FINITE SETTLING TIME STABILIZATION:
The Robust SISO Case

E. Milonidis[†], A Kostarigka[‡], N Karcanias[†]

[†]School of Engineering and Mathematical Sciences,
City University London,
Northampton Square, London EC1V 0HB
e-mail: e.milonidis@city.ac.uk n.karcanias@city.ac.uk

[‡]School of Engineering, Aristotle University of Thessaloniki,
Thessaloniki, Greece
e-mail: akost@auth.gr

Abstract

This paper deals with the problem of robustness to multiplicative plant perturbations for the case of *Finite Settling Time Stabilization* (FSTS) of SISO, linear, discrete–time systems. FSTS is a generalization of the deadbeat control and as in the case of deadbeat control the main feature of FSTS is the placement of all closed–loop poles at the origin of the *z*–plane. This makes FSTS sensitive to plant perturbations hence, the need of robust design. An efficient robustness index is introduced and the problem is reduced to a finite linear program where all the benefits of the simplex method, such as effectiveness, efficiency and ability to provide complete solution to the optimization problem, can be exploited.

1. Introduction

The concept of *Finite Settling Time Stabilization* was introduced by Karcanias and Milonidis [4], [5] as a generalization to the deadbeat control. FSTS simply requires that in a unity feedback system as shown in Fig. (2.1), all internal and external signals settle to a steady state value in finite time after a step is applied to any of the system inputs and for any initial condition.

As in the case of deadbeat regulation, the main feature of the FSTS controllers is that they place the poles of the closed-loop system at the origin of the *z*–plane. This makes the FSTS problem sensitive to plant parameter variations and the need for robust design arises naturally. The case of robust output deadbeat tracking with internal stability has been treated in the framework of one– and two–parameter controllers by Zhao and Kimura [10]–[13]. In this paper, we consider the robust FSTS problem rather than the output deadbeat, where in addition to the external signals all the internal signals settle to steady state values (a necessary requirement for ripple-free response). Within this framework, the robust FSTS is treated as a linear optimization problem, whereas in the case of output deadbeat, by Zhao and Kimura, the problem is reduced to quadratic optimization.

In the next section we give some background results in terms of the problem formulation and a basic mathematical notation. We define formally the FSTS problem and derive the parametrization of the family of all causal FSTS controllers. This leads to the solvability conditions for tracking a family of signals in FST sense as a solution to a linear algebra problem. In section 3, a robustness index in the form of a l^1-norm is introduced and this reduces the robust FSTS problem to multiplicative plant variations, to a linear program. We finish in section 4 with the presentation of an algorithm and an example.

Throughout the paper we will use the following abbreviations. BIBO: Bounded Input Bounded Output; FST: Finite Settling Time; FSTS: FST Stabilization; SISO: Single Input Single Output. We shall also denote by: $d = z^{-1}$ the delay operator (indeterminate), $\mathbb{R}$ the set of real numbers, $\mathbb{R}[\![d]\!]$ the set of formal power series over $\mathbb{R}$, $\mathbb{R}^{\circ}(d)$ the set of causal sequences over $\mathbb{R}$, $*$ the convolutory multiplication, | the property of divisibility, $\|\cdot\|_p$ the *p*-norm, $g_p(\hat{h})$ the gain of the l^p-operator $\hat{h}$, and $\partial(t)$ the degree of the polynomial *t*. Finally, *n* and *d* ***with subscript***, denote polynomial numerator and denominator factors respectively of a rational function; i.e. the expression " $p = n_p / d_p$ is a coprime polynomial fraction in d " is equivalent to " $p(d) = n_p(d)/d_p(d)$, $n_p(d), d_p(d) \in \mathbb{R}[d]$ and $n_p(d), d_p(d)$ are coprime".

2. Finite Settling Time Stabilization: Background Results

Problem Formulation

In this section the finite settling time stabilization problem is defined. A complete parametrization of the family of all causal FSTS controllers is given and also the solution to tracking in FST sense (i.e. zero steady state error in finite time) for a family of signals is presented. All proofs are omitted; they can be found in [4] and [9].

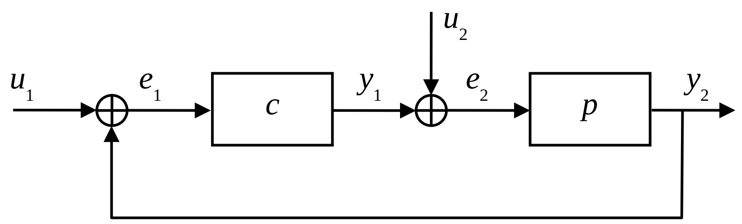

Fig. 2.1: The unity feedback configuration

Definition 2.1: ([4], [9]) The unity feedback system of Fig. (2.1) exhibits a *finite settling time* response, if for a step change in any of the inputs u_1, u_2 and for any initial condition, all the signals e_1, e_2 or y_1, y_2 settle to a new steady state value in finite time. □

Theorem 2.1: ([4], [9]) Consider the closed-loop system of Fig. (2.1) and let $p = n_p / d_p$, $c = n_c / d_c$ be coprime polynomial fractions in d of the plant and the controller transfer functions respectively. Then, the solution of the FST problem exists *if and only if*

$$\delta(p,c) := n_p n_c + d_p d_c \ \in \mathbb{R} \backslash \{0\} \tag{2.1}$$

Moreover, the family of all causal FST controllers is given by

$$\begin{gathered}\mathscr{F}(p) = \{(n_{c,} d_c) : n_c = x + t d_p,\, d_c = y - t n_p,\ t \in \mathbb{R}[d] \text{ and} \\ y(0) - t(0) n_p(0) \neq 0 \text{ if } n_p(0) \neq 0\}\end{gathered} \tag{2.2}$$

where x, y is a particular solution pair of the Diophantine equation

$$n_p x + d_p y = 1 \tag{2.3}$$

□

Corollary 2.1: ([9]) It can be easily shown, using condition (2.1), that the transfer function matrix $\mathbf{W}(p,c)$ from $\mathbf{u} = [u_1 \quad u_2]^T$ to $\mathbf{y} = [y_1 \quad y_2]^T$ in the FST case is

$$\mathbf{W}(p,c) = \begin{bmatrix} d_p n_c & -n_p n_c \\ 1 - d_p d_c & n_p d_c \end{bmatrix} \tag{2.4}$$

□

According to Theorem (2.1) the parametrization of the family $\mathscr{F}(p)$ requires only the derivation of one particular solution (x, y) of the Diophantine equation (2.3). One such particular solution is given by the following proposition.

Proposition 2.1 *(Prime FSTS controller):* ([4]) Let $p = n_p / d_p$ be the transfer function of the plant with (n_p, d_p) $\mathbb{R}[d]-$coprime and $\partial(n_p) = m,\ \partial(d_p) = n$. Then, there always exists a unique FSTS controller $\tilde{c} = \tilde{n}_c / \tilde{d}_c$ with $\partial(\tilde{n}_c) = n - 1$ and $\partial(\tilde{d}_c) = m - 1$. The controller $\tilde{c} = \tilde{n}_c / \tilde{d}_c$ will be referred to as the *prime* FSTS controller and is the minimum McMillan degree controller. The parameter vectors of $\tilde{n}_c,\ \tilde{d}_c$ are given as a solution of the following system of equations

$$\left[\begin{array}{c} g_0 \\ \vdots \\ g_{n-1} \\ \hline f_0 \\ \vdots \\ f_{m-1} \end{array}\right] = \left[\begin{array}{cccc:cccc} b_0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ b_1 & b_0 & \ddots & \vdots & a_1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ \vdots & & & b_0 & \vdots & & & 1 \\ b_m & & & \vdots & a_n & & & a_1 \\ 0 & b_m & & \vdots & 0 & a_n & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & b_m & 0 & \cdots & 0 & a_n \end{array}\right]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.5}$$

where

$$\begin{aligned} n_p &= b_0 + b_1 d + \cdots + b_m d^m \\ d_p &= 1 + a_1 d + \cdots + a_n d^n \\ \tilde{n}_c &= g_0 + g_1 d + \cdots + g_{n-1} d^{n-1} \\ \tilde{d}_c &= f_0 + f_1 d + \cdots + f_{m-1} d^{m-1} \end{aligned}$$

□

Theorem 2.2 *(FST Tracking):* ([4], [9]) Let $p = n_p / d_p$ be the transfer function of the plant and $c = n_c / d_c$ be the transfer function of any FSTS controller, with all fractions involved being coprime polynomial fractions. Suppose also that the input $u_1 = n_r / d_r$ belongs to a specified class of signals. Then, the output y_2 tracks the input u_1 in a finite time *if and only if* $d_r \mid d_p d_c$. □

Remark 2.1: The tracking condition of Theorem (2.2) can be written as

$$d_p d_c = q d_r \tag{2.6}$$

where $q \in \mathbb{R}[d]$ can be considered as a *tracking parameter*. Using the parametrization equations (2.2) for d_c and the tracking condition (2.6) we have the resulting equation

$$q d_r + t n_p d_p = y d_p \tag{2.7}$$

where t is the free parameter that specifies the family of all FSTS controllers and in the case of FST tracking is given as the non-unique solution of equation (2.7). In that respect, t will be used to accomplish robust performance of the system. □

The development of the robustness analysis of the FSTS problem requires some results on the norms and the gain of an operator and we will use the terminology from Dahleh and Pearson [1], [2].

Mathematical Notation

Consider the set of formal power series $\mathbb{R}[\![d]\!]$ in one indeterminate d over $\mathbb{R}$. Then

$$\forall f = \{ f_i \} = \sum_{i=0}^{\infty} f_i d^i \in \mathbb{R}[\![d]\!] \tag{2.8}$$

the expressions

$$\| f \|_p := \left\{ \sum_{i=0}^{\infty} | f_i |^p \right\}^{1/p}, \; 1 \le p < \infty \tag{2.9}$$

$$\| f \|_\infty := \sup_i | f_i | \tag{2.10}$$

define the p-norms of f. The space of all sequences f such that $\| f \|_p$ is defined, i.e. $\| f \|_p < \infty$, is denoted by l^p.

A sequence $f \in \mathbb{R}[\![d]\!]$ can represent the impulse response of a linear, time–invariant discrete–time system (in fact $f \in \mathbb{R}^{\circ}(d) \subset \mathbb{R}[\![d]\!]$ [6], [7]). Clearly, the system is BIBO–stable, if and only if f is a l^1 sequence. We recall that the series (2.8) is formal and d is an indeterminate and not a variable. If f is a l^1 sequence, the series (2.8) are summable for some $d \in \mathbb{C}$ and f may also represent a function of the complex variable d. In system theory terms, $f = \{ f_i \}$ is the recurrent impulse response of a lumped linear time–invariant system whereas $\hat{f}(d)$ which is no more than the z-transform of $f = \{ f_i \}$ with $d = z^{-1}$, is the rational transfer function of the system. In this case, and because $\mathbb{R}$ is an infinite field, *recurrent sequences* and *rational functions* are *isomorphic*. Thus, f and $\hat{f}$ represent the same algebraic entities and may not be distinguished from each other.

Let $\mathbf{A}$ denote the space of all functions with elements BIBO–stable functions. Therefore, for every $\hat{h}(d) \in \mathbf{A}$, $d \in \mathbb{C}$, $h = \{ h_i \} \in l^1$ is the impulse response of a linear system. If l^∞ is the space of all bounded sequences we can regard $\mathbf{A}$ as the space of bounded linear time–invariant operators on l^∞, i.e., $\forall \hat{h} \in \mathbf{A}$, $f \in l^\infty$, then

$$\hat{h} : l^\infty \to l^\infty \text{ and } \hat{h}\{ f \} = h * f \tag{2.11}$$

We can define the induced norm of the operator $\hat{h}$ on $\mathbf{A}$ as

$$\left\| \hat{h} \right\|_{\mathbf{A}} := \sup_{\| f \|_\infty \neq 0} \frac{\left\| \hat{h}\{ f \} \right\|_\infty}{\| f \|_\infty} = \sum_{i=0}^{\infty} | h_i | := \| h \|_1 \tag{2.12}$$

We can generalize the previous concepts by defining the *gain* of the operator $\hat{h}$ as

$$g_p\left(\hat{h}\right) := \sup_{\|f\|_p \neq 0} \frac{\left\|\hat{h}\{f\}\right\|_p}{\|f\|_p} < \infty \tag{2.13}$$

if and only if $\hat{h}$ is a map form l^p to l^p, and in that case $\hat{h}$ is said to be $l^p-stable$, $1 \leq p \leq \infty$. From the definition of the gain of the operator we have

$$g_\infty\left(\hat{h}\right) = \left\|\hat{h}\right\|_{\mathbf{A}} = \|h\|_1 \quad \text{and} \quad g_2\left(\hat{h}\right) = \|h\|_\infty \tag{2.14}$$

Furthermore, the l^∞–induced norm on $\hat{h}$ bounds from above all other l^p–induced norms, or equivalently

$$g_p\left(\hat{h}\right) \leq g_\infty\left(\hat{h}\right) = \left\|\hat{h}\right\|_{\mathbf{A}},\ 1 \leq p < \infty \tag{2.15}$$

3. Robust Finite Settling Time Stabilization

Consider the unity feedback scheme of Fig. (2.1) and let p_0 denote the transfer function of the nominal plant and p the dynamics of the actual plant. Under multiplicative perturbation we may have

$$p - p_0 = \delta p \cdot p,\quad \delta p \ \ l^\infty - \text{stable} \tag{3.1}$$

Suppose that $\delta h := \left(h - h_0\right)/h$ is the relative error between the actual transfer function h from u_1 to y_2 and the corresponding nominal one, h_0. Then, it can be shown (see Appendix) that

$$\delta h = (1 - h_0)\delta p \tag{3.2}$$

Therefore,

$$g_p\left(\delta\hat{h}\right) \leq g_p\left(1 - \hat{h}_0\right) g_p\left(\delta\hat{p}\right) \tag{3.3}$$

and since the induced l^∞–norm bounds from above all the other induced norms we may choose as robustness index ρ, the induced l^∞–norm of $1 - \hat{h}_0$, i.e.

$$\rho := g_\infty\left(1 - \hat{h}_0\right) = \left\|1 - h_0\right\|_1 \tag{3.4}$$

In the case of FST stabilization, $h_0 = w_{21}(p,c) = 1 - d_{p_0} d_c$ and the robustness index becomes

$$\rho = \left\|d_{p_0} d_c\right\|_1$$

As a consequence of the above discussion and Remark (2.1), the solution to the robust FSTS problem may be given by the next theorem.

Theorem 3.1 *(Robust FSTS):* Let $p_0 = n_{p_0}/d_{p_0}$ and $c = n_c/d_c$ be the transfer functions of the nominal plant and controller respectively. If $u_1 = n_r/d_r$, $\|u_1\|_\infty < \infty$ is the input to be tracked, then the robust FSTS to multiplicative perturbations can be described by the following linear program

$$\begin{aligned} &\text{minimize} \quad \left\|d_{p_0} d_c\right\|_1 = \left\|d_{p_0}\left(y - t n_{p_0}\right)\right\|_1 \\ &\text{subject to} \quad q d_r + t n_{p_0} d_{p_0} = y d_{p_0} \end{aligned} \tag{3.5}$$

for some particular $\nu = \partial(t)$ for which the equation

$$q d_r + t n_{p_0} d_{p_0} = y d_{p_0} \tag{3.6}$$

has a solution, and *y* is a particular solution of the equation $n_{p_0} x + d_{p_0} y = 1$.

Proof: If x, y is a particular solution pair of the Diophantine equation $n_{p_0} x + d_{p_0} y = 1$ then the family of FSTS controllers that track the input $u_1 = n_r/d_r$, $\|u_1\|_\infty < \infty$ is given by

$$n_c = x + t d_{p_0},\ d_c = y - t n_{p_0},\ t \in \mathbb{R}[d]$$

and *t* satisfies equation (3.6). For robustness, if $\nu = \partial(t)$

$$\rho = \left\|d_{p_0} d_c\right\|_1 = \left\|d_{p_0}\left(y - t n_{p_0}\right)\right\|_1 \text{ must be } \textit{minimum}.$$

This results to the optimization problem (3.5). □

Remark 3.1: Due to the nature of the linear programming the optimal solution $\begin{bmatrix} q^* & t^* \end{bmatrix}^T$ for a particular $\nu = \partial(t)$ is a suboptimal solution to the optimization problem with $\lambda = \partial(t) > \nu$. Therefore, $\rho_\nu^* = \left\|d_{p_0} d_c^*\right\|_1$ is a monotonically decreasing function of ν, i.e.

$$\rho_\nu^* \le \rho_\lambda^*,\ \nu \ge \lambda$$

Hence, the robustness performance of the closed-loop system can be improved by increasing the settling time of its response. □

An important part of the solution of the robust FST tracking is the solution of the Diophantine equation (3.6) and the specification of a $\nu = \partial(t)$, for which such a solution exists. The rest of this section deals with this problem.

Lemma 3.1: Let $p_0 = n_{p_0}/d_{p_0}$ be the transfer function of the nominal plant and $c = n_c/d_c$ be the transfer function of any tracking FSTS controller such that the closed-loop system of Fig. (2.1) tracks the input $u_1 = n_r/d_r$. If d_{r_p} is the common factor of d_r and d_{p_0}, and d_{r_c} and $d_{p_{0r}}$ are the remaining factors of d_r and d_{p_0} respectively, then $\left(d_{r_c}, n_{p_0}d_{p_0}/d_{r_p}\right) = \left(d_{r_c}, n_{p_0}d_{p_{0r}}\right)$ are $\mathbb{R}[d]-$coprime.

Proof: From Theorem (2.2), the condition for FST tracking is $d_r \mid d_{p_0}d_c$, or

$$d_{p_0}d_c = qd_r \tag{3.7}$$

If d_{r_p} is the common factor of d_r and d_{p_0}, then according to equation (3.7)

$$d_r = d_{r_p}d_{r_c} \tag{3.8}$$

where

$$d_{p_0} = d_{r_p}d_{p_{0r}} \text{ and } d_c = d_{r_c}d_{c_r} \tag{3.9}$$

Since $c = n_c/d_c$ is a tracking FST controller, (n_c, d_c) satisfy the Diophantine equation $n_{p_0}n_c + d_{p_0}d_c = 1$. Hence, (d_c, n_{p_0}) are $\mathbb{R}[d]-$coprime and since d_{r_c} is a factor of d_c, then

$$\left(d_{r_c}, n_{p_0}\right) \text{ are } \mathbb{R}[d]-\text{coprime} \tag{3.10}$$

Also,

$$\left(d_{r_c}, d_{p_{0r}}\right) \text{ are } \mathbb{R}[d]-\text{coprime} \tag{3.11}$$

by construction (equations (3.8) and (3.9)). Thus, from (3.10) and (3.11) FST tracking requires

$$\left(d_{r_c}, n_{p_0}d_{p_{0r}}\right) \text{ should be } \mathbb{R}[d]-\text{coprime}$$ □

Remark 3.2: Lemma (3.1) covers all divisibility cases $d_r \mid d_{p_0}d_c$ for FST tracking. Indeed:

1. If d_r and d_{p_0} do not have common factors, d_r should divide d_c for FST tracking and equations (3.8) and (3.9) become $d_{r_c} = d_r$ and $d_{p_{0r}} = d_{p_0}$. In that case, for FST tracking we need
$$\left(d_r, n_{p_0}d_{p_0}\right) \text{ to be } \mathbb{R}[d]-\text{coprime}$$
2. If $d_r \mid d_{p_0}$ then $d_{r_c} = 1$ and $d_r = d_{r_p}$. In that case, for FST tracking we need
$$\left(1, n_{p_0}d_{p_{0r}}\right) \text{ to be } \mathbb{R}[d]-\text{coprime}$$
which is always true. In fact, all FSTS controllers are FST tracking controllers as $d_{p_0}d_c = qd_r$ is satisfied $\forall d_c \in \mathscr{F}(p_0)$ in this case. □

If d_r and d_{p_0} have, in general, a common factor d_{r_p}, then $d_r = d_{r_p}d_{r_c}$, $d_{p_0} = d_{r_p}d_{p_{0r}}$ and the Diophantine equation (3.6) is reduced to

$$qd_{r_c} + tn_{p_0}d_{p_{0r}} = yd_{p_{0r}} \tag{3.12}$$

by dividing both sides of equation (3.6) by d_{r_p}. By allowing d_{r_p} to take also the "extreme" forms of $d_{r_p}=1$, or $d_{r_p}=d_r$ we include the cases of $\left(d_r, d_{p_0}\right)$ to be coprime, or $d_r \mid d_{p_0}$ respectively. Therefore, the solution of the Diophantine equation (3.6) is reduced to the solution of the Diophantine equation (3.12) for all cases of FST tracking, and the solvability conditions are given by the following theorem.

Theorem 3.2: Let $p_0 = n_{p_0} / d_{p_0}$ and $\tilde{c} = \tilde{n}_c / \tilde{d}_c$ be the transfer functions of the nominal plant and the prime FSTS controller respectively. Suppose also $u_1 = n_r / d_r$ is the input to be tracked and d_{r_p} is the common factor of d_r and d_{p_0}, i.e. $d_r = d_{r_p} d_{r_c}$ and $d_{p_0} = d_{r_p} d_{p_{0r}}$. If $\partial\left(d_{r_c}\right) = l$, then a unique solution to equation

$$q d_{r_c} + t n_{p_0} d_{p_{0r}} = \tilde{d}_c d_{p_{0r}} \tag{3.13}$$

exists for

1. $\partial(t) = l-1$, if $l \geq 1$ and $\left(d_{r_c}, n_{p_0} d_{p_{0r}}\right)$ are $\mathbb{R}[d]-$coprime
2. $\partial(t) = -\infty$, if $l = 0$

Proof:

1. $\partial\left(d_{r_c}\right) = l \geq 1$, i.e. d_r has common factors at least with d_c, and $\left(d_{r_c}, n_{p_0} d_{p_{0r}}\right)$ are $\mathbb{R}[d]-$coprime. Let

$$\begin{aligned} q &= q_0 + q_1 d + \cdots + q_\mu d^\mu \\ t &= t_0 + t_1 d + \cdots + t_\nu d^\nu \\ d_{r_c} &= w_0 + w_1 d + \cdots + w_l d^l \\ r &= n_{p_0} d_{p_{0r}} = r_0 + r_1 d + \cdots + r_{m+\xi} d^{m+\xi} \\ s &= \tilde{d}_c d_{p_{0r}} = s_0 + s_1 d + \cdots + s_{m+\xi-1} d^{m+\xi-1} \end{aligned} \tag{3.14}$$

Since $\tilde{d}_c$ is the denominator of the prime FSTS controller, then $\partial\left(\tilde{d}_c\right) = m-1$. Therefore, $\partial\left(n_{p_0} d_{p_{0r}}\right) = m+\xi > \partial\left(\tilde{d}_c d_{p_{0r}}\right) = (m-1)+\xi$ and equation (3.13) has potentially a solution for any $\partial(t) \geq 0$ with the necessary condition

$$\partial\left(q d_{r_c}\right) = \partial\left(t n_{p_0} d_{p_{0r}}\right), \text{ or } \mu + l = \nu + (m+\xi) \tag{3.15}$$

Equation (3.13) can be written as

$$\begin{bmatrix} d_{r_c} & n_{p_0} d_{p_{0r}} \end{bmatrix} \begin{bmatrix} q \\ t \end{bmatrix} = \tilde{d}_c d_{p_{0r}} \tag{3.16}$$

and using expressions (3.14) it becomes

$$
\underbrace{\left[\begin{array}{cccc:cccc}
w_0 & 0 & \cdots & 0 & r_0 & 0 & \cdots & 0 \\
w_1 & w_0 & \ddots & \vdots & r_1 & r_0 & \ddots & \vdots \\
\vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\
\vdots & & & w_0 & \vdots & & & r_0 \\
w_l & & & w_1 & r_{m+\xi} & & & r_1 \\
0 & w_l & & \vdots & 0 & r_{m+\xi} & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & \cdots & 0 & w_l & 0 & \cdots & 0 & r_{m+\xi}
\end{array}\right]}_{T_{(l+1+\mu)\times(\mu+1+\nu+1)}}
\begin{bmatrix} q_0 \\ \vdots \\ q_\mu \\ \hdashline t_0 \\ \vdots \\ t_\nu \end{bmatrix}
=
\begin{bmatrix} s_0 \\ \vdots \\ s_{m+\xi-1} \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix}
\tag{3.17}
$$

If $\nu = \partial(t) = l-1$, then according to equation (3.15) $\mu = \partial(q) = m+\xi-1$. In that case the Toeplitz matrix T becomes a square matrix with order $m+\xi+l$. T is the Sylvester matrix of the coprime polynomials d_{r_c} and $r = n_{p_0} d_{p_{0r}}$ and it is invertible to that extend [8]. Therefore, the system of equations (3.17) has a unique solution for $\nu = \partial(t) = l-1$.

2. $\partial\left(d_{r_c}\right) = l = 0$, i.e. $d_{r_c} = 1$ without loss of generality and d_r has no common factors with d_c and divides d_{p_0}. In that case all FSTS controllers are FST tracking controllers and also equation (3.13) becomes

$$
q + t n_{p_0} d_{p_{0r}} = \tilde{d}_c d_{p_{0r}} \tag{3.18}
$$

Equation (3.18) has the unique solution $t = 0$ and $q = \tilde{d}_c d_{p_{0r}}$, if $\left(\partial(t) = -\infty\right)$ and the corresponding FST tracking controller is the prime FSTS controller. □

Before we proceed with the presentation of the robust FSTS algorithm, we finish this section with a note on the stability of the perturbed closed–loop system.

Remark 3.3: The FST controller of Theorem (3.1) does not necessarily guarantee stability of the perturbed closed–loop transfer function. Indeed, resolving equation $\delta h = \left(h - h_0\right)/h$ for *h*, we have that

$$
h = \left(1 - \delta h\right)^{-1} h_0
$$

For stability, $\left(1-\delta h\right)^{-1}$ must be stable, and this is possible ([3]), if

$$
g_\infty\left(\delta \hat{h}\right) = g_\infty\left(\left(1 - \hat{h}_0\right)\delta \hat{p}\right) < 1 \tag{3.19}
$$

This is a sufficient condition and a $\nu = \partial(t)$ for the robust FSTS problem could be chosen such that (3.19) is satisfied and so closed–loop stability is guaranteed. □

4. The Robust FSTS Algorithm and an Example

In this section we give an outline of the algorithm for the solution of the robust FST tracking problem and we illustrate its use by an example.

Assume that the transfer function of the nominal plant in the feedback configuration of Fig. (2.1) is given by $p = n_{p_0} / d_{p_0}$, (n_{p_0}, d_{p_0}) $\mathbb{R}[d]-$coprime, and the signal to be FST tracked is $u_1 = n_r / d_r$. The algorithm for the robust FST tracking is as follows:

Robust FST Algorithm

Step 1 Input n_{p_0}, d_{p_0}, d_r. Compute the common factor, d_{r_p}, of d_r and d_{p_0}, and factorize d_r and d_{p_0} as $d_r = d_{r_p} d_{r_c}$ and $d_{p_0} = d_{r_p} d_{p_{0r}}$. *Proceed* to *Step 2* if $(d_{r_c}, n_{p_0} d_{p_{0r}})$ are $\mathbb{R}[d]-$coprime. Otherwise *end* the procedure.

Step 2 Assign $m := \partial\left(n_{p_0}\right), n := \partial\left(d_{p_0}\right), \xi := \partial\left(d_{p_{0r}}\right), l := \partial\left(d_{r_c}\right)$

Step 3 Compute the *prime* FSTS controller $\tilde{c} = \tilde{n}_c / \tilde{d}_c$ (equations (2.5))

Step 4 Select $\nu := \partial\left(t\right) \geq \partial\left(d_{r_c}\right) = l$. (Start with $\nu = \partial\left(d_{r_c}\right) = l$ rather than $\nu = \partial\left(d_{r_c}\right) - 1 = l - 1$ because equation (3.13) has a unique solution for $\nu = l - 1$ and there is no need for optimization in that case)

Step 5 Compute $\mu := \partial\left(q\right) = m + \xi + \nu - l$

Step 6 Solve the linear program

$$\begin{aligned} &\text{minimize} \quad \left\| d_{p_0} d_c \right\|_1 = \left\| d_{p_0} \left(\tilde{d}_c - t n_{p_0} \right) \right\|_1 \\ &\text{subject to} \quad q d_{r_c} + t n_{p_0} d_{p_{0r}} = \tilde{d}_c d_{p_{0r}} \end{aligned} \tag{4.1}$$

with respect to *t* and *q*

Step 7 Compute the robust FST tracking controller

$$c^* = n_c^* / d_c^* : n_c^* = \tilde{n}_c + t^* d_{p_0},\ d_c^* = \tilde{d}_c - t^* n_{p_0}$$

Step 8 To improve robustness, increase the degree of *t* by one, i.e. $\nu := \nu + 1$ and go to *Step 5*. Otherwise *end* the procedure.

We should note that although it is claimed that the optimization problem (4.1), or Theorem (3.1) constitutes a linear program, this is not straightforward by its formulation. Indeed, the objective function in the optimization problem (4.1) is

$$\left\| d_{p_0} \left(\tilde{d}_c - t n_{p_0} \right) \right\|_1 = \sum_{i=0}^{m+n+\nu} \left| \left(d_{p_0} \left(\tilde{d}_c - t n_{p_0} \right)_i \right) \right| \tag{4.2}$$

where $d_{p_0} \left(\tilde{d}_c - t n_{p_0} \right)_i$ is the *i*th coefficient of $d_{p_0} \left(\tilde{d}_c - t n_{p_0} \right)$, and it does not represent a linear function with respect to *t*.

Remark 4.1: Due to relationship (4.2), problem (4.1) is not a linear problem as it stands. To alleviate that, we introduce new variables $b_i \geq 0,\ i = 0,\ldots,m+n+\nu$ such that

$$\left| d_{p_0} \left(\tilde{d}_c - tn_{p_0} \right)_i \right| \leq b_i \tag{4.3}$$

Introducing inequalities (4.3), to the optimization problem (4.1) we have

$$\begin{aligned} &\text{minimize} && \sum_{i=0}^{m+n+\nu} b_i \\ &\text{subject to} && qd_{r_c} + tn_{p_0} d_{p_0} = \tilde{d}_c d_{p_0} \\ &&& \left(tn_{p_0} d_{p_0} \right)_i + b_i \geq \left(\tilde{d}_c d_{p_0} \right)_i,\ i = 0,\ldots,m+n+\nu \\ &&& \left(tn_{p_0} d_{p_0} \right)_i - b_i \leq \left(\tilde{d}_c d_{p_0} \right)_i,\ i = 0,\ldots,m+n+\nu \\ &&& b_i \geq 0,\ i = 0,\ldots,m+n+\nu \end{aligned} \tag{4.4}$$

Hence, the optimization problem (4.4) constitutes a linear program equivalent to optimization problem (4.1) and we are using this linear program (4.4) in the algorithm for robust FSTS. □

Before illustrating the robust FSTS algorithm with an example we will elaborate briefly about the nature of the robustness index ρ and a possible strategy of stopping criteria; a full investigation of the latter is beyond the scope of this work and a matter of further research on the rate of convergence which is directly related to the nature of the specific linear programme in each problem case.

Remark 4.2: The robustness index ρ is given by the following alternative relationships:

$$\rho := \left\| 1 - h_0 \right\|_1 = \left\| \left(1 + p_0 c \right)^{-1} \right\|_1 = \left\| s_0 \right\|_1$$

where s_0 is the sensitivity of the nominal configuration of Fig. (2.1) and is not more than the nominal error transfer function from u_1 to e_1. Therefore, using $\min\{\rho\}$ as an upper bound of the l^∞ induced norm of the relative difference between the nominal and actual transfer functions from input u_1 to output y_2 is equivalent to the minimisation of the l^1–norm of the nominal error transfer function from u_1 to e_1. □

Remark 4.3: The number of iterations *k* of the robust algorithm is linearly related to the degree $\nu = \partial(t)$ of the free parameter $t \in \mathbb{R}[d]$ and to that extend to the complexity of the FST controller (McMillan degree) and the duration of the nominal error response. Indeed, if

$$m := \partial\left(n_{p_0}\right), n := \partial\left(d_{p_0}\right), \xi := \partial\left(d_{p_{0r}}\right), l := \partial\left(d_{r_c}\right)$$

and at the *k*-th iteration

$$\nu := \partial(t), \mu := \partial(q) = m + \xi + \nu - l \tag{4.5}$$

Then

1. $k = \partial(t) - l + 1 = \nu - l + 1$ (4.6)
2. The McMillan degree $\delta_{\mathfrak{M}}(c)$ of the FST tracking controller $c(d)$ is given by
$$\delta_{\mathfrak{M}}(c) = \partial(t) + \max\{m, n\} = k + l - 1 \ \ [4]$$
3. The duration of the nominal error response $e_1(d) \in \mathbb{R}[d]$ to $u_1(d) = n_r(d)/d_r(d)$ is given by $\partial(e_1)+1$, i.e. e_1 becomes exactly zero after $\tau = \partial(e_1)+1$ steps. This relates to the number k of iterations as follows:
$$e_1 = d_c d_{p_0} u_1 \text{, and due to Equation (2.6) } e_1 = q d_r \frac{n_r}{d_r} = q n_r$$
Therefore
$$\partial(e_1) = \partial(q) + \partial(n_r) = k + m + \xi - 1 + \partial(n_r) \text{ due to equations (4.5) and (4.6)}$$
So
$$\tau = \partial(e_1) + 1 = k + m + \xi + \partial(n_r)$$

□

Taking into consideration that the optimum robustness index ρ_ν^* is monotonically decreasing with respect to $\nu = \partial(t)$ we can establish the following proposition.

Proposition 4.1: Depending on the nature of the robust FST problem, we can use alternatively, or simultaneously, the following stopping criteria for the robust FST algorithm:

1. $\rho_\nu^* \le \rho_{\min}$, where $\rho_{\min}$ is an upper bound of the l^1–norm of the nominal sensitivity.
2. $k \le k_{\max}$, where $k_{\max}$ is an absolute maximum of the number of iterations, or is related to an upper bound on the McMillan degree of the FST controller, or the settling time on the error signal in the nominal case.

□

We illustrate now the robust FSTS algorithm by the following example.

Robust FSTS Example

Consider the unity feedback configuration of Fig. (2.1) with the nominal plant
$$p_0(d) = \frac{n_{p_0}(d)}{d_{p_0}(d)} = \frac{-0.0132d - 0.0139d^2}{1 - 2.1889d + 1.1618d^2} \tag{4.7}$$
The system is subjected to a parabolic input of the form
$$u_1(d) = u_r(d) = \frac{n_r(d)}{d_r(d)} = \frac{d(d+1)}{(1-d)^3} \tag{4.8}$$
which has to be robustly tracked in FST sense.

Step 1 Input n_{p_0}, d_{p_0}, d_r; (d_r, d_{p_0}) are $\mathbb{R}[d]-$coprime, therefore $d_{p_{0r}} = d_{p_0}$, $d_{p_r} = 1$ and $d_{r_c} = d_r = (1-d)^3$. $(d_{r_c}, n_{p_0} d_{p_{0r}}) = (d_r, n_{p_0} d_{p_0})$ are $\mathbb{R}[d]-$coprime, so *proceed* to *Step 2*.

Step 2
$$m := \partial\left(n_{p_0}\right) = 2, n := \partial\left(d_{p_0}\right) = 2, \xi := \partial\left(d_{p_{0r}}\right) = \partial\left(d_{p_0}\right) = 2$$
$$l := \partial\left(d_{r_c}\right) = \partial\left(d_r\right) = 3$$

Step 3 The *prime* FSTS controller is $\tilde{c} = \dfrac{\tilde{n}_c(d)}{\tilde{d}_c(d)} = \dfrac{g_0 + g_1 d}{f_0 + f_1 d}$ where

$$\begin{bmatrix} g_0 \\ g_1 \\ \hline f_0 \\ f_1 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ -0.0132 & 0 & -2.1889 & 1 \\ -0.0139 & -0.0132 & 1.1618 & -2.1889 \\ 0 & -0.0139 & 0 & 1.1618 \end{array}\right]^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} -105.3836 \\ 66.6854 \\ \hdashline 1 \\ 0.7978 \end{array}\right] \quad (4.9)$$

Therefore, $\tilde{c} = \dfrac{\tilde{n}_c(d)}{\tilde{d}_c(d)} = \dfrac{-105.3836 + 66.6854d}{1 + 0.7978d}$

Step 4 $\nu = \partial\left(t\right) = \partial\left(d_{r_c}\right) = l = 3$

Step 5 $\mu = \partial\left(q\right) = m + n_{pr} + \nu - l = 4$

Step 6 Solve the linear program

$$\begin{aligned} &\text{minimize} && \sum_{i=0}^{4+\nu} b_i \\ &\text{subject to} && q d_r + t n_{p_0} d_{p_0} = \tilde{d}_c d_{p_0} \\ & && \left(t n_{p_0} d_{p_0}\right)_i + b_i \ge \left(\tilde{d}_c d_{p_0}\right)_i,\ i = 0, \dots, 4+\nu \\ & && \left(t n_{p_0} d_{p_0}\right)_i - b_i \le \left(\tilde{d}_c d_{p_0}\right)_i,\ i = 0, \dots, 4+\nu \\ & && b_i \ge 0,\ i = 0, \dots, 4+\nu \end{aligned} \quad (4.10)$$

For $\nu = \partial\left(t\right) = 3$ and $\mu = \partial\left(q\right) = 4$,

$t^*(d) = -151.9 + 41.49d + 102.7d^2 - 58.59d^3$

Step 7 The robust FST tracking controller is

$$c^* = n_c^* / d_c^* : n_c^* = \tilde{n}_c + t^* d_{p_0},\ d_c^* = \tilde{d}_c - t^* n_{p_0}$$

and in the case of $\nu = \partial\left(t\right) = 3$ and $\mu = \partial\left(q\right) = 4$

$$c^* = \frac{n_c^*}{d_c^*} = \frac{-257.3 + 440.7d - 164.6d^2 - 235.2d^3 + 247.7d^4 - 68.15d^5}{1 - 1.207d - 1.563d^2 + 1.933d^3 + 0.6532d^4 - 0.8154d^5} \quad (4.11)$$

and $\rho^* = \left\| d_{p_0} d_c^* \right\|_1 = 19.4733$

Step 8 If $\nu = 50$, *end* the procedure (48 iterations, just to observe the tendency of the robustness index ρ^*). Otherwise, $\nu := \nu + 1$ and *go* to *Step 5*.

The optimum robustness index $\rho^* = \left\| d_{p_0} d_c^* \right\|_1$ as a function of $\nu = \partial(t^*)$, is shown in Fig. (4.1) and it is a decreasing function with respect to $\nu = \partial(t^*)$, as expected.

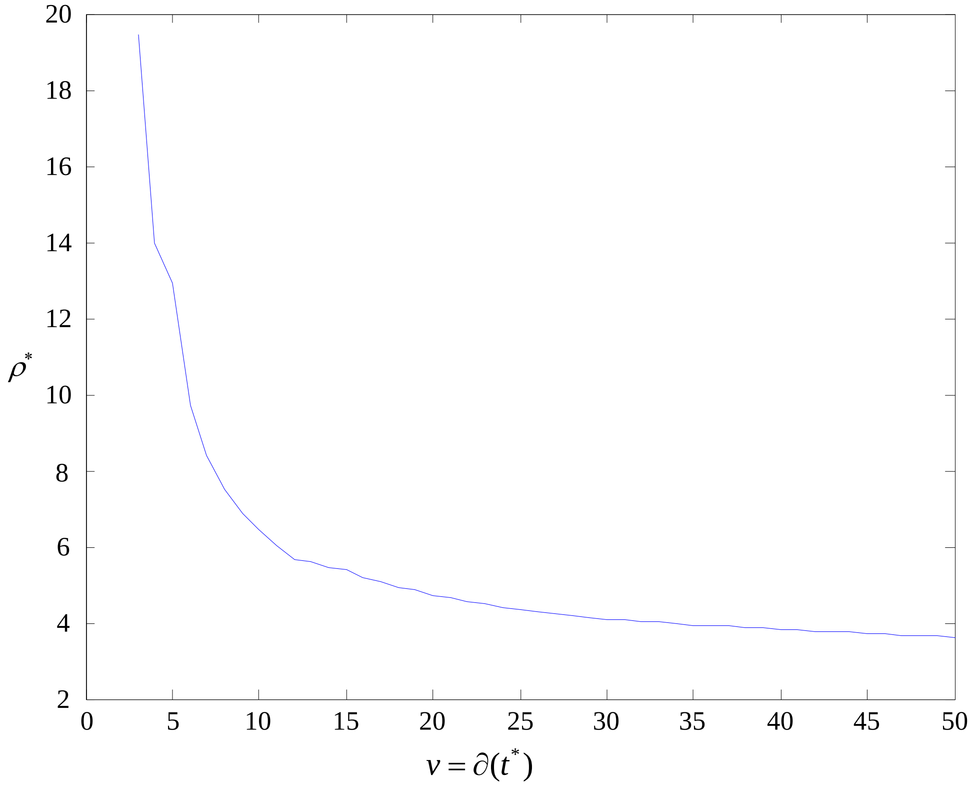

Fig.4.1: Robustness index as a function of the degree of t^*

In order to examine the robustness of the feedback system we apply to the nominal plant multiplicative perturbations of the form

$$\delta p = \frac{b}{1-ad}, \quad |a| < 1 \tag{4.12}$$

We consider the error responses of the perturbed plant, for the following cases

(a)	$\nu = \partial(t^*) = 3$	$a = 0.8$	$b = 0.05$
(b)	$\nu = \partial(t^*) = 3$	$a = 0.8$	$b = 0.10$
(c)	$\nu = \partial(t^*) = 6$	$a = 0.8$	$b = 0.05$
(d)	$\nu = \partial(t^*) = 6$	$a = 0.8$	$b = 0.10$
(e)	$\nu = \partial(t^*) = 9$	$a = 0.8$	$b = 0.10$
(f)	$\nu = \partial(t^*) = 9$	$a = 0.8$	$b = 0.25$

Table 4.1: Different cases of multiplicative perturbations and robust FST design

The resulting system responses for each case are shown in Figures (4.2)-(4.4). The robustness of the system improves with settling time. With $\nu = \partial(t^*) = 3$ and $\nu = \partial(t^*) = 6$ the system can withstand disturbances expressed by a maximum value of b=0.10, whereas with $\nu = \partial(t^*) = 9$, the value of b increases to 0.25.

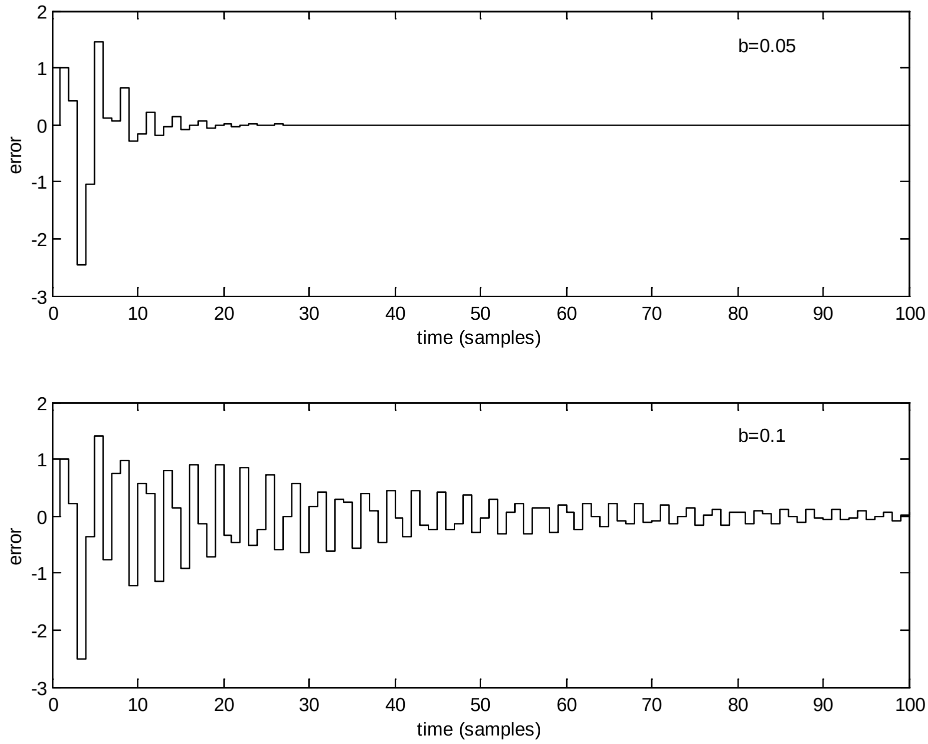

Fig. 4.2: Error responses of the disturbed system with $\nu = \partial(t^*) = 3$

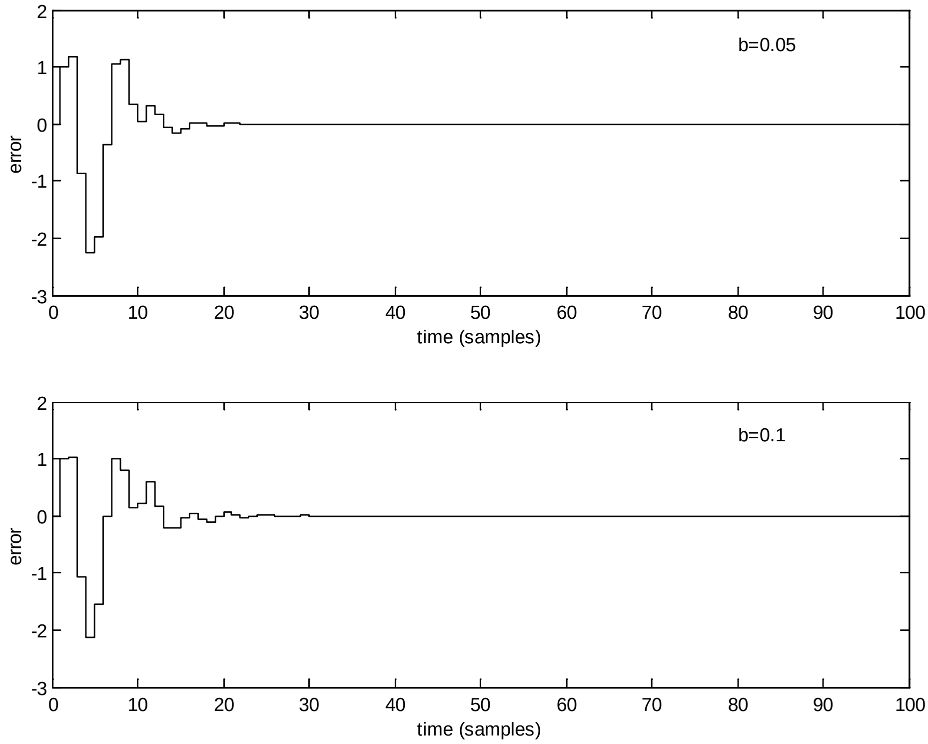

Fig.4.3: Error responses of the disturbed system with $\nu = \partial(t^*) = 6$

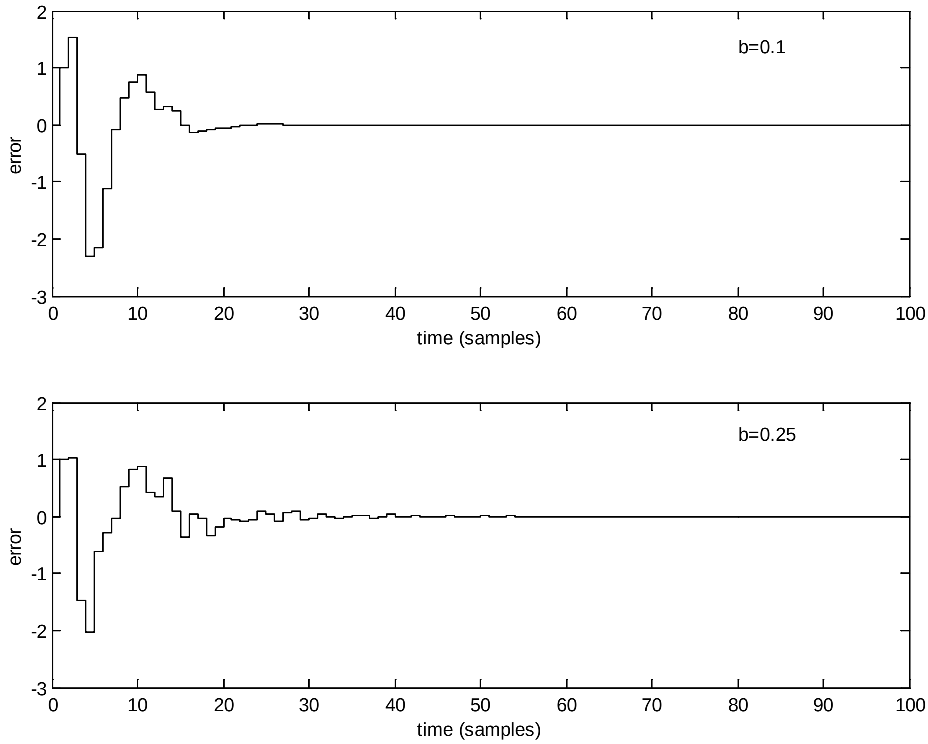

Fig. 4.4: Error responses of the disturbed system with $\nu = \partial(t^*) = 9$

From the error responses of the robust system it seems that the closed–loop system achieves a “close” to an FST response especially as the robustness increases. Dynamically we may assume that this means that the *z*–poles of the robust system are kept close to the origin so it resembles a Finite Impulse Response system. In fact this is not what is happening as the pole–zero maps indicate for the different perturbed plants of Table (4.1).

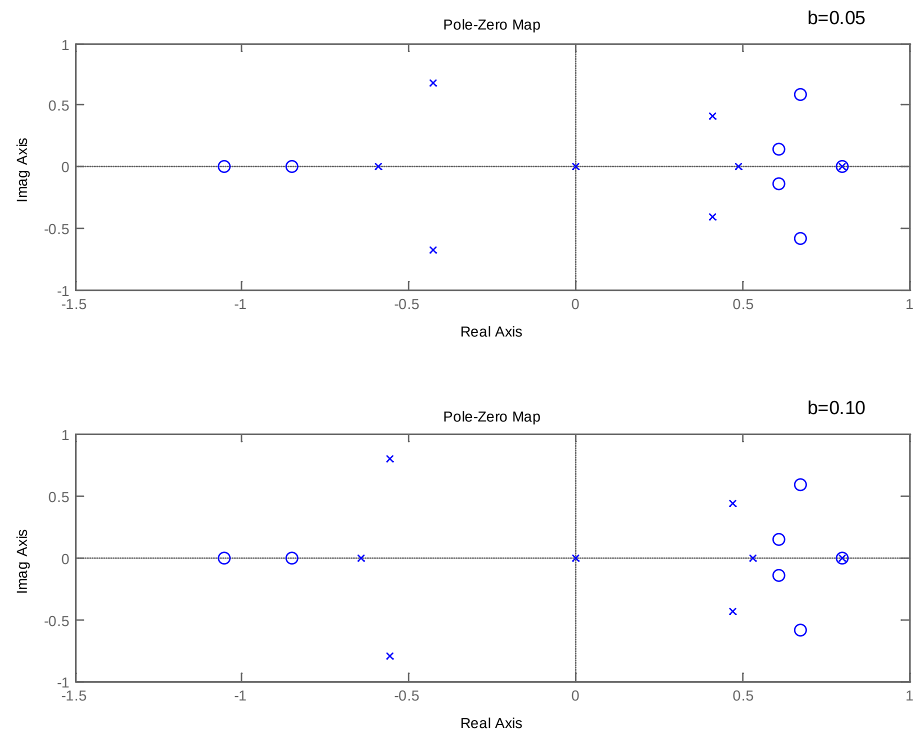

Fig. 4.5: Pole–zero map of the disturbed system with $\nu = \partial(t^*) = 3$

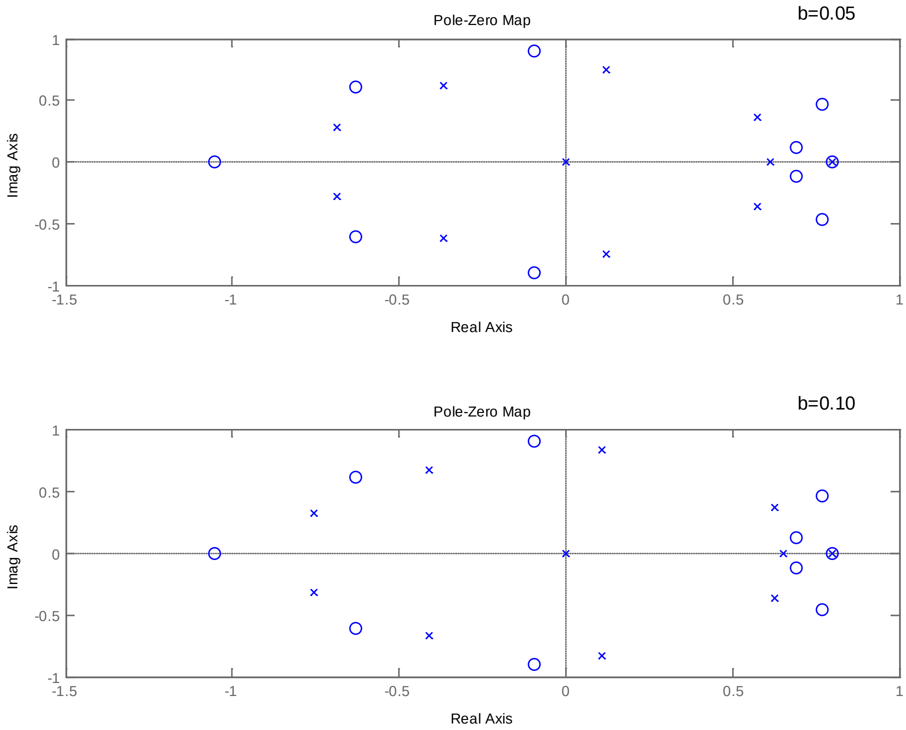

Fig. 4.6: Pole–zero map of the disturbed system with $\nu = \partial(t^*) = 6$

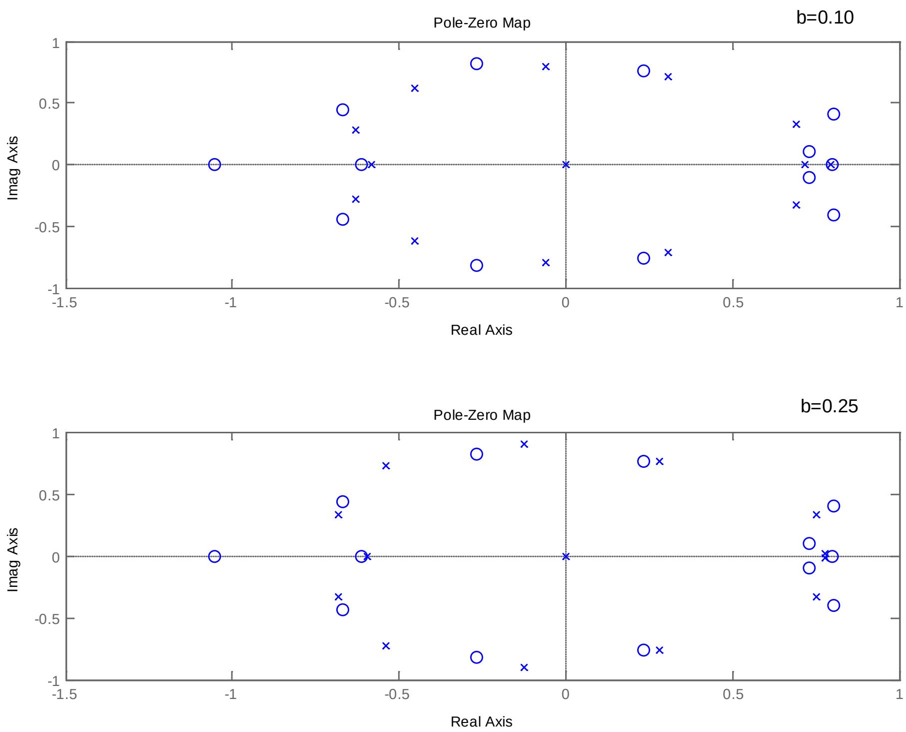

Fig. 4.7: Pole–zero map of the disturbed system with $\nu = \partial(t^*) = 9$

Instead of the poles of the robust system being kept close to the origin, there is only one “dominant” pole at the origin. The rest of the poles and zeros are “scattered” in such a way so that the residues of the non–zero modes are kept small and the system behaves “close” to an FSTS system. As the robustness increases, by increasing the degree of the robust controller, the non–zero poles move closer to the zeros with some resulting to almost pole–zero cancelations.

Remark 4.4: Tracking FSTS controllers observe the *“internal model principle”* due to the divisibility condition $d_r \mid d_c d_{p_0}$. This principle is still valid in the disturbed case, when the input dynamics are not partially part of the dynamics of the nominal plant, resulting in a zero steady–state error response, but it is violated otherwise. If zero steady–state error is a requirement, then the input dynamics should always be included in the dynamics of the FSTS controller to the expense of the controller complexity and nominal settling time. □

5. Conclusions

Finite Settling Time Stabilization (FSTS) is sensitive to plant parameter variations and model inaccuracy. In the present work we considered multiplicative plant perturbations and the robustness problem within the FSTS framework. Initially, the class of all FST stabilizing controllers that track a specific input is derived in terms of the constraint condition on the free parameter in a YBJK parametrization. The fact that all tracking FSTS controllers are parametrized linearly with respect to the free parameter, gives rise to an efficient robustness index which leads to the minimization of the l^1-norm of the sensitivity function. The robust FSTS problem is further reduced to a linear programme where all the benefits of the linear programming can be exploited.

Based on these results, a design procedure is proposed and an optimal robust FST stabilizing controller is extracted. The effectiveness of the method is illustrated by a numerical example. Due to the nature of the linear program, the robustness of the system can be improved to the expense of the finite settling time. The problem of robust MIMO FSTS is under consideration.

Appendix

We give a proof of relationship (3.2). Consider the unity feedback scheme of Fig. (2.1) and let p_0 denote the transfer function of the nominal plant and p the dynamics of the actual plant. Under multiplicative perturbation we may have

$$p - p_0 = \delta p \cdot p, \quad \delta p \;\; l^\infty - \text{stable} \tag{A.1}$$

Suppose that

$$\delta h := \left(h - h_0\right) / h \tag{A.2}$$

is the relative error between the actual transfer function h from u_1 to y_2 and the corresponding nominal one, h_0. Then

$$\delta h = (1-h_0)\delta p \tag{A.3}$$

Indeed

$$\begin{aligned} h-h_0 &= \frac{pc}{1+pc} - \frac{p_0c}{1+p_0c} = \frac{(p_0-p)c}{(1+p_0c)(1+pc)} \\ &= \frac{\delta p \cdot pc}{(1+p_0c)(1+pc)} \quad \text{(due to (A.1))} \\ &= \frac{1}{1+p_0c}\delta p \cdot h = \left(1-\frac{p_0c}{1+p_0c}\right)\delta p \cdot h \\ &= \left(1-h_0\right)\delta p \cdot h \end{aligned}$$

Therefore $$\delta h := \left(h-h_0\right)/h = (1-h_0)\delta p$$

References

[1] Dahleh M.A. and J.B. Pearson, "l_1 optimal feedback controllers for MIMO discrete – time systems". *IEEE Trans. Auto. Control*, AC-32, April 1987, pp. 314-322.

[2] Dahleh M.A. and J.B. Pearson, "Optimal rejection of persistent disturbances, robust stability, and mixed sensitivity minimization". *IEEE Trans. Auto. Control*, AC-33, No. 8, Aug. 1988, pp. 722-731.

[3] Desoer C.A. and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York Academic, 1975.

[4] Karcanias N. and E. Milonidis, "Total finite settling time stabilization for discrete time SISO systems". *Proc. 5th IMA Control Theory Conf.*, Univ. of Strathclyde Glasgow, 1988. Also, in *The Mathematics of Control Theory*. (N. K. Nichols and D. H. Owens, (Ed.)), Clarendon Press, Oxford, pp. 71-85, 1992.

[5] Karcanias N. and E. Milonidis, "Finite settling time stabilization of a family of discrete time plants". *Kybernetica*, vol. 27, no. 4, pp. 171-183, 1991.

[6] Kučera V., *Discrete Linear Control: The polynomial equation approach*, John Wiley and Sons Inc., London, 1979.

[7] Kučera V., *Analysis and Design of Discrete Linear Control*, Hemel Hempstead, UK: Prentice Hall International (UK) Ltd, 1991.

[8] Kung S.-Y., T. Kailath and M. Morf, "A generalized resultant matrix for polynomial matrices", *CDC'76*, Florida, Dec. 1976, pp. 892-895.

[9] Milonidis E. and N. Karcanias, "Finite settling time stabilisation for multivariable discrete – time systems: a polynomial equation approach". *Inter. Journal of Control*, Vol. 79, No. 12, Dec. 2006, pp. 1565-1580.

[10] Zhao Y. and H. Kimura. "Dead–beat control with robustness". *Int. J. Control*, Vol. 43, No. 5, 1986, pp.1427-1440.

[11] Zhao Y. and H. Kimura. "Multivariable dead–beat control with robustness". *Int. J. Control*, Vol. 47, No. 1, 1988, pp. 229-255.
[12] Zhao Y. and H. Kimura. "Two–degrees–of–freedom dead–beat control system with robustness". *Int. J. Control*, Vol. 48, No. 1, 1988, pp. 303-315.
[13] Zhao Y. and H. Kimura. "Two–degrees–of–freedom dead–beat control system with robustness: Multivariable case". *Int. J. Control*, Vol. 49, No. 2, 1989, pp. 667-679.