmatrix}, \tag{2.34}$$

the relation (2.9) is read off directly by comparing (2.33) with (2.34). The Coxeter numbers are $h = \tilde{h} = 10$ and the set of exponents separate according to (2.12) into

$$\{1,3,5,5,7,9\} = \{1,5,9\} \cup \{3,5,7\}. \tag{2.35}$$

Let us now see how ω acts on the orbits Ω_i and how the map (2.27) is realized. We choose $\sigma = \sigma_1\sigma_4\sigma_3\sigma_6\sigma_2\sigma_5$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\sigma}_2$ for the Coxeter element of D_6 and H_3, respectively. The corresponding orbits Ω_i and $\tilde{\Omega}_i$ are computed by successive actions of σ and $\tilde{\sigma}$, respectively, on the simple roots. One realizes that the map ω relates them indeed as specified in (2.27). See appendix A for the explicit computation of the orbits.

Once more we may check how the root system Δ can be mapped into $\tilde{\Delta} \cup \phi\tilde{\Delta}$ orbit by orbit. First we relate $\mathcal{W}$ to $\tilde{\mathcal{W}}$ and verify (2.18) for the generating relations of $\tilde{\mathcal{W}}$, which for H_3 are known to be $\tilde{\sigma}_i^2 = \tilde{\sigma}_{i+\tilde{\ell}}^2 = 1$ for $i = 1,2,3$ together with

$$(\tilde{\sigma}_2\tilde{\sigma}_3)^5 = (\tilde{\sigma}_1\tilde{\sigma}_2)^3 = (\tilde{\sigma}_1\tilde{\sigma}_3)^2 = 1\ . \tag{2.36}$$

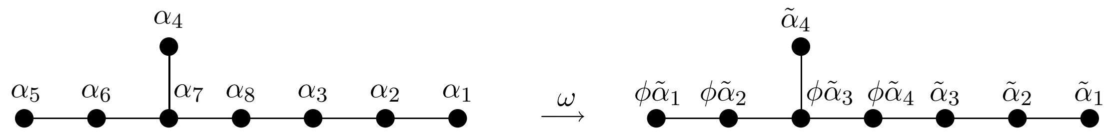
We verify that (2.36) corresponds to

$$(\sigma_2\sigma_5\sigma_3\sigma_6)^5 = (\sigma_1\sigma_4\sigma_2\sigma_5)^3 = (\sigma_1\sigma_4\sigma_3\sigma_6)^2 = 1\ . \tag{2.37}$$

By the same reasoning as in the H_2-case it also follows that $(\sigma_i\sigma_{i+\tilde{\ell}})^2 = 1$.

2.5 The embedding $H_4 \hookrightarrow E_8$

Also in this case we first fix our conventions for naming the simple roots in order to guarantee that we can realize the map ω as defined in (2.2). Here we label according to the Coxeter graph



The corresponding Cartan matrix of E_8 together with its construction from inner products in $\tilde{\Delta}$ in agreement with (2.5) is

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} = R \begin{pmatrix} \tilde{K} & \phi\tilde{K} \\ \phi\tilde{K} & \phi^2\tilde{K} \end{pmatrix} . \tag{2.38}$$

The Cartan matrix of H_4 reads

$$\tilde{K} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\phi \\ 0 & 0 & -\phi & 2 \end{pmatrix} . \tag{2.39}$$

Now the Coxeter numbers are $h = \tilde{h} = 30$ and the set of exponents separate according to (2.12) into

$$\{1, 7, 11, 13, 17, 19, 23, 29\} = \{1, 11, 19, 29\} \cup \{7, 13, 17, 23\}. \tag{2.40}$$

In order to see how ω acts on the orbits Ω_i and how the map (2.27) is realized, we choose $\sigma = \sigma_1\sigma_5\sigma_3\sigma_7\sigma_2\sigma_6\sigma_4\sigma_8$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\sigma}_2\tilde{\sigma}_4$ for the Coxeter element of E_8 and H_4, respectively. We may then compute the corresponding orbits Ω_i, $\tilde{\Omega}_i$ by successive actions of σ, $\tilde{\sigma}$ and the simple roots and realize that the map ω relates them indeed as specified in (2.27). See appendix for the orbits. The individual reflections are related as (2.18) and the generating relations for the Coxeter group are

$$(\tilde{\sigma}_6\tilde{\sigma}_7)^5 = (\tilde{\sigma}_4\tilde{\sigma}_5)^3 = (\tilde{\sigma}_4\tilde{\sigma}_6)^3 = 1 \ . \tag{2.41}$$

which correspond to

$$(\sigma_2\sigma_6\sigma_3\sigma_7)^5 = (\sigma_4\sigma_8\sigma_1\sigma_5)^3 = (\sigma_4\sigma_8\sigma_2\sigma_6)^3 = 1 \ . \tag{2.42}$$

By the same reasoning as in the H_2-case it also follows that $(\sigma_i\sigma_{i+\tilde{\ell}})^2 = 1$.

3. Affine Toda field theories

3.1 Generalities

In this section we will demonstrate how one may construct an affine Toda field theory related to a non-crystallographic Coxeter group $\tilde{\mathcal{W}}$ from a theory related to a crystallographic Coxeter group $\mathcal{W}$ by means of the discussed embedding. We start by taking G to be a Lie group with $H \subset G$ a maximal Torus and $\mathbf{h}$ its Cartan subalgebra. Then the affine Toda field theory Lagrangian can be expressed as [30, 31]

$$\mathcal{L} = \frac{1}{\beta^2} \operatorname{Tr}\left(\frac{1}{2}\, \partial_\mu g^{-1} \partial^\mu g - m^2 g E g^{-1} E^\dagger\right), \tag{3.1}$$

where $g = \exp(\beta\Phi) \in H$, β is a coupling constant and m a mass scale. The regular element $E = \eta \cdot h'$ with conjugate $E^\dagger$ can be expanded in the Cartan subalgebra in apposition $h'_i \in H'$ [35]. We normalize the trace according to the Cartan-Weyl basis, that is $\operatorname{Tr}(E_\alpha E_{-\alpha}) = 1$. Notice that one can not write down a Lagrangian of the type (3.1) and relate it directly to a non-crystallographic Coxeter group, since there is no proper Lie group and Lie algebra associated to them.

Conventionally one introduces ℓ scalar fields ϕ_i by expanding the field Φ in the Cartan subalgebra $\Phi = \sum_{i=1}^{\ell} \phi_i H_i$ with Cartan Weyl generators $H_i \in \mathbf{h}$ [30, 31]. Developing now the Lagrangian (3.1) in powers of the coupling constant, it follows that the term of zeroth order in β, i.e. the quadratic term in ϕ_i becomes

$$-\frac{1}{2}\sum_{i=1}^{\ell} m_i^2\ |\phi_i|^2 = -\frac{1}{2} m^2 \sum_{i=1}^{\ell} |\eta \cdot \alpha_i|^2\ |\phi_i|^2, \tag{3.2}$$

such that the mass of particle i can be identified as $m_i = m|\eta \cdot \alpha_i|$. Here η is the eigenvector of the Coxeter element with eigenvalue $\exp(2\pi i/h)$. Proceeding to the first order term in β, the constant in front of the cubic terms in the fields divided by the symmetry factor 3! is taken to define the three-point coupling. It is computed [31] to

$$C_{ijk} = \frac{4\beta m^2}{\sqrt{-h}} \varepsilon_{ijq} \Delta_{ijk} \tag{3.3}$$

where $\Delta_{ijk} = \sqrt{s(s-m_i)(s-m_j)(s-m_k)}$ with $s = (m_i + m_j + m_k)/2$ is Heron's formula for the area of the triangle formed by the masses of the three fusing particles i, j, k. The structure constants ε_{ijq} result from the Lie algebraic commutator $[E_{\gamma_i}, E_{\sigma^q \gamma_j}] = \varepsilon_{ijq} E_{\gamma_i + \sigma^q \gamma_j} = \varepsilon_{ijq} E_{\sigma^{\bar{q}} \gamma_{\bar{k}}}$ and for simply laced Lie algebras are normalized to $\varepsilon_{ijq} = \pm 1, 0$. The $\gamma_i = c_i \alpha_i$ are the coloured simple roots introduced in section 2.2. In other words the three-point coupling C_{ijk} is non-vanishing if and only if the commutator of step operators related to roots in the orbits Ω_i and Ω_j is non-zero. Through this reasoning the fusing rule [33]

$$C_{ijk} \neq 0 \qquad \Leftrightarrow \qquad \gamma_i + \sigma^q \gamma_j + \sigma^p \gamma_k = 0. \tag{3.4}$$

can be related to the ATFT-Lagrangian.

Having computed these classical quantities one may ask the question towards a formulation of the corresponding quantum field theory. A first glimpse of its nature is obtained from a perturbative computation of the mass corrections. For ATFT the logarithmic divergencies of the self-energy corrections can be removed simply by normal ordering [36, 11]. Then the one loop corrections to the mass of the particle k are just the finite contributions resulting from a bubble graph, which were found [9, 10, 11] to be

$$\delta m_k^2 = -i \frac{\sum_{ij}(m_k^2)}{(2\pi)^2}, \tag{3.5}$$

where the sum extends over all intermediate contributions

$$\sum\nolimits_{ij}(m_k^2) = i\pi \frac{C_{ijk}^2(\pi - \theta_{ij}^k)}{m_i m_j \sin\theta_{ij}^k}, \tag{3.6}$$

with θ_{ij}^k being the fusing angle for the process $i + j \to \bar{k}$.

One should note that instead of the Lie algebraic formulation for the affine Toda field theory Lagrangian (3.1), one can in principle also start with a Lagrangian for which the trace in (3.1) is already computed in the adjoint representation, such that it involves only roots rather than Lie algebraic quantities

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{\beta^2}\sum_{i=0}^{\ell} n_i \exp(\beta\alpha_i\cdot\Phi). \tag{3.7}$$

Here $\alpha_0 = -\sum_{i=0}^{\ell} n_i\alpha_i$ is the negative of the highest root and n_i the Kac labels. In this case the expansion of the Lagrangian in β yields

$$\left(M^2\right)_{ij} = m^2\sum_{p=0}^{\ell} n_p\alpha_p^i\alpha_p^j \qquad \text{and} \qquad C_{ijk} = \beta m^2\sum_{p=0}^{\ell} n_p\alpha_p^i\alpha_p^j\alpha_p^k. \tag{3.8}$$

In most cases the two formulations are equivalent, however one is often more advantageous than the other. For instance, (3.7) does not allow for a generic case independent treatment as it relies on a special choice of the basis for the roots α_i needed to ensure that the mass matrix becomes diagonal. In addition (3.1) yields an explanation [30, 31] for the fact that the masses of ATFT organize into the Perron-Frobenius vector of the Cartan matrix and a generic derivation [31] of the fusing rule (3.4). For our present purposes it is important to note that (3.1) relies on the existence of a Lie algebra, whereas (3.7) only requires a root system. Thus in principle we could write down a Lagrangian of the type (3.7) for non-crystallographic Coxeter groups, but the formulation of an equivalent Lie algebraic version would be impossible. In addition, classical integrability for ATFT is established by means of a Lax pair formulation in terms of Lie algebraic quantities [37, 38]. Therefore it would remain an open issue whether for non-crystallographic Coxeter groups (3.7) corresponds to integrable models or not. Here our formulation will not be a Lagrangian of the form (3.7) involving a non-crystallographic root system, but rather a reduced system which exploits the previously discussed embedding structure with regard to the Cartan subalgebra in apposition. The crystallographic Lie algebra structure is preserved in the reduction procedure, from which we can see immediately that the new theories also possess Lax pairs ensuring the classical integrability.

3.2 Reduction from crystallographic to non-crystallographic ATFT

Somewhat analogue to the folding procedure [28], we may now alter the above expansion of Φ in order to define a new theory, which in this case only contains $\tilde{\ell} = \ell/2$ scalar fields $\tilde{\varphi}_i$. This is achieved by expanding

$$\Phi = \sum_{i=1}^{\ell} \mu(\varphi_i) H_i, \qquad \text{with} \;\; \mu(\varphi_i) = \frac{1}{\phi\sqrt{3}} \begin{cases} \tilde{\varphi}_i & \text{for} \quad 1 \leq i \leq \tilde{\ell} = \ell/2 \\ \phi\tilde{\varphi}_{i-\tilde{\ell}} & \text{for} \quad \tilde{\ell} < i \leq \ell. \end{cases} \tag{3.9}$$

Here the map μ is inspired by the previously introduced map $\tilde{\omega}$ (2.7). At this point one could have also defined μ in such a way that it multiplies the fields φ_i for $\tilde{\ell} < i \leq \ell$ by ϕ^{-1}. However, the choice (3.9) is distinct by the structure emerging below for the three-point coupling. Other possibilities are also conceivable. Having an alternative expansion for the fields Φ as in (3.9), we re-consider first the quadratic term (3.2) in the expansion of the Lagrangian, which now becomes

$$-\frac{1}{2}\sum_{i=1}^{\ell} m_i^2 \; |\mu(\tilde{\varphi}_i)|^2 = -\frac{1}{2}\sum_{i=1}^{\tilde{\ell}} \frac{1}{3}(\phi^{-2} m_i^2 + m_{i+\tilde{\ell}}^2) \; |(\tilde{\varphi}_i)|^2. \tag{3.10}$$

We take this identity as the defining relation for the masses of the $\tilde{\ell}$ new scalar fields $\tilde{\phi}_i$

$$\tilde{m}_i^2 = \frac{1}{3}(\phi^{-2} m_i^2 + m_{i+\tilde{\ell}}^2) = m_i^2 \;. \tag{3.11}$$

In (3.11) we made use of the fact that the masses m_i can be identified as the components of the Perron-Frobenius eigenvector of K, that is $m_i = (y_1)_i$. Subsequently we employ (2.14) which implies that $m_i = \phi m_{i+\tilde{\ell}}$ and then $\phi^{-2} + \phi^2 = 3$. We also notice the fact that the masses $\tilde{m}_i$ admit a "genuine" interpretation as belonging to an affine Toda field theory related to a non-crystallographic Coxeter group, since (2.14) also implies that the masses $\tilde{m}_i$ are the components of the Perron-Frobenius eigenvector of $\tilde{K}(\phi)$.

Similarly we proceed further and read off the next order term in β to define a new three-point coupling $\tilde{C}_{ijk}$. We compute

$$\begin{aligned} \tilde{C}_{ijk} = {} & \phi^3 C_{(i+\ell)(j+\tilde{\ell})(k+\tilde{\ell})} + \phi^2 \left(C_{i(j+\tilde{\ell})(k+\tilde{\ell})} + C_{(i+\tilde{\ell})j(k+\tilde{\ell})} + C_{(i+\tilde{\ell})(j+\tilde{\ell})k} \right) \\ & + \phi \left(C_{(i+\tilde{\ell})jk} + C_{i(j+\tilde{\ell})k} + C_{ij(k+\tilde{\ell})} \right) + C_{ijk}, \end{aligned} \tag{3.12}$$

for $1 \leq i, j, k, \leq \tilde{\ell}$. The identification of the Coxeter element (2.18) translates the fusing rule (3.4) into

$$\tilde{C}_{ijk} \neq 0 \qquad \Leftrightarrow \qquad \omega\gamma_i + \tilde{\sigma}^q \omega\gamma_j + \tilde{\sigma}^p \omega\gamma_k = 0, \tag{3.13}$$

$$\Leftrightarrow \qquad \phi^{t_i}\tilde{\gamma}_i + \phi^{t_j}\tilde{\sigma}^q\tilde{\gamma}_j + \phi^{t_k}\tilde{\sigma}^p\tilde{\gamma}_k = 0, \tag{3.14}$$

where the t_i are the integers introduced in equation (2.5). The relation (3.14) is entirely expressed in quantities belonging to the non-crystallographic Coxeter group. In comparison with the fusing rule related to ATFT for simply laced Lie algebras (3.4), it had to be "ϕ-deformed", somewhat similar to the q-deformed versions of the fusing rule needed for

ATFT associated with non-simply laced Lie algebras [39]. Note that besides the two non-equivalent solutions related as $q \to h+1/2(c_i-c_j)-q$ and $p \to h+1/2(c_i-c_j)-p$ [34, 39], in (3.14) there could be more solutions corresponding to different values of the integers t_i, t_j, t_k.

Alternatively, we may compute the masses and three-point couplings from the Lagrangian (3.7). Using the expansion (3.9) in (3.7) yields the same quantities. We obtain now

$$\left(\tilde{M}^2\right)_{ij} = m^2 \sum_{p=0}^{\ell} n_p \hat{\alpha}_p^i \hat{\alpha}_p^j \qquad \text{and} \qquad \tilde{C}_{ijk} = \beta m^2 \sum_{p=0}^{\ell} n_p \hat{\alpha}_p^i \hat{\alpha}_p^j \hat{\alpha}_p^k. \tag{3.15}$$

where we have folded the ℓ-components of the root α_p^i with $1 \leq i \leq \ell$ into an $\tilde{\ell}$-component vector

$$\hat{\alpha}_p^i = \frac{1}{3}(\phi^{-1}\alpha_p^i + \alpha_p^{i+\tilde{\ell}}) \ , \tag{3.16}$$

for $1 \leq i \leq \tilde{\ell}$. We stress that this is not the same as writing down (3.7) for a non-crystallographic root system, since it still involves ℓ roots, albeit now represented in $\mathbb{R}^{\tilde{\ell}}$. Having computed the three-point couplings for the reduced theory, we may compute the mass renormalisation from (3.6) as

$$\sum\nolimits_{ij} (\tilde{m}_k^2) = i\pi \frac{\tilde{C}_{ijk}^2 (\pi - \theta_{ij}^k)}{\tilde{m}_i \tilde{m}_j \sin\theta_{ij}^k}. \tag{3.17}$$

This will shed light on the possible form of the scattering matrix for ATFT related to non-crystallographic Coxeter groups.

3.3 From A_4 to H_2-affine Toda field theory

Let us now make the above general formulae more explicit. When ignoring the overall mass scale, the masses of A_4-ATFT can be brought into the simple form

$$m_1 = m_2 = 1 \qquad \text{and} \qquad m_3 = m_4 = \phi. \tag{3.18}$$

Keeping the same normalization for the overall mass scale, the identity (3.11) yields for the classical masses of the H_2-ATFT

$$\tilde{m}_1 = m_1 \qquad \text{and} \qquad \tilde{m}_2 = m_2. \tag{3.19}$$

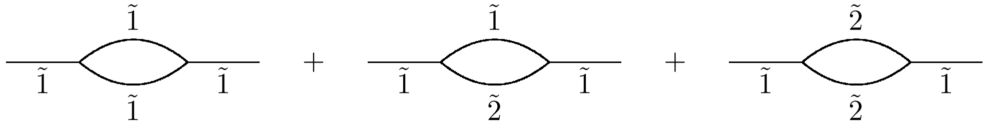
According to (3.3) and (3.12) we then compute the three point couplings C_{ijk} and $\tilde{C}_{ijk}$ together with their corresponding fusing rules, which result from the tables provided in section 2.3

$C_{113} = \frac{4i}{\sqrt{5}}\Delta_{113}$	$\gamma_1 + \sigma\gamma_1 + \sigma^3\gamma_3 = 0$	$\tilde{C}_{111} = 3\phi C_{113} = \frac{4i}{\sqrt{5}}\sqrt{9+12\phi}\tilde{\Delta}_{111}$
$C_{144} = \phi C_{113}$	$\gamma_1 + \sigma^2\gamma_4 + \sigma^4\gamma_4 = 0$	$\tilde{C}_{122} = \phi^3 C_{113}$
$C_{224} = -C_{113}$	$\gamma_2 + \sigma\gamma_2 + \sigma^3\gamma_4 = 0$	$\tilde{C}_{222} = -\tilde{C}_{111}$
$C_{233} = -\phi C_{113}$	$\gamma_2 + \sigma\gamma_3 + \sigma^3\gamma_3 = 0$	$\tilde{C}_{211} = -\tilde{C}_{122}$.

(3.20)

We did not report the factor βm^2 in each of the couplings. Here $\tilde{\Delta}_{ijk}$ is the area of the triangle bounded by the masses $\tilde{m}_i, \tilde{m}_j, \tilde{m}_k$. Note that now the factor of proportionality between $|\tilde{C}_{ijk}|$ and $\tilde{\Delta}_{ijk}$ is no longer universal as in (3.3). This is sufficient information to carry out the perturbation theory up to order β^2. The mass corrections to $\tilde{m}_1$ according to (3.17) are computed by summing the convergent contributions to the one-loop corrections, which up to the symmetry factors are

$\Sigma(\tilde{m}_1^2) =$

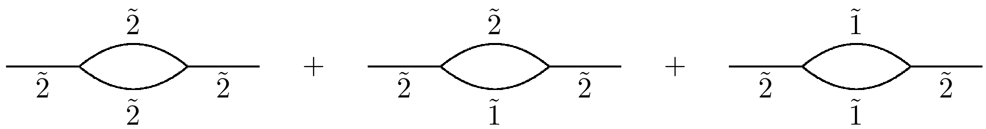


We omitted the usual arrows indicating the time direction and assume throughout this paper that they all run to the right hand side. We computed the symmetry factors by applying Wick's theorem and include them into formula (3.17), such that

$$\sum\nolimits_{ij}(\tilde{m}_1^2) = i\pi \frac{\frac{\pi}{3}}{\sin\frac{2\pi}{3}} \left(18\tilde{C}_{111}^2 + 36\tilde{C}_{112}^2 + 18\tilde{C}_{221}^2\right) . \tag{3.21}$$

where we assume that the particles are conjugate to each other, that is $\bar{1} = 2$, $\bar{2} = 1$ where the bar indicates the anti-particle. Similarly, the mass corrections to $\tilde{m}_2$ result from summing

$\Sigma(\tilde{m}_2^2) =$



which gives

$$\sum\nolimits_{ij}(\tilde{m}_2^2) = i\pi \frac{\frac{\pi}{3}}{\sin\frac{2\pi}{3}} \left(18\tilde{C}_{222}^2 + 36\tilde{C}_{221}^2 + 18\tilde{C}_{112}^2\right) . \tag{3.22}$$

Assembling this according to (3.5) yields the important fact that the classical mass ratios are conserved in the quantum field theory

$$\frac{\delta m_1^2}{m_1^2} = \frac{\delta m_2^2}{m_2^2} = \frac{\sqrt{3}}{4}(5 + 3\phi)\tilde{C}_{111}^2 . \tag{3.23}$$

This means at first order perturbation theory the masses renormalise equally. As the masses of both particles coincide and they undergo the same fusing processes, they appear to be indistinguishable at this stage. This possibly hints towards a non-diagonal scattering theory, that means a theory in which backscattering is possible. However, it remains to be seen whether there exist higher charges in this theory which make the particles distinct.

3.4 From D_6 to H_3-affine Toda field theory

The masses of D_6-ATFT are known for a long time and can be brought into the form (1.3). Keeping the same normalization for the overall mass scale, the identity (3.11) yields for the classical masses of the H_3-ATFT

$$\tilde{m}_1 = \phi^{-1}, \qquad \tilde{m}_2 = \sqrt{1 + \phi^{-2}}, \qquad \tilde{m}_3 = 1. \tag{3.24}$$

According to (3.3) and (3.12) we then compute the three point couplings C_{ijk} and $\tilde{C}_{ijk}$ together with their corresponding fusing rules, which result from the tables provided in the appendix

$C_{112} = 4i/\sqrt{10}\Delta_{112}$	$\gamma_1 + \sigma\gamma_1 + \sigma^6\gamma_2 = 0$	
$C_{442} = \phi^3 C_{112}$	$\gamma_4 + \sigma^3\gamma_4 + \sigma^7\gamma_2 = 0$	$\tilde{C}_{112} = (10\phi + 7)\tilde{\Delta}_{112}$
$C_{445} = \phi^2 C_{112}$	$\gamma_4 + \sigma\gamma_4 + \sigma^6\gamma_5 = 0$	
$C_{665} = -\phi^5 C_{112}$	$\gamma_6 + \sigma^3\gamma_6 + \sigma^7\gamma_5 = 0$	
$C_{332} = \phi^3 C_{112}$	$\gamma_3 + \sigma^3\gamma_3 + \sigma^7\gamma_2 = 0$	$\tilde{C}_{332} = -(1 + 5\phi)\tilde{\Delta}_{332}$
$C_{335} = \phi^2 C_{112}$	$\gamma_3 + \sigma\gamma_3 + \sigma^6\gamma_5 = 0$	
$C_{126} = \phi^2 C_{112}$	$\gamma_1 + \sigma^2\gamma_2 + \sigma^6\gamma_6 = 0$	$\tilde{C}_{123} = 5.56758\tilde{\Delta}_{123}$
$C_{156} = \phi^3 C_{112}$	$\gamma_1 + \sigma^4\gamma_5 + \sigma^8\gamma_6 = 0$	
$C_{225} = (\phi^4 - 1) C_{112}$	$\gamma_2 + \sigma^2\gamma_2 + \sigma^6\gamma_5 = 0$	$\tilde{C}_{222} = -8.6253\tilde{\Delta}_{222}$
$C_{255} = (\phi - \phi^5) C_{112}$	$\gamma_2 + \sigma^3\gamma_5 + \sigma^7\gamma_5 = 0$	
$C_{134} = \phi^2 C_{112}$	$\gamma_1 + \sigma^3\gamma_3 + \sigma^7\gamma_4 = 0$	$\tilde{C}_{113} = 2\phi^2\tilde{\Delta}_{113}$
$C_{346} = \phi^3 C_{112}$	$\gamma_3 + \sigma^8\gamma_4 + \sigma^4\gamma_6 = 0$	$\tilde{C}_{331} = 2\phi^3\tilde{\Delta}_{331}.$

(3.25)

Here we did not report the factor of $i\beta m^2 4/\sqrt{10}$ in $\tilde{C}$. The fusing rules reduce according to (3.13), for instance

$$\gamma_1 + \sigma\gamma_1 + \sigma^6\gamma_2 = 0 \;\Rightarrow\; \tilde{\gamma}_1 + \tilde{\sigma}\tilde{\gamma}_1 + \tilde{\sigma}^6\tilde{\gamma}_2 = 0, \tag{3.26}$$

$$\gamma_4 + \sigma^3\gamma_4 + \sigma^7\gamma_2 = 0 \;\Rightarrow\; \phi\tilde{\gamma}_1 + \phi\tilde{\sigma}^3\tilde{\gamma}_1 + \tilde{\sigma}^7\tilde{\gamma}_2 = 0, \tag{3.27}$$

$$\gamma_4 + \sigma\gamma_4 + \sigma^6\gamma_5 = 0 \;\Rightarrow\; \phi\tilde{\gamma}_1 + \phi\tilde{\sigma}\tilde{\gamma}_1 + \phi\sigma^6\gamma_2 = 0. \tag{3.28}$$

Note that we can construct the solution (3.28) trivially from (3.26) simply by multiplying it with ϕ. However, (3.27) can not be obtained from (3.26) or (3.28) in such a manner and has to be regarded as independent.

As described in the previous section, we compute the mass renormalisation to

$$\sum\nolimits_{ij}(\tilde{m}_1^2) = i\pi\left(\frac{36\tilde{C}_{112}^2\frac{\pi}{10}}{\tilde{m}_1\tilde{m}_2\sin\frac{9\pi}{10}} + \frac{36\tilde{C}_{123}^2\hat{\phi}}{\tilde{m}_2\tilde{m}_3\sin(\pi-\hat{\phi})} + \frac{18\tilde{C}_{133}^2\frac{2\pi}{10}}{\tilde{m}_3\tilde{m}_3\sin\frac{8\pi}{10}} + \frac{36\tilde{C}_{113}^2\frac{2\pi}{10}}{\tilde{m}_1\tilde{m}_3\sin\frac{8\pi}{10}}\right) \tag{3.29}$$

$$\sum\nolimits_{ij}(\tilde{m}_2^2) = i\pi\left(\frac{18\tilde{C}_{222}^2\frac{\pi}{3}}{\tilde{m}_2\tilde{m}_2\sin\frac{2\pi}{3}} + \frac{36\tilde{C}_{123}^2\frac{\pi}{2}}{\tilde{m}_1\tilde{m}_3\sin\frac{\pi}{2}} + \frac{18\tilde{C}_{112}^2\frac{8\pi}{10}}{\tilde{m}_1\tilde{m}_1\sin\frac{2\pi}{10}} + \frac{18\tilde{C}_{233}^2\frac{4\pi}{10}}{\tilde{m}_3\tilde{m}_3\sin\frac{6\pi}{10}}\right) \tag{3.30}$$

$$\sum\nolimits_{ij}(\tilde{m}_3^2) = i\pi\left(\frac{36\tilde{C}_{233}^2\frac{3\pi}{10}}{\tilde{m}_2\tilde{m}_3\sin\frac{7\pi}{10}} + \frac{36\tilde{C}_{123}^2\check{\phi}}{\tilde{m}_2\tilde{m}_3\sin(\pi-\check{\phi})} + \frac{18\tilde{C}_{113}^2\frac{4\pi}{10}}{\tilde{m}_1\tilde{m}_1\sin\frac{6\pi}{10}} + \frac{36\tilde{C}_{133}^2\frac{4\pi}{10}}{\tilde{m}_1\tilde{m}_3\sin\frac{6\pi}{10}}\right) \tag{3.31}$$

where we abbreviate $\hat{\phi} = \arctan\phi^{-1}$, $\check{\phi} = \arctan\phi$. From this it follows then that classical mass ratios are not conserved in the quantum field theory

$$\frac{\delta m_1^2}{m_1^2} = 196.996\ldots \qquad \frac{\delta m_2^2}{m_2^2} = 647.392\ldots \qquad \text{and} \qquad \frac{\delta m_3^2}{m_3^2} = 924.343\ldots \tag{3.32}$$

This means that the scattering matrix for H_3-ATFT can not be of the simple form as for ATFT related to simply laced Lie algebras.

3.5 From E_8 to H_4 affine Toda field theory

The masses of E_8-ATFT in the form (1.1) indicate the underlying H_4 structure and (3.11) yields for the classical masses for the H_4-ATFT

$$\tilde{m}_1 = m_1, \qquad \tilde{m}_2 = m_2, \qquad \tilde{m}_3 = m_3, \qquad \tilde{m}_4 = m_4. \tag{3.33}$$

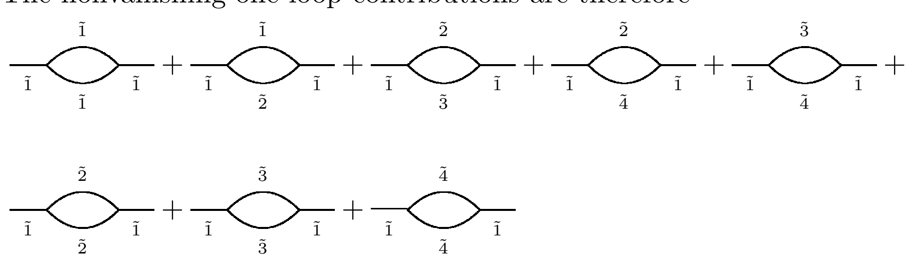
Similarly as in the previous section we compute the three point couplings C_{ijk} and $\tilde{C}_{ijk}$ from (3.3) and (3.12) together with their corresponding fusing rules

$C_{111} = \frac{4i}{\sqrt{30}}\ 0.433013$ $C_{511} = \frac{4i}{\sqrt{30}}\ 0.475528$ $C_{551} = \phi C_{511}$ $C_{555} = \phi^2 C_{111}$	$\gamma_1 + \sigma^{10}\gamma_1 + \sigma^{20}\gamma_1 = 0$ $\gamma_5 + \sigma^{12}\gamma_1 + \sigma^{18}\gamma_1 = 0$ $\gamma_5 + \sigma^{12}\gamma_5 + \sigma^{21}\gamma_1 = 0$ $\gamma_5 + \sigma^{10}\gamma_5 + \sigma^{20}\gamma_5 = 0$	$\tilde{C}_{111} = 31.3768\ \tilde{\Delta}_{111}$
$C_{211} = \frac{4i}{\sqrt{30}}\ 0.103956$ $C_{521} = -\phi^2 C_{321}$ $C_{655} = \phi^2 C_{211}$	$\gamma_2 + \sigma^{14}\gamma_1 + \sigma^{15}\gamma_1 = 0$ $\gamma_5 + \sigma^{13}\gamma_2 + \sigma^{23}\gamma_1 = 0$ $\gamma_6 + \sigma^{14}\gamma_5 + \sigma^{15}\gamma_5 = 0$	$\tilde{C}_{211} = 37.1363\ \tilde{\Delta}_{211}$
$C_{321} = -\frac{4i}{\sqrt{30}}\ 0.307324$ $C_{631} = -C_{432}/\phi$ $C_{653} = C_{432}$ $C_{765} = -\phi^2 C_{321}$	$\gamma_3 + \sigma^{15}\gamma_2 + \sigma^{16}\gamma_1 = 0$ $\gamma_6 + \sigma^{13}\gamma_3 + \sigma^{20}\gamma_1 = 0$ $\gamma_6 + \sigma^{9}\gamma_5 + \sigma^{17}\gamma_3 = 0$ $\gamma_7 + \sigma^{15}\gamma_6 + \sigma^{16}\gamma_5 = 0$	$\tilde{C}_{321} = 22.6132\ \tilde{\Delta}_{321}$
$C_{421} = \frac{4i}{\sqrt{30}}\ 0.972789$ $C_{641} = -\phi^2 C_{321}$ $C_{861} = \phi^3 C_{321}$ $C_{865} = \phi^2 C_{421}$	$\gamma_4 + \sigma^{13}\gamma_2 + \sigma^{19}\gamma_1 = 0$ $\gamma_6 + \sigma^{14}\gamma_4 + \sigma^{17}\gamma_1 = 0$ $\gamma_8 + \sigma^{14}\gamma_6 + \sigma^{18}\gamma_1 = 0$ $\gamma_8 + \sigma^{13}\gamma_6 + \sigma^{19}\gamma_5 = 0$	$\tilde{C}_{421} = 9.92482\ \tilde{\Delta}_{421}$
$C_{431} = -\frac{4i}{\sqrt{30}}\ 1.09848$ $C_{831} = C_{421}/\phi$ $C_{871} = -C_{421}$ $C_{875} = -\phi^2 C_{431}$	$\gamma_4 + \sigma^{13}\gamma_3 + \sigma^{24}\gamma_1 = 0$ $\gamma_8 + \sigma^{14}\gamma_3 + \sigma^{16}\gamma_1 = 0$ $\gamma_8 + \sigma^{14}\gamma_7 + \sigma^{27}\gamma_1 = 0$ $\gamma_8 + \sigma^{13}\gamma_7 + \sigma^{24}\gamma_5 = 0$	$\tilde{C}_{431} = 8.65727\ \tilde{\Delta}_{431}$
$C_{441} = \frac{4i}{\sqrt{30}}\ 1.17616$ $C_{854} = -C_{421}$ $C_{885} = -\phi^2 C_{441}$	$\gamma_4 + \sigma^{13}\gamma_4 + \sigma^{21}\gamma_1 = 0$ $\gamma_8 + \sigma^{13}\gamma_5 + \sigma^{16}\gamma_4 = 0$ $\gamma_8 + \sigma^{13}\gamma_8 + \sigma^{21}\gamma_5 = 0$	$\tilde{C}_{441} = -14.4209\ \tilde{\Delta}_{441}$
$C_{541} = C_{421}/\phi$ $C_{554} = -\phi^3 C_{321}$	$\gamma_5 + \sigma^{14}\gamma_4 + \sigma^{26}\gamma_1 = 0$ $\gamma_5 + \sigma^{7}\gamma_5 + \sigma^{19}\gamma_4 = 0$	$\tilde{C}_{411} = 5.35386$
$C_{771} = -C_{432}$	$\gamma_7 + \sigma^{14}\gamma_7 + \sigma^{22}\gamma_1 = 0$	$\tilde{C}_{331} = -4.27444\ \tilde{\Delta}_{331}$
$C_{222} = -\frac{4i}{\sqrt{30}}\ 1.71313$ $C_{622} = -\frac{4i}{\sqrt{30}}\ 1.88133$ $C_{662} = \phi C_{622}$ $C_{666} = -\phi^2 C_{222}$	$\gamma_2 + \sigma^{10}\gamma_2 + \sigma^{20}\gamma_2 = 0$ $\gamma_6 + \sigma^{18}\gamma_2 + \sigma^{12}\gamma_2 = 0$ $\gamma_6 + \sigma^{12}\gamma_6 + \sigma^{21}\gamma_2 = 0$ $\gamma_6 + \sigma^{10}\gamma_6 + \sigma^{20}\gamma_6 = 0$	$\tilde{C}_{222} = -9.1965\ \tilde{\Delta}_{222}$
$C_{322} = -\frac{4i}{\sqrt{30}}\ 1.96731$ $C_{762} = -C_{432}$ $C_{766} = -\phi^2 C_{322}$	$\gamma_3 + \sigma^{12}\gamma_2 + \sigma^{19}\gamma_2 = 0$ $\gamma_7 + \sigma^{14}\gamma_6 + \sigma^{18}\gamma_2 = 0$ $\gamma_7 + \sigma^{12}\gamma_6 + \sigma^{19}\gamma_6 = 0$	$\tilde{C}_{322} = 3.75947\ \tilde{\Delta}_{322}$

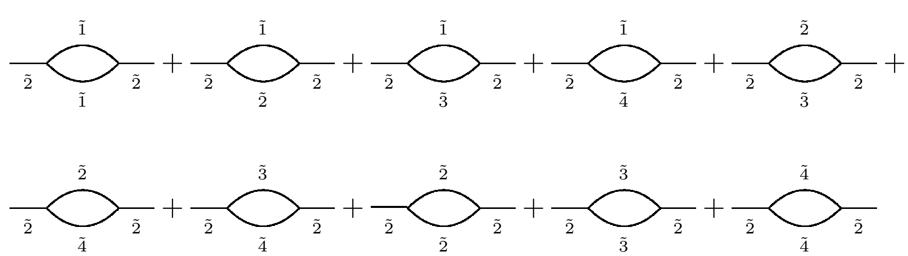
$C_{522} = C_{432}/\phi$ $C_{652} = -\phi^3 C_{321}$	$\gamma_5 + \sigma^{10}\gamma_2 + \sigma^{21}\gamma_2 = 0$ $\gamma_6 + \sigma^{12}\gamma_5 + \sigma^{17}\gamma_2 = 0$	$\tilde{C}_{122} = 9.55253\ \tilde{\Delta}_{122}$
$C_{822} = \phi^2 C_{321}$	$\gamma_8 + \sigma^{14}\gamma_2 + \sigma^{16}\gamma_2 = 0$	$\tilde{C}_{422} = -0.683318\ \tilde{\Delta}_{422}$
$C_{432} = \frac{4i}{\sqrt{30}}\ 2.37859$ $C_{832} = \phi^2 C_{431}$ $C_{863} = -\phi^3 C_{431}$ $C_{876} = \phi^2 C_{432}$	$\gamma_4 + \sigma^{11}\gamma_3 + \sigma^{22}\gamma_2 = 0$ $\gamma_8 + \sigma^{12}\gamma_3 + \sigma^{19}\gamma_2 = 0$ $\gamma_8 + \sigma^{11}\gamma_6 + \sigma^{19}\gamma_3 = 0$ $\gamma_8 + \sigma^{11}\gamma_7 + \sigma^{22}\gamma_6 = 0$	$\tilde{C}_{432} = 15.2555\ \tilde{\Delta}_{432}$
$C_{732} = -C_{432}/\phi$ $C_{772} = -\phi^3 C_{431}$	$\gamma_7 + \sigma^{14}\gamma_3 + \sigma^{17}\gamma_2 = 0$ $\gamma_7 + \sigma^{13}\gamma_7 + \sigma^{22}\gamma_2 = 0$	$\tilde{C}_{332} = 2.68177\ \tilde{\Delta}_{332}$
$C_{333} = -\frac{4i}{\sqrt{30}}\ 3.78439$ $C_{733} = \frac{4i}{\sqrt{30}}\ 4.15597$ $C_{773} = \phi C_{733}$ $C_{777} = \phi^2 C_{333}$	$\gamma_3 + \sigma^{10}\gamma_3 + \sigma^{20}\gamma_3 = 0$ $\gamma_7 + \sigma^{12}\gamma_3 + \sigma^{18}\gamma_3 = 0$ $\gamma_7 + \sigma^{12}\gamma_7 + \sigma^{21}\gamma_3 = 0$ $\gamma_7 + \sigma^{10}\gamma_7 + \sigma^{20}\gamma_7 = 0$	$\tilde{C}_{333} = 7.1965\ \tilde{\Delta}_{333}$
$C_{433} = -\frac{4i}{\sqrt{30}}\ 3.24742$ $C_{743} = -\phi^2 C_{431}$ $C_{877} = \phi^2 C_{433}$	$\gamma_4 + \sigma^{9}\gamma_3 + \sigma^{20}\gamma_3 = 0$ $\gamma_7 + \sigma^{13}\gamma_4 + \sigma^{17}\gamma_3 = 0$ $\gamma_8 + \sigma^{9}\gamma_7 + \sigma^{20}\gamma_7 = 0$	$\tilde{C}_{433} = -9.22437\ \tilde{\Delta}_{433}$
$C_{553} = -C_{421}$	$\gamma_5 + \sigma^{4}\gamma_5 + \sigma^{17}\gamma_3 = 0$	$\tilde{C}_{113} = -2.5468$
$C_{444} = -\frac{4i}{\sqrt{30}}\ 2.50428$ $C_{844} = \frac{4i}{\sqrt{30}}\ 2.75016$ $C_{884} = \phi C_{844}$ $C_{888} = -\phi^2 C_{444}$	$\gamma_4 + \sigma^{10}\gamma_4 + \sigma^{20}\gamma_4 = 0$ $\gamma_8 + \sigma^{12}\gamma_4 + \sigma^{18}\gamma_4 = 0$ $\gamma_8 + \sigma^{12}\gamma_8 + \sigma^{21}\gamma_4 = 0$ $\gamma_8 + \sigma^{10}\gamma_8 + \sigma^{20}\gamma_8 = 0$	$\tilde{C}_{444} = 29.3768\ \tilde{\Delta}_{444}$
$C_{644} = \phi^2 C_{431}$	$\gamma_6 + \sigma^{11}\gamma_4 + \sigma^{19}\gamma_4 = 0$	$\tilde{C}_{244} = -2.13686\ \tilde{\Delta}_{244}$
$C_{744} = C_{421}/\phi$ $C_{874} = \phi^3 C_{431}$	$\gamma_7 + \sigma^{15}\gamma_4 + \sigma^{16}\gamma_4 = 0$ $\gamma_8 + \sigma^{12}\gamma_7 + \sigma^{23}\gamma_4 = 0$	$\tilde{C}_{344} = -8.34233\ \tilde{\Delta}_{344}$

Also here we did not report the overall factor of $i\beta m^2 4/\sqrt{30}$ in $\tilde{C}$. Note that $\tilde{C}_{411}$ and $\tilde{C}_{311}$ have no classical mass triangle associated to them and therefore yield no poles in the propagators. The nonvanishing one-loop contributions are therefore

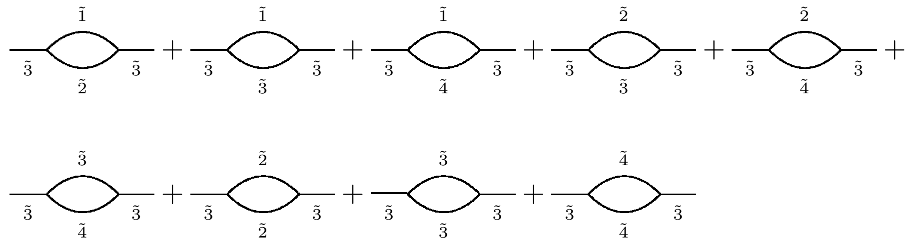
$\Sigma(\tilde{m}_1^2) \quad =$



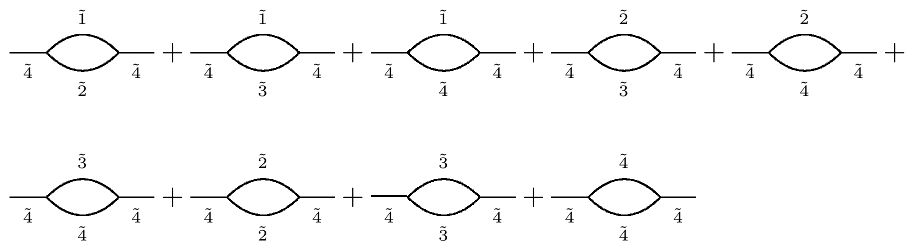
$\Sigma(\tilde{m}_2^2) \quad =$



$\Sigma(\tilde{m}_3^2) \quad =$



$\Sigma(\tilde{m}_4^2) \quad =$



From this it follows that the classical mass ratios are not conserved in the quantum field theory

$$\frac{\delta m_1^2}{m_1^2} = 54045.1\ldots \qquad \frac{\delta m_2^2}{m_2^2} = 68239.3\ldots \qquad \frac{\delta m_3^2}{m_3^2} = 11488.2\ldots \qquad \frac{\delta m_4^2}{m_4^2} = 2914.28\ldots \tag{3.34}$$

Hence we have the same type of behaviour under renormalisation as in the H_3-ATFT obtained from the reduction of the D_6-ATFT.

4. Conclusions

With regard to previously studied ATFT, the embedding of non-crystallographic into crystallographic Coxeter groups leads to an explanation for the fact that in some theories the masses can be organised into pairs such that one mass differs from the other only by a factor of ϕ. This also holds for the higher charges.

We showed that it is possible to construct ATFT related to non-crystallographic Coxeter groups despite the fact that there is no Lie algebra associated to them. The construction is possible since one may exploit the embedding of non-crystallographic into crystallographic Coxeter groups, making use of the fact that the latter do possess a Lie algebraic structure and thus preserving integrability. Unlike the folding from simply laced Lie algebras to non-simply laced Lie algebras [28], the resulting theories we obtain here are not equivalent to a direct formulation of the theory in terms of non-crystallographic Coxeter groups. In this context the reduction procedure is vital for consistency and not merely an additional structure. It is of course possible to write down a Lagrangian of the form (3.7) involving directly roots of $\tilde{\Delta}$, but it remains to be seen whether such theories are consistent and especially if they are classically integrable.

With regard to the quantum field theory, our computations showed for the H_3, H_4-ATFT that the masses of these new theories do not renormalise with an overall factor, i.e. $\delta m_k^2/m_k^2$ is not a universal constant for all particle types k, preventing that the classical mass ratios can be maintained in the quantum field theory. Remarkably this was true for ATFT related to simply laced Lie algebras, which allowed for a relatively straightforward construction of the scattering matrices [10]. For ATFT related to non-simply laced Lie algebras this was found no longer to be true, such that different types of scenarios had to be devised. One is to have floating masses such that depending on the coupling constant the masses flow from one Lie algebra in the weak limit to its Langlands dual in the strong coupling limit [40, 41, 42, 43, 44, 45, 46, 39]. The other alternative proposal was to introduce additional Fermions into the model [47, 48, 49, 50], which compensate for the unequal mass shifts. From our analysis it is clear that the construction of a consistent quantum field theory for the proposed H_3, H_4-theories has to be modelled along the line of the construction of theories related to non-simply laced algebras due to unequal mass renormalisations for each individual particle. It remains to be seen in future work, which of the prescriptions will be successful in this context.

As we showed, the behaviour under renormalisation is different for the H_2-ATFT, where the classical mass ratios remain preserved up to first order perturbation theory. Despite this, it is not immediately obvious how to write down a scattering matrix to all orders in perturbation theory.

Our detailed analysis of the embedding of non-crystallographic into crystallographic Coxeter groups allows one to apply the aforementioned reduction method to a wide range of application in physics, chemistry and biology, where Coxeter groups play a role. For instance in our forthcoming publication [51] we will apply this method to the generalized Calogero-Moser models.

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Appendix

A. The orbits of H_3 and D_6

Successive action of $\sigma = \sigma_1\sigma_4\sigma_3\sigma_6\sigma_2\sigma_5$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\sigma}_2$ yields

	Ω_1	$\omega(\Omega_1) = \tilde{\Omega}_1$	Ω_4	$\omega(\Omega_4) = \phi\tilde{\Omega}_1$
σ^0	α_1	$\tilde{\alpha}_1$	α_4	$\phi\tilde{\alpha}_1$
σ^1	$\alpha_2 + \alpha_6$	$\tilde{\alpha}_2 + \phi\,\tilde{\alpha}_3$	$\alpha_3 + \alpha_5 + \alpha_6$	$\phi\tilde{\alpha}_2 + \phi^2\,\tilde{\alpha}_3$
σ^2	$\alpha_3 + \alpha_4 + \alpha_5$	$\phi\tilde{\alpha}_1 + \phi\tilde{\alpha}_2 + \tilde{\alpha}_3$	$\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$\phi^2\tilde{\alpha}_1 + \phi^2\tilde{\alpha}_2 + \phi\tilde{\alpha}_3$
σ^3	$\alpha_5 + \alpha_6$	$\phi\,(\tilde{\alpha}_2 + \tilde{\alpha}_3)$	$\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6$	$\phi^2\,(\tilde{\alpha}_2 + \tilde{\alpha}_3)$
σ^4	$\alpha_1 + \alpha_2$	$\tilde{\alpha}_1 + \tilde{\alpha}_2$	$\alpha_4 + \alpha_5$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2)$
σ^5	$-\alpha_1$	$-\tilde{\alpha}_1$	$-\alpha_4$	$-\phi\tilde{\alpha}_1$

	Ω_2	$\omega(\Omega_2) = \tilde{\Omega}_2$	Ω_5
σ^0	$-\alpha_2$	$-\tilde{\alpha}_2$	$-\alpha_5$
σ^1	$\alpha_1 + \alpha_2 + \alpha_6$	$\tilde{\alpha}_1 + \tilde{\alpha}_2 + \phi\tilde{\alpha}_3$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
σ^2	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	$\phi\tilde{\alpha}_1 + \phi\tilde{\alpha}_2 + \phi^2\tilde{\alpha}_3$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6$
σ^3	$\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$	$\phi\tilde{\alpha}_1 + 2\phi\tilde{\alpha}_2 + \phi^2\tilde{\alpha}_3$	$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6$
σ^4	$\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6$	$\tilde{\alpha}_1 + \phi^2\tilde{\alpha}_2 + \phi\tilde{\alpha}_3$	$\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$
σ^5	α_2	$\tilde{\alpha}_2$	α_5

	Ω_3	$\omega(\Omega_3) = \tilde{\Omega}_3$	Ω_6
σ^0	α_3	$\tilde{\alpha}_3$	α_6
σ^1	$\alpha_4 + \alpha_5 + \alpha_6$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$
σ^2	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6$	$\tilde{\alpha}_1 + \phi^2\tilde{\alpha}_2 + \phi^2\tilde{\alpha}_3$	$\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6$
σ^3	$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$	$\phi(\tilde{\alpha}_1 + \phi\tilde{\alpha}_2 + \tilde{\alpha}_3)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$
σ^4	$\alpha_3 + \alpha_5$	$\phi\tilde{\alpha}_2 + \tilde{\alpha}_3$	$\alpha_2 + \alpha_5 + \alpha_6$
σ^5	$-\alpha_3$	$-\tilde{\alpha}_3$	$-\alpha_6$

We did not write here the additional $\tilde{\sigma}^i$ in the first column. The identities $\omega(\Omega_4) = \phi\tilde{\Omega}_1$, $\omega(\Omega_4) = \phi\tilde{\Omega}_1$and $\omega(\Omega_6) = \phi\tilde{\Omega}_3$ follow upon using (1.2).

B. The orbits of H_4 and E_8

Successive action of $\sigma = \sigma_1\sigma_5\sigma_3\sigma_7\sigma_2\sigma_6\sigma_4\sigma_8$ and $\tilde{\sigma} = \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\sigma}_2\tilde{\sigma}_4$ yields

	Ω_1	$\omega(\Omega_1) = \tilde{\Omega}_1$	Ω_5
σ^0	α_1	$\tilde{\alpha}_1$	α_5
σ^1	$\alpha_2 + \alpha_3$	$\tilde{\alpha}_2 + \tilde{\alpha}_3$	$\alpha_6 + \alpha_7$
σ^2	$\alpha_7 + \alpha_8$	$\phi(\tilde{\alpha}_3 + \tilde{\alpha}_4)$	$\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8$
σ^3	$\alpha_4 + \alpha_5 + \alpha_7 + \alpha_8$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + \tilde{\alpha}_4$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5$ $+\alpha_6 + \alpha_7 + \alpha_8$
σ^4	$\alpha_3 + \alpha_6 + \alpha_7 + \alpha_8$	$\phi\tilde{\alpha}_2 + \phi^2\tilde{\alpha}_3 + \phi\tilde{\alpha}_4$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$ $+2\alpha_7 + \alpha_8$
σ^5	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8$	$\tilde{\alpha}_1 + \tilde{\alpha}_2 + \phi^2(\tilde{\alpha}_3 + \tilde{\alpha}_4)$	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ $+2\alpha_7 + \alpha_8$
σ^6	$\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_4) + \phi^2(\tilde{\alpha}_2 + \tilde{\alpha}_3)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ $+\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$
σ^7	$\alpha_4 + \alpha_6 + 2\alpha_7 + \alpha_8$	$\phi\tilde{\alpha}_2 + 2\phi\tilde{\alpha}_3 + \phi^2\tilde{\alpha}_4$	$\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$ $+\alpha_6 + 2\alpha_7 + 2\alpha_8$
σ^8	$\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2) + \phi^2(\tilde{\alpha}_3 + \tilde{\alpha}_4)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ $+\alpha_5 + \alpha_6 + 2\alpha_7 + 2\alpha_8$
σ^9	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8$	$\tilde{\alpha}_1 + \phi\tilde{\alpha}_4 + \phi^2(\tilde{\alpha}_2 + \tilde{\alpha}_3)$	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ $+2\alpha_6 + 2\alpha_7 + \alpha_8$
σ^{10}	$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7 + \alpha_8$	$\tilde{\alpha}_2 + \phi^2(\tilde{\alpha}_3 + \tilde{\alpha}_4)$	$\alpha_3 + \alpha_4 + \alpha_6 + 2\alpha_7$ $+2\alpha_8$
σ^{11}	$\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$	$\phi(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ $+\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$
σ^{12}	$\alpha_4 + \alpha_6 + \alpha_7$	$\phi(\tilde{\alpha}_2 + \tilde{\alpha}_3) + \tilde{\alpha}_4$	$\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7$ $+\alpha_8$
σ^{13}	$\alpha_3 + \alpha_8$	$\tilde{\alpha}_3 + \phi\tilde{\alpha}_4$	$\alpha_4 + \alpha_7 + \alpha_8$
σ^{14}	$\alpha_1 + \alpha_2$	$\tilde{\alpha}_1 + \tilde{\alpha}_2$	$\alpha_5 + \alpha_6$
σ^{15}	$-\alpha_1$	$-\tilde{\alpha}_1$	$-\alpha_5$

	Ω_2	$\omega(\Omega_2)=\tilde{\Omega}_2$	Ω_6
σ^0	$-\alpha_2$	$\tilde{\alpha}_2$	$-\alpha_6$
σ^1	$\alpha_1+\alpha_2+\alpha_3$	$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3$	$\alpha_5+\alpha_6+\alpha_7$
σ^2	$\alpha_2+\alpha_3+\alpha_7+\alpha_8$	$\tilde{\alpha}_2+\phi^2\,\tilde{\alpha}_3+\phi\tilde{\alpha}_4$	$\alpha_3+\alpha_4+\alpha_6+2\,\alpha_7+\alpha_8$
σ^3	$\alpha_4+\alpha_5+\alpha_6+2\,\alpha_7+\alpha_8$	$\phi(\tilde{\alpha}_1+\tilde{\alpha}_2+2\tilde{\alpha}_3+\phi\tilde{\alpha}_4)$	$\alpha_1+\alpha_2+2\,\alpha_3+\alpha_4$ $+\alpha_5+\alpha_6+2\,\alpha_7+2\,\alpha_8$
σ^4	$\alpha_3+\alpha_4+\alpha_5$ $+2(\alpha_6+\alpha_7)+\alpha_8$	$\phi(\tilde{\alpha}_1+2\,\tilde{\alpha}_2)+\phi^3\tilde{\alpha}_3+\phi^2\tilde{\alpha}_4$	$\alpha_1+2\,\alpha_2+2\,\alpha_3+\alpha_4$ $+\alpha_5+2\,\alpha_6+3\,\alpha_7+2\,\alpha_8$
σ^5	$\alpha_1+\alpha_2+\alpha_4+\alpha_6$ $+2(\alpha_3+\alpha_7+\alpha_8)$	$\tilde{\alpha}_1+\phi^2(\tilde{\alpha}_2+2\tilde{\alpha}_3)+\phi^3\tilde{\alpha}_4$	$\alpha_2+2\,\alpha_3+2\,\alpha_4+\alpha_5$ $+2\,\alpha_6+4\,\alpha_7+3\,\alpha_8$
σ^6	$\alpha_1+\alpha_4+\alpha_5+\alpha_6$ $+2(\alpha_2+\alpha_3+\alpha_7+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+2\tilde{\alpha}_3+\phi\tilde{\alpha}_4)+(2+\phi)\tilde{\alpha}_2$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$ $+2\alpha_5+3\alpha_6+4\alpha_7+3\alpha_8$
σ^7	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$ $+3\,\alpha_7+2(\alpha_6+\alpha_8)$	$\phi\,\tilde{\alpha}_1+(\phi^4-1)\tilde{\alpha}_3+\phi^3(\tilde{\alpha}_2+\tilde{\alpha}_4)$	$\alpha_1+2\,\alpha_2+3\,\alpha_3+2\,\alpha_4$ $+\alpha_5+3\,\alpha_6+4\,\alpha_7+3\,\alpha_8$
σ^8	$\alpha_3+\alpha_5+3\,\alpha_7$ $+2(\alpha_4+\alpha_6+\alpha_8)$	$\phi\,(\tilde{\alpha}_1+\tilde{\alpha}_2+2\phi\tilde{\alpha}_4)+(\phi^4-1)\,\tilde{\alpha}_3$	$\alpha_1+2\,\alpha_2+3\,\alpha_3+2\,\alpha_4$ $+\alpha_5+2\,\alpha_6+4\,\alpha_7+4\,\alpha_8$
σ^9	$\alpha_1+\alpha_2+\alpha_4+\alpha_5$ $+2(\alpha_3+\alpha_6+\alpha_7+\alpha_8)$	$\phi^2\,(\tilde{\alpha}_1+2\tilde{\alpha}_3)+\phi^3\,(\tilde{\alpha}_2+\tilde{\alpha}_4)$	$\alpha_1+2\,\alpha_2+2\,\alpha_3+2\,\alpha_4$ $+2\,\alpha_5+3\,\alpha_6+4\,\alpha_7+3\,\alpha_8$
σ^{10}	$\alpha_1+2(\alpha_2+\alpha_3)+\alpha_4$ $+\alpha_6+2(\alpha_7+\alpha_8)$	$\tilde{\alpha}_1+(2+\phi)\,\tilde{\alpha}_2+\phi^2(2\tilde{\alpha}_3+\phi\tilde{\alpha}_4)$	$\alpha_2+2\,\alpha_3+2\,\alpha_4+\alpha_5$ $+3\,\alpha_6+4\,\alpha_7+3\,\alpha_8$
σ^{11}	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$ $+\alpha_6+2\,\alpha_7+2\,\alpha_8$	$\phi\,\tilde{\alpha}_1+\phi^2\,\tilde{\alpha}_2+\phi^3\,(\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_1+\alpha_2+2\,\alpha_3+2\,\alpha_4$ $+\alpha_5+2\,\alpha_6+3\,\alpha_7+3\,\alpha_8$
σ^{12}	$\alpha_4+\alpha_5+2(\alpha_6+\alpha_7)+\alpha_8$	$\phi\tilde{\alpha}_1+2\,\phi\,(\tilde{\alpha}_2+\tilde{\alpha}_3)+\phi^2\tilde{\alpha}_4$	$\alpha_1+2\,\alpha_2+2\,\alpha_3+\alpha_4$ $+\alpha_5+2\,\alpha_6+2\,\alpha_7+2\,\alpha_8$
σ^{13}	$\alpha_3+\alpha_4+\alpha_6+\alpha_7+\alpha_8$	$\phi\tilde{\alpha}_2+\phi^2\,(\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_2+\alpha_3+\alpha_4+\alpha_6+2(\alpha_7+\alpha_8)$
σ^{14}	$\alpha_1+\alpha_2+\alpha_3+\alpha_8$	$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\phi\tilde{\alpha}_4$	$\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8$
σ^{15}	α_2	$\tilde{\alpha}_2$	α_6

	Ω_3	$\omega(\Omega_3)=\tilde{\Omega}_3$	Ω_7
σ^0	α_3	$\tilde{\alpha}_3$	α_7
σ^1	$\alpha_1+\alpha_2+\alpha_3+\alpha_7+\alpha_8$	$\tilde{\alpha}_1+\tilde{\alpha}_2+\phi^2\tilde{\alpha}_3+\phi\tilde{\alpha}_4$	$\alpha_3+\alpha_4+\alpha_5$ $+\alpha_6+2\,\alpha_7+\alpha_8$
σ^2	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$ $+\alpha_6+2\,\alpha_7+\alpha_8$	$\phi\tilde{\alpha}_1+\phi^2\tilde{\alpha}_2+\phi^3\tilde{\alpha}_3+\phi^2\tilde{\alpha}_4$	$\alpha_1+\alpha_2+\alpha_4+\alpha_5$ $+3\,\alpha_7+2(\alpha_3+\alpha_6+\alpha_8)$
σ^3	$\alpha_3+\alpha_4+\alpha_5$ $+3\,\alpha_7+2(\alpha_6+\alpha_8)$	$\phi(\tilde{\alpha}_1+2\tilde{\alpha}_2)+(\phi^4-1)\tilde{\alpha}_3+\phi^3\tilde{\alpha}_4$	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_6)$ $+\alpha_5+4\alpha_7+3(\alpha_3+\alpha_8)$
σ^4	$\alpha_1+\alpha_2+2(\alpha_3+\alpha_4)$ $+\alpha_5+3\,\alpha_7+2(\alpha_6+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+\phi\tilde{\alpha}_2+\phi^2\tilde{\alpha}_3+2\tilde{\alpha}_4)$	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_5)$ $+3(\alpha_3+\alpha_6)+5\,\alpha_7+4\,\alpha_8$
σ^5	$\alpha_1+\alpha_4+2(\alpha_2+\alpha_6)$ $+\alpha_5+3(\alpha_3+\alpha_7+\alpha_8)$	$\phi^2\,(\tilde{\alpha}_1+2\,\tilde{\alpha}_2+3\,\tilde{\alpha}_3)$ $+\left(\phi^4-1\right)\tilde{\alpha}_4$	$\alpha_1+2\,\alpha_2+3(\alpha_3+\alpha_4)$ $+2\,\alpha_5+6\,\alpha_7+4(\alpha_6+\alpha_8)$
σ^6	$2(\alpha_2+\alpha_3+\alpha_4+\alpha_6)$ $+\alpha_1+\alpha_5+4\,\alpha_7+3\,\alpha_8$	$\phi^2(\tilde{\alpha}_1+2\tilde{\alpha}_2+2\phi\tilde{\alpha}_3+\phi^2\tilde{\alpha}_4)$	$\alpha_1+2(\alpha_2+\alpha_5)+3\,\alpha_4$ $+4(\alpha_3+\alpha_6)+6\,\alpha_7+5\,\alpha_8$
σ^7	$\alpha_2+2(\alpha_3+\alpha_4+\alpha_5)$ $+4\,\alpha_7+3(\alpha_6+\alpha_8)$	$\phi(\tilde{\alpha}_1+2\phi^2\tilde{\alpha}_3+\phi^3\tilde{\alpha}_4)$ $+\left(\phi^4-1\right)\tilde{\alpha}_2$	$2(\alpha_1+\alpha_5)+3(\alpha_2+\alpha_4)$ $+4(\alpha_3+\alpha_6)+6\,\alpha_7+5\,\alpha_8$
σ^8	$\alpha_1+\alpha_2+2(\alpha_3+\alpha_4)$ $+\alpha_5+4\,\alpha_7+3(\alpha_6+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+2\phi\tilde{\alpha}_3+\phi^2\tilde{\alpha}_4)$ $+\left(\phi^4-1\right)\tilde{\alpha}_2$	$\alpha_1+3(\alpha_2+\alpha_4)+2\,\alpha_5$ $+4(\alpha_3+\alpha_6)+6\,\alpha_7+5\,\alpha_8$
σ^9	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_6)$ $+\alpha_5+3(\alpha_3+\alpha_7+\alpha_8)$	$\phi^2\,(\tilde{\alpha}_1+2\,\tilde{\alpha}_2+3\,\tilde{\alpha}_3)+\phi^4\tilde{\alpha}_4$	$\alpha_1+2(\alpha_2+\alpha_5)+4\,\alpha_6$ $+3(\alpha_3+\alpha_4)+6\,\alpha_7+5\,\alpha_8$
σ^{10}	$\alpha_1+2(\alpha_2+\alpha_3+\alpha_6)$ $+\alpha_4+\alpha_5+3(\alpha_7+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+2\tilde{\alpha}_2+\phi^2\tilde{\alpha}_3)$ $+\left(\phi^4-1\right)\tilde{\alpha}_4$	$\alpha_1+2\,(\alpha_2+\alpha_5)+4\,\alpha_6$ $+3(\alpha_3+\alpha_4)+5\,\alpha_7+4\,\alpha_8$
σ^{11}	$\alpha_2+\alpha_3+\alpha_5+3\alpha_7$ $+2(\alpha_4+\alpha_6+\alpha_8)$	$\phi\,\tilde{\alpha}_1+\phi^3\tilde{\alpha}_2+(\phi^4-1)\tilde{\alpha}_3+2\phi^2\tilde{\alpha}_4$	$\alpha_1+2(\alpha_2+\alpha_4)+\alpha_5$ $+3(\alpha_3+\alpha_6)+4(\alpha_7+\alpha_8)$
σ^{12}	$\alpha_3+\alpha_4+\alpha_5$ $+2(\alpha_6+\alpha_7+\alpha_8)$	$\phi\tilde{\alpha}_1+2\phi\tilde{\alpha}_2+\phi^3\,(\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_1+2(\alpha_2+\alpha_3+\alpha_4)$ $+\alpha_5+2\,\alpha_6+3(\alpha_7+\alpha_8)$
σ^{13}	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_6+\alpha_7+\alpha_8$	$\tilde{\alpha}_1+\phi^2\,(\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$ $+2(\alpha_6+\alpha_7+\alpha_8)$
σ^{14}	$\alpha_2+\alpha_3+\alpha_8$	$\tilde{\alpha}_2+\tilde{\alpha}_3+\phi\tilde{\alpha}_4$	$\alpha_4+\alpha_6+\alpha_7+\alpha_8$
σ^{15}	$-\alpha_3$	$-\tilde{\alpha}_3$	$-\alpha_7$

	Ω_4	$\omega(\Omega_4)=\tilde{\Omega}_4$	Ω_8
σ^0	$-\alpha_4$	$-\tilde{\alpha}_4$	$-\alpha_8$
σ^1	$\alpha_4+\alpha_7$	$\phi\tilde{\alpha}_3+\tilde{\alpha}_4$	$\alpha_3+\alpha_7+\alpha_8$
σ^2	$\alpha_3+\alpha_5+\alpha_6+\alpha_7+\alpha_8$	$\phi(\tilde{\alpha}_1+\tilde{\alpha}_2+\phi\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_5+\alpha_6+2\,\alpha_7+\alpha_8$
σ^3	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_6+2\,\alpha_7+\alpha_8$	$\tilde{\alpha}_1+\phi^2(\tilde{\alpha}_2+\tilde{\alpha}_4)+\phi^3\tilde{\alpha}_3$	$\alpha_2+\alpha_4+\alpha_5+3\,\alpha_7$ $+2(\alpha_3+\alpha_6+\alpha_8)$
σ^4	$\alpha_2+\alpha_4+\alpha_5+\alpha_6$ $+2(\alpha_3+\alpha_7+\alpha_8)$	$\phi\tilde{\alpha}_1+\phi^2\,(\tilde{\alpha}_2+2\,\tilde{\alpha}_3+\phi\tilde{\alpha}_4)$	$\alpha_1+\alpha_2+2(\alpha_3+\alpha_4+\alpha_6)$ $+\alpha_5+4\,\alpha_7+3\,\alpha_8$
σ^5	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_5+3\alpha_7+2(\alpha_6+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+\phi\,\tilde{\alpha}_2+\phi\tilde{\alpha}_4)$ $+(\phi^4-1)\tilde{\alpha}_3$	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_5)$ $+3(\alpha_3+\alpha_6+\alpha_8)+4\,\alpha_7$
σ^6	$\alpha_2+2(\alpha_3+\alpha_4)+\alpha_5$ $+3\,\alpha_7+2(\alpha_6+\alpha_8)$	$\phi(\tilde{\alpha}_1+\phi^2\tilde{\alpha}_2+\phi^3\tilde{\alpha}_3+\phi\tilde{\alpha}_4)$	$\alpha_1+2(\alpha_2+\alpha_4)+\alpha_5$ $+3(\alpha_3+\alpha_6)+5\,\alpha_7+4\,\alpha_8$
σ^7	$\alpha_1+\alpha_2+2(\alpha_3+\alpha_6)$ $+\alpha_4+\alpha_5+3(\alpha_7+\alpha_8)$	$\phi^2(\tilde{\alpha}_1+\phi\tilde{\alpha}_2+\phi^2\tilde{\alpha}_3)$ $+(\phi^4-1)\tilde{\alpha}_4$	$\alpha_1+2(\alpha_2+\alpha_5)+5\alpha_7$ $+3(\alpha_3+\alpha_4+\alpha_6)+4\,\alpha_8$
σ^8	$\alpha_1+2(\alpha_2+\alpha_3+\alpha_4)$ $+\alpha_5+2(\alpha_6+\alpha_8)+3\alpha_7$	$\phi^2(\tilde{\alpha}_1+2\tilde{\alpha}_2+\phi^2\tilde{\alpha}_3+2\tilde{\alpha}_4)$	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_5)$ $+3\,\alpha_3+5\,\alpha_7+4(\alpha_6+\alpha_8)$
σ^9	$\alpha_2+\alpha_4+\alpha_5$ $+2(\alpha_6+\alpha_3)+3(\alpha_7+\alpha_8)$	$\phi\tilde{\alpha}_1+\phi^3\tilde{\alpha}_2+\phi^4\tilde{\alpha}_3$ $+(\phi^4-1)\tilde{\alpha}_4$	$\alpha_1+2\,\alpha_2+\alpha_5+5\alpha_7$ $+3(\alpha_3+\alpha_4+\alpha_6)+4\,\alpha_8$
σ^{10}	$\alpha_1+\alpha_2+\alpha_3+\alpha_5$ $+2(\alpha_4+\alpha_6+\alpha_8)+3\alpha_7$	$\phi^2(\tilde{\alpha}_1+2\tilde{\alpha}_4)+\phi^3\tilde{\alpha}_2$ $+\left(\phi^4-1\right)\tilde{\alpha}_3$	$\alpha_1+2(\alpha_2+\alpha_4+\alpha_5)$ $+3(\alpha_3+\alpha_6)+4(\alpha_7+\alpha_8)$
σ^{11}	$\alpha_2+\alpha_4+\alpha_5$ $+2(\alpha_3+\alpha_6+\alpha_7+\alpha_8)$	$\phi\,\tilde{\alpha}_1+\phi^3(\tilde{\alpha}_2+\tilde{\alpha}_4)$ $+2\phi^2\,\tilde{\alpha}_3$	$\alpha_1+2(\alpha_2+\alpha_3+\alpha_4)$ $+\alpha_5+4\,\alpha_7+3(\alpha_6+\alpha_8)$
σ^{12}	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_6+2(\alpha_7+\alpha_8)$	$\tilde{\alpha}_1+\phi^2\,\tilde{\alpha}_2+\phi^3\,(\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_2+2(\alpha_3+\alpha_4+\alpha_6)$ $+\alpha_5+3(\alpha_7+\alpha_8)$
σ^{13}	$\alpha_2+\alpha_3+\alpha_4+\alpha_5$ $+\alpha_6+\alpha_7+\alpha_8$	$\phi\,\tilde{\alpha}_1+\phi^2\,(\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$ $+\alpha_5+2(\alpha_6+\alpha_7+\alpha_8)$
σ^{14}	$\alpha_6+\alpha_7+\alpha_8$	$\phi(\tilde{\alpha}_2+\tilde{\alpha}_3+\tilde{\alpha}_4)$	$\alpha_2+\alpha_3+\alpha_4+\alpha_6+\alpha_7+\alpha_8$
σ^{15}	α_4	$\tilde{\alpha}_4$	α_8

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