# Form factors of integrable Heisenberg (higher) spin chains

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**Abstract**

We present determinant formulae for the form factors of spin operators of general integrable $XXX$ Heisenberg spin chains for arbitrary (finite dimensional) spin representations. The results apply to any "mixed" spin chains, such as alternating spin chains, or to spin chains with magnetic impurities.

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# 1  Introduction

Correlation functions are the main quantities characterizing quantum theories. From an experimental point of view, measurable physical quantities are directly related to dynamical correlation functions (or equivalently to their Fourier transform, the structure functions), for example in neutron scattering experiments on ferromagnetic crystals [1, 2]. In the context of two dimensional integrable models various approaches to the computation of correlation functions and form factors have been developed over the years. It started first with models equivalent to free Fermions for which considerable works have been necessary to obtain full answer (see e.g. [3, 4, 5, 6, 7, 8, 9]). Going beyond the free Fermion case has been a major challenge for the last twenty years.

For integrable massive 1+1-dimensional quantum field theories, form factors are accessible (using some hypothesis) from the bootstrap form factor program pioneered in the late 70's [10, 11, 12]; the analytical summation of their series corresponding to the correlation functions of local operators remains however an open question, although accurate numerical results exists for many theories and correlation functions.

For integrable quantum spin chains [13, 14, 15] and lattice models [16], the first attempts to go beyond free Fermion models relied on the Bethe ansatz techniques [17, 18] and was undertaken by A. G. Izergin and V. E. Korepin (see e.g. [13] and references therein). Their approach yields formulae for the correlation functions [13, 19, 20, 21] written as vacuum expectation values of some determinants depending on so-called "dual fields" which were introduced to overcome the huge combinatorial sums arising in particular from the action of local operators on Bethe states. However these formulae are not completely explicit, since these "dual fields" cannot be eliminated from the final result.

In the last fifteen years, two main approaches to a more explicit computation of form factors and correlation functions have been developed, mainly for lattice models.

One of these approaches was initiated by M. Jimbo, T. Miwa and their collaborators [22, 23, 24] and enables, using some hypothesis, to compute form factors and correlation functions of quantum spin chains of infinite length (and in their massive regime) by expressing them in terms of traces of $q$-deformed vertex operators over an irreducible highest weight representation of the corresponding quantum affine algebra. These traces turn out to satisfy an axiomatic system of equations called q-deformed Knizhnik-Zamolodchikov (q-KZ) equations, the solutions of which can be expressed in terms of multiple integral formulae. Using these equations similar formulae can be obtained in the massless regime. Recently, a more algebraic representation for the solution of these q-deformed Knizhnik-Zamolodchikov equations have been obtained for the XXX and XXZ (and conjectured for the XYZ) spin 1/2 chains; in these representations, all elementary blocks of the correlation functions can be expressed in terms of some transcendental functions [25, 26, 27]. A detailed review of the approach can be found in [14].

The second approach has been developed by N. Kitanine, J. M. Maillet and V. Terras [28, 29, 30]. It combines the algebraic Bethe ansatz techniques [17, 18] with the solution of the so-called quantum inverse scattering problem [28, 29]. It leads in particular to explicit determinant formulae for form factors of the finite Heisenberg spin 1/2 XXX and XXZ chains and to their correlation functions as explicit multiple sums. The solution of the inverse scattering problem means in practice finding an explicit realization of the local operators of a large variety of quantum spin chains in terms of the quantum monodromy matrix entries appearing in the algebraic Bethe ansatz framework and containing in particular the creation operators for Bethe eigenstates of the chain. Hence the computation of elementary blocks of the correlation functions reduces to a soluble algebraic problem in the Yang-Baxter algebra generated by the monodromy

matrix entries. Elementary blocks of correlation functions of the infinite XXZ spin 1/2 chain both in the massive and massless regime as well as in the presence of a magnetic field [30] have been computed in terms of multiple integrals. At zero magnetic field it gives a complete proof of the multiple integral representations obtained in [22, 24] both for massive and massless regimes. Hence, together with the works [22, 24], it also gives a proof that correlation functions of the XXZ (inhomogeneous) chain indeed satisfy (reduced) q-deformed Knizhnik-Zamolodchikov equations. In addition this method has proven to be effective in dealing with spin-spin correlation functions in the presence of a magnetic field [31, 32], dynamical correlation functions [33] and at non zero temperature [34], cases which were out of reach of the vertex operator method. A recent review of this approach can be found in [15].

So far, we have only mentioned various works dealing with spin 1/2 chains. However, considerable effort has been made to extend the results mentioned above to the higher spin integrable chains [35, 36, 37, 38]. Integral formulae for the correlation functions of the spin 1 XXZ chain have been obtained in [39, 40, 41]. The correlations of spin chains of arbitrary spin have also been studied: integral formulae for the form factors of local operators of the XXZ chain were obtained in [42] and for the correlation functions of the XYZ chain in [43]. From the Bethe ansatz method, the solution of the quantum inverse scattering problem has been given also for higher spin cases in [29] leading in [44] to integral formulae for the correlation functions of the spin 1 XXX spin chain.

The aim of the present article is to employ the Bethe ansatz approach and the solution of the quantum inverse scattering problem for higher spins to obtain determinant formulae for the form factors of spin operators of general integrable Heisenberg quantum spin chains in arbitrary (mixed) finite dimensional spin representations. More precisely, we will consider integrable Heisenberg spin chains with different finite dimensional spin representations at each site. With regard to their physical properties (in particular the characteristics of the zero temperature ground state), these "mixed" spin chains can be subdivided into two groups: those where spins are mixed in fixed proportions and those where a particular type of spin dominates while other spins can be regarded as impurities. Examples of the first type of chains are alternating spin chains. The second type are spin chains with magnetic impurities. The eigenstates, spectrum and thermodynamic properties of both classes of models have been studied within the Bethe ansatz framework. The alternating spin 1- spin 1/2 chain has been studied in [45, 46, 47]. More general combinations of spins have been dealt with in [48, 49]. The spin 1/2 chain with one spin $s$ impurity has been studied in [50] and especially in the context of the Kondo problem [51, 52]. The more general situation of a spin $s$ XXX chain with one spin $s'$ impurity was analyzed in [53] (the special case $s = 1$ has been studied in [54, 55]). The effect of impurities in Heisenberg quantum spin chains has also been the object of experimental investigation, as shown for example in [56].

In this article we will compute the form factors of local spin operators for the general integrable XXX (mixed) spin chains and express our formulae in terms of determinants of elementary functions, similar to those found in [28] for the spin 1/2 case.

This paper is organized as follows: In section 2 we review the algebraic Bethe ansatz framework for the general XXX quantum spin chains. In section 3 we derive closed formulae for all form factors of spin operators of the XXX chain in arbitrary spin representations. We employ our formulae to compute the total magnetization of the chain. In section 4 we give our conclusions and discuss some perspectives. Some lengthy computations are given in appendices: appendix A contains proofs of two identities involving the higher spin eigenvalues of the transfer matrix which we have employed in our form factor computations and appendix B presents an alternative formula for the solution of the quantum inverse scattering problem for local spin

operators useful in form factor computations.

## 2 Algebraic Bethe ansatz for XXX quantum spin chains

In this paper we will consider general integrable XXX quantum spin chains of length $N$ and periodic boundary conditions $S_{N+i}^\alpha = S_i^\alpha$ for $\alpha = z, \pm$. From the algebraic Bethe ansatz view point, the fundamental object characterizing such chains is the quantum monodromy matrix,

$$
\begin{aligned}
T_{0;1\ldots N}^{(\frac{1}{2})}(\lambda; \{\xi\}) &= R_{0N}^{(\frac{1}{2}, s_N)}(\lambda - \xi_N) \cdots R_{0j}^{(\frac{1}{2}, s_j)}(\lambda - \xi_j) \cdots R_{01}^{(\frac{1}{2}, s_1)}(\lambda - \xi_1) \\
&= \begin{pmatrix} A(\lambda; \{\xi\}) & B(\lambda; \{\xi\}) \\ C(\lambda; \{\xi\}) & D(\lambda; \{\xi\}) \end{pmatrix}_0,
\end{aligned}
\tag{2.1}
$$

given as a tensor product of $R$-matrices solutions of the Yang-Baxter equations [57, 58] which express the quantum integrability of the system,

$$
R_{0j}^{(\frac{1}{2}, s_j)}(\lambda - \xi_j) \in \mathrm{End}(V_0^{(\frac{1}{2})} \otimes V_j^{(s_j)}),
\tag{2.2}
$$

where $V^{(s)}$ are (2s+1)-dimensional vector spaces, usually chosen as $\mathbb{C}^{2s+1}$. The quantities $\{\xi\} = \{\xi_1, \ldots, \xi_N\}$ are arbitrary inhomogeneity parameters attached to the sites of the chain. The spins $s_1, \ldots, s_N$ at each site of the chain are belonging to arbitrary finite dimensional representations in (2.1) which can be a priori different for different sites of the chain. The auxiliary space, indicated by the 0 index, has been chosen to be 2-dimensional, which allows us to rewrite $T(\lambda)$ in the standard matrix form with operator entries $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}^*$. As a consequence of the Yang-Baxter equations for the $R$-matrices, these operators satisfy a Yang-Baxter quadratic algebra and the traces of the above constructed monodromy matrices, namely the transfer matrices,

$$
t^{(1/2)}(\mu) = (A + D)(\mu),
\tag{2.3}
$$

commute among themselves for arbitrary spectral parameters $\mu, \lambda$, namely,

$$
[t^{(1/2)}(\mu), t^{(1/2)}(\lambda)] = 0,
\tag{2.4}
$$

leading to a large commutative subalgebra. Thus, the above constructed monodromy matrix describes a general XXX integrable quantum chain having different (finite dimensional) spin representations at each site of chain.

A particular example of $R$-matrices is the one corresponding to the spin 1/2 representation, the well-known $4 \times 4$ matrix

$$
R^{(\frac{1}{2}, \frac{1}{2})}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\tag{2.5}
$$

where the functions $b(\lambda)$ and $c(\lambda)$ are

$$
b(\lambda) = \frac{\lambda}{\lambda + \eta} \quad \text{and} \quad c(\lambda) = \frac{\eta}{\lambda + \eta}.
\tag{2.6}
$$

---

*We will usually not make explicit the dependence of the operators $A$, $B$, $C$ and $D$ and the monodromy matrix (2.1) on the inhomogeneity parameters $\{\xi\}$.

and $\eta$ is an arbitrary parameter. The matrix (2.5) is normalized in such a way that the following unitarity condition is automatically satisfied

$$R_{12}^{(\frac{1}{2},\frac{1}{2})}(\lambda)R_{21}^{(\frac{1}{2},\frac{1}{2})}(-\lambda) = I_{12}, \tag{2.7}$$

where $I$ is the identity matrix. $R$-matrices and monodromy matrices of higher spin representations can be constructed from (2.6) by means of the fusion procedure developed in [35]. It gives,

$$R^{(\frac{1}{2},s)}(\lambda) = \frac{1}{\lambda + \eta\left(s + \frac{1}{2}\right)} \begin{pmatrix} \lambda + \eta\left(S^z + \frac{1}{2}\right) & \eta S^- \\ \eta S^+ & \lambda + \eta\left(\frac{1}{2} - S^z\right) \end{pmatrix}, \tag{2.8}$$

where $S^{\pm}, S^z$ are generators of the $su(2)$ algebra in the spin $s$ representation with standard commutation relations.

## 2.1 The Bethe ansatz equations

Having introduced $R$- and $T$-matrices we can now proceed to characterize the energy eigenstates of the chain. The key observation there is that the local Hamiltonian operator can be identified as a simple function of the transfer matrix. Hence finding its spectrum reduces to constructing the common spectrum of transfer matrices $t^{(1/2)}(\mu)$ for arbitrary values of the spectral parameter $\mu$ (since they all commute). The algebraic Bethe ansatz method starts by constructing a particular eigenstate of the transfer matrix, called the reference state $|0\rangle$, which is annihilated by the operator $C(\lambda)$. In order to identify such a state we notice that for every vector space $V_j$ one can find a vector $|0\rangle_j$ of dimension $2s_j + 1$ which fulfils the following equality

$$R_{0j}^{(\frac{1}{2},s_j)}(\lambda - \xi_j)|0\rangle_j = \begin{pmatrix} 1 & * \\ 0 & \frac{\lambda - \xi_j^- - s_j\eta}{\lambda - \xi_j^- + s_j\eta} \end{pmatrix} |0\rangle_j. \tag{2.9}$$

Here, and in the following, we use the notation $u^{\pm} = u \pm \eta/2$ for any complex number $u$. The reference state $|0\rangle$ is nothing but the tensor product

$$|0\rangle = \bigotimes_{j=1}^{N} |0\rangle_j \qquad \text{with} \qquad |0\rangle_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_j, \tag{2.10}$$

that is a completely ferromagnetic state with all spins up. The monodromy matrix (2.1) acts on the reference state as

$$T_{0;1\dots N}^{(1/2)}(\lambda)|0\rangle = \begin{pmatrix} 1 & * \\ 0 & d(\lambda) \end{pmatrix} |0\rangle, \tag{2.11}$$

namely

$$A(\lambda)|0\rangle = |0\rangle, \quad C(\lambda)|0\rangle = 0 \quad \text{and} \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \tag{2.12}$$

with

$$d(\lambda) = \prod_{j=1}^{N} \frac{\lambda - \xi_j^- - s_j\eta}{\lambda - \xi_j^- + s_j\eta} = \prod_{j=1}^{N} \left[ \prod_{k=1}^{2s_j} b(\lambda - \xi_j^- - (k - s_j)\eta) \right]. \tag{2.13}$$

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It follows from the definition above that $d(\lambda)$ has zeros at $\lambda = \xi_j^- + s_j\eta$ , and poles at $\lambda = \xi_j^- - s_j\eta$ for all $j = 1, \ldots, N$. Let us now proceed to the description of the spectrum. Eigenvectors of the transfer matrix other than $|0\rangle$ will be constructed within the Bethe ansatz framework by acting successively on $|0\rangle$ with operators $B(\lambda_i)$. In general such eigenvectors will be of the form

$$|\Psi(\{\lambda\})\rangle = B(\lambda_1)\ldots B(\lambda_\ell)|0\rangle. \tag{2.14}$$

We want to select out those states (2.14) which are eigenstates of the transfer matrix and hence of the Hamiltonian. The condition that (2.14) is a common eigenstate of (2.3) for all values of $\mu$ is a set of coupled algebraic equations for the spectral parameters $\lambda_i$ describing the state (2.14) which are known as Bethe ansatz equations

$$\prod_{k=1}^{\ell} \frac{b(\lambda_j - \lambda_k)}{b(\lambda_k - \lambda_j)} = -d(\lambda_j), \tag{2.15}$$

The eigenvalues of $t^{(1/2)}(\mu)$ corresponding to the eigenvectors (2.14) fulfilling (2.15) are given by

$$\tau^{(1/2)}(\mu, \{\lambda\}) = \prod_{j=1}^{\ell} b^{-1}(\lambda_j - \mu) + d(\mu) \prod_{j=1}^{\ell} b^{-1}(\mu - \lambda_j). \tag{2.16}$$

## 2.2   Fusion

The mechanism of fusion for quantum spin chains was first developed in [35] for the XXX spin chain (and later on in [59] for the XXZ spin chain). It provides a procedure for constructing higher spin objects ($R$-, monodromy and transfer matrices) starting with spin $1/2$ objects. The fusion identities for the quantum monodromy matrix can be written as follows

$$P_{12}T_1^{(\frac{1}{2})}(x^- + s\eta)T_2^{(s-\frac{1}{2})}(x^-)P_{12} = \begin{pmatrix} T_{\langle 12\rangle}^{(s)}(x) & 0 \\ * & \chi(x + (s-1)\eta)T_{(12)}^{(s-1)}(x - \eta) \end{pmatrix}, \tag{2.17}$$

where for the sake of simplicity we dropped the quantum indices $1, \ldots, N$ of the monodromy matrices (2.1) and the monodromy matrix associated to the spin $s$ in the auxiliary (first) space can be computed as a product over $R^{(s,s_j)}$ matrices as,

$$T_{0;1\ldots N}^{(s)}(\lambda; \{\xi\}) = R_{0N}^{(s,s_N)}(\lambda - \xi_N)\cdots R_{0j}^{(s,s_j)}(\lambda - \xi_j)\cdots R_{01}^{(s,s_1)}(\lambda - \xi_1) \tag{2.18}$$

with,

$$R_{0j}^{(s,s_j)}(\lambda - \xi_j) \in \mathrm{End}(V_0^{(s)} \otimes V_j^{(s_j)}). \tag{2.19}$$

The matrix

$$P_{12} = P_{\langle 12\rangle}^+ \oplus P_{(12)}^-, \tag{2.20}$$

is a direct sum of projectors $P^{\pm}$ onto the vector spaces $V_{\langle 12\rangle}^{(s)} \sim \mathbb{C}^{2s+1}$ and $V_{(12)}^{(s-1)} \sim \mathbb{C}^{2s-1}$ resulting from the tensor decomposition

$$V_1^{(s-\frac{1}{2})} \otimes V_2^{(\frac{1}{2})} \simeq V_{\langle 12\rangle}^{(s)} \oplus V_{(12)}^{(s-1)}, \tag{2.21}$$

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with $V_0^{(\frac{1}{2})} \sim \mathbb{C}^2$ and $V_{\hat{0}}^{(s-\frac{1}{2})} \sim \mathbb{C}^{2s}$. Further

$$\chi(u) = A(u^+)D(u^-) - B(u^+)C(u^-), \tag{2.22}$$

is the quantum determinant of the transfer matrix (2.1), which commutes with all operators $A, B, C, D$. A completely analog expression holds when replacing $T$-matrices by $R$-matrices which permits the construction of (2.8) starting with (2.5). A direct consequence of (2.17) is the following recursive relation involving transfer matrices of various spins,

$$t^{(s)}(x) = t^{(1/2)}(x^- + s\eta)t^{(s-1/2)}(x^-) - \chi(x + (s-1)\eta)t^{(s-1)}(x - \eta), \tag{2.23}$$

and therefore,

$$\tau^{(s)}(x) = \tau^{(1/2)}(x^- + s\eta)\tau^{(s-1/2)}(x^-) - d(x^- + (s-1)\eta)\tau^{(s-1)}(x - \eta), \tag{2.24}$$

for the corresponding eigenvalues on a Bethe state $|\Psi(\{\lambda\})\rangle$ and where for simplicity we have dropped the explicit dependency of the eigenvalues $\tau$ on the spectral parameters $\lambda$ of the Bethe state. In [59] it was also realized that, as a consequence of (2.23), a generating functional for monodromy matrices of higher spin could be constructed as

$$F(z) = \frac{1}{1 - zt^{(1/2)}(u) + z^2\chi(u^+)} = \sum_{k=0}^{\infty} z^k t^{(k/2)}(u^- + k\eta/2). \tag{2.25}$$

Here we take $t^{(0)}(x) = 1$ for any values of $x$, and define $z$ as a shift operator which acts on an operator $t(x)$ as

$$t(x)z = zt(x + \eta). \tag{2.26}$$

There are some differences between (2.25) and the formula given in [59], which are due to the different normalization chosen for the $R$-matrices in this paper. With the help of (2.23) we can easily see that indeed (2.25) generates any higher spin monodromy matrices. One must only match the terms on the l.h.s. and r.h.s. which contain the same powers of $z$, paying special attention to the fact that $z$ is a shift operator. For example, expanding the l.h.s. of (2.25) we find the following first terms

$$F(z) = 1 + zt^{(1/2)}(u) + \left[ zt^{(1/2)}(u)zt^{(1/2)}(u) - z^2\chi(u^+) \right] + \dots \tag{2.27}$$

The $\mathcal{O}(1)$ and $\mathcal{O}(z)$ terms match trivially those on the r.h.s. of (2.25) whereas the $\mathcal{O}(z^2)$ term is

$$\left[ zt^{(1/2)}(u) \right]^2 - z^2\chi(u^+) = z^2 \left[ t^{(1/2)}(u + \eta)t^{(1/2)}(u) - \chi(u^+) \right] = z^2 t^{(1)}(u^+), \tag{2.28}$$

which exactly agrees with (2.23) when taking $s = 1$. Similarly, one can establish the agreement with (2.23) at every order in $z$.

The existence of the generating function $F(z)$ translates into the existence of a generating function for the eigenvalues of the operators $t^{(s)}(u)$. They appear as coefficients of the corresponding Laurent series in $z$ for the generating function. This allowed the authors of [59] to find explicit formulae for the eigenvalues which, translated into our present normalization, take the form

$$\tau^{(s)}(u, \{\lambda\}) = \sum_{\alpha=0}^{2s} C_\alpha^{(s)}(u) \prod_{p=1}^{\ell} \frac{(u^+ - \lambda_p + s\eta)(u^- - \lambda_p - s\eta)}{(u^+ - \lambda_p + (\alpha - s)\eta)(u^- - \lambda_p + (\alpha - s)\eta)}, \tag{2.29}$$

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with

$$C_\alpha^{(s)}(u) = \prod_{k=\alpha}^{2s-1} d(u^+ + (k-s)\eta) \qquad \text{and} \qquad C_{2s}^{(s)}(u) = 1. \tag{2.30}$$

It is now easy to check that (2.29) indeed solves (2.24). Similar relations hold for the XXZ case as well.

## 2.3 The inverse scattering problem

In the context of the algebraic Bethe ansatz approach, the inverse scattering problem is understood as the problem of expressing quantum local operators of the chain in terms of the operators $A, B, C, D$ with generate the Bethe states. The solution to this problem for a large variety of spin chains was provided in [28, 29]. In particular, the reconstruction formulae for the spin operators of the XXX chain in an arbitrary spin $s_j$ representation were found to be

$$S_j^\alpha = \left[\prod_{k=1}^{j-1} t^{(s_k)}(\xi_k)\right] \Lambda_\alpha^{(s_j)}(\xi_j) \left[\prod_{k=1}^{j} t^{(s_k)}(\xi_k)^{-1}\right], \qquad \alpha = \pm, z \tag{2.31}$$

with

$$\begin{aligned}
\Lambda_\alpha^{(s)}(u) &:= \mathrm{Tr}_0\left(S_0^\alpha T_0^{(s)}(u)\right) \\
&= \sum_{k=1}^{2s} t^{(s-\frac{k}{2})}\left(u + \frac{k\eta}{2}\right) \Lambda_\alpha^{(\frac{1}{2})}(u^- + (k-s)\eta) t^{(\frac{k-1}{2})}\left(u^- + \frac{(k-2s)\eta}{2}\right) \tag{2.32} \\
&= \sum_{k=1}^{2s} t^{(\frac{k-1}{2})}\left(u^- + \frac{(k-2s)\eta}{2}\right) \Lambda_\alpha^{(\frac{1}{2})}(u^- + (k-s)\eta) t^{(s-\frac{k}{2})}\left(u + \frac{k\eta}{2}\right). \tag{2.33}
\end{aligned}$$

The expression (2.32) was derived in [29] using (2.17), whereas the non-trivial equivalence between the expressions (2.32) and (2.33) is established in appendix B of the present manuscript. The existence of these two expressions will be very useful for later form factor computations.

## 3 Form factors

Let us start by introducing several formulae which we will use in the course of our computations. First of all, we will need the action of the operators $A$ and $D$ on a generic quantum state $|\Psi(\{\lambda\})\rangle := \prod_{k=1}^{\ell} B(\lambda_k)|0\rangle$,

$$\begin{aligned}
A(x)|\Psi(\{\lambda\})\rangle &= \prod_{k=1}^{\ell} b^{-1}(\lambda_k - x)|\Psi(\{\lambda\})\rangle \\
&\quad - \sum_{p=1}^{\ell} c(\lambda_p - x - \eta) \prod_{k\neq p} b^{-1}(\lambda_k - \lambda_p) B(x) \prod_{k\neq p} B(\lambda_k)|0\rangle, \tag{3.1} \\
D(x)|\Psi(\{\lambda\})\rangle &= d(x) \prod_{k=1}^{\ell} b^{-1}(x - \lambda_k)|\Psi(\{\lambda\})\rangle \\
&\quad + \sum_{p=1}^{\ell} d(\lambda_p) c(\lambda_p - x - \eta) \prod_{k\neq p} b^{-1}(\lambda_p - \lambda_k) B(x) \prod_{k\neq p} B(\lambda_k)|0\rangle. \tag{3.2}
\end{aligned}$$

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We see that this action is divided into two kinds of terms, which are traditionally refer to as "direct" and "indirect" terms. Direct terms are those which leave the original state unchanged up to a scalar factor while in indirect terms one of the parameters $\lambda_p$ has been replaced by the parameter $x$.

Secondly, we need to compute scalar products of quantum states, either of two Bethe states or of one Bethe and one generic state. The formula for the scalar product of two Bethe states was originally obtained in [60, 19]. Later on it was proven [61] that a completely analog formula also holds for the scalar product of a Bethe state and an arbitrary state. Finally, the same formula was re-derived in [28] with the help of the $F$-basis introduced in [62]. It takes the following form,

$$\langle \psi(\{\mu\})|\psi(\{\lambda\})\rangle := S_\ell(\{\mu\}, \{\lambda\}) = \frac{\det H(\{\mu\}, \{\lambda\})}{\prod_{i<j}(\lambda_i - \lambda_j)(\mu_j - \mu_i)} = S_\ell(\{\lambda\}, \{\mu\}), \tag{3.3}$$

where $H(\{\mu\}, \{\lambda\})$ is a $\ell \times \ell$ matrix of components

$$H_{ab} = \frac{\eta}{\mu_a - \lambda_b}\left[\prod_{i\neq a}(\mu_i - \lambda_b + \eta) - d(\lambda_b)\prod_{i\neq a}(\mu_i - \lambda_b - \eta)\right], \tag{3.4}$$

and $\{\mu\} = \{\mu_1, \ldots, \mu_\ell\}$ and $\{\lambda\} = \{\lambda_1, \ldots, \lambda_\ell\}$ are a Bethe state and an arbitrary state, respectively.

Finally, we would like to recall a property of determinants which we will employ in the computation of the magnetization below: given two $\ell \times \ell$ matrices $\mathcal{X}$ and $\mathcal{Y}$ such that all rows of $\mathcal{Y}$ are identical (rank 1 matrix) the following equality holds,

$$\det(\mathcal{X} + \mathcal{Y}) = \det \mathcal{X} + \sum_{p=1}^\ell \det \mathcal{X}^{(p)}, \tag{3.5}$$

where

$$\mathcal{X}_{ab}^{(p)} = \mathcal{X}_{ab} \quad \text{for} \quad b \neq p, \tag{3.6}$$

$$\mathcal{X}_{ap}^{(p)} = \mathcal{Y}_{ap}. \tag{3.7}$$

## 3.1   The form factors of $S_j^z$

We define the non-vanishing form factors of the local operator $S_j^z$ in the spin $s_j$ representation as

$$F_\ell^z(j, \{\mu\}, \{\lambda\}) = \langle \psi(\{\mu\})| S_j^z |\psi(\{\lambda\})\rangle, \tag{3.8}$$

with $\{\mu\}$ and $\{\lambda\}$ being two sets of $\ell$ Bethe roots, therefore characterizing two Bethe states.

Inserting (2.31) with (2.32) and $\alpha = z$ into (3.8) we obtain the following sum of matrix elements

$$F_\ell^z(j, \{\mu\}, \{\lambda\}) = \frac{\phi_{j-1}(\{\mu\})}{2\phi_j(\{\lambda\})} \sum_{k=1}^{2s_j} \left[\tau^{(s_j - \frac{k}{2})}\left(\xi_j + \frac{k\eta}{2}, \{\mu\}\right)\right.$$
$$\left.\times \langle \psi(\{\mu\})| (A - D)(\nu_j(k))t^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k - 2s_j)\eta}{2}\right)|\psi(\{\lambda\})\rangle\right], \tag{3.9}$$

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where the functions

$$\phi_j(\{\mu\}) = \prod_{k=1}^{j} \tau^{(s_k)}(\xi_k, \{\mu\}), \tag{3.10}$$

originate from the actions

$$\langle\psi(\{\mu\})| \prod_{k=1}^{j-1} t^{(r_k)}(\xi_k) = \phi_{j-1}(\{\mu\})\langle\psi(\{\mu\})|, \tag{3.11}$$

$$\prod_{k=1}^{j} t^{(r_k)}(\xi_k)^{-1} |\Psi(\{\lambda\})\rangle = \phi_j(\{\lambda\})^{-1} |\Psi(\{\lambda\})\rangle, \tag{3.12}$$

and we introduced the variable

$$\nu_j(k) = \xi_j^- + (k - s_j)\eta. \tag{3.13}$$

In (3.9) we have not yet replaced the operators $t^{((k-1)/2)}$ by their eigenvalues on the Bethe state $|\Psi\{\lambda\}\rangle$ for reasons which will become apparent below. Rewriting now $A - D = 2A - (A + D)$ and recalling the general action of the operator $A$ on a Bethe state given in (3.1) we find that (3.9) is equivalent to

$$F_\ell^z(j, \{\mu\}, \{\lambda\}) = \frac{\phi_{j-1}(\{\mu\})}{2\phi_j(\{\lambda\})} \left[ g(j, \{\mu\})S_\ell(\{\mu\}, \{\lambda\}) \right. \tag{3.14}$$

$$-2\sum_{k=1}^{2s_j}\sum_{p=1}^{\ell} \left[ \tau^{(s_j - \frac{k}{2})}\left(\xi_j + \frac{k\eta}{2}, \{\mu\}\right) \tau^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k - 2s_j)\eta}{2}, \{\lambda\}\right) \right.$$

$$\times \left. \left. \frac{\eta}{\mu_p - \nu_j(k)} \prod_{k \neq p} b^{-1}(\mu_k - \mu_p)S_\ell(\{\lambda\}, \{\mu, \mu_p \to \nu_j(k)\}) \right] \right].$$

Here $g(j, \{\mu\})$ is the function defined in equation (A.1) of appendix A. The term proportional to $g(j, \{\mu\})$ collects all contributions which are "direct" in the sense indicated after (3.1)-(3.2). The form of this term results from acting with both operators $t^{(s_j - k/2)}$ and $t^{(k-1)/2}$ on the Bethe state $|\Psi(\{\mu\})\rangle$. By carrying out the computation in this way we achieve that the direct contribution to the form factor is proportional to the function $g(j, \{\mu\})$ and we can take advantage of the identity (A.11) proven in appendix A.

The matrix element $S(\{\lambda\}, \{\mu, \mu_p \to \nu_j(k)\})$ in (3.14) is the scalar product (3.3) with the parameter $\mu_p$ replaced by $\nu_j(k)$. It can be written as

$$S(\{\mu, \mu_p \to x\}, \{\lambda\}) = \prod_{j \neq p} \frac{(\mu_p - \mu_j)}{(x - \mu_j)} \frac{\det H^{(p)}(x)}{\prod_{i < j}(\lambda_i - \lambda_j)(\mu_j - \mu_i)}, \tag{3.15}$$

where $H^{(p)}(x)$ is a matrix such that $H_{ab}^{(p)} = H_{ab}(\{\lambda\}, \{\mu\})$ for $b \neq p$ and

$$H_{ap}^{(p)}(x) = \frac{\eta}{\lambda_a - x} \left[ \prod_{i \neq a}(\lambda_i - x + \eta) - d(x) \prod_{i \neq a}(\lambda_i - x - \eta) \right] \tag{3.16}$$

$$= \frac{t(\lambda_a, x)}{b(\lambda_a - x)} \prod_{i=1}^{\ell}(\lambda_i - x) \left[ \prod_{i \neq a} b^{-1}(\lambda_i - x) - d(x) \prod_{i \neq a} b^{-1}(x - \lambda_i) \right],$$

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with

$$t(\lambda, x) = \frac{\eta}{(\lambda - x)(\lambda - x + \eta)} = -i\partial_\lambda p^{(\frac{1}{2})}(\lambda - x^-), \tag{3.17}$$

and $p^{(1/2)}(\lambda)$ being the momentum defined in (A.6). Inserting (3.15) into (3.14) with $x = \nu_j(k)$ we obtain the following expression for the form factors

$$F_\ell^z(j, \{\mu\}, \{\lambda\}) = \frac{\phi_{j-1}(\{\mu\})}{2\phi_j(\{\lambda\})} \frac{\left[ g(j, \{\mu\}) \det H - 2 \sum_{p=1}^\ell \prod_{k=1}^\ell (\mu_k - \mu_p + \eta) \det \tilde{H}^{(p)}(\xi_j) \right]}{\prod_{i<j}(\lambda_i - \lambda_j)(\mu_j - \mu_i)}, \tag{3.18}$$

where

$$\tilde{H}_{ab}^{(p)} = H_{ab}(\{\lambda\}, \{\mu\}) \qquad \text{for} \qquad b \neq p, \tag{3.19}$$
$$\tilde{H}_{ap}^{(p)} = f^{(p)}(j, \{\mu\}, \{\lambda\}) \tag{3.20}$$

and $f^{(p)}(j, \{\mu\}, \{\lambda\})$ and $g(j, \{\mu\})$ are the functions introduced at the beginning of the appendix A. Thanks to the identities (A.3) and (A.4) this expression can be simplified to

$$F_\ell^z(j, \{\mu\}, \{\lambda\}) = \frac{\phi_j(\{\mu\})}{\phi_j(\{\lambda\})} \frac{s_j \det H - \sum_{p=1}^\ell \prod_{k=1}^\ell (\mu_k - \mu_p + \eta) \det \mathcal{Z}^{(p)}(\xi_j)}{\prod_{i<j}(\lambda_i - \lambda_j)(\mu_j - \mu_i)}, \tag{3.21}$$

with

$$\mathcal{Z}_{ab}^{(p)} = H_{ab}(\{\lambda\}, \{\mu\}) \qquad \text{for} \qquad b \neq p, \tag{3.22}$$
$$\mathcal{Z}_{ap}^{(p)} = -i\partial_a p^{(s_j)}(\lambda_a - \xi_j^-) \prod_{k=1}^\ell \frac{\lambda_k - \xi_j^- + s_j\eta}{\mu_k - \xi_j^- + s_j\eta}. \tag{3.23}$$

Notice that, since the entries $\mathcal{Z}_{ap}^{(p)}$ depend on both indices $a, p$ we can not write (3.21) in terms of a single determinant by employing (3.5). However, the factor $\prod_{k=1}^\ell (\mu_k - \mu_p + \eta)$ will be canceled when normalizing by the norm of the Bethe states, which allows a particularly simple form for the magnetization. We compute this quantity in the next section. Let us also remark that the computation of (3.8) can be done in two different ways, which of course must lead to the same result: in (3.14) we decided to make the operators $(A - D)(\nu_j(k))$ act on the Bethe state $\{\mu\}$, however we could as well have chosen to act on the Bethe state $\{\lambda\}$. If we do that and employ the same reconstruction formula (2.32) as before, we will obtain an expression which looks rather different from (3.21) and cannot be simplified in any obvious way. However, we can alternatively employ the reconstruction formula (2.33) to do the same computation. In that case we will obtain a formula completely analogous to (3.21) where the roles of $\{\lambda\}$ and $\{\mu\}$ are just exchanged.

## 3.2 Magnetization

One interesting physical quantity we can now compute is the magnetization at site $j$ which is defined as

$$\langle S_j^z \rangle = \frac{F_\ell^z(j, \{\lambda\}, \{\lambda\})}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle}, \tag{3.24}$$

The norm of the Bethe state is given in terms of the Gaudin matrix $\Phi'(\{\lambda\})$ as follows [60]

$$\langle\psi(\{\lambda\})|\psi(\{\lambda\})\rangle \;\; = \;\; \eta^\ell \prod_{a\neq b} b^{-1}(\lambda_a - \lambda_b)\det\Phi'(\{\lambda\}), \tag{3.25}$$

with

$$\Phi'_{ab} \;\; = \;\; -\frac{\partial}{\partial\lambda_b}\ln\left(\frac{1}{d(\lambda_a)}\prod_{\substack{k=1\\k\neq a}}^{\ell}\frac{b(\lambda_a - \lambda_k)}{b(\lambda_k - \lambda_a)}\right), \tag{3.26}$$

and can be obtained from the scalar product formula (3.3) in the limit $\lambda_a \to \mu_a$ for all $a = 1\dots\ell$. In particular, it is easy to prove that for two identical Bethe states, the entries of the matrix $H(\{\lambda\})$ defined in (3.4) become proportional to those of the Gaudin matrix as

$$H_{ab} = \prod_{i=1}^{\ell}(\lambda_i - \lambda_b + \eta)\Phi'_{ab}. \tag{3.27}$$

and therefore

$$\det H(\{\lambda\}) = \prod_{i,j=1}^{\ell}(\lambda_i - \lambda_j + \eta)\det\Phi'(\{\lambda\}). \tag{3.28}$$

Using these identities and (3.18) we obtain

$$\langle S_j^z\rangle = \frac{s_j\det\Phi' - \sum_{a=1}^{\ell}\det\mathcal{M}^{(a)}}{\det\Phi'}, \tag{3.29}$$

where the matrix $\mathcal{M}^{(a)}$ is such that

$$\mathcal{M}_{ij}^{(a)} \;\; = \;\; \Phi'_{ij} \qquad \text{for} \qquad j \neq a, \tag{3.30}$$
$$\mathcal{M}_{ia}^{(a)} \;\; = \;\; -i\partial_{\lambda_a}p^{(s_j)}(\lambda_a - \xi_j^-), \tag{3.31}$$

where $p^{(s_j)}(\lambda)$ is the momentum (A.6) defined in appendix A. Since $\mathcal{M}_{ia}^{(a)}$ is independent of the value of the index $i$, the identity (3.5) can be used to bring (3.29) into the form

$$\langle S_j^z\rangle = s_j\frac{\det(\Phi' - \mathcal{M}/s_j)}{\det\Phi'}, \tag{3.32}$$

where $\mathcal{M}$ is a rank 1 matrix whose entries are $\mathcal{M}_{ab} = \mathcal{M}_{ab}^{(b)}$. The total magnetization of the chain can be computed as

$$\mu_{\text{tot}} = \sum_{j=1}^{N}\langle S_j^z\rangle = s_0\frac{\det(\Phi' - \mathcal{M}_{\text{tot}}/s_0)}{\det\Phi'}, \tag{3.33}$$

where $\mathcal{M}_{\text{tot}}$ is a rank 1 matrix with entries

$$(\mathcal{M}_{\text{tot}})_{ia} = \sum_{j=1}^{N} -i\partial_{\lambda_a}p^{(s_j)}(\lambda_a - \xi_j^-), \tag{3.34}$$

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and $s_0$ is the total spin of the completely ferromagnetic state $|0\rangle$

$$s_0 = \sum_{j=1}^{N} s_j. \tag{3.35}$$

The total magnetization can be computed by noticing the following property of the Gaudin matrix (3.26)

$$\sum_{b=1}^{\ell} \Phi'_{ab} = (\mathcal{M}_{\text{tot}})_{ka}. \tag{3.36}$$

Therefore

$$\mu_{\text{tot}} = s_0 \det\left(I - \frac{\mathcal{U}}{s_0}\right) = s_0 - \ell, \tag{3.37}$$

where $I$ denotes the identity matrix and $\mathcal{U}$ is the matrix,

$$\mathcal{U}_{ab} = 1 \qquad \forall \qquad a, b. \tag{3.38}$$

This computation provides a consistency check of the general formula (3.18) as it reproduces the values of the magnetization known for particular cases. For example, for the pure spin $1/2$ XXX chain in the ground state we have $\ell = N/2$ and $s_0 = N/2$ so that the total magnetization vanishes (see e.g. [28]). Another well known example is the spin $1/2$ XXX chain with one spin $s$ impurity. In this case the ground state is also characterized by $\ell = N/2$ roots, but the total spin of the chain is $s_0 = (N-1)/2 + s$. Therefore the total magnetization becomes $s - 1/2$, in agreement with the value computed in [52].

## 3.3 The form factors of $S_j^+$

In this section we compute the form factors of the operator $S_j^+$ of the quantum XXX spin chain. That is a spin generator sitting at site $j$ of the chain and living in the spin $s_j$ representation. The only non-vanishing form factors are

$$F_\ell^+(j, \{\lambda\}, \{\mu\}) \;\;=\;\; \langle\psi(\{\lambda\})| \, S_j^+ \, |\psi(\{\mu\})\rangle \,, \tag{3.39}$$

where $|\Psi(\{\mu\})\rangle$ and $|\Psi(\{\lambda\})\rangle$ are two Bethe states with $\ell + 1$ and $\ell$ Bethe roots, respectively. From (2.31) and (2.32) we obtain

$$F_\ell^+(j, \{\lambda\}, \{\mu\}) = \frac{\phi_{j-1}(\{\lambda\})}{\phi_j(\{\mu\})} \sum_{k=1}^{2s_j} \left[\tau^{(s_j - \frac{k}{2})}\left(\xi_j + k\eta/2, \{\lambda\}\right)\right.$$
$$\times \tau^{(\frac{k-1}{2})}\left(\xi_j^- + (k - 2s_j)\eta/2, \{\mu\}\right) S_{\ell+1}(\{\mu\}, \{\lambda, \nu_j(k)\})\Big], \tag{3.40}$$

where $\nu_j(k)$ is again the variable introduced in (3.13) and $S_{\ell+1}(\{\mu\}, \{\lambda, \nu_j(k)\})$ is the scalar product introduced in (3.3) with $\{\lambda, \nu_j(k)\}$ the set of $\ell + 1$ variables $\{\lambda_1, \ldots, \lambda_\ell, \nu_j(k)\}$ in the given order, that is $\nu_j(k)$ is the variable number $\ell + 1$ (notice that the ordering of variables is important in (3.3)). Employing the definition (3.3) we have

$$S_{\ell+1}(\{\mu\}, \{\lambda, \nu_j(k)\}) = \frac{\det H(\{\mu\}, \{\lambda, \nu_j(k)\})}{\prod_{1 \leq i < j \leq \ell+1}(\mu_j - \mu_i) \prod_{1 \leq i < j \leq \ell}(\lambda_i - \lambda_j)} \prod_{i=1}^{\ell}(\lambda_i - \nu_j(k))^{-1}, \tag{3.41}$$

where $H$ is a matrix with components $H_{ab}$ given by (3.4), except for the column $\ell + 1$ whose entries are given by (3.16) with $x = \nu_j(k)$. This means that (3.40) can be rewritten as

$$F_\ell^+(j, \{\lambda\}, \{\mu\}) = \frac{\phi_{j-1}(\{\lambda\})}{\phi_j(\{\mu\})} \frac{\det H^+(\xi_j)}{\prod_{1 \le i < j \le \ell+1} (\mu_j - \mu_i) \prod_{1 \le i < j \le \ell} (\lambda_i - \lambda_j)}, \tag{3.42}$$

where

$$H_{ab}^+ = H_{ab}(\{\mu\}, \{\lambda\}) \qquad \text{for} \qquad b \ne \ell + 1, \tag{3.43}$$
$$H_{a\ell+1}^+ = f^{(a)}(j, \{\lambda\}, \{\mu\}), \tag{3.44}$$

with $f^{(a)}(j, \{\lambda\}, \{\mu\})$ being the function (A.2) defined in the appendix. Exploiting the identities proven in appendix A we can bring the form factors into the simple form

$$F_\ell^+(j, \{\lambda\}, \{\mu\}) = \frac{\phi_{j-1}(\{\lambda\})}{\phi_{j-1}(\{\mu\})} \frac{\prod_{k=1}^{\ell+1} (\mu_k - \xi_j^- - s_j \eta)}{\prod_{k=1}^{\ell} (\lambda_k - \xi_j^- - s_j \eta)} \frac{\det \mathcal{C}(\xi_j)}{\prod_{1 \le i < j \le \ell+1} (\mu_j - \mu_i) \prod_{1 \le i < j \le \ell} (\lambda_i - \lambda_j)}, \tag{3.45}$$

with

$$\mathcal{C}_{ab} = H_{ab}(\{\mu\}, \{\lambda\}) \qquad \text{for} \qquad b \ne \ell + 1, \tag{3.46}$$
$$\mathcal{C}_{a\ell+1} = -i \partial_{\mu_a} p^{(s_j)}(\mu_a - \xi_j^-). \tag{3.47}$$

## 3.4 The form factors of $S_j^-$

In this section we compute the form factors of the operator $S_j^-$ of the XXX spin chain in the spin $s_j$ representation

$$F_\ell^-(j, \{\mu\}, \{\lambda\}) = \langle \psi(\{\mu\})| S_j^- |\psi(\{\lambda\})\rangle . \tag{3.48}$$

We can compute these form factors along the same lines of the previous section. However if we take (2.31) with (2.32) as starting point for our computations we will obtain closed formulae for (3.48) which lack the simplicity of (3.45). It is in fact more convenient to use (2.33) instead of (2.32). Using formula (2.33) and proceeding as in the previous section it is not difficult to prove that the form factors (3.48) are related to (3.45) as

$$F_\ell^+(j, \{\lambda\}, \{\mu\}) = \frac{\phi_{j-1}(\{\lambda\})\phi_j(\{\lambda\})}{\phi_j(\{\mu\})\phi_{j-1}(\{\mu\})} F_\ell^-(j, \{\mu\}, \{\lambda\}). \tag{3.49}$$

# 4 Conclusions and outlook

In this paper we have obtained general expressions for the form factors of all spin operators $\{S^z, S^\pm\}$ of the integrable XXX quantum (higher) spin chain. Our formulae hold for any spin representation of the operators as well as for any spin configuration at the remaining sites of the chain. These results can be extended to the anisotropic XXZ spin chain as well.

In view of the very general nature of our formulae we expect they will be useful for the study of a number of interesting physical systems, whose thermodynamic properties have been already extensively studied in the literature but whose correlation functions remain unknown. Amongst

these systems alternating spin chains, first studied in [45, 46, 48] and more recently in [49], and impurity systems, such as the Kondo model [51, 52] and the systems considered in [50, 53, 63], seem specially interesting examples.

Recent results for the XXX and XXZ spin 1/2 chain show that determinant formulae for the form factors, similar to those obtained here, can be successfully employed for numerical computations of spin-spin dynamical correlation functions of finite chains, which can match very precisely experimental data [64, 65]. The form factors computed here might serve for the numerical study of the correlation functions of impurity systems.

From the analytical point of view, the most natural continuation of this work is the computation of correlation functions for mixed spin chains along the lines of [30, 31, 32]. Further interesting generalizations of this would be the study of dynamical correlation functions in the spirit of [33] and the computation of correlation functions at finite temperature, generalizing the program initiated in [34].

# A   Proof of two identities for the eigenvalues of higher spin transfer matrices

In this appendix we prove two identities allowing for a compact formula for form factors. Let us consider two Bethe states $\{\lambda\} = \{\lambda_1, \ldots, \lambda_\ell\}$ and $\{\mu\} = \{\mu_1, \ldots, \mu_{\tilde{\ell}}\}$ with $\ell$ and $\tilde{\ell}$ Bethe roots respectively and define the functions

$$
g(j, \{\lambda\}) = \sum_{k=1}^{2s_j} \left[ \tau^{(s_j - \frac{k}{2})} \left( \xi_j + \frac{k\eta}{2}, \{\lambda\} \right) \tau^{(\frac{k-1}{2})} \left( \xi_j^- + \frac{(k - 2s_j)\eta}{2}, \{\lambda\} \right) \right.
$$

$$
\left. \times \left[ \prod_{p=1}^{\ell} b^{-1}(\lambda_p - \nu_j(k)) - d(\nu_j(k)) \prod_{p=1}^{\ell} b^{-1}(\nu_j(k) - \lambda_p) \right] \right], \tag{A.1}
$$

and

$$
f^{(a)}(j, \{\lambda\}, \{\mu\}) = \sum_{k=1}^{2s_j} \left[ \tau^{(s_j - \frac{k}{2})} \left( \xi_j + \frac{k\eta}{2}, \{\lambda\} \right) \tau^{(\frac{k-1}{2})} \left( \xi_j^- + \frac{(k - 2s_j)\eta}{2}, \{\mu\} \right) \right. \tag{A.2}
$$

$$
\times \frac{t(\mu_a, \nu_j(k))}{b(\mu_a - \nu_j(k))} \left[ \prod_{p\neq a} b^{-1}(\mu_p - \nu_j(k)) - d(\nu_j(k)) \prod_{p\neq a} b^{-1}(\nu_j(k) - \mu_p) \right] \frac{\prod\limits_{p=1}^{\tilde{\ell}} (\mu_p - \nu_j(k))}{\prod\limits_{p=1}^{\ell} (\lambda_p - \nu_j(k))} .
$$

with $\nu_j(k)$ given in (3.13). The aim of this appendix is to prove that

$$g(j, \{\lambda\}) = 2s_j \tau^{(s_j)}(\xi_j, \{\lambda\}), \tag{A.3}$$

$$f^{(a)}(j, \{\lambda\}, \{\mu\}) = \frac{\displaystyle\prod_{p=1}^{\tilde{\ell}} (\mu_p - \xi_j^- + s_j \eta)}{\displaystyle\prod_{p=1}^{\ell} (\lambda_p - \xi_j^- - s_j \eta)} \sum_{k=1}^{2s_j} t(\mu_a, \nu_j(k)), \tag{A.4}$$

with,

$$\sum_{k=1}^{2s_j} t(\mu_a, \nu_j(k)) = -i\partial_{\mu_a} p^{(s_j)}(\mu_a - \xi_j^-), \tag{A.5}$$

and

$$p^{(s_j)}(\lambda) = i \log \left[ \frac{\lambda - s_j \eta}{\lambda + s_j \eta} \right]. \tag{A.6}$$

is the momentum of a spin $s_j$ pseudo-particle of "rapidity" $\lambda$. In particular, the identities above imply

$$f^{(a)}(j, \{\lambda\}, \{\lambda\}) = \tau^{(s_j)}(\xi_j, \{\lambda\}) \left[ -i\partial_{\lambda_a} p^{(s_j)}(\lambda_a - \xi_j^-) \right], \tag{A.7}$$

Let us commence our proof by writing down the explicit expression of the eigenvalues following from eq.(2.29)

$$\tau^{(s_j - \frac{k}{2})} \left( \xi_j + \frac{k\eta}{2}, \{\lambda\} \right) = \sum_{\alpha=0}^{2s_j - k} \left[ C_\alpha^{(s_j - \frac{k}{2})} \left( \xi_j + \frac{k\eta}{2} \right) \right.$$
$$\left. \times \prod_{p=1}^{\ell} \frac{(\xi_j^+ + s_j \eta - \lambda_p)(\nu_j(k) - \lambda_p)}{(\nu_j(k) + (\alpha+1)\eta - \lambda_p)(\nu_j(k) + \alpha\eta - \lambda_p)} \right], \tag{A.8}$$

$$\tau^{(\frac{k-1}{2})} \left( \xi_j^- + \frac{(k - 2s_j)\eta}{2}, \{\mu\} \right) = \sum_{\beta=0}^{k-1} \left[ C_\beta^{(\frac{k-1}{2})} \left( \xi_j^- + \frac{(k - 2s_j)\eta}{2} \right) \right.$$
$$\left. \times \prod_{p=1}^{\tilde{\ell}} \frac{(\nu_j(k) - \mu_p)(\xi_j^- - s_j \eta - \mu_p)}{(\xi_j^+ - (s_j - \beta)\eta - \mu_p)(\xi_j^- - (s_j - \beta)\eta - \mu_p)} \right], \tag{A.9}$$

whose product enters the two formulae we want to prove. Before we insert these formulae in (A.1)-(A.2) it is useful to note that in (A.8) the only non vanishing term is the one corresponding to $\alpha = 2s_j - k$. It is easy to see that any other terms will contain a factor $d(\xi_j^- + s_j \eta) = 0$ and never contain the singular factor $d(\xi_j^- - s_j \eta)$. Likewise it is straightforward to argue that such factors will never appear in $C_\beta^{((k-1)/2)}(\xi_j^- + (k - 2s_j)\eta/2)$ for any of the allowed values of $\beta$ and

$k$, and therefore each term in the $\beta$-sum is non-vanishing and non-singular. Therefore

$$\tau^{(s_j - \frac{k}{2})}\left(\xi_j + k\eta/2, \{\lambda\}\right) \tau^{(\frac{k-1}{2})}\left(\xi_j^- + (k - 2s_j)\eta/2, \{\mu\}\right)$$

$$= \frac{\prod\limits_{p=1}^{\tilde{\ell}} (\mu_p - \xi_j^- + s_j\eta)}{\prod\limits_{p=1}^{\ell} (\lambda_p - \xi_j^- - s_j\eta)} \sum_{\beta=0}^{k-1} C_\beta^{(\frac{k-1}{2})}(\xi_j^- + (k - 2s_j)\eta/2)$$

$$\times \frac{\left[\prod\limits_{p=1}^{\tilde{\ell}} (\mu_p - \nu_j(k))\right]\left[\prod\limits_{p=1}^{\ell} (\lambda_p - \nu_j(k))\right]}{\prod\limits_{p=1}^{\tilde{\ell}} (\mu_p - \xi_j^+ + (s_j - \beta)\eta)(\mu_p - \xi_j^- + (s_j - \beta)\eta)}, \tag{A.10}$$

where we used the second property stated in (2.30), i.e. $C_{2s_j-k}^{(s_j-k/2)}(\xi_j + k\eta/2) = 1$. Using these formulae, the equalities (A.1) and (A.2) are equivalent to proving :

$$\sum_{k=1}^{2s_j}\sum_{\beta=0}^{k-1}\left[C_\beta^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k - 2s_j)\eta}{2}\right)\prod_{p=1}^{\ell}\frac{(\lambda_p - \nu_j(k))^2}{h(\lambda_p - \xi_j^+ + (s_j - \beta)\eta)}\right.$$

$$\left.\times\left[\prod_{p=1}^{\ell} b^{-1}(\lambda_p - \nu_j(k)) - d(\nu_j(k))\prod_{p=1}^{\ell} b^{-1}(\nu_j(k) - \lambda_p)\right]\right] = 2s_j, \tag{A.11}$$

and

$$\sum_{k=1}^{2s_j}\sum_{\beta=0}^{k-1}\left[C_\beta^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k - 2s_j)\eta}{2}\right)\prod_{p=1}^{\tilde{\ell}}\frac{(\mu_p - \nu_j(k))^2}{h(\mu_p - \xi_j^+ + (s_j - \beta)\eta)}\right.$$

$$\times\left[\prod_{p\neq a} b^{-1}(\mu_p - \nu_j(k)) - d(\nu_j(k))\prod_{p\neq a} b^{-1}(\nu_j(k) - \mu_p)\right]$$

$$\left.\times\frac{t(\mu_a, \nu_j(k))}{b(\mu_a - \nu_j(k))}\right] = -i\partial_{\lambda_a} p^{(s_j)}(\lambda_a - \xi_j^-), \tag{A.12}$$

respectively, with

$$h(\lambda) := (\lambda)(\lambda + \eta). \tag{A.13}$$

We commence by establishing (A.11). Let us consider those terms in (A.11) which do not contain any factors proportional to the function $d(x)$. The only terms of this kind correspond to taking $\beta = k - 1$ and only the first contribution on the second line of (A.11), that is

$$\sum_{k=1}^{2s_j}\prod_{p=1}^{\ell}\frac{(\lambda_p - \xi_j^- + (s_j - k)\eta)(\lambda_p - \xi_j^- - (k - s_j - 1)\eta)}{(\lambda_p - \xi_j^+ + (s_j - k + 1)\eta)(\lambda_p - \xi_j^- + (s_j - k + 1)\eta)} = 2s_j, \tag{A.14}$$

therefore it remains to prove that all remaining terms cancel each other. Those remaining terms are

$$\sum_{k=2}^{2s_j}\sum_{\beta=0}^{k-2} C_\beta^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k-2s_j)\eta}{2}\right)\prod_{p=1}^{\ell}\frac{h(\lambda_p - \nu_j(k))}{h(\lambda_p - \xi_j^+ + (s_j-\beta)\eta)}$$

$$-\sum_{k=1}^{2s_j}\sum_{\beta=0}^{k-1} C_\beta^{(\frac{k}{2})}\left(\xi_j + \frac{(k-2s_j)\eta}{2}\right)\prod_{p=1}^{\ell}\frac{h(\lambda_p - \nu_j(k) - \eta)}{h(\lambda_p - \xi_j^+ + (s_j-\beta)\eta)}, \qquad (A.15)$$

where we have used the identity

$$C_\beta^{(\frac{k-1}{2})}\left(\xi_j^- + \frac{(k-2s_j)\eta}{2}\right)d(\nu_j(k)) = C_\beta^{(\frac{k}{2})}\left(\xi_j + \frac{(k-2s_j)\eta}{2}\right). \qquad (A.16)$$

Replacing the sum over $k$ by a sum over $k' = k - 1$ in the first line of (A.15) we see that both terms are essentially identical but for the extra term $k = 2s_j$ contributing in the second sum. However such term is vanishing since $C_\beta^{(s_j)}(\xi_j^- + (k - 2s_j)\eta/2) = 0$ and therefore we have proven that indeed all terms containing the function $d(x)$ in (A.11) cancel each other. With this we have established (A.11) and therefore (A.1). The proof of the identity (A.12) can be carried out by following exactly the same steps, namely considering separately those terms which contain $d$-functions and those which do not.

# B   An alternative formula for the solution of the inverse scattering problem for XXX spin chains

The solution of the inverse scattering problem for XXX spin chains and arbitrary spin representations was given in [29]. This solution was recalled in formulae (2.31) and (2.32) and employed in order to obtain closed expressions for higher spin form factors of the operators $S^z$ and $S^+$. However we have observed that in some cases the form factor formulae obtained from (2.32) simplify considerably (thanks to the identities proven in appendix A), whereas in other cases such simplifications do not occur. For example, a simple formula for all form factors of the operator $S^+$ can be obtained form (2.31)-(2.32), but the formula we would obtain for the form factors of $S^-$ from (2.32) is much more complicated. This asymmetry seems rather unnatural and one may suppose that the formulae for the $S^-$ form factors can be further simplified and recasted into analogous formulae as those for the $S^+$ form factors. This should be doable by using properties of determinants as well as the Bethe ansatz equations; however it turns out to be a rather non-trivial proof. An alternative way of finding simpler formulae for the form factors of $S^-$ is to start with a different (but equivalent) reconstruction formula, that is (2.33). In this appendix we will show that (2.32) and (2.33) are indeed equivalent. The key tool are once again the fusion relations (2.17) and one particular property of these noticed in [35].

## B.1   An alternative version of the fusion relations

Let us start by decomposing (2.17) into the following two equations

$$P_{a_1\{a\}}^+ T_{a_1}^{(\frac{1}{2})}(x^- + s\eta)T_{\{a\}}^{(s-\frac{1}{2})}(x^-)P_{a_1\{a\}}^+ = T_{\langle a_1\{a\}\rangle}^{(s)}(x), \qquad (B.1)$$

$$P_{a_1\{a\}}^- T_{a_1}^{(\frac{1}{2})}(x^- + s\eta)T_{\{a\}}^{(s-\frac{1}{2})}(x^-)P_{a_1\{a\}}^- = \chi(x + (s-1)\eta)T_{(a_1\{a\})}^{(s-1)}(x - \eta). \qquad (B.2)$$

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With respect to (2.17) we have slightly changed our notation and introduced the index $\{a\}$ to indicate that the space $V^{(s-1/2)}_{\{a\}} \sim \mathbb{C}^{2s-1}$ is isomorphic to the space $V^{(s-1/2)}_{\langle a_2 \ldots a_{2s} \rangle}$ resulting from the fusion of $2s - 1$ spin $1/2$ quantum spaces

$$V^{(\frac{1}{2})}_{a_j} \sim \mathbb{C}^2, \qquad j = 2, \ldots, 2s. \tag{B.3}$$

We can now employ fusion successively in order to express (B.1) solely in terms of spin $1/2$ quantum monodromy matrices. By doing so, we obtain the following expression

$$P^+_{a_1 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(\frac{1}{2})}_{a_j}(x^+ + (s-j)\eta) P^+_{a_1 \ldots a_{2s}} = T^{(s)}_{\langle a_1 \ldots a_{2s} \rangle}(x). \tag{B.4}$$

It was proven in [35] that, for XXX spin chains, the operator $P^+_{a_1 \ldots a_{2s}}$ is a complete symmetrizer on all indices $a_1, \ldots, a_{2s}$. Let us now consider the formula (B.2). We can also express this relation entirely in terms of spin $1/2$ monodromy matrices

$$P^-_{a_1 \{a\}} P^+_{a_2 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(\frac{1}{2})}_{a_j}(x^+ + (s-j+1)\eta) P^+_{a_2 \ldots a_{2s}} P^-_{a_1 \{a\}} = \chi(x+s\eta) T^{(s-1)}_{(a_1 \{a\})}(x). \tag{B.5}$$

As before the operator $P^+_{a_2 \ldots a_{2s}}$ is completely symmetric on the indices $\{a_2 \ldots a_{2s}\}$. On the other hand the operator $P^-_{a_1 \{a\}}$ is antisymmetric on the index $a_1$ (see [35]). The identities (B.4)-(B.5) can be further manipulated by employing the following property of the projectors $P^\pm$ which was stated in [35]

$$P^\pm_{a_1 \ldots a_n} R^{(\frac{1}{2}, \frac{1}{2})}_{a_1 a_{n+1}}(\pm n\eta) P^\pm_{a_2 \ldots a_{n+1}} = \pm P^\pm_{a_1 \ldots a_{n+1}}. \tag{B.6}$$

This property allowed the authors of [35] to prove that

$$P^+_{a_1 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(\frac{1}{2})}_{a_j}(x^+ + (s-j)\eta) P^+_{a_1 \ldots a_{2s}} = P^+_{a_1 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(\frac{1}{2})}_{a_j}(x^- - (s-j)\eta) P^+_{a_1 \ldots a_{2s}}, \tag{B.7}$$

and

$$P^-_{a_1 \{a\}} P^+_{a_2 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(1/2)}_{a_j}(x^+ + (s-j)\eta) P^+_{a_2 \ldots a_{2s}} P^-_{a_1 \{a\}}$$

$$= P^-_{a_1 \{a\}} P^+_{a_2 \ldots a_{2s}} \prod_{j=1}^{2s} T^{(1/2)}_{a_j}(x^- - (s-j)\eta) P^+_{a_2 \ldots a_{2s}} P^-_{a_1 \{a\}}. \tag{B.8}$$

The equalities (B.7)-(B.8) imply that the fusion relation (2.17) still holds if in the arguments on the r.h.s. $\eta$ is replaced by $-\eta$, that is

$$P_{12} T^{(\frac{1}{2})}_1(x^+ - s\eta) T^{(s-\frac{1}{2})}_2(x^+) P_{12} = \begin{pmatrix} T^{(s)}_{\langle 12 \rangle}(x) & 0 \\ * & \chi(x + (s-1)\eta) T^{(s-1)}_{(12)}(x - \eta) \end{pmatrix}. \tag{B.9}$$

This equivalence was stated in equation (18) of [35] for $R$-matrices, but it is easy to prove that it implies a similar property of the monodromy matrices.

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**The $s = 1$ case:** As an example, let us prove (B.7) for $s = 1$. In this case (B.4) becomes simply

$$P^+_{a_1 a_2} T^{(\frac{1}{2})}_{a_1}(x^+) T^{(\frac{1}{2})}_{a_2}(x^-) P^+_{a_1 a_2} = T^{(1)}_{\langle a_1 a_2 \rangle}(x), \tag{B.10}$$

and the relation (B.6) can be written as

$$R^{(\frac{1}{2},\frac{1}{2})}_{a_1 a_2}(\pm n\eta) = \pm P^{\pm}_{a_1 a_2}, \tag{B.11}$$

for $n = 1$. The $RTT$-relations

$$R^{(\frac{1}{2},\frac{1}{2})}_{a_1 a_2}(\lambda) T^{(\frac{1}{2})}_{a_1}(\lambda + \mu) T^{(\frac{1}{2})}_{a_2}(\mu) = T^{(\frac{1}{2})}_{a_2}(\mu) T^{(\frac{1}{2})}_{a_1}(\lambda + \mu) R^{(\frac{1}{2},\frac{1}{2})}_{a_1 a_2}(\lambda), \tag{B.12}$$

at $\lambda = \eta$ and $\mu = x^-$ imply therefore

$$P^+_{a_1 a_2} T^{(\frac{1}{2})}_{a_1}(x^+) T^{(\frac{1}{2})}_{a_2}(x^-) = T^{(\frac{1}{2})}_{a_2}(x^-) T^{(\frac{1}{2})}_{a_1}(x^+) P^+_{a_1 a_2}. \tag{B.13}$$

Multiplying (B.13) from the right and from the left by $P^+_{a_1 a_2}$ we obtain

$$P^+_{a_1 a_2} T^{(\frac{1}{2})}_{a_1}(x^+) T^{(\frac{1}{2})}_{a_2}(x^-) P^+_{a_1 a_2} = P^+_{a_1 a_2} T^{(\frac{1}{2})}_{a_2}(x^-) T^{(\frac{1}{2})}_{a_1}(x^+) P^+_{a_1 a_2}, \tag{B.14}$$

which is precisely (B.7) for $s = 1$.

## B.2 An alternative formula for the solution of the inverse problem

In order to solve the inverse problem we need to find a systematic way of computing the traces

$$\Lambda^{(s_j)}_\alpha := \text{Tr}_0 \left( S^\alpha_0 T^{(s_j)}_{0;1\ldots N}(\xi_j) \right), \qquad \alpha = \pm, z. \tag{B.15}$$

To do that, the key idea [29] is to use the fusion relations (B.9) in order to obtain recursive formulae similar to (2.23) for (B.15). Recall that for the general XXX chain in finite dimensional representations, the co-product for local spin operator is trivial and therefore we can write the spin matrices in the two following ways

$$S^\alpha_{12} = I_1 \otimes S^\alpha_2 + S^\alpha_1 \otimes I_2 \simeq S^\alpha_{\langle 12 \rangle} \oplus S^\alpha_{(12)}, \qquad \alpha = \pm, z. \tag{B.16}$$

Multiplying (B.9) by $S^\alpha_{12}$ and taking thereafter the trace over the spaces 1 and 2 on the l.h.s. and over the fused spaces $\langle 12 \rangle$ and $(12)$ on the r.h.s. we obtain the following relations

$$t^{(\frac{1}{2})}(x^+ - s\eta)\Lambda^{(s-\frac{1}{2})}_\alpha(x^+) + \Lambda^{(\frac{1}{2})}_\alpha(x^+ - s\eta)t^{(s-\frac{1}{2})}(x^+)$$
$$= \Lambda^{(s)}_\alpha(x) + \chi(x + (s-1)\eta)\Lambda^{(s-1)}_\alpha(x - \eta). \tag{B.17}$$

In addition, taking directly the trace on (B.9) we obtain also new fusion relations for the traces of the transfer matrices

$$t^{(1/2)}(x^+ - s\eta)t^{(s-1/2)}(x^+) = t^{(s)}(x) + \chi(x + (s-1)\eta)t^{(s-1)}(x - \eta). \tag{B.18}$$

The recursive relations (B.17) are solved by

$$\Lambda^{(s)}_\alpha(u) = \sum_{k=1}^{2s} t^{(s-\frac{k}{2})}\left(u - \frac{k\eta}{2}\right) \Lambda^{(\frac{1}{2})}_\alpha(u^+ - (k-s)\eta) t^{(\frac{k-1}{2})}\left(u^+ - \frac{(k-2s)\eta}{2}\right). \tag{B.19}$$

This can be proven by substituting $\Lambda_\alpha^{(s-1)}$ and $\Lambda_\alpha^{(s-1/2)}$ in (B.17) by the corresponding formulae from (B.19) and using thereafter the fusion relations (B.18). This is exactly the same solution (2.32) up to the replacement $\eta \to -\eta$ in the arguments. By introducing a new index $p = 2s-k+1$ we can rewrite (B.19) as

$$\Lambda_\alpha^{(s)}(u) = \sum_{p=1}^{2s} t^{(\frac{p-1}{2})}\left(u^- + \frac{(p-2s)\eta}{2}\right) \Lambda_\alpha^{(\frac{1}{2})}(u^- + (p-s)\eta) t^{(s-\frac{p}{2})}\left(u + \frac{p\eta}{2}\right), \qquad (B.20)$$

which is identical to (2.32) up to the exchange of the transfer matrices.

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