Boundary form factors of the sinh-Gordon model with Dirichlet boundary conditions at the self-dual point

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Abstract

In this manuscript we present a detailed investigation of the form factors of boundary fields of the sinh-Gordon model with a particular type of Dirichlet boundary condition, corresponding to zero value of the sinh-Gordon field at the boundary, at the self-dual point. We follow for this the boundary form factor program recently proposed by Z. Bajnok, L. Palla and G. Takacs [1], extending the analysis of the boundary sinh-Gordon model initiated there. The main result of the paper is a conjecture for the structure of all n-particle form factors of the boundary operators $\partial_x \phi$ and $(\partial_x \phi)^2$ in terms of elementary symmetric polynomials in certain functions of the rapidity variables. In addition, form factors of boundary "descendant" fields have been constructed.

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1 Introduction

In the context of 1+1-integrable quantum field theories (QFTs) form factors are defined as tensor valued functions, representing matrix elements of some local operator $\mathcal{O}(x)$ located at the origin between a multi-particle *in*-state and the vacuum

$$F_n^{\mathcal{O}|\mu_1\cdots\mu_n}(\theta_1,\ldots,\theta_n) := \langle 0|\mathcal{O}(0)Z_{\mu_1}(\theta_1)Z_{\mu_2}(\theta_2)\ldots Z_{\mu_n}(\theta_n)|0\rangle_{\text{in}} . \qquad (1.1)$$

Here $|0\rangle$ represents the vacuum state and $Z_{\mu_i}(\theta_i)$ are vertex operators which provide a generalization of Bosonic or Fermionic algebras and allow for the definition of a space of physical states. They carry indices μ_i, which are quantum numbers characterizing the various particle species, and depend on the parameters θ_i, which are called rapidities. In 1+1 dimensions the energy and momentum of the particles can be expressed in terms of the rapidity variables as

$$E_i = m_i \cosh\theta_i \qquad \text{and} \qquad P_i = m_i \sinh\theta_i, \qquad (1.2)$$

where m_i is the mass of the particle considered. The braiding relations amongst the operators $Z_{\mu_i}(\theta_i)$ and their Hermitian conjugates are known as Faddeev-Zamolodchikov algebra [2, 3]. For theories without back-scattering (diagonal theories) they are given by

$$Z_{\mu_i}(\theta_i)Z_{\mu_j}(\theta_j) = S_{\mu_i\mu_j}(\theta_{ij})Z_{\mu_j}(\theta_j)Z_{\mu_i}(\theta_i), \qquad (1.3)$$

$$Z_{\mu_i}^\dagger(\theta_i)Z_{\mu_j}^\dagger(\theta_j) = S_{\mu_i\mu_j}(\theta_{ij})Z_{\mu_j}^\dagger(\theta_j)Z_{\mu_i}^\dagger(\theta_i), \qquad (1.4)$$

$$Z_{\mu_i}(\theta_i)Z_{\mu_j}^\dagger(\theta_j) = S_{\mu_i\mu_j}(-\theta_{ij})Z_{\mu_j}^\dagger(\theta_j)Z_{\mu_i}(\theta_i) + 2\pi\delta_{\mu_i\mu_j}\delta(\theta_{ij}), \qquad (1.5)$$

where $S_{\mu_i\mu_j}(\theta_{ij})$ is the 2-particle scattering matrix and we introduced the variable $\theta_{ij} = \theta_i - \theta_j$.

The form factor program for integrable models was pioneered by P. Weisz and M. Karowski [4, 5] and thereafter developed to a large extent by F. A. Smirnov, who also formulated some of the consistency equations for form factors [6]. In these seminal works, the fundamental properties of form factors in 1+1-dimensional theories were established (see [7] for a recent review). It was found that the form factors of local operators can be obtained as the solutions to a set of consistency equations whose origin is based on physically-motivated requirements. The solution to these equations allows in principle the computation of all n-particle form factors associated to any local field of the massive QFT. Following this program, form factors of a large class of integrable models have been computed (see e.g. [4]-[29] and references therein). Once the form factors of a certain operator are known they can be used for many interesting applications like the computation of correlation functions (see e.g. [9], [16]-[19] and [22, 23]), the identification of the operator content of the perturbed conformal field theory (CFT) (see e.g. [14, 30, 31] and [22]-[27]) and the explicit computation of quantities which characterize the underlying CFT (see e.g. [32, 33, 17, 31] and [20]-[23]).

So far, we have briefly recalled the program for bulk integrable QFTs, that is QFTs without boundaries or defects. Given that most realistic physical systems possess boundaries and/or defects, it is both interesting and natural to ask the question whether or not the integrability of certain 1+1 dimensional QFTs may be preserved in the presence of boundaries or defects. The study of the conditions under which that question can be answered in the affirmative has attracted a lot of attention in the last two decades, with many works devoted to the study of integrable QFTs with boundaries [34]-[38] and defects [39, 40, 41]. A natural further step is trying to extend the bulk form factor program just recalled to boundary and defect theories. Here we want to concentrate on the boundary case. Given an integrable QFT with a boundary there are two possible approaches to the problem of computing correlation functions:

Boundary at $t = 0$: we may place the boundary at $t = 0$ extending along the space direction (t represents the time coordinate). In this case it is natural to characterize the presence of the boundary by means of a boundary state $|B\rangle$. The Hilbert space is exactly the same as for the bulk theory, and therefore form factors can be completely characterized by those in the bulk. The only additional information needed is an explicit description of the boundary state, which has been provided by S. Ghoshal and A.B. Zamolodchikov [36],

$$|B\rangle := \exp\left(\frac{1}{4\pi}\int_{-\infty}^{\infty} R_{\mu_i}(\frac{i\pi}{2} - \theta)Z_{\mu_i}(-\theta)Z_{\mu_i}(\theta)d\theta\right)|0\rangle. \tag{1.6}$$

Here $R_{\mu_i}(\theta)$ is the reflection amplitude off the boundary, which we take to be diagonal in this case (particles just reflect off the boundary with a certain probability, without changing their species). Expanding the exponential above, boundary form factors can be expressed in terms of bulk form factors, although the former will then be an infinite sum of the latter. This sum is in general very involved, but can be performed analytically for some correlation functions of free theories [42]. This picture can be generalized to the defect case, by replacing the boundary state above by a defect state. A realization of this state and explicit form factor computations have been carried out for free theories [39, 40] with different types of defects. More recently it has been proven in [43] that a defect theory may always be regarded alternatively as a boundary theory by means of the so-called "folding trick". Exploiting this correspondence, the authors also constructed the defect state for integrable models with purely transmitting defects.

Boundary at $x = 0$: we may place the boundary at $x = 0$ extending in the time direction (x represents the position coordinate). In this case the Hilbert space will change with respect to the bulk case, as only half of the physical states remain linearly independent in the presence of a boundary. This is so since in addition to (1.3)-(1.5) the further constraint

$$Z_{\mu_i}(\theta_i) = R_{\mu_i}(\theta_i)Z_{\mu_i}(-\theta_i), \tag{1.7}$$

holds. Recently, a generalization of the bulk form factor program for theories with boundaries and for boundary operators (that is, operators defined on the boundary) has been proposed [1], and its validity tested against previously known results for various QFTs. It has been established that the relations (1.3)-(1.5) and (1.7) together with the analytical properties of S- and R-matrices lead to a set of consistency equations for the boundary form factors, which can be solved for particular models along similar lines as for bulk theories. Some of these solutions were found in [1], although in most cases only for low particle numbers. The main aim of this paper is to extend some of the results in [1] by carrying out a detailed study of the boundary form factor solutions of the sinh-Gordon model with a particular type of Dirichlet boundary conditions (corresponding to vanishing value of the sinh-Gordon field at the boundary) at the self-dual point.

The paper is organized as follows: In section 1 we review the boundary form factor program as proposed in [1]. In section 2 we recall the S- and R-matrices of the sinh-Gordon model with Dirichlet boundary conditions and vanishing boundary field. We also present the form factor equations found in [1] for this particular model. In section 3 we re-write these equations in terms of elementary symmetric polynomials and analyze their special properties at the self-dual point. In section 4 we provide new explicit solutions to the form factor equations up to 16-particles and formulate a conjecture for the form of all n-particle form factors of the boundary operators $\partial_x\phi$ and $(\partial_x\phi)^2$. In section 5 we find solutions to the form factor consistency equations for other boundary fields. We summarize our conclusions in section 6. Finally, we list various explicit form factor formulae in the appendix.

2 The boundary form factor program

In this section we will review the main aspects of the boundary form factor program proposed in [1], for diagonal scattering theories possessing a single particle spectrum and no particle bound states (such as the sinh-Gordon model). For such a model, the particle indices μ_i in (1.1) can be dropped and the boundary form factor axioms written as

$$
\begin{align}
F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots \theta_n) &= S(\theta_{i\,i+1})F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots \theta_n), \tag{2.1}\\
F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1}, \theta_n) &= R_n(\theta_n)F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{n-1}, -\theta_n), \tag{2.2}\\
F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n) &= R_n(i\pi - \theta_1)F_n^{\mathcal{O}}(2\pi i - \theta_1, \theta_2, \ldots, \theta_n), \tag{2.3}
\end{align}
$$

The first two axioms follow from the braiding relations (1.3) and (1.7), whereas the last axiom expresses crossing symmetry in the presence of a boundary. In addition we have the kinematical residue equations which generalize the corresponding ones in the bulk

$$
- i\mathrm{Res}_{\bar{\theta}_0 = \theta_0} F_{n+2}^{\mathcal{O}}(\theta_0 + i\pi, \bar{\theta}_0, \theta_1 \ldots, \theta_n) = \left(1 - \prod_{i=1}^{n} S(\theta_{0i})S(\hat{\theta}_{0i})\right) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n), \tag{2.4}
$$

with $\hat{\theta}_{ij} := \theta_i + \theta_j$. It was argued in [1] that a second kinematical relation which has no counterpart in the bulk form factor program emerges whenever the reflection amplitude $R(\theta)$ has a pole at $\theta = i\pi/2$. We will not report this equation here, since it is trivial for the particular model we want to study later.

As in the bulk case, further recursive relations exist in the presence of particle bound states but we will not review them here. Mimicking the solution procedure developed for the bulk case and diagonal scattering matrices, an ansatz for the solutions of the above equations can be made, which solves axioms (2.1)-(2.3) by construction and at the same time makes explicit the form factor pole structure

$$
F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) = H_n^{\mathcal{O}} Q_n^{\mathcal{O}}(y_1, \ldots, y_n) \prod_{i=1}^{n} r(\theta_i) \prod_{1 \le i < j \le n} \frac{f(\theta_{ij})f(\hat{\theta}_{ij})}{y_i + y_j}. \tag{2.5}
$$

Here $y_i = 2\cosh\theta_i$, $H_n^{\mathcal{O}}$ is a constant and $Q_n^{\mathcal{O}}(y_1, \ldots, y_n)$ is an entire function of the variables $y_1, \ldots, y_n$. The function $f(\theta)$ is the same minimal (2-particle) form factor found in the bulk case and $r(\theta)$ is the minimal 1-particle form factor. Integral representations for these minimal form factors can be readily obtained from the representations of the S-matrix and reflection amplitude $R(\theta)$, respectively.

3 Boundary form factors of the sinh-Gordon model with Dirichlet boundary conditions

The particular example we want to study in this letter is the sinh-Gordon theory with a particular type of Dirichlet boundary conditions. The sinh-Gordon model is one of the simplest integrable models possessing a non-trivial scattering matrix since its spectrum consists of a single particle and no particle bound states occur. The bulk scattering matrix [44, 45, 46] is a pure CDD factor given by

$$
S(\theta) = -(-B)_\theta (B - 2)_\theta, \tag{3.1}
$$

in terms of the blocks

$$
(x)_\theta = \frac{\sinh\frac{1}{2}\left(\theta + \frac{i\pi x}{2}\right)}{\sinh\frac{1}{2}\left(\theta - \frac{i\pi x}{2}\right)}. \tag{3.2}
$$

The parameter $B \in [0, 2]$ is the effective coupling constant which is related to the coupling constant β in the sinh-Gordon Lagrangian [47, 48]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2} \cosh(\beta \phi), \tag{3.3}$$

where m is a mass scale and ϕ is the sinh-Gordon field, as

$$B(\beta) = \frac{2\beta^2}{8\pi + \beta^2}, \tag{3.4}$$

under CFT normalization [3]. The S-matrix is obviously invariant under the transformation $B \rightarrow 2-B$, a symmetry which is also referred to as week-strong coupling duality, as it corresponds to $B(\beta) \rightarrow B(\beta^{-1})$ in (3.4). The point $B = 1$ is known as the self-dual point. When an integrable boundary is introduced and Dirichlet boundary conditions imposed the corresponding reflection amplitude $R(\theta)$ was found to be [49]

$$R(\theta) = (1)_\theta \left(1 + \frac{B}{2}\right)_\theta \left(2 - \frac{B}{2}\right)_\theta \frac{(E-1)_\theta}{(E+1)_\theta}. \tag{3.5}$$

This expression was obtained by analytical continuation of the sine-Gordon breather reflection amplitudes calculated in [50]. In (3.5) E is a continuous parameter which is proportional to the boundary value of the sinh-Gordon field as

$$E = \frac{4iB\phi(0)}{\beta}. \tag{3.6}$$

At the special value $E = \phi(0) = 0$ which we will consider in this manuscript, the reflection amplitude (3.5) has no pole at $\theta = i\pi/2$ and therefore the ansatz (2.5) captures the full pole structure of the boundary form factors. In addition, since Dirichlet boundary conditions imply a fixed value of $\phi(0)$, the boundary operator content of the theory consists solely of products of space-derivatives of the boundary field like $\partial_x \phi$, $(\partial_x \phi)^2$, that is, derivatives in the direction perpendicular to the boundary. The recursive equation obtained by substituting (2.5) in the kinematic residue equation (2.4) was derived in [1] and takes the form

$$Q_{n+2}^{\mathcal{O}}(-y, y, y_1, \ldots, y_n) = -P_n(y, y_1, \ldots, y_n)Q_n^{\mathcal{O}}(y_1, \ldots, y_n). \tag{3.7}$$

The polynomial P_n was also given in [1]. Here we present it in a slightly different form, in terms of elementary symmetric polynomials in the variables $y_1, \ldots, y_n$,

$$P_n(y, y_1, \ldots, y_n) = \frac{i(-\omega_+ \omega_-)^n}{\omega_+ - \omega_-} \sum_{k,p=0}^{n} (-i\omega_-)^{-k}(-i\omega_+)^{-p} \sigma_k^n \sigma_p^n \sin(\pi(k-p)/2), \tag{3.8}$$

which are defined through the generating function*

$$\prod_{k=1}^{n}(z + y_k) = \sum_{k=0}^{n} z^{n-k} \sigma_k^n, \tag{3.9}$$

The variables $\omega_\pm$ are given by

$$\omega_\pm = 2 \cosh(\theta \pm i\pi B/2). \tag{3.10}$$

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*In what follows we will employ the simpler notation $\sigma_k^n = \sigma_k$, since the number of variables n on which the polynomials depend is usually obvious from the context in which they occur.

3.1 Recursive relations at the self-dual point

In this section we initiate the presentation and discussion of the new results obtained in this manuscript. Whereas in [1] computations up to $n = 5$ particle form factors and for generic values of B where carried out, here we have concentrated our analysis on the special case $B = 1$, that is the self-dual point. In this section we will show how at this point a new factorized structure is found, both of the form factor equations and their solutions, which will allow us to find form factor solutions up to much higher particle numbers ($n = 16$) and to put forward a plausible hypothesis for the structure of the solutions of the form factor equations for arbitrary particle numbers. Let us start by noticing that for $B = 1$ the variables $\omega_\pm$ become proportional to each other

$$\omega_\pm = \pm 2i \sinh\theta, \tag{3.11}$$

and this allows us after some simple algebra to bring P_n into the factorized form

$$P_n(y, y_1, \ldots, y_n) = P_n^e(y, y_1, \ldots, y_n) P_n^o(y, y_1, \ldots, y_n), \tag{3.12}$$

with

$$P_n^e(y, y_1, \ldots, y_n) = (-i\omega_+)^{2\left[\frac{n}{2}\right]} \sum_{p=0}^{n} \sigma_p^n (-i\omega_+)^{-p} \cos(\pi p/2),$$

$$P_n^o(y, y_1, \ldots, y_n) = (-1)^{n+1} (-i\omega_+)^{2\left[\frac{n+1}{2}\right]-1} \sum_{p=0}^{n} \sigma_p^n (-i\omega_+)^{-p} \sin(\pi p/2), \tag{3.13}$$

where $[x]$ denotes the integer part of x. Clearly, the polynomials P_n^e and P_n^o involve only elementary symmetric polynomials of even and odd degree, respectively. In addition, the coefficients of the symmetric polynomials in P_n^e and P_n^o are always even powers of $-i\omega_+$, that is, P_n^e and P_n^o are polynomials in $(i\omega_+)^2 = y^2 - 4$, of which the first few examples are

$$\begin{array}{llll}
P_1^e &=& 1, & \qquad P_1^o &=& \sigma_1, \\
P_2^e &=& (i\omega_+)^2 - \sigma_2, & \qquad P_2^o &=& -\sigma_1, \\
P_3^e &=& (i\omega_+)^2 - \sigma_2, & \qquad P_3^o &=& (i\omega_+)^2 \sigma_1 - \sigma_3, \\
P_4^e &=& (i\omega_+)^4 - (i\omega_+)^2 \sigma_2 + \sigma_4, & \qquad P_4^o &=& -(i\omega_+)^2 \sigma_1 + \sigma_3.
\end{array} \tag{3.14}$$

It was noticed in [1] that for $E = 0$ the symmetry of the Lagrangian under $\phi \to -\phi$, where ϕ is the sinh-Gordon fundamental field, is preserved in the boundary theory. This means that, depending on the operator $\mathcal{O}$ considered, only the even or odd particle form factors will be non-vanishing, exactly as in the bulk theory [13, 14, 29].

In [1] the first non-vanishing form factors of two operators identified as $\partial_x\phi$ and $(\partial_x\phi)^2$ were computed. Setting $B = 1$, the solutions found become

$$Q_1^{\partial_x\phi} = 1, \tag{3.15}$$

$$Q_2^{(\partial_x\phi)^2} = -\sigma_1, \tag{3.16}$$

$$Q_3^{\partial_x\phi} = -\sigma_1, \tag{3.17}$$

$$Q_4^{(\partial_x\phi)^2} = \sigma_1^2(4 + \sigma_2), \tag{3.18}$$

$$Q_5^{\partial_x\phi} = \sigma_1(4\sigma_1 + \sigma_3)(4 + \sigma_2). \tag{3.19}$$

Here we have presented the solutions for both operators in order of increasing particle number to make certain features more apparent: first of all, all Q_n factorize into a factor containing only

elementary symmetric polynomials of even degree and another factor containing only those of odd degree. Secondly, some of these factors repeat in various form factors (like $4 + \sigma_2$ which appears both in $Q_4^{(\partial_x \phi)^2}$ and $Q_5^{\partial_x \phi}$), so that some sort of pattern seems to emerge. In this letter strong evidence will be provided that these two features are actually general: all functions $Q_n^{\partial \phi}, Q_n^{(\partial_x \phi)^2}$ factorize and a general pattern for the n-particle solutions exists, such that once the form factors of $\partial_x \phi$ are known also those of $(\partial_x \phi)^2$ are automatically determined and viceversa.

Recalling that the P-function (3.8) factorizes in a similar way, our results suggest that the recursive equations (3.7) can be split up into two simpler recursive equations involving elementary symmetric polynomials of even and odd degree, respectively

$$Q_n^{\mathcal{O}}(y_1, \ldots, y_n) = Q_n^{\mathcal{O}|e}(y_1, \ldots, y_n) Q_n^{\mathcal{O}|o}(y_1, \ldots, y_n), \tag{3.20}$$

with

$$\begin{aligned}
Q_{n+2}^{\mathcal{O}|e}(-y, y, y_1, \ldots, y_n) &= P_n^e(y, y_1, \ldots, y_n) Q_n^{\mathcal{O}|e}(y_1, \ldots, y_n), \\
Q_{n+2}^{\mathcal{O}|o}(-y, y, y_1, \ldots, y_n) &= -P_n^o(y, y_1, \ldots, y_n) Q_n^{\mathcal{O}|o}(y_1, \ldots, y_n).
\end{aligned} \tag{3.21}$$

In this new notation, the solutions above become:

$$\begin{aligned}
Q_1^{\partial_x \phi|e} &= 1, & Q_1^{\partial_x \phi|o} &= 1, \\
Q_2^{(\partial_x \phi)^2|e} &= -1, & Q_2^{(\partial_x \phi)^2|o} &= \sigma_1, \\
Q_3^{\partial_x \phi|e} &= 1, & Q_3^{\partial_x \phi|o} &= -\sigma_1, \\
Q_4^{(\partial_x \phi)^2|e} &= 4 + \sigma_2, & Q_4^{(\partial_x \phi)^2|o} &= \sigma_1^2, \\
Q_5^{\partial_x \phi|e} &= -(4 + \sigma_2), & Q_5^{\partial_x \phi|o} &= -\sigma_1(4\sigma_1 + \sigma_3).
\end{aligned} \tag{3.22}$$

This factorization simplifies greatly the process of finding solutions of (3.7) for higher particle numbers. From (3.21) and (3.13) two important characteristics of the solutions of (3.21) can be deduced:

- the functions $Q_{2n}^{\partial_x \phi|e}$ and $Q_{2n+1}^{(\partial_x \phi)^2|e}$ always coincide, but for the number of variables entering the symmetric polynomials and a sign. The reason is that P_{2n}^e and P_{2n+1}^e are in fact the same in terms of elementary symmetric polynomials and that the input values $Q_1^{(\partial_x \phi)^2|e} = -Q_2^{\partial \phi|e} = 1$ are the same but for a sign.

- Similarly, $Q_{2n+1}^{(\partial_x \phi)^2|o}$ and $Q_{2n+2}^{\partial_x \phi|o}$ also coincide but for an extra $-\sigma_1$ factor in $Q_{2n+2}^{\partial_x \phi|o}$. This is again due to the fact that the corresponding P-functions, P_{2n+1}^o and P_{2n+2}^o are identical up to a sign in terms of elementary symmetric polynomials and that the initial values $Q_2^{\partial_x \phi|o} = \sigma_1$ and $Q_1^{(\partial_x \phi)^2|o} = 1$ differ only by a σ_1 factor which satisfies the first equation in (4.1).

These features can be found on the few examples listed above. Most importantly, they allow us to conclude that, at the self-dual point, the knowledge of all non-vanishing form factors of the operator $(\partial_x \phi)^2$ automatically implies the knowledge of all non-vanishing form factors of the operator $\partial_x \phi$ through the relations

$$\begin{aligned}
Q_{2n}^{(\partial_x \phi)^2|e} &= -Q_{2n+1}^{\partial_x \phi|e}, & \text{for} \quad n \in \mathbb{Z}^+, \tag{3.23} \\
Q_{2n}^{(\partial_x \phi)^2|o} &= -\sigma_1 Q_{2n-1}^{\partial_x \phi|o}, & \text{for} \quad n \in \mathbb{Z}^+. \tag{3.24}
\end{aligned}$$

Note that these equalities are only true when the functions are expressed in terms of elementary symmetric polynomials. The latter will obviously depend on different particle numbers on the r.h.s. and l.h.s. of the equations (and therefore, on different numbers of variables).

4 Solutions for higher particle numbers

We shall now attempt to find new solutions to (3.21) in terms of elementary symmetric polynomials in the standard way employed previously for several bulk theories (see e.g. [13, 14, 10, 22, 23]). This solution procedure can be rather tedious as the number of terms on the r.h.s. of (3.21) increases rapidly with the particle number. From these terms one must be able to "reconstruct" the l.h.s. of the equation when the two first variables are not y and $-y$ but generic. This reconstruction can be performed by employing the following relations between elementary symmetric polynomials depending on $n + 2$ and n variables

$$\sigma_k(-y, y, y_1, \ldots, y_n) \quad = \quad -y^2 \sigma_{k-2}(y_1, \ldots, y_n) + \sigma_k(y_1, \ldots, y_n), \tag{4.1}$$

with $\sigma_{-1} = 0$ and $\sigma_0 = 1$. The relations (4.1) allow us to make a guess about the possible origin of each of the terms on the r.h.s. of (3.21). For example, if a term σ_2^n appears it can only originate from a term σ_2^{n+2} in the l.h.s. and if a term $-y^2 \sigma_n^n$ appears, then it can originate either from a σ_{n+2}^{n+2} or a $\sigma_2^{n+2} \sigma_n^{n+2}$ term on the r.h.s. By consistently accounting for each term on the r.h.s. of (3.21) a unique polynomial solution to the equations can be found. Solving the recursive equations systematically in this way we have found explicit solutions up to 16-particle form factors. All the solutions found exhibit the following general structure:

$$Q_{2n}^{(\partial_x \phi)^2 | e} \quad = \quad (-1)^{n+1} Q_{2n-2}^{(\partial_x \phi)^2 | e} \mu_{2n-2} + \sigma_{2n} R_{2n}^e, \tag{4.2}$$

$$Q_{2n}^{(\partial_x \phi)^2 | o} \quad = \quad (-1)^n Q_{2n-2}^{(\partial_x \phi)^2 | o} \mu_{2n-3} + \sigma_1^2 \sigma_{2n-1} R_{2n}^o, \tag{4.3}$$

Let us now explain the various terms appearing in (4.2)-(4.3):

- The first term on the r.h.s. of the equations shows a recursive relation between $2n$- and $(2n - 2)$-particle solutions of (3.21). However, the polynomials $Q_{2n-2}^{(\partial_x \phi)^2 | e}$ and $Q_{2n-2}^{(\partial_x \phi)^2 | o}$ appearing on the r.h.s. of the above equations are not any more the solutions of (3.21). They are identical to them in terms of elementary symmetric polynomials, but these are now polynomials on all $2n$ variables involved on the l.h.s. of the equation.

- The functions μ_i are defined as follows:

$$\mu_0 \quad = \quad 1, \qquad \mu_1 = \sigma_1,$$
$$\mu_i \quad = \quad 4\mu_{i-2} + \sigma_i, \quad \text{for} \quad i > 1, \tag{4.4}$$

in terms of elementary symmetric polynomials. Equivalently

$$\mu_i = \sum_{k=0}^{[i/2]} 4^k \sigma_{i-2k}, \tag{4.5}$$

where, as before $[x]$ stands for the integer part of x. The emergence of such particular combinations of elementary symmetric polynomials is rather natural given their reduction properties, which can be deduced from (4.1):

$$\mu_k(-y, y, y_1, \ldots, y_n) \quad = \quad (4 - y^2) \mu_{k-2}(y_1, \ldots, y_n) + \sigma_k(y_1, \ldots, y_n), \tag{4.6}$$

where $\mu_{-1} = 0$ and $\mu_0 = 1$. As we can see the variable y appears always in the combination $4 - y^2$, which is precisely the same variable entering the polynomials P^e and P^o.

7

- Further, the conjectured expressions (4.2)-(4.3) involve the additional terms $\sigma_{2n}R_{2n}^e$ and $\sigma_1^2\sigma_{2n-1}R_{2n}^o$, respectively. One can easily convince oneself from the previous definitions, that these are in fact the only terms containing the elementary symmetric polynomials σ_{2n} and σ_{2n-1}, respectively. Therefore $\sigma_{2n}R_{2n}^e$ and $\sigma_1^2\sigma_{2n-1}R_{2n}^o$ are the terms containing the elementary symmetric polynomial of maximum degree. This definition makes it straightforward to reconstruct the functions R_{2n}^e and R_{2n}^o for particular cases up to high particle numbers. From (4.2), (4.3), (3.21), (4.1) and (4.6) it follows that they must satisfy the following reduction properties:

$$-y^2 R_{2n}^e(-y,y,y_1,\ldots,y_{2n-2})\sigma_{2n-2}^{2n-2} = P_{2n-2}^e Q_{2n-2}^{(\partial_x\phi)^2|e}(y_1,\ldots,y_{2n-2})$$
$$+(-1)^n Q_{2n-2}^{(\partial_x\phi)^2|e}(-y,y,y_1,\ldots,y_{2n-2})((4-y^2)\mu_{2n-4}^{2n-2}+\sigma_{2n-2}^{2n-2}),\tag{4.7}$$

$$y^2(\sigma_1^{2n-2})^2 R_{2n}^o(-y,y,y_1,\ldots,y_{2n-2})\sigma_{2n-3}^{2n-2} = P_{2n-2}^o Q_{2n-2}^{(\partial_x\phi)^2|o}(y_1,\ldots,y_{2n-2})$$
$$+(-1)^n Q_{2n-2}^{(\partial_x\phi)^2|o}(-y,y,y_1,\ldots,y_{2n-2})((4-y^2)\mu_{2n-5}^{2n-2}+\sigma_{2n-3}^{2n-2}),\tag{4.8}$$

where, in order to avoid confusion, we have made the variable dependence explicit and

$$\sigma_k^{2n-2} = \sigma_k(y_1,\ldots,y_{2n-2}) \quad\text{and}\quad \mu_k^{2n-2} = \mu_k(y_1,\ldots,y_{2n-2}),\tag{4.9}$$

for any values of k. Notice that, once the conjectured formulae (4.2)-(4.3) are assumed to hold, R_{2n}^e and R_{2n}^o are the only unknowns, which means an enormous simplification of the original problem. Explicit formulae for R_{2n}^e and R_{2n}^o up to $n=8$ can be found in the appendix.

In addition to the structure just outlined, a simple recipe allows to relate the solutions $Q_{2n}^{(\partial_x\phi)^2|e}$ and $Q_{2n+2}^{(\partial_x\phi)^2|o}$. The latter can always be obtained from the former by performing the replacements $\sigma_k \to \sigma_{k+1}$ and $\mu_k \to \mu_{k+1}$ and multiplying by a global factor $-\sigma_1^2$. Once these transformations have been done, we must still introduce as many factors σ_1 as to achieve that each term in $Q_{2n+2}^{(\partial_x\phi)^2|o}$ is exactly a product of $n+1$ symmetric polynomials. For example, if a term σ_8 would appear in $Q_8^{(\partial_x\phi)^2|e}$, then it would imply a term $-\sigma_1^2\sigma_1\sigma_9$ in $Q_{10}^{(\partial_x\phi)^2|e}$. Thus, provided we know $Q_{2n}^{(\partial_x\phi)^2|e}$ we can systematically determine $Q_{2n}^{(\partial_x\phi)^2|o}$. From these solutions, as concluded above (3.23)-(3.24), also all functions $Q_{2n+1}^{\partial_x\phi|e}$ and $Q_{2n+1}^{\partial_x\phi|e}$ can be obtained.

5 Solutions for other boundary fields

In this section we will address the problem of finding solutions of the form factor equations for fields other than $\partial_x\phi$ and $(\partial_x\phi)^2$. It is worth recalling that the fields $\partial_x\phi$ and $(\partial_x\phi)^2$ were identified in [1] by analyzing the asymptotic behaviour of their form factors. Although this identification seems plausible, it would be very interesting to strengthen it with some numerical work.

A particular way of looking for new solutions of the form factor equations is to find entire functions $I_n^s(y_1,\ldots,y_n)$ such that if $F_n^{\mathcal{O}}(y_1,\ldots,y_n)$ is a solution, then the product

$$F_n^{\mathcal{O}'}(y_1,\ldots,y_n) = I_n^s(y_1,\ldots,y_n)F_n^{\mathcal{O}}(y_1,\ldots,y_n),\tag{5.1}$$

is also one. In the bulk case, such type of form factors (with variables $y_i = e^{\theta_i} + e^{-\theta_i}$ replaced by $x_i = e^{\theta_i}$) have been related to "descendent" fields for the first time in the work of J. L. Cardy and G. Mussardo [8] for the Ising model. The name descendent must be understood in the sense

that if $\mathcal{O}$ is related to a spinless primary field of the underlying CFT in the ultraviolet limit, then $\mathcal{O}'$ would be related to a descendent field of $\mathcal{O}$ in the same limit. Which particular class of descendent fields $\mathcal{O}'$ relates to is determined by the scaling properties of the function I_n^s under a rapidity shift, that is, the spin of $\mathcal{O}'$ which we call s. Solutions to the bulk form factor equations of the type (5.1) have been also found for the sinh-Gordon [13] and Yang-Lee [51] models.

In our case, the link between form factors of the type (5.1) and descendent fields in the conformal limit is a priory less clear, as the operator content of the CFT at the boundary is not classified in terms of spin representations. However, we may still attempt to find solutions of the form (5.1). They will be related to new fields of the QFT, that is higher derivatives of ϕ which is very natural to relate to descendent fields at conformal level. In this case the superscript s in I_n^s is just the leading order behaviour of I_n^s when all rapidities tend to infinity. The analysis is in fact completely analog to the one presented in [13] for the bulk sinh-Gordon model. We just need to replace the variables x_i by the variables y_i. First of all, it is trivial to see that the form factor (5.1) will automatically satisfy all consistency equations (2.1)-(2.3) provided $I_n^s(y_1, \ldots, y_n)$ can be entirely expressed in terms of elementary symmetric polynomials in the variables $\{y_1, \ldots, y_n\}$. Secondly, if (5.1) is to satisfy the kinematic residue equation (2.4) then I_n^s must satisfy the following reduction property:

$$I_{n+2}^s(-y, y, y_1, \ldots, y_n) = I_n^s(y_1, \ldots, y_n). \tag{5.2}$$

This is exactly the same equation found in [8, 13]. A basis of the space of solutions to (5.2) has been found to be

$$I_n^{2s-1} = (-1)^{s+1} \det \mathcal{I}, \tag{5.3}$$

with $\mathcal{I}$ being a matrix of entries

$$\mathcal{I}_{1j} = \sigma_{2j-1} \qquad \text{and} \qquad \mathcal{I}_{ij} = \sigma_{2j-2i+2}, \tag{5.4}$$

for $j = 1, \ldots, s$ and $i = 2, \ldots, s$. As in the bulk case $\det \mathcal{I}$ is of order $2s-1$ in terms of elementary symmetric polynomials.

To finish this section it is worth mentioning that it would be very interesting to establish rigorously a correspondence between the number of solutions to the form factor equations for fixed s and the number of descendent fields of the underlying CFT at level s. For this one should proceed along the lines of the analysis carried out in [8] for the bulk Ising model. It is however to be expected that such analysis becomes more involved for the present model.

6 Conclusions and outlook

In this paper we have studied the form factors of two boundary operators of the sinh-Gordon model with a particular type of Dirichlet boundary conditions, corresponding to vanishing boundary value of the sinh-Gordon field, at the self-dual point. We have found that both the form factor recursive equations (3.7) and their solutions exhibit distinct properties at $B = 1$: they both factorize into one factor containing only elementary symmetric polynomials of even degree and another term containing only those of odd degree. This factorization has allowed us to obtain explicit solutions of the equations up to remarkably high particle numbers and to provide a conjecture for the structure of the form factor solutions for arbitrary particle numbers. Unfortunately we have not yet been able to provide a mathematically rigorous proof of that structure, although the many examples computed strongly support it.

The finding of closed formulae for all n-particle form factors of local fields is a highly complicated task which has only been successfully carried out for few bulk scattering theories and

operators [10, 11, 13, 14, 17, 22, 23]. Even when closed formulae have been found, rigorous proofs of their general validity have only been given in some cases [13, 22, 23]. A crucial element of these proofs has been the fact that the form factor solutions were given in terms of determinants whose entries were elementary symmetric polynomials. It is this determinant structure and the possibility of employing general properties of determinants which made these proofs possible. The formulae conjectured in the present paper (4.2)-(4.3) are however not of determinant form, so that general proofs should go along very different lines. In addition, it is worth noticing an important difference with respect to the bulk form factor program with regard to the structure of the form factor solutions: boundary fields do not possess a definite spin (since translation invariance is broken) and therefore the transformation under a rapidity shift of the allowed terms in the form factor formulae is not constrained. This means that not all terms in the functions Q_n need to have the same degree. As a result, formulae tend to be more involved and the form factor's general structure is harder to identify. It would be very interesting to investigate whether or not some structures of the boundary form factors may be universal or at least common to many models. By this we mean structures such as the determinant formulae mentioned above for bulk theories. For the case at hand, it would be of course desirable to find a general pattern for the functions R_n^e, R_n^o, which may also facilitate proofs.

In addition, new solutions to the form factor consistency equations related to local operators other than $\partial_x \phi$ and $(\partial_x \phi)^2$ have been found. We have seen that the form factors of these fields are related to the ones of $\partial_x \phi$ and $(\partial_x \phi)^2$ by a multiplicative factor (5.1). The same kind of structure has been previously identified for the form factors of the bulk Ising [8], sinh-Gordon [13] and Yang-Lee models [51] and has been related to descendant fields at the conformal level. It is natural to assume that this interpretation also holds here, however two interesting open problems remain: first, a numerical study of the short-distance behaviour of correlation functions of these fields should be carried out in order to make the identification with descendents at conformal level more clear. Second, a rigorous study of the correspondence between form factor solutions with a particular asymptotic behaviour and descendent fields for fixed level should be carried out along the lines of [8].

We would like to conclude by saying that the program proposed in [1] is very recent and has so far only been pursued for the few models treated in the original paper and for the particular case considered here in more detail. A lot of work remains to be done to develop the boundary form factor program to a degree comparable to that achieved for bulk theories. Obviously computing boundary form factors of other integrable models would be interesting, particularly so for non-diagonal theories such as the sine-Gordon model. For the latter theory boundary form factors have in fact been computed in [52] by a different approach but it would still be interesting to employ the method of [1] for the same computation as a consistency check and/or as a possible way to obtain alternative (maybe more explicit) representations of the form factors. It would also be very interesting to exploit these form factors solutions in similar ways as has been done for bulk theories: computing correlation functions, investigating the operator content and computing characteristic quantities of the underlying boundary CFT. Some effort in this direction has been made in [1].

A Formulae for the polynomials R_{2n}^e and R_{2n}^o

For the solutions already found in [1] and listed in (3.22), it is easy to see that $R_2^e = R_2^o = R_4^e = R_4^o = 0$. However, for higher particle form factors non-trivial R-functions are found, which become more involved as the particle number increases. Expressions become simpler when written in terms of the polynomials:

$$\alpha_i = \sum_{k=0}^{[i/2]} (k+1)4^k \sigma_{i-2k}, \qquad \beta_i = \frac{1}{2} \sum_{k=0}^{[i/2]} (k+1)(k+2)4^k \sigma_{i-2k} \tag{A.1}$$

We have constructed the following solutions†:

$$
\begin{aligned}
R_6^e &= -1, \\
R_8^e &= \alpha_2^2, \\
R_{10}^e &= (8(2+\alpha_2) - \alpha_4)\alpha_6 + \beta_2(\alpha_4^2 - 2\alpha_8) + \alpha_{10}, \\
R_{12}^e &= -\beta_2\alpha_6(\beta_4\alpha_6 - ((16+\alpha_2)\alpha_8 - 3\alpha_{10})) + \alpha_6^3 - \beta_2^2(8(2\beta_2+\alpha_4)\alpha_8 - 2\beta_4\alpha_{10} + \beta_2\alpha_{12}), \\
R_{14}^e &= (8+\alpha_2)^2\alpha_8^3 + \alpha_8^2\left(\alpha_6^2 - \alpha_2\alpha_6\beta_4 - 4\beta_2\beta_4^2\right) + 2\alpha_6^2\left(4\alpha_{10} - \alpha_{12}\right)\beta_4 \\
&\quad -8\alpha_{10}\beta_4^2\left((2+\alpha_2)\alpha_6 + 2\beta_2\beta_4\right) + \left(64\left(28 + \alpha_2\left(6+\beta_2\right)\right) + \alpha_4^2\right)\alpha_8\alpha_{10}\beta_2 \\
&\quad + (4\left(128 + \alpha_2\left(48 + 5\alpha_2\right)\right)\alpha_4 - 2\left(8+\alpha_2\right) - \alpha_6\left(4\alpha_2 + \beta_4\right))\alpha_8\alpha_{10} \\
&\quad + \left(1792 + 4\alpha_2\left(16\left(6+\beta_2\right) + \alpha_4\right) - 12\alpha_0\alpha_6 + \alpha_8\right)\alpha_{10}^2 + \left(8\alpha_{12} - \alpha_{14}\right)\beta_2\beta_4^3 \\
&\quad + \left(2\alpha_2\alpha_{12} + \alpha_{14}\right)\alpha_6\beta_4^2 + \alpha_{10}\alpha_{12}\left(\left(4-\alpha_2\right)\alpha_4 + 3\alpha_6 - 4\left(7\alpha_2^2 + 84\beta_2 - 48\right)\right) \\
&\quad + \left(8+\alpha_2\right)\left(\alpha_6 - \left(8+3\beta_2\right)\beta_4\right)\alpha_8\alpha_{12} + \left(208 + 3\alpha_2\left(16+\alpha_2\right)\right)\alpha_{12}^2 \\
&\quad + \left(16\left(16+\alpha_4\right) + \alpha_2\left(64 + 3\beta_4\right) - 2\alpha_6\right)\alpha_8\alpha_{14} + \left(8\left(8+\alpha_2\right) - \alpha_4\right)\alpha_{10}\alpha_{14} \\
&\quad - \left(16 + 3\beta_2\right)\alpha_{12}\alpha_{14} + \alpha_{14}^2,
\end{aligned}
$$

(A.2)

(A.3)

---

†Notice that the formula for R_{16}^e given in the next page can only be found in the ArXiv version of the manuscript. For shortness it has been removed in J. Phys. A. published version of the manuscript. This is however the only difference between the two documents.

$$
\begin{aligned}
R^e_{16} \;=\; & ((\alpha_2\alpha_4 - \alpha_6 - 16\,(4 + 3\beta_2))\alpha_6 - (8 + \alpha_2)^2)\alpha_8^2\alpha_{10}^2 + (16\,(7 + \alpha_2) + \alpha_4)\,\alpha_6\alpha_{10}^3 \\
& + 2\,(8 + \alpha_2)\,\alpha_8\alpha_{10}^3 - \alpha_{10}^4 - 2^8\,(64 + \beta_2\,(16 + \beta_2\,(20 + 17\beta_2)))\,\alpha_4\alpha_{10}\alpha_{12} \\
& - 128((-8 + \beta_2\,(2 + 7\,\beta_2))\,\alpha_4^2 + 2\,(16 + \beta_2\,(-4 + \beta_2\,(-3 + 2\,\beta_2)))\,\alpha_6)\alpha_{10}\alpha_{12} \\
& - 4(16\,(-8 + \alpha_2\,(14 + 3\alpha_2)) + (4 + 5\alpha_2)\,\alpha_4)\alpha_4\alpha_6\alpha_{10}\alpha_{12} + \alpha_6^3\alpha_{10}\alpha_{12} \\
& - (4\,(-160 + (-24 + \alpha_2)\,\alpha_2) + (-16 + \alpha_2)\,\alpha_4)\alpha_6^2\alpha_{10}\alpha_{12} \\
& + (8 + \alpha_2)\,(8\,(16\,(8 + \alpha_4) + \alpha_2\,(48 + 2\alpha_2 + \alpha_4)) + (24 + \alpha_2)\,\alpha_6)\,\alpha_8\alpha_{10}\alpha_{12} \\
& + (16\,(80 + \beta_2\,(32 + \beta_2\,(11 + \beta_2))) + 2\alpha_2\,(12 + \alpha_2)\,\alpha_4 - (40 + 3\alpha_2)\,\alpha_6)\alpha_{10}^2\alpha_{12} \\
& + 4\alpha_6\,(96\,(80 + \alpha_4) + \alpha_2\,(2368 - \alpha_2\,(-192 + \alpha_4) + 12\alpha_4 + \alpha_6))\,\alpha_{12}^2 \\
& + 12(8 + \alpha_2)^2\,(12 + \alpha_2)\,\alpha_8\alpha_{12}^2 - (8 + \alpha_2)\,(12 + \alpha_2)\,(24 + \alpha_2)\,\alpha_{10}\alpha_{12}^2 \\
& - 2(2^8\beta_2\,(-80 + (-8 + \alpha_2)\,\alpha_2)\,\alpha_4 - \alpha_6^2\left(\alpha_6 + 4\,(4 + \beta_2)^2 - (-8 + \beta_2)\,\beta_4\right))\alpha_8\alpha_{14} \\
& + \left(32\,(64 + \beta_2\,(2 + \beta_2)\,(8 + 5\alpha_2)) + \alpha_4\left(-2^9 + 44\alpha_2^2 + 3\alpha_2\,(64 + \alpha_4)\right)\right)\alpha_6\alpha_{10}\alpha_{14} \\
& - (28\,(8 + \alpha_2) + 3\beta_4)\,\alpha_6^2\alpha_{10}\alpha_{14} - 8\,(8 + \alpha_2)\,(80 + \alpha_2\,(22 + \alpha_2))\,\alpha_8\alpha_{10}\alpha_{14} \\
& - (8 + \alpha_2)\,((24 + \alpha_2)\,\alpha_4 + 4\alpha_6)\alpha_8\alpha_{10}\alpha_{14} - 16\,(\beta_2\,(8 + 3\beta_2\,(4 + \beta_2)) - 32)\,\alpha_4\alpha_{14}^2 \\
& + (4\,(-32 + 2\alpha_4 + \alpha_6) - \alpha_2\,(4\,(8 + \alpha_2) + \beta_4))\,\alpha_{10}^2\alpha_{14} \\
& + 4\left(32\left(2^9 + \beta_2\,(4 + \beta_2)\,(88 + \beta_2\,(20 + 9\beta_2))\right) + \beta_2\,(40 + 7\alpha_2)\,\alpha_4^2\right)\alpha_{12}\alpha_{14} \\
& + 16\alpha_4\,(-320 + 3\beta_2\,(-16 + \beta_2\,(20 + 7\beta_2)))\,\alpha_{12}\alpha_{14} \\
& - (4\,(896 + \beta_2\,(528 + \beta_2\,(112 + \beta_2))) - (-192 + (-32 + \alpha_2)\,\alpha_2)\,\alpha_4)\alpha_6\alpha_{12}\alpha_{14} \\
& - (((\alpha_2 - 8)\,\alpha_6^2 + 3(8 + \alpha_2)^2\,(16 + \alpha_2)\,\alpha_8) - (8 + \alpha_2)\,(72 + 5\alpha_2)\,\alpha_{10})\alpha_{12}\alpha_{14} \\
& - 16\left(2^{10} + \beta_2\,(896 + \beta_2\,(416 + \beta_2\,(108 + 13\,\beta_2)))\right)\alpha_{14}^2 \\
& + (2432 + 24\alpha_4 + \alpha_2\,(688 + 52\alpha_2 + 5\alpha_4) - 2\alpha_6)\,\alpha_6\alpha_{14}^2 + 3(8 + \alpha_2)^2\,\alpha_8\alpha_{14}^2 \\
& - 4\,(8 + \alpha_2)\,\alpha_{10}\alpha_{14}^2 + 2(8 + \alpha_2)^2\,((8 + \alpha_2)\,\alpha_8 - \alpha_{10})\,\alpha_{10}\alpha_{16} \\
& + (8 + \alpha_2)^2\left(-8\left(2^8 + \alpha_2\,(80 + 8\alpha_2 + \alpha_4)\right) + (24 + \alpha_2)\,\alpha_6\right)\alpha_{12}\alpha_{16} \\
& + (8 + \alpha_2)^2\,(1152 + 8\alpha_4 + \alpha_2\,(336 + 28\alpha_2 + 3\alpha_4) - 4\alpha_6)\,\alpha_{14}\alpha_{16} - (8 + \alpha_2)^4\alpha_{16}^2 \\
& - 2^{10}\,(896 + \alpha_2\,(496 + \alpha_2\,(100 + 7\alpha_2)))\,\alpha_{10}\alpha_{12}\beta_2 - 32\alpha_4^3\beta_2(2\alpha_{10}\alpha_{12} + \alpha_8\alpha_{14}) \\
& - (16 + 3\alpha_2)\,\alpha_4^2\alpha_{14}^2\beta_2 + 2^{14}\alpha_8\alpha_{14}\beta_2^2\,(2 + \beta_2) - 2^8\,\alpha_4^2\alpha_8\alpha_{14}\,(-8 + \alpha_2\beta_2) \\
& + 64\left(-2^8\,(88 + 3\alpha_4) - 2\alpha_2\,(7936 + 2\alpha_2\,(1120 + \alpha_2\,(144 + 7\alpha_2)))\right)\alpha_{12}^2 \\
& + 64\left((320 + \alpha_2\,(72 + 5\alpha_2))\,\alpha_4 - (6 + \alpha_2)\,\alpha_4^2\beta_2\right)\alpha_{12}^2 + 12\alpha_4^2\beta_2\alpha_{10}\alpha_{14}\beta_4 \\
& + \left(-16\,(16 + \alpha_2\,(9 + \alpha_2))\,\alpha_4 - \alpha_4^2\beta_2 - 64\,(16 + \beta_2\,(8 + \beta_2\,(4 + \beta_2)))\right)\alpha_{10}^3 \\
& - 16\,((2 + \alpha_2)\,\alpha_4 - 8\,(28 + 5\alpha_2))\,\alpha_6\alpha_8\alpha_{14}\beta_4 + 4\,((\alpha_4 - 32)\,\beta_2 - 64)\,\beta_4\alpha_8\alpha_{10}^2 \\
& + 4\,(4\alpha_4\,(\beta_2\,(12 + 11\beta_2) - 32) + 32\,(64 + \beta_2\,(32 + \beta_2\,(18 + 5\beta_2))))\,\alpha_{10}\alpha_{14}\beta_4 \\
& + 8\,(-4\,(48 + \alpha_4) + \alpha_2\,(-32 + 2\alpha_2 + \alpha_4) - \alpha_6)\,\alpha_6^2\alpha_8\alpha_{12} + 2(8 + \alpha_2)^2\,\alpha_8^2\alpha_{14}\beta_6 \\
& + 64\,((2\,(-48 + \alpha_4) + \alpha_2\,(-8 + 2\alpha_2 + \alpha_4)))\,\beta_4\alpha_6\alpha_8\alpha_{12} - 8(8 + \alpha_2)^2\,\alpha_8^2\alpha_{12}\beta_6 \\
& - (8 + \alpha_2)\,(8\,(48 + \alpha_4) + \alpha_2\,(16\,(9 + \alpha_2) + 3\alpha_4) - 4\alpha_6)\,\alpha_{10}\alpha_{16}\beta_6 \\
& - (8 + \alpha_2)^2\alpha_8\alpha_{16}\beta_6^2 + (-\alpha_6 + \beta_2\beta_4)\,\beta_6^2\left(4\alpha_{10}^2 + (16\alpha_{12} - 8\alpha_{14} + \alpha_{16})\,\beta_6\right). \qquad \text{(A.4)}
\end{aligned}
$$

The polynomials R^o_{2n} can be obtained from the polynomials R^e_{2n-2} by substituting $\alpha_k \to \alpha_{k+1}$, $\beta_k \to \beta_{k+1}$, multiplying by a global factor -1 and introducing as many factors σ_1 as needed in order to achieve that every term in R^o_{2n} is a product of exactly $n-3$ elementary symmetric

polynomials. For example:

$$
\begin{aligned}
R_6^o &= 0, \\
R_8^o &= \sigma_1, \\
R_{10}^o &= -\alpha_3^2, \\
R_{12}^o &= -(8(2\alpha_1 + \alpha_3) - \alpha_5)\alpha_1\alpha_7 - \beta_3(\alpha_5^2 - 2\alpha_1\alpha_9) + \alpha_1^2\alpha_{11}, \\
R_{14}^o &= \beta_3\alpha_7(\beta_5\alpha_7 - ((16\alpha_1 + \alpha_3)\alpha_9 - 3\alpha_1\alpha_{11})) + \alpha_1\alpha_7^3 + \beta_3^2(8(2\beta_3 + \alpha_5)\alpha_9 - 2\beta_5\alpha_{11} + \beta_3\alpha_{13}),
\end{aligned}
\tag{A.5}
$$

and so on. Notice that $\alpha_1 = \sigma_1$.

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