Minimal areas from q-deformed oscillator algebras

Andreas Fring•, Laure Gouba◦ and Bijan Bagchi∗

• *Centre for Mathematical Science, City University London, Northampton Square, London EC1V 0HB, UK*

◦ *National Institute for Theoretical Physics (NITheP), Stellenbosch 7600, South Africa*

∗ *Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700 009, India*

E-mail: *a.fring@city.ac.uk, gouba@sun.ac.za,bbagchi123@rediffmail.com*

Abstract: We demonstrate that dynamical noncommutative space-time will give rise to deformed oscillator algebras. In turn, starting from some q-deformations of these algebras in a two dimensional space for which the entire deformed Fock space can be constructed explicitly, we derive the commutation relations for the dynamical variables in noncommutative space-time. We compute minimal areas resulting from these relations, i.e. finitely extended regions for which it is impossible to resolve any substructure in form of measurable knowledge. The size of the regions we find is determined by the noncommutative constant and the deformation parameter q. Any object in this type of space-time structure has to be of membrane type or in certain limits of string type.

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1. Introduction

The idea to extend the quantization procedure from canonical variables to space-time itself [1] traces back over sixty years. In recent years this general possibility has become more and more appealing, especially in the context of quantum field theories as such type of space-time structures will introduce natural cut-offs and theories on them are therefore renormalized by construction [2, 3]. In addition, almost all possible theories of quantum gravity require non-Minkowskian space-time in one form or another [4, 5, 6, 7, 8].

One of the interesting consequences of these type of space-time structures is that in many cases they lead to modifications of Heisenberg's uncertainty relations, which in turn result in the emergence of minimal lengths. This means in such spaces one has almost inevitably definite fundamental distances below which no substructure can be resolved [9, 10, 11, 12, 13, 14, 15, 16, 17]. Recently some of us proposed [18] a consistent dynamical noncommutative space-time structure in a two dimensional space which leads to a fundamental length in one direction, implying that objects in these spaces are of string type. Here we provide a different type of dynamical noncommutative space-time implying a fundamental length in each of the two directions, thus giving rise to minimal areas for which any substructures is beyond measurable knowledge. In our construction procedure we will not only postulate the deformed Heisenberg canonical commutation relations and check their consistency, but we will also derive them from some more extensively studied and more fundamental structure, namely q-deformed oscillator algebras for which the entire Fock space can be constructed explicitly [12, 13, 14].

In section 2 we commence with various consistent deformations of Heisenberg's canonical commutation relations and investigate the consequences on the commutation relations of the associated oscillator algebra. We find that the latter are almost inevitably deformed. In section 3 we take this fact into account and reverse the setting by starting instead from a well suited q-deformed oscillator algebra and derive from it Heisenberg's uncertainty relations for the dynamical variables. In section 4 we briefly recall the standard argument leading to minimal length and compute the minimal area for a selected algebra. Our conclusions and an outlook to further open problems are stated in section 5.

2. Creation and annihilation operators from noncommutative space-time

2.1 Oscillator algebras in flat noncommutative space-time

Noncommutative flat space-time in two dimensions manifests itself in the following modification of Heisenberg's canonical commutation relations for the dynamical variables

$$\begin{array}{lll} [x_0, y_0] = i\theta, & [x_0, p_{x_0}] = i\hbar, & [y_0, p_{y_0}] = i\hbar, \\ [p_{x_0}, p_{y_0}] = 0, & [x_0, p_{y_0}] = 0, & [y_0, p_{x_0}] = 0. \end{array} \tag{2.1}$$

Restricting the noncommutative constant to be real, i.e. $\theta \in \mathbb{R}$, ensures that x_0 and y_0 are Hermitian operators. We now wish to find a representation for creation and annihilation operators in terms of the dynamical variables $x_0, y_0, p_{x_0}, p_{y_0}$ satisfying the standard commutation relations for a Fock space representation

$$[a_i, a_j^\dagger] = \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad \text{for } i, j = 1, 2. \tag{2.2}$$

In order to reduce the number of unknown coefficients in a possible Ansatz for the $a_i, a_i^\dagger$ we may take the properties of the dynamical variables under a $\mathcal{PT}$-transformation as a guiding principle. These type of considerations have proved to be very fruitful, allowing even a consistent formulation of non-Hermitian systems with real eigenvalues, see e.g. [19, 20, 21] for a review or [22, 23] for recent special issues. For this purpose we note that the relations (2.1) are $\mathcal{P}_x\mathcal{T}$-symmetric and $\mathcal{P}_y\mathcal{T}$-symmetric in the sense that they remain invariant under a simultaneous reflection in the x_0-direction together with a time reversal and under a simultaneous reflection in the y_0-direction together with a time reversal, respectively,

$$\begin{array}{llllll} \mathcal{P}_x: & x_0 \mapsto -x_0, & y_0 \mapsto y_0, & p_{x_0} \mapsto -p_{x_0}, & p_{y_0} \mapsto p_{y_0}, & \\ \mathcal{P}_y: & x_0 \mapsto x_0, & y_0 \mapsto -y_0, & p_{x_0} \mapsto p_{x_0}, & p_{y_0} \mapsto -p_{y_0}, & \\ \mathcal{T}: & x_0 \mapsto x_0, & y_0 \mapsto y_0, & p_{x_0} \mapsto -p_{x_0}, & p_{y_0} \mapsto -p_{y_0}, & i \mapsto -i, \\ \mathcal{P}_x\mathcal{T}: & x_0 \mapsto -x_0, & y_0 \mapsto y_0, & p_{x_0} \mapsto p_{x_0}, & p_{y_0} \mapsto -p_{y_0}, & i \mapsto -i, \\ \mathcal{P}_y\mathcal{T}: & x_0 \mapsto x_0, & y_0 \mapsto -y_0, & p_{x_0} \mapsto -p_{x_0}, & p_{y_0} \mapsto p_{y_0}, & i \mapsto -i. \end{array} \tag{2.3}$$

We demand now to have a definite transformation property for the $a_i, a_i^\dagger$, that is we would like them to be either even or odd under a $\mathcal{P}_{x,y}\mathcal{T}$-transformation, i.e. $a_i \mapsto a_i$, $a_i^\dagger \mapsto a_i^\dagger$ or $a_i \mapsto -a_i$, $a_i^\dagger \mapsto -a_i^\dagger$, such that we can use this property to reduce the total number of constants. Assuming that the dependence on the $x_0, y_0, p_{x_0}, p_{y_0}$ is still linear, the general operators of the form

$$
\begin{aligned}
a_1 &:= \alpha_1 x_0 + i\alpha_2 y_0 + i\alpha_3 p_{x_0} + \alpha_4 p_{y_0}, \qquad & a_1^\dagger &:= \alpha_1 x_0 - i\alpha_2 y_0 - i\alpha_3 p_{x_0} + \alpha_4 p_{y_0}, \\
a_2 &:= \alpha_5 x_0 + i\alpha_6 y_0 + i\alpha_7 p_{x_0} + \alpha_8 p_{y_0}, \qquad & a_2^\dagger &:= \alpha_5 x_0 - i\alpha_6 y_0 - i\alpha_7 p_{x_0} + \alpha_8 p_{y_0},
\end{aligned} \tag{2.4}
$$

with unknown constants $\alpha_1, \ldots, \alpha_8 \in \mathbb{R}$ for the time being, are $\mathcal{P}_x\mathcal{T}$-odd: $a_i \mapsto -a_i$, $a_i^\dagger \mapsto -a_i^\dagger$ and $\mathcal{P}_y\mathcal{T}$-even: $a_i \mapsto a_i$, $a_i^\dagger \mapsto a_i^\dagger$ when using the realization (2.3). The reverse scenario is simply achieved by $\alpha_j \mapsto i\alpha_j$ for $j = 1, \ldots, 8$.

The operators defined in (2.4) satisfy the commutation relations (2.2) provided that the following four constraints on the constants hold

$$
\alpha_1 = \frac{\alpha_6}{2\hbar\Delta}, \qquad \alpha_4 = \frac{\theta\alpha_6 + \hbar\alpha_7}{2\hbar^2\Delta}, \qquad \alpha_5 = -\frac{\alpha_2}{2\hbar\Delta}, \qquad \alpha_8 = -\frac{\theta\alpha_2 + \hbar\alpha_3}{2\hbar^2\Delta}, \tag{2.5}
$$

where we abbreviated $\Delta := \alpha_3\alpha_6 - \alpha_2\alpha_7 \neq 0$[1]. This means we have still four almost entirely free parameters left. Inverting the relations (2.4) while keeping the constraints (2.5), we can express the coordinates and the momenta in terms of the creation and annihilation operators

$$
\begin{aligned}
x_0 &= (\theta\alpha_2 + \hbar\alpha_3)(a_1 + a_1^\dagger) + (\theta\alpha_6 + \hbar\alpha_7)(a_2 + a_2^\dagger), \quad & y_0 &= \tfrac{i\alpha_7}{2\Delta}(a_1 - a_1^\dagger) - \tfrac{i\alpha_3}{2\Delta}(a_2 - a_2^\dagger), \\
p_{x_0} &= -\tfrac{i\alpha_6}{2\Delta}(a_1 - a_1^\dagger) + \tfrac{i\alpha_2}{2\Delta}(a_2 - a_2^\dagger), & p_{y_0} &= -\hbar\alpha_2(a_1 + a_1^\dagger) - \hbar\alpha_6(a_2 + a_2^\dagger).
\end{aligned} \tag{2.6}
$$

It is easily verified that these operators obey (2.1) when using (2.2).

2.2 Oscillator algebras from string type noncommutative space-time

Let us now carry out a similar analysis for the situation when the underlying space-time is dynamical, i.e. the constant θ becomes position and possibly also momentum dependent. A set of consistent commutation relations for such a scenario was introduced in [18]

$$
\begin{aligned}
&[x, y] = i\theta(1 + \tau y^2), \qquad & &[x, p_x] = i\hbar(1 + \tau y^2), \qquad & &[y, p_y] = i\hbar(1 + \tau y^2), \\
&[p_x, p_y] = 0, & &[x, p_y] = 2i\tau y(\theta p_y + \hbar x), & &[y, p_x] = 0.
\end{aligned} \tag{2.7}
$$

Defining the analogues to the creation and annihilation operators and keeping the dependence on the dynamical variables similar as in (2.4)

$$
\begin{aligned}
\hat{a}_1 &:= \alpha_1 x + i\alpha_2 y + i\alpha_3 p_x + \alpha_4 p_y, \qquad & \hat{a}_1^\dagger &:= \alpha_1 x - i\alpha_2 y - i\alpha_3 p_x + \alpha_4 p_y, \\
\hat{a}_2 &:= \alpha_5 x + i\alpha_6 y + i\alpha_7 p_x + \alpha_8 p_y, \qquad & \hat{a}_2^\dagger &:= \alpha_5 x - i\alpha_6 y - i\alpha_7 p_x + \alpha_8 p_y,
\end{aligned} \tag{2.8}
$$

[1]For the specific choice

$$
\alpha_1 = \alpha_2 = -\frac{\lambda_1}{\hbar\sqrt{K_1}}, \qquad \alpha_3 = -\alpha_4 = -\frac{1}{\sqrt{K_1}}, \qquad \alpha_5 = -\alpha_6 = \frac{\lambda_2}{\hbar\sqrt{K_2}}, \qquad \alpha_7 = \alpha_8 = \frac{1}{\sqrt{K_2}},
$$

we recover the representation found in [24] when comparing with equations (57) and (58) therein and identifying the quantities λ_1, λ_2 and K_1, K_2 which are defined in equation (56) and (59), respectively.

we can compute the resulting commutation relations. Keeping the constraints (2.5) and setting in addition $\alpha_3 = 0$ we find that the standard commutation relations are deformed

$$[\hat{a}_i, \hat{a}_i^\dagger] = 1 + \frac{\tau}{4\alpha_2^2}\left(\hat{a}_1\hat{a}_1^\dagger + \hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_1^\dagger\right) \qquad \text{for } i = 1, 2 \tag{2.9}$$

$$[\hat{a}_1, \hat{a}_2] = [\hat{a}_1, \hat{a}_2^\dagger] = [\hat{a}_1^\dagger, \hat{a}_2] = [\hat{a}_1^\dagger, \hat{a}_2^\dagger] = \frac{\tau}{4\alpha_2^2}\left(\hat{a}_1\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger\right). \tag{2.10}$$

The asymmetry between $i = 1$ and $i = 2$ in (2.9) appears odd at first sight in the light of (2.8), but it is a consequence of the non-symmetric nature of (2.7) and our choice $\alpha_3 = 0$. Clearly when the deformation parameter τ vanishes we obtain the usual Fock space commutation relations (2.2).

2.3 Oscillator algebras from membrane type noncommutative space-time

We propose now a new type of deformation for the flat noncommutative space-time (2.1)

$$\begin{aligned} &[\tilde{x}, \tilde{y}] = i\theta + i\tau\left(\tilde{x}^2 + \tilde{y}^2\right), && [\tilde{x}, \tilde{p}_x] = i\hbar + i\tfrac{\tau\hbar}{\theta}\left(\tilde{x}^2 + \tilde{y}^2\right), \quad [\tilde{x}, \tilde{p}_y] = 0, \\ &[\tilde{p}_x, \tilde{p}_y] = i\tau\left[2\tfrac{\hbar}{\theta}(\tilde{y}\tilde{p}_x - \tilde{x}\tilde{p}_y) - \tilde{p}_x^2 - \tilde{p}_y^2\right], && [\tilde{y}, \tilde{p}_y] = i\hbar + i\tfrac{\tau\hbar}{\theta}\left(\tilde{x}^2 + \tilde{y}^2\right), \quad [\tilde{y}, \tilde{p}_x] = 0. \end{aligned} \tag{2.11}$$

In the same manner as for (2.7) we may verify that these commutation relations are consistent in the sense that the Jacobi identities are satisfied. Using the standard arguments to find a minimal length, we observe that the $\tilde{x}, \tilde{y}$-commutator implies a minimal length in the $\tilde{x}$ as well as in the $\tilde{y}$-direction, which means the underlying object, whose substructure we can not determine, is of a membrane structure. Once again we define creation and annihilation type operators analogously to (2.4) keeping the dependence on the dynamical variables the same. When specifying the coefficients such that

$$\begin{aligned} &\tilde{a}_1 := \sqrt{\tfrac{1-\tau}{2\theta}}(\tilde{x} + i\tilde{y}), && \tilde{a}_1^\dagger := \sqrt{\tfrac{1-\tau}{2\theta}}(\tilde{x} - i\tilde{y}), \\ &\tilde{a}_2 := \sqrt{\tfrac{1-\tau}{2\theta}}\left[\tilde{x} - i\tilde{y} + \tfrac{\theta}{\hbar}(\tilde{p}_y + i\tilde{p}_x)\right], && \tilde{a}_2^\dagger := \sqrt{\tfrac{1-\tau}{2\theta}}\left[\tilde{x} + i\tilde{y} + \tfrac{\theta}{\hbar}(\tilde{p}_y - i\tilde{p}_x)\right], \end{aligned} \tag{2.12}$$

we find the commutation relations

$$\tilde{a}_i\tilde{a}_j^\dagger - \left(\frac{1+\tau}{1-\tau}\right)^{\delta_{ij}}\tilde{a}_j^\dagger\tilde{a}_i = \delta_{ij}, \quad [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0, \quad [\tilde{a}_i, \tilde{a}_j] = 0, \qquad \text{for } i, j = 1, 2. \tag{2.13}$$

As expected (2.2) is recovered for $\tau \to 0$. These relations are very reminiscent of the q-deformed oscillator algebra studied in this context for instance in [9, 10, 11, 12, 13, 14, 15, 16].

This example and the one in the previous subsection indicate that dynamical space-time relations will naturally lead to deformed Fock spaces. As we have seen some of them have a very convenient and well studied structure, as (2.13), whereas others are rather awkward such as (2.9) and (2.10). Let us therefore now reverse the scenario and deform first the Fock space relations in a "nice" way and subsequently compute the corresponding commutation relations for the dynamical variables.

3. Noncommutative space-time from q-deformed creation and annihilation operators

Resembling the relations (2.13) we q-deform the relations in (2.2) by defining a new set of creation and annihilation operators $A_1, A_1^\dagger, A_2, A_2^\dagger$ satisfying

$$A_i A_j^\dagger - q^{2\delta_{ij}} A_j^\dagger A_i = \delta_{ij}, \quad [A_i^\dagger, A_j^\dagger] = 0, \quad [A_i, A_j] = 0, \qquad \text{for } i,j = 1,2. \tag{3.1}$$

There exist various other possibilities to deform the relations (2.2) which still lead to constructable Fock spaces, such as for instance using different qs in the first relation of (3.1), i.e. $q^{2\delta_{ij}} \to q_i^{2\delta_{ij}}$ or replacing the δ_{ij} on the right hand side of the first relation by $q^{g(A_i^\dagger A_i)}$ with $g(x)$ being an arbitrary function as in [11, 16]. Guided by the limit $q \to 1$ in which we should recover the relations (2.6) and the properties of these operators under a $\mathcal{PT}$-transformation, we expand the new set of deformed canonical variables X, Y, P_x, P_y linearly in terms of the $A_1, A_1^\dagger, A_2, A_2^\dagger$ as

$$\begin{aligned} X &= \kappa_1(A_1^\dagger + A_1) + \kappa_2(A_2^\dagger + A_2), \qquad & P_x &= i\kappa_3(A_1^\dagger - A_1) + i\kappa_4(A_2^\dagger - A_2), \\ Y &= i\kappa_5(A_1^\dagger - A_1) + i\kappa_6(A_2^\dagger - A_2), \qquad & P_y &= \kappa_7(A_1^\dagger + A_1) + \kappa_8(A_2^\dagger + A_2). \end{aligned} \tag{3.2}$$

The constants $\kappa_1, \ldots, \kappa_8 \in \mathbb{R}$ are unknown for the time being. Inverting the relations (3.2) we may express the deformed creation and annihilation operators in terms of the deformed canonical variables

$$\begin{aligned} A_1 &= \tfrac{\kappa_8}{\lambda}X + i\tfrac{\kappa_4}{\mu}Y - i\tfrac{\kappa_6}{\mu}P_x - \tfrac{\kappa_2}{\lambda}P_y, \qquad & A_1^\dagger &= \tfrac{\kappa_8}{\lambda}X - i\tfrac{\kappa_4}{\mu}Y + i\tfrac{\kappa_6}{\mu}P_x - \tfrac{\kappa_2}{\lambda}P_y, \\ A_2 &= -\tfrac{\kappa_7}{\lambda}X - i\tfrac{\kappa_3}{\mu}Y + i\tfrac{\kappa_5}{\mu}P_x + \tfrac{\kappa_1}{\lambda}P_y, \qquad & A_2^\dagger &= -\tfrac{\kappa_7}{\lambda}X + i\tfrac{\kappa_3}{\mu}Y - i\tfrac{\kappa_5}{\mu}P_x + \tfrac{\kappa_1}{\lambda}P_y, \end{aligned} \tag{3.3}$$

where we abbreviated $\lambda := 2(\kappa_1\kappa_8 - \kappa_2\kappa_7) \neq 0$ and $\mu := 2(\kappa_4\kappa_5 - \kappa_3\kappa_6) \neq 0$. Using the representation (3.2) together with (3.1) we compute

$$[X,Y] = 2i(\kappa_1\kappa_5 + \kappa_2\kappa_6) + 2i(q^2-1)(\kappa_1\kappa_5 A_1^\dagger A_1 + \kappa_2\kappa_6 A_2^\dagger A_2), \tag{3.4}$$

$$[X,P_x] = 2i(\kappa_1\kappa_3 + \kappa_2\kappa_4) + 2i(q^2-1)(\kappa_1\kappa_3 A_1^\dagger A_1 + \kappa_2\kappa_4 A_2^\dagger A_2), \tag{3.5}$$

$$[Y,P_y] = -2i(\kappa_5\kappa_7 + \kappa_6\kappa_8) + 2i(1-q^2)(\kappa_5\kappa_7 A_1^\dagger A_1 + \kappa_6\kappa_8 A_2^\dagger A_2), \tag{3.6}$$

$$[P_x,P_y] = -2i(\kappa_3\kappa_7 + \kappa_4\kappa_8) + 2i(1-q^2)(\kappa_3\kappa_7 A_1^\dagger A_1 + \kappa_4\kappa_8 A_2^\dagger A_2), \tag{3.7}$$

$$[X,P_y] = 0, \tag{3.8}$$

$$[Y,P_x] = 0. \tag{3.9}$$

Next we employ the relations (3.3) and evaluate

$$\begin{aligned} A_1^\dagger A_1 = {} & \frac{\kappa_8^2}{\lambda^2}X^2 + \frac{\kappa_4^2}{\mu^2}Y^2 + \frac{\kappa_6^2}{\mu^2}P_x^2 + \frac{\kappa_2^2}{\lambda^2}P_y^2 - \frac{2\kappa_8\kappa_2}{\lambda^2}XP_y - \frac{2\kappa_4\kappa_6}{\mu^2}YP_x \\ & + i\frac{\kappa_4\kappa_8}{\lambda\mu}[X,Y] + i\frac{\kappa_4\kappa_2}{\lambda\mu}[Y,P_y] - i\frac{\kappa_6\kappa_8}{\lambda\mu}[X,P_x] - i\frac{\kappa_6\kappa_2}{\lambda\mu}[P_x,P_y], \end{aligned} \tag{3.10}$$

$$\begin{aligned} A_2^\dagger A_2 = {} & \frac{\kappa_7^2}{\lambda^2}X^2 + \frac{\kappa_3^2}{\mu^2}Y^2 + \frac{\kappa_5^2}{\mu^2}P_x^2 + \frac{\kappa_1^2}{\lambda^2}P_y^2 - \frac{2\kappa_7\kappa_1}{\lambda^2}XP_y - \frac{2\kappa_3\kappa_5}{\mu^2}YP_x \\ & + i\frac{\kappa_3\kappa_7}{\lambda\mu}[X,Y] + i\frac{\kappa_3\kappa_1}{\lambda\mu}[Y,P_y] - i\frac{\kappa_5\kappa_7}{\lambda\mu}[X,P_x] - i\frac{\kappa_5\kappa_1}{\lambda\mu}[P_x,P_y]. \end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into the right hand sides of (3.4)-(3.7) we obtain four equations for the four unknown commutators $[X,Y]$, $[X,P_x]$, $[Y,P_y]$ and $[P_x,P_y]$. Solving these equations, the resulting dynamical noncommutative relations are

$$
\begin{aligned}
[X,Y] &= i\theta + i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_2\kappa_6\kappa_7^2+\kappa_1\kappa_5\kappa_8^2}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}X^2+\frac{\kappa_2\kappa_6\kappa_3^2+\kappa_1\kappa_5\kappa_4^2}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}Y^2\right. \qquad (3.12)\\
&\quad +\frac{\kappa_5\kappa_6(\kappa_2\kappa_5+\kappa_1\kappa_6)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}P_x^2+\frac{\kappa_1\kappa_2(\kappa_2\kappa_5+\kappa_1\kappa_6)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}P_y^2\\
&\quad \left.-\frac{2\kappa_1\kappa_2(\kappa_6\kappa_7+\kappa_5\kappa_8)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}XP_y-\frac{2\kappa_5\kappa_6(\kappa_2\kappa_3+\kappa_1\kappa_4)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}YP_x\right],\\
[X,P_x] &= i\hbar + i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_2\kappa_4\kappa_7^2+\kappa_1\kappa_3\kappa_8^2}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}X^2+\frac{\kappa_3\kappa_4(\kappa_2\kappa_3+\kappa_1\kappa_4)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}Y^2\right. \qquad (3.13)\\
&\quad +\frac{\kappa_2\kappa_4\kappa_5^2+\kappa_1\kappa_3\kappa_6^2}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}P_x^2+\frac{\kappa_1\kappa_2(\kappa_2\kappa_3+\kappa_1\kappa_4)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}P_y^2\\
&\quad \left.-\frac{2\kappa_1\kappa_2(\kappa_4\kappa_7+\kappa_3\kappa_8)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}XP_y-\frac{2\kappa_3\kappa_4(\kappa_2\kappa_5+\kappa_1\kappa_6)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}YP_x\right],\\
[Y,P_y] &= i\hbar - i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_7\kappa_8(\kappa_6\kappa_7+\kappa_5\kappa_8)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}X^2+\frac{\kappa_6\kappa_8\kappa_3^2+\kappa_5\kappa_7\kappa_4^2}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}Y^2\right. \qquad (3.14)\\
&\quad +\frac{\kappa_5\kappa_6(\kappa_6\kappa_7+\kappa_5\kappa_8)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}P_x^2+\frac{\kappa_6\kappa_8\kappa_1^2+\kappa_5\kappa_7\kappa_2^2}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}P_y^2\\
&\quad \left.-\frac{2\kappa_7\kappa_8(\kappa_2\kappa_5+\kappa_1\kappa_6)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}XP_y-\frac{2\kappa_5\kappa_6(\kappa_4\kappa_7+\kappa_3\kappa_8)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}YP_x\right],\\
[P_x,P_y] &= -i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_7\kappa_8(\kappa_4\kappa_7+\kappa_3\kappa_8)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}X^2+\frac{\kappa_3\kappa_4(\kappa_4\kappa_7+\kappa_3\kappa_8)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}Y^2\right. \qquad (3.15)\\
&\quad +\frac{\kappa_4\kappa_8\kappa_5^2+\kappa_3\kappa_7\kappa_6^2}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}P_x^2+\frac{\kappa_4\kappa_8\kappa_1^2+\kappa_3\kappa_7\kappa_2^2}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}P_y^2\\
&\quad \left.-\frac{2\kappa_7\kappa_8(\kappa_2\kappa_3+\kappa_1\kappa_4)}{(\kappa_2\kappa_7-\kappa_1\kappa_8)^2}XP_y-\frac{2\kappa_3\kappa_4(\kappa_6\kappa_7+\kappa_5\kappa_8)}{(\kappa_4\kappa_5-\kappa_3\kappa_6)^2}YP_x\right].
\end{aligned}
$$

For the constant terms of these commutators we have implemented here the constraints

$$\kappa_1\kappa_5+\kappa_2\kappa_6 = \frac{\theta}{4}\left(1+q^2\right), \qquad (3.16)$$

$$\kappa_1\kappa_3+\kappa_2\kappa_4 = \frac{\hbar}{4}\left(1+q^2\right), \qquad (3.17)$$

$$\kappa_5\kappa_7+\kappa_6\kappa_8 = -\frac{\hbar}{4}\left(1+q^2\right), \qquad (3.18)$$

$$\kappa_3\kappa_7+\kappa_4\kappa_8 = 0, \qquad (3.19)$$

in order to ensure that the limit $q\to 1$ for the relations (3.12)-(3.15) will yield the standard commutation relations for noncommutative flat space-time (2.1). The relations (3.8) and (3.9) remain of course unchanged.

3.1 Some special limits

Keeping all the constants generic in the algebra (3.12)-(3.15) will make the handling very cumbersome. However, using the fact that we still have four κs free at our disposal allows us to extract some special limiting cases in order to obtain some more tractable algebras.

3.1.1 Dependent X and Y directions

Considering (3.2) the first natural limit is to reduce the number of free parameters to four, e.g. $\kappa_1, \ldots, \kappa_4$, and introduce some dependence for the coefficients in the Y-direction on those in the X-direction. Considering the representation (3.3) we impose

$$\kappa_5 = \kappa_1, \qquad \kappa_6 = -\kappa_2, \qquad \kappa_7 = -\kappa_3 \qquad \text{and} \qquad \kappa_8 = \kappa_4, \tag{3.20}$$

such that without activating the constraints (3.16)-(3.19) the eight unknown constants are already limited to four. The four constraints (3.16)-(3.19) are not independent for these choices as (3.17) and (3.18) become identical. The remaining three constraints read

$$\kappa_1^2 - \kappa_2^2 = \frac{\theta}{4}\left(1+q^2\right), \qquad \kappa_1\kappa_3 + \kappa_2\kappa_4 = \frac{\hbar}{4}\left(1+q^2\right) \quad \text{and} \quad \kappa_3^2 = \kappa_4^2, \tag{3.21}$$

which means we have still one constant at our disposal. The algebra (3.12)-(3.15), (3.8) and (3.9) simplifies to

$$[X,Y] = i\theta + i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_1\kappa_4 - \kappa_2\kappa_3}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(X^2+Y^2) - \frac{2\kappa_1\kappa_2}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(XP_y - YP_x)\right], \tag{3.22}$$

$$[X,P_x] = ih + i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(X^2+Y^2) + \frac{\kappa_1\kappa_2}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(P_x^2+P_y^2)\right], \tag{3.23}$$

$$[Y,P_y] = ih + i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(X^2+Y^2) + \frac{\kappa_1\kappa_2}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(P_x^2+P_y^2)\right], \tag{3.24}$$

$$[P_x,P_y] = -i\frac{q-q^{-1}}{q+q^{-1}}\left[\frac{\kappa_1\kappa_4 - \kappa_2\kappa_3}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(P_x^2+P_y^2) - \frac{2\kappa_3\kappa_4}{\kappa_1\kappa_4 + \kappa_2\kappa_3}(XP_y - YP_x)\right], \tag{3.25}$$

$$[X,P_y] = 0, \tag{3.26}$$

$$[Y,P_x] = 0. \tag{3.27}$$

The conditions $\lambda \neq 0$, $\mu \neq 0$ now coincide and have translated into $\kappa_1\kappa_4 + \kappa_2\kappa_3 \neq 0$. Our choice of constants has achieved that the terms XP_y and YP_x have combined into the angular momentum operator L_z.

3.2 Membrane and string type relations

As one of the κs is still not fixed we can simplify the commutation relations (3.22)-(3.27) further by setting $\kappa_2 = 0$, such that all three unknown left are fixed by the remaining three relations

$$\kappa_1^2 = \frac{\theta}{4}\left(1+q^2\right), \qquad \kappa_1\kappa_3 = \frac{\hbar}{4}\left(1+q^2\right) \quad \text{and} \quad \kappa_3^2 = \kappa_4^2. \tag{3.28}$$

We may now implement the constraints (3.28) in the algebra (3.22)-(3.27) and eliminate all constants κ_i being left with a purely q-deformed algebra

$$[X,Y] = i\theta + i\frac{q-q^{-1}}{q+q^{-1}}\left(X^2+Y^2\right), \tag{3.29}$$

$$[X,P_x] = i\hbar + i\frac{q-q^{-1}}{q+q^{-1}}\frac{\hbar}{\theta}\left(X^2+Y^2\right), \tag{3.30}$$

$$[Y,P_y] = i\hbar + i\frac{q-q^{-1}}{q+q^{-1}}\frac{\hbar}{\theta}\left(X^2+Y^2\right), \tag{3.31}$$

$$[P_x,P_y] = i\frac{q^{-1}-q}{q^{-1}+q}\left[P_x^2+P_y^2+2\frac{\hbar}{\theta}(XP_y-YP_x)\right], \tag{3.32}$$

$$[X,P_y] = 0, \tag{3.33}$$

$$[Y,P_x] = 0. \tag{3.34}$$

These relations reduce to (2.11) for $q=\pm\sqrt{(1+\tau)/(1-\tau)}$. Notice further that the q-deformation and the θ-deformation originally introduced in the space-space commutation relations have become intrinsically linked through the constraints. We can no longer take the limit $\theta\to 0$ separately without taking also the limit $q\to 0$. However, the limit $q\to 0$ may still be taken separately and we recover (2.1).

We named these relations "membrane type" as the relation (3.29) will give rise to a minimal length in the X and Y direction in a simultaneous measurement as we will explain in more detail below. As it stands, the relation (3.29) will lead to the same minimal length in either direction. This is by no means unavoidable and can be overcome by taking another limit of the algebra (3.12)-(3.15), (3.8) and (3.9). Setting for instance $\kappa_2=\kappa_6=0$ without any additional constraints besides (3.16)-(3.19), which in this case read

$$\kappa_1\kappa_5=\frac{\theta}{4}\left(1+q^2\right),\quad \kappa_1\kappa_3=\frac{\hbar}{4}\left(1+q^2\right),\quad \kappa_5\kappa_7=-\frac{\hbar}{4}\left(1+q^2\right),\quad \kappa_3\kappa_7=-\kappa_4\kappa_8. \tag{3.35}$$

the algebra simplifies considerably

$$[X,Y] = i\theta + i\frac{q-q^{-1}}{q+q^{-1}}\left(\frac{\kappa_5}{\kappa_1}X^2+\frac{\kappa_1}{\kappa_5}Y^2\right), \tag{3.36}$$

$$[X,P_x] = i\hbar + i\frac{q-q^{-1}}{q+q^{-1}}\left(\frac{\kappa_3}{\kappa_1}X^2+\frac{\kappa_1\kappa_3}{\kappa_5^2}Y^2\right), \tag{3.37}$$

$$[Y,P_y] = i\hbar - i\frac{q-q^{-1}}{q+q^{-1}}\left(\frac{\kappa_5\kappa_7}{\kappa_1^2}X^2+\frac{\kappa_7}{\kappa_5}Y^2\right), \tag{3.38}$$

$$\begin{aligned}[P_x,P_y] = -i\frac{q-q^{-1}}{q+q^{-1}}\Bigg[&(\kappa_4\kappa_7+\kappa_3\kappa_8)\left(\frac{\kappa_7}{\kappa_8\kappa_1^2}X^2+\frac{\kappa_3}{\kappa_4\kappa_5^2}Y^2\right)\\ &+\frac{\kappa_8}{\kappa_4}P_x^2+\frac{\kappa_4}{\kappa_8}P_y^2-2\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}YP_x-2\frac{\kappa_3\kappa_8}{\kappa_4\kappa_5}XP_y\Bigg],\end{aligned} \tag{3.39}$$

$$[X,P_y] = 0, \tag{3.40}$$

$$[Y,P_x] = 0. \tag{3.41}$$

We notice that in (3.36) we have now different coefficients in front of the X^2 and Y^2-terms and may achieve unequal minimal length in either direction, although they are not entirely independent being related by the first relation in (3.35).

Taking now a less trivial limit, we may obtain string like relations from (3.36)-(3.41) similar to those proposed in [18]. Parameterizing $q = e^{2\tau\kappa_5^2}$ with $\tau \in \mathbb{R}^+$ and taking the limit $\kappa_5 \to 0$ we obtain yet simpler relations. As we have still many free parameters left in (3.39) we have several choices. With respect to the constraints (3.35) we can take for instance $\kappa_3 = \hbar/\theta\kappa_5$, $\kappa_4 = \hbar^2/\theta\kappa_5$, $\kappa_8 = (1+q^2)/(4\kappa_5)$ and derive the simple "string type" relations

$$\begin{array}{lll} [X,Y] = i\theta\left(1+\tau Y^2\right), & [X,P_x] = i\hbar\left(1+\tau Y^2\right), & [X,P_y] = 0, \\ [P_x,P_y] = i\tau\frac{\hbar^2}{\theta}Y^2, & [Y,P_y] = i\hbar\left(1+\tau Y^2\right), & [Y,P_x] = 0. \end{array} \tag{3.42}$$

Arguing in the same way as in [18], we obtain now from the first relation in (3.42) a minimal length in the Y-direction in a simultaneous X,Y-measurement as the commutator $[X,Y]$ is identical. The remaining commutators are, however, different.

There are of course plenty of other possible limits compatible with the constraints (3.16)-(3.19), which we do not present here.

4. Minimal areas and minimal lengths

As mentioned, one of the interesting physical consequences of noncommutative space-time, especially when it is dynamical, is the emergence of minimal lengths in simultaneous measurements of two observables. The standard noncommutative space-time relations (2.1) give rise to additional uncertainties similar to the usual Heisenberg uncertainty relations, meaning for instance that the two position operators x_0 and y_0 can never be known with complete precision at *the same time*, where θ plays the role of $\hbar$ when compared with the conventional relations. When the underlying algebra becomes a dynamical noncommutative space-time structure the consequences are more severe and one finds that the position operators X or Y can *never* be known, that is even when giving up the entire knowledge about the canonical conjugate partner Y or X, respectively. Thus X or Y are said to be bound by some absolute minimal length ΔX_0 or ΔY_0, which is the highest possible precision to which these quantities can be resolved.

Minimal lengths have been known and studied for some time [9, 10, 11, 12, 13, 14, 15, 16] in simultaneous x,p-measurements as a consequence of a deformation of the x,p-commutator. In [18] it was demonstrated explicitly that they also result in simultaneous x,y-measurements as a consequence of the dynamical noncommutativity of space-time. Whereas the algebra investigated in [18] only gave rise to a minimal length in one direction, i.e. "string like" objects, we demonstrate here that the algebras provided in section 3 will lead to minimal lengths in two direction, i.e. minimal areas. Objects in these type of spaces are "membrane like", meaning that there exists a finitely extended region about whose substructure it is impossible to obtain any measurable knowledge.

Following the standard arguments we will now compute these quantities by starting with the well known relation

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A,B] \rangle\right|, \tag{4.1}$$

which holds for any two observables A and B, which are Hermitian with respect to the standard inner product. In order to determine the range of validity for this inequality we simply have to minimize $f(\Delta A, \Delta B) := \Delta A \Delta B - \frac{1}{2} |\langle [A,B] \rangle|$ as a function of ΔB to find the absolute minimal length ΔA_0. This means we need to solve the two equations $\partial_{\Delta B} f(\Delta A, \Delta B) = 0$ and $f(\Delta A, \Delta B) = 0$ for $\Delta A =: \Delta A_{\min}$ and subsequently compute the smallest value for $\Delta A_{\min}$ in order to obtain the absolute minimal length ΔA_0. In case we obtain minimal length for both of these observables we define the minimal area and its smallest possible value of four times the product, that is $\Delta(AB)_{\min}$ and $\Delta(AB)_0$, respectively.

For definiteness we choose now $\theta \in \mathbb{R}^+$ and carry out the analysis for the algebra (3.36)-(3.41) starting with a simultaneous X,Y-measurement. When $q^2 > 1$ the imaginary parts of all terms of the commutator $[X,Y]$ are positive due to the first constraint in (3.35). The absolute value for $|\langle [X,Y] \rangle|$ is therefore simply $\mathrm{Im} \langle [X,Y] \rangle$. When $q^2 < 1$ we use $|A - B| \geq A - B$ for $A, B > 0$ to drop the absolute value. Using furthermore that the mean-squared deviation about the expectation value $\langle X \rangle$ is given by $\Delta X^2 = \langle X^2 \rangle - \langle X \rangle^2$ and similarly for $X \leftrightarrow Y$, we compute

$$\Delta X_{\min} = \frac{\sqrt{|q^2-1| \left(\kappa_1^2 \langle X \rangle^2 + \kappa_5^2 \langle Y \rangle^2 \right) + \theta (q^4 - 1) \kappa_1 \kappa_5}}{2q\kappa_5}, \tag{4.2}$$

$$\Delta Y_{\min} = \frac{\sqrt{|q^2-1| \left(\kappa_5^2 \langle X \rangle^2 + \kappa_1^2 \langle Y \rangle^2 \right) + \theta (q^4 - 1) \kappa_1 \kappa_5}}{2q\kappa_1}, \tag{4.3}$$

such that the absolute minimal lengths result to

$$\Delta X_0 = \frac{\kappa_1}{q} \sqrt{|q^2-1|} \qquad \text{and} \qquad \Delta Y_0 = \frac{\kappa_5}{q} \sqrt{|q^2-1|}, \tag{4.4}$$

hen $\langle X \rangle = \langle Y \rangle = 0$. Together with the first constraint in (3.35) the absolute minimal area in the X,Y-plane results to

$$\Delta(XY)_0 = \theta \left| q^2 - q^{-2} \right|. \tag{4.5}$$

This means the size of the minimal area is independent of the free parameters κ_1 and κ_5. We can also make ΔY_0 a function of ΔX_0 and compute for given ΔX_0 the corresponding minimal length ΔY_0 or vice versa. Note that it is impossible to achieve any of the minimal lengths to vanish without the other becoming infinitely large. We illustrate this in figure 1, where we plot $\Delta Y_0(\Delta X_0) = \pm \theta \left| q^2 - q^{-2} \right| / (4 \Delta X_0)$ for a specific value of θ and various values of q. The two minimal areas indicated in the figure have the same size.

For a simultaneous X, P_x-measurement we compute similarly the minimal momentum in the X-direction

$$(\Delta P_x)_{\min} = \frac{\sqrt{(q^2-1)^2 (\langle Y \rangle^2 + \langle Y^2 \rangle) \kappa_3^2 \kappa_1^2 + \hbar |q^4 - 1| \kappa_1 \kappa_3 \kappa_5^2 + \langle X \rangle^2 (q^2-1)^2 \kappa_3^2 \kappa_5^2}}{(q^2+1)\kappa_1 \kappa_5}, \tag{4.6}$$

such that the corresponding absolute value turns out to be

$$(\Delta P_x)_0 = 2\kappa_3 \frac{\sqrt{|q^2-1|}}{q^2+1}. \tag{4.7}$$

There is no minimal length for X in this case as we can tune ΔX to be as small as we wish by enlarging ΔP_x.

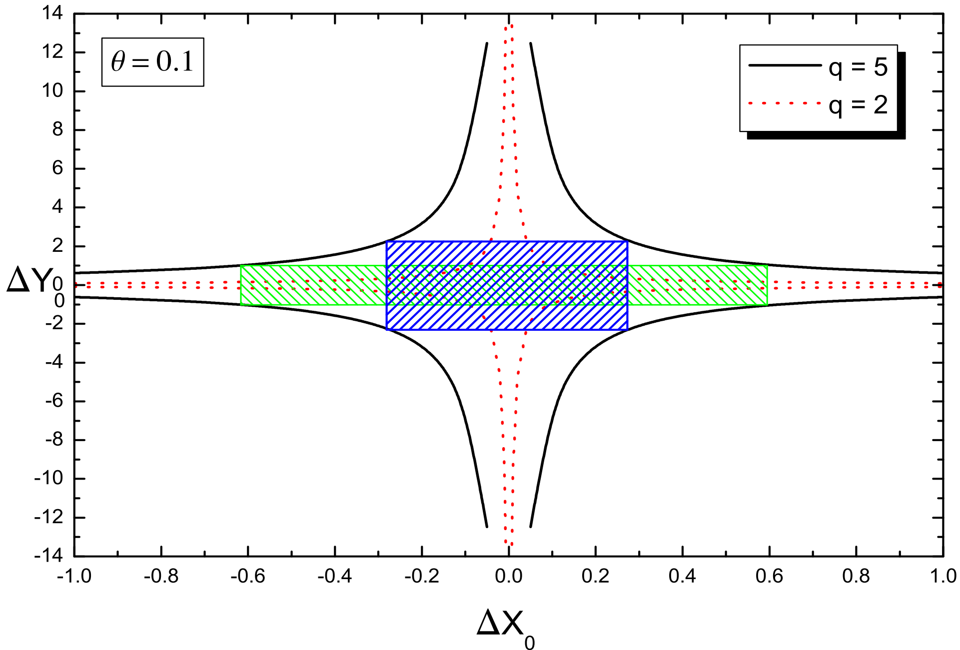

Figure 1: Minimal areas in the XY-plane.

Similarly we compute for a simultaneous Y, P_y-measurement the minimal momentum in the Y-direction

$$(\Delta P_y)_{\min} = \frac{\sqrt{(q^2-1)^2(\langle X\rangle^2 + \langle X^2\rangle)\kappa_7^2\kappa_5^2 + \hbar\,|1-q^4|\,\kappa_5\kappa_7\kappa_1^2 + \langle Y\rangle^2 (q^2-1)^2\kappa_1^2\kappa_7^2}}{(q^2+1)\kappa_1\kappa_5}, \tag{4.8}$$

with corresponding absolute value

$$(\Delta P_y)_0 = 2\kappa_7 \frac{\sqrt{|q^2-1|}}{q^2+1}. \tag{4.9}$$

By the same reasoning as in the previous case there is also no minimal length for Y in this case as ΔY can be taken to be as small as desiredh by enlarging ΔP_y.

The analysis for a simultaneous P_x, P_y-measurement is less straightforward due to the appearance of the angular momentum term. we first note that

$$\begin{aligned} |\langle[P_x, P_y]\rangle| \geq \left|\frac{q^2-1}{q^2+1}\right| \Bigg[& |\kappa_4\kappa_7 + \kappa_3\kappa_8| \left(\frac{\kappa_7}{\kappa_8\kappa_1^2}\left\langle X^2\right\rangle - \left|\frac{\kappa_3}{\kappa_4\kappa_5^2}\right|\left\langle Y^2\right\rangle\right) \\ & + \frac{\kappa_8}{\kappa_4}\left\langle P_x^2\right\rangle + \frac{\kappa_4}{\kappa_8}\left\langle P_y^2\right\rangle - 2\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\left|\langle Y P_x\rangle\right| - 2\frac{\kappa_3\kappa_8}{\kappa_4\kappa_5}\left|\langle X P_y\rangle\right|\Bigg], \end{aligned} \tag{4.10}$$

where for definiteness we assumed that $\kappa_3^2 < \kappa_4^2$. Using next the estimate $|\langle AB\rangle| \leq \Delta A \Delta B + |\langle A\rangle\langle B\rangle|$ we compute

$$\Delta P_x \Delta P_y \geq \frac{1}{2}\left|\frac{q^2-1}{q^2+1}\right| \left[\frac{\kappa_8}{\kappa_4}\Delta P_x^2 + \frac{\kappa_4}{\kappa_8}\Delta P_y^2 - 2\left|\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\right|\Delta Y \Delta P_x - 2\frac{\kappa_3\kappa_8}{\kappa_4\kappa_5}\Delta X \Delta P_y + \lambda\right], \tag{4.11}$$

with

$$\lambda = \frac{\kappa_8}{\kappa_4}\langle P_x\rangle^2 + \frac{\kappa_4}{\kappa_8}\langle P_y\rangle^2 + |\kappa_4\kappa_7 + \kappa_3\kappa_8|\left(\frac{\kappa_7}{\kappa_8\kappa_1^2}\langle X\rangle^2 - \left|\frac{\kappa_3}{\kappa_4\kappa_5^2}\right|\langle Y\rangle^2\right) \tag{4.12}$$
$$-2\left|\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\right| |\langle Y\rangle\langle P_x\rangle| - 2\frac{\kappa_3\kappa_8}{\kappa_4\kappa_5}|\langle X\rangle\langle P_y\rangle| .$$

When varying the inequality (4.11) in the same manner as the expressions above we find

$$(\Delta P_x)_{\min} = -\frac{|q^4-1|}{4q^2}\frac{\kappa_3\kappa_8}{\kappa_4\kappa_5}\Delta X - \frac{(q^2-1)^2}{4q^2}\left|\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\right|\frac{\kappa_4}{\kappa_8}\Delta Y \tag{4.13}$$
$$\pm\frac{|q^2-q^{-2}|}{4}\sqrt{\frac{\kappa_3^2\kappa_8^2\Delta X^2}{\kappa_5^2\kappa_4^2} + \frac{\kappa_7^2\kappa_4^4\Delta Y^2}{\kappa_1^2\kappa_8^4} + \frac{2\left|\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\right|\kappa_3\Delta X\Delta Y}{\kappa_5|q^2-1|(q^2+1)^{-1}} + \frac{4q^2\lambda\kappa_4}{\kappa_8(q^2-1)^2}}.$$

and

$$(\Delta P_y)_{\min} = -\frac{(q^2-1)^2}{4q^2}\frac{\kappa_3}{\kappa_1\kappa_4^2\kappa_5^2}\Delta X - \frac{|1-q^4|}{4q^2}\left|\frac{\kappa_4\kappa_7}{\kappa_1\kappa_8}\right|\frac{1}{\kappa_1\kappa_5\kappa_8^2}\Delta Y \tag{4.14}$$
$$\pm\frac{|q^2-q^{-2}|}{4}\sqrt{\frac{\kappa_3^2\kappa_8^4\Delta X^2}{\kappa_4^4\kappa_5^2} + \frac{\kappa_4^2\kappa_7^2\Delta Y^2}{\kappa_1^2\kappa_8^2} + \frac{2\left|\frac{\kappa_7\kappa_8}{\kappa_1\kappa_4}\right|\kappa_3\Delta X\Delta Y}{\kappa_5(q^2-1)^2|1-q^4|^{-1}} + \frac{4q^2\lambda\kappa_8}{\kappa_4(q^2-1)^2}}.$$

We can minimize this expression further with a subsequent X, Y-measurement. This is, however, a matter of interpretation if one would like to view measurements as a pairwise succession or whether this should be considered as a simultaneous measurement of four quantities. A further option would be to exploit the explicit occurrence of the L_z-operator and take this complication here as a hint that the angular momentum variables are possibly a more natural set of variables. We leave this problem for future investigations. Similar expressions are obtained for the choice $\kappa_3^2 > \kappa_4^2$.

5. Conclusions

We have demonstrated that dynamical noncommutative space-time relations will inevitably lead to deformed oscillator algebras. Taking some well studied oscillator algebras with the useful property that the entire Fock spaces associated to them is explicitly constructable as a starting point, we derived some very general commutation relations (3.12)-(3.15) for the dynamical variables. Since these relations are rather cumbersome, we investigated some specific limits leading to simplified and more tractable variants, whose properties can be discussed more transparently. All of these special limits led to minimal lengths in the two dimensional space and mostly to minimal areas which we have calculated explicitly (4.5).

There are some obvious further problems following from our considerations. First of all it would be very interesting to explore the consequences of taking different types of deformations as starting points and derive the resulting dynamical commutation relations. Secondly it would be interesting to consider explicit models on these type space-time structures and thirdly but not last a generalization to three dimensional space would be highly interesting. The latter will almost inevitably lead to minimal volumes.

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