Geometric and Algebraic Properties of Minimal Bases of Singular Systems

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Abstract

For a general singular system $S_e[E, A, B]$ with an associated pencil $T(S)$, a complete classification of the right polynomial vector pairs $(\underline{x}(s), \underline{u}(s))$, connected with the $\mathcal{N}_r\{T(s)\}$ rational vector space, is given according to the proper-nonproper property, characterising the relationship of the degrees of those two vectors. An integral part of the classification of right pairs is the development of the notions of canonical and normal minimal bases for $\mathcal{N}_r\{T(s)\}$ and $\mathcal{N}_r\{R(s)\}$ rational vector spaces, where $R(s)$ is the state restriction pencil of $S_e[E, A, B]$. It is shown that the notions of canonical and normal minimal bases are equivalent; the first notion characterises the pure algebraic aspect of the classification, whereas the second is intimately connected to the real geometry properties and the underlying generation mechanism of the proper and nonproper state vectors $\underline{x}(s)$. The results presented, highlight both the algebraic and geometric properties of the partitioning of the set of reachability indices; the classification of all proper and nonproper polynomial vectors $\underline{x}(s)$ induces a corresponding classification for the reachability spaces to proper-nonproper and results related to the possible dimensions feedback-spectra assignment properties of them are also given. The classification of minimal bases introduces new feedback invariants for singular systems, based on the real geometry of polynomial minimal bases, and provides an extension of the standard theory for proper systems [2] to the case of singular systems.
Nicos Karcanias dedicates this paper to Alistair MacFarlane FRS who has motivated him to explore the relationships between geometric and algebraic methods in feedback control. Alistair MacFarlane was throughout his career interested in exploring the links between geometry and frequency response methods.

1 Introduction

The algebraic notion of polynomial bases of rational vector spaces [3] is central in the study of structure and solvability of linear system problems [3], [4],[5], [6], [7], [8], [9] (and references therein). In the context of state space theory, important rational vector spaces emerge as null spaces of matrix pencils [10] [12] the polynomial vectors in such vector spaces characterise important system theory concepts, such as those of controllability subspaces [2],[11], [12], [25], [26] and their Forney degrees define corresponding types of invariant indices, such as the controllability, observability indices [8], output nulling indices [13] etc. The recent developments in the theory of singular systems [14], and especially in the area of feedback invariants and canonical forms [15],[24], under different types of state space transformations, has motivated the study of a further classification of the polynomial vectors associated with the matrix pencil generated rational vector spaces. For proper (regular state space) systems, there is an important relationship between the degrees of the state and input polynomial vectors of the right null space of the state controllability pencil, known as "plus one" property [2]. It has been observed that the "plus one" property breaks down in the case of singular systems and this has led to the classification of the input and state polynomial vector pairs into proper (satisfying the "plus one" property) and non-proper (violating the "plus one" property) [15], [16], [17], [19]. The classification of state-input polynomial pairs into proper-nonproper, introduces a classification of minimal indices (reachability indices) which defines new invariants under the general feedback transformation group [15]; this classification has been studied in terms of the properties of the recursive subspace algorithms [16], [19] in terms of the elementary transformations which lead to the definition of feedback canonical forms [15], and in terms of the properties of canonical minimal bases [17] associated with the singular system. The aim of this paper is to fully develop the algebraic and geometric properties of canonical minimal bases of singular systems, introduce new feedback invariants, discuss methods for their construction and provide a complete classification of all state polynomial vectors and their associated invariant subspaces.
The current work is within the general framework of developing the algebraic tools linked to matrix pencil theory which enable the characterisation of geometric concepts using as vehicle matrix pencils [27], [22], [23] and the properties of minimal bases [3], [17], [21].

The present work is based on the properties of the real geometry and invariant spaces associated with rational vector spaces [20], and in particular those corresponding to matrix pencils [21], [23]. The analysis is greatly simplified by considering the input-restricted system [22], which enables the development of the properties of state polynomial vectors of $S(E, A, B)$, whereas the nonproperness is simply characterised by the fact that nonproper state vectors are generated by vectors in $\mathcal{N}_r(E)$. The overall classification of minimal bases according to the proper-nonproper property, is based on the study of the partitioning of the family of high coefficient invariant spaces of the system by $\mathcal{N}_r(E)$ which in turn implies an invariant partitioning of the ordered minimal bases and leads in a natural way to the definition of normal minimal bases; the latter are shown to be equivalent to the notion of canonical minimal bases [17] and their construction reveals the partitioning of the reachability indices into the nonproper-proper sets, which goes hand in hand with the decomposition of the maximal prime state module $\mathcal{M}^*$ as a direct sum of a nonproper and a proper maximal submodule $\mathcal{M}_E^*$ and $\tilde{\mathcal{M}}_E^*$ respectively.The structure of the $\mathcal{M}_E^*$, $\tilde{\mathcal{M}}_E^*$ is described by the nesting of the corresponding nonproper, proper prime submodules, each one of them characterising the family of nonproper, proper vectors of a given maximal degree. It is shown that $\mathcal{M}^*$, $\mathcal{M}_E^*$ are uniquely defined, but not $\tilde{\mathcal{M}}_E^*$; however, any $\tilde{\mathcal{M}}_E^*$ is characterised by the same set of dynamical indices, which are the proper reachability indices of the system. The supporting spaces of the polynomial vectors in $\mathcal{M}_E^*$, $\tilde{\mathcal{M}}_E^*$ define the nonproper-proper reachability spaces of the system; the properties of such spaces are intimately connected to the structure of the canonical minimal bases, from which both feedback invariant and spectrum assignment properties of these spaces may be inferred. Although the present study is based on the $S(E, A, B)$ singular system, the results may be readily extended to the case of $S(E, A, C)$, $S(E, A, B, C)$ systems.

The paper is structured as follows: in Section 2 we provide some background definitions and results on the relationship between the invariants of the state-input system pencil and the corresponding restriction pencil, whereas in Section 3 we expand the previous results, introduce the classification of state-input polynomial pairs into proper-nonproper, and define the notion of canonical bases. In Section 4 we summarise some results from [21] on the structure of matrix pencil generated rational vector spaces and in particular those properties related to the real geometry and invariants of such spaces, as well as the generation of polynomial vectors. In section 5 we consider the problem of generating complete sets of polynomial vectors based on a given system of progenitor spaces; the results in Section 5 are specialised to the case where the progenitor spaces are subspaces of $\mathcal{N}_r(E)$ and this leads to the presentation of the main results, given in Section 6 and dealing with the properties, invariant spaces and modules associated with the normal bases. The relationship between normal and canonical bases is also examined in Section 6, where methods for constructing such bases are also considered.

Throughout the paper we denote: R, C, $R(s)$, the real, complex numbers, rational functions respectively. $R^n[s]$, $R^{m\times n}[s]$ are the $n \times 1$, $m \times n$ vectors, matrices with elements from the ring of polynomials $R[s]$. If $T(s) = sF - G \in R^{m\times n}[s]$ is a matrix pencil, then $\Psi(T) \triangleq \{D_\infty(T); D_f(T); I_c(T); I_r(T)\}$ denotes the set of Kronecker invariants of $T(s)$ [10], and in particular, $D_\infty(T)$, $D_f(T)$ are the sets of infinite, finite elementary divisors (IED, FED) and $I_c(T)$, $I_r(T)$ are the sets of column, row minimal indices (CMI, RMI). If H is a matrix (map), then $\rho(H)$ denotes its rank, $\mathcal{N}_r\{H\}$, $\mathcal{N}_l\{H\}$, $\mathcal{R}\{H\}$ denote the right, left null space and range correspondingly. If $\mathcal{V}$ is a vector space, then $\dim\mathcal{V}$ denotes its dimension, V a basis matrix for $\mathcal{V}$ and $\underline{v}$ a general vector. If $t(s) \in R[s]$, then $\vartheta[t(s)]$ denotes its degree. Finally, if n is a positive integer, then $\underline{n} = \{1, 2, \ldots, n\}$.

2 Statement of the Problem and Background Definition and Results

We consider the family of singular systems S_e described by

$$S_\mathrm{e} : E\dot{\underline{x}} = A\underline{x} + B\underline{u}, E, A \in R^{n\times n}, B \in R^{n\times l}, \rho(B) = l < n \tag{1}$$

The family of such systems is denoted by $\Sigma_{l,n}$ and every system in $\Sigma_{l,n}$ is defined by the triplet (E, A, B), or algebraically in terms of the associated system pencil.

$$T(s) = [sE - A, -B] \in R^{n\times(n+l)}[s] \tag{2}$$

We shall denote by $\Pi_{l,n}$ the family of pencils corresponding to $\Sigma_{l,n}$. Consider now the set of ordered pairs

$$H_D = \{h = (W, Q(s)) : W \in R^{n\times n}, |W| \neq 0 \tag{3}$$

$$Q(s) = \begin{bmatrix} V & 0 \\ F_P + sF & G \end{bmatrix} \in R^{(n+l)\times(n+l)}[s],$$

$$|Q(s)| = c \in R - \{0\}, V \in R^{n\times n}, G \in R^{l\times l}\} \tag{4}$$

and define on H_D the composition rule $* : H_D \times H_D \to H_D : \forall h_i = (W_i, Q_i(s)), i = 1, 2$ we have

$$h_1 * h_2 = (W_1, Q_1(s)) * (W_2, Q_2(s)) = (W_1W_2, Q_2(s)Q_1(s)) \tag{5}$$

$(H_D, *)$ is a group, referred to as *Linear Dynamic Feedback Group* and will be simply denoted by H_D. If $F_D = 0$, or $F_P = F_D = 0$ then the resulting subgroups of H_D are denoted by H, H_C and are referred to as state-feedback-(or Brunovsky-)group, coordinate transformation group respectively. The action of H_D on $\Sigma_{l,n}$ is defined as an action on $\Pi_{l,n}$ by: $\circ : H_D \times \Pi_{l,n} \to \Pi_{l,n} : \forall h \in H_D$ and $T(s) \in \Pi_{l,n}$, then

$$h \circ T(s) = T'(s) = W[sE - A, -B] \begin{bmatrix} V & 0 \\ F_P + sF_D & G \end{bmatrix} = [sE' - A', -B'] \tag{6}$$

We shall denote by $H_D(S_\mathrm{e}), H(S_\mathrm{e}), H_C(S_\mathrm{e})$ the equivalence classes or orbits of S_e, or corresponding $T(s)$, under the action of the H_D, H, H_C respectively. Note that every $h \in H_D$ is uniquely defined by the ordered set $\overline{h} = (W, V, G, F_P, F_D)$ and thus the elements of H_D, H, H_C may be denoted by $\overline{h} = (W, V, G, F_P, F_D), \overline{h} =$

$(W,V,G,F_P),\overline{h}=(W,V,G)$ respectively ($\bar{\cdot}$ denotes the ordered set representation).

If $N\in R^{(n+l)\times n}, B^{\dagger}\in R^{l\times n}$ is pair of a left annihilator, left inverse of B respectively ($NB=0, B^{\dagger}B=I_l, \rho(N)=n-l$), then (1) may be equivalently represented [22],[23], by:

$$S_{\mathrm{er}}: NE\underline{x}=NA\underline{x} \tag{7}$$

$$\underline{u}=B^{\dagger}\{E\underline{x}-A\underline{x}\} \tag{8}$$

The differential system S_{e} defined by (7) is known as input-space restricted state model and represents the state trajectory generating mechanism of S_{e} [23]. In fact, for every solution of S_{e}^{r} (generated by an initial condition) the input which together with the initial condition generates this trajectory of S_{e} is defined in (8). The pencil $R(s)=sNE-NA$ is known as a *restriction pencil* [22],[23] and its importance is demonstrated by the following result.

Lemma 2.1. *For any $S_e\in\Sigma_{l,n}$ and any pair $\overline{h}_C=(W,I_n,I_l)\in H_C$ reduces $T(s)$ to*

$$h_C\circ T(s)=WT(s)=\begin{bmatrix} sNE-NA & 0\\ sB^{\dagger}-B^{\dagger}A & -I_l\end{bmatrix}, W=\begin{bmatrix} N\\ B^{\dagger}\end{bmatrix}\in R^{n\times n}, |W|\neq 0 \tag{9}$$

Furthermore, there exists $\overline{h}=(W',V',I_l)\in H_C$ such that

$$\overline{h}_C\circ T(s)=W'T(s)\begin{bmatrix} V' & 0\\ 0 & I_l\end{bmatrix}=\begin{bmatrix} R_k(s) & 0\\ s\,B'^{\dagger}\,E'-B'^{\dagger}\,A' & I_l\end{bmatrix}, W'=\begin{bmatrix} N'\\ B'^{\dagger}\end{bmatrix} \tag{10}$$

where $(N',B'^{\dagger})$ is another pair of left annihilator and inverse of B and $R_k(s)$ is the Kronecker canonical form of $R(s)=sNE-NA$.

Proof. It is clear that for every $(N,B^{\dagger})$ pair $W\in R^{n\times n}, |W|\neq 0$, and that $WT(s)$ is in the form (9). If we select any $(N,B^{\dagger})$ pair, then there exist (L,V) pair of strict equivalence transformations such that $L(sNE-NA)V=R_k(s)$ is in Kronecker form. Clearly from (9) we have

$$\begin{bmatrix} L & 0\\ 0 & I_{n-l}\end{bmatrix}\begin{bmatrix} sNE-NA & 0\\ sB^{\dagger}E'-B^{\dagger}A & I_l\end{bmatrix}\begin{bmatrix} V & 0\\ 0 & I_l\end{bmatrix}=\begin{bmatrix} R_k(s) & 0\\ sB^{\dagger}E'-B^{\dagger}A' & -I_l\end{bmatrix} \tag{11}$$

and since $LN=N'$ is another left annihilator ($L\in R^{(n-l)\times(n-l)}, |L|\neq 0$) the result is established.

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We shall refer to (9) as a *restriction form* and to (10) as a *normal restriction form* of $T(s)$. Clearly, these forms are not uniquely defined, however, the normal restriction form may be considered as a "pseudo-canonical", since $R_K(s)$ is in Kronecker form. Using the above lemma we may state:

Proposition 2.1. *For any $S_e \in \Sigma_{l,n}$, let $\overline{h}_C = (W', V', I_l) \in H_C$ be a transformation that reduces $T(s)$ to its normal restriction form $h_C \circ T(s)$ as in (10), where W' is defined by the $(N', B'^{\dagger})$ pair. Then, if*

$$\overline{h} = (W', V', I_l, -B'^{\dagger} AV') \in H, \overline{h}_D = (W', V', I_l, -B'^{\dagger} AV', B'^{\dagger} EV') \in H_D \tag{12}$$

$$h \circ T(s) = \begin{bmatrix} R_K(s) & 0 \\ s\,B'^{t}\,EV' & -I_l \end{bmatrix} = \tilde{T}(s) \in H(S_e) \tag{13}$$

$$h_D \circ T(s) = \begin{bmatrix} R_K(s) & 0 \\ 0 & -I_l \end{bmatrix} = \tilde{T}^*(s) \in H_D(S_e) \tag{14}$$

■

The proof follows immediately from Lemma 2.1 and the definition. The form defined by (12) is a "pseudo-canonical" form under the H group, since $B'^{\dagger} EV'$ is not uniquely defined. From (14) we have the following important result.

Theorem 2.1. *Let $S_e \in \Sigma_{l,n}$ and let $\Psi(R)$ be the set of strict equivalence invariants of $R(s) = sNE - NA$. The set $\{l, \Psi(R)\}$ is a complete invariant of the $H_D(S_e)$ class and $T^*(s)$ in (14) is a canonical form.*

Proof. The invariance of $\Psi(R)$ is already known (see [22],[23]). If $S_e, S'_e \in \Sigma_{l,n}$ and they have the same $\Psi(R)$, then by Proposition (2.1) we have that there exist $h, h'_D \in H_D$ such that $h_D \circ T(s) = h'_D \circ T'(s)$ and thus $S'_e \in H_D(S_e)$. Clearly then, $T^*(s)$ is a canonical form.

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An alternative proof of the completeness property has been given in [15]. The above results demonstrate that S_e characterise the orbit $H_D(S_e)$ and not just the individual system S_e; S_e^r may thus be also referred to as a "feedback free" description. The invariants of $R(s)$ completely characterise the orbit $H_D(S_e)$, but they are not complete for the $H(S_e), H_C(S_e)$ orbits. The study of relationships between the invariants of $R(s)$ and $T(s)$ is an essential part in our effort to establish the relationships between the sets of CMI, RMI and corresponding minimal bases of the pencils $R(s)$ and $T(s)$ and thus provide a deeper understanding of their algebraic, geometric and feedback properties. Some general properties of the invariants of $R(s)$ and $T(s)$ are examined next.

3 Relationships Between the Invariants of the System and Restriction Pencils: Preliminary Results

In this section, some general properties of the $R(s)$ and $T(s)$ pencils (describing relationships between their invariants) are first derived and then a classification of the pairs of the polynomial vectors $(\underline{x}(s), \underline{u}(s))$ associated with $\mathcal{N}_r\{T(s)\}$ is introduced. The results described here provide the background for the development of the algebraic and geometric classification of CMI and associated minimal bases discussed in the following sections. Throughout the paper we assume that $\rho = \rho_{R(s)}\{T(s)\} \leqslant n$ and it is not necessarily equal to n. Systems with $\rho = n$ are called *normal*, whereas those with $\rho < n$ are called *degenerate* [18];in the following we make no distinction between those two cases. We first note:

Remark 3.1. The pencil $T(s)$ and any of its restriction forms are strict equivalent and thus they have the same set of invariants.

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Some general relationships between $R(s)$ and $T(s)$ are given below.

Proposition 3.1. *For the pencils $T(s), R(s)$ the following properties hold true:*

(i) *If $r = \rho_{R(s)}\{R(s)\}$, then $\rho = \rho_{R(s)}\{T(s)\} = r + l$.*

(ii) *$D_f(R) = D_f(T)$.*

(iii) *Every minimal basis matrix $Y_T(s)$ of $\mathcal{N}_l\{T(s)\}$ may be expressed as $Y_T(s) = Y_R(s)N$ where $Y_R(s)$ is a minimal basis matrix for $\mathcal{N}_l\{(R(s))\}$ and vice versa. Thus, $I_r(R) = I_r(S)$.*

(iv) *$\dim \mathcal{N}_r\{T(s)\} = \dim \mathcal{N}_r\{R(s)\} = n + l - \rho = n - r = p$ and thus $R(s)$ and $T(s)$ have the same number of CMI.*

Proof. (i), (ii), (iv). By Proposition (2.1), $T(s)$ and $T^*(s)$ are $R[s]-$equivalent and thus they have the same Smith form and rank over $R(s)$. From the block diagonal structure of $T^*(s)$, parts (i) and (ii) follow immediately. Part (iv) is a consequence of part (i).
(iii). If $\underline{y}(s)^t \in \mathcal{N}_l\{T(s)\}$, then we may write

$$\underline{y}(s)^t = [\underline{\tilde{y}}_1(s)^t, \underline{\tilde{y}}_2(s)^t] \begin{bmatrix} N \\ B^\dagger \end{bmatrix}, W = \begin{bmatrix} N \\ B^\dagger \end{bmatrix} \tag{15}$$

where $(N, B^\dagger)$ is any pair of annihilator and inverse defined on B and thus $\underline{y}(s)^t T(s) = 0$ implies

$$\underline{\tilde{y}}_1(s)^t, \underline{\tilde{y}}_2(s)^t] \begin{bmatrix} sNE - NA & 0 \\ sB^\dagger E - B^\dagger A & -I_l \end{bmatrix} = 0 \Leftrightarrow \left\{ \begin{array}{r} \underline{\tilde{y}}_1(s)^t(sNE - NA) = 0 \\ \underline{\tilde{y}}_2(s)^t = 0 \end{array} \right. \tag{16}$$

and thus

$$\underline{y}(s)^t = \underline{\tilde{y}}_1(s)^t N, \underline{\tilde{y}}_1(s)^t R(s) = 0 \tag{17}$$

From (15) and (17) and the fact that $W \in R^{n \times n}, |W| \neq 0$, it follows that any two minimal bases $Y_T(s), Y_R(s)$ for $T(s)$ and $R(s)$ are related by

$$Y_T(s) = [Y_R(s), 0]W = Y_R(s)N \tag{18}$$

and thus they also have the same set of RMI.

$\blacksquare$

It is clear from the above result that as far as FED and RMI the two pencils carry the same (or equivalent) information. The relationship between their CMI and associated minimal bases is examined in this paper, whereas the respective relationships between their IED has been considered in [23]. As far as the number of the IED of $R(s)$ and $T(s)$ we have the following result.

Proposition 3.2. *Let* $S_e = (E, A, B) \in \Sigma_{l,n}$ *and denote by* $\rho = \rho_{R(s)}\{T(s)\}$,$r = \rho_{R(s)}\{R(s)\}$. *If* $n_{IED}(\cdot)$ *denotes the numbers of the IED of a pencil, then:*

$$n_{IED}(T) = \rho - \rho(E) \tag{19}$$

$$n_{IED}(R) = r - \rho(NE) = \rho - l - \rho([E, B]) \tag{20}$$

$$n_{IED}(T) = n_{IED}(R) + \rho([E, B]) - \rho(E) \tag{21}$$

and

$$n_{IED}(T) = n_{IED}(R), \text{ if and only if , } I_m(B) \subseteq I_m(E) \tag{22}$$

Proof. For any pencil $P(s) = sF - G, n_{RMI}(P) + n_{IED}(P) = \dim N_l(F)$, where $n_{RMI}(\cdot), n_{IED}(\cdot)$ denote total numbers of RMI, IED respectively. From the above identity and part (i) of Proposition 3.1, conditions (15) the first two parts of (20) are established. Note that $\rho([E, B]) = l + \rho(NE)$ and thus the last part of (20) follows. The rest of the result follows from (21) and (22).

$\blacksquare$

Throughout the rest of the paper we consider the properties of the rational vector spaces $\mathcal{X}_T \triangleq \mathcal{N}_r\{t(s)\}, \mathcal{X}_R \triangleq \mathcal{N}_r\{R(s)\}$ defined over $R(s)$. It is known [3], that the study of properties of rational vector spaces may be reduced to a study of polynomial vector bases; thus, in the following we consider the properties of polynomial vectors in $\mathcal{X}_T$.

Consider a pair $(\underline{x}(s), \underline{u}(s)), \underline{x}(s) \in R^n[s], \underline{u}(s) \in R^l[s]$ such that:

$$[sE - A, -B] \begin{bmatrix} \underline{x}(s) \\ \underline{u}(s) \end{bmatrix} = 0 \Leftrightarrow T(s)\underline{z}(s) = 0 \tag{23}$$

The pair $(\underline{x}(s), \underline{u}(s))$ will be called a *right pair* of S_e and (23) may be equivalently expressed as:

$$(sNE - NA)\underline{x}(s) = \underline{0} \tag{24}$$

$$\underline{u}(s) = B^{\dagger}(sE - A)\underline{x}(s) \tag{25}$$

Remark 3.2. From (25) we have that $\vartheta[\underline{u}(s)] \leqslant \vartheta[\underline{x}(s)] + 1$, where $\vartheta[\cdot]$ denotes the degree of the polynomial vector (max of degrees of its elements). For proper systems,$|E| \neq 0$ and thus the above relationship holds only as equality [2]; for singular systems however the sign "$\leqslant$" holds in general.

Definition 3.1. Let $\underline{z}(s) \in \mathcal{X}_T \cap R^{(n+l)}[s]$ and $(\underline{x}(s), \underline{u}(s))$ be the associated right pair. The vector $\underline{z}(s)$, or the pair $(\underline{x}(s), \underline{u}(s))$ is called *proper*, if $\vartheta[\underline{u}(s)] = \vartheta[\underline{x}(s)] + 1$ and *nonproper*, if $\vartheta[\underline{u}(s)] \leqslant \vartheta[\underline{x}(s)]$.

For proper systems, it is known [2] that all pairs are proper; the properties of the proper pairs are intimately related to important dynamic and feedback system properties [2], [11], [22]. The extension of the classical results (developed for proper systems) to the singular systems case requires the development of the algebraic and geometric aspects of the proper-nonproper pair classification, which is considered here.

By Remark 3.2, it follows that a right pair $(\underline{x}(s), \underline{u}(s))$ with $\vartheta[\underline{x}(s)] = k$ may be represented as:

$$\underline{x}(s) = \underline{x}_0 + s\,\underline{x}_1 + \ldots + s^k\,\underline{x}_k, \underline{x}_k \neq 0, \underline{u}(s) = \underline{u}_0 + s\,\underline{u}_1 + \ldots + s^{k+1}\,\underline{u}_{k+1} \quad (26)$$

We shall denote by $\underline{x}^h = \underline{x}_k \triangleq [\underline{x}(s)]_h, \underline{x}^l = \underline{x}_0\,[\underline{x}(s)]_l$ and shall refer to k as the *essential degree* and $\underline{x}^h, \underline{x}^l$ as the *generators*, *cogenerators* of the pair. The characterisation of the proper-nonproper pairs is given in the following result.

Proposition 3.3. *Let $(\underline{x}(s), \underline{u}(s))$ be a right pair with generator $\underline{x}^h = \underline{x}_k$ and essential degree k.*

(i) *The pair is proper, if and only if there exists $\underline{u}_{k+1} \in R^l \neq 0$, such that*

$$E\,\underline{x}_k = B\,\underline{u}_{k+1} \quad (27)$$

(ii) *The pair is nonproper, if and only if $\underline{x}_k \in N_r(E)$. Furthermore*

$$\vartheta[\underline{u}(s)] = \vartheta[\underline{x}(s)] - \mu = k - \mu, \mu = 0, 1, \ldots \quad (28)$$

if and only if the following conditions are satisfied

$$E\,\underline{x}_k = 0, E\,\underline{x}_{k-1} = A\,\underline{x}_k, \ldots, E\,\underline{x}_{k-\mu} = A\,\underline{x}_{k-\mu}, E\,\underline{x}_{k-\mu+1} - A\,\underline{x}_{k-\mu} = B\,\underline{u}_{k-\mu} \neq 0 \quad (29)$$

■

The proof follows immediately from the definition. From the above result we have some further properties:

Remark 3.3. If $(N, B^\dagger)$ is any pair of a left annihilator and inverse of B, then the right pair $(\underline{x}(s), \underline{u}(s))$ with generator $\underline{x}_h$ is

(i) proper, if and only if

$$N E\,\underline{x}_h = \underline{0}, B^\dagger E\,\underline{x}_h \neq \underline{0} \quad (30)$$

(ii) nonproper, if and only if

$$N E \, \underline{x}_h = \underline{0}, B^{\dagger} E \, \underline{x}_h = \underline{0} \tag{31}$$

$\blacksquare$

Remark 3.4. The generators $\underline{x}_h$ of proper pairs are vectors such that $0 \neq E \, \underline{x}_h \in \mathcal{R}(B)$, whereas those of the nonproper pairs are vectors in $\mathcal{N}_r(E)$. However, not every vector in $\mathcal{N}_r(E)$ generates a right nonproper pair.

$\blacksquare$

Remark 3.5. If $(\underline{x}(s), \underline{u}(s))$ is a nonproper pair and $\vartheta[\underline{u}(s)] = k - \mu = \vartheta[\underline{x}(s)] - \mu, \mu = 0, 1, \ldots$ then the vectors $\{\underline{x}_k, \underline{x}_{k+1}, \ldots, \underline{x}_{k+\mu}\}$ define a Jordan chain at $s = \infty$ for the pair (or pencil sE-A), if they are linearly independent.

From the above remark we have a characterisation of the possible degrees of $\underline{u}(s)$ vectors.

Remark 3.6. Let $\{q_i, i \in \underline{\tau}\}$ be the set of degrees of IED of $sE - A$. If $(\underline{x}(s), \underline{u}(s))$ is a nonproper pair with essential degree k, then

$$k + 1 < \vartheta[\underline{u}(s)] \leqslant k + 1 - q, q \in \{q_1, \ldots, q_\tau\} \tag{32}$$

$\blacksquare$

The question that naturally arises is whether the property of a pair to be proper, or nonproper is invariant under transformations of the H, or H_D type. Note that if $h \in H_D$ and $\overline{h} = (W, V, G, F_P, F_D)$, then the pair defined by $(\underline{x}(s), \underline{u}(s))$, where

$$\underline{x}'(s) = V \underline{x}(s), \ \underline{u}'(s) = F_P \underline{x}(s) + s F_D \underline{x}(s) + G \underline{u}(s) \tag{33}$$

will be referred to as the $h-$*equivalent pair* of $(\underline{x}(s), \underline{u}(s))$ and it is a right pair for the system associated with $T'(s) = h \circ T(s)$. The preservation, or non-preservation of the classification of $h-$equivalent pairs, will be referred to as invariance or non-invariance of the proper, nonproper property under H, or H_D group action. The following result is readily established and stated without proof.

Proposition 3.4. *Let $S_e \in \Sigma_{l,n}$ and $(\underline{x}(s), \underline{u}(s))$ be a right pair of S_e. Then,*

(i) *If $(\underline{x}(s), \underline{u}(s))$ is proper, for all ${S'}_e \in H(S_e)$ the equivalent pairs are also proper.*

(ii) *If $(\underline{x}(s), \underline{u}(s))$ is nonproper, then for all ${S'}_e \in H(S_e)$ the equivalent pairs are also nonproper. Furthermore, if ${S'}_e$ is a generic element of $H(S_e)$ and $(\underline{x}'(s), \underline{u}'(s))$ is the equivalent pair, then $\vartheta[\underline{x}'(s)] = \vartheta[\underline{u}'(s)]$.*

Remark 3.7. The proper-nonproper classification of right pairs is not invariant under H_D-transformations. Thus, as it has been already stated by Theorem (2.1), this classification does not provide any new invariants under H_D-equivalence; however, it may provide new invariants under the $H-$ transformation group.

$\blacksquare$

Proposition 3.4 suggests that since the classification is invariant under the $H-$group, it introduces some new invariants for the $H(S_e)$ class. Defining the properties of this new set of invariants is the aim of the paper. The classification of proper-nonproper pairs may be reduced to an equivalent problem of classifying minimal bases [3], since any polynomial vector may be expressed as a linear combination of vectors of such bases. Thus, we may define [17]:

Definition 3.2. Let $\hat{Z}(s) = [\ldots, \hat{z}_i(s), \ldots] \in R^{(n+m)\times p}[s]$ be a minimal basis matrix (MBM) [3] of $\mathcal{X}_T, \dim \mathcal{X}_T = p$, where $\hat{z}_i(s) = [\underline{x}_i\,(s)^t, \underline{u}_i\,(s)^t]^t$.

(i) If we partition $\hat{Z}(s)$, according to the partitioning of $T(s)$, i.e.

$$\hat{Z}(s) = \left[\begin{array}{c} \hat{X}(s) \\ \hat{U}(s) \end{array} \right], \hat{X}(s) \in R^{n\times p}[s], \hat{U}(s) \in R^{l\times p}[s] \tag{34}$$

then $\hat{X}(s), \hat{U}(s)$ are referred to as *state-*, *input-parts* and $\hat{Z}(s)$ as $(n,l)-$partitioned;

(ii) If $\hat{Z}(s)$ is $(n,l)-$*partitioned* and its columns are ordered such as

$$\hat{Z}(s) = [Z(s); \tilde{Z}(s)] = \left[\begin{array}{cc} X(s) & \tilde{X}(s) \\ U(s) & \tilde{U}(s) \end{array} \right] = \left[\begin{array}{c} \hat{X}(s) \\ \hat{U}(s) \end{array} \right] \tag{35}$$

where the pairs corresponding to the columns of $\tilde{Z}(s), Z(s)$ are proper, nonproper respectively, and the columns in $\tilde{Z}(s), Z(s)$ are ordered according to ascending degrees, then $\hat{Z}(s)$ is called $E-$*ordered*;

(iii) If $\hat{Z}(s)$ is an $E-$*ordered* basis and its state part $\hat{X}(s)$ is a minimal basis itself, then $\hat{Z}(s)$, as well as $\hat{X}(s)$ are called *canonical*.

Obviously, every MBM of $\mathcal{X}_T$ may be reordered and become an $E-$ordered MBM; however, any $E-$ordered MBM is not necessarily canonical. The classification and study of properties of canonical MBMs is one of the main objectives of this paper. Some preliminary properties of the $(n,l)-$partitioned MBMs of $\mathcal{X}_T$ are discussed next. We first note that for any $\hat{Z}(s)$ matrix, $T(s)\hat{Z}(s) = 0$ is equivalent to

$$(sNE - NA)\hat{X}(s) = 0 \tag{36}$$

$$\hat{U}(s) = B^{\dagger}(sE - A)\hat{X}(s) \tag{37}$$

From the above we have:

Proposition 3.5. *Let $\hat{Z}(s) = [\hat{X}(s)^t, \hat{U}(s)^t]^t \in R^{(n+l)\times p}[s]$ be an $(n,l)-$partitioned matrix, such that $T(s)\hat{Z}(s) = 0$. Then,*

(i) *$\hat{Z}(s)$ is an $R[s]-$basis of $\mathcal{X}_T^r$, if and only if $\hat{X}(s)$ is an $R[s]-$basis of $\mathcal{X}_R$.*

(ii) *$\hat{Z}(s)$ is a least degree basis [5] of $\mathcal{X}_T$, if and only if $\hat{X}(s)$ is a least degree basis of $\mathcal{X}_R$.*

Proof.

(i) If $\hat{Z}(s)$ is a basis matrix for $\mathcal{X}_T$, but $\hat{X}(s)$ loses rank over $R[s]$, then $\exists \underline{v}(s) \in R^p[s]$ such that $\hat{X}(s)\underline{v}(s) = 0$; thus, from (37) follows that $\hat{Z}(s)\underline{v}(s) = \underline{0}$ which leads to a contradiction. By Proposition (3.1), it is clear that $\dim \mathcal{X}_T = \dim \mathcal{X}_R = n + l - \rho = n - r = p$ and thus $\hat{X}(s)$ is a basis of $\mathcal{X}_R$, since it satisfies (37), and has τ independent columns. If $\hat{X}(s)$ is a basis matrix of $\mathcal{X}_R$, then it has rank p and obviously $\hat{Z}(s)$ has also rank p. By Prop. (3.1), the sufficiency is established.

(ii) If $\hat{X}(s)$ is least degree, i.e. has no finite zeros, then obviously $\hat{Z}(s)$ has no finite zeros. Assume now that $\hat{Z}(s)$ is least degree, but $\hat{X}(s)$ is not; since $\hat{X}(s)$ has finite zeros, say $z \in C$ is one of them, then $\exists \underline{v} \in C^p$ such that $\hat{X}(z)\underline{v} = 0$; by (37) and for $s = z$ we have that $\hat{U}(z)\underline{v} = 0$ and thus $\hat{Z}(z)\underline{v} = 0$ which leads to a contradiction. This completes the proof.

■

Remark 3.8. If $\hat{Z}(s)$ defines a canonical minimal basis (MB) of $\mathcal{X}_T$, then its state part $\hat{X}(s)$ defines an MB for $\mathcal{X}_R$. If $\hat{X}(s)$ defines a MB of $\mathcal{X}_R$ and $\hat{U}(s)$ is defined by (37), then $\hat{Z}(s) = [\hat{X}(s)^t, \hat{U}(s)^t]^t$ is a least degree basis of $\mathcal{X}_T$ but not necessarily a MB.

■

The question that naturally arises is to determine the conditions under which an $E-$ordered MB is also canonical, or equivalently the conditions under which an MB of $\mathcal{X}_R$ may be extended to an MB of $\mathcal{X}_T$. Note that the key tool for this investigation are the geometric properties of the generators of the right pairs; the properties of the generators form an integral part of the theory of algebraic and geometric invariants of matrix pencils [20] which are summarised in the following section.

4 Geometric and Algebraic Invariants of Rational Vector Spaces Associated with Matrix Pencils

It is evident from the characterisation of the proper, non-proper pairs that the geometry of the space of generators of such pairs, plays an essential role in the classification process. The geometry of the generators spaces is part of a theory of geometric and algebraic invariants of rational vector spaces [20], and matrix pencils in particular [21]. In this section we review some of the basic concepts and results from the algebraic and geometric theory of invariants associated with matrix pencil generated minimal bases, which are of importance in our present task; a full treatment of the topic may be found in [20],[21].

Let $\mathcal{X}$ be an $R(s)-$vector space of $R^q(s)$ with $dim\mathcal{X} = p$. The set $\mathcal{M}^\star$ of all polynomial vectors $\underline{x}(s)$ of $X, \underline{x}(s) \in R^q[s] \cap X$, is an $R[s]$ maximal Noetherian module [1]. A basis matrix $X(s) \in R^{q\times p}[s]$ which has no finite zeros and it is column reduced [8] is called a *Minimal Basis Matrix* (MBM) [3]. Minimal bases of $\mathcal{X}$ are not uniquely defined, but all of them have the same column degrees, known as *Forney dynamical indices*; this set is an invariant of $\mathcal{X}$ [3] and may be represented as an ordered set, i.e.

$$I(\mathcal{X}) \triangleq \{(\varepsilon_i, \rho_i), i \in \underline{\mu}, 0 \leqslant \varepsilon_1 < \ldots < \varepsilon_\mu\} \tag{38}$$

where ε_i denotes the distinct values of the degrees and ρ_i the multiplicity of ε_i. $I(\mathcal{X})$ may be referred to as the *index* and the set $(I(\mathcal{X})) = \{(\varepsilon_i, i \in \underline{\mu})\}$ as the *list* of $\mathcal{X}$; clearly, $p = dim\mathcal{X} = \sum_{i=1}^{\mu} \rho$. Any MBM of $\mathcal{X}, X(s)$, is called an *ordered*-MBM (OMBM), if it may be expressed as

$$X(s) = [X_1(s), \ldots, X_\mu(s)], X_i(s) = [\ldots, \underline{x}_{ij}(s), \ldots] \in R^{q\times \rho_i}[s] \tag{39}$$

where $\vartheta[\underline{x}_{ij}(s)] = \varepsilon_i, \forall j \in \underline{\rho}_i, i \in \underline{\mu}$. Any two OMBMs are related as shown below [6], [7]:

Lemma 4.1. *Let $X(s), X'(s) \in R^{q\times p}[s]$ and assume that $X(s)$ is an OMBM of $\mathcal{X}$ with index $I(\mathcal{X})$ as in (38). $X'(s)$ is an OMBM of $\mathcal{X}$ if and only if there exists $W(s) \in R^{p\times p}[s], R[s]-$unimodular, such that*

$$X'(s) = X(s)W(s) \tag{40}$$

where $W(s)$ is an $I(X)-$structured matrix defined by:

$$W(s) = \begin{array}{c} \begin{array}{ccccc} \rho_1 \text{ —} & \rho_2 \text{ —} & \rho_3 \text{ —} & \ldots \text{ —} & \rho_\mu \\ \leftrightarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow \end{array} \\ \left[\begin{array}{ccccc} W_1 & W_{12}(s) & W_{13}(s) & \ldots & W_{1\mu}(s) \\ 0 & W_2 & W_{23}(s) & \ldots & W_{2\mu}(s) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & W_{\mu-1,\mu}(s) \\ 0 & \ldots & \ldots & 0 & W_\mu \end{array}\right] \end{array} \begin{array}{l} \updownarrow \rho_1 \\ \updownarrow \rho_2 \\ \vdots \\ \updownarrow \rho_{\mu-1} \\ \updownarrow \rho_\mu \end{array} \tag{41}$$

with $W_i \in R^{\rho_i\times\rho_i}, i \in \underline{\mu}, |W_i| \neq 0$, and $W_{ij}(s) = s^{p_{ij}}W_{ij} + \ldots + W'_{ij} \in R^{\rho_i\times\rho_j}[s], p_{ij} \leqslant \varepsilon_j - \varepsilon_i$, but otherwise arbitrary.

■

If $T(s) = [\underline{x}_1(s), \ldots, \underline{x}_p(s)] \in R^{p\times q}[s]$ and $\underset{i}{\underline{x}}^h$, $\underset{i}{\underline{x}}^\ell$ denote the high-low coefficient vectors of $\underline{x}_i(s)$, then we shall denote by $[T(s)]_h = T^h = [\underline{x}_1^h, \ldots, \underline{x}_p^h]$ and by $[T(s)]_l = T^l = [\underline{x}_1^l, \ldots, \underline{x}_p^l]$. For any OMBM we may define:

Definition 4.1. Let $\mathcal{X} \in R^q(s)$ be an $R[s]-$vector-space with index $I(\mathcal{X})$ as in (38) and let $X(s) = [X_1(s), \ldots, X_\mu(s)] \in R^{p\times q}[s]$ be an OMBM of $\mathcal{X}$. If we denote by $X^{i)}(s) = [X_1(s), \ldots, X_i(s)]$, we may define:

(i) The set of $R[s]-$*prime modules* $\{\mathcal{M}_i, i \in \underline{\mu}\}$, by

$$\mathcal{M}_i = colsp_{R[s]}\{X^{i)}(s)\}, i = 1, 2, \ldots, \mu \tag{42}$$

(ii) If $X^{i,h} = [X^{i)}(s)]_h, X_{i,l} = [X^{i)}(s)]_l \in R^{p\times q}$, we define the sets of high-low coefficient spaces $\{\mathcal{P}_i, i \in \underline{\mu}\}, \{\hat{\mathcal{P}}_i, i \in \underline{\mu}\}$ by

$$\mathcal{P}_i = col.sp\{X^{i,h}\}, \hat{\mathcal{P}}_i\, col.sp_R\{X^{i,l}\}, i = 1, 2, \ldots, \mu \tag{43}$$

If $\mathcal{V}$ is any subspace of $\mathcal{P}_i$ (or $\hat{\mathcal{P}}_i$), where $\mathcal{V} \notin \mathcal{P}_{i-1}(\hat{\mathcal{P}}_{i-1})$, then ε_i is called the *order* of $\mathcal{V}$ and is denoted by $\gamma(\mathcal{V}) = \varepsilon_i$.

(iii) If $X_k(s) = [\underline{x}_{k,1}(s), \ldots, \underline{x}_{k,\rho_k}], k \in \underline{\mu}, \vartheta[\underline{x}_{k,j}(s)] = \varepsilon_k, \forall j \in \underline{\rho_k}$, and $\underline{x}_{k,j}(s) = \underline{x}_0^{k,j} + \ldots + s^{\varepsilon_k}\, \underline{x}_{\varepsilon_k}^{k,j}$, then $\mathcal{S}_j^k = sp_R\{\underline{x}_0^{k,j}, \ldots, \underline{x}_{\varepsilon_k}^{k,j}\}$ is the *supporting space* of $\underline{x}_{k,j}(s)$. We may define the set of *prime spaces* $\{\mathcal{R}_i, i \in \underline{\mu}\}$ by

$$\mathcal{R}_i = \sum_{k=1}^{i} \sum_{j=1}^{\rho_k} S_j^k, \forall i = 1, 2, \ldots, \mu \tag{44}$$

The importance of the above concepts is described below.

Theorem 4.1. *[20] Let $\mathcal{X} \in R^q(s), dim\mathcal{X} = q, I(\mathcal{X})$ be its index (as in (38)) and let $\{\mathcal{M}_i, i \in \underline{\mu}\}, \{\mathcal{P}_i, i \in \underline{\mu}\}, \{\hat{\mathcal{P}}_i, i \in \underline{\mu}\}, \{\mathcal{R}_i, i \in \underline{\mu}\}$ be the sets associated with an OMBM $X(s)$ of $\mathcal{X}$. The following properties hold true:*

(i) *The modules $\mathcal{M}_i$ and the spaces $\mathcal{P}_i, \hat{\mathcal{P}}_i, \mathcal{R}_i$ are invariants of $\mathcal{X}$, for all $i = 1, \ldots, \mu$ and they satisfy the chain conditions*

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_\mu = \mathcal{M}^\star \tag{45}$$

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_\mu = \mathcal{P}^\star, \hat{\mathcal{P}}_1 \subset \hat{\mathcal{P}}_2 \subset \ldots \subset \hat{\mathcal{P}}_\mu = \hat{\mathcal{P}}^\star \tag{46}$$

$$\mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_\mu = \mathcal{R}^\star \tag{47}$$

(ii) *If $X(s)$ is any OMBM of $\mathcal{X}$ and $X^{i)}(s), X^{i,h}, X^{i,l}$ are the associated matrices (see Definition 4.1), then $\mathcal{M}_i$ has $X^{i)}(s)$ as an OMBM and dynamical indices $I_i = \{(\varepsilon_j, \rho_j), j \in \underline{i}, 0 \leqslant \varepsilon_1 < \ldots < \varepsilon_i\}$. The spaces $\mathcal{P}_i, \hat{\mathcal{P}}_i, \mathcal{R}_i$ are invariants of $\mathcal{M}_i$ and $X^{i,h}, X^{i,l}$ are bases matrices of $\mathcal{P}_i, \hat{\mathcal{P}}_i$ respectively. Furthermore $\mathcal{P}_i, \hat{\mathcal{P}}_i$ are proper subspaces of $\mathcal{R}_i$ and*

$$dim\mathcal{P}_i = dim\,\hat{\mathcal{P}}_i = \sum_{j=1}^{i} \rho_j = p_i, \forall i \in \underline{\mu} \tag{48}$$

The spaces $\mathcal{P}^\star, \hat{\mathcal{P}}^\star, \mathcal{R}^\star$ are called the *maximal-high*, *maximal-low*, *maximal prime-spaces* of $\mathcal{X}$, whereas $\mathcal{M}^\star$ is the maximal $R[s]-$module in $\mathcal{X}$. The index I_i will be referred to as the $i-th$ *partial index* of $\mathcal{X}$ and $X^{i)}(s)$ as the $i-th$ *partial OMBM* of the OMBM $X(s)$ of $\mathcal{X}$. Note that by invariants of $\mathcal{X}$ (or $\mathcal{M}$) we mean "basis free, independent" invariants. Some further properties highlighting the significance of the above concepts are given below [20].

Corollary 4.1. *Let $\underline{x}(s) = s^k\,\underline{x}^h + \ldots + \underline{x}^l \in X$ and $\mathcal{S}_{\mathcal{X}} = sp\{\underline{x}^h, \ldots, \underline{x}^l\}$ be its supporting space. The following properties hold true:*

(i) *If $\underline{x}(s) \in \mathcal{M}_i$, then $\mathcal{S}_{\underline{x}} \in \mathcal{R}_i, \underline{x}^h \in \mathcal{P}_i$ and $\underline{x}^l \in \hat{\mathcal{P}}_i$; furthermore, $\mathcal{R}_i$ is the minimal subspace that contains all $\mathcal{S}_{\underline{x}}$, for which $\underline{x}(s) \in \mathcal{M}_i$.*

(ii) *For every $\underline{x}^h \in \mathcal{P}_i(\underline{x}^l \in \hat{\mathcal{P}}_i)$, there exists $\underline{x}(s) \in \mathcal{M}_i$ such that $[\underline{x}(s)]_h = \underline{x}^h([\underline{x}(s)]_l = \underline{x}^l)$.*

■

The above results also apply to the case of matrix pencil generated rational vector spaces. Thus if $sF-G \in R^{t\times q}[s]$, we shall denote by $X_r = N_r\{sF-G\} \in R^q(s)(X_l = N_l\{sF-G\} \in R^t(s))$ and the results apply also to $\mathcal{X}_r(\mathcal{X}_l)$. Matrix pencil generated rational vector spaces have richer properties as it is shown next. We first define:

Definition 4.2. Let $<\mathcal{X}>= \{\underline{x}_i(s) : \underline{x}_i(s) \in R^q[s], \vartheta[\underline{x}_i(s)] = d_i, i \in \underline{\tau}\}$ and let $\mathcal{S}_i$ be the supporting space of $\underline{x}_i(s), \forall i \in \underline{\tau}$. We define:

(i) $\underline{x}_i(s)$ as *prime*, if $dim\mathcal{S}_i = d_i + 1$.

(ii) $<\mathcal{X}>$ as a *proper set*, if the corresponding set $\{\mathcal{S}_i, i \in \underline{\tau}$ is linearly independent; otherwise, it is called *nonproper*.

(iii) $<\mathcal{X}>$ as a *complete set*, if it is proper and every $\underline{x}_i(s) \in <\mathcal{X}>$ is prime.

Remark 4.1. Every complete set $<\mathcal{X}>$ is linearly independent over $R[s]$, the corresponding matrix $X(s) = [\underline{x}_1(s), \ldots, \underline{x}_\tau(s)]$ has no zeros and it is column reduced; that is, it is a minimal basis. The converse is not always true; that is any minimal basis of any rational vector space, does not necessarily define a complete set.

■

We consider next the space $\mathcal{X}_r, dim\mathcal{X}_r = p$ and with index $I(\mathcal{X}_r)$. Note that $I(\mathcal{X}_r)$ is the ordered set of CMI of $sF-G$ and thus we may also denote $I(\mathcal{X}_r)$ by $I_c(F,G)$. Some further properties of $\mathcal{X}_r$ are described below [21].

Theorem 4.2. *Let $sF-G \in R^{t\times q}[s], I_c(F,G) = \{(d_i, r_i), i \in \underline{n}, 0 \leqslant d_1 < \ldots < d_n\}$ and $\{\mathcal{R}_i, i \in \underline{n}\}$ be the set of prime spaces $\mathcal{X}_r$. Then,*

(i) *Any $R[s]-$minimal basis of $\mathcal{X}_r$ is complete.*

(ii) *If $X(s) = [\dots, X_i(s), \dots], X_i(s) = [\dots, \underline{x}_{ij}(s), \dots], \vartheta[\underline{x}_{ij}(s)] = d_i, j \in \underline{r}_i, i \in \underline{n}$, is any OMBM of $\mathcal{X}_r$ and $\mathcal{S}^i_j$ denotes the supporting spaces of $\underline{x}_{ij}(s)$, then*

$$\mathcal{R}_k = S^1_1 \oplus \dots \oplus S^1_{r_1} \oplus \dots \oplus S^k_1 \oplus \dots \oplus S^k_{r_k} \tag{49}$$

$$dim\mathcal{R}_k = \sum_{i=1}^{k} r_i(d_i+1) = \tau_k \tag{50}$$

■

The completeness of any OMBM of $\mathcal{X}_r$ is the distinguishing feature of matrix pencil generated rational vector spaces, and has important implications on the relationships between $\mathcal{P}_k, \hat{\mathcal{P}}_k, \mathcal{R}_k$, as well as the construction of their bases. Some further properties of the $\mathcal{R}^\star, \mathcal{P}^\star, \hat{\mathcal{P}}^\star$ spaces are stated next [21]:

Corollary 4.2. *Let $sF - G$ be a singular pencil and $\mathcal{R}^\star, \mathcal{P}^\star, \hat{\mathcal{P}}^\star$ be the spaces associated with X_r. Then,*

$$\mathcal{N}_r\{G\} \cap \mathcal{R}^\star = \hat{\mathcal{P}}^\star, \mathcal{N}_r\{F\} \cap \mathcal{R}^\star = \mathcal{P}^\star \tag{51}$$

$$\mathcal{N}_r\{G\} \cap \mathcal{N}_r\{F\} = \hat{\mathcal{P}}^\star \cap \mathcal{P}^\star = \mathcal{S} \tag{52}$$

Furthermore, if d_1 is the smallest CMI and r_i its multiplicity,then $\mathcal{S} \neq \{0\}$ and $dim\mathcal{S} = r_1$ if and only if $d_1 = 0$.

■

Some further invariant spaces of X_r may be introduced by using the structure of Toeplitz representations of OMBMs [21]. The Toeplitz representation of OMBMs is essential in deriving some of the properties of the polynomial vectors in $\mathcal{X}_r$ and its most essential features are summarised below [20], [21]:

Definition 4.3. Let $X(s) = [X_1(s), \dots, X_n(s)]$ be an OMBM of $\mathcal{X}$ with index $I(\mathcal{X}) = \{(d_i, r_i), i \in \underline{n}\}$ and let $X_j(s) = s^{d_j} X^j_{d_j} + \dots + sX^j_1 + X^j_0 \in R^{q\times r_j}[s], \forall j \in \underline{\mu}$. We may define:

(i) For every $k \geqslant d_j$, with $f_j = k - d_j \geqslant 0$, we define

$$T^k_{d_j}(X_j) = \underbrace{\begin{bmatrix} X^j_{d_j} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ X^j_0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & X^j_{d_j} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X^j_0 \end{bmatrix}}_{f_j \text{ - blocks}} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} f_j \text{ - blocks} \in R^{(k+1)q\times(f_j+1)r_j} \tag{53}$$

as the $k-$th Toeplitz matrix of $X_j(s)$.

(ii) For all $k : d_j \leqslant k < d_{j+1}$ we define the matrix

$$T_I^k(X) = [T_{d_1}^k(X_1); \dots ; T_{d_j}^k(X_j)] \in R^{(k+1)q \times \phi_k^j} \tag{54}$$

where $\phi_k^j = \sum_{i=1}^{j} (k+1-d_i)r_i$, as the *k-Toeplitz matrix* of $X(s)$. If $k = d_n$, then $T_I^{d_n} = (X)T_I^n(X)$ is called the *normal Toeplitz matrix* (nTm) of $X(s)$ and has dimensions $(d_n+1)q \times \phi$, where $\phi = \phi_{d_n}^n = \sum_{i=1}^{n} (d_n + 1 - d_i)r_i$ is called the *dynamic order* of $\mathcal{X}$.

Lemma 4.2. *[20] For any OMBM $X(s)$ of $\mathcal{X}$ and $k : d_i = max\{d_j \leqslant k\}$,*

$$\rho(T_I^k(\mathcal{X})) = \phi_k^i = \sum_{j=1}^{i} (k+1-d_j)r_j \tag{55}$$

The importance of the Toeplitz representation of OMBMs is illustrated by the following result [20]:

Proposition 4.1. *Let $X(s) = [X_1(s), \dots, X_\mu(s)]$ be an OMBM of $\mathcal{X} \in R^q(s), dim\mathcal{X} = p$ with $I(\mathcal{X}) = \{(d_i, r_i), i \in \underline{n}\}$. If $\underline{x}(s) = s^k\, \underline{x}_k + \dots + s\, \underline{x}_1 + \underline{x}_0 \in R^q[s], \underline{x}_k \neq 0$, then $\underline{x}(s) \in \mathcal{X}$, if and only if either of the following equivalent conditions hold true:*

(i) *$\underline{x}(s)$ may be expressed as*

$$\underline{x}(s) = \sum_{i=1}^{\mu} X_i(s)\, \underline{a}_i(s), \underline{a}_i(s) \in R^{r_i}[s] \tag{56}$$

where $\vartheta[\underline{a}_i(s)] \leqslant k - d_i$, if $k \geqslant d_i$ and $\underline{a}_i(s) = \underline{0}$ if $k < d_i$.

(ii) *If we denote by $\underline{z}(\underline{x}) = [\underline{x}_k^t, \dots, \underline{x}_1^t, \underline{x}_0^t]^t \in R^{q(k+1)}$, then*

$$\underline{z}(\underline{x}) = [T_{d_1}^k(X_1), \dots, T_{d_i}^k(X_i)]\, \underline{a}_{i,k} \tag{57}$$

where $d_i = max\{d_j : k \geqslant d_j\}$ and $\underline{a}_{i,k}$ some real vector.

■

The above result expresses the conditions for polynomial vectors to be in $\mathcal{X}$; in particular, part (ii) expresses a fundamental isomorphism between vectors of $R^q[s] \cap \mathcal{X}$ and real vectors in the column space of $k-$Toeplitz matrices. Some further properties of the polynomial vectors of $R^q[s]$ are examined next.

5 Generation of Complete Sets of Polynomial Vectors with Given High Coefficient Spaces

For a given matrix pencil $sF - G \in R^{t\times q}[s]$, any minimal basis of $\mathcal{X}_r = \mathcal{N}_r\{sF - G\}$ defines a complete set of polynomial vectors.The study of the generation and properties of complete sets of polynomial vectors of $\mathcal{X}_r$, which have a given high coefficient space, is motivated by the need to classify the proper-nonproper pairs. This problem is examined in this section. The results provide the necessary tools needed for the classification of minimal bases of singular systems, when the pencil considered is the restriction pencil $R(s) = sNE - NA$. The general case of the pencil $sF - G$ is considered here and the results also apply to $R(s)$. We use the same definitions and notation for the invariants of $\mathcal{X}_r$ as was done for the general case of a rational vector space $\mathcal{X}$. Specifically, $I(\mathcal{X}_r) = I = \{(d_i, r_i), i \in \underline{\eta}, 0 \leqslant d_1 < \cdots < d_\eta\}$ denotes the index,$(I) = \{d_i, i \in \underline{\eta}\}$ the list and $[I] = \{\ldots, \overline{d_i, \ldots, d_i : r_i}, i \in \underline{\eta}\}$ the *explicit* representation of I. If $X(s) = [X_1(s), \ldots, X_n(s)]$ is an OMBM of $\mathcal{X}_r, \mathcal{X}_i(s) \in R^{q\times r_i}[s]$, then $\{\mathcal{M}_i, i \in \underline{\eta}\}, \{\mathcal{P}_i, i \in \underline{\eta}\}, \{\hat{\mathcal{P}}_i, i \in \underline{\eta}\}, \{\mathcal{R}_i, i \in \underline{\eta}\}$ will denote the corresponding sets of invariant-modules, spaces of $\mathcal{X}_r$.

The generation of complete sets of polynomial vectors of $\mathcal{X}_r$ having a given generator and common degree is considered first.

Proposition 5.1. *Let $I_k = \{(d_i, r_i) i \in \underline{k}\}$ be the $k-th$ partial index of $\mathcal{X}_r$ and $\mathcal{M}_k, \mathcal{P}_k$ be the corresponding prime module, space. The following properties hold true:*

(i) *For every $\underline{x}_r \in \mathcal{P}_k$, there exists $\underline{x}(s) = \underline{x}_0 + \ldots + s^r\, \underline{x}_r \in \mathcal{M}_k$ with $r = d_k$; furthermore, if $\underline{x}_r \in \mathcal{P}_k$ and $\underline{x}_r \notin \mathcal{P}_{k-1}$ any vector $\underline{x}(s)$ in $\mathcal{M}_k$ generated by $\underline{x}_r$ has $\vartheta[\underline{x}(s)] = d_k$ and it is prime.*

(ii) *If $\{\underline{x}_r^j, j \in \underline{\mu}, \mu \leqslant r_k\}$ is a linearly independent set such that $\mathcal{V}_r = sp_R\{\underline{x}_r^j, j \in \underline{\mu}\} \in \mathcal{P}_k, \mathcal{V}_r \cap \mathcal{P}_{k-1} = \{0\}$ then there exists a vector set $\{\underline{x}_j(s) = \underline{x}_0^j + \ldots + s^r\, \underline{x}_r^j, r = d_k, j \in \underline{\mu} : \underline{x}_j(s) \in \mathcal{M}_k, \forall j \in \underline{\mu}\}$; furthermore, any such vector set is complete.*

Proof. (i) Let $X(s) = [\ldots, X_i(s), \ldots], X_i(s) = s^{d_i} X_{d_i}^i + \ldots + sX_1^i + X_0^i, i \in \underline{n}$

be an OMBM of $\mathcal{X}_r$ and let $T_I^{d_k}(X)$ be the d_k-th Toeplitz matrix, where

$$T_I^{d_k}(X) = \begin{bmatrix} X_{d_1}^1 & 0 & \dots & 0 & \dots & X_{d_k}^k \\ X_{d_1-1}^1 & X_{d_1}^1 & \ddots & \vdots & \dots & X_{d_k-1}^k \\ \vdots & \ddots & \ddots & 0 & & \vdots \\ X_0^1 & \ddots & \ddots & X_{d_1}^1 & \dots & \vdots \\ 0 & \ddots & \ddots & X_{d_1-1}^1 & & \vdots \\ \vdots & \ddots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & X_0^1 & \dots & X_0^k \end{bmatrix} = \begin{bmatrix} T_{d_k}^1 & \dots & T_{d_k}^k \\ T_{d_k-1}^1 & \dots & T_{d_k-1}^k \\ \vdots & \dots & \vdots \\ \\ \\ \vdots & \dots & \vdots \\ T_0^1 & \dots & T_0^k \end{bmatrix} \tag{58}$$

If $\underline{x}_r \in \mathcal{P}_k$, there exist vectors $\{\underline{b}_{0,1}, \dots, \underline{b}_{0,k}\}$ such that

$$\underline{x}_r = [X_{d_1}^1; \dots ; X_{d_k}^1] \begin{bmatrix} \underline{b}_{0,1} \\ \vdots \\ \underline{b}_{0,k} \end{bmatrix} \tag{59}$$

with the above selection of $\underline{b}_{0,1}, \dots, \underline{b}_{0,k}$ we may always define a polynomial vector $\underline{x}(s) = s^r\, \underline{x}_r + \dots + s\, \underline{x}_1 + \underline{x}_0, r = d_k$, the coefficients of which are defined by

$$\begin{bmatrix} \underline{x}_r \\ \vdots \\ \underline{x}_0 \end{bmatrix} = T_I^{d_k}(X) \begin{bmatrix} \underline{a}_1 \\ \vdots \\ \underline{a}_k \end{bmatrix}, \underline{a}_i = \begin{bmatrix} \underline{b}_{0,i} \\ \underline{b}_{1,i} \\ \vdots \\ \underline{b}_{d_k-d_i,i} \end{bmatrix}, i \in \underline{k} \tag{60}$$

where $\underline{b}_{j,i}, j \neq 0$ are appropriate dimension vectors, but otherwise arbitrary; the dimensions of these vectors are consistent with the block structure of $T_I^{d_k}$. By Proposition 4.1, $\underline{x}(s) \in \mathcal{M}_k, \underline{x}_r = [\underline{x}(s)]_h \in \mathcal{P}_k$ and $\vartheta[\underline{x}(s)] = d_k$.

If $\underline{x}(s)$ is constructed as above and $\underline{x}_r \notin \mathcal{P}_{k-1}$, then clearly $\underline{b}_{0,k} \neq \underline{0}$, since $[X_{d_1}^1, \dots, X_{d_i}^i]$ is a basis matrix for$\mathcal{P}_i$. If $\underline{x}(s)$ is not prime, there exist scalars $\{c_i, i = 0, 1, \dots, r, r = d_k\}$ not all of them zero, such that $\sum_{i=0}^r c_i\, \underline{x}_i = \underline{0}$; by (58) and the structure of $T_I^{d_k}(X)$, it follows that the latter is equivalent to

$$X_{d_k}^k\, \underline{f}_{d_k}^k + \dots + X_0^k\, \underline{f}_0^k + \dots + X_{d_1}^1\, \underline{f}_{d_1}^1 + \dots + X_0^1\, \underline{f}_0^1 = \underline{0} \tag{61}$$

where $\underline{f}_j^i$ are appropriate combinations of the $\underline{b}_{i,j}$ vectors, defined from the c_i and the Toeplitz structure. Since $X(s)$ is an OMBM associated with a pencil, it defines a complete set of polynomials and thus (59) yields $\underline{f}_j^i = \underline{0}$ for all $j = 0, \dots, d_i$ and $i \in \underline{k}$. From the structure of $T_I^{d_k}(X)$ it follows that

$$\underline{f}_j^k = c_j\, \underline{b}_{0,k} = 0, \forall j = 0, 1, \dots, d_k \tag{62}$$

and since $\underline{b}_{0,k} \neq 0$, we have that $c_j = 0, \forall j = 0, 1, \ldots, r = d_k$ which contradicts the linear dependence assumption. Thus, any $\underline{x}(s)$ constructed by (57) and (58) is prime, as long as $\underline{x}_r \in \mathcal{P}_k$ and $\underline{x}_r \notin \mathcal{P}_{k-1}$.
(ii) By part(i), we may always define a set $\{\underline{x}_j(s) : \underline{x}_j(s) = \underline{x}_0^j + \ldots + s^r \underline{x}_r^j, r = d_k, \underline{x}_j(s) \in \mathcal{M}_k, j \in \underline{\mu}\}$ of prime vectors which are generated by $\underline{x}_r^j \in \mathcal{P}_k$, but with $\underline{x}_r^j \notin \mathcal{P}_{k-1}$. If $P(s) = [\underline{x}_1(s), \ldots, \underline{x}_\mu(s)]$, then $[P(s)]_h = P^h = [\underline{x}^1, \ldots, \underline{x}_r^\mu]$ and it is clear that $\rho_{R(s)}(P(s)) = \rho(P^h) = \mu$ and thus the $\{\underline{x}_j(s), j \in \underline{\mu}\}$ set is linearly independent. To prove that the set is complete, assume the opposite, that is the set $\{\underline{x}_i^j, i = 0, 1, \ldots, r, j \in \underline{\mu}\}$ of all coefficient vectors is linearly dependent; then, there exist scalars $\{c_{j,i}, i = 0, 1, \ldots, r, j \in \underline{\mu}\}$, not all of them zero, such that

$$\sum_{j=1}^{\mu} \sum_{i=0}^{r} c_{j,i}\, \underline{x}_i^j = \underline{0} \tag{63}$$

where for every $j \in \underline{\mu}$, the vectors $\underline{x}_i^j, i = 0, 1, \ldots, r = d_k$ are defined in terms of (58), which together with (56) and (61) leads to

$$\begin{aligned} &T_r^1\{\sum_{j=1}^{\mu} c_{j,r}\, \underline{a}_{j,1}\} + T_r^2\{\sum_{j=1}^{\mu} c_{j,r}\, \underline{a}_{j,2}\} + \ldots + T_r^k\{\sum_{j=1}^{\mu} c_{j,r}\, \underline{a}_{j,k}\} + \\ &+T_{r-1}^1\{\sum_{j=1}^{\mu} c_{j,r-1}\, \underline{a}_{j,1}\} + T_{r-1}^2\{\sum_{j=1}^{\mu} c_{j,r-1}\, \underline{a}_{j,2}\} + \ldots + T_{r-1}^k\{\sum_{j=1}^{\mu} c_{j,r-1}\, \underline{a}_{j,k}\} + \\ &+T_0^1\{\sum_{j=1}^{\mu} c_{j,r_0}\, \underline{a}_{j,1}\} + T_0^2\{\sum_{j=1}^{\mu} c_{j,r_0}\, \underline{a}_{j,2}\} + \ldots + T_0^k\{\sum_{j=1}^{\mu} c_{j,r_0}\, \underline{a}_{j,k}\} \end{aligned} \tag{64}$$

where the matrices T_j^i are defined by the partitioning of $T_I^{d_k}(X)$ as shown in (56). Note that for all $t = 0, 1, \ldots, r, T_t^k = X_t^k$ and that

$$sp_R\{[T_r^1, \ldots, T_r^{k-1}, \ldots, T_0^1, \ldots, T_0^{k-1}]\} \cap sp_R\{[T_r^k, \ldots, T_0^k]\} = \{0\} \tag{65}$$

thus, from (63), (62) and the completeness of $X(s)$ we have:

$$\sum_{j=1}^{\mu} c_{j,t}\, \underline{a}_{j,k} = 0, t = 0, 1, \ldots, r = d_k \tag{66}$$

Note that since $\underline{x}_r^j \notin \mathcal{P}_{k-1}$, it follows that for $\forall j \in \underline{\mu}, \underline{a}_{j,k} \neq \underline{0}$. We shall prove next that the set $\{\underline{a}_{j,k}, j \in \underline{\mu}\}$ is linearly independent. Let us assume that (64) holds true for at least a set of non-zero $c_{j,t}$ corresponding to a fixed t. Then,

$$\begin{aligned} \underline{x}_r^\star = \sum_{j=1}^{\mu} c_{j,t}\, \underline{x}_r^j = [X_{d_1}^1, 0, \ldots, 0](\sum_{j=1}^{\mu} c_{j,t}\, \underline{a}_{j,1}) + \ldots + \\ +[X_{d_{k-1}}^{k-1}, 0, \ldots, 0](\sum_{j=1}^{\mu} c_{j,t}\, \underline{a}_{j,k-1}) + [X_{d_k}^k](\sum_{j=1}^{\mu} c_{j,t}\, \underline{a}_{j,k}) \in P_{k-1}, \end{aligned} \tag{67}$$

since (64) holds true and $[X^1_{d_1}, \ldots, X^{k-1}_{d_{k-1}}]$ is a basis matrix of $\mathcal{P}_{k-1}$. Given that there exists an $\underline{x}^\star_r \in \mathcal{V}_r, \underline{x}^\star_r \neq \underline{0}$ and $\underline{x}^\star_r \in \mathcal{P}_{k-1}$, it follows that $\mathcal{V}_r \cap \mathcal{P}_{k-1} \neq \{0\}$ and this leads to a contradiction; thus, the vectors $\{\underline{a}_{j,k}, j \in \underline{\mu}\}$ are linearly independent and (64) implies that $c_{j,t} = 0, \forall j \in \underline{\mu}, t = 0, 1, \ldots, r$. The latter clearly shows that the set $\{\underline{x}_j(s), j \in \underline{\mu}\}$ is complete.

■

The linearly independent set or vectors $B_i = \{\underline{x}^j_{i,d_i}, j \in \underline{\mu}_i\}$ for which $\mathcal{V}_i = sp_R\{\underline{x}^j_{i,d_i}, j \in \underline{\mu}_i\}$ satisfies the conditions

$$\mathcal{V}_i \subseteq \mathcal{P}_i, \mathcal{V}_i \cap \mathcal{P}_{i-1} = \{0\} \text{ and } \mu_i = dim\mathcal{V}_i \leqslant r_i \tag{68}$$

will be called a $(d_i, \mu_i)-$*progenitor set* and $\mathcal{V}_i$ a $(d_i, \mu_i)-$*progenitor space*. From the proof of Proposition 5.1 we also have the following results:

Remark 5.1. If $B_i = \{\underline{x}^j_{i,d_i}, j \in \underline{\mu}_i\}$ is a $(d_i, \mu_i)-$progenitor set there exist families of complete polynomial vector sets T_i each one of them generated by B_i. Every set $T_i = \{\underline{x}^j_i(s) = s^{d_i}\, \underline{x}^j_{i,d_i} + \ldots + s\, \underline{x}^j_{i,1} + \underline{x}^j_{i,0}, j \in \underline{\mu}_i\}$ generated by B_i is given by

$$\begin{bmatrix} \underline{x}^j_{i,d_i} \\ \vdots \\ \underline{x}^j_{i,1} \\ \underline{x}^j_{i,0} \end{bmatrix} = T^{d_i}_I(X) \begin{bmatrix} \underline{a}^j_1 \\ \vdots \\ \vdots \\ \underline{a}^j_i \end{bmatrix}, \underline{a}^j_k = \begin{bmatrix} \underline{b}^j_{0,k} \\ \underline{b}^j_{1,k} \\ \vdots \\ \underline{b}^j_{d_i-d_k,k} \end{bmatrix}, k \in \underline{i}, j \in \underline{\mu} \tag{69}$$

where $T^{d_i}_I(X)$ is the d_i-Toeplitz matrix of any OMBM of $\mathcal{X}_r$, the vectors $\{\underline{b}^j_{0,1}, \ldots, \underline{b}^j_{0,i}\}$ are uniquely defined by

$$\underline{x}^j_{i,d_i} = [X^1_{d_1}; \ldots; X^i_{d_i}] \begin{bmatrix} \underline{b}^j_{0,1} \\ \cdots \\ \underline{b}^j_{0,i} \end{bmatrix} \tag{70}$$

and the rest of the vectors $\underline{b}^j_{a,k}, a = 1, \ldots, d_j - d_k, j \in \underline{\mu}_k$ arbitrary.

■

Remark 5.2. If $B_i = \{\underline{x}^j_{i,d_i}, j \in \underline{\mu}_i\}$ is a $(d_i, \mu_i)-$progenitor set and

$$\underline{x}^j_{i,d_i} = [X^1_{d_i}, 0, \ldots, 0]\, \underline{a}^j_1 + [X^2_{d_2}, 0, \ldots, 0]\, \underline{a}^j_2 + \ldots + [X^i_{d_i}]\, \underline{a}^j_i \tag{71}$$

then the set of vectors $\{\underline{a}^j_i, j \in \underline{\mu}_i\}$ is linearly independent.

■

The above two remarks follow immediately from the proof of Proposition 5.1. Any vector set T_i defined as in Remark 5.2 will be referred to as a $(d_i, \mu_i)-$*normal set* and the modules $\mathcal{N}_i = sp_{R[s]}\{T_i\}$ as the corresponding $(d_i, \mu_i)-$*normal module* generated by T_i. For the $\mathcal{N}_i$ modules we have the following properties:

Corollary 5.1. *Let B_i be a $(d_1, \mu_i)-$progenitor set and let $\mathcal{V}_i, T_i, \mathcal{N}_i$ be the corresponding space, set, module respectively. Then,*

(i) *$\mathcal{N}_i$ is a maximal $R[s]-$module of rank $\mu_i \leqslant r_i$, indices $I(\mathcal{N}_i) = \{(d_i, \mu_i)\}$, high coefficient space, $\mathcal{V}_i$ and $\mathcal{T}_i$ is an OMB.*

(ii) *$\mathcal{N}_i \subseteq \mathcal{M}_i$ and $\mathcal{N}_i \cap \mathcal{M}_{i-1} = \{0\}$.*

Proof. (i) The completeness of the set T_i implies that T_i defines a minimal basis (Remark 4.1); thus, the $\mathcal{N}_i$ module generated by T_i is a maximal Noetherian and has indices $\{(d_i, \mu_i)\}$, those defined T_i.
(ii) By Proposition 5.1 it is clear that $\mathcal{N}_i \subseteq \mathcal{M}_i$. To prove that $\mathcal{N}_i \cap \mathcal{M}_{i-1} = \{0\}$, assume that there exists $\underline{x}(s) \in \mathcal{N}_i$ such that $\underline{x}(s) \in \mathcal{M}_{i-1}$. If $\underline{x}(s) = \underline{x}_0 + \ldots + s^k\, \underline{x}_k, k \geqslant d_i$, then clearly $\underline{x}_k \in \mathcal{V}_i$, since $\mathcal{V}_i$ is the highest coefficient space of $\mathcal{N}_i$ and $\underline{x}_k \in \mathcal{P}_{i-1}$ since $\underline{x}(s) \in \mathcal{M}_{i-1}$; however, the last two conditions imply that $\mathcal{V}_i \cap \mathcal{P}_{i-1} \neq \{0\}$ which leads to a contradiction.

■

Proposition 5.1 provides the means for a parametrisation of all $(d_i, \mu_i)-$normal sets generated by a given B_i set. To extend the above result to the case of general progenitor sets we need some further definitions and notation.

Definition 5.1. Let $I = \{(d_i, r_i), i \in \underline{\eta}\}$ be the index and $\{P\} = \{P_i, i \in \underline{\eta}\}$ be the set of prime spaces of X_r.

(i) Any subset of I, will be called a *character* of I and will be denoted by $J = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}, d_{\nu_i} \in (I), \mu_{\nu_i} \leqslant r_{\nu_i}, 0 \leqslant d_{\nu_i} < \ldots < d_{\nu_\rho}\}$ if $J \neq \emptyset$. Two characters $J, \tilde{J}$, are said to be *complementary* if $[J] \cup [\tilde{J}] = [I]$

(ii) Any set of spaces $\{\mathcal{Y}\} = \{\mathcal{Y}_i, i \in \underline{\eta}\}$ defined by

$$\mathcal{P}_1 = \mathcal{Y}_1, \mathcal{P}_i = \mathcal{P}_{i-1} \oplus Y_i, i = 2, \ldots, n, dim\mathcal{Y}_i = r_i \tag{72}$$

will be called a *system of generator spaces* (SGSP) of $\mathcal{X}_r, I$ is its *index* and $\mathcal{Y}_i$ is a $(d_i, r_i)-$*generator space.*

(iii) Let $\{\mathcal{Y}\} = \{\mathcal{Y}_i, i \in \underline{n}\}$ be an SGSP and define the sets of subspaces

$$\{\mathcal{V}\} = \{\mathcal{V}_i : V_i \subseteq \mathcal{Y}_i, dim\mathcal{V}_i = \mu_i \geqslant 0, i \in \underline{\eta}\}$$

$$\{\tilde{\mathcal{V}}\} = \{\tilde{\mathcal{V}}_i : \tilde{\mathcal{V}}_i \subseteq \mathcal{Y}_i, dim\, \tilde{\mathcal{V}}_i = \tau_i \geqslant 0, i \in \underline{\eta}\} \tag{73}$$

such that $\mu_i + \tau_i = r_i, \forall i \in \underline{\eta}$ and

(a) If $\mu_i > 0, \tau_i > 0$, then $\mathcal{Y}_i = \mathcal{V}_i \oplus \tilde{\mathcal{V}}_i$.

(b) if $\mu_i = r_i$ (or $\tau_i = r_i$) and $\tau_i = 0$ (or $\mu_i = 0$), then $\mathcal{V}_i = \mathcal{Y}_i$ (or $\tilde{\mathcal{V}}_i = \mathcal{Y}_i$) and $\tilde{\mathcal{V}}_i = \{0\}(\mathcal{V}_i = \{0\})$.

The spaces $\mathcal{V}_i, \tilde{\mathcal{V}}_i$ are called complementary *d_i-order progenitor spaces* (d_i-PSP) and $\{\mathcal{V}\} = \{\mathcal{V}_i, i \in \underline{\eta}\}$, $\{\tilde{\mathcal{V}}\} = \{\tilde{\mathcal{V}}_i, i \in \underline{\eta}\}$ are said to be complementary *systems of progenitor spaces* (CPSP). The SGSP set $\{\mathcal{Y}\} = \{\mathcal{Y}_i, i \in \underline{\eta}\}$ that generates $\mathcal{V}, \tilde{\mathcal{V}}$ is called the *parent* SGSP set. The subsets of $\mathcal{V}, \tilde{\mathcal{V}}$ defined by

$$\langle \mathcal{V} \rangle = \{\mathcal{V}_k : \mathcal{V}_k \in \{\mathcal{V}\}, \neq \{0\}, \dim \mathcal{V}_k = \mu_k, \gamma(\mathcal{V}_k) = d_k, k \in \{v_1, \dots, v_\rho\}\}$$

$$\left\langle \tilde{\mathcal{V}} \right\rangle = \{\tilde{\mathcal{V}}_p : \tilde{\mathcal{V}}_p \in \{\tilde{\mathcal{V}}\}, \neq \{0\}, \dim \tilde{\mathcal{V}}_p = \tau_\rho, \gamma(\tilde{\mathcal{V}}_p) = d_p, p \in \{\sigma_1, \dots, \sigma_\pi\}\} \tag{74}$$

are called the *flags* of the $\mathcal{V}, \tilde{\mathcal{V}}$ respectively; the sets $tr\{\mathcal{V}\} = \{v_1, \dots, v_\rho\} \in \{\eta\}$, $tr\{\tilde{\mathcal{V}}\} = \{\sigma_1, \dots, \sigma_\pi\} \in \{\eta\}$, $\{\eta\} = \{1, 2, \dots, n\}$ are called the *traces* of $\mathcal{V}, \tilde{\mathcal{V}}$ and $J = \{(d_{v_i}, \mu_{v_i}), \mu_{v_i} > 0, i \in \underline{\rho}\}$, $\bar{J} = \{(d_{\sigma_j}, \tau_{\sigma_j}), \tau_{\sigma_j} > 0, j \in \underline{\pi}\}$ are their *characteristics*. The sets $\mathcal{V}, \tilde{\mathcal{V}}$, will be called the *completions* of $\langle \mathcal{V} \rangle, < \tilde{\mathcal{V}} >$ respectively.

(iv) An SPSP $\{V\}$ is called complete, if $= \mathcal{Y}_i, \forall i \in \underline{\eta}$, where $\{\mathcal{Y}\} = \{\mathcal{Y}_i, i \in \underline{\eta}\}$ is an SGSP of $\mathcal{X}_r$.

■

Remark 5.3. An SGSP set $\{\mathcal{Y}_i, i \in \underline{\eta}\}$ is not uniquely defined, but all such systems have the same index I, and $\overline{\mathcal{P}}^\star = \mathcal{Y}_1 \oplus \dots \oplus \mathcal{Y}_\eta$. For the subspaces $\mathcal{V}_i$ of Y_i we have that

$$\mathcal{V}_i \subseteq \mathcal{P}_i, \text{ and } \mathcal{V}_i \cap \mathcal{P}_{i-1} = \{0\}, \text{ where } \mathcal{P}_0 = \{0\} \tag{75}$$

It is clear, that the characteristics of the complementary flags $< \mathcal{V} >, < \tilde{\mathcal{V}} >$ define complementary characters of I. From Remark 5.3 and Proposition 5.1 follows that we may extend Definition 5.3 as follows:

Definition 5.2. Let $(\{\mathcal{V}\}, \{\tilde{\mathcal{V}}\})$ be a pair of complementary SPSP of X_r with characteristics $J = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}\}, \tilde{J} = \{(\sigma_{\nu_j}, \tau_{\nu_j}), j \in \underline{\pi}\}$ and flags $< \mathcal{V} >= \{\mathcal{V}_{\nu_i}, i \in \rho\}, < \tilde{\mathcal{V}} >= \{\tilde{\mathcal{V}}_{\sigma_j}, j \in \underline{\pi}\}$.

(i) A basis set B_{ν_i} of $\mathcal{V}_{\nu_i}$ is called a $(d_{\nu_i}, \mu_{\nu_i})-$*progenitor set* and $< B >= \{B_{\nu_i}, i \in \underline{\rho}\}$ a $J-$*system of progenitor sets* (J-SPS). A $\tilde{J}-$SPS $< \tilde{B} >= \{B_{\sigma_j}, j \in \underline{\pi}\}$ is defined similarly for $\{\tilde{\mathcal{V}}\}$ and $(< B >, < \tilde{B} >)$ will be referred to as *complementary* $(J, \tilde{J})-$SPS.

(ii) If $T_{\nu_i} = \{\underline{x}_j^{\nu_i}(s) : \underline{x}_j^{\nu_i}(s) \in \mathcal{M}_{\nu_i}, \vartheta[\underline{x}_j^{\nu_i}] = d_{\nu_i}, j \in \underline{\mu}_{\nu_i}\}$ is a (d_{ν_i}, μ_{ν_i}) normal set generated by B_{ν_i} (see Proposition 5.1), then $\mathcal{N}_{\nu_i} = sp_{R[s]}\{T_{\nu_i}\}$ is called the $(d_{\nu_i}, \mu_{\nu_i})-$*normal module* generated by T_{ν_i}. We shall refer to $< T >= \{\tilde{T}_{\nu_i}, i \in \underline{\rho}\}$ as $J-$*systems of normal sets* ($J-$-SNS), $J-$*system of normal modules* ($J-$SNM) respectively of $\{\mathcal{V}\}$. We may define similarly the sets $< \tilde{T} >, < \tilde{\mathcal{N}} >$ from $< \tilde{\mathcal{V}} >$ and the pairs $(< T >, < \tilde{T} >), (< \mathcal{N} >, < \tilde{\mathcal{N}} >)$ will be called *complementary* $(J, \tilde{J})-$SNS, $(J, \tilde{J})-$SNM respectively.

■

The properties of the sets defined above are examined next.

Lemma 5.1. *Let $\{\mathcal{V}\}$ be an SPSP of $\mathcal{X}_R$ and $< \mathcal{V} >= \{\mathcal{V}_{\nu_i}, i \in \underline{\rho}\}$ its flag. The set $< \mathcal{V} >$ is linearly independent.*

Proof. The result is proved by induction. Assume that $\mathcal{V}_{\nu_1} \cap \mathcal{V}_{\nu_2} \neq \{0\}$. Since $\mathcal{V}_{\nu_1} \subseteq \mathcal{P}_{\nu_1}$ and $\mathcal{P}_{\nu_1} \subseteq \mathcal{P}_{\nu_2 - 1}$, it follows that $\mathcal{V}_{\nu_1} \subseteq P_{\nu_2 - 1}$ and $\mathcal{V}_{\nu_1} \cap \mathcal{V}_{\nu_2} \neq \{0\}$ implies that $\mathcal{P}_{\nu_2 - 1} \cap \mathcal{V}_{\nu_2} \neq \{0\}$; however, by definition $\mathcal{V}_{\nu_2} \cap \mathcal{P}_{\nu_2 - 1} = \{0\}$ and this leads to a contradiction. Thus,$\mathcal{V}_{\nu_1} \cap \mathcal{V}_{\nu_2} = \{0\}$ and $\mathcal{V}_{\nu_1}, \mathcal{V}_{\nu_2}$ are independent. Consider now $\mathcal{V}_{\nu_1,\nu_2} = \mathcal{V}_{\nu_1} \oplus \mathcal{V}_{\nu_2}$, which is clearly a subspace of $\mathcal{P}_{\nu_2}$ and assume that $\mathcal{V}_{\nu_1,\nu_2} \cap \mathcal{V}_3 \neq \{0\}$. Note that $\mathcal{V}_{v_1,v_2} \subseteq \mathcal{P}_{v_2}$ also implies $\mathcal{V}_{v_1,v_2} \subseteq \mathcal{P}_{v_3 - 1}$ and this implies $\mathcal{P}_{\nu_3 - 1} \cap \mathcal{V}_{\nu_3} \neq \{0\}$; however, by definition $\mathcal{P}_{\nu_3 - 1} \cap \mathcal{V}_{\nu_3} = \{0\}$ and thus we are led to a contradiction. Thus, $\mathcal{V}_{\nu_1,\nu_2} \cap \mathcal{V}_{\nu_3} = \{0\}$ and the spaces $\{\mathcal{V}_{\nu_1}, \mathcal{V}_{\nu_2}, \mathcal{V}_{\nu_3}\}$ are linearly independent. Consider next the space $\mathcal{V}_{\nu_1,\nu_2,\nu_3} = \mathcal{V}_{\nu_1} \oplus \mathcal{V}_{\nu_2} \oplus \mathcal{V}_{\nu_3}$. The induction is rather obvious and following similar arguments we prove that $\mathcal{V}_{\nu_1,\nu_2,\nu_3} \cap \mathcal{V}_{\nu_4} = \{0\}$.

■

Remark 5.4. Let $\{\mathcal{V}\}, \{\tilde{\mathcal{V}}\}$ be a pair of complementary SPSP and $< \mathcal{V} >= \{\mathcal{V}_{\nu_i}, i \in \underline{\rho}\}, < \tilde{\mathcal{V}} >= \{\mathcal{V}_{\sigma_j}, j \in \underline{\pi}\}$ their corresponding flags. The set $\{< \mathcal{V} >, < \tilde{\mathcal{V}} >\}$ is linearly independent and

$$\mathcal{P}^\star = \mathcal{V}_{\nu_i} \oplus \cdots \oplus \mathcal{V}_{\nu_\rho} \oplus \tilde{\mathcal{V}}_{\sigma_1} \oplus \tilde{\mathcal{V}}_{\sigma_\pi} \tag{76}$$

■

The properties of $< T >, < \mathcal{N} >$ sets generated by B examined next.

Theorem 5.1. *Let $\{\mathcal{V}\}$ be an SPSP of $\mathcal{X}_r, < \mathcal{V} >= \{\mathcal{V}_{\nu_i}, i \in \underline{\rho}\}$ its flag, $< B >= \{B_{\nu_i}, i \in \underline{\rho}\}$ a J-SPS for $< V >$ and $J = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}\}$ be the characteristic of $\{V\}$. The following properties hold true:*

(i) *An J-SNS $< T >= \{T_{\nu_i}, i \in \underline{\rho}\}$, generated by $< B >$ is linearly independent, as well as the corresponding J-SNM $< \mathcal{N} >= \{\mathcal{N}_{\nu_i}, i \in \underline{\rho}\}$.*

(ii) *If $\mathcal{N}^\star = \mathcal{N}_{\nu_1} \oplus \cdots \oplus \mathcal{N}_{\nu_\rho}$ then $\mathcal{N}^\star$ is a maximal $R[s]-$module with J index and $< T >= \{T_{\nu_i}, i \in \underline{\rho}\}$ is an OMB of $\mathcal{N}^\star$.*

Proof. (i) By Proposition 5.1, it follows that for each $i \in \underline{\rho}$ the sets T_{ν_i} are linearly independent. To prove that $< T >$ is independent it is adequate to prove that $< \mathcal{N} >$ is independent; this is proved by induction as shown below:

(a) Assume that $\mathcal{N}_{\nu_i} \cap \mathcal{N}_{\nu_2} \neq \{0\}$. By Corollary 5.1, $\mathcal{N}_{\nu_1} \cap M_{\nu_1} = \{0\}$ and since $\mathcal{M}_{\nu_1} \subseteq \mathcal{M}_{\nu_2 - 1}$, it follows that $\mathcal{N}_{\nu_1} \subseteq \mathcal{M}_{\nu_2-1}$; thus, $\mathcal{N}_{\nu_1} \cap \mathcal{N}_{\nu_2} \neq \{0\}$ implies that $\mathcal{N}_{\nu_2} \cap \mathcal{M}_{\nu_2-1} \neq \{0\}$. By Corollary 5.1, $\mathcal{N}_{\nu_2} \cap \mathcal{M}_{\nu_2-1} = \{0\}$ and thus we are led to a contradiction. So we have that $\mathcal{N}_{\nu_1} \cap \mathcal{N}_{\nu_2} = \{0\}$.

(b) Consider now the module $\mathcal{N}_{\nu_1,\nu_2} = \mathcal{N}_{\nu_1} \oplus \mathcal{N}_{\nu_2} \subseteq \mathcal{M}_{\nu_2}$ and assume that $\mathcal{N}_{\nu_1,\nu_2} \cap \mathcal{N}_{\nu_3} \neq \{0\}$. Since $\mathcal{N}_{\nu_1,\nu_2} \subseteq \mathcal{M}_{\nu_2} \subseteq \mathcal{M}_{\nu_3-1}$, it follows that $\mathcal{N}_{\nu_3} \cap \mathcal{M}_{\nu_3-1} \neq \{0\}$, which clearly leads to a contradiction (see Corollary 5.1). Thus $\{\mathcal{N}_{\nu_1}, \mathcal{N}_{\nu_2}, \mathcal{N}_{\nu_3}\}$ are linearly independent. The general step of the induction follows along similar lines.

(ii) The module $\mathcal{N}^\star = \mathcal{N}_{\nu_1} \oplus \cdots \oplus \mathcal{N}_{\nu_\rho}$ is well defined and $< \mathcal{T} >= \{\mathcal{T}_{\nu_i}, i \in \underline{\rho}\}$ is a basis of $\mathcal{N}^\star$. To prove this result, it has to be shown that $< T >$ is a minimal basis; it is adequate to show that the set $< T >$ is complete. Thus let $T_{\nu_i} = \{\underline{x}^j_{\nu_i}(s) = s^{d_{\nu_i}}\, \underline{x}^j_{\nu_i, d_{\nu_i}} + \ldots + s\, \underline{x}^j_{\nu_i,1} + \underline{x}^j_{\nu_i,0}, j \in \underline{\mu}_{\nu_i}\}$ and $X(s) = [\ldots, X_i(s), \ldots]$ be any OMBM of X_r; by Remark 5.1 we have that

$$\begin{bmatrix} \underline{x}^j_{\nu_i, d_{\nu_i}} \\ \vdots \\ \underline{x}^j_{\nu_i,1} \\ \underline{x}^j_{\nu_i,0} \end{bmatrix} = T_I^{d_{\nu_\rho}}(X) \begin{bmatrix} \underline{a}^j_{\nu_i,1} \\ \vdots \\ \underline{a}^j_{\nu_i,\nu_\rho - 1} \\ \underline{a}^j_{\nu_i,\nu_\rho} \end{bmatrix}, i \in \underline{\eta}, j \in \underline{\mu}_{\nu_i} \tag{77}$$

where the Toeplitz matrix is defined for the largest of d_{ν_i}, the d_{ν_ρ}. By Remark 5.2, the sets $\{\underline{a}^j_{\nu_\rho,\nu_\rho}, j \in \underline{\mu}_{\nu_i}\}$ are linearly independent for all $i \in \underline{\eta}$. The result is now proved by induction as follows:

(a) Consider the set $\{T_{\nu_i}, T_{\nu_2}\}$ and assume it not to be complete. The set $\{\underline{x}^j_{\nu_i,k}, k = 0, 1, \ldots, d_{\nu_i}, j \in \underline{\mu_{\nu_i}}, i = 1, 2\}$ is linearly dependent and there exist scalars $c^j_{\nu_i,k}$, not all of them zero, such that

$$\sum_{j=1}^{\mu_{\nu_1}} \sum_{k=0}^{d_{\nu_1}} c^j_{\nu_1,k}\, \underline{x}^j_{\nu_1,k} + \sum_{j=1}^{\mu_{\nu_2}} \sum_{k=0}^{d_{\nu_2}} c^j_{\nu_2,k}\, \underline{x}^j_{\nu_2,k} = 0 \tag{78}$$

Note that using (73) and the partitioned form of $T_I^{d_{\nu_2}}(X)$, defined by (56), we can express $\underline{x}^j_{\nu_i,k}$ in terms of the $\underline{a}^j_{\nu_i,p}, p = 1, \ldots, \nu_2, i = 1, 2$ vectors and the block $T^p_\tau, p = 1, \ldots, \nu_2, \tau = 0, 1, \ldots, d_{\nu_2}$. Taking into account the completeness of $X(s)$ that $T^{\nu_2}_\tau = X^{\nu_2}_\tau, \tau = 0, 1, \ldots, \nu_2$ and using similar arguments as in the proof of Proposition 5.1 (arguments following the analysis of (61)), it follows that the coefficients of $X^{\nu_2}_\tau, \tau = 0, 1, \ldots, d_{\nu_2}$ (in the resulting equation after the substitution) should be zero, i.e.

$$\sum_{j=1}^{\mu_{\nu_2}} c^j_{\nu_2,k}\, \underline{a}^j_{\nu_2,k} = 0, \forall k = 0, 1, \ldots, d_{\nu_2} \tag{79}$$

however, by Remark 5.2, $\{\underline{a}^j_{\nu_2,\nu_2}, j \in \underline{\mu}_{\nu_2}\}$ is linearly independent and thus (78) implies $c^j_{\nu_2,k} = 0, \forall j \in \underline{\mu_{\nu_2}}, k = 0, 1m \ldots, d_{\nu_2}$. The latter conditions reduce (77) to

$$\sum_{j=1}^{\mu_{\nu_1}} \sum_{k=0}^{d_{\nu_1}} c^j_{\nu_1,k} \, \underline{x}^j_{\nu_1,k} = 0 \tag{80}$$

The set T_{ν_1} is however complete, as it has been established by Proposition 5.1, and thus $c^j_{\nu_i,k} = 0, \forall k = 0, 1, \ldots, d_{\nu_i}, j \in \underline{\mu_{\nu_i}}, i = 1, 2$ which contradicts the dependence of the vectors. Thus $\{T_{\nu_1}, T_{\nu_2}\}$ is complete. The general step of the induction follows along similar lines.

■

For a given SPSP $\{\mathcal{V}\}$ with flag $< \mathcal{V} >$, trace $tr\{\mathcal{V}\}$ and characteristic J, we may define sets $< B >, < T >, < \mathcal{N} >$, although not in a unique manner, which are characterised by the $tr\{\mathcal{V}\}$ and J indices; any set $\Omega(\mathcal{V}) = \{< \mathcal{V} >, < B >, < T >, < \mathcal{N} >\}$ will be called a $V(J)-$system generated by $< V >$. A $V(J)-$system is called *complete*, if $< \mathcal{V} >$ is complete, i.e. $< \mathcal{V} >= \{\mathcal{Y}\}$ is an SGSP. For complete sets $J = I$ and we also have the result:

Corollary 5.2. *If $\Omega(\mathcal{V}) = \{< \mathcal{V} >, < B >, < T >, < \mathcal{N} >\}$ is complete, then*

(i) *For the set $< \mathcal{V} >= \{\mathcal{V}_i, i \in \underline{\eta}\}$ we have:*

$$\mathcal{P}_1 = \mathcal{V}_1, \mathcal{P}_k = \mathcal{P}_{k-1} \oplus \mathcal{V}_k, k = 2, \ldots, \eta \tag{81}$$

(ii) *The set $< T >=< T_i, i \in \underline{\eta} >$ defines an OMB of $\mathcal{X}_r$ with index I.*

(iii) *For the set of modules $< \mathcal{N} >= \{\mathcal{N}_i, i \in \underline{\eta}\}$ we have:*

$$\mathcal{M}_1 = \mathcal{N}_1, \mathcal{M}_k = \mathcal{M}_{k-1} \oplus \mathcal{N}_k, k = 2, \ldots, \eta \tag{82}$$

If $\Omega(\mathcal{V}) = \{< \mathcal{V} >, < B >, < T >, < \mathcal{N} >\}, \Omega(\tilde{\mathcal{V}}) = \{< \tilde{\mathcal{V}} >, < \tilde{B} >, < \tilde{T} >, < \tilde{\mathcal{N}} >\}$ are systems for which $< V >, < \tilde{V} >$ are complementary, then they are called *complementary* and clearly $(< T >, < \tilde{T} >), (< \mathcal{N} >, < \tilde{\mathcal{N}} >)$ are also complementary. We may define the union of complementary sets, as follows:

Definition 5.3. For the $\Omega(\mathcal{V}), \Omega(\tilde{\mathcal{V}})$ complementary systems we may define their *union* as the set

$$\Omega(\mathcal{V}, \tilde{\mathcal{V}}) = \{< \mathcal{V}; \tilde{\mathcal{V}} >, < B; \tilde{B} >, < T; \tilde{T} >, < \mathcal{N}; \tilde{\mathcal{N}} >\} \tag{83}$$

where $< \mathcal{V}; \tilde{\mathcal{V}} > \{\mathcal{V}^\star_i, i \in \underline{\eta}\} = \{\mathcal{Y}_i, i \in \underline{\eta}\}$ is the parent SGSP set of spaces and $< B; \tilde{B} >$ the corresponding naturally ordered bases of $< V; \tilde{V} >$ formed from the bases of $< \mathcal{V} >, < \tilde{\mathcal{V}} >$; the sets $< T; \tilde{T} >= \{T^\star_i, i \in \underline{\eta}\}, < \mathcal{N}; \tilde{\mathcal{N}} >= \{\mathcal{N}^\star_i, i \in \underline{\eta}\}$ are defined by:

(a) If $i \in tr\{\mathcal{V}\} = \{\nu_1, \ldots, \nu_\rho\}$ and $i \in tr\{\tilde{\mathcal{V}}\} = \{\sigma_1, \ldots, \sigma_\pi\}$, then:

$$T_i^\star = \{T_i; \tilde{T}_i\} \text{ and } \mathcal{N}_i^\star = \mathcal{N}_i \oplus \tilde{\mathcal{N}}_i \tag{84}$$

(b) If $i \in tr\{\mathcal{V}\}$ and $i \notin tr\{\tilde{\mathcal{V}}\}$, then $T_i^\star = T_i$ and $\mathcal{N}_i^\star = \mathcal{N}_i$.

(c) If $i \notin tr\{\mathcal{V}\}$ and $i \in tr\{\tilde{\mathcal{V}}\}$, then $T_i^\star = \tilde{T}_i$ and $\mathcal{N}_i^\star = \tilde{\mathcal{N}}_i$.

From the above definition it is clear that:

Remark 5.5. The union of two complementary sets $\Omega(\mathcal{V}), \Omega(\tilde{\mathcal{V}})$ is a set $\Omega(\mathcal{V}; \tilde{\mathcal{V}})$ which is complete.

■

The results presented in this section provide the means for the classification of proper-nonproper vectors which is examined next.

6 A Classification of Minimal Bases and Indices of Singular Systems

The characterisation of the vectors $\underline{x}(s) \in \mathcal{X}_r \cap R^n[s]$, where $R(s) = sNE - NA$, according to whether $\underline{x}_h = [\underline{x}(s)]_h \in \mathcal{N}_r(E)$, or $\notin \mathcal{N}_r(E)$ introduces a classification of the OMBs of $\mathcal{X}_r$ according to the above property of the generator spaces. The notion of a normal minimal basis of $\mathcal{X}_r$ is introduced here, using geometric properties and it is shown to be an equivalent notion to that of a canonical basis of $\mathcal{X}_T$. The properties of normal MBs are studied and new invariants are introduced under the Brunovsky group. The most important of these invariants is the partitioning of the index $I_C(R)$, and thus also of $I_C(T)$ into two complementary subsets characterising the proper, nonproper properties of pairs. The construction of normal MBs using algebraic and geometric tools is finally also considered here.

It is assumed that $I_C(R) = I = \{(d_i, r_i), i \in \underline{\eta}, 0 \leqslant d_1 < \ldots < d_\eta\}$ is the index of X_r, or set of CMF of $R(s)$ and that $\{\mathcal{P}_i, i \in \underline{\eta}\}$ is the set of high spaces of X_r. The definition of special systems of progenitor spaces is the starting point of our study here. For the system S_e we introduce the following family of spaces

$$\mathcal{Q}_i = \mathcal{P}_i \cap \mathcal{N}_r(E), i = 1, 2, \ldots, \eta \tag{85}$$

The family $\{\mathcal{Q}_i, i \in \underline{\eta}\}$ is uniquely defined by S_e and since $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ it follows that $\mathcal{Q}_i \subseteq \mathcal{Q}_{i+1}, i \in \underline{\eta}$. From this we have:

Remark 6.1. There exists a set of indices $\Theta_E = \{\nu_1, \ldots, \nu_\rho\} \subseteq \{\eta\}$:

$$\{0\} = \ldots = \mathcal{Q}_{\nu_1 - 1} \subset \mathcal{Q}_{\nu_1} = \ldots = \mathcal{Q}_{\nu_2 - 1} \subset \mathcal{Q}_{\nu_2} = \ldots = \mathcal{Q}_{\nu_\rho - 1} \subset \mathcal{Q}_{\nu_\rho} = \ldots = \mathcal{Q}_\eta \tag{86}$$

■

The set of indices $\Theta_E = \{\nu_1, \ldots, \nu_\rho\}$ is referred to as the $E-$trace of $\{\eta\}$; the above property motivates the following definition:

Definition 6.1. Let I be the index of $X_r; \Theta_E = \{\nu_1, \ldots, \nu_\rho\}$ be the $E-$trace and let $\mu_{\nu_i} = dim Q_{\nu_i} - dim Q_{\nu_i - 1}, i = 1, \ldots, \rho, Q_{\nu_0} = \{0\}$.

(i) The set of indices $J_E = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}, 0 \leqslant d_{\nu_i} < \ldots < d_{\nu_\rho}\}$ is called the $E-$characteristic of $I = \{(d_i, r_i), i \in \underline{\eta}\}$.

(ii) We define as an $E-$system of *generator spaces* any set of spaces $< V_E >$ $\{V_{\nu_1}, \ldots, V_{\nu_\rho}\}$ defined by:

$$\mathcal{V}_{\nu_1} = \mathcal{Q}_{\nu_1}, \mathcal{Q}_{\nu_i} = \mathcal{Q}_{\nu_i - 1} \oplus \mathcal{V}_{\nu_i}, i = 2, \ldots, \rho \tag{87}$$

Proposition 6.1. *Let $< \mathcal{V}_E >= \{\mathcal{V}_{\nu_1}, \ldots, \mathcal{V}_{\nu_\rho}\}$ be an $E-$system of vector spaces. Then,*

$$\mathcal{V}_{\nu_i} \in \mathcal{P}_{\nu_i} \text{ and } \mathcal{V}_{\nu_i} \cap \mathcal{P}_{\nu_i - 1} = \{0\}, \forall i \in \underline{\rho} \tag{88}$$

Furthermore $\mathcal{V}_{\nu_i}$ is uniquely defined, $\gamma(\mathcal{V}_{\nu_i}) = d_{\nu_i}$ and $dim\mathcal{V}_{\nu_i} = \mu_{\nu_i}, \forall i \in \underline{\rho}$.

Proof. By definition $\mathcal{V}_{\nu_i} \subseteq \mathcal{Q}_{\nu_i} \forall i \in \underline{\rho}$ with equality holding only for $i = 1$. Since $\mathcal{Q}_{\nu_i} = \mathcal{N}_r(E) \cap \mathcal{P}_{\nu_i} \subseteq \mathcal{P}_{\nu_i}$ it is clear that $\mathcal{V}_{\nu_i} \subseteq \mathcal{P}_{\nu_i}, i \in \underline{\rho}$. Note that $\mathcal{V}_{\nu_1} = \mathcal{Q}_{\nu_1}$ and $\mathcal{Q}_{\nu_1 - 1} = \{0\}$. If $V_{\nu_1} \cap \mathcal{P}_{\nu_1 - 1} \neq \{0\}$, then there exists $\underline{x} \in \mathcal{V}_{\nu_1} \cap \mathcal{P}_{\nu_1 - 1}, \underline{x} \neq \underline{0}$, such that $\underline{x} \in N_r(E)$; thus, since $\underline{x} \in \mathcal{P}_{\nu_1 - 1}$ it follows that $\mathcal{Q}_{\nu_1 - 1} = \mathcal{N}_r \cap \mathcal{P}_{\nu_1 - 1} \neq \{0\}$, which contradicts the assumption that $\mathcal{Q}_{\nu_1 - 1} = \{0\}$. It is thus proved that $\mathcal{V}_{\nu_1} \cap \mathcal{P}_{\nu_1 - 1} = \{0\}$. Assume now that $\mathcal{V}_{\nu_i} \cap \mathcal{P}_{\nu_i - 1} \neq \{0\}, i = 2, \ldots, \rho$. Then, there exists $\underline{x} \in \mathcal{V}_{\nu_i} \cap \mathcal{P}_{\nu_i - 1}, \underline{x} \neq \underline{0}$, and thus $\underline{x} \in \mathcal{N}_r(E), \underline{x} \in \mathcal{P}_{\nu_i - 1}$ and $\underline{x} \in \mathcal{Q}_{\nu_i - 1}$; however, $\mathcal{Q}_{\nu_i - 1} = \mathcal{Q}_{\nu_{i-1}}$ (by (82)) and the existence of $x \in \mathcal{Q}_{\nu_{i-1}}$ and $\underline{x} \in \mathcal{V}_{\nu_i}, \underline{x} \neq \underline{0}$, contradicts the independence of $\mathcal{Q}_{\nu_{i-1}}, \mathcal{V}_{\nu_i}$ implied from (83). Thus, $\underline{x} = \underline{0}$ and $\mathcal{V}_{\nu_i} \cap \mathcal{P}_{\nu_i - 1} = \{0\}$. The rest of the proof follows from the definitions.

■

The set $< \mathcal{V}_E >$ may be completed to the set of all indices $\{\eta\} = \{1, 2, \ldots, \eta\}$ by defining:

$$\{\mathcal{V}_E\} = \{\mathcal{V}_i, i \in \underline{\eta} : \mathcal{V}_i = \{0\}, \text{ if } i \notin \Theta_E, \mathcal{V}_i = \mathcal{V}_{\nu_j}, \text{ if } i = \nu_j \in \Theta_E\} \tag{89}$$

$\{\mathcal{V}_E\}$ is called the $\eta-$*completion* of $< \mathcal{V}_E >$ defined by (83). In the context of Definition 5.1, $\{\mathcal{V}_E\}$ is a SPSP and will be called an $E-$*system of progenitor spaces* (E-SPSP); the set $< \mathcal{V}_E >$ is then the flag of $\{\mathcal{V}_E\}$ and may be formally described as:

$$< \mathcal{V}_E >= \{\mathcal{V}_k : \mathcal{V}_k \in \{\mathcal{V}_E\} \neq \{0\}, dim\mathcal{V}_k = \mu_k, \gamma(\mathcal{V}_k) = d_k, k \in \Theta_E\} \tag{90}$$

Remark 6.2. An E-SPSP $\{\mathcal{V}_E\}$ with $< \mathcal{V}_E >$ flag is not uniquely defined apart from its first non-zero element $\mathcal{V}_{\nu_1}$ the dimensions $dim\mathcal{V}_{\nu_i} = \mu_{\nu_i} \leqslant r_{\nu_i}, i \in \underline{\rho}$ and the $E-$trace $\Theta_E = \{\nu_1, \ldots, \nu_\rho\}$. For every system $< \mathcal{V}_E >$, the spaces $\mathcal{V}_{\nu_1}, i \in \underline{\rho}$ are maximal dimension spaces of R^n which satisfy the conditions

$$\mathcal{V}_{\nu_i} \in \mathcal{N}_r(E) \cap \mathcal{P}_{\nu_i} = \mathcal{Q}_{\nu_i}, \mathcal{V}_{\nu_i} \cap \mathcal{Q}_{\nu_i - 1} = \{0\} \tag{91}$$

$$\mathcal{Q}_\eta = \mathcal{N}_r(E) \cap \mathcal{P}_\eta = \mathcal{V}_{\nu_1} \oplus \ldots \oplus \mathcal{V}_{\nu_\rho} \tag{92}$$

■

The above remark readily follows from the definitions and the results so far. In the following, we shall denote by $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$, the extended direct sum of two vector spaces defined by $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$ if $\mathcal{X}, \mathcal{Y} \neq \{0\}$, $\mathcal{Z} = \mathcal{X}$ if $\mathcal{Y} = \{0\}$ and $\mathcal{Z} = \mathcal{Y}$ if $\mathcal{X} = \{0\}$. Using this notion, we may define for any E-SPSP $\{\mathcal{V}_E\} = \{\mathcal{V}_i, i \in \underline{\eta}\}$ a complementary $\{\tilde{\mathcal{V}}_E\} = \{\tilde{\mathcal{V}}_i, i \in \underline{\eta}\}$ in the following manner:

$$\mathcal{P}_1 = \mathcal{V}_1 \dot{\oplus} \tilde{\mathcal{V}}_1, \mathcal{P}_i = \mathcal{P}_{i-1} \oplus (\mathcal{V}_i \dot{\oplus} \tilde{\mathcal{V}}_i), i = 2, 3, \ldots, \eta \tag{93}$$

The set $\{\tilde{\mathcal{V}}_E\}$ defined as above, is called an *$E-$dual-system of progenitor spaces* (Ed-SPSP) and its flag (subset containing the nonzero spaces) is defined by

$$< \tilde{\mathcal{V}}_E >= \{\tilde{\mathcal{V}}_p : \mathcal{V}_p \in \{\tilde{\mathcal{V}}_E\} \neq \{0\}, dim\, \tilde{\mathcal{V}}_p = \tau_p, \gamma(\tilde{\mathcal{V}}_p) = d_p, p \in \tilde{\Theta}_E\} \tag{94}$$

where $\tilde{\Theta}_E = \{\sigma_1, \ldots, \sigma_\pi\}$ is a subset of $\{\eta\} = \{1, 2, \ldots, \eta\}$ referred to as the *$E-$dual trace*. The set of indices $\tilde{J}_E = \{(d_{\sigma_i}, \tau_{\sigma_i}), i \in \underline{\pi}, 0 \leqslant d_{\sigma_1} < \ldots < d_{\sigma_\pi}\}$ is called the E-dual characteristic of $I = \{(d_i, r_i), i \in \underline{\eta}\}$. From the definition of Ed-SPSP we have:

Remark 6.3. If $\{\tilde{\mathcal{V}}_E = \{\tilde{\mathcal{V}}_i, i \in \underline{n}\}$ is an Ed-SPSP with $< \tilde{\mathcal{V}}_E >$ flag, then $\{\tilde{V}_E\}$ is not uniquely defined, but the E-dual trace $\tilde{\Theta}_E = \{\sigma_1, \ldots, \sigma_\pi\}$ and the E-dual character $\tilde{J}_E = \{(d_k, \tau_k), k \in \tilde{\Theta}_E\}$ are uniquely defined. Furthermore, the $J_E, \tilde{J}_E$ characters are complementary, that is $[J_E] \cup [\tilde{J}_E] = [I]$, where I is the index of X_R; thus $\tilde{J}_E, \tilde{\Theta}_E$ are uniquely defined from J_E, Θ_E respectively.

■

For any E-SPSP $\{\mathcal{V}_E\}$ we can always define an Ed-SPSP $\{\tilde{V}_E\}$ (not in a unique manner); the pair $(\{\mathcal{V}_E\}, \{\tilde{\mathcal{V}}_E\})$ is clearly complementary and shall be referred to as an *$E-$pair of SPSP*. For any such pair we have the following properties:

Remark 6.4. Let $(\{\mathcal{V}_E\}, \{\tilde{\mathcal{V}}_E\})$ be an E-pair of SPSP. For $\forall p \in \tilde{\Theta}_E$ the space $\tilde{\mathcal{V}}_p \in < \tilde{\mathcal{V}}_E >$ is a maximal dimension space that satisfies the conditions

$$\tilde{\mathcal{V}}_p \subset \mathcal{P}_p, \tilde{\mathcal{V}}_p \cap \mathcal{P}_{p-1} = \{0\} \text{ and } \tilde{\mathcal{V}}_p \cap \mathcal{N}_r(E) = \{0\}, \forall p \in \tilde{\Theta}_E \tag{95}$$

Furthermore, if $< \mathcal{V}_E >= \{\mathcal{V}_{\nu_i}, i \in \underline{\rho}\}, < \tilde{\mathcal{V}}_E >= \{\tilde{\mathcal{V}}_{\sigma_j}, j \in \underline{\pi}\}$ and define

$$\mathcal{Q}_E = \mathcal{V}_{\nu_1} \oplus \ldots \oplus \mathcal{V}_{\nu_\rho}\ ,\ \tilde{\mathcal{Q}}_E\, \tilde{\mathcal{V}}_{\sigma_1} \oplus \ldots \oplus \tilde{\mathcal{V}}_{\sigma_\pi} \tag{96}$$

then

$$\mathcal{P}_\eta = \mathcal{Q}_E \oplus \tilde{\mathcal{Q}}_E \tag{97}$$

where $\mathcal{Q}_E$ is the maximal subspace of $\mathcal{P}_\eta$ that intersects with $\mathcal{N}_r(E)$ and thus it is uniquely defined; $\tilde{\mathcal{Q}}_E$ is a maximal dimension subspace of $\mathcal{P}_\eta$ for which $\tilde{\mathcal{Q}}_E \cap \mathcal{N}_r(E) = \{0\}$; $\tilde{\mathcal{Q}}_E$ is not uniquely defined, but its dimension is uniquely defined.

■

Any E-pair of SPSP $(\{\mathcal{V}_E\}, \{\tilde{\mathcal{V}}_E\})$ with $(<\mathcal{V}_E>, <\tilde{\mathcal{V}}_E>)$ pair of flags is characterised by $(J_E, \tilde{J}_E)$ characteristics, referred to as the *pair of E-characteristics* of I. Using the results of the previous section, we may define for any $(<\mathcal{V}_E>, <\tilde{\mathcal{V}}_E>)$ pair the sets

$$\Omega(\mathcal{V}_E) = \{<\mathcal{V}_E>; <B_E>; <T_E>; <\mathcal{N}_E>\} \tag{98}$$

$$\Omega(\tilde{\mathcal{V}}_E) = \{<\tilde{\mathcal{V}}_E>; <\tilde{B}_E>; <\tilde{T}_E>; <\tilde{\mathcal{N}}_E>\} \tag{99}$$

where $(<B_E>, <\tilde{B}_E>), (<T_E>, <\tilde{T}_E>), (<\mathcal{N}_E>, <\tilde{\mathcal{N}}_E>)$ are E-pairs of complementary systems of progenitor sets (SPS), normal sets (SNS), and normal modules (SNM) respectively (see Definition 5.1). We may refer to $\Omega(\mathcal{V}_E), \Omega(\tilde{\mathcal{V}}_E)$ as an *$E-$non-proper*, *$E-$proper system* respectively of S_E for reasons that will become clear next, and to their union (Definition 5.7)

$$\Omega(V_E, \tilde{V}_E) = \{<V_E; \tilde{V}_E>; <B_E; \tilde{B}_E>; <T_E; \tilde{T}_E>; <N_E; \tilde{N}_E>\} \tag{100}$$

as an E-system of S_E.

Remark 6.5. An E-system $\Omega(\mathcal{V}_E; \tilde{\mathcal{V}}_E)$ is not uniquely defined on a given S_E; however, all such systems have the same pair of E-characteristics $(J_E, \tilde{J}_E)$.

■

We may state now one of the main results of this section, which demonstrates the importance of the E-systems.

Theorem 6.1. *Let $S_e \in \Sigma_{l,n}$ and $\Omega(\mathcal{V}_E, \tilde{\mathcal{V}}_E)$ be an E-system with a pair of E-characteristics $(J_E, \tilde{J}_E)$. The following properties hold true:*

(i) *The pair of E-characteristics $(J_E, \tilde{J}_E)$ are invariants of the Brunovsky orbit $H(S_E)$.*

(ii) *If $(<T_E>, <\tilde{T}_E>)$ is the E-pair complementary SNS of $\Omega(V_E, \tilde{V}_E)$, where $<T_E>= \{T_{\nu_i}, i \in \underline{\rho}\}, <\tilde{T}_E>= \{T_{\sigma_j}, j \in \underline{\pi}\}$ then:*

(a) *$<T_E; \tilde{T}_E>= \{T_{\nu_1}; \ldots; T_{\nu_\rho}; \tilde{T}_{\sigma_1}; \ldots; \tilde{T}_{\sigma_\pi}\}$ is a minimal basis for $N_r\{R(s)\}$.*

(b) *If* $\underline{x}_j^{\nu_i}(s) \in T_{\nu_i}, j \in \underline{\mu}_{\nu_i}, i \in \underline{\rho}, \vartheta\{\underline{x}_j^{\nu_i}(s)\} = d_{\nu_i}, \underline{\tilde{x}}_j^{\sigma_i}(s) \in \tilde{T}_{\sigma_i}, j \in \underline{\tau}_{\sigma_i}, \vartheta\{\underline{x}_j^{\sigma_i}(s)\} = d_{\sigma_i}$ *and define the vectors*

$$\underline{u}_j^{\nu_i}(s) = B^{\dagger}(sE-A)\,\underline{x}_j^{\nu_i}(s), \underline{z}_j^{\nu_i}(s) = \left[\begin{array}{c} \underline{u}_j^{\nu_i}(s) \\ \underline{x}_j^{\nu_i}(s) \end{array}\right], \forall j \in \underline{\mu}_{\nu_i}, i \in \underline{\rho} \tag{101}$$

$$\underline{\tilde{u}}_j^{\sigma_i}(s) = B^{\dagger}(sE-A)\,\underline{\tilde{x}}_j^{\sigma_i}(s), \underline{z}_j^{\sigma_i}(s) = \left[\begin{array}{c} \underline{\tilde{x}}_j^{\sigma_i}(s) \\ \underline{\tilde{u}}_j^{\sigma_i}(s) \end{array}\right], \forall j \in \underline{\tau}_{\sigma_i}, i \in \underline{\pi} \tag{102}$$

then the pairs $(\underline{x}_j^{\nu_i}(s), \underline{u}_j^{\nu_i}(s))$ *for al* $j \in \underline{\mu}_{\nu_i}, i \in \underline{\rho}$ *are non-proper; the pairs* $(\underline{\tilde{x}}_j^{\sigma_i}(s), \underline{\tilde{u}}_j^{\sigma_i}(s))$ *for all* $j \in \underline{\tau}_{\sigma_i}, i \in \underline{\pi}$ *are proper and the set defined by*

$$Z_E = \{Z_{\nu_i}; \ldots; Z_{\nu_\rho}; \tilde{Z}_{\sigma_1}; \ldots; \tilde{Z}_{\sigma_\pi}\} \tag{103}$$

$$Z_{\nu_i} = \{\underline{z}_j^{\nu_i}(s), j \in \underline{\mu}_{\nu_i}\}, \tilde{Z}_{\sigma_i} = \{\underline{\tilde{z}}_j^{\sigma_i}(s), j \in \underline{\tau}_{\sigma_i} \tag{104}$$

is a minimal basis for $\mathcal{N}_r\{T(s)\}$*.*

Proof. (i) The $J_E, \tilde{J}_E$ sets may be computed from the dimensions of the set of $\{\mathcal{Q}_i : i \in \underline{\eta}\}$ spaces. For the general $S'_E = H(S_e)$ we have

$$[sE' - A', -B'] = W[sE-A, -B]\left[\begin{array}{cc} V & 0 \\ F & G \end{array}\right] \tag{105}$$

If N is a left annihilator of B, then we may select a left annihilator of B' as $N' = NW^{-1}$ and thus

$$R'(s) = sN'E' - N'A' = NW^{-1}\{W(sE-A)V - WBF\} = (sNE-NA)V = R(s)V \tag{106}$$

If $\{P_1, \ldots, P_n\}$ are the high coefficient spaces of $R(s)$ and $\{P_i, i \in \underline{n}\}$ are bases matrices for them, then the high coefficient spaces for $R'(s)$ are defined by $P'_i = col.sp\{V^{-1}P_i\}, i \in \underline{n}$. If $E^{\perp}$ is a basis matrix for $N_r(E)$, then $V^{-1}E^{\perp} = E^{'\perp}$ is a basis matrix for $N_r(E')$. Note that the spaces $Q_i = N_r(E) \cap P_i, Q'_i = N_r(E') \cap P'_i, i \in \underline{n}$ are defined by:

$$Q_i = N_r\{\left[\begin{array}{c} P_i \\ E \end{array}\right]\}, Q'_i = N_r\{\left[\begin{array}{c} P'_i \\ E' \end{array}\right]\} = N_r\{\left[\begin{array}{cc} V^{-1} & 0 \\ 0 & V^{-1} \end{array}\right]\left[\begin{array}{c} P_i \\ E \end{array}\right]\} \tag{107}$$

and thus $dimQ_i = dim\,Q'_i, \forall i \in \underline{n}$. Given that $J_E, \tilde{J}_E$ are completely defined by $dimQ_i, i \in \underline{n}$, it follows that $J_E = J'_E, J_E = \tilde{J}'_E$, where $(J'_E, \tilde{J}'_E)$ are the $E-$characteristics associated with S'_e.
(ii)

(a) Since $(< V_E >, < \tilde{V}_E >)$ is an $E-$pair of SPSP, then the $E-$system $\Omega(V_E, \tilde{V}_E)$ is complete and by Corollary 5.5, the set $< T_E, \tilde{T}_E >$ defines an OMB of X_r.

(b) The pairs $(\underline{x}_j^{\nu_i}(s), \underline{u}_j^{\nu_i}(s))$ for $\forall j \in \underline{\mu}_{\nu_i}, i \in \underline{\rho}$ are non-proper, and the high coefficient vector $[\underline{x}_j^{\nu_i}(s)] \in V_{\nu_i} \subset N_r(E)$ whereas the pairs $(\underline{x}_j^{\sigma_i}(s), \underline{u}_j^{\sigma_i}(s))$ for $\forall j \in \underline{\tau}_{\sigma_i}, i \in \underline{\pi}$ are proper since $[\underline{x}_j^{\sigma_{\nu_i}(s)}]_h \in \tilde{V}_{\sigma_i}$ and thus not in $N_r(\tilde{E})$. Consider now the ordered matrix

$$\hat{Z}(s) = [Z_{\nu_i}(s); \ldots; Z_{\nu_\rho}(s); \tilde{Z}_{\sigma_1}(s); \ldots; \tilde{Z}_{\sigma_\pi}(s)] = [Z_{\eta p}(s), Z_p(s)] \quad (108)$$

$$= \begin{bmatrix} X_{\nu_1}(s); \ldots; X_{\nu_\rho}(s) & \tilde{X}_{\sigma_1}; \ldots; \tilde{X}_{\sigma_\pi}(s)(106) \\ U_{\nu_1}(s); \ldots; U_{\nu_\rho}(s) & \tilde{U}_{\sigma_1}; \ldots; \tilde{U}_{\sigma_\pi}(s) \end{bmatrix} = \begin{bmatrix} X(s) & \tilde{X}(s)(107) \\ U(s) & \tilde{U}(s) \end{bmatrix} = \begin{bmatrix} \hat{X}(s) \\ \hat{U}(s) \end{bmatrix} \quad (109)$$

constructed from the pairs $(\underline{x}_j^{\nu_i}(s), \underline{u}_j^{\nu_i}(s)), \forall j \in \underline{\mu}_{\nu_i}, i \in \underline{\rho}, (\underline{x}_j^{\sigma_i}(s), \underline{u}_j^{\sigma_i}(s)), \forall j \in \underline{\tau}_{\sigma_i}(s), i \in \underline{\pi}$. By Proposition 3.5, it follows that since

$$\hat{u}(s) = B^t(sE - A)\hat{X}(s) \quad (110)$$

and $\hat{X}(s)$ is an OMBM of X_r it follows that $\hat{Z}(s)$ is an OMBM, it suffices to show that it is column reduced. Let $X_h[X(s)]_h, \tilde{U}_h\,[\tilde{U}(s)]_h, \hat{Z}_h\,[\hat{Z}(s)]_h$ be the high coefficient matrices. Since $(X(s), U(s))$ are non-proper and $(\tilde{X}(s), \tilde{U}(s))$ are proper, it follows that:

$$\hat{Z}_h \begin{bmatrix} X_h & 0 \\ D & \tilde{U}_h \end{bmatrix} \quad (111)$$

where D is some appropriate matrix. However, $\tilde{U}(s) = B^t(sE - A)\tilde{X}(s)$ and since $(\tilde{X}(s), \tilde{U}(s))$ is proper, $\tilde{U}_h = B^tE\,\tilde{X}_h, \tilde{X}_h\,[\tilde{X}(s)]_h$ it follows that

$$\hat{Z}_h = \begin{bmatrix} X_h & 0 \\ D & B^tE\,\tilde{X}_h \end{bmatrix} \quad (112)$$

From the block-diagonal structure of $\hat{Z}_h$, it follows that since X_h is full rank, then $\hat{Z}_h$ has full rank if and only if $B^tE\,\tilde{X}_h$ has full rank. Note that $B^tE\,\tilde{X}_h \in R^{l\times(l-\tilde{q})}, \tilde{q} = \sum_{i=1}^{\pi} \tau_{\sigma_i}$ and thus loss of rank of $B^tE\,\tilde{X}_h$ implies that there exists $\underline{v} = 0$ such that $B^tE\hat{X} = 0$ by the definition of $\tilde{X}_h$ we also have that $NE\,\tilde{X}_h = 0$ and thus $NE\,\tilde{X}_h\,\underline{v} = 0$. From the last two it follows that

$$\begin{bmatrix} B^tE\,\tilde{X}_h \\ NE\,\tilde{X}_h \end{bmatrix} \underline{v} = 0 \Leftrightarrow \begin{bmatrix} B^t \\ N \end{bmatrix} E\,\tilde{X}_h\,\underline{v} = 0 \Leftrightarrow E\,\tilde{X}_h\,\underline{v} = 0 \quad (113)$$

since the column of $\tilde{X}_h$ are linearly independent (111) implies that $N_r(E) \cap sp\{\tilde{X}_h\} \neq \{0\}$; however $sp\{\tilde{X}_h\} = \tilde{Q}_E$ and by Remark 6.5 $N_r(E) \cap \tilde{Q}_E = \{0\}$, which leads to a contradiction. Thus,$\hat{Z}_h$ has full rank and $\hat{Z}(s)$ is an OMBM of X_T.

■

The above result establishes the existence of canonical minimal bases of X_T. The bases $< T_E; \tilde{T}_E >, Z_E$ for X_R, X_T respectively, defined by the above result, have a special structure and shall be referred to as *state-*, *composite-*, *normal* OMBs correspondingly. A state-normal OMB may always be extended to a composite-normal OMB, which is always canonical; the reverse problem, that is whether every canonical is also normal is examined next.

Proposition 6.2. *Let $\hat{Z}(s)$ be a canonical MB of X_T with a state input partitioning $(X(s), U(s); \tilde{X}(s), \tilde{U}(s))$, where $X(s) \in R^{n\times p}[s], \tilde{X}(s) \in R^{n\times(l-p)}, l = dimX_R = dimX_T$. If $X_h[\tilde{X}(s)]_h$, then:*

$$\rho(E\,\tilde{X}_h) = l - p \tag{114}$$

$$sp\{X_h\} = N_r(E) \cap P_n andsp\{\tilde{X}_h\} \cap N_r(E) = \{0\} \tag{115}$$

Proof. Since $(X(s), U(s))$ is non-proper and $(\tilde{X}(s), \tilde{U}(s))$ proper, then

$$\hat{Z}_h = [\hat{Z}(s)]_h = \begin{bmatrix} X_h & 0 \\ D & \tilde{U}_h \end{bmatrix} \tag{116}$$

where $\tilde{U}_h = B^t E\,\tilde{X}_h$, since $\tilde{U}(s) = B^t(sE - A)\tilde{X}(s)$ and from the properness none of the columns of $\tilde{X}_h$ is in $N_r(E)$. Since $Z(s)$ is column proper and $\rho(X_h) = p$, condition (116) implies that $\rho(\tilde{U}_h) = \rho(B^t E\,\tilde{X}_h) = l - p$. Let us now assume that $\rho(E\,\tilde{X}_h) < l-p$. Then, there exists $\underline{v} \neq 0$ such that $E\,\tilde{X}_h\,\underline{v} = 0$ from which it follows that

$$NE\,\tilde{X}_h\,\underline{v} = 0, B^t E\,\tilde{X}_h\,\underline{v} = \underline{0} \tag{117}$$

The first of (116) is automatically satisfied (since $NE\,\tilde{X}_h = 0$), whereas the second implies that the $l \times (l-p)$ matrix $B^t E\,\tilde{X}_h$ loses rank, which contradicts the column properness of $\hat{Z}(s)$. Thus,$\rho(E\,\tilde{X}_h) = 2p$.
Note that $[X_h, \tilde{X}_h]$ is a basis matrix for P_h and thus $\forall \underline{x} \in P_h$ may be expressed as

$$\underline{x} = X_h\,\underline{v}_1 + \tilde{X}_h\,\underline{v}_2 \tag{118}$$

and thus the dimension of $N_r(E) \cap P_n$ is defined by the number of independent vectors of the type (117) in $N_r(E)$, i.e. the solutions of

$$E\underline{x} = 0 = EX_h\,\underline{v}_1 + E\,\tilde{X}_h\,\underline{v}_2 = E\,\tilde{X}_h\,\underline{v}_2 \tag{119}$$

By (113) however, $\rho(E\,\tilde{X}_h) = l - p$ implies that $\underline{v}_1 = 0$ and thus the dimension of $N_r(E) \cap P_n$ is the dimension of $\underline{v}_1$, i.e. p. However, it is clear that $sp\{X_h\} \subseteq N_r(E) \cap P_n$ and since $dimsp\{X_h\} = dimN_r(E) \cap P_n = p$ we have that $sp\{X_h\} = N_r(E) \cap P_n$. Given that $\rho(E\,\tilde{X}_h) = \rho(\tilde{X}_h) = l - p$, it follows that $sp\{\tilde{X}_h\} \cap N_r(E) = \{0\}$.

■

The above result is also valid for degenerate systems, i.e. systems with $\tau = dimX_T = dimX_R > l$ as long as $l \geqslant \tau - p, p = dimQ_E$. The reverse problem, that is the link of canonical and normal MBs is considered next.

Corollary 6.1. *Let $\hat{Z}(s)$ be an $E-$ordered MB of X_T with state part $\hat{X}(s) = [X(s), \tilde{X}(s)]$, where $(X(s), U(s))$ corresponds to non-proper pairs and $(\tilde{X}(s), \tilde{U}(s))$ to proper pairs. The state part $\hat{X}(s)$ is a normal MB of X_R, if and only if $\hat{Z}(s)$ is a canonical MB of X_T.*

Proof. By Theorem 6.2 it follows that if $\hat{X}(s)$ is normal, then it is an MB for X_R and may be extended to an MB $\hat{Z}(s)$ of X_T, which immediately makes $\hat{Z}(s)$ canonical; this proves that $\hat{X}(s)$ normal implies $\hat{Z}(s)$ canonical. To prove that $\hat{Z}(s)$ canonical implies $\hat{X}(s)$ normal, we argue as follows: $\hat{Z}(s)$ canonical implies that $\hat{X}(s)$ is a minimal basis. Let us partition the ordered $\hat{X}(s)$ as

$$\hat{X}(s) = [X_{\nu'_1}(s), \ldots, X_{\nu'_\rho}(s); \tilde{X}_{\sigma'_1}(s), \ldots, \tilde{X}_{\sigma'_\pi}(s)] = [X(s), \tilde{X}(s)] \tag{120}$$

where $X_{\nu'_1}(s) \in R^{n\times\mu'_{\nu'_i}}[s], \tilde{X}_{\sigma'_j}(s) \in R^{n\times\tau'_{\sigma'_j}}[s]$ correspond to blocks of non-proper, proper vectors with degrees $d_{\nu'_i}, d_{\sigma'_j}$ respectively. The (119) partitioning of $\hat{X}(s)$ defines sets of indices

$$J' = \{(d_{\nu'_i}, \mu'_{\nu'_i}), i \in \underline{\rho}\}, \tilde{J}' = \{(d_{\sigma'_j}, \tau'_{\sigma'_j}), j \in \underline{\pi}\} \tag{121}$$

where it is also assumed that the blocks in $X(s), \tilde{X}(s)$ are ordered according to ascending degrees. We may now order the columns of $\hat{X}(s)$ according to ascending degrees, that is:

$$X'(s) = [X'_1(s); \ldots; X'_n(s)] \tag{122}$$

where $X'_i(s) \in R^{n\times r_i}[s]$ and all columns have the same degree d_i. These blocks are formed as:

(i) If $i = \nu'_k$ and $\nu'_k \notin \{\sigma'_1, \ldots, \sigma'_\pi\}$, then $X'_i(s) = X'_{\nu'_k}$;

(ii) If $i = \sigma'_k$ and $\sigma'_k \notin \{\nu'_1, \ldots, \nu'_{\rho'}\}$, then $X'_i(s) = \tilde{X}_{\sigma'_k}(s)$;

(iii) If $i = \nu'_k = \sigma'_k$, then $X'_i(s) = [X_{\nu'_k}(s), \tilde{X}_{\sigma'_k}(s)]$.

If we now denote by $X_h^{'i}, X_h^{'j}, X_h^{'k}$ the high coefficient matrices of $X'_i(s), X_j(s), \tilde{X}_k(s)$ and denote

$$X_h^{i)}[\ldots, X_h^j, \ldots], \tilde{X}_h^{i)}[\ldots, \tilde{X}_h^j, \ldots]\forall j \leqslant i \tag{123}$$

then since $X'(s)$ is an OMB of $X_R, P_i sp[X_h^{'1}, \ldots, X_h^{'i}]$ and $Q_i = N_r(E) \cap P_i = sp\{x_h^{i)}\}$, since $E[\tilde{X}_h^{\sigma_1}; \ldots; \tilde{X}_h^{\sigma_\pi}]$ has full rank (by Proposition 115) and thus $E\,\tilde{X}_h^{i)}$ has also full rank. From the construction of Q_i it follows that the set of indices for which $Q_{i-1} \subset Q_i$ are precisely the indices $\{\nu'_1, \ldots, \nu'_\rho\}$. For each such index it is clear by the construction of the bases of Q_i that

$$Q_{\nu_i} = Q_{\nu_i - 1} \notin sp\{X_h^{\nu'_i}\} \tag{124}$$

and thus the set $\{sp\{X_h^{\nu'_i}\}, i \in \underline{\rho}\}$ is an E-SPSP, with the set $\{sp\{\tilde{X}_h^{\sigma'_i}\}, i \in \underline{\pi}\}$ being its complementary set. Thus $J = J'$ and $\tilde{J} = \tilde{J}'$.

■

An immediate consequence of Proposition 6.3 and Corollary 6.4 is the following result characterising canonical minimal bases.

Corollary 6.2. *Let $\hat{X}(s) = [X(s); \tilde{X}(s)]$ be an MBM of X_r, where $[X(s)]_h = [\ldots, \underline{x}^i, \ldots] = X_h, \underline{x}_h^i \in N_r(E), [\tilde{X}(s)]_h = [\ldots, \underline{\tilde{x}}_h^j, \ldots] = \tilde{X}_h, \underline{\tilde{x}}_h^j \notin N_r(E)$ and the columns in $X(s), \tilde{X}(s)$ are ordered according to ascending degrees. The MB $\hat{X}(s)$ is normal, if and only if*

$$sp\{X_h\} = N_r(E) \cap P_n \text{ and } sp\{\tilde{X}_h\} \cap N_r(E) = \{0\} \qquad (125)$$

The above results show that the notions of canonical and normal OMBs are equivalent. The natural partitioning of the indices I of X_R into the pair of $E-$characteristics $J_E, \tilde{J}_E$ introduces new variants for the $H(S_E)$ orbit and this is expressed by the following result.

Corollary 6.3. *Let $(J_E, \tilde{J}_E)$ be the pair of $E-$characteristics of X_R, where $J_E = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}, 0 \leqslant d_{\nu_i} < \ldots < d_{\nu_\rho}\}, \tilde{J}_E = \{(d_{\sigma_i}, \tau_{\sigma_i}), i \in \underline{\pi}, 0 \leqslant d_{\sigma_1} < \ldots < d_{\sigma_\pi}\}$. Then,*

(i) *$(J_E, \tilde{J}_E)$ defines a partitioning of the set of CMI, $I_C(R)$ in the sense:*

$$I_C(R) = J_E \dot{\cup} \tilde{J}_E \qquad (126)$$

(ii) *If $\tilde{J}_E^\star = \{(d_{\sigma_i+1}, \tau_{\sigma_i}), i \in \underline{\pi}\}$, then the pair $(J_E, \tilde{J}_E^\star)$ defines a partitioning of the set of CMI, $I_C(T)$, in the sense:*

$$I_C(T) = J_E \dot{\cup} \tilde{J}_E^\star \qquad (127)$$

The above partitioning of $I_C(T)$ is an invariant of $H(S_E)$ and common to all canonical MBs of X_T.

Note that $\dot{\cup}$ indicates "extended union" in the sense that if (d_i, p_i) is in J_E, or $\tilde{J}_E$, then (d_i, p_i) is in the union; if $(d_i, \mu_i) \in J_E$ and $(d_i, \tau_i) \in \tilde{J}_E$, then $(d_i, \mu_i + \tau_i)$ appears in the union. The sets of indices are $J_E, \tilde{J}_E^\star$ shall be called the *non-proper-*, *proper indices* of S_E respectively. Procedures for computing these indices are suggested by the way they have been defined. In fact, we distinguish a geometric approach based on the geometric invariants of X_R and an algebraic approach based on the properties of canonical minimal bases.
Geometric Procedure: Let $\{P_i, i \in \underline{\pi}\}$ be the set of high coefficient spaces (computed from any OMB of X_R (see [21])) and define the sequence of spaces $\{Q_i : Q_i = N_r(E) \cap P_i, i \in \underline{n}\}$. Given that $Q_1 \subseteq \ldots \subseteq Q_n$, we may compute the $E-$trace $\theta_E = \{\nu_1, \ldots, \nu_\rho\}$ for which $Q_{\nu_i - 1} \subset Q_{\nu_i}$. For every $\nu_i \in \theta_E$ we define J_E by

$$J_E = \{(d_{\nu_i}, \mu_{\nu_i}) : \nu_i \in \theta_E, d_{\nu_i} = \gamma(P_{\nu_i}), \mu_{\nu_i} = dimQ_{\nu_i} = dimQ_{\nu_i - 1}\} \qquad (128)$$

The $E-$dual characteristic $\tilde{J}_E$ is then defined as the complementary of J_E with respect to the index $I = \{(d_i, r_i), i \in \underline{n}\}$ of X_R.
The algebraic approach relies on the construction of canonical or normal MBs of X_R from any OMB. We may state the following result.

Proposition 6.3. *Let $X'(s) = [X'_1(s), \ldots, X'_n(s)]$ be an OMBM of X_R, with index $I = \{(d_i, r_i), i \in \underline{n}, 0 \leqslant d_1 < \ldots < d_n\}$ and let $(J_E, \tilde{J}_E)$ be the $E-$pair of characteristics of I, where $J_E = \{(d_{\nu_i}, \mu_{\nu_i}), i \in \underline{\rho}\}, \tilde{J}_E = \{(d_{\sigma_j}, \tau_{\sigma_j}), j \in \underline{\pi}\}$. There always exists an $I-$structured $R[s]-$unimodular matrix $W(s)$ such that*

$$X'(s)W(s) = \hat{X}(s) = [\hat{X}_1(s), \ldots, \hat{X}_n(s)] \tag{129}$$

where $\hat{X}(s)$ is a normal MB of X_R such that:

(i) *If $i = \nu_k$ and $i \notin \{\sigma_1, \ldots, \sigma_\pi\}$, then $\hat{X}_i(s) = X_{\nu_k}(s)$;*

(ii) *If $i = \sigma_k$ and $i \notin \{\nu_1, \ldots, \nu_\rho\}$, then $\hat{X}_i(s) = \tilde{X}_{\sigma_k}(s)$;*

(iii) *If $i = \nu_k = \sigma_k$, then $\hat{X}_i(s) = [X_{\nu_k}(s); \tilde{X}_{\sigma_k}(s)]$.*

and $X_i(s), \tilde{X}_j(s)$ are the blocks associated with the $J_E, \tilde{J}_E$ characteristics respectively.

Proof. The proof is constructive and the two general steps are described below. We shall denote by $X^h[X(s)]_h$.

Step 1: Let ν_i be the smallest integer such that $P_{\nu_i} \cap N_r(E) \neq \{0\}$ whereas $P_{\nu_1 - 1} \cap N_r(E) = \{0\}$. Two different bases for P_{ν_1} may be defined by the matrices

$$P_{\nu_i} = [X_1'^h; \ldots; X_{\nu_1-1}'^h; X_{\nu_1}'^h], \overline{P}_{\nu_1} = [X_1'^h; \ldots; X_{\nu_1-1}'^h; X_{\nu_1}'^h; \tilde{X}_{\nu_1}^h] \tag{130}$$

where $X_{\nu_1}^h$ is a basis for $P_{\nu_1} \cap N_r(E)$ and $\tilde{X}_{\nu_1}^h$ a complementary block such that $sp\{\tilde{X}_{\nu_1}^h\} \in P_{\nu_1}, sp\{\tilde{X}_{\nu_1}^h\} \cap N_r(E) = \{0\}$. Clearly:

$$\overline{P}_{\nu_1} = P_{\nu_1} \begin{bmatrix} I & & & & Q_1 \\ & I & & 0 & Q_2 \\ & & \ddots & & \vdots \\ & 0 & & I & Q_{\nu_1-1} \\ & & 0 & & Q_{\nu_1} \end{bmatrix}, Q_{\nu_1} \in R^{r_{\nu_1} \times r_{\nu_1}}, |Q_{\nu_1}| \neq 0 \tag{131}$$

If we now define the matrix

$$W_{\nu_1}(s) = \begin{bmatrix} I_{r_1} & 0 & & Q_1 s^{d_{\nu_1} - d_1} & \\ & \ddots & & \vdots & 0 \\ 0 & & I_{r_{\nu_1-1}} & Q_{\nu_1-1} s^{d_{\nu_1} - d_{\nu_1} - 1} & \\ & & 0 & Q_{\nu_1} & \\ & & & 0 & I \end{bmatrix} \tag{132}$$

it is clear that

$$X^{\nu_1}(s) = X'(s)W_{\nu_1}(s) = [\overline{X}^{\nu_1}(s); X'_{\nu_1+1}(s), \ldots, X'_n(s)] \tag{133}$$

$$\overline{X}^{\nu_1}(s) = [\tilde{X}_1(s); \ldots; \tilde{X}'_{\nu_1-1}; X_{\nu_1}(s); \tilde{X}_{\nu_1}(s)] \tag{134}$$

where the blocks $X_{\nu_1}(s), \tilde{X}_{\nu_1}(s)$ are generated by $X^h_{\nu_1}, \tilde{X}^h_{\nu_1}$ respectively and $\tilde{X}_i(s) = X'_i(s), i = 1, \ldots, \nu_1-1$. Since $P_{\nu_1-1} \cap N_r(E) = \{0\}$, it follows that the matrix $X^{\nu_1}(s)$ satisfies the conditions of a normal matrix up to the ν_1 index. The matrix $W_{\nu_1}(s)$ is clearly an $I-$structured $R[s]-$unimodular.

Step 2: Assume now that ν_2 is the next index for which $Q_{\nu_2-1} \subset Q_{\nu_2}$, where $Q_i = N_r(E) \cap P_i$. Starting from the $X^{\nu_1}(s)$ OMBM we consider two different basis matrices for P_{ν_2} as shown below

$$P_{\nu_2} = [\overline{P}_{\nu_1}; \ldots; X^{'h}_{\nu_2-1}; X^{'h}_{\nu_2}], \overline{P}_{\nu_2} = [\overline{P}_{\nu_1}; \ldots; X^{'h}_{\nu_2-1}; X^{'h}_{\nu_2}; \tilde{X}^h_{\nu_2}] \tag{135}$$

where since $Q_{\nu_1+1} = \ldots = Q_{\nu_2-1}$ it follows that the spaces $sp\{X_i^{'h}\}, i = \nu_1+1, \ldots, \nu_2-1$ have no intersection with $N_r(E), X^h_{\nu_2}$ is a matrix such that $sp\{X^h_{\nu_1}\} \oplus sp\{X^h_{\nu_2}\} = Q_{\nu_2} = N_r(E) \cap P_{\nu_2}$ and $\tilde{X}^h_{\nu_2}$ is a complementary column block such that $sp\{\tilde{X}^h_{\nu_2}\} \cap N_r(E) = \{0\}$ and $sp\{\tilde{X}^h_{\nu_2}\} \in P_{\nu_2}$ but not in P_{ν_2-1}. Clearly,

$$\overline{P}_{\nu_2} = P_{\nu_2} \begin{bmatrix} I & & & Q'_1 \\ & \ddots & 0 & \vdots \\ 0 & & I & Q'_{\nu_2-1} \\ & 0 & & Q_{\nu_2} \end{bmatrix}, Q_{\nu_2} \in R^{r_{\nu_2} \times r_{\nu_2}}, |Q_{\nu_2}| \neq 0 \tag{136}$$

If we now define the matrix

$$W_{\nu_2}(s) = \begin{bmatrix} I_{r_1} & 0 & Q_1 s^{d_{\nu_2}-d_1} & \\ & \ddots & \vdots & 0 \\ 0 & I & Q'_{\nu_2-1} s^{d_{\nu_2}-d_{\nu_2}-1} & \\ & & Q_{\nu_2} & \\ & 0 & I & \end{bmatrix} \tag{137}$$

It is clear that

$$X^{\nu_2}(s) = X^{\nu_1}(s)W_{\nu_2}(s) = [\overline{X}^{\nu_2}(s); X'_{\nu_2+1}(s); \ldots; X'_n(s)] \tag{138}$$

$$\overline{X}^{\nu}(s) = [\overline{X}^{\nu_1}(s); \tilde{X}_{\nu_1+1}(s); \ldots; \tilde{X}_{\nu_2-1}(s); X_{\nu_2}(s); \tilde{X}_{\nu_2}(s)] \tag{139}$$

where the blocks $X_{\nu_2}(s), \tilde{X}_{\nu_2}(s)$ are generated by $X^h_{\nu_2}, \tilde{X}^h_{\nu_2}$ respectively and $\tilde{X}_i(s) = X'_i(s), i = \nu_1+1, \ldots, \nu_2+1$. Since $sp\{X_i^{'h}\} \cap N_r(E) = \{0\}$ for a normal matrix up to the ν_2 index. It is clear that $W_{\nu_2}(s)$ is an $I-$structured $R[s]-$unimodular. The general step of the construction is obvious.

■

The above result provides algebraic means for the reduction of any OMB of X_R to a normal OMB. Given that the proof of Proposition 6.7 is constructive, the general steps used there, also provide an algorithmic procedure for the construction of $I-$structured $R[s]-$unimodular matrices that reduces any OMB to a normal one.
The construction of a normal OMB from any OMB of X_R provides the means for parametrising all proper and non-proper pairs for the linear system S_E. Given that such pairs characterise spaces of the linear system with reachability-controllability properties [25], [26], the current algebraic construction extends the algebraic characterisation of controllability subspaces [2], [11] to the case of singular systems by defining them as the span of the coefficient vectors of the state parts of the proper, non-proper pairs. This algebraic characterisation based on the proper, non-proper pair characterisation is crucial in defining the families of proper and non-proper reachability spaces for singular systems [26].

7 Conclusions

Singular system properties are intimately linked to the theory of invariants of associated matrix pencils. The theory of minimal bases of the right (left) null spaces of such pencils is critical in defining the properties of system invariants, of the column minimal indices and row minimal indices type, whereas the properties of the vectors of the minimal bases define important geometric invariants (controllability spaces [2], [11],[12], [26]). An important difference between the case of regular and singular systems is the classification of the state-input pairs into proper and non-proper which lead to the notion of canonical minimal bases of singular systems. This paper has developed the algebraic and geometric properties of canonical minimal bases, introduced new feedback invariants and provided algorithmic procedures for the computation of the invariants and the construction of such bases. The link of the proper- non-proper classification of right pairs to the definition of proper- non-proper reachability spaces [26] and the construction of normal OMBs are central to the parameterisation of the families of the proper and non-proper reachability spaces for singular systems. These spaces have assignability of spectra features and the study of these properties under different feedback schemes (state, state and derivative feedback etc) is currently under investigation. The use of the new invariants (proper-non-proper classification of column indices) has already being demonstrated in the construction of canonical forms under the different transformation groups [15]; the feedback canonical form derived in [15] (generalization of the Kronecker form for singular systems) was constructed using elementary transformations. The current results on the construction of normal OMBs provide alternative means for the systematic construction of the transformation that can be used for this derivation and this is a topic under investigation. Of particular interest is the further classification of canonical bases in terms of properties of the pivot indices [3],[8] and the study of the implications in the development of canonical forms under the general coordination transformation group. The significance of the pivot indices [8], [24], [28] to problems of system identification suggests that such extensions to the case of singular systems may provide the basis for a geometric characterisation of proper- non-proper pivot indices by providing additional insight to such issues. A detailed study of the classification of minimal bases using the further properties of the pivot indices and the investigation of such additional invariants to the properties of reachability and controllability subspaces for the case of singular systems [25],[26] is a subject for further research. The results on normal OMBs and their construction may be useful in the development of explicit solutions to control synthesis of singular systems [27], [28] by providing geometric tools for the construction of the feedback transformations.

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