Disturbance and Input – Output Decoupling of Singular Systems

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Abstract

The disturbance decoupling and the simulteneous disturbance and input – output decoupling problems for singular systems are considered in the context of the matrix fraction description (MFD) of the system. Solvability conditions are obtained in terms of the composite matrix of a column reduced MFD of the system, a characterisation of the fixed poles of both problems is given and it is shown that the remaining poles can be arbitrarily assigned.

Key words: singular systems, decoupling, disturbance rejection, matrix fraction description .

1 Introduction

Disturbance decoupling is one of the most studied problems in control. For the class of state space systems with proper or strictly proper transfer functions numerous papers were published over the last decades and several aspects of disturbance decoupled systems have been investigated (see for example [4] , [22] where the problem was tackled by using geometric control theory, [6] where a structural approach was followed for the solution of diagonal and disturbance decoupling of a state space system). In [17] the problem of fixed poles of disturbance decoupling was solved by using the geometric approach. In [14] the disturbance decoupling problem with input - output decoupling was solved in the frequency domain. The fixed poles of the latter problem were also considered in [5]. Disturbance decoupling for singular or implicit systems has attracted the attention of researchers and several papers have been published: [7] was the first paper to tackle the problem. Other papers where important work was done are [2], [1] solved the problem and studied the stabilizability of the closed loop system by using arguments and tools based on the state space systems. In [3] and [15] the disturbance decoupling problem has been considered for the case of implicit systems.

The aim of the present paper is to provide a frequency domain approach to the disturbance and input – output decoupling for singular systems. The approach used follows along the lines of [21] where the block decoupling problem was considered. Although block decoupling and disturbance decoupling are different design goals, they have a major similarity when they are defined in the context of matrix fraction description (MFD) of the transfer function of the system: in both problems the desired resulting system has the property that certain rows of the numerator matrix lie in the rational vector space spanned by certain rows of the denominator matrix. This similarity naturally leads to similar methodologies for the solution of the above problems, individually and in combination. Frequency domain approach allows the use of common tools for problems of different nature. This is an advantage, comparing to dominant state space approaches used.

The MFD representations of nonproper systems have the characteristic that some of the pivot indices [8], [9] of a column reduced composite matrix of the MFD appear in the numerator matrix in contrast to the class of strictly proper (state space) systems where all pivot indices appear in the denominator matrix. In this way we have a classification of the pivot indices into proper and nonproper [20]. When nonproper pivot indices exist (i.e. the transfer function of the system is non proper), state feedback can alter the "denominator matrix" of the system in such that its column (row in the case of left MFD) highest order coefficient matrix can change. This is a consequence of the fact that for the case of singular systems feedback can change the stucture at infinity.

The treatment of the problem and the methodology followed in the present paper is based on the above property of singular systems and the existence of non - proper controllability indices [12], [16] when the system is singular. Necessary and sufficient conditions are obtained for the existence of a solution to the disturbance decoupling

and simultaneous disturbance and input – output decoupling problems. The conditions are easily testable and can be derived from the MFD of the disturbed system. The proof of the sufficiency of the solvability conditions provides a constructive way for selecting the feedback matrices solving the problem. For both disturbance decoupling and simultaneous disturbance and input – output decoupling problems the set of fixed poles is characterised in terms of the MFD of the system in a way analogous to that of state space systems (see [14], [5]).

In what follows the disturbance decoupling problem for singular systems will be referred to as DDSS while the combined disturbance and input – outout decoupling as DDDSS. The following notation will be used: The row (column) high order coefficient matrix [9] of a polynomial matrix $P(s)$ will be denoted by $[P]_{hr}$ ($[P]_{hc}$). The row span over $\mathbb{R}$ ($\mathbb{R}(s)$) of a matrix P will be denoted by $span_{\mathbb{R}}\{P\}$ ($span_{\mathbb{R}(s)}\{P\}$). The notation $(N(s), D(s))$ will be used when we refer to a system with composite matrix $T(s) = [N^T(s), D^T(s)]^T$. A singular system with matrices E, A, B, C will be denoted by (E, A, B, C) and the feedback law $u = Fx + Gv$ will be referred to as feedback pair (F, G). The matrix $[T]_{hc}$ will be written as $[N_{hc}^T, D_{hc}^T]^T$, $N_{hc} \in \mathbb{R}^{m\times\ell}$, $D_{hc} \in \mathbb{R}^{\ell\times\ell}$.

2 DDSS Problem Statement and Preliminaries

Consider the singular system

$$E\dot{x} = Ax + Bu + \Xi\xi,\ y = Cx,\ det(sE - A) \neq 0 \qquad (1)$$

where $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times\ell}$, $C \in \mathbb{R}^{m\times n}$ and $\Xi \in \mathbb{R}^{n\times d}$ i.e. the system has n states, ℓ inputs, m outputs and d disturbance inputs. Matrix E may be singular. Our goal is the determination of the necessary and sufficient conditions for the elimination of the influence of the disturbance $\xi(t)$ on the output $y(t)$ by means of state feedback of the type

$$u = Fx + Gv,\ det(G) \neq 0,\ det(sE - A - BF) \neq 0 \quad (2)$$

and the method of construction of such feedback laws. Throughout the paper it will be assumed that the system (1) is reachable, i.e. $[sE - A, B, \Xi]$ has no finite zeros and $[E, B, \Xi]$ has full row rank. It will also be assumed that B is monic and $m \leq \ell$. From (1) and (2) it follows

$$\begin{bmatrix} u \\ \xi \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} x + \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ \xi \end{bmatrix} \qquad (3)$$

or

$$\begin{bmatrix} u \\ \xi \end{bmatrix} = \hat{F}x + \hat{G} \begin{bmatrix} v \\ \xi \end{bmatrix} \qquad (4)$$

Notice that $\hat{G}$ is invertible as long as G is invertible. Let

$$H(s) = N(s)D^{-1}(s) \qquad (5)$$

be a coprime and column reduced MFD of the transfer function of (1). Then we have (see [20], [21]) that the closed loop MFD has "numerator" and "denominator"

$$N(s) = CS(s), \quad D_c(s) = \hat{G}^{-1}[D(s) - \hat{F}S(s)] \qquad (6)$$

where $S(s)$ =diag$\{[1, s, \cdots, s^{r_i-1}]^T\}$, with r_i, $i = 1, \cdots, \ell+d$ being the reachability indices of $(E, A, [B\ \Xi])$. Then from (3)

$$D_c(s) = \begin{bmatrix} G^{-1} & 0 \\ 0 & I \end{bmatrix} \left[D(s) - \begin{bmatrix} F \\ 0 \end{bmatrix} S(s) \right] \qquad (7)$$

Now partitioning $D(s)$ conformably to the block partitioning of $\hat{G}$, $\hat{F}$ i.e. if

$$D(s) = \begin{bmatrix} D_u(s) \\ D_\xi(s) \end{bmatrix},\ D_u(s) \in \mathbb{R}^{\ell\times(\ell+d)}[s] \\ D_\xi(s) \in \mathbb{R}^{d\times(\ell+d)}[s] \qquad (8)$$

it follows from (7) that

$$D_c(s) = \begin{bmatrix} G^{-1}\left[D_u(s) - FS(s)\right] \\ D_\xi(s) \end{bmatrix} \triangleq \begin{bmatrix} D_v(s) \\ D_\xi(s) \end{bmatrix} \qquad (9)$$

The meaning of the above is that state feedback of type (2) on system (1) affects only the top ℓ rows of the closed loop system denominator matrix.

Definition 1 *The row span (over $\mathbb{R}$) of the row high order coefficient matrix $[P]_{hr}$ of a row reduced polynomial matrix [9] $P(s)$ is referred to as the highest degree characteristic space of the rational vector space spanned by the rows of $P(s)$ and is denoted by $\mathscr{L}\{P(s)\}$.* □

Some useful properties of the highest degree characteristic space are given below

Lemma 2 *[13], [11], [10] (i) All row reduced bases of a rational vector space have the same highest degree characteristic space. (ii) If $[P]_{hr}$ is the high order coefficient matrix of $P(s)$ then $span_{\mathbb{R}}\{[P]_{hr}\} \subseteq \mathscr{L}\{\mathsf{P(s)}\}$. (iii) If $P_1(s)$ and $P_2(s)$ are two polynomial matrices such that $span_{\mathbb{R}(s)}\{P_1(s)\} \subseteq span_{\mathbb{R}(s)}\{P_2(s)\}$ then $\mathscr{L}\{P_1(s)\} \subseteq \mathscr{L}\{P_2(s)\}$.* □

A consequence of the above lemma is

Proposition 3 *If (1) under feedback (2) is disturbance decoupled, then*

$$\mathscr{L}\{N(s)\} \subseteq \mathscr{L}\{D_v(s)\} \qquad (10)$$

that is, when the system is disturbance decoupled, the rows of the numerator matrix $N(s)$ are spanned only by the rows of the matix $D_v(s)$ in (9).

□

The pivot indices (p.i.) of a column reduced basis of a rational vector space play an important role in the paper. Their definition is the following

Definition 4 *[8] Let V be a column reduced basis of a vector space over $\mathbb{R}(s)$ with ordered column degrees $v_1 \leq \cdots \leq v_\ell$ The pivot indices $q_1 \cdots q_\ell$ are defined as follows: Let V have be n_1 columns with degree ν_1. Find the first (lowest index) n_1 rows of V_{hc} such that the $n_1 \times n_1$ submatrix of V_{hc} so defined is nonsingular. The indices of these rows, in order, form the first n_1 pivot indices $q_1 \cdots q_{n_1}$. Delete these n_1 columns and n_1 rows from V and repeat the above procedure to find the next group of pivot indices, corresponding to the columns with the next distinct index value; and so forth.* □

Pivot indices of $T(s) = [N^T(s), D^T(s)]^T$ can be classified into two types [20]:

Definition 5 *Let $q_1, \cdots, q_\ell + d$ denote the pivot indices of $[N^T(s), D^T(s)]^T$. Then q_i is called* **proper** *if $q_i > m$ and* **nonproper** *if $q_i \leq m$.* □

The entries (q_i, i) of $T(s)$ will be referred to as *pivot elements* and are classified into proper and nonproper pivot elements according to the above definition. Furthermore the rows of $T(s)$ that contain pivot elements will be referred to as *pivot rows*. The matrix $T(s)$ is a basis of the vector space spanned by its columns. Throughout the rest of the paper it will be assumed that the given system has τ nonproper p.i.

Definition 6 *The integers p_i, $\overline{p}_i$, τ are defined as follows:*

(i) *p_i are the column indices of $T(s)$ such that the corresponding p.i. q_{p_i} are proper.*
(ii) *$\overline{p}_i$ are the column indices of $T(s)$ such that the corresponding p.i. $q_{\overline{p}_i}$ are nonproper.*
(iii) *τ is the number of the rows of $N(s)$ containing nonproper pivot elements of $T(s)$.* □

Proposition 7 *The DDSS is solvable if and only if there exist sate feedback of type (2) such that $N(s) = H_v(s)G^{-1}[D_u(s) - FS(s)]$ or*

$$N(s) = H_v(s)D_v(s) \tag{11}$$

Proof: Let the system be disturbance decoupled by the pair (F, G). Then, for the the closed loop system we have

$$y(s) = [H_v(s), H_\xi(s)] \begin{bmatrix} v(s) \\ \xi(s) \end{bmatrix} \tag{12}$$

In the above we must have $H_\xi(s) = 0$ because otherwise there is no guarantee that $\xi(s)$ has no influence on the output of the system for any $u(s)$, i.e. the system is not disturbance decoupled. Thus, for a given system (1) the transfer function of the closed loop system is

$$N(s)D_c^{-1}(s) = \left[H_v(s), 0_{(m\times d)}\right], \quad H_v(s) \in \mathbb{R}^{m\times\ell}(s) \tag{13}$$

or, from (8)

$$N(s) = H_v(s)D_v(s) \tag{14}$$

Conversely let feedback of type (2) be such that (14) holds true. The closed loop numerator is

$$N(s) = H_v(s)D_v(s) + H_\xi(s)D_\xi(s) \tag{15}$$

The regularity requirement imposed by (2) means that the matrix $[D_v^T(s)\ D_\xi^T(s)]^T$ is invertible and therefore $H_\xi(s) = 0$ in (15) which means that the system is disturbance decoupled. □

Proposition 8 *If system (1) is disturbance decoupled, then $D_v(s)$ contains at least as many nonpivot rows of $T(s)$ as the number of pivot rows of $T(s)$ contained in $N(s)$.*

Proof: Let τ and β be the numbers of pivot rows of $N(s)$ and nonpivot rows of $D_v(s)$ respectively. Since the system is disturbance decoupled and $D_v(s)$ has full row rank (because of the invertibilty of $D(s)$) it follows from (14) that $rank[N^T(s)\ D_v^T(s)]^T = \ell$. The pivot rows form a linearly independent set of rational vectors and

$$\ell \geq \tau + (\ell - \beta) \Rightarrow \beta \geq \tau$$

□

Remark 9 *All the rows of $D_\xi(s)$ contain pivot rows of $T(s)$ when the system is disturbance decoupled.* □

3 Solvability condition of DDSS

In this section the necessary and sufficient solvability condition of the disturbance decoupling problem is obtained. The proof of the sufficiency of the condition provides a constructive method for the choice of the disturbance decoupling feedback law. Let the composite matrix of the closed loop system be $T_c(s) = [N^T(s), D_c^T(s)]^T$ and define the matrices

$$\begin{gathered} U(s) = N(s)diag(s^{\sigma-\sigma_i}),\ \overline{U}(s) = M(s)U(s) \\ \overline{N}(s) = M(s)N(s) \end{gathered} \tag{16}$$

where $M(s)$ is a unimodular matrix such that $M(s)U(s)$ is row reduced. Let N_α be the row high order coefficient

matrix of $\overline{U}(s)$ in (16). Note that N_α is a feedback invariant of the system and therefore is the same for open and closed loop systems. Also define

$$D_v(s) = \begin{bmatrix} D^{np}(s) \\ D^p(s) \end{bmatrix}, \quad N(s) = \begin{bmatrix} N^{np}(s) \\ N^p(s) \end{bmatrix} \tag{17}$$

where $D^p(s)$ and $N^{np}(s)$ consist of these rows of $D_v(s)$ and $N(s)$ respectively, containing pivot elements of $T_c(s)$ (note that the nonproper pivot elements of the closed loop system are those of the uncompensated). From proposition 8 it follows that $D^{np}(s)$ has τ rows. Now let the rows of $[T_c]_{hc}$ corresponding to $D_v(s)$ be written (upon row reordering) , according to the partitioning of (17), as

$$[D]_{hc} = \begin{bmatrix} [D^{np}]_{hc} \\ [D^p]_{hc} \end{bmatrix} \tag{18}$$

The following is needed for the main result:

Proposition 10 *The systems with composite matrices*

$$T(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}, \quad \overline{T}(s) = \begin{bmatrix} \overline{N}(s) \\ D(s) \end{bmatrix} \tag{19}$$

where the rows of $N(s)$ and $\overline{N}(s)$, span the same vector space over $\mathbb{R}(s)$, yield the same set of solutions to the disturbance decoupling problem.

Proof: A solution for the first system will have a closed loop denominator matrix of the form (9). This means that the rows of $N(s)$ lie in the span of of $D_v(s)$ and therefore the rows of $\overline{N}(s) = M(s)N(s)$ also lie in the span of of $D_v(s)$. Thus a solution for the first system is also a solution of the second one and vice-versa. □

Let r_i and σ_i, $i = 1, \cdots, \ell + d$ be the reachability indices (r.i.) the controllability indices (c.i.) respectively [16], of the triple $(E, A, [B\ \Xi])$.

Proposition 11 *[20] Let $H_c(s) = N_c(s)D_c^{-1}(s)$ be the closed loop transfer function (t.f.) of the system (1) under the law (2). Then $N_c(s) = N(s)$, and $D_c(s) = \hat{G}^{-1}[D(s) - \hat{F}S(s)]$, where $S(s) = diag\{[1, s, \cdots, s^{r_i - 1}]^T\}$, $i = 1, \cdots, \ell + d$, i.e. the degrees of the columns $v_i(s)$ of the matrix $V(s) = FS(s)$ are*

$$deg\ v_i(s) \leq \begin{cases} \sigma_i - 1 & if\ \sigma_i\ is\ proper \\ \sigma_i & if\ \sigma_i\ is\ nonproper \end{cases} \tag{20}$$

□

For the classification of c.i. into proper and nonproper see [12], [16]. Note that in the case where some r.i. are zero then the corresponding column of $S(s)$ is the zero vector of dimension equal to the sum of the nonzero r.i.

Theorem 12 *Necessary and sufficient condition for the solvability of the disturbance decoupling problem is that $D^{np}(s)$ in (17) has τ rows and*

$$span_{\mathbb{R}}\{N_\alpha\} \subseteq span_{\mathbb{R}}\{\begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix}\} \tag{21}$$

Where J is the matrix formed by the $q_{\overline{p}_j}$, rows (the non proper pivot rows) of $[T]_{hc}$.

Proof: Necessity will be proven first. The requirement that $D^{np}(s)$ in (17) has τ rows readily follows from Proposition 8. Matrix J is the row high order coefficient of the $q_{\overline{p}_j}$ rows of $U(s)$. Thus, from Lemma 2

$$\mathscr{L}\{U(s)\} \supseteq span_{\mathbb{R}}\{J\} \tag{22}$$

Consider the matrix $[J^T(s), T_p^T(s)]^T$ formed by the pivot rows of $T(s)$, where $J(s)$ and $T_p(s)$ contain the nonproper pivot and proper rows respectively. Its high order coefficient matrix has the form

$$\begin{bmatrix} J \\ T_p \end{bmatrix} \tag{23}$$

where T_p is formed by the proper pivot rows of T_{hc}. From the definition of the pivot rows it follows that this matrix is square and invertible. Observe that the rows of $[D^p]_{hc}$ defined in (18) are also rows of T_p. Then $[D^p]_{hc}$ has full row rank and

$$span_{\mathbb{R}}\{[D^p]_{hc}\} \cap span_{\mathbb{R}}\{J\} = \{0\} \tag{24}$$

Further, we have

$$span_{\mathbb{R}}\{[D^p]_{hc}\} = span_{\mathbb{R}}\{[D^p diag(s^{\sigma - \sigma_i})]_{hr}\} \tag{25}$$

Now, since the system is disturbance decoupled it follows from (14) and Lemma 2 that

$$\mathscr{L}\{U(s)\} \subseteq \mathscr{L}\{D_v(s) diag(s^{\sigma - \sigma_i})\} \tag{26}$$

Then, from (22), (24) and Proposition 8 it follows that

$$\mathscr{L}\{D_v(s) diag(s^{\sigma - \sigma_i})\} = span_{\mathbb{R}}\{\begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix}\} \tag{27}$$

and, since $\mathscr{L}\{U(s)\} = span_{\mathbb{R}}\{N_\alpha\}$, we have

$$span_{\mathbb{R}}\{N_\alpha\} \subseteq span_{\mathbb{R}}\{\begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix}\} \tag{28}$$

which proves the necessity.

Sufficiency will be proven by constructing a feedback pair (F, G) that disturbance decouples $(\overline{N}(s), D(s))$

which has the same set of solutions of the disturbance decoupling problem with the original system (see Proposition 10). As a first step, we apply a preliminary feedback pair (F_1, G_1) such that the matrix $[D]_{hc}$ (see (18)) of the resulting system has full row rank and is equal to $[J^T, [D^p]_{hc}^T]^T$. Note that this is always possible because $D^{np}(s)$ has τ rows and Proposition 11. From (21) it follows that there exists constant full row rank matrix G_{11} such that $N_\alpha = G_{11}[J^T, [D^p]_{hc}^T]^T$. Then there exists matrix Y such that $[N_\alpha^T, Y^T]^T$ has full row rank. Then we can write

$$\begin{bmatrix} N_\alpha \\ Y \end{bmatrix} = \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix} \begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix} \triangleq G_2^{-1} \begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix} \tag{29}$$

We use G_2 as further input transformation. The resulting system has $D'_v(s) = G_2^{-1}G_1^{-1}[D_u(s) + F_1S(s)]$ with $[D]_{hc} = [(N_\alpha)^T, Y^T]^T$ which is epic. Let $D'_v(s)$ be partitioned as

$$D'_v(s) = \begin{bmatrix} D'_1(s) \\ D'_2(s) \end{bmatrix} \tag{30}$$
$$D'_1(s) \in \mathbb{R}^{m\times(\ell+d)}[s], D'_2(s) \in \mathbb{R}^{(\ell-m)\times(\ell+d)}[s]$$

Next we are going to apply state feedback on the system $(\overline{N}(s), D'_v(s))$ such that the resulting system is distubance decoupled. Consider the equation

$$D'_1(s) + F'S(s) = H^{-1}(s)\overline{N}(s) \tag{31}$$

where $H^{-1}(s) = diag\{\frac{\mu_j(s)}{\nu_j(s)}\}$, $j = 1, \cdots, m$ with $\mu_j(s)$, $\nu_j(s)$ monic polynomials. Equation (31) describes a diagonally decoupled square system, since $H(s)$ is diagonal matrix. The solvability of this equation with respect to F' was considered in [20] where it was shown that (31) is solvable with respect to F' only if the following hold

$$deg\{\mu_j(s)\} - deg\{\nu_j(s)\} = f_j,\ \nu_j(s)|\varphi_j(s) \text{ if } f_j \neq 0 \tag{32}$$

$$\begin{array}{l} 0 \leq deg\{\nu_j(s)\} - deg\{\mu_j(s)\} \leq \\ deg\{\varphi_j(s)\},\ \mu_j(s)|\varphi_j(s) \text{ if } f_j = 0 \end{array} \tag{33}$$

where $\varphi_j(s)$ is the g.c.d. of the entries of the i–th row of $\overline{N}(s)$ and f_i, $i = 1, \cdots, m$ are the orders of the infinite zeros of $\overline{n}^i(s)diag(s^{-\sigma_j})$, $j = 1, \cdots, \ell$ with $\overline{n}^i(s)$ is the i–th row of $\overline{N}(s)$. From (30), (31) it follows that

$$\begin{bmatrix} H(s),\ 0_{m\times(\ell-m)} \end{bmatrix} \begin{bmatrix} D'_1(s) + F'S(s) \\ D'_2(s) \end{bmatrix} = \overline{N}(s)\} \tag{34}$$

If $F'' = [(F')^T, 0_{(\ell-m)\times\ell}^T]^T$, we have that $[(D'(s) + F''S(s))diag\{s^{s-s_i}\}]_{hr} = [(N_\alpha)^T, Y^T]^T$ is epic. Furthermore from (29) and Remark 9 it follows that the collumn high order coefficient matrix of the denominator matrix $D_c(s)$ of the overall system (see (9)) has full rank. Thus the regularity reguirement for $sE - A - BF$ is fulfilled. The t.f. of the closed loop system $\overline{T}(s)$ is

$$\overline{H}_c(s) = \begin{bmatrix} H(s),\ 0_{m\times(\ell-m)}, 0_{m\times d} \end{bmatrix} \tag{35}$$

which clearly means that the pair $(G_1G_2, -G_1G_2F'' - F_1)$ disturbance decouples system (1). The t.f. of the system obtained by applying this feedback pair to to (1) is $H_c(s) = M^{-1}(s)\overline{H}_c(s)$. □

Remark 13 *If, in the construction of the above decoupling pair, we choose $\mu_j(s) = \varphi_j(s)$ and $\nu_i(s)$ such that $deg\{\nu_i(s)\} = deg\{\varphi_i(s)\}$, we obtain elimination of the infinite poles since in that case, $deg\{det(D'(s) + F''S(s))\} = \sum_{i=1}^{\ell} \sigma_i = rankE$.* □

4 Fixed poles of DDSS

In the previous section, the solution of the problem i.e. the pair (F, G) is constrained in order to meet the requirement of disturbance decoupling. This means that some of the system poles may not be assignable and remain fixed for any solution of the disturbance decoupling. Here, the issue of fixed poles is investigated, a characterisation of these poles is given and it is shown that the rest of the poles can be arbitrarily assigned. The following Proposition is necessary for the proof of the main result of this section.

Lemma 14 *[19] If a singular system (E, A, B) is reachable with nonzero r.i., then it can always be transformed by restricted system equivalence transformations [18], such that the pencil $[sE - A, B]$ has the form*

$$[sE - A, B] = \begin{bmatrix} L(s) & 0 \\ sK - \Lambda & I \end{bmatrix} \tag{36}$$

where $L(s) = diag\{L_{r_i-1}(s)\}, L_{r_i-1}(s) = s[I_{r_i-1}, 0_{r_i-1\times 1}] - [0_{r_i-1\times 1}, I_{r_i-1}]$, $i = 1, \cdots, \ell$. Furthermore, the numerator and denominator of the MFD of the t.f. of the system are $CS(s)$ and $(sK - \Lambda)S(s)$ respectively. □

Let $Q(s)$ be the greatest common left divisor (g.c.l.d.) of the entries of the columns of $N(s)$ and write

$$N(s) = Q(s)\hat{N}(s) \tag{37}$$

The fixed poles are given by the following result:

Theorem 15 *Let system (1) be disturbance decoupled by state feedback and regular input transformation (2). Then the fixed poles of the system are the zeros of the matrix*

$$\begin{bmatrix} \hat{N}(s) \\ D_\xi(s) \end{bmatrix} \tag{38}$$

where $D_\xi(s)$ is defined in (8).

Proof: Since the system is disturbance decoupled it follows from (14) that

$$H_v(s)D_v(s) = N(s) = Q(s)\hat{N}(s) \tag{39}$$

The rows of $\hat{N}(s)$ have no finite zeros. Now, since $H_v(s)$ has full row rank and $\ell \geq m$, it follows that there exists unimodular matrix $R(s)$ such that

$$H_v(s)R(s) = [\tilde{H}(s), 0] \tag{40}$$

where $\tilde{H}(s)$ is square and invertible rational matrix. Then

$$[\tilde{H}(s), 0] \begin{bmatrix} \tilde{D}(s) \\ \overline{D}(s) \end{bmatrix} = N(s) = Q(s)\hat{N}(s) \tag{41}$$

where $[\tilde{D}^T(s), \overline{D}^T(s)]^T = R^{-1}(s)D_v(s)$. The matrix $\tilde{D}(s)$ has full row rank because $N(s)$ has full row rank. Furthermore, (41) yields

$$\tilde{D}(s) = \tilde{H}^{-1}(s)Q(s)\hat{N}(s) = \Phi(s)\hat{N}(s) \tag{42}$$

Matrix $\Phi(s)$ is polynomial because $\hat{N}(s)$ has no finite zeros (its Smith form is $[I\ 0]$). Then the denominator of the closed loop disturbance decoupled system can be written as

$$D_c(s) = \begin{bmatrix} \Phi(s)\hat{N}(s) \\ \overline{D}(s) \\ D_\xi(s) \end{bmatrix} = \begin{bmatrix} \tilde{D}(s) \\ \overline{D}(s) \\ D_\xi(s) \end{bmatrix} \tag{43}$$

which clearly means that the zeros of $[\hat{N}^T(s), D_\xi^T(s)]^T$ are poles of the closed loop disturbance decoupled system. Let $(N(s), D'(s))$ be MFD pair of the system which is disturbance decoupled according to the constuctive proof of the sufficiency part of Theorem 12. In what follows, the complete feedback law (3) is going to be used. The state feedback used so far was of the form

$$\begin{bmatrix} F_A \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

Let $(N(s), D'(s))$ have a generalised state space representation $(\tilde{E}, \tilde{A}, \tilde{B}, \Xi, \tilde{C})$. On this system apply state feedback of the form

$$\begin{bmatrix} 0 \\ F_B \\ 0 \end{bmatrix} \tag{45}$$

Then the poles of the resulting system are the zeros of the pencil

$$s\tilde{E} - \tilde{A} - [B_1\ B_2\ \Xi] \begin{bmatrix} 0 \\ F_B \\ 0 \end{bmatrix} = s\tilde{E} - \tilde{A} - B_2F_B \tag{46}$$

where $\tilde{B} = [B_1\ B_2]$. Thus the pole placement of properies $(\tilde{E}, \tilde{A}, \tilde{B})$ under feedback (45) are those of $(\tilde{E}, \tilde{A}, B_2)$ under full state feedback F_B. Since the system has no zero r.i. we can assume according to Lemma 14 that

$$[s\tilde{E} - \tilde{A}, \tilde{B}, \Xi] = \begin{bmatrix} L(s) & 0 & 0 & 0 \\ sK_1 - \Lambda_1 & I & 0 & 0 \\ sK_2 - \Lambda_2 & 0 & I & 0 \\ sK_3 - \Lambda_3 & 0 & 0 & I \end{bmatrix} \tag{47}$$

where $(sK_1 - \Lambda_1)S(s) = \tilde{D}(s)$ $(sK_2 - \Lambda_2)S(s) = \overline{D}(s)$ and $(sK_3 - \Lambda_3)S(s) = D_\xi(s)$. In this coordinate frame

$$[s\tilde{E} - \tilde{A}, B_2] = \begin{bmatrix} L(s) & 0 \\ sK_1 - \Lambda_1 & 0 \\ sK_2 - \Lambda_2 & I \\ sK_3 - \Lambda_3 & 0 \end{bmatrix} \tag{48}$$

The above has the same zero structure to

$$\begin{bmatrix} L(s) \\ sK_1 - \Lambda_1 \\ sK_3 - \Lambda_3 \end{bmatrix} \tag{49}$$

The uncontrollable poles of $(\tilde{E}, \tilde{A}, B_2)$ are the zeros of pencil (49). Consider the unimodular matrix $\hat{S}(s) = [S(s)\ \overline{S}(s)]$ where

$$\overline{S}(s) = diag\{\overline{S}_i(s)\}$$

$$\overline{S}_i(s) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hline -1 & 0 & \cdots & 0 \\ -s & -1 & 0 \ \cdots & 0 \\ \vdots & \vdots & \ddots \ \ddots & \vdots \\ -s^{r_i-3} & -s^{r_i-4} & \cdots \ -1 & 0 \\ -s^{r_i-2} & -s^{r_i-3} & \cdots \ -s & -1 \end{bmatrix} \tag{50}$$

Then

$$\begin{bmatrix} L(s) \\ sK_1 - \Lambda_1 \\ sK_3 - \Lambda_3 \end{bmatrix} \hat{S}(s) = \begin{bmatrix} 0 & I \\ \tilde{D}(s) & (sK_1 - \Lambda_1)\overline{S}(s) \\ D_\xi(s) & (sK_3 - \Lambda_3)\overline{S}(s) \end{bmatrix} \tag{51}$$

From the above it is clear that the poles of $(\tilde{E}, \tilde{A}, B_2)$ that cannot be shifted by state feedback F_B are the zeros of $[\tilde{D}^T(s), D_\xi^T(s)]^T$ which, in turn are the poles of $(\tilde{E}, \tilde{A}, \tilde{B})$ which cannot be shifted by feedback (45). From (43) it follows

$$\begin{bmatrix} \tilde{D}(s) \\ D_\xi(s) \end{bmatrix} = \begin{bmatrix} \Phi(s)\hat{N}(s) \\ D_\xi(s) \end{bmatrix} \tag{52}$$

Equation (43) yields that the zeros of $\Phi(s)$ are zeros of $\tilde{D}(s)$. If the disturbance decoupling pair is selected as described in and Remark 13, all the poles of the closed loop system are finite and the feedback pair can be chosen such that the poles of $H(s)$, i.e. the zeros of $\Phi(s)$ in (52), are assigned by appropriate choice of $\mu_i(s)$ and

$\nu_i(s)$ in (32), (33). Thus, all the poles of the original system, except the poles given by the zeros of the matrix (38) can be arbitrarily assigned by a state feedback of the form

$$\begin{bmatrix} F_A \\ F_B \\ 0 \end{bmatrix}$$

Since feedback for disturbance decoupling has to be as above (i.e. the lower submatrix of the above must be zero matrix), the result follows. □

Remark 16 *In the above theorem it was assumed that all the r.i. of the system are nonzero. This assumption does not affect the generality since in the case where some r.i. are zero and, under the assumption that B is monic, we have (upon reordering of the disturbance variables) that*

$$[s\tilde{E} - \tilde{A}, \tilde{B}, \Xi] = \begin{bmatrix} L(s) & 0 & 0 & 0 & 0 \\ sK_1 - \Lambda_1 & I & 0 & 0 & \Xi_1 \\ sK_2 - \Lambda_2 & 0 & I & 0 & \Xi_2 \\ sK_3 - \Lambda_3 & 0 & 0 & I & 0 \end{bmatrix} \tag{53}$$

which means that the above pencil has the same Smith zeros with (47). Therefore the aruments following (47) are identical for the case of zero r.i.

5 The DDDSS problem

In this section the combined problem of simultaneous disturbance and input – output decoupling is considered. The problem is stated as follows: Given system (1) find state feedback and regular input transformation (2) such that the closed loop system has transfer function.

$$\begin{aligned} &N(s)D^{-1}(s) = \left[\Delta(s), 0_{(m\times d)}\right] \\ &\Delta(s) = diag\{\Delta_i(s)\} \in \mathbb{R}^{m\times\ell}(s) \end{aligned} \tag{54}$$

where $\Delta_i(s) \in \mathbb{R}^{m_i\times\ell_i}(s)$, $\sum_{i=1}^{k} \ell_i = \ell$, $\sum_{i=1}^{k} m_i = m$, $i = 1, \cdots, k$. Let $N(s)$ be partitioned according to the above into k blocks i.e. $N(s) = bl\{N_i(s)\},\ i = 1, \cdots, k$. Then the corresponding partitioning of $D_v(s)$ is $D_v(s) = bl\{D_v^i(s)\}$. Next define the matrices

$$\widetilde{U}(s) = bl\{\widetilde{U}_i(s)\} = bl\{N_i(s)\}diag(s^{\sigma-\sigma_i}) \tag{55}$$

$$\widehat{U}(s) = bl\{\widehat{U}_i(s)\} = bl\{\hat{M}_i(s)\widetilde{U}_i(s)\} = \hat{M}(s)\widetilde{U}(s) \tag{56}$$

$$N^{\dagger}(s) = \hat{M}(s)N(s) = bl\{N_i^{\dagger}(s)\} \tag{57}$$

$$\hat{N}_\alpha = bl\{\hat{N}_\alpha^i\} \tag{58}$$

where $\hat{M}_i(s)$ are unimodular matrices such that $\hat{M}_i(s)\widetilde{U}_i(s),\ i = 1, \cdots, k$, are row reduced and $\hat{N}_\alpha^i = [\widehat{U}_i(s)]_{hr}$. Also denote by $[D_v^i]_{hc}$ the corresponding blocks of the column high order coefficient $[D]_{hc}$ of $D_v(s)$ in the sense it is defined in (18). The solvability conditions of the combined problem are given below:

Theorem 17 *Necessary and sufficient conditions for the solvability of DDDSS is that $D^{np}(s)$ in (17) has τ rows and the following hold true*

$$rank(\hat{N}_\alpha) = m \tag{59}$$

$$span_{\mathbb{R}}\{\hat{N}_\alpha\} \subseteq span_{\mathbb{R}}\{\begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix}\} \tag{60}$$

Proof: By using arguments similar to those of Theorem 12 we are going to consider the system $(N^{\dagger}(s), D(s))$ which is equivalent to the original system as far as the solvability of DDDP is concerned. First the necessity is proven. Since the row span (over $\mathbb{R}(s)$) of $N^{\dagger}(s)diag(s^{\sigma-\sigma_i})$ is a subspace of the row span of $D_v^i(s)diag(s^{\sigma-\sigma_i})$ it follows from Lemma 2 that the row span over $\mathbb{R}$ of $\hat{N}_\alpha^i$ is a subspace of the row span of $[D_v^i]_{hc}$. The rows of $[D]_{hc} = bl\{[D_v^i]_{hc}\}$ are lineraly independent over $\mathbb{R}$ and the necessity of (59) and (60) follows.

The sufficiency can be proven in a way similar to that of the sufficiency part of Theorem 12 where $\overline{N}(s)$ is replaced by $N^{\dagger}(s)$ (see [21] for details). The resulting transfer function of the closed loop system will be

$$H_c(s) = \left[\hat{M}^{-1}(s)\overline{H}(s),\ 0_{m\times(\ell-m)}, 0_{m\times d}\right] \tag{61}$$

where $\overline{H}(s) = diag\{\overline{H}_i(s)\}$, $\overline{H}_i(s) = diag\{\frac{\nu_j(s)}{\mu_j(s)}\}$, $i = 1, \cdots, k,\ j = 1, \cdots, m_i$ with $\mu_j(s)$ monic polynomials derived from (32), (33) with $N^{\dagger}(s)$ in place of $\overline{N}(s)$, which completes the proof. □

Since input - output decoupling is an additional constraint to the disturbance decoupling in the system of this section, it is expected that the set of fixed poles is different. Let

$$N(s) = diag\{Q_i(s)\}\ bl\{\hat{N}_i(s)\} = Q\hat{N}(s),\ i = 1, \cdots, k \tag{62}$$

where $Q_i(s)$ is a greatest common left divisor (g.c.l.d.) of the columns of $N_i(s)$. Note that $Q(s)$ and $\hat{N}(s)$ are different from those in (37) but the same notation is used for the sake of simplicity. Write

$$\hat{N}(s) = \Gamma(s)\tilde{N}(s) \tag{63}$$

where $\Gamma(s)$ is a g.c.l.d. of the columns of $\hat{N}(s)$. The following theorem gives the characterisation of the fixed poles of the DDDSS:

Theorem 18 *The fixed poles of DDDSS are:*

(i) The zeros of $\Gamma(s)$

(ii) The zeros of $\begin{bmatrix} \tilde{N}(s) \\ D_\xi(s) \end{bmatrix}$

Proof: Since the system is input - output decoupled it follows from (62) that

$$\Delta_i(s)D_u^i(s) = N_i(s) = Q_i(s)\hat{N}_i(s) \tag{64}$$

Notice that $\hat{N}_i(s)$ has no finite zeros. Now, since $\Delta_i(s)$ has full row rank and $\ell_i \geq m_i$, it follows that there exist unimodular matrices $R_i(s)$ such that

$$\Delta_i(s)R_i(s) = [\tilde{\Delta}_i(s), 0] \tag{65}$$

where $\tilde{\Delta}_i(s)$, $i = 1, \cdots, k$ are square and invertible rational matrices. Then

$$[\tilde{\Delta}_i(s), 0] \begin{bmatrix} \tilde{D}_i(s) \\ \overline{D}_i(s) \end{bmatrix} = N_i(s) = Q_i(s)\hat{N}_i(s) \tag{66}$$

where $[\tilde{D}_i^T(s), \overline{D}_i^T(s)]^T = R_i^{-1}(s)D_v^i(s)$. Thus

$$\tilde{D}_i(s) = \tilde{\Delta}_i^{-1}(s)Q_i(s)\hat{N}_i(s) = \Phi_i(s)\hat{N}_i(s) \tag{67}$$

Matrix $\hat{N}_i(s)$ has full row rank and no finite zeros and therefore its Smith form is $[I\ 0]$ which means that, $\Phi_i(s)$ is polynomial. Then

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \left[\begin{array}{c} diag\{Q_i(s)\}\ bl\{\hat{N}_i(s)\} \\ \hline diag\{\Phi_i(s)\}\ bl\{\hat{N}_i(s)\} \\ bl\{\overline{D}_i(s)\} \\ D_\xi(s) \end{array}\right] = \left[\begin{array}{c} diag\{Q_i(s)\}\ bl\{\hat{N}_i(s)\} \\ \hline \tilde{D}(s) \\ \overline{D}(s) \\ D_\xi(s) \end{array}\right] \tag{68}$$

where $\tilde{D}(s) = bl\{\Phi_i(s)\}\ bl\{\hat{N}_i(s)\}$ and $\overline{D}(s) = bl\{\overline{D}_i(s)\}$. From (68), (63) it follows that

$$det(D(s)) = det(\Phi(s))det((\Gamma(s))det(\begin{bmatrix} \tilde{N}(s) \\ \overline{D}(s) \\ D_\xi(s) \end{bmatrix}) \tag{69}$$

where $\Phi(s) = diag\{\Phi_i(s)\}$ which proves (i). The proof of (ii) can be carried out by working similarly to Theorem 15 and Theorem 4 of [21]. □

Remark 19 *The results on fixed poles of DDP and DDDP for singular systems obtained are consistent with the results of [14] and [5] for disturbance rejection of state – space systems.*

Example 20 Consider the system with

$$T(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} N(s) \\ D_u(s) \\ D_\xi(s) \end{bmatrix} =$$

$$\left[\begin{array}{cccccc} 0 & s^2-1 & s+1 & s+1 & -2s-2 & 0 \\ 1 & s^2+s+1 & 1 & s & s+2 & s^2-s \\ 0 & 3 & -1 & s^3+7 & s^2-2 & 0 \\ \hline 1 & s & s-1 & s^2 & s & -s \\ 0 & 0 & s^2+1 & s+2 & s-3 & 1 \\ s & 3 & 1 & 2 & 1 & -2 \\ s+6 & 21 & 1 & -5 & -3 & 30 \\ \hline 0 & s-1 & 1 & 1 & 1 & s^4-s-2 \\ 0 & 3 & -1 & s-53 & s^3+78 & 0 \end{array}\right]$$

For this system $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = 2$, $\sigma_4 = \sigma_5 = 3$, $\sigma_6 = 4$, $\tau = 1$, $p_1 = 1$, $p_2 = 3$, $p_3 = 5$, $p_4 = 6$, $\overline{p}_1 = 1$, $\overline{p}_2 = 4$, $q_{p_1} = 6$, $q_{p_2} = 5$, $q_{p_3} = 9$, $q_{p_4} = 8$, $q_{\overline{p}_1} = 1$, $q_{\overline{p}_2} = 3$. We are going to consider disturbance and input - output decoupling with $m_1 = 2$ and $m_2 = 1$. We have

$$N(s) = Q(s)\hat{N}(s) = \left[\begin{array}{cc|c} s+1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \times$$
$$\left[\begin{array}{cccccc} 0 & s-1 & 1 & 1 & -2 & 0 \\ 1 & s^2+s+1 & 1 & s & s+2 & s^2-s \\ \hline 0 & 3 & -1 & s^3+7 & s^2-2 & 0 \end{array}\right]$$

$$N^\dagger(s) = \hat{M}(s)N(s) = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \times$$
$$\left[\begin{array}{cccccc} 1 & s+2 & -s & -1 & 3s+4 & s^2-s \\ 0 & s^2-1 & s+1 & s+1 & -2s-2 & 0 \\ \hline 0 & 3 & -1 & s^3+7 & s^2-2 & 0 \end{array}\right]$$

$$\hat{N}(s) = \Gamma(s)\tilde{N}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & s+2 \end{bmatrix} \times$$
$$\left[\begin{array}{cccccc} 0 & s-1 & 1 & 1 & -2 & 0 \\ 1 & s^2+s+1 & 1 & s & s+2 & s^2-s \\ 0 & 1 & 0 & s^2-2s+4 & s-2 & 0 \end{array}\right]$$

$f_1 = 1$, $f_2 = 0$, $f_3 = 0$, $\varphi_1 = 1$, $\varphi_2 = s+1$, $\varphi_3 = 1$ and

$$\begin{bmatrix} J \\ [D^p]_{hc} \end{bmatrix} = \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \ \hat{N}_\alpha = \left[\begin{array}{cccccc} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

From the above matrices it follows that the system is diturbance and input-output decouplable. If the desired transfer function is

$$\overline{H}(s) = \left[\begin{array}{cc|cc|cc} (s+6)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{s+1}{s+7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

i.e

$$H_c(s) = \hat{M}^{-1}(s)\overline{H}(s) = \left[\begin{array}{ccc|cc|cc} 0 & & \frac{s+1}{s+7} & 0 & 0 & 0 & 0 \\ (s+6)^{-1} & & \frac{s+1}{s+7} & 0 & 0 & 0 & 0 \\ \hline 0 & & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

we and up with the following feedback matrices:

$$G_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ F_1 S(s) = \begin{bmatrix} 0 & s^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_2^{-1} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$F'' S(s) = \left[\begin{array}{c} F_A \\ \hline 0 \end{array}\right] S(s) =$$

$$\left[\begin{array}{cccc} 1 & 9-7\,s & -1+7\,s & s^2-1 \\ 1 & 7-5\,s & -8 & -7-s+s^2 \\ -6 & -21 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] /$$

$$\left[\begin{array}{cc} -24-22\,s-3\,s^2 & 29+5\,s-5\,s^2-s^3 \\ 14+3\,s & -s \\ 6-s^2 & -32 \\ \hline 0 & 0 \end{array}\right]$$

Matrix $[\tilde{N}^T(s), D_\xi^T(s)]^T$ has a zero at $s = -4$, which is fixed a pole of the closed loop system. The poles of the c.l. system are $\{-2, -4, -5, -6, -7, 0.989, 1.699, 2.345 \pm 3.712j, 0.2557994100 \pm 1.7j, -0.112 \pm 0.787j, -0.653, -2.022\}$. Poles $\{-6, -7, \}$ are the poles of the transfer function and are selected as described in Theorem 12. The zero of $\Gamma(s)$ at $s = -2$ is also a fixed pole of the system. All remaining poles can be shifted to the desired locations by appropriate selection of F_B (see (45)). If the desired location of these poles is at $s = -1$ i.e. the characteristic polynomial of the closed loop system is $(s+2)(s+4)(s+6)(s+7)(s+1)^{11}$ we can choose F_B such that

$F_B S(s) =$

$$[-\frac{26122}{529},\ -\frac{4564943}{5290}\,s^2 + \frac{46606469}{2645}\,s - \frac{17198631}{1058},$$

$$-\frac{4361913}{5290}\,s + \frac{17636897}{1058},\ \frac{2072613}{5290}\,s - \frac{38413009}{5290},$$

$$0,\ \frac{62734}{1587}\,s^3 + \frac{203587}{1587}\,s + \frac{141329}{1587}]$$

6 Conclusions

The problems of disturbance decoupling and simultaneous disturbance and input – output decoupling for singular systems have been solved by using an approach based on the matrix fraction description of the disturbed system. The treatment of the problems is based on the additional feature of the existence of non proper pivot indices when the system is nonproper. Necessary and sufficent conditions for the solvability of the problems by means of state feedback have been derived. The proof of the sufficiency part of the solvability conditions provides a constuctive way for selecting the disturbance decoupling feedback pair. The characterisation of the fixed poles of the decoupled system has been given and is has been shown that it is consistent to the one given for the state – space systems. The MFD based algebraic approach of the paper provides a unification of the formulation of different problems, which are usually tackled by pure state-space methodologies and tools, and the ability of using detailed invariants, the pivot indices, which are very natural within the algebraic framework.

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