# A version of Alperin's weight conjecture for finite category algebras

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**Abstract**

Let $p$ be a prime and $k$ an algebraically closed field of characteristic $p$. We construct a functor $\mathcal{C} \to \mathcal{O}_\mathcal{C}$ on the category of finite categories with the property that if $G = \mathcal{C}$ is a finite group, then $\mathcal{O}_\mathcal{C}$ is the orbit category of $p$-subgroups of $G$. This leads to an extension of Alperin's weight conjecture to any finite category $\mathcal{C}$, stating that the number of isomorphism classes of simple $k\mathcal{C}$-modules should be equal to that of the weight algebra $W(k\mathcal{O}_\mathcal{C})$ of $\mathcal{O}_\mathcal{C}$. We show that the versions of Alperin's weight conjecture for finite groups and for finite categories are in fact equivalent.

## 1 Introduction

Throughout this paper we fix a prime number $p$ and an algebraically closed field $k$ of characteristic $p$. A category $\mathcal{C}$ is called *finite* if its morphism class is a finite set.

**Definition 1.1.** Let $\mathcal{C}$ be a finite category. The *$p$-transporter category of $\mathcal{C}$* is the finite category $\mathcal{T}_\mathcal{C}$ defined as follows. The objects of $\mathcal{T}_\mathcal{C}$ are the pairs $(X, Q)$ consisting of an object $X$ of $\mathcal{C}$ and a not necessarily unitary $p$-subgroup $Q$ of the monoid $\mathrm{End}_\mathcal{C}(X)$. For any two objects $(X, Q)$, $(Y, R)$, the morphism set $\mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X, Q), (Y, R))$ is the set of all triples $(s, Q, R)$ where $s : X \to Y$ is a morphism in $\mathcal{C}$ satisfying $s = s \circ 1_Q = 1_R \circ s$ and $s \circ Q \subseteq R \circ s$. The composition of morphisms in $\mathcal{T}_\mathcal{C}$ is induced by that in $\mathcal{C}$.

The identity morphism of an object $(X, Q)$ in $\mathcal{T}_\mathcal{C}$ is $(1_Q, Q, Q)$. On occasion, and if no confusion arises, we will denote a morphism $(s, Q, R)$ in $\mathcal{T}_\mathcal{C}$ again by $s$ and simply specify that $s$ is viewed as a morphism in $\mathcal{T}$ rather than $\mathcal{C}$. Allowing for nonunitary subgroups $Q$ of $\mathrm{End}_\mathcal{C}(X)$ in the definition of objects of $\mathcal{T}$ means that the unit element $1_Q$ of $Q$ need not be equal to $\mathrm{Id}_X$ but can be any idempotent endomorphism of $X$. The condition $s = s \circ 1_Q = 1_R \circ s$ in this definition implies that $\mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X, Q), (Y, R))$ can be identified to a subset of $1_R \circ \mathrm{Hom}_\mathcal{C}(X, Y) \circ 1_Q$. With this identification, the morphism set $\mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X, Q), (Y, R))$ is an $R$-$Q$-subbiset of $1_R \circ \mathrm{Hom}_\mathcal{C}(X, Y) \circ 1_Q$ with respect to the actions induced by precomposing with morphisms in $Q$ and composing with morphisms in $R$. The condition $s \circ Q \subseteq R \circ s$ in the above definition implies that $R \circ s \circ Q = R \circ s$; that is, any $R$-$Q$-orbit in $\mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X, Q), (Y, R))$ is in fact an $R$-orbit.

**Definition 1.2.** Let $\mathcal{C}$ be a finite category. The *$p$-orbit category of $\mathcal{C}$* is the finite category $\mathcal{O}_\mathcal{C}$ defined as follows. The objects of $\mathcal{O}_\mathcal{C}$ are the same as those of $\mathcal{T}_\mathcal{C}$. For any two objects $(X, Q)$, $(Y, R)$ of $\mathcal{O}_\mathcal{C}$, the morphism set $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}((X, Q), (Y, R))$ is the set of all $R$-$Q$-orbits

$R\backslash\mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X,Q),(Y,R))/Q$ of morphisms in $\mathcal{T}_\mathcal{C}$. The composition of morphisms in $\mathcal{O}_\mathcal{C}$ is induced by that in $\mathcal{T}_\mathcal{C}$.

For a finite-dimensional $k$-algebra $A$, denote by $\ell(A)$ the number of isomorphism classes of simple $A$-modules. Following [8] and [9], given a finite category $\mathcal{C}$, the isomorphism classes of simple $k\mathcal{C}$-modules are parametrised by isomorphism classes of pairs $(e,T)$, with $e$ an idempotent endomorphism of some object $X$ in $\mathcal{C}$ and $T$ a simple $kG_e$-module, where $G_e$ is the group of all invertible elements in the monoid $e \circ \mathrm{End}_\mathcal{C}(X) \circ e$. Such a pair $(e,T)$ is called a *weight* if the simple $kG_e$-module $T$ is in addition projective. The associated *weight algebra* $W(k\mathcal{C})$ is an algebra of the form $W(k\mathcal{C}) = c \cdot k\mathcal{C} \cdot c$ for some idempotent $c$ which acts as the identity on all simple $k\mathcal{C}$-modules parametrised by a weight and which annihilates all simple modules not parametrised by a weight; we review this in §2 below. The group-theoretic version of Alperin's weight conjecture in [1] is equivalent to the statement that there should be an equality $\ell(kG) = \ell(W(k\mathcal{O}_G))$, for any finite group $G$. This leads to the obvious extension of Alperin's weight conjecture to finite categories:

**Conjecture 1.3.** *For any finite category $\mathcal{C}$ we have $\ell(k\mathcal{C}) = \ell(W(k\mathcal{O}_\mathcal{C}))$ .*

Theorem 1.4 below implies that the versions of Alperin's weight conjecture for finite groups and for finite categories are in fact equivalent. This equivalence holds more generally for twisted group algebras and twisted category algebras with a 2-cocycle which is extendible to the orbit category. There are canonical functors from $\mathcal{C}$ to $\mathcal{T}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ sending an object $X$ in $\mathcal{C}$ to $(X,\{\mathrm{Id}_X\})$, and a morphism in $\mathcal{C}$ to its obvious images in $\mathcal{T}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$, respectively. In particular, every 2-cocycle $\alpha$ in $Z^2(\mathcal{O}_\mathcal{C};k^\times)$ restricts to a 2-cocycle in $Z^2(\mathcal{C};k^\times)$, again denoted by $\alpha$.

**Theorem 1.4.** *Let $\mathcal{C}$ be a finite category and $\alpha \in Z^2(\mathcal{O}_\mathcal{C};k^\times)$. If for any idempotent endomorphism $e$ in $\mathcal{C}$ we have $\ell(k_\alpha G_e) = \ell(W(k_\alpha \mathcal{O}_{G_e}))$, then $\ell(k_\alpha\mathcal{C}) = \ell(W(k_\alpha\mathcal{O}_\mathcal{C}))$.*

**Corollary 1.5.** *Alperin's weight conjecture holds for all finite categories if and only if it holds for all finite groups.*

For a fixed finite category $\mathcal{C}$, the converse of Theorem 1.4 could be false in the rather extreme scenario where $\mathcal{C}$ has more than one maximal subgroup for whom Alperin's weight conjecture is false, in such a way that the sum total of errors at individual subgroups is zero. Note that the formulation for twisted category algebras requires the 2-cocycle $\alpha$ to be the restriction to $\mathcal{C}$ of a 2-cocycle of $\mathcal{O}_\mathcal{C}$ along the canonical functor $\mathcal{C} \to \mathcal{O}_\mathcal{C}$. It is not clear whether the map $H^2(\mathcal{O}_\mathcal{C};k^\times) \to H^2(\mathcal{C};k^\times)$ induced by the canonical functor $\mathcal{C} \to \mathcal{O}_\mathcal{C}$ is injective or surjective in general. The corresponding canonical functor $\mathcal{C} \to \mathcal{T}_\mathcal{C}$ sending $X$ to $(X,\{\mathrm{Id}_X\})$ poses no problem:

**Proposition 1.6.** *Let $\mathcal{C}$ be a finite category. The canonical functor $\mathcal{C} \to \mathcal{T}_\mathcal{C}$ induces a graded isomorphism $H^*(\mathcal{T}_\mathcal{C};k^\times) \cong H^*(\mathcal{C};k^\times)$.*

This will be proved in 4.1 below. It seems less clear what happens under the functor $\mathcal{T}_\mathcal{C} \to \mathcal{O}_\mathcal{C}$ in general. We have the following special case for $EI$-categories (that is, categories in which all endomorphisms are isomorphisms):

**Proposition 1.7.** *Let $\mathcal{C}$ be a finite $EI$-category. Then $\mathcal{T}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are $EI$-categories, and the canonical functor $\mathcal{C} \to \mathcal{O}_\mathcal{C}$ induces a graded isomorphism $H^*(\mathcal{O}_\mathcal{C};k^\times) \cong H^*(\mathcal{C};k^\times)$.*

This will be proved at the end of section 4, using some of the properties of regular functors between $EI$-categories from [7].

**Remarks 1.8.** Let $\mathcal{C}$ be a finite category.

**(1)** Aside from the canonical functor $\mathcal{C} \to \mathcal{T}_\mathcal{C}$ sending an object $X$ in $\mathcal{C}$ to $(X, \{\mathrm{Id}_X\})$ and sending a morphism $s : X \to Y$ in $\mathcal{C}$ to $s$ viewed as a morphism from $(X, \{\mathrm{Id}_X\})$ to $(Y, \{\mathrm{Id}_Y\})$, there is also a canonical functor $\mathcal{T}_\mathcal{C} \to \mathcal{O}_\mathcal{C}$ which is the identity on objects and which sends a morphism $s : (X, Q) \to (Y, R)$ in $\mathcal{T}_\mathcal{C}$ to the $R$-$Q$-orbit of $s$. There is a canonical 'forgetful' functor $\mathcal{T}_\mathcal{C} \to \mathcal{C}$ which sends an object $(X, Q)$ in $\mathcal{T}_\mathcal{C}$ to $X$ and which sends a morphism $(s, Q, R) : (X, Q) \to (Y, R)$ in $\mathcal{T}_\mathcal{C}$ to the underlying morphism $s : X \to Y$.

**(2)** If $p$ is coprime to the orders of all subgroups of endomorphism monoids of $\mathcal{C}$, then $\mathcal{T}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are both isomorphic to the idempotent completion $\mathcal{C}'$ of $\mathcal{C}$. In that case, $W(k\mathcal{O}_\mathcal{C}) \cong k\mathcal{C}'$ is Morita equivalent to $k\mathcal{C}$, and hence Alperin's weight conjecture holds trivially for $k\mathcal{C}$. In general, the idempotent completion of $\mathcal{C}$ is isomorphic to the full subcategories of $\mathcal{T}$ and of $\mathcal{O}$, consisting of the objects of the form $(X, \{e\})$, where $X$ is an object in $\mathcal{C}$ and $e$ an idempotent endomorphism of $X$.

**(3)** If $\mathcal{C}$ is a finite group $G$, viewed as a category with one object, then $\mathcal{T}$ is the transporter category of $p$-subgroups of $G$ and $\mathcal{O}$ the corresponding orbit category.

**(4)** The correspondences sending a finite category $\mathcal{C}$ to its $p$-transporter category $\mathcal{T}_\mathcal{C}$ and $p$-orbit category $\mathcal{O}_\mathcal{C}$ extend to functors $\mathbf{T}$ and $\mathbf{O}$, respectively, on the category **cat** of finite categories: if $\Phi : \mathcal{C} \to \mathcal{D}$ is a functor between finite categories $\mathcal{C}$, $\mathcal{D}$, then $\Phi$ induces a functor $\mathcal{T}_\Phi : \mathcal{T}_\mathcal{C} \to \mathcal{T}_\mathcal{D}$ sending an object $(X, Q)$ in $\mathcal{T}_\mathcal{C}$ to the object $(\Phi(X), \Phi(Q))$ in $\mathcal{T}_\mathcal{D}$, with the obvious maps induced by $\Phi$ on morphism sets. Similarly, $\Phi$ induces a functor $\mathcal{O}_\Phi : \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ sending an object $(X, Q)$ in $\mathcal{O}_\mathcal{C}$ to the object $(\Phi(X), \Phi(Q))$ in $\mathcal{O}_\mathcal{D}$. The canonical functors $\mathcal{C} \to \mathcal{T}_\mathcal{C} \to \mathcal{O}_\mathcal{C}$ are natural in $\mathcal{C}$, hence induce natural transformations $\mathrm{Id}_{\mathbf{cat}} \to \mathbf{T} \to \mathbf{O}$ of functors on **cat**.

**(5)** The above version of Alperin's weight conjecture for categories raises the question whether the reformulation, due to Knörr and Robinson [5], of Alperin's weight conjecture for blocks in terms of alternating sums can be extended to twisted category algebras along similar lines.

**Examples 1.9.** Brauer algebras, Temperley-Lieb algebras, partition algebras, and their cyclotomic analogues can be interpreted as twisted monoid algebras (cf. [12]). The 2-cocycles of the underlying monoids for these algebras satisfy the hypotheses of 4.6 below; in particular, they are constant on maximal subgroups (so that their restrictions to maximal subgroups represent the trivial classes) and they extend to the associated orbit categories. Using the fact, due to Alperin and Fong [2], that Alperin's weight conjecture holds for symmetric groups it is easy to see that the maximal subgroups of the underlying diagram monoids satisfy Alperin's weight conjecture, and hence so do these diagram algebras.

Alperin's weight conjecture for block algebras of finite groups leads to considering twisted category algebras over (the subcategory of centrics in) fusion systems (see e. g. [6]). It is not clear, whether there is an analogous extension to block algebras of finite category algebras; for the sake of completeness, we include a generalisation of fusion systems and their orbit categories from finite groups to finite categories.

**Definition 1.10.** Let $\mathcal{C}$ be a finite category. The *$p$-fusion category of* $\mathcal{C}$ is the finite category $\mathcal{F}_\mathcal{C}$ defined as follows. The objects of $\mathcal{F}_\mathcal{C}$ are the same as those of $\mathcal{T}_\mathcal{C}$. For any two objects $(X, Q)$,

$(Y,R)$ of $\mathcal{F}_\mathcal{C}$, the morphism set $\mathrm{Hom}_{\mathcal{F}_\mathcal{C}}((X,Q),(Y,R))$ is the set of all group homomorphisms $\varphi : Q \to R$ for which there exists an element $(s,Q,R) \in \mathrm{Hom}_{\mathcal{T}_\mathcal{C}}((X,Q),(Y,R))$ satisfying $s \circ u = \varphi(u) \circ s$ for all $u \in Q$. The composition of morphisms in $\mathcal{F}_\mathcal{C}$ is induced by the usual composition of group homomorphisms.

The group homomorphism $\varphi$ satisfying $s\circ u = \varphi(u)\circ s$ for all $u \in Q$ is not necessarily uniquely determined by $s$. Clearly any inner automorphism of $Q$ is an automorphism of $(X,Q)$ in $\mathcal{F}_\mathcal{C}$, and hence $\mathrm{Hom}_{\mathcal{F}_\mathcal{C}}((X,Q),(Y,R))$ is an $\mathrm{Inn}(R)$-$\mathrm{Inn}(Q)$-biset through (pre-)composition with inner automorphisms of $Q$ and $R$. If $u \in Q$ and $s$, $\varphi$ are as before, then $s \circ c_u = c_{\varphi(u)} \circ s$, where $c_u$, $c_{\varphi(u)}$ are the inner automorphisms of $Q$, $R$ given by conjugation with $u$, $\varphi(u)$, respectively, and hence an $\mathrm{Inn}(R)$-$\mathrm{Inn}(Q)$-orbit in $\mathrm{Hom}_{\mathcal{F}_\mathcal{C}}((X,Q),(Y,R))$ is an $\mathrm{Inn}(R)$-orbit.

**Definition 1.11.** Let $\mathcal{C}$ be a finite category. The *$p$-fusion orbit category of* $\mathcal{C}$ is the finite category $\bar{\mathcal{F}}_\mathcal{C}$ defined as follows. The objects of $\mathcal{F}_\mathcal{C}$ are the same as those of $\mathcal{T}_\mathcal{C}$. For any two objects $(X,Q)$, $(Y,R)$ of $\mathcal{F}_\mathcal{C}$, the morphism set $\mathrm{Hom}_{\bar{\mathcal{F}}_\mathcal{C}}((X,Q),(Y,R))$ is the set of $\mathrm{Inn}(R)$-$\mathrm{Inn}(Q)$-orbits in $\mathrm{Hom}_{\mathcal{F}_\mathcal{C}}((X,Q),(Y,R))$. The composition of morphisms in $\bar{\mathcal{F}}_\mathcal{C}$ is induced by that in $\mathcal{F}_\mathcal{C}$.

# 2 Background material

Let $\mathcal{C}$ be a finite category. The set of idempotent endomorphisms of objects in $\mathcal{C}$ is partially ordered, with partial order given by $e \le f$ whenever $e$, $f$ are idempotent endomorphisms of an object $X$ in $\mathcal{C}$ satisfying $e = e\circ f = f \circ e$. Two idempotent endomorphisms $e$, $f$ of objects $X$, $Y$, respectively, are called *isomorphic* if there are morphisms $s : X \to Y$ and $t : Y \to X$ satisfying $t \circ s = e$ and $s \circ t = f$. In that case, $s$ and $t$ can be chosen such that $s = f \circ s = s \circ e$ and $t = e \circ t = t \circ f$; indeed, using that $e$, $f$ are idempotents, we have $e = t \circ s = e \circ t \circ s \circ t \circ s \circ e = (e\circ t\circ f)\circ(f\circ s\circ e)$, and a similar argument yields $f = (f\circ s\circ e)\circ(e\circ t\circ f)$. Let $\alpha$ be a 2-cocycle in $Z^2(\mathcal{C}; k^\times)$; that is, $\alpha$ is a map sending any two morphisms $s$, $t$ in $\mathrm{Mor}(\mathcal{C})$ for which $t\circ s$ is defined to an element $\alpha(t,s)$ in $k^\times$ such that for any three morphisms $s$, $t$, $u$ for which the compositions $t\circ s$ and $u\circ t$ are defined, we have the 2*-cocycle identity* $\alpha(u, t\circ s)\alpha(t,s) = \alpha(u\circ t, s)\alpha(u,t)$. The *twisted category algebra* $k_\alpha\mathcal{C}$ is the $k$-vector space having the morphism set $\mathrm{Mor}(\mathcal{C})$ as a $k$-basis, with a $k$-bilinear multiplication given by $t \cdot s = \alpha(t,s)(t \circ s)$ if $t \circ s$ is defined, and $t \cdot s = 0$, otherwise. The 2-cocycle identity is equivalent to the associativity of this multiplication. The isomorphism class of $k_\alpha\mathcal{C}$ depends only on the class of $\alpha$ in $H^2(\mathcal{C}; k^\times)$, with $k^\times$ here understood as a constant contravariant functor on $\mathcal{C}$. If $\alpha$ represents the trivial class, then $k_\alpha\mathcal{C} \cong k\mathcal{C}$, the ususal category algebra of $\mathcal{C}$ over $k$. For any idempotent endomorphism $e$ of an object $X$ in $\mathcal{C}$ we denote by $G_e$ the group of all invertible elements in the monoid $e\circ\mathrm{End}_\mathcal{C}(X)\circ e$. The restriction of $\alpha$ to the groups $G_e$ is abusively again denoted by $\alpha$. Note that the image in $k_\alpha\mathcal{C}$ of an idempotent endomorphism $e$ of an object in $\mathcal{C}$ is not necessarily an idempotent; more precisely, the square of $e$ in $k_\alpha\mathcal{C}$ is equal to $e \cdot e = \alpha(e,e)e$, and hence $\hat{e} = \alpha(e,e)^{-1}e$ is an idempotent in $k_\alpha\mathcal{C}$. For any $k_\alpha\mathcal{C}$-module $U$, the space $eU = \hat{e}U$ is an $\hat{e}k_\alpha\mathcal{C}\hat{e}$-module, hence restricts to a $k_\alpha G_e$-module. By [8, 5.2, 5.4], if $e$, $f$ are isomorphic idempotents in $\mathcal{C}$, then there is an algebra isomorphism $k_\alpha G_e \cong k_\alpha G_f$ which is uniquely determined up to an inner automorphism. Two pairs $(e,U)$, $(f,V)$ consisting of idempotents $e \in \mathrm{End}_\mathcal{C}(X)$, $f \in \mathrm{End}_\mathcal{C}(Y)$, a $k_\alpha G_e$-module $U$ and a $k_\alpha G_f$-module $V$, are called *isomorphic* if the idempotents $e$, $f$ are isomorphic and if the isomorphism classes of $U$, $V$ correspond to each other through the induced isomorphism $k_\alpha G_e \cong k_\alpha G_f$. We need the following parametrisation of simple $k_\alpha\mathcal{C}$-modules from [8]:

**Theorem 2.1** ([8, Theorem 1.2]). *Let $\mathcal{C}$ be a finite category and let $\alpha \in Z^2(\mathcal{C}; k^\times)$. The map sending a simple $k_\alpha\mathcal{C}$-module $S$ to the pair $(e, eS)$, where $e$ is an idempotent endomorphism in $\mathcal{C}$, minimal with respect to $eS \neq \{0\}$, induces a bijection between the set of isomorphism classes of simple $k_\alpha\mathcal{C}$-modules and the set of isomorphism classes of pairs $(e, T)$ consisting of an idempotent endomorphism $e$ in $\mathcal{C}$ and a simple $k_\alpha G_e$-module $T$.*

The following terminology from [8] singles out those simple $k_\alpha\mathcal{C}$-modules which are parametrised by isomorphism classes of pairs $(e, T)$ with $T$ a projective simple $k_\alpha G_e$-module:

**Definition 2.2** ([8, 1.4]). Let $\mathcal{C}$ be a finite category and $\alpha \in Z^2(\mathcal{C}; k^\times)$. A *weight of* $k_\alpha\mathcal{C}$ is a pair $(e, T)$ consisting of an idempotent endomorphism $e$ of an object $X$ in $\mathcal{C}$ and a projective simple $k_\alpha G_e$-module $T$. A *weight algebra* $W(k_\alpha\mathcal{C})$ of $k_\alpha\mathcal{C}$ is a $k$-algebra of the form $W(k_\alpha\mathcal{C}) = c \cdot k_\alpha\mathcal{C} \cdot c$, where $c$ is an idempotent in $k_\alpha\mathcal{C}$ with the property that $cS = S$ for every simple $k_\alpha\mathcal{C}$-module $S$ parametrised by a weight, and $cS' = \{0\}$ for every simple $k_\alpha\mathcal{C}$-module $S'$ which is not parametrised by a weight.

The idempotent $c$ is unique up to conjugacy in $k_\alpha\mathcal{C}$, and the number $\ell(W(k_\alpha\mathcal{C}))$ of isomorphism classes of simple $W(k_\alpha\mathcal{C})$-modules is equal to the number of isomorphism classes of weights of $k_\alpha\mathcal{C}$. See [9, Theorem 1.11] for a sufficient criterion for weight algebras to be quasi-hereditary.

# 3 Simple modules and weight algebras of $p$-orbit categories

Let $\mathcal{C}$ be a finite category, with $p$-transporter category $\mathcal{T} = \mathcal{T}_\mathcal{C}$ and associated $p$-orbit category $\mathcal{O} = \mathcal{O}_\mathcal{C}$. The canonical functor $\mathcal{T} \to \mathcal{O}$ is the identity on objects, and surjective on morphisms between any pair of objects in $\mathcal{T}$. For any object $(X, Q)$ in $\mathcal{T}$, the kernel of the canonical monoid homomorphism $\mathrm{End}_\mathcal{T}((X, Q)) \to \mathrm{End}_\mathcal{O}((X, Q))$ can be identified with $Q$. For any two objects $(X, Q)$, $(Y, R)$ in $\mathcal{T}$, the canonical map $\mathrm{Hom}_\mathcal{T}((X, Q), (Y, R)) \to \mathrm{Hom}_\mathcal{O}((X, Q), (Y, R))$ induces a bijection of sets $R\backslash\mathrm{Hom}_\mathcal{T}((X, Q), (Y, R)) \cong \mathrm{Hom}_\mathcal{O}((X, Q), (Y, R))$. The results of this section can be formulated for any functor between finite categories with the above properties, but in view of the intended application - the proof of Theorem 1.4 at the end of this section - we have chosen to formulate them for the canonical functor from $\mathcal{T}$ to $\mathcal{O}$.

**Lemma 3.1.** *Let $(X, Q)$ be an object in $\mathcal{T}$ and $(e, Q, Q)$ an idempotent in $\mathrm{End}_\mathcal{T}((X, Q))$. The following hold.*

*(i) We have $e \leq 1_Q$ and $e \circ Q = e \circ Q \circ e$.*

*(ii) The set $e \circ Q$ is a $p$-subgroup of $\mathrm{End}_\mathcal{C}(X)$ with unit element $e$, and the map sending $u \in Q$ to $e \circ u$ is a surjective group homomorphism $Q \to e \circ Q$.*

*(iii) Let $u \in Q$. Then $u \circ e$ is an idempotent in $\mathrm{End}_\mathcal{C}(X)$ if and only if $e \circ u = e$. In that case the idempotents $u \circ e$ and $e$ in $\mathrm{End}_\mathcal{C}(X)$ are isomorphic, and the idempotent endomorphisms $(e, Q, Q)$ and $(u \circ e, Q, Q)$ in $\mathrm{End}_\mathcal{T}((X, Q))$ are isomorphic.*

*(iv) The idempotent endomorphism $(e, Q, Q)$ of the object $(X, Q)$ in $\mathcal{T}$ is isomorphic to the identity morphism $(e, e \circ Q, e \circ Q)$ of the object $(X, e \circ Q)$ in $\mathcal{T}$. In particular, the category $\mathcal{T}$ is idempotent complete.*

*Proof.* Since $(e, Q, Q)$ is an endomorphism of $(X, Q)$ in $\mathcal{T}$, we have $e = 1_Q \circ e = e \circ 1_Q$ and $e \circ Q \subseteq Q \circ e$. The first condition is equivalent to $e \leq 1_Q$. The second condition implies that for any $u \in Q$ we have $e \circ u = e \circ u \circ e$, where we use that $e$ is an idempotent. Thus, for $u, v \in Q$ we have $(e \circ u) \circ (e \circ v) = (e \circ u \circ e) \circ v = e \circ u \circ v$. This shows (i) and (ii). Let $u \in Q$. If $e \circ u = e$, then $(u \circ e) \circ (u \circ e) = u \circ e$. Conversely, if $(u \circ e) \circ (u \circ e) = u \circ e$, then composing with $e$ implies that $e \circ u \circ e$ is an idempotent. By (i) we have $e \circ u \circ e = e \circ u$, which is an element of the finite group $e \circ Q$, so this is an idempotent if and only if $e \circ u = e$. Thus the morphisms $e$ and $u \circ e$ define in this case isomorphisms between the idempotents as stated in (iii). The equality $e \circ Q \circ e = e \circ Q$ implies that $e$ defines a morphism $(e, Q, e \circ Q) : (X, Q) \to (X, e \circ Q)$ in $\mathcal{T}$ as well as a morphism $(e, e \circ Q, Q) : (X, e \circ Q) \to (X, Q)$ in $\mathcal{T}$. Statement (iv) follows. $\square$

Idempotents in $\mathcal{O}$ lift to idempotents in $\mathcal{T}$ (this is a special cases of the well-known fact that idempotents lift through surjective homomorphisms between finite monoids):

**Lemma 3.2.** *Let $(X, Q)$ be an object in $\mathcal{T}$. Identify endomorphisms of $(X, Q)$ in $\mathcal{T}$ and in $\mathcal{O}$ with their canonical images in $\mathrm{End}_{\mathcal{C}}(X)$ and $Q \backslash \mathrm{End}_{\mathcal{C}}(X)/Q$, respectively. Let $f$ be an idempotent in $\mathrm{End}_{\mathcal{O}}((X, Q))$. There is an idempotent $e$ in $\mathrm{End}_{\mathcal{T}}((X, Q))$ such that $f = Q \circ e \circ Q$.*

*Proof.* Let $s \in \mathrm{End}_{\mathcal{T}}((X, Q))$ such that $f = Q \circ s \circ Q$. Since $\mathrm{End}_{\mathcal{T}}((X, Q))$ is a finite monoid, there is a positive integer $n$ such that $e = s^n$ is an idempotent. Since $f$ is an idempotent, we have $f = f^n = Q \circ e \circ Q$, as required. $\square$

Isomorphic idempotents in $\mathcal{O}$ lift to isomorphic idempotents in $\mathcal{T}$:

**Lemma 3.3.** *Let $(X, Q)$ and $(Y, R)$ be objects in $\mathcal{T}$. Identify morphisms in $\mathcal{T}$ between these objects with their images in the morphism set of $\mathcal{C}$. Let $e \in \mathrm{End}_{\mathcal{T}}((X, Q))$ and $f \in \mathrm{End}_{\mathcal{T}}((Y, R))$ be idempotent endomorphisms. Denote by $\bar{e} = Q \circ e \circ Q = Q \circ e$ and $\bar{f} = R \circ f \circ R = R \circ f$ the canonical images of $e$ and $f$ in $\mathrm{End}_{\mathcal{O}}((X, Q))$ and $\mathrm{End}_{\mathcal{O}}((Y, R))$, respectively. Then $e$ and $f$ are isomorphic idempotents in $\mathcal{T}$ if and only if $\bar{e}$ and $\bar{f}$ are isomorphic idempotents in $\mathcal{O}$.*

*Proof.* Suppose that $\bar{e}$ and $\bar{f}$ are isomorphic in $\mathcal{O}$. That is, there are morphisms $s : (X, Q) \to (Y, R)$ and $t : (Y, R) \to (X, Q)$ in $\mathcal{T}$ such that $\bar{e} = \bar{t} \circ \bar{s}$ and $\bar{f} = \bar{s} \circ \bar{t}$, where $\bar{s}$, $\bar{t}$ are the images of $s$, $t$ in $\mathcal{O}$. Thus there are $u \in Q$ and $v \in R$ such that $t \circ s = u \circ e$, and $s \circ t = v \circ f$ in $\mathcal{C}$. Since $s$, $t$ are morphisms in $\mathcal{T}$, we have $s \circ Q \subseteq R \circ s$ and $t \circ R \subseteq Q \circ t$. Let $n$ be a positive integer such that both $(t \circ s)^n$ and $(s \circ t)^n$ are idempotents in $\mathcal{C}$, hence in $\mathcal{T}$. Set $t' = (t \circ s)^{n-1} \circ t$. Then $(t \circ s)^n = t' \circ s$ and $(s \circ t)^n = s \circ t'$. Write $(u \circ e)^n = u' \circ e$ for some $u' \in Q$ and $(v \circ f)^n = v' \circ f$ for some $v' \in R$. Then $t' \circ s = u' \circ e$ and $s \circ t' = v' \circ f$ are isomorphic idempotents in $\mathcal{T}$. By Lemma 3.1 (iii), the idempotent $u' \circ e$ is isomorphic to $e$, and the idempotent $v' \circ f$ is isomorphic to $f$ in $\mathcal{T}$. Together it follows that $e$ and $f$ are isomorphic idempotents in $\mathcal{T}$. The converse is trivial. $\square$

Maximal subgroups of endomorphism monoids in $\mathcal{O}$ lift to $\mathcal{T}$:

**Lemma 3.4.** *Let $(X, Q)$ be an object in $\mathcal{T}$. Identify $\mathrm{End}_{\mathcal{T}}((X, Q))$ with its canonical image in $\mathrm{End}_{\mathcal{C}}(X)$. Let $e$ be an idempotent in $\mathrm{End}_{\mathcal{T}}((X, Q))$. Denote by $\bar{e} = Q \circ e \circ Q$ the image of $e$ in $\mathrm{End}_{\mathcal{O}}((X, Q))$ and by $G_e$ the group of invertible elements in the monoid $e \circ \mathrm{End}_{\mathcal{C}}(X) \circ e$.*

*(i) The group of invertible elements in the monoid $e \circ \mathrm{End}_{\mathcal{T}}((X, Q)) \circ e$ is equal to $N_{G_e}(e \circ Q)$.*

*(ii) The group of invertible elements in the monoid $\bar{e} \circ \mathrm{End}_{\mathcal{O}}((X, Q)) \circ \bar{e}$ is equal to $N_{G_e}(e \circ Q)/e \circ Q$.*

*Proof.* Let $s$, $t$ be endomorphisms of $(X,Q)$ in $e \circ \mathrm{End}_{\mathcal{T}}((X,Q)) \circ e$ which are inverse to each other. Then $s$, $t$ are in particular inverse to each other when considered as elements of the monoid $e \circ \mathrm{End}_{\mathcal{C}}(X) \circ e$, hence in $G_e$. Moreover we have $s \circ Q \subseteq Q \circ s$ and $t \circ Q \subseteq Q \circ t$. The first inclusion precomposed with $t$ and composed with $e$ yields $s \circ Q \circ t \subseteq e \circ Q \circ e = e \circ Q$, where the last equality uses Lemma 3.1 (i). The second inclusion composed with $s$ yields $e \circ Q \subseteq s \circ Q \circ t$. Together we get $s \circ Q \circ t = e \circ Q$. Since $s = s \circ e$, this equality is equivalent to $s \circ (e \circ Q) \circ t = e \circ Q$, whence (i). Let now $s$, $t$ be endomorphisms of $(X,Q)$ in $e \circ \mathrm{End}_{\mathcal{T}}((X,Q)) \circ e$ whose images $\bar{s}$, $\bar{t}$ in $\bar{e} \circ \mathrm{End}_{\mathcal{O}}((X,Q)) \circ \bar{e}$ are inverse to each other. Then $t \circ s = u \circ e$ for some $u \in Q$. Since $t = e \circ t$, this implies $t \circ s = e \circ u \circ e = e \circ u$. Thus $e \circ u^{-1} \circ t$ is the inverse of $s$ in $e \circ \mathrm{End}_{\mathcal{T}}((X,Q)) \circ e$, and hence $s$ and $t$ are in $N_{G_e}(e \circ Q)$ by (i). Statement (ii) follows. □

**Lemma 3.5.** *Let $\mathcal{E}$ be a set of representatives of the isomorphism classes of idempotents in $\mathcal{C}$. For any $e \in \mathcal{E}$, denote by $X_e$ the object in $\mathcal{C}$ of which $e$ is an idempotent endomorphism, by $G_e$ the subgroup of invertible elements of the monoid $e \circ \mathrm{End}_{\mathcal{C}}(X_e) \circ e$ and by $\mathcal{X}_e$ a set of representatives of the $G_e$-conjugacy classes of $p$-subgroups of $G_e$. Then the following hold.*

*(i) The set $\{(X_e, Q) \mid e \in \mathcal{E},\ Q \in \mathcal{X}_e\}$ is a set of representatives of the isomorphism classes of objects in $\mathcal{T}$.*

*(ii) The set $\{(e, Q, Q) \mid e \in \mathcal{E},\ Q \in \mathcal{X}_e\}$ is a set of representatives of the isomorphism classes of idempotents in $\mathcal{T}$.*

*Proof.* Suppose that $(X,Q)$ and $(Y,R)$ are isomorphic objects in $\mathcal{T}$. This is equivalent to the existence of morphisms $s : X \to Y$ and $t : Y \to X$ in $\mathcal{C}$ satisfying $s \circ 1_Q = 1_R \circ s = s$, $t \circ 1_R = 1_Q \circ t = t$, $s \circ Q \subseteq R \circ s$, $t \circ R \subseteq Q \circ t$, $t \circ s = 1_Q$, and $s \circ t = 1_R$. In particular, the idempotents $e = 1_Q$ and $f = 1_R$ are isomorphic in $\mathcal{C}$. Moreover, we have $Q = t \circ s \circ Q \subseteq t \circ R \circ s \subseteq Q \circ t \circ s = Q$, and hence these inclusions are all equalities. Thus $Q = t \circ R \circ s$, and precomposition with $t$ yields $Q \circ t = t \circ R$. Similarly, $s \circ R = Q \circ s$. Applied to $e = f$ this shows that $(X_e, Q) \cong (X_e, R)$ in $\mathcal{T}$ if and only if $Q$, $R$ are conjugate in $G_e$. This shows that the objects in the set described in (i) are pairwise nonisomorphic, and one easily checks that every object of $\mathcal{T}$ is isomorphic to an object in this set, whence (i). If $Q \in \mathcal{X}_e$, then $e = 1_Q$, and hence $(e, Q, Q)$ is the identity morphism of the object $(X_e, Q)$ in $\mathcal{T}$. Thus (ii) follows from (i) combined with the fact that every idempotent endomorphism in $\mathcal{T}$ is isomorphic to the identity morphism of some object in $\mathcal{T}$ by Lemma 3.1 (iv). □

*Proof of Theorem 1.4.* The parametrisation of the isomorphism classes of simple $k_\alpha\mathcal{C}$-modules in [8] implies that

$$\ell(k_\alpha\mathcal{C}) = \sum_{e \in \mathcal{E}} \ell(k_\alpha G_e) \ .$$

If Alperin's weight conjecture holds for the twisted finite group algebras $k_\alpha G_e$, then this sum is equal to the sum

$$\sum_{e \in \mathcal{E}} \sum_{Q \in \mathcal{X}_e} z(k_\alpha N_{G_e}(Q)/Q) \ ,$$

where $z(k_\alpha N_{G_e}(Q)/Q)$ is the number of isomorphism classes of projective simple $k_\alpha N_{G_e}(Q)/Q$-modules. It remains to show that this is also the number of isomorphism classes of weights of $k_\alpha\mathcal{O}$. In this double sum, $e$ runs over $\mathcal{E}$ and $Q$ over $\mathcal{X}_e$. By Lemma 3.5, this implies that the triples $(e, Q, Q)$ run over a set of representatives of the isomorphism classes of idempotents in $\mathcal{T}$. Lemma 3.3 implies that the images of the triples $(e, Q, Q)$ in the morphism set of $\mathcal{O}$ run over a set of representatives of the isomorphism classes of idempotents in $\mathcal{O}$. By Lemma 3.4, the maximal subgroup determined by the image of any such $(e, Q, Q)$ in $\mathcal{O}$ is $N_{G_e}(Q)/Q$, and hence $z(k_\alpha N_{G_e}(Q)/Q)$ is equal to the number of isomorphism classes of weights of $\mathcal{O}$ associated with the image of $(e, Q, Q)$ in $\mathcal{O}$. Thus the above double sum is equal to the number of isomorphism classes of weights of $k_\alpha\mathcal{O}$, hence equal to $\ell(W(k_\alpha\mathcal{O}))$ as stated. □

# 4 On the cohomology of $p$-transporter categories

**Proposition 4.1.** *Let $\mathcal{C}$ be a finite category, and set $\mathcal{T} = \mathcal{T}_\mathcal{C}$. The canonical functor $\Phi : \mathcal{C} \to \mathcal{T}$ induces a graded isomorphism $H^*(\mathcal{T}; k^\times) \cong H^*(\mathcal{C}; k^\times)$.*

*Proof.* This follows from combining the Lemmas 4.2 and 4.3 below. □

**Lemma 4.2.** *Let $\mathcal{C}$ be a finite category, and set $\mathcal{T} = \mathcal{T}_\mathcal{C}$. Denote by $\mathcal{T}_1$ the full subcategory of $\mathcal{T}$ whose objects are of the form $(X, \{e\})$, where $X$ is an object in $\mathcal{C}$ and $e$ is an idempotent in $\mathrm{End}_\mathcal{C}(X)$. The following hold.*

*(i) The inclusion functor $\mathcal{T}_1 \to \mathcal{T}$ has a right adjoint $\Phi : \mathcal{T} \to \mathcal{T}_1$ sending an object $(X, Q)$ in $\mathcal{T}$ to $(X, \{1_Q\})$.*

*(ii) For any functor $\mathcal{F} : \mathcal{T}_1^{op} \to Ab$, the functor $\Phi$ induces an isomorphism $H^*(\mathcal{T}_1; \mathcal{F}) \cong H^*(\mathcal{T}; \mathcal{F} \circ \Phi)$.*

*(iii) The inclusion functor $\mathcal{T}_1 \to \mathcal{T}$ induces an isomorphism $H^*(\mathcal{T}; k^\times) \cong H^*(\mathcal{T}_1; k^\times)$.*

*Proof.* The map sending an object $(X, Q)$ in $\mathcal{T}$ to $(X, \{1_Q\})$ extends in an obvious way to a functor $\Phi : \mathcal{T} \to \mathcal{T}_1$. Let $(X, e)$ be an object in $\mathcal{T}_1$ and let $(Y, R)$ be an object in $\mathcal{T}$. Both $\mathrm{Hom}_{\mathcal{T}_1}((X, \{e\}), \Phi(Y, R))$ and $\mathrm{Hom}_\mathcal{T}((X, \{e\}), (Y, R))$ correspond bijectively to the set $1_R \circ \mathrm{Hom}_\mathcal{C}(X, Y) \circ e$. It follows that $\Phi$ is right adjoint to the inclusion functor $\mathcal{T}_1 \to \mathcal{T}$. This shows (i). Statement (ii) is a special case of more general base change properties of functor cohomology; see [3, 3.1], [4, 5.1]. The inclusion $\mathcal{T}_1 \to \mathcal{T}$ composed with the functor $\Phi : \mathcal{T} \to \mathcal{T}_1$ is the identity functor on $\mathcal{T}_1$, hence induces the identity on $H^*(\mathcal{T}_1; k^\times)$. It follows from (ii) applied to the constant functor $k^\times$ that $\Phi$ induces an isomorphism $H^*(\mathcal{T}_1; k^\times) \cong H^*(\mathcal{T}; k^\times)$, and hence the inclusion functor $\mathcal{T}_1 \to \mathcal{T}$ induces the inverse of this isomorphism, implying (iii). □

The category $\mathcal{T}_1$ is canonically isomorphic to the idempotent completion of $\mathcal{C}$. The next observation is a restatement of the well-known fact that a finite category and its idempotent completion have Morita equivalent category algebras (we leave the proof to the reader).

**Lemma 4.3.** *With the notation as in the previous lemma, for any functor $\mathcal{F} : \mathcal{T}_1^{op} \to Ab$, the canonical functor $\Psi : \mathcal{C} \to \mathcal{T}_1$ sending an object $X$ in $\mathcal{C}$ to the object $(X, \{\mathrm{Id}_X\})$ in $\mathcal{T}_1$ induces an isomorphism $H^*(\mathcal{T}_1; \mathcal{F}) \cong H^*(\mathcal{C}; \Psi \circ \mathcal{F})$. In particular, $\Psi$ induces an isomorphism $H^*(\mathcal{T}_1; k^\times) \cong H^*(\mathcal{C}; k^\times)$.*

An *EI*-category is a small category $\mathcal{C}$ with the property that any endomorphism of an object is an isomorphism; this concept is due to Lück [10]. The set $[\mathcal{C}]$ of isomorphism classes of an *EI*-category $\mathcal{C}$ is partially ordered via $[X] \leq [Y]$ whenever $\mathrm{Hom}_\mathcal{C}(X, Y)$ is nonempty, where

$[X]$, $[Y]$ are the isomorphism classes of objects $X$, $Y$ in $\mathcal{C}$. Following some terminology in [6], an $EI$-category $\mathcal{C}$ is called *regular* if for any two objects $X$, $Y$ in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is nonempty, the group $\mathrm{Aut}_{\mathcal{C}}(X)$ acts regularly (that is, transitively and freely) on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. Let $\mathcal{C}$, $\mathcal{D}$ be $EI$-categories and $\Phi : \mathcal{C} \to \mathcal{D}$ a functor. Following [7, 5.1], the functor $\Phi$ is called *regular* if $\Phi$ induces an isomorphism of partially ordered sets $[\mathcal{C}] \cong [\mathcal{D}]$, for any two objects $X$, $Y$ in $\mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(\Phi(X),\Phi(Y))$ induced by $\Phi$ is surjective, and for any two objects $X$, $Y$ in $\mathcal{C}$ such that the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is nonempty, the group $K(X) = \ker(\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{D}}(\Phi(X))$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ through composition of morphisms and induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)/K(X) \cong \mathrm{Hom}_{\mathcal{D}}(\Phi(X),\Phi(Y)) \ .$$

An $EI$-category $\mathcal{C}$ is regular if and only if the canonical functor $\mathcal{C} \to [\mathcal{C}]$ is regular. By a result of Słomińska [11, 1.5], the cohomology of an $EI$-category $\mathcal{C}$ with constant coefficients is invariant under passage to its *subdivision* $S(\mathcal{C})$, which is a category defined as follows. The objects of $S(\mathcal{C})$ are faithful functors $\sigma : [m] \to \mathcal{C}$, where $m$ is a nonnegative integer and the totally ordered set $[m] = \{0 < 1 < .. < m\}$ is viewed as category in the obvious way; a morphism in $S(\mathcal{C})$ from $\sigma$ to another object $\tau : [n] \to \mathcal{C}$ is a pair $(\alpha, \mu)$ consisting of an injective order preserving map $\alpha : [m] \to [n]$ and an isomorphism of functors $\mu : \sigma \cong \tau \circ \alpha$. The composition of $(\alpha, \mu)$ with another morphism $(\beta, \nu)$ from $\tau$ to $\rho : [r] \to \mathcal{C}$ is defined by $(\beta, \nu) \circ (\alpha, \mu) = (\beta \circ \alpha, (\nu\alpha) \circ \mu)$, where $\nu\alpha : \tau \circ \alpha \cong \rho \circ \beta \circ \alpha$ is induced by precomposing $\nu$ with $\alpha$. The pair $(\alpha, \mu)$ induces a group homomorphism $\mathrm{Aut}_{S(\mathcal{C})}(\tau) \to \mathrm{Aut}_{S(\mathcal{C})}(\sigma)$ mapping $(\mathrm{Id}_{[n]}, \gamma)$ to $(\mathrm{Id}_{[m]}, \mu^{-1} \circ (\gamma\alpha) \circ \mu)$, for any automorphism $\gamma$ of the functor $\tau$, where $\gamma\alpha$ is the induced automorphism of $\tau \circ \alpha$. Clearly $S(\mathcal{C})$ is again an $EI$-category. Moreover, $S(\mathcal{C})$ is regular, and every morphism in $S(\mathcal{C})$ is a monomorphism. The proof of the following result is similar to that of Proposition [7, 6.2].

**Proposition 4.4.** *Let $\mathcal{C}$ be a finite EI-category. Set $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ and $\mathcal{O} = \mathcal{O}_{\mathcal{C}}$. The canonical functor $\mathcal{T} \to \mathcal{O}$ induces a regular functor $\Psi : S(\mathcal{T}) \to S(\mathcal{O})$, and for any object $\sigma \in S(\mathcal{T})$, the kernel $K(\sigma) = \ker(\mathrm{Aut}_{S(\mathcal{T})}(\sigma) \to \mathrm{Aut}_{S(\mathcal{O})}(\Psi(\sigma)))$ is a finite $p$-group.*

*Proof.* The canonical functor $\mathcal{T} \to \mathcal{O}$ preserves nonisomorphisms, hence induces a functor $\Psi : S(\mathcal{T}) \to S(\mathcal{O})$ which in turn induces an isomorphism of partially ordered sets $[S(\mathcal{T})] \cong [S(\mathcal{O})]$. Let $\sigma : [m] \to \mathcal{T}$ and $\tau : [n] \to \mathcal{T}$ be objects in $S(\mathcal{T})$ and denote by $\bar{\sigma}$, $\bar{\tau}$ their images in $S(\mathcal{O})$ under $\Psi$. That is, $\sigma(i) = \bar{\sigma}(i) = (X_i, Q_i)$ for some object $X_i$ in $\mathcal{C}$ and a $p$-subgroup $Q_i$ of $\mathrm{Aut}_{\mathcal{C}}(X_i)$, where $0 \leq i \leq m$. Similarly, $\tau(j) = \bar{\tau}(j) = (Y_j, R_j)$ for some object $Y_j$ in $\mathcal{C}$ and some $p$-subgroup $R_j$ of $\mathrm{Aut}_{\mathcal{C}}(Y_j)$, where $0 \leq j \leq n$. Let $(\alpha, \bar{\mu}) : \bar{\sigma} \to \bar{\tau}$ be a morphism in $S(\mathcal{O})$; that is, $\alpha : [m] \to [n]$ is an order preserving map and $\bar{\mu} : \bar{\sigma} \cong \bar{\tau} \circ \alpha$ is a natural isomorphism. Explicitly, $\bar{\mu}$ consists of a compatible family of isomorphisms $\bar{\mu}_i : \bar{\sigma}(i) = (X_i, Q_i) \cong \bar{\tau}(\alpha(i)) = (Y_{\alpha(i)}, R_{\alpha(i)})$; that is, for $0 \leq i < m$ we have $\bar{\mu}_{i+1} \circ \bar{\sigma}(i < i+1) = \bar{\tau}(\alpha(i) < \alpha(i+1)) \circ \bar{\mu}_i$. Since the functor $\mathcal{T} \to \mathcal{O}$ is surjective on morphisms, there are isomorphisms $\mu_i : \sigma(i) \to \tau(\alpha(i))$ in $\mathcal{T}$ which lift the $\bar{\mu}_i$, but this family need not be a natural transformation. More precisely, for $0 \leq i < m$ there is an element $v_i \in R_{\alpha(i)}$ such that $v_{i+1} \circ \mu_{i+1} \circ \sigma(i < i+1) = \tau(\alpha(i) < \alpha(i+1)) \circ \mu_i$. An easy inductive argument shows that after replacing $\mu_{i+1}$ by $v_{i+1} \circ \mu_{i+1}$ we get a natural isomorphism $\mu : \sigma \cong \tau \circ \alpha$ lifting $\bar{\mu}$, which shows that the functor $S(\mathcal{T}) \to S(\bar{\mathcal{O}})$ is surjective on morphisms. If $\mu$, $\mu'$ are two morphisms in $S(\mathcal{T})$ from $\sigma$ to $\tau$ then by the regularity of $S(\mathcal{T})$ there is a unique automorphism $\beta$ of $\sigma$ such that $\mu' = \mu \circ \beta$. Thus, if in addition the images

$\bar{\mu}$, $\bar{\mu}'$ of $\mu$, $\mu'$ in $S(\mathcal{O})$ are equal then the image $\bar{\beta}$ in $S(\mathcal{O})$ satisfies $\bar{\mu} = \bar{\mu} \circ \bar{\beta}$. Since every morphism in $S(\mathcal{O})$ is a monomorphism this implies that $\bar{\beta} = \mathrm{Id}_{\bar{\sigma}}$, and hence $\beta$ belongs to the kernel $K(\sigma)$ of the canonical map from $\mathrm{Aut}_{S(\mathcal{T})}(\sigma)$ to $\mathrm{Aut}_{S(\mathcal{O})}(\bar{\sigma})$, which shows that indeed the functor $S(\mathcal{T}) \to S(\mathcal{O})$ is regular. The statement on $K(\sigma)$ being a finite $p$-group follows from the fact that for any object $(X,Q)$ in $\mathcal{T}$, the canonical map $\mathrm{Aut}_{\mathcal{T}}((X,Q)) \to \mathrm{Aut}_{\mathcal{O}}((X,Q))$ has as kernel the $p$-group $Q$. $\square$

*Proof of Proposition 1.7.* Set $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$ and $\mathcal{O} = \mathcal{O}_{\mathcal{C}}$. The fact that $\mathcal{T}$ and $\mathcal{O}$ are $EI$-categories is a trivial verification. It follows from Proposition 4.1 that in order to prove Proposition 1.7, it suffices to show that the canonical functor $\mathcal{T} \to \mathcal{O}$ induces an isomorphism $H^*(\mathcal{O}; k^\times) \cong H^*(\mathcal{T}; k^\times)$. As mentioned earlier, by [11, 1.5], we may replace $\mathcal{T}$ and $\mathcal{O}$ by their subdivisions. Since $H^q(Q; k^\times)$ is trivial for any positive integer $q$ and any finite $p$-group $Q$, it follows from Proposition 4.4 that the hypotheses of Theorem [7, 5.6] are satisfied with $S(\mathcal{T})$, $S(\mathcal{O})$, $k^\times$ instead of $\mathcal{C}$, $\mathcal{D}$, $A$, respectively. The conclusion of Theorem [7, 5.6] yields $H^*(S(\mathcal{T}); k^\times) \cong H^*(S(\mathcal{O}); k^\times)$, and hence $H^*(\mathcal{T}; k^\times) \cong H^*(\mathcal{O}; k^\times)$ as stated. $\square$

**Lemma 4.5** ([8, 4.2])**.** *Let $\mathcal{C}$ be a finite category, $\alpha \in Z^2(\mathcal{C}; k^\times)$, $s : X \to Y$ a morphism, $e \in \mathrm{End}_{\mathcal{C}}(X)$ and $f \in \mathrm{End}_{\mathcal{C}}(Y)$ idempotent endomorphisms satisfying $s = s \circ e = f \circ s$. Then $\alpha(s,e) = \alpha(e,e)$ and $\alpha(f,s) = \alpha(f,f)$.*

**Proposition 4.6.** *Let $\mathcal{C}$ be a finite category and $\alpha \in Z^2(\mathcal{C}; k^\times)$. Suppose that for any two any idempotent endomorphisms $e$, $f$ of objects $X$, $Y$, respectively, any morphism $s \in f \circ \mathrm{Hom}_{\mathcal{C}}(X,Y) \circ e$ and any $x \in G_e$, $y \in G_y$ we have $\alpha(s,x) = \alpha(s,e)$ and $\alpha(y,s) = \alpha(f,s)$. Then $\alpha$ extends to a 2-cocycle of $\mathcal{O}_{\mathcal{C}}$ through the canonical functor $\mathcal{C} \to \mathcal{O}_{\mathcal{C}}$.*

*Proof.* Let $s : (X,Q) \to (Y,R)$ and $t : (Y,R) \to (Z,S)$ be two morphisms in $\mathcal{T}_{\mathcal{C}}$. In order to show that $\alpha$ induces a 2-cocycle on $\mathcal{O}_{\mathcal{C}}$, it suffices to show that $\alpha(t,s)$ depends only on the $R$-orbit of $s$ and the $S$-orbit of $t$; that is, it suffices to show that for any $v \in R$ and any $w \in S$ we have

$$\alpha(t,s) = \alpha(t, v \circ s) \ = \alpha(w \circ t, s)$$

Let $w \in S$. Applying the 2-cocycle identity to $w$, $t$, $s$ yields

$$\alpha(w \circ t, s)\alpha(w,t) = \alpha(w, t \circ s)\alpha(t,s) \ .$$

By the assumptions and Lemma 4.5 we have $\alpha(w,t) = \alpha(1_S, t) = \alpha(1_S, 1_S) = \alpha(w, t \circ s)$, so that cancelling $\alpha(1_S, 1_S)$ yields $\alpha(w \circ t, s) = \alpha(t,s)$. Let $v \in R$. Applying the 2-cocycle identity to $t$, $v$, $s$ yields

$$\alpha(t, v \circ s)\alpha(v,s) = \alpha(t \circ v, s)\alpha(t,v) \ .$$

As before, we have $\alpha(v,s) = \alpha(1_R, 1_R) = \alpha(t,v)$, hence $\alpha(t, v \circ s) = \alpha(t \circ v, s)$. Since $t$ defines a morphism from $(Y,R)$ to $(Z,S)$, there is $w \in S$ such that $t \circ v = w \circ t$. Thus $\alpha(t, v \circ s) = \alpha(t \circ v, s) = \alpha(w \circ t, s) = \alpha(t,s)$, completing the proof. $\square$

# References


[1] J. L. Alperin, *Weights for finite groups*, Proc. Symp. Pure Math. **47** (1987), 369–379.

[2] J. L. Alperin and P. Fong, *Weights for symmetric an general linear groups*, J. Algebra **131** (1990), 2–22.

[3] S. Jackowski, J. McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), 113–132.

[4] S. Jackowski, J. McClure, R. Oliver, *Homotopy classifications of self-maps of BG via G-actions II*, Ann. Math. **135** (1992), 227–270.

[5] R. Knörr and G. R. Robinson, *Some remarks on a conjecture of Alperin*, J. London Math. Soc. **39** (1989), 48–60.

[6] M. Linckelmann, Alperin's weight conjecture in terms of equivariant Bredon cohomology. *Math. Z.* **250** (2005), 495–513.

[7] M. Linckelmann, *On $H^*(\mathcal{C}; k^\times)$ for fusion systems.* Homology, Homotopy and Appl. bf 11 (2009), 203–218.

[8] M. Linckelmann and M. Stolorz, *On simple modules over twisted finite category algebras*, Proc. Amer. Math. Soc. **140** (2012), 3725–3737.

[9] M. Linckelmann and M. Stolorz, *Quasi-hereditary twisted finite category algebras*, preprint 2012, to appear.

[10] W. Lück, *Transformation Groups and Algebraic K-Theory*, Springer Lecture Notes Math. **1408** (1989).

[11] J. Słomińska, *Homotopy colimits on EI-categories*, Lecture Notes in Mathematics, Springer Verlag, Berlin **1474** (1991), 273–294.

[12] S. Wilcox, *Cellularity of diagram algebras as twisted semigroup algebras*, J. Algebra **309** (2007), 10–31.