Non-Hermitian Hamiltonians of Lie algebraic type

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Abstract: We analyse a class of non-Hermitian Hamiltonians, which can be expressed in terms of bilinear combinations of generators in the $sl_2(\mathbb{R})$-Lie algebra or their isomorphic $su(1,1)$-counterparts. The Hamlitonians are prototypes for solvable models of Lie algebraic type. Demanding a real spectrum and the existence of a well defined metric, we systematically investigate the constraints these requirements impose on the coupling constants of the model and the parameters in the metric operator. We compute isospectral Hermitian counterparts for some of the original non-Hermitian Hamiltonian. Alternatively we employ a generalized Bogoliubov transformation, which allows to compute explicitly real energy eigenvalue spectra for these type of Hamiltonians, together with their eigenstates. We compare the two approaches.

1. Introduction

Non-Hermitian operators in a complex Hilbert space have been studied in the mathematical literature [1, 2, 3, 4] for a long time. Also in various contexts of physics non-Hermitian Hamiltonians have frequently occurred over the years. Besides those having complex eigenvalue spectra, and thus describing dissipative systems, some with real eigenvalues have been considered too. For instance in the study of strong interactions at high energies in form of Regge models [5, 6], in integrable quantum field theories in form of affine Toda field theories with complex coupling constants [7, 8], in condensed matter physics in the context of the XXZ-spin chain [9] and recently also in a field theoretical scenario in the quantization procedure of strings on an $AdS_5 \times S^5$- background [10]. Various attempts to understand these sort of Hamiltonians have been made over the years, e.g. [2, 3, 4, 11, 12], which may be traced back more than half a century [13]. A more systematic study and revival of such type of Hamiltonians was initiated roughly ten years ago [14], for reviews and special issues see e.g. [15, 16, 17, 18, 19]. Meanwhile some concrete experimental settings have been proposed in which properties of these models can be tested [20]. Here we wish to focus on the large subclass of non-Hermitian Hamiltonians with real eigenvalues of Lie algebraic type.

Many interesting and important physical Hamiltonians may be cast into a Lie algebraic formulation. For very general treatments one can take these formulations as a starting point and generic frameworks, such that particular models simply result as specific choices of representations[1]. The virtue of this kind of approach is that it allows for a high degree of universality and it has turned out to be especially fruitful in the context of integrable and solvable models, the former implying that the amount of conserved quantities equals the degrees of freedom in the system and the latter referring to a situation in which the spectra can be determined explicitly. Here we will extend such type of treatment to pseudo-Hermitian or more precisely, and more useful, to quasi-Hermitian Hamiltonian systems. More specifically, we wish to consider non-Hermitian Hamiltonian systems for which an exact similarity transformation can be explicitly constructed, such that it transforms it into a Hermitian one. We refer to them as *solvable quasi Hermitian* (SQH) Hamiltonian systems, see e.g. [21, 22, 23, 24, 25, 26, 27, 28] for explicit models of this type. The main virtues of these models is that they obviously possess real eigenvalue spectra [4], due to the fact that quasi-Hermitian systems are directly related to Hermitian Hamiltonian systems. Alternatively, and very often equivalently, one may explain the reality of the spectra of some non-Hermitian Hamiltonians when one encounters unbroken $\mathcal{PT}$-symmetry, which in the recent context was first pointed out in [29]. Unbroken specifies here that both the Hamiltonian and the wavefunction remain invariant under a simultaneous parity transformation $\mathcal{P}: x \to -x$ and time reversal $\mathcal{T}: t \to -t$. Noting that the $\mathcal{PT}$-operator is just a specific example of an anti-linear operator this is known for a long time [30].

Our manuscript is organised as follows: In section 2 we introduce the basic ideas of Hamiltonians of Lie algebraic type, focussing especially on the two isomorphic cases $sl_2(\mathbb{R})$ and $su(1,1)$. Section 3 is devoted to the systematic construction of similarity transformations towards isospectral Hermitian counterparts and metric operators. In section 4 we employ generalized Bogoliubov transformations to compute real eigenvalue spectra for the Hamiltonians of Lie algebraic type and compare our results with the findings of section 3. In section 5 we comment on some explicit realisations, which are useful in order to relate to some specific physical models. Our conclusions are stated in section 6.

2. Hamiltonians of Lie algebraic type

The notion of *quasi-exactly solvable* operators was introduced by Turbiner [31] demanding that their action on the space of polynomials leaves it invariant. More specifically when taking the operator to be a Hamiltonian operator H acting on the space of polynomials of order n as H: $V_n \mapsto V_n$, it preserves by definition the entire flag $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n \subset \ldots$ Models respecting this property are referred to as *exactly solvable*. Whenever these type of Hamiltonians can be written in terms of bilinear combinations of first-order differential operators generating a finite dimensional Lie algebra, it is said they are of *Lie algebraic type* [32].

In order to be more concrete we have to identify V_n as the representation space of some specific Lie algebra. The simplest choice is to involve the only rank one Lie algebra

[1] See section 5 for concrete examples, such as the BCS-Hamiltonian of supersymmetry and others.

$sl_2(\mathbb{C})$. It is well known that this algebra contains the compact real form $su(2)$ and the non-compact real form $sl_2(\mathbb{R})$, which is isomorphic to $su(1,1)$, see for instance [33, 34]. We will focus here on these two choices.

2.1 Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type

The three generator J_0, J_1 and J_2 of $sl_2(\mathbb{R})$ satisfy the commutation relations $[J_1, J_2] = -iJ_0$, $[J_0, J_1] = iJ_2$ and $[J_0, J_2] = -iJ_1$, such that the operators J_0, $J_\pm = J_1 \pm J_2$ obey

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0, \qquad \text{and} \qquad J_0^\dagger, J_\pm^\dagger \notin \{J_0, J_\pm\}. \tag{2.1}$$

As possible realisation for this algebra one may take for instance the differential operators

$$J_- = \partial_x, \qquad J_0 = x\partial_x - \frac{n}{2}, \qquad J_+ = x^2\partial_x - nx, \qquad n \in \mathbb{Z}, \tag{2.2}$$

allegedly attributed to Sophus Lie, see e.g. [31]. Clearly the action of this algebra on the space of polynomials

$$V_n = \text{span}\{1, x, x^2, x^3, x^4, ..., x^n\} \tag{2.3}$$

leaves it invariant. According to the above specified notions, a quasi-exactly solvable Hamiltonian of Lie algebraic type is therefore of the general form

$$H_J = \sum_{l=0,\pm} \kappa_l J_l + \sum_{n,m=0,\pm} \kappa_{nm} : J_n J_m :, \qquad \kappa_l, \kappa_{nm} \in \mathbb{R}, \tag{2.4}$$

where we introduced the ordering

$$: J_n J_m := \begin{cases} J_n J_m & \text{for } n \geq m \\ 0 & \text{for } n < m \end{cases} \tag{2.5}$$

to avoid unnecessary double counting[2]. This means the Hamiltonian H_J involves nine real constants κ, plus a possible overall shift in the energy. It is evident from the representation (2.2) that when $\kappa_+ = \kappa_{++} = \kappa_{+0} = 0$ the model becomes exactly solvable in the sense specified above. For the given representation (2.2) the $\mathcal{PT}$-symmetry may be implemented trivially by rescaling $J_\pm \to \tilde{J}_\pm = \pm iJ_\pm$ and $J_0 \to \tilde{J}_0 = J_0$, which leaves the algebra (2.1) unchanged. Taking the algebra in this representation will leave the real vector space of $\mathcal{PT}$-symmetric polynomials

$$V_n^{\mathcal{PT}} = \text{span}\{1, ix, x^2, ix^3, x^4, ..., e^{i\pi n/2}x^n\} \tag{2.6}$$

invariant. Since by construction the Hamiltonian $H_{\tilde{J}}$ and the wavefunctions are $\mathcal{PT}$-symmetric, as they are polynomials in $V_n^{\mathcal{PT}}$, the eigenvalues for these systems must be real by construction [29, 30]. Nonetheless, to determine the explicit similarity transformation remains a challenge.

A simple explicit example for $H_{\tilde{J}}$ with $\kappa_{00} = -4$, $\kappa_+ = -2\zeta = \kappa_-$, $\zeta \in \mathbb{R}$ an overall energy shift by $M^2 + \zeta^2$ and all remaining coefficients equal to zero was recently studied by

[2]By setting some of the arrangements to zero our normal ordering prescription differs slightly from the ordinary one, but this is simply convention here and has no bearing on our analysis.

Bagchi et al [35, 36]. The Hamiltonian arises as a gauged version from the $\mathcal{PT}$-symmetric potential $V(x) = -[\zeta \sinh 2x - iM]^2$. The first energy levels together with their corresponding wavefunctions were constructed and the typical real energy spectrum for unbroken $\mathcal{PT}$-symmetry and complex conjugate pairs for broken $\mathcal{PT}$-symmetry was found. However, even for this simple version of (2.4) a general treatment leading to the complete eigenvalue spectrum and a well defined metric has not been carried out.

As we indicated, the representation (2.2) is ideally suited with regard to the question of solvability. However, the Hermiticity properties for the J's are not straightforward to determine within a Lie algebraic framework, since the Hermitian conjugates of the J's can not be written in terms of the original generators. This feature makes the representation (2.2) rather unsuitable for the determination of the Hermiticity properties of the Hamiltonian $H_{\tilde{J}}$ in generality. The implication is that we may carry out our programme only for specific representations using directly some concrete operator expressions or equivalently Moyal products of functions [37, 38, 27], see section 3.1 for an example, and not in a generic representation independent way. An additional undesired feature is that the Hamiltonian $H_{\tilde{J}}$ in terms the representation (2.2) does not allow to capture many of the important and interesting physical models. We will therefore consider a slightly different type of algebra.

2.2 Hamiltonians of $su(1,1)$-Lie algebraic type

The above mentioned problems do not occur when we express our Hamiltonian in terms of the isomorphic $su(1,1)$-Lie algebra, whose generators K_0, K_1 and K_2 satisfy the same commutation relations $[K_1, K_2] = -iK_0$, $[K_0, K_1] = iK_2$ and $[K_0, K_2] = -iK_1$. Consequently the operators K_0, $K_\pm = K_1 \pm K_2$ satisfy an isomorphic algebra to (2.1)

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0 \qquad \text{and} \qquad K_0^\dagger = K_0, K_\pm^\dagger = K_\mp. \tag{2.7}$$

In analogy to (2.4) we may then consider a Hamiltonian of Lie algebraic type in terms of the $su(1,1)$-generators

$$H_K = \sum_{l=0,\pm} \mu_l K_l + \sum_{n,m=0,\pm} \mu_{nm} : K_n K_m :, \qquad \mu_l, \mu_{nm} \in \mathbb{R}, \tag{2.8}$$

where we have used the same conventions for the ordering as in equation (2.5). In general this Hamiltonian is not Hermitian, that is when the constants $\mu_+ \neq \mu_-$, $\mu_{++} \neq \mu_{--}$ or $\mu_{+0} \neq \mu_{0-}$ we have $H_K^\dagger \neq H_K$. Our main aim is now to identify a subset of Hamiltonians H_K, which despite being non-Hermitian possess a real eigenvalue spectrum.

There are various types of representations in terms of differential operators for this algebra as for instance the multi-boson representation

$$K_0 = k_0(N), \qquad K_+ = k_+(N)(a^\dagger)^n, \qquad K_- = k_-(N)(a)^n, \tag{2.9}$$

where the $a, a^\dagger$ are the usual bosonic annihilation and creation operators with $N = a^\dagger a$ being the number operator. The $k_0(N)$, $k_\pm(N)$ are functions of the latter and may be determined recursively for any number of bosons n involved [39]. The simplest case $n = 1$

yields the Holstein-Primakoff representation [40] with $K_0 = N + \frac{1}{2}$, $K_+ = \sqrt{N}a^\dagger$ and $K_- = a\sqrt{N}$. For $n = 2$ one obtains the very well known two boson representation

$$K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}), \qquad K_+ = \frac{1}{2}a^\dagger a^\dagger, \qquad K_- = \frac{1}{2}aa. \tag{2.10}$$

Differential operators in x-space are then obtained by the usual identification $a = (\omega\hat{x} + i\hat{p})/\sqrt{2\omega}$ and $a^\dagger = (\omega\hat{x} - i\hat{p})/\sqrt{2\omega}$ with the operators $\hat{x}$, $\hat{p} = -i\partial_x$ and $\omega \in \mathbb{R}$.

The part of the Hamiltonian H_K linear in the generators K corresponds to the Hamiltonian recently studied by Quesne [41], who constructed an explicit metric operator for this Hamiltonian together with its Hermitian isospectral partner. For the particular representation (2.10) this reduces to the so-called Swanson Hamiltonian [42], for which various metric operators were constructed previously by Musumbu et al [26]. Here we shall extend the analysis to the case involving bilinear combinations, staying as generic as possible without appealing to any particular representation.

3. Construction of a metric operator and Hermitian counterpart

Our key aim is now to construct a well defined metric operator, i.e. a linear, invertible, Hermitian and positive operator acting in the Hilbert space, such that H becomes a self-adjoint operator with regard to this metric. Our starting point will be the assumption that there exists a similarity transformation, which maps the non-Hermitian Hamiltonian H adjointly to a Hermitian Hamiltonian h

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \ \text{ with } \rho = \eta^\dagger \eta. \tag{3.1}$$

There exist also variations of these properties, which lead to less stringent conclusions. We summarize the most common ones in the following table

	$H^\dagger = \eta^\dagger\eta H(\eta^\dagger\eta)^{-1}$	$H^\dagger\rho = \rho H$	$H^\dagger = \rho H \rho^{-1}$
positivity of ρ	✓	✓	×
Hermiticity of ρ	✓	✓	✓
invertibility of ρ	✓	×	✓
terminology for H	(3.1)	quasi-Hermiticity[2]	pseudo-Hermiticity[43]
spectrum of H	real[4]	could be real[2]	real[4]
definite metric	guaranteed	guaranteed	not conclusive

We should stress that this is the most frequently used terminology and at times it is mixed up and people imply different properties by using the same names. Making no assumption on the positivity of the ρ in (3.1), the relation on the right hand side constitutes the well known *pseudo-Hermiticity* condition, see e.g. [43, 44, 45], when the operator ρ is linear, invertible and Hermitian. In case the operator ρ is positive but not invertible this condition is usually referred to as *quasi-Hermiticity*[3] [2, 3, 4, 12]. With regard to the

[3]Surprisingly the early literature on the subject, such as [2, 3, 4], is entirely ignored in recent publications and statements such as "the terminology quasi Hermitian was coined in [12]", see e.g. [46], are obviously incorrect. The term *quasi-Hermitian operator* was first introduced by Dieudonné in 1960 [2]. Relaxing the requirement of invertibility the operators become *symmetrizable operators* for which there exists an extensive even earlier literature, see e.g. [1] and references therein.

properties of discrete spectra of H the difference is irrelevant as both conditions may be used to establish its reality. However, in the case of pseudo-Hermiticity this is guaranteed whereas in the case of quasi-Hermiticity one merely knowns that it could be real. With regard to the construction of a metric operator the difference becomes also important, since pseudo-Hermiticity may lead to an indefinite metric, whereas quasi-Hermiticity will guarantee the existence of a positive definite metric.

Naturally we expect to find many solutions if the Hermitian Hamiltonian h is not specified concretely. In other words when given the non-Hermitian Hamiltonian H the similarity transformation is not unique when the only requirement for h is its Hermiticity. The ambiguities indicate the existence of a symmetry, see e.g. section 3.3 in [27]. However, uniqueness may be achieved by specifying either the concrete form of h or any other irreducible observable [12].

3.1 Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type

Let us start by considering first the Hamiltonian $H_{\tilde{J}}$ in (2.4) in the context of the above mentioned programme and try to solve the equation

$$h_{\tilde{J}} = \eta H_{\tilde{J}} \eta^{-1} = h_{\tilde{J}}^{\dagger} \tag{3.2}$$

for η. As a general ansatz we start with the non-Hermitian operator

$$\eta = e^{2\varepsilon[\tilde{J}_0 + \lambda(\tilde{J}_+ + \tilde{J}_-)]} \neq \eta^{\dagger} \qquad \varepsilon, \lambda \in \mathbb{R}. \tag{3.3}$$

Unlike as in the case when $\eta = \eta^{\dagger}$, see section 3.2, we do not need to worry here about the positivity of η, since the decomposition of the metric operator $\rho = \eta^{\dagger}\eta$ ensures it to be positive.

One of the simplest cases to consider for expressions of $H_{\tilde{J}}$ is the purely linear one, i.e. when all κ_{nm} vanish. In principle this Hamiltonian fits into the class of general $\mathcal{PT}$-symmetric Hamiltonians considered in [27], when the constants therein are identified as $\alpha_3 = -\kappa_+$, $\alpha_4 = -\kappa_-$, $\alpha_6 = -(n+1)\kappa_0/2$, $\alpha_8 = \kappa_0$, $\alpha_{10} = -(n+1)\kappa_+$ and all remaining constants are taken to be zero. However, none of the exactly solvable models obtained in there matches with $H_{\tilde{J}}$. Nonetheless, relaxing the condition $\eta = \eta^{\dagger}$ as in the ansatz (3.3) allows to construct an exact Hermitian isospectral counterpart. An example of how to transform the non-Hermitian Hamiltonian $H_{\tilde{J}}$ to a Hermitian Hamiltonian is given when the parameters in the model are related as

$$\kappa_0 = \pm 2\sqrt{\kappa_-\kappa_+} \qquad \text{and} \qquad \frac{\tanh\chi}{\chi/\varepsilon} = \pm\frac{\sqrt{\kappa_+}}{\sqrt{\kappa_+} + 2\lambda\sqrt{\kappa_-}}, \tag{3.4}$$

with $\chi = \varepsilon\sqrt{1-4\lambda^2}$. The Hermitian Hamiltonian counterpart is subsequently computed to

$$h_{\tilde{J}} = \left(\pm\frac{1}{2\lambda}\kappa_0 + \kappa_+ + \kappa_-\right)\tilde{J}_-. \tag{3.5}$$

Another interesting simple example is obtained by setting all terms involving the generator $\tilde{J}_+$ to zero, that is taking $\kappa_+ = \kappa_{++} = \kappa_{+0} = \kappa_{+-} = 0$. In this case we are led to the relations

$$\kappa_0 = -(n+1)\kappa_{00}, \qquad \kappa_- = -\frac{n}{\lambda}\mu_{00}, \qquad \kappa_{--} = \frac{\mu_{00}}{\lambda^2}, \qquad \kappa_{0-} = \frac{2}{\lambda}\kappa_{00} \tag{3.6}$$

together with

$$\tanh\chi = \chi/\varepsilon. \tag{3.7}$$

The non-Hermitian Hamiltonian $H_{\tilde{J}}$ is then transformed to the Hermitian Hamiltonian

$$h = \kappa_{00}\tilde{J}_0^2 - \kappa_0\tilde{J}_0 \tag{3.8}$$

with $0 < |\lambda| < \frac{1}{2}$. These examples demonstrate that it is possible to carry out the above mentioned programme for some specific realisations of the $sl_2(\mathbb{R})$-Lie algebra, albeit not in complete generality and in a generic representation independent manner.

3.2 Hamiltonians of $su(1,1)$-Lie algebraic type

We shall now see that the Hamiltonians H_K in (2.8) allow for a more general treatment as the problems of the previous section may be circumvented. In analogy to (3.2) let us therefore solve the equation

$$h_K = \eta H_K \eta^{-1} = h_K^\dagger \tag{3.9}$$

for η.

To start with we take a similar operator ansatz for the similarity transformation as the one chosen in [26, 41]

$$\eta = \exp(2\varepsilon K_0 + 2\nu_+ K_+ + 2\nu_- K_-), \tag{3.10}$$

where the parameters $\varepsilon, \nu_+, \nu_-$ are left variable for the time being. Hermiticity for this operator η may be guaranteed when we take from the very beginning $\nu_+ = \nu$, $\nu_- = \nu^*$ and $\varepsilon \in \mathbb{R}$ together with the Hermiticity conditions for the Lie algebraic generators as specified in (2.7). Noting that the eigenvalue spectrum of η is given by $\exp[(n+1/2)\sqrt{\varepsilon^2 - 4\nu_+\nu_-}]$ one can ensure the positivity of η when $\varepsilon^2 > 4\nu_+\nu_-$. One should note that these properties are only essential for the metric operator $\eta^\dagger\eta$. making now the more restrictive assumption that η is Hermitian, we can obtain a linear, invertible positive Hermitian metric operator η^2 when $\varepsilon^2 - 4\,|\nu|^2 > 0$. Following [26] it is convenient to introduce the variable $\theta = \sqrt{\varepsilon^2 - 4|\nu|^2}$.

Besides the restriction we impose on η by demanding it to be Hermitian, we could have been also more generic by making a more general ansatz for the expressions for η, such as for instance allowing in addition bilinear combinations in the arguments of the exponential. In fact, as we will show in section 4, we are certain that more general types of metric operators must exist. Another very natural version of this ansatz would be to start with a Gauss or Iwasawa decomposed expression for η.

Using the ansatz (3.10), we have to compute its adjoint action on H_K in order to solve (3.9). In fact, the adjoint action of η on each of the $su(1,1)$-generators can be computed exactly. We find

$$\eta K_l \eta^{-1} = t_{l0} K_0 + t_{l-} K_- + t_{l+} K_+ \qquad \text{for } l = 0, \pm, \tag{3.11}$$

where the constant coefficients are

$$\begin{aligned} t_{00} &= 1 - 8|\nu|^2 \frac{\sinh\theta^2}{\theta^2}, \quad t_{\pm\pm} = \left(\cosh\theta \pm \varepsilon \frac{\sinh\theta}{\theta}\right)^2, \quad t_{\pm\mp} = 4(\nu_\pm)^2 \frac{\sinh\theta^2}{\theta^2}, \\ t_{0\pm} &= \mp 2\nu_\mp \frac{\sinh\theta}{\theta}\left(\cosh\theta \pm \varepsilon \frac{\sinh\theta}{\theta}\right), \qquad t_{\pm 0} = \pm 4\nu_\pm \frac{\sinh\theta}{\theta}\left(\cosh\theta \pm \varepsilon \frac{\sinh\theta}{\theta}\right). \end{aligned} \tag{3.12}$$

These expressions agree with the result in [41].

With the help of these exact relations we evaluate the adjoint action of η on the Hamiltonian H_K

$$\eta H_K \eta^{-1} = \sum_{l=0,\pm} \hat{\mu}_l K_l + \sum_{n,m=0,\pm} \hat{\mu}_{nm} : K_n K_m : . \tag{3.13}$$

It is evident from (3.11) that the general structure of the Hamiltonian will not change, albeit with a different set of constants $\hat{\mu}$, which are rather lengthy and we will therefore not report them here explicitly. However, they simplify when we impose the constraint that the resulting Hamiltonian ought to be Hermitian. The condition (3.9) leads to the six constraints

$$\hat{\mu}_0 = \hat{\mu}_0^*, \qquad \hat{\mu}_{00} = \hat{\mu}_{00}^*, \qquad \hat{\mu}_{+-} = \hat{\mu}_{+-}^*, \tag{3.14}$$

$$\hat{\mu}_+ = \hat{\mu}_-^*, \qquad \hat{\mu}_{++} = \hat{\mu}_{--}^*, \qquad \hat{\mu}_{+0} = \hat{\mu}_{0-}^*. \tag{3.15}$$

The first set of three equations (3.14) on the reality of $\hat{\mu}_0$, $\hat{\mu}_{00}$ and $\hat{\mu}_{+-}$ is simply satisfied by the condition $\nu = \nu^*$. Introducing the variables

$$\lambda = \frac{\nu}{\varepsilon} \qquad \text{and} \qquad Y = \varepsilon \frac{\tanh\theta}{\theta} \tag{3.16}$$

the remaining three equations (3.15) may be converted into simpler, albeit still lengthy, equations

$$\begin{aligned} 0 = {} & \mu_+ - \mu_- + 2Y\left[\mu_+ + \mu_- + 2\lambda\left(\mu_{++} + \mu_{--} - \mu_0 - \mu_{00}\right)\right] \\ & + 12\,Y^2\lambda\left[\mu_{++} - \mu_{--} + \lambda\left(\mu_{0-} - \mu_{+0}\right)\right] \\ & - 2\,Y^3\left\{\mu_+ + \mu_- - 2\lambda\left[\mu_0 + \mu_{00} + 3\left(\mu_{--} + \mu_{++}\right)\right]\right. \\ & \left. - \lambda^2\left[8\,\mu_{0-} - 4\left(\mu_- + \mu_+ - 2\,\mu_{+0}\right)\right] + 8\,\lambda^3\left(\mu_{++} + \mu_{--} + \mu_0 - \mu_{00} - 2\,\mu_{+-}\right)\right\} \\ & + Y^4\left(1 - 4\,\lambda^2\right)\left\{\mu_- - \mu_+ + 4\,\lambda\left[\mu_{++} - \mu_{--} + \lambda\left(\mu_{0-} - \mu_- + \mu_+ - \mu_{+0}\right)\right]\right\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} 0 = {} & \mu_{++} - \mu_{--} - 2Y\left[\lambda\left(\mu_{0-} + \mu_{+0}\right) - 2\left(\mu_{--} + \mu_{++}\right)\right] \\ & + 6\,Y^2\left[\mu_{++} - \mu_{--} + \lambda\left(\mu_{0-} - \mu_{+0}\right)\right] \\ & - 2\,Y^3\left[3\,\lambda\left(\mu_{+0} + \mu_{0-}\right) + 4\,\lambda^3\left(\mu_{+0} + \mu_{0-}\right) - 8\,\lambda^2\left(\mu_{00} + \mu_{+-}\right) - 2\left(\mu_{++} + \mu_{--}\right)\right] \\ & + Y^4\left(1 - 4\,\lambda^2\right)\left\{\mu_{++} - \mu_{--} - 2\,\lambda\left[\mu_{+0} - \mu_{0-} + 2\,\lambda\left(\mu_{--} - \mu_{++}\right)\right]\right\}, \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
0 = {} & \mu_{+0} - \mu_{0-} + 2\,Y\,\left[\mu_{0-} + \mu_{+0} + 4\,\lambda\,\left(\mu_{++} + \mu_{--} - \mu_{00} - \mu_{+-}\right)\right] \\
& +24\,Y^2\,\left[\lambda(\mu_{++} - \mu_{--}) + \lambda^2\,\left(\mu_{0-} - \mu_{+0}\right)\right] \\
& -2\,Y^3\,\left\{\mu_{+0} + \mu_{0-} - 4\,\lambda\,\left[\mu_{00} + \mu_{+-} + 3(\mu_{++} + \mu_{--})\right] - 12\lambda^2\,\left(\mu_{0-} + \mu_{+0}\right)\right. \\
& \left. +\ 16\lambda^3\,\left(\mu_{++} + \mu_{--} - \mu_{00} - \mu_{+-}\right)\right\} \\
& +Y^4\,\left(1 - 4\,\lambda^2\right)\,\left\{\mu_{0-} - \mu_{+0} + 4\,\lambda\,\left[\lambda\,\left(\mu_{0-} - \mu_{+0}\right) + 2\,\left(\mu_{++} - \mu_{--}\right)\right]\right\}.
\end{aligned}
\tag{3.19}
$$

We will now systematically discuss the solutions for these three equations together with their implications on the metric operator and the corresponding isospectral pairs of Hamiltonians.

3.2.1 Non-Hermitian linear term and Hermitian bilinear combinations

The simplest modification with regard to the purely linear case, treated previously in [26, 41], is to perturb it with Hermitian bilinear combinations. This means we may assume the equalities $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$ in order to determine the relations between the remaining constants from (3.17), (3.18) and (3.19). We find that (3.18) and (3.19) are solved solely by demanding

$$
\mu_{++} = \mu_{--} = \frac{\lambda^2(\mu_{00} + \mu_{+-})}{1 + 2\lambda^2} \qquad \text{and} \qquad \mu_{+0} = \mu_{0-} = \frac{2\lambda(\mu_{00} + \mu_{+-})}{1 + 2\lambda^2}, \tag{3.20}
$$

without any further constraint on Y. Solving subsequently equation (3.17) for Y yields the constraint

$$
\frac{\tanh 2\theta}{\theta/\varepsilon} = \frac{\lambda(\mu_- - \mu_+)}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - 2\mu_{++}}. \tag{3.21}
$$

Considering (3.10) we note that the positivity of η^2 requires $|\lambda| < 1/2$ as a further restriction on the domain of λ. Notice that when we send all coefficients μ_{nm} with $n, m \in \{0, \pm\}$ resulting from bilinear combinations to zero we recover precisely the constraint found in [26], see equation (9) therein. These equations parametrize the metric and are enough to compute the Hermitian counterpart via equation (3.9). We will not report the expression here as they are rather lengthy and can be obtained as a reduction from the more general setting to be treated below.

3.2.2 Hermitian linear term and non-Hermitian bilinear combinations

Reversing the situation of the preceding subsection we may consider the Hamiltonian H_K with Hermitian linear part, i.e. $\mu_+ = \mu_-$, and non-Hermitian part involving bilinear combinations. In this case we can solve the equations (3.17), (3.18) and (3.19) by

$$
\mu_+ = \mu_- = \lambda(\mu_0 + \mu_{00} - \mu_{++} - \mu_{--}), \tag{3.22}
$$

$$
\lambda(\mu_{+0} - \mu_+) = \lambda^2(\mu_{+-} - \mu_0) + \mu_{++}, \tag{3.23}
$$

$$
\lambda(\mu_{0-} - \mu_-) = \lambda^2(\mu_{+-} - \mu_0) + \mu_{--}, \tag{3.24}
$$

together with

$$
\frac{\tanh 2\theta}{\theta/\varepsilon} = \frac{\mu_{++} - \mu_{--}}{2\lambda\mu_+ + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}. \tag{3.25}
$$

This case does not reduce to any case treated in the literature before.

Let us now embark on the general setting in which the linear as well as the terms in H_K involving bilinear combinations are taken to be non-Hermitian. We will find two different types of solutions, one being reducible to the foregoing two cases and the other being intrinsically non-Hermitian and not reducible to any of the previous cases. Reducible is meant in the sense that the limit of the relevant parameters going to zero is well defined.

3.2.3 Generic non-Hermitian reducible Hamiltonian

Taking now H_K to be genuinely non-Hermitian, we find that the equations (3.17), (3.18) and (3.19) are solved subject to the three constraints

$$\begin{aligned} \mu_{++} - \mu_{--} &= \lambda(\mu_{+0} - \mu_{0-}) \\ \mu_{--} - \lambda\mu_{0-} &= \lambda^2(\mu_{++} + \mu_{--} - \mu_{+-} - \mu_{00}) \\ 2\mu_+\mu_{--} - \mu_-(\mu_{++} + \mu_{--}) &= \lambda[(\mu_{++} - \mu_{--})(\mu_{++} + \mu_{--} - \mu_0 - \mu_{00}) + \mu_{0-}(\mu_+ - \mu_-)] \end{aligned} \tag{3.26}$$

together with

$$\frac{\tanh 2\theta}{\theta/\varepsilon} = \frac{\lambda(\mu_- - \mu_+) + \mu_{++} - \mu_{--}}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}. \tag{3.27}$$

We note that taking $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$ or $\mu_+ = \mu_-$ these constraints reduce precisely to the ones previously treated in the sections 3.2.2 or 3.2.3, respectively. A further interesting specialization of this general case is the one involving purely bilinear combinations, which may be obtained for $\mu_- = \mu_+ = \mu_0$ in (3.26) and (3.27). For the situation in which the Hamiltonian does not contain any generators of the type K_-, i.e. $\mu_- = \mu_{--} = \mu_{0-} = 0$ we find

$$\lambda\mu_{+0} = \mu_{++}, \quad \mu_{00} = \mu_{++} - \mu_0, \quad \mu_{+-} = \mu_0, \quad \varepsilon = \frac{\operatorname{arctanh}\sqrt{1-4\lambda^2}}{2\sqrt{1-4\lambda^2}}, \tag{3.28}$$

and when H_K does not contain any generators of the type K_+, i.e. $\mu_+ = \mu_{++} = \mu_{0+} = 0$ the equations simplify to

$$\lambda\mu_{0-} = \mu_{--}, \quad \mu_{00} = \mu_{--} - \mu_0, \quad \mu_{+-} = \mu_0, \quad \varepsilon = -\frac{\operatorname{arctanh}\sqrt{1-4\lambda^2}}{2\sqrt{1-4\lambda^2}}. \tag{3.29}$$

Another trivial consistency check is obtained when we add to the Swanson model a multiple of the Casimir operator $C = K_0^2 - \{K_+, K_-\}/2$ and consider

$$H_C = H_l + \kappa C \ = (\mu_0 + \kappa)K_0 + \mu_+K_+ + \mu_-K_- + \kappa K_0^2 - \kappa K_+K_- \qquad \text{for } \kappa \in \mathbb{R}. \tag{3.30}$$

Since the Casimir operator is Hermitian and commutes with η no further constraint should result from this modification when compared with the non-Hermitian linear case. In fact, the linear case together with the constraining equations will produce the Casimir operator. Starting with the latter case and replacing $\mu_0 \to \mu_0 + \kappa$, we can interpret $\kappa = -\mu_{+-}$ according to (3.27). When $\mu_{+-} \neq 0$ we can satisfy the constraints (3.26) by $\mu_{+-} = -\mu_{00}$

and setting all remaining μ's with double subscripts to be zero, which is obviously satisfied by (3.30), together with

$$\frac{\tanh 2\theta}{\theta/\varepsilon} = \frac{\mu_- - \mu_+}{\mu_- + \mu_+ - 2\lambda\mu_0}. \tag{3.31}$$

We conclude this section by making use of the constraining equation (3.27) and re-express the operator η in (3.10) purely as as a function of $\lambda \in [-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$

$$\eta(\lambda) = \exp\left[\frac{K_0 + \lambda(K_+ + K_-)}{\sqrt{1-4\lambda^2}} \operatorname{arctanh} F(\lambda)\right], \tag{3.32}$$

where

$$F(\lambda) := \sqrt{1-4\lambda^2}\frac{\lambda(\mu_- - \mu_+) + \mu_{++} - \mu_{--}}{\lambda(\mu_- + \mu_+) + 2\lambda^2(\mu_{+-} - \mu_0) - (\mu_{++} + \mu_{--})}, \tag{3.33}$$

subject to the constraints (3.26).

Hermitian counterpart Using the explicit solution (3.32) we can compute the Hermitian counterpart h_K using the formula (3.9). As expected from similar calculations previously carried out in this context the explicit non-Hermitian Hamiltonian turns out to be rather complicated when compared to the fairly simple non-Hermitian Hamiltonian (2.8). Nonetheless, it may be computed exactly and we find the coefficients in (3.13) to be given by

$$\hat{\mu}_0 = A_0 + 2(\mu_- - \mu_+)\lambda B_0, \tag{3.34}$$

$$\hat{\mu}_+ = \hat{\mu}_- = \lambda A_+ + \frac{1}{2}\left[\mu_- - \mu_+ + 2(\mu_{--} - \mu_{++})\lambda\right] B_0, \tag{3.35}$$

$$\hat{\mu}_{00} = \frac{1}{2\lambda}A_{00} + 2(\mu_{--} - \mu_{++})B_0, \tag{3.36}$$

$$\hat{\mu}_{+-} = A_{+-} + (\mu_{--} - \mu_{++})B_0, \tag{3.37}$$

$$\hat{\mu}_{++} = \hat{\mu}_{--} = \lambda A_{++} + \frac{1}{2}(\mu_{--} - \mu_{++})B_0, \tag{3.38}$$

$$\hat{\mu}_{+0} = \hat{\mu}_{0-} = 2A_{++} + \left(\frac{1+4\lambda^2}{2\lambda}\right)(\mu_{--} - \mu_{++})B_0, \tag{3.39}$$

where we further abbreviated

$$A_0 = \mu_0 - \frac{2\lambda}{1-4\lambda^2}\frac{(\mu_- - \mu_+)(\mu_{--} + 3\mu_{++} - 2\lambda\mu_{+0})}{(\mu_{--} - \mu_{++})}, \tag{3.40}$$

$$\begin{aligned} A_{00} = \frac{1}{(1-4\lambda^2)(\mu_{--} - \mu_{++} - \lambda(\mu_- - \mu_+))}\big[&2(\mu_{--}\mu_+ - \mu_{++}\mu_-) \\ &-2\lambda(\mu_{--} - \mu_{++})(\mu_0 + \mu_{++} + \mu_{--}) + 2\lambda^2(\mu_- - \mu_+)(\mu_{--} + \mu_{++} + \mu_{+-}) + \\ &-8\lambda^3(\mu_{--} - \mu_{++})(\mu_{--} + \mu_{++} - \mu_{+-}) + 8\lambda^4(\mu_- - \mu_+)(\mu_{--} + \mu_{++} - \mu_{+-})\big], \end{aligned} \tag{3.41}$$

$$\begin{aligned} A_{+-} = \frac{1}{(1-4\lambda^2)(\mu_{--} - \mu_{++} - \lambda(\mu_- - \mu_+))}\big\{&(\mu_{++} - \mu_{--})(\mu_{--} + \mu_{++} - \mu_{+-}) \\ &-\lambda\left[\mu_{+-}(\mu_- - \mu_+) - (\mu_- + \mu_+)(\mu_{--} - \mu_{++})\right] \\ &-2\lambda^2(\mu_{--} - \mu_{++})(\mu_{+-} + \mu_0) + 4\lambda^3(\mu_- - \mu_+)\mu_{+-}\big\}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} A_+ = \frac{1}{(1-4\lambda^2)\lambda(\mu_{--} - \mu_{++})}\big\{&-\lambda\left[\mu_{--}^2 - (\mu_- - \mu_+)\mu_{+0} + 2\mu_{--}\mu_{++} - 3\mu_{++}^2\right] \\ &+\mu_+(\mu_{--} + \mu_{++}) - 2\mu_-\mu_{++} - 2\lambda^2(\mu_{--} - \mu_{++})(\mu_- + \mu_- - \mu_{+0})\big\}, \end{aligned} \tag{3.43}$$

$$A_{++} = \frac{\lambda\mu_{+0} - \mu_{++} - 2\lambda^2(\mu_{--} + \mu_{++})}{(1-4\lambda^2)\lambda}, \tag{3.44}$$

$$B_0 = 2\frac{\sqrt{2\mu_{++}(\mu_{--} + \mu_{++}) - \lambda(\mu_{--} + 3\mu_{++})\mu_{+0} + \lambda^2\left[\mu_{+0}^2 + (\mu_{--} - \mu_{++})^2\right]}}{(1-4\lambda^2)(\mu_{--} - \mu_{++})}. \tag{3.45}$$

Clearly this Hamiltonian does not constitute an obvious starting point, whereas the non-Hermitian Hamiltonian H_J is fairly simple and natural to consider. We could also express the Hermitian version in a simple fashion by solving (3.34)-(3.45) for the μs, such that instead H_J would acquires a complicated form. However, the construction procedure itself is only meaningful in the direction $H_J \to h_J$ and not $h_J \to H_J$.

3.2.4 Generic non-Hermitian non-reducible Hamiltonian

Remarkably in contrast to the previously analysed purely linear case there exists a second non-equivalent type of solution. We find that (3.17), (3.18) and (3.19) are also solved by the four constraints

$$\mu_+ - \mu_- = 2\lambda(\mu_{++} - \mu_{--}), \tag{3.46}$$

$$\mu_{+0} - \mu_{0-} = 2(\mu_+ - \mu_-), \tag{3.47}$$

$$\mu_{+0} = 2\mu_+ + 2(\mu_{+-} - \mu_0)\lambda, \tag{3.48}$$

$$\mu_+ = \lambda(\mu_0 + \mu_{00} + 2\mu_{++}) - 2\lambda^2(\mu_- - \mu_+ + \mu_{+0}) \quad , \tag{3.49}$$

together with

$$\frac{\tanh 4\theta}{\theta/\varepsilon} = \frac{\mu_{--} - \mu_{++}}{\mu_{--} + \mu_{++} + \lambda(\mu_+ - \mu_- - \mu_{+0})}. \tag{3.50}$$

Notice that this solution can not be reduced to the cases of a non-Hermitian linear term plus Hermitian bilinear combination or a Hermitian linear term plus a non-Hermitian bilinear combination as discussed in sections 3.2.3 or 3.2.2, respectively. This is seen from (3.46) and

(3.47) as $\mu_+ = \mu_-$ implies $\mu_{++} = \mu_{--}$, $\mu_{+0} = \mu_{0-}$ and vice versa, such that it is impossible to convert one part into a Hermitian one while keeping the other non-Hermitian.

As for the foregoing set of constraints there are some interesting subcases. For instance, we can consider again the situation where the Hamiltonian does not contain any generators of the type K_-, i.e. $\mu_- = \mu_{--} = \mu_{0-} = 0$. Then the constraints simplify to

$$\begin{aligned} 2\lambda\mu_{++} = \mu_+, \qquad \mu_{+0} = 2\mu_+, \qquad \mu_{00} = 2\lambda\mu_+ - \mu_0, \qquad \mu_{+-} = \mu_0, \\ \varepsilon = -\frac{1}{4\sqrt{1-4\lambda^2}}\,\mathrm{arctanh}\left(\frac{\sqrt{1-4\lambda^2}}{1-2\lambda^2}\right). \end{aligned} \tag{3.51}$$

Similarly, if the Hamiltonian does not contain any generators of the type K_+, i.e., $\mu_+ = \mu_{++} = \mu_{0+} = 0$, the constraints reduce to

$$\begin{aligned} 2\lambda\mu_{--} = \mu_-, \qquad \mu_{0-} = 2\mu_-, \qquad \mu_{00} = 2\lambda\mu_- - \mu_0, \qquad \mu_{+-} = \mu_0 \\ \varepsilon = \frac{1}{4\sqrt{1-4\lambda^2}}\,\mathrm{arctanh}\left(\frac{\sqrt{1-4\lambda^2}}{1-2\lambda^2}\right). \end{aligned} \tag{3.52}$$

Note that also for this reduced case the solutions (3.28) and (3.51) as well as (3.29) and (3.52) are different.

As before we can also in this case use of the constraining equation (3.27) and re-express the operator η in (3.10) purely as a function of $\lambda \in [-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$

$$\eta(\lambda) = \exp\left[\frac{K_0 + \lambda(K_+ + K_-)}{2\sqrt{1-4\lambda^2}}\,\mathrm{arctanh}\,G(\lambda)\right], \tag{3.53}$$

where

$$G(\lambda) := \sqrt{1-4\lambda^2}\frac{(\mu_{--} - \mu_{++})}{\mu_{--} + \mu_{++} + \lambda(\mu_+ - \mu_- - \mu_{+0})}, \tag{3.54}$$

subject to the constraints (3.46) and (3.49).

Hermitian counterpart Using again the explicit solution (3.32) we can compute the Hermitian counterpart h_K using the formula (3.53). Once more the expressions are quite cumbersome

$$\hat{\mu}_0 = C_0 + 4\lambda^2 D_0, \tag{3.55}$$
$$\hat{\mu}_+ = \hat{\mu}_- = C_+ + 2\lambda D_0, \tag{3.56}$$
$$\hat{\mu}_{00} = 2(C_{00} + 4\lambda^2 D_0), \tag{3.57}$$
$$\hat{\mu}_{+-} = C_{+-} + 4\lambda^2 D_0, \tag{3.58}$$
$$\hat{\mu}_{++} = \hat{\mu}_{--} = \lambda C_{++} + (1-2\lambda^2)D_0, \tag{3.59}$$
$$\hat{\mu}_{+0} = \hat{\mu}_{0-} = 2(C_{++} + 2\lambda D_0), \tag{3.60}$$

where further abbreviated

$$C_0 = \frac{\mu_0 - \lambda(\mu_- - \mu_+) - 4\lambda^2(\mu_{++} + \mu_0) + 2\lambda^3(\mu_- + \mu_+) + 4\lambda^4(\mu_{+-} - \mu_0)}{1 - 4\lambda^2}, \quad (3.61)$$

$$C_{00} = \frac{\mu_{00}}{2} - \frac{2\lambda^2\left[\mu_{--} + \mu_{++} - \lambda\mu_{+0} - 2\lambda^2(\mu_{--} - \mu_{++})\right]}{1 - 4\lambda^2}, \quad (3.62)$$

$$C_{+-} = C_0 + \mu_{+-} - \mu_0, \quad (3.63)$$

$$C_+ = \frac{\mu_+ - 2\lambda\mu_{++} - \lambda^2(\mu_- + \mu_+) + 2\lambda^3(\mu_{+-} - \mu_0)}{1 - 4\lambda^2}, \quad (3.64)$$

$$C_{++} = \frac{\mu_{+0} - 4\lambda\mu_{++} - 2\lambda^2\mu_{+0} - 4\lambda^3(\mu_{--} - \mu_{++})}{2(1 - 4\lambda^2)}, \quad (3.65)$$

$$D_0 = \frac{1}{2\ \left(4\,\lambda^2 - 1\right)} \left\{4\,\mu_{--}\,\mu_{++} + \lambda^2\ \left[\mu_{+0}^2 + 8\,\mu_{++}\ \left(\mu_{++} - \mu_{--}\right)\right]\right. \quad (3.66)$$
$$\left. -2\,\lambda\,\mu_{+0}\ \left(\mu_{--} + \mu_{++}\right) + 4\,\lambda^3\,\mu_{+0}\ \left(\mu_{--} - \mu_{++}\right) + 4\,\lambda^4\left(\mu_{--} - \mu_{++}\right)^2\right\}^{1/2}.$$

Again this demonstrates the general feature that some fairly simple non-Hermitian Hamiltonians possess quite complicated isospectral Hermitian counterparts.

3.2.5 A simpler metric, the case $\lambda = 0$

In the previous discussion we have excluded the case $\lambda = 0$, which equals $\nu = 0$ in our ansatz for the metric (3.10). This case may be dealt with separately and in fact is fairly easy, as η simplifies considerably because it only depends on the generator K_0. In this situation also the constraints turn out to be far simpler

$$\mu_{--}\mu_+^2 = \mu_{++}\mu_-^2, \qquad \mu_{--}\mu_{+0}^2 = \mu_{++}\mu_{0-}^2 \qquad \text{and} \qquad \varepsilon = \frac{1}{8}\ln\frac{\mu_{--}}{\mu_{++}} \quad (3.67)$$

and even the Hermitian counterpart Hamiltonian becomes fairly compact too

$$h_\varepsilon = \mu_0 K_0 + \mu_+ e^{2\varepsilon}(K_+ + K_-) + \mu_{00}K_0^2 + \mu_{+-}K_+K_- + \mu_{++}e^{4\varepsilon}(K_+^2 + K_-^2) \quad (3.68)$$
$$+\mu_{+0}e^{2\varepsilon}(K_+K_0 + K_0K_-).$$

This suggests that the simple metric $\eta = e^{2\varepsilon K_0}$ may be employed as an easy transformation also for other more complicated Hamiltonians.

3.2.6 Two further simple cases $\lambda = \pm 1/2$

Finally let us also investigate the other boundary values for the parameter λ, that is $\lambda = \pm 1/2$. In this case the constraints are

$$\mu_{++} = \pm(\mu_+ - 2\mu_-) + \frac{(\mu_- - \mu_+)\mu_{+-}}{\mu_{+0} - \mu_{0-} + 2(\mu_- - \mu_+)} \pm \frac{(\mu_- - \mu_+)(\mu_{0-} - 2\mu_- \pm \mu_0)}{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)} \tag{3.69}$$
$$+\frac{\mu_0 + \mu_{00}}{2} + \frac{(\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+))(\mu_{+0} - 2(\mu_0 + \mu_{00} - 2\mu_-))}{4(\mu_{0-} - 2\mu_- \pm (\mu_0 - \mu_{+-}))},$$

$$\mu_{--} = \mp\mu_+ + \frac{(\mu_- - \mu_+)\mu_{+-}}{\mu_{+0} - \mu_{0-} + 2(\mu_- - \mu_+)} + \frac{(\mu_- - \mu_+)(\mu_0 + \mu_{0-} - 2\mu_-)}{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)} \tag{3.70}$$
$$+\frac{\mu_0 + \mu_{00}}{2} + \frac{(2(\mu_0 + \mu_{00}) \mp (\mu_{0-} + 4\mu_+))(\mu_{0-} - \mu_{0+} + 2(\mu_+ - \mu_-))}{4(\mu_{+0} - 2\mu_+ \pm (\mu_0 - \mu_{+-}))},$$

$$\varepsilon = \frac{\mu_{0-} - \mu_{+0} - 2(\mu_- - \mu_+)}{2(\mu_{0-} + \mu_{+0} - 2(\mu_+ + \mu_-) \pm 2(\mu_0 - \mu_{+-}))}. \tag{3.71}$$

The general Hermitian counterpart turns out to have a very complicated form, but there are some simple special cases, such as

$$H_{\frac{1}{2}} = K_+ - K_- - K_0 + K_0^2 + K_+K_- + K_+K_0 + K_0K_- + \frac{11}{2}K_+^2 + \frac{1}{2}K_-^2 \tag{3.72}$$

which is mapped into the Hermitian form

$$h_{\frac{1}{2}} = \mp\frac{13}{16}(K_+ + K_-) - \frac{23}{16}K_0 + \frac{5}{8}K_0^2 + \frac{13}{16}K_+K_- \mp \frac{11}{8}(K_+K_0 + K_0K_-) + \frac{61}{32}(K_+^2 + K_-^2) \tag{3.73}$$

with $\varepsilon = \mp\frac{1}{4}$ for $\lambda = \pm\frac{1}{2}$.

4. Generalised Bogoliubov transformation

Bogoliubov transformations were first introduced with the purpose to understand the pairing interaction in superconductivity [47] and have been generalized thereafter in many different ways, as for instance in [48]. In the present context they have been applied by Swanson [42] as an alternative method to establish the reality of the spectrum of a non-Hermitian Hamiltonian. Instead of constructing an explicit similarity transformation one can make a constraining assumption about the form of its Hermitian counterpart. The simplest assumption to make is that the counterpart is of a harmonic oscillator type. We will now demonstrate how the Hamiltonian H_K can be transformed into such a form by means of a generalized Bogoliubov transformation. Following [42], we define for this purpose two new operators c and d via

$$\begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} \beta & -\delta \\ -\alpha & \gamma \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{C}. \tag{4.1}$$

Demanding that these operators commute in the same manner as the annihilation and creation operators $a, a^\dagger$, i.e. $[d, c] = 1$, and that they may be reduced to the former in a well defined limit yields the constraints

$$\beta\gamma - \alpha\delta = 1 \qquad \text{and} \qquad \beta, \gamma \neq 0, \tag{4.2}$$

on the complex parameters $\alpha, \beta, \gamma, \delta$. Note that we do not require the same Hermiticity conditions as for the conventional operators $a = (a^\dagger)^\dagger$, that is in general we have $c \neq d^\dagger$. For our purposes we also require a definite behaviour under the $\mathcal{PT}$-transformation. Noting that $\mathcal{PT}$: $a, a^\dagger \to -a, -a^\dagger$ implies that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ or $i\mathbb{R}$, such that $\mathcal{PT}$: $c, d \to -c, -d$ or c, d, respectively. In fact, we shall see below that demanding unbroken $\mathcal{PT}$-symmetry requires the cofficients $\alpha, \beta, \gamma, \delta$ to be purely complex.

We may now simply invert the relations (4.1)

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} d \\ c \end{pmatrix} \tag{4.3}$$

and express the generators $K_0, K_\pm$ in the concrete two-boson representation (2.10) in terms of these new operators

$$K_0 = \frac{1}{2}(\gamma\beta + \delta\alpha)cd + \frac{1}{2}\delta\beta c^2 + \frac{1}{2}\gamma\alpha d^2 + \frac{1}{2}\delta\alpha + \frac{1}{4}, \tag{4.4}$$

$$K_+ = \alpha\beta(cd + \frac{1}{2}) + \frac{1}{2}\beta^2 c^2 + \frac{1}{2}\alpha^2 d^2, \tag{4.5}$$

$$K_- = \gamma\delta(cd + \frac{1}{2}) + \frac{1}{2}\delta^2 c^2 + \frac{1}{2}\gamma^2 d^2, \tag{4.6}$$

which by construction satisfy the same $su(1,1)$-commutation relations (2.7). Naturally we can now define the analogues of the generators $K_0, K_\pm$ in terms of the operators c, d

$$\check{K}_0 = \frac{1}{2}(cd + \frac{1}{2}), \qquad \check{K}_+ = \frac{1}{2}cc, \qquad \check{K}_- = \frac{1}{2}dd, \tag{4.7}$$

such that

$$\begin{pmatrix} K_0 \\ K_+ \\ K_- \end{pmatrix} = \begin{pmatrix} \gamma\beta + \delta\alpha & \beta\delta & \alpha\gamma \\ 2\alpha\beta & \beta^2 & \alpha^2 \\ 2\gamma\delta & \delta^2 & \gamma^2 \end{pmatrix} \begin{pmatrix} \check{K}_0 \\ \check{K}_+ \\ \check{K}_- \end{pmatrix}. \tag{4.8}$$

Inverting the relation (4.8), upon using (4.2) we may also express the $\check{K}_0, \check{K}_\pm$ in terms of $K_0, K_\pm$

$$\begin{pmatrix} \check{K}_0 \\ \check{K}_+ \\ \check{K}_- \end{pmatrix} = \begin{pmatrix} \gamma\beta + \delta\alpha & -\gamma\delta & -\alpha\beta \\ -2\gamma\alpha & \gamma^2 & \alpha^2 \\ -2\delta\beta & \delta^2 & \beta^2 \end{pmatrix} \begin{pmatrix} K_0 \\ K_+ \\ K_- \end{pmatrix}. \tag{4.9}$$

Replacing in H_K the generators $K_0, K_\pm$ by the newly defined generators $\check{K}_0, \check{K}_\pm$ we can transform the Hamiltonian into the form

$$H_K = \sum_{l=0,\pm} \check{\mu}_l \check{K}_l + \sum_{n,m=0,\pm} \check{\mu}_{nm} : \check{K}_n \check{K}_m : . \tag{4.10}$$

Notice that due to the identity $8\check{K}_+\check{K}_- = 8\check{K}_0\check{K}_0 - 8\check{K}_0 + 1$ not all coefficients $\check{\mu}_l$, $\check{\mu}_{nm}$ are uniquely defined. However, this ambiguity will not play any role in our analysis as the relevant equations will be insensitive to these redefinitions. Demanding that the Hamiltonian in terms of the new generators $\check{K}_0$, $\check{K}_\pm$ acquires the form of a harmonic oscillator plus a

Casimir operator means we have to set the constants $\hat{\mu}_+$, $\hat{\mu}_-$, $\hat{\mu}_{++}$, $\hat{\mu}_{--}$, $\hat{\mu}_{+0}$, $\hat{\mu}_{0-}$ to zero. Expressing these constraints through the original constants in (2.8) yields the equations

$$\begin{aligned} \mu_{++}y^4+\mu_{+0}y^3+(\mu_{+-}+\mu_{00})y^2+\mu_{0-}y+\mu_{--} &= 0, \\ \mu_{--}z^4+\mu_{0-}z^3+(\mu_{+-}+\mu_{00})z^2+\mu_{+0}z+\mu_{++} &= 0, \end{aligned} \tag{4.11}$$

$$\mu_{+0}y^3z+4\mu_{++}y^3+2(\mu_{+-}+\mu_{00})y^2z+3\mu_{+0}y^2+3\mu_{0-}yz+2(\mu_{+-}+\mu_{00})y+4\mu_{--}z+\mu_{0-}=0,$$
$$\mu_{0-}yz^3+4\mu_{--}z^3+2(\mu_{+-}+\mu_{00})yz^2+3\mu_{0-}z^2+3\mu_{+0}yz+2(\mu_{+-}+\mu_{00})z+4\mu_{++}y+\mu_{+0}=0,$$

$$\begin{aligned} (\mu_{0-}-\mu_-)yz^3+(2\mu_{+-}+\mu_{00}-\mu_0)yz^2+2\mu_{--}z^3+(2\mu_{+0}-\mu_+)yz+(\mu_-+\mu_{0-})z^2 \\ +(\mu_0+\mu_{00})z+2\mu_{++}y+\mu_+ = 0 \\ (\mu_{+0}-\mu_+)y^3z+(2\mu_{+-}+\mu_{00}-\mu_0)y^2z+2\mu_{++}y^3+(2\mu_{0-}-\mu_-)yz+(\mu_++\mu_{+0})y^2 \\ +(\mu_0+\mu_{00})y+2\mu_{--}z+\mu_- = 0 \end{aligned}$$

where we abbreviated

$$y=\frac{\alpha}{\gamma} \qquad \text{and} \qquad z=\frac{\delta}{\beta}. \tag{4.12}$$

We will now systematically solve the six equations (4.11). When α, $\delta \neq 0$ the equations reduce to the simpler form

$$z^2(\mu_{00}+\mu_{+-}) = \mu_{++}(1+4yz+y^2z^2), \tag{4.13}$$
$$z^2(\mu_{+-}-\mu_0) = \mu_{++}(1+yz)^2+\mu_+z^3+\mu_-z, \tag{4.14}$$
$$\mu_{--}z^2 = \mu_{++}y^2, \tag{4.15}$$
$$\mu_{+0}z = -2\mu_{++}(1+yz), \tag{4.16}$$
$$\mu_-z = \mu_+y, \tag{4.17}$$
$$\mu_{0-}z = \mu_{+0}y. \tag{4.18}$$

Similarly as in section 3 the solutions fall into different classes distinguished by vanishing linear or bilinear combinations.

4.1 Genuinely non-Hermitian non-reducible Hamiltonian

We start to solve the six constraints (4.13)-(4.18) for the generic case by demanding $\mu_+, \mu_- \neq 0$ and $\mu_{++}, \mu_{--}, \mu_{+0}, \mu_{0-} \neq 0$. We find the unique solution

$$\mu_- = \frac{y}{z}\mu_+, \qquad \mu_{--} = \frac{\mu_-^2}{\mu_+^2}\mu_{++}, \qquad \mu_{0-} = \frac{\mu_-}{\mu_+}\mu_{+0}, \qquad y = \frac{\pm\vartheta-\mu_{+0}/4}{\mu_{++}}, \tag{4.19}$$

$$\mu_{+-} = \mu_0-\frac{\mu_+\mu_{+0}}{2\mu_{++}}+\frac{\mu_{+0}^2}{4\mu_{++}}, \qquad \mu_{00} = -\mu_0+\frac{\mu_+\mu_{+0}}{2\mu_{++}}+\frac{2\mu_-\mu_{++}}{\mu_+}, \tag{4.20}$$

with the abbreviation $\vartheta := \sqrt{\mu_{+0}^2/16-\mu_{++}^2\mu_+/\mu_-}$. The Hamiltonian H_K in (2.8) or in other words the Hamiltonians $H_{\check{K}}$ in (4.10) can now be expressed entirely in terms of the number operator $\check{N}=cd$ and acquires the simple form

$$H_K = \frac{\vartheta^2}{\mu_{++}}(\check{N}^2+\check{N})\pm\frac{\vartheta(\mu_{+0}-2\mu_+)}{2\mu_{++}}\left(\check{N}+\frac{1}{2}\right)+\frac{3\mu_0}{16}-\frac{3\mu_+\mu_{+0}}{32\mu_{++}}+\frac{\mu_{+0}^2}{16\mu_{++}}-\frac{5\mu_-\mu_{++}}{8\mu_+}. \tag{4.21}$$

In analogy to the harmonic oscillator case we may now easily construct the eigensystem for this Hamiltonian. Defining the states $|\check{n}\rangle = (n!)^{-1/2} c^n |\check{0}\rangle$ with $d|\check{0}\rangle = 0$ we have $\check{N}|\check{n}\rangle = n|\check{n}\rangle$. Note that demanding $\mathcal{PT}$-symmetry for the states $|\check{n}\rangle$ requires that $\mathcal{PT}$: $c \to c$, which in turn implies $\alpha, \beta, \gamma, \delta \in i\mathbb{R}$. Demanding that the eigenspectrum is real and bounded from below imposes the further constraints

$$\mu_{++} > 0 \qquad \text{and} \qquad \mu_- \mu_{+0}^2 > 16 \mu_{++}^2 \mu_+ . \tag{4.22}$$

It is now interesting to compare this result with our previous construction for the isospectral counterpart in section 3.2.4. Using the constraints (4.19) and (4.20) we may solve the conditions (3.46)-(3.49) for the similarity transformation needed to be able to construct a well defined metric operator by

$$\mu_{+0} = 2\mu_+ \qquad \text{and} \quad \lambda = \frac{\mu_+^2}{2(\mu_+ + \mu_-)\mu_{++}}, \tag{4.23}$$

such that (3.50) acquires the form

$$\frac{\tanh 4\theta}{\theta/\varepsilon} = \frac{2(\mu_-^2 - \mu_+^2)\mu_{++}^2}{2(\mu_-^2 + \mu_+^2)\mu_{++}^2 - \mu_+^4}. \tag{4.24}$$

Thus upon these constraints, the two constructions coincide, if besides (4.22) we also demand that $\mu_+^2 \leq \mu_{++} \left|\mu_+ + \mu_-\right|$ since $|\lambda| \leq 1/2$. This means that in this situation we do not only have an explicit similarity transformation, a well defined metric and a Hermitian counterpart, but in addition we know the exact eigenspectrum and eigenfunctions. Relaxing these conditions it also implies that there must be a larger class of similarity transformations not covered by the ansatz (3.10) for the operator η. As already mentioned we might be loosing out on some possibilities by demanding η to be Hermitian. A further natural generalisation would be to include also bilinear combinations into the argument of the exponential in the expression for η.

4.2 Hermitian linear term and non-Hermitian bilinear combinations

It seems natural that we mimic the same cases as for the construction of the metric in section 3. However, when tuning the linear term to be Hermitian by demanding $\mu_+ = \mu_-$ the constraints (4.15), (4.17) and (4.18) imply that $\mu_{++} = \mu_{--}$ and $\mu_{+0} = \mu_{0-}$, such that also the terms involving bilinear combinations becomes Hermitian. The case $\mu_+ = \mu_- = 0$ is special since the last equation in (4.20) yields $\mu_+/\mu_- = (\mu_0 + \mu_{00})/(2\mu_{++})$. Using this and still demanding that $\mu_{++}, \mu_{--}, \mu_{+0}, \mu_{0-} \neq 0$, the solutions to (4.13)-(4.18) become

$$\mu_{--} = \frac{(\mu_0 + \mu_{00})^2}{4\mu_{++}}, \qquad \mu_{0-} = \frac{\mu_{+0}}{2\mu_{++}}(\mu_0 + \mu_{00}), \qquad y = \frac{\pm\bar{\vartheta} - \mu_{+0}/4}{\mu_{++}}, \tag{4.25}$$

$$\mu_{+-} = \mu_0 + \frac{\mu_{+0}^2}{4\mu_{++}}, \qquad z = y\frac{2\mu_{++}}{\mu_0 + \mu_{00}}, \tag{4.26}$$

with the abbreviation $\bar{\vartheta} := \sqrt{\mu_{+0}^2/16 - \mu_{++}(\mu_0 + \mu_{00})/2}$. The Hamiltonian H_K in (2.8), (4.10) can be expressed again entirely in terms of the number operator and acquires the

simple form

$$H_K \;= \frac{\bar{\vartheta}^2}{\mu_{++}}(\check{N}^2+\check{N}) \pm \frac{\bar{\vartheta}\mu_{+0}}{2\mu_{++}}\left(\check{N}+\frac{1}{2}\right)+\frac{\mu_{+0}^2}{16\mu_{++}}-\frac{5}{16}\mu_{00}-\frac{\mu_0}{8}. \tag{4.27}$$

The requirement that the eigenspectrum is real and bounded from below yields in this case the additional constraints

$$\mu_{++}>0 \qquad \text{and} \qquad \mu_{+0}^2>8\mu_{++}\left(\mu_0+\mu_{00}\right). \tag{4.28}$$

Interestingly when demanding (4.25) and (4.26), we can not solve the constraints in section 3 and therefore can not construct a metric with the ansatz (3.10) in this case.

4.3 Non-Hermitian linear case and Hermitian bilinear combinations

Reversing the setting of the previous section we may now demand the bilinear combinations to be Hermitian, $\mu_{++}=\mu_{--}$ and $\mu_{+0}=\mu_{0-}$. This is equally pathological as now the linear term becomes also Hermitian by (4.15), (4.17) and (4.18). Nonetheless, a non-trivial limit is obtained with $\mu_{++}=\mu_{--}=\mu_{+0}=\mu_{0-}=0$ and requiring $\mu_+,\mu_-\neq 0$. We may then solve (4.13)-(4.18) by

$$\mu_-=\frac{y}{z}\mu_+, \qquad \mu_{+-}=-\mu_{00}, \qquad y=\frac{\pm\tilde{\vartheta}-(\mu_0+\mu_{00})/2}{\mu_+}, \tag{4.29}$$

with the abbreviation $\tilde{\vartheta}:=\sqrt{(\mu_0+\mu_{00})^2/4-\mu_+\mu_-}$. Once again the Hamiltonian H_K in (2.8), (4.10) can be expressed entirely in terms of the number operator simplifying it to

$$H_K \;= \pm\tilde{\vartheta}\left(\check{N}+\frac{1}{2}\right)-\frac{3\mu_{00}}{16}. \tag{4.30}$$

The eigenspectrum is real and bounded from below when we discard the minus sign in (4.30) and impose the condition

$$(\mu_0+\mu_{00})^2>4\mu_+\mu_-. \tag{4.31}$$

When setting $\mu_{00}=\mu_{+-}=0$ these expressions reduce precisely to those found in [42] for the purely linear case.

Comparing now with the construction in section 3.2.4, we find that (3.20) is solved by the conditions (4.29), if we further demand that

$$\mu_{00}+\mu_{+-}=0, \tag{4.32}$$

such that (3.21) becomes

$$\frac{\tanh 2\theta}{\theta/\varepsilon}=\frac{\mu_--\mu_+}{\mu_-+\mu_+-2\lambda(\mu_{00}+\mu_0)}. \tag{4.33}$$

We may also put further restrictions on the generalized Bogoliubov transformation (4.1) itself by setting some of the constants to zero.

4.4 Asymmetric generalized Bogoliubov transformation with $\delta = 0$

Let us now set the α in (4.1) to zero. Then the equations (4.11) are solved by

$$\mu_+ = \mu_{++} = \mu_{+0} = 0, \quad \mu_{0-} = -\frac{2\mu_{--}}{y}, \quad \mu_{00} = -\mu_0 - \frac{\mu_-}{y}, \quad \mu_{+-} = \frac{\mu_{--}}{y^2} - \mu_{00}. \quad (4.34)$$

In this situation the transformed Hamiltonian H_K (4.10) can be expressed as

$$H_K \ = \frac{\mu_{0-}^2}{16\mu_{--}}(\check{N}^2 - \check{N}) + \frac{\mu_{0-}\mu_-}{4\mu_{--}}\left(\check{N} + \frac{1}{8}\right) + \frac{3\mu_0}{16}. \quad (4.35)$$

Once again we may compare with the construction in section 3.2.4. The operator η can be constructed when we demand

$$\mu_{+-} = \mu_0 \qquad \text{and} \qquad \lambda = -\frac{1}{2y} \quad (4.36)$$

together with

$$\varepsilon = \frac{1}{4\sqrt{1-\frac{1}{y^2}}}\text{ArcTanh}\left(\frac{2y^2\sqrt{1-\frac{1}{y^2}}}{1-2y^2}\right). \quad (4.37)$$

The meaningful interval $\lambda \in [-1/2, 1/2]/\{0\}$ is now translated into the condition $y \in [-1, 1]/\{0\}$.

4.5 Asymmetric generalized Bogoliubov transformation with $\alpha = 0$

We may also put further constraints on the transformation (4.1) itself. Then the equations (4.11) are solved by

$$\mu_- = \mu_{--} = \mu_{0-} = 0, \quad \mu_{+0} = -\frac{2\mu_{++}}{z}, \quad \mu_{00} = -\mu_0 - \frac{\mu_+}{z}, \quad \mu_{+-} = \frac{\mu_{++}}{z^2} - \mu_{00}. \quad (4.38)$$

Now the transformed Hamiltonian H_K (4.10) can be expressed as

$$H_K \ = \frac{\mu_{+0}^2}{16\mu_{++}}(\check{N}^2 - \check{N}) + \frac{\mu_{+0}\mu_+}{4\mu_{++}}\left(\check{N} + \frac{1}{8}\right) + \frac{3\mu_0}{16}. \quad (4.39)$$

The comparison with the construction in section 3.2.4 yields now that the operator η can be constructed when we demand

$$\mu_{+-} = \mu_0 \qquad \text{and} \qquad \lambda = -\frac{1}{2z} \quad (4.40)$$

together with

$$\varepsilon = \frac{1}{4\sqrt{1-\frac{1}{z}}}\text{ArcTanh}\left(\frac{2z^2\sqrt{1-\frac{1}{z^2}}}{1-2z^2}\right) \quad (4.41)$$

Now $\lambda \in [-1/2, 1/2]/\{0\}$ is translated to the condition $z \in [-1, 1]/\{0\}$.

As a trivial consistency we observe that for $\alpha = \delta = 0$, i.e. when $y = z = 0$, the transformation (4.1) becomes the identity and we have the vanishing of all coefficients

except for μ_0, μ_{00} and μ_{+-}. Thus the initial Hamiltonian is already Hermitian and just corresponds to the harmonic oscillator displaced by a Casimir operator. The configuration when the constants μ_+, μ_-, μ_{++}, μ_{--}, μ_{+0} and μ_{0-} vanish is obviously of little interest.

For completeness we also comment on the case $yz = -1$ for which we may also find an explicit solution. However, in this situation the coefficients in front of $\check{K}_0^2$ and $\check{K}_0$ are not positive and consequently this scenario is of little physical relevance.

5. Some concrete realisations

Let us finish our generic discussion with a few comments related to some concrete realisations of the algebras discussed. The most familiar representation of the $su(1,1)$ is probably the aforementioned two-boson representation (2.10), but also the realisation for $n = 1$ in (2.10) plays an important role for instance in the study of the Jaynes-Cummings model [49]. With the usual identifications for the creation and annihilation operators in terms of differential operators in x-space it is then straightforward to express H_K in terms maximally quartic in the position and momentum operators, albeit not in its most general form,

$$H_{xp} = \gamma_0 + \gamma_1\hat{x}^2 + \gamma_2\hat{p}^2 + \gamma_3\hat{x}^4 + \gamma_4\hat{p}^4 + \imath\gamma_5\hat{x}\hat{p} + \gamma_6\hat{x}^2\hat{p}^2 + \imath\gamma_7\hat{x}\hat{p}^3 + \imath\gamma_8\hat{x}^3\hat{p}. \tag{5.1}$$

The coefficients γ_i in (5.1) and the μ_l, $\mu_{n,m}$ in (2.8) are related as

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{pmatrix} = \frac{1}{16} \begin{pmatrix} 0 & -4 & 4 & -2 & 3 & 3 & 1 & 2 & -2 \\ 4 & 4 & 4 & 0 & -6 & 6 & -4 & -5 & 1 \\ 4 & -4 & -4 & 0 & 6 & -6 & -4 & -1 & 5 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & -8 & 8 & -4 & 12 & 12 & -4 & 4 & -4 \\ 0 & 0 & 0 & 2 & -6 & -6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & -4 & 4 & 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_+ \\ \mu_- \\ \mu_{00} \\ \mu_{++} \\ \mu_{--} \\ \mu_{+-} \\ \mu_{+0} \\ \mu_{0-} \end{pmatrix}. \tag{5.2}$$

Since the determinant of the matrix in (5.2) is non-vanishing we may also express the μ_l, $\mu_{n,m}$ in terms of the γ_i, which then translates the constraining equations and the coefficient occurring in the Hermitian counterparts too. It is interesting to note that the argument $2\varepsilon[K_0 + \lambda(K_+ + K_-)]$ in the exponential of the operator η becomes $\varepsilon[\frac{1}{2}(\hat{x}^2 + \hat{p}^2) + \lambda(\hat{x}^2 - \hat{p}^2) + 1]$, such that at the boundaries of the interval in which λ takes its values $\lambda = 1/2$ and $\lambda = -1/2$ the operator and therefore the metric becomes a function only of $\hat{x}$ and $\hat{p}$, respectively.

There are plenty of other representations. An interesting one is for instance one mentinoned in [41]

$$K_0 = \frac{1}{4\xi}\left(-\frac{d^2}{dr^2} + \frac{g}{r^2} + \xi^2 r^2\right), \tag{5.3}$$

$$K_\pm = \frac{1}{4\xi}\left(\frac{d^2}{dr^2} - \frac{g}{r^2} + \xi^2 r^2 \mp \xi\left(2r\frac{d}{dr} + 1\right)\right), \tag{5.4}$$

which may also be used to relate to Calogero models [50]. Using this representation H_K may be expressed as a differential operator in r

$$H_R = \rho_0 + \rho_1 \frac{d^4}{dr^4} + \rho_2 r \frac{d^3}{dr^3} + \rho_3 r^2 \frac{d^2}{dr^2} + \rho_4 \frac{d^2}{dr^2} + \rho_5 \frac{1}{r^2}\frac{d^2}{dr^2} + \rho_6 r^3 \frac{d}{dr} + \\ + \rho_7 r \frac{d}{dr} + \rho_8 \frac{1}{r}\frac{d}{dr} + \rho_9 \frac{1}{r^3}\frac{d}{dr} + \rho_{10} r^4 \frac{d}{dr} + \rho_{11} r^2 + \rho_{12}\frac{1}{r^2} + \rho_{13}\frac{1}{r^4} \tag{5.5}$$

and its corresponding Hermitian counterpart, using the constraints (3.26), (3.27), (3.27), (3.27), is given by

$$h_R = \tilde{\rho}_0 + \tilde{\rho}_1 \frac{d^4}{dr^4} + \tilde{\rho}_3 r^2 \frac{d^2}{dr^2} + \tilde{\rho}_4 \frac{d^2}{dr^2} + \tilde{\rho}_5 \frac{1}{r^2}\frac{d^2}{dr^2} + \\ + \tilde{\rho}_7 r \frac{d}{dr} + \tilde{\rho}_9 \frac{1}{r^3}\frac{d}{dr} + \tilde{\rho}_{10} r^4 \frac{d}{dr} + \tilde{\rho}_{11} r^2 + \tilde{\rho}_{12}\frac{1}{r^2} + \tilde{\rho}_{13}\frac{1}{r^4} \tag{5.6}$$

The ρs and $\tilde{\rho}$s may be computed explicitly, but this is not relevant for our purposes here. Keeping only linear terms in K in the Hamiltonian, we obtain

$$H = \frac{(\mu_0 - \mu_+ - \mu_-)}{4\xi}\left(-\frac{d^2}{dr^2} + \frac{g}{r^2}\right) + \frac{\mu_- - \mu_+}{4}\left(1 + 2r\frac{d}{dr}\right) + \frac{(\mu_0 + \mu_- + \mu_-)}{4}\xi r^2, \tag{5.7}$$

which under the action of

$$\eta = \exp\left[\frac{1}{\sqrt{1-4\lambda^2}}\text{ArcTanh}\left(\frac{(\mu_- - \mu_+)\sqrt{1-4\lambda^2}}{\mu_- + \mu_+ - 2\lambda\mu_0}\right)[K_0 + \lambda(K_+ + K_-)]\right] \tag{5.8}$$

transforms into

$$h = \frac{a+b}{\xi}\left(\frac{d^2}{dr^2} - \frac{g}{r^2}\right) + \left(\frac{1+2\lambda}{1-2\lambda}a + 3b\right)\xi r^2 \tag{5.9}$$

with parameters given by

$$a = \frac{1}{2(1+2\lambda)}\sqrt{\mu_+\mu_- - \lambda\mu_0(\mu_+ + \mu_-) + \lambda^2(\mu_0^2 + (\mu_- - \mu_+)^2)}, \tag{5.10}$$

$$b = \frac{\mu_0 - 2\lambda(\mu_+ + \mu_-)}{4(1-4\lambda^2)}. \tag{5.11}$$

As already mentioned mild varations of the representation (5.3), (5.4) can be used to obtain multi-particle systems, such as Calogero models. An easier multi-particle model is the two-mode Bose-Hubbard model [51] or the description of a charged particle in a magnetic field [52], which result when taking as representation

$$K_0 = \frac{1}{2}\left(a_2^\dagger a_2 - a_1^\dagger a_1\right), \qquad K_+ = a_2^\dagger a_1, \qquad K_- = a_1^\dagger a_2, \tag{5.12}$$

where the $a_i^\dagger, a_i$ are the creation and annihilation of the i-th bosonic particle. It is straightforward to apply the above programme also to this type of system.

As a variation of the above idea we may also study multi-particle $\mathcal{PT}$-symmetric Hamiltonians, for which we do not mix different particle types implicitly within $su(1,1)$-generators, i.e. taking direct sums of Fock speces, but consider instead systems of the type

$su(1,1) \oplus su(1,1)$, such as

$$\begin{aligned} H_m = \mu_0^{(1)} K_0^{(1)} + \mu_+^{(1)} K_+^{(1)} + \mu_-^{(1)} K_-^{(1)} + \mu_0^{(2)} K_0^{(2)} + \mu_+^{(2)} K_+^{(2)} + \mu_-^{(2)} K_-^{(2)} + \mu_{00} K_0^{(1)} K_0^{(2)} \\ + \mu_{+-} K_+^{(1)} K_-^{(2)} + \mu_{-+} K_-^{(1)} K_+^{(2)} + \mu_{++} K_+^{(1)} K_+^{(2)} + \mu_{--} K_-^{(1)} K_-^{(2)} + + \mu_{+0} K_+^{(1)} K_0^{(2)} \\ + \mu_{0-} K_0^{(1)} K_-^{(2)} + \mu_{0+} K_0^{(1)} K_+^{(2)} + \mu_{-0} K_-^{(1)} K_0^{(2)} \end{aligned} \tag{5.13}$$

with the superscripts in the $K^{(i)}$ indicate the particle type. We may start with an ansatz of a similar type

$$\eta = \exp[2\varepsilon_1 (K_0^{(1)} + \lambda_1 K_+^{(1)} + \lambda_1 K_-^{(1)}) + 2\varepsilon_2 (K_0^{(2)} + \lambda_2 K_+^{(2)} + \lambda_2 K_-^{(2)})] \tag{5.14}$$

and it is then straightforward to show that the constraints

$$\mu_{00} = \frac{\mu_{++}}{\lambda_1 \lambda_2}, \qquad \mu_{-+} = \mu_{+-} = \mu_{--} = \mu_{++} \tag{5.15}$$

$$\mu_{0+} = \mu_{0-} = \frac{\mu_{++}}{\lambda_1}, \qquad \mu_{+0} = \mu_{-0} = \frac{\mu_{++}}{\lambda_2}, \tag{5.16}$$

with

$$\frac{\tanh 2\theta_i}{\theta_i/\varepsilon_i} = \frac{\mu_-^{(i)} - \mu_+^{(i)}}{\mu_-^{(i)} + \mu_+^{(i)} - 2\lambda_i \mu_0^{(i)}} \qquad \text{for } i = 1, 2 \tag{5.17}$$

convert the Hamiltonian H_m into a Hermitian one. Note that despite the fact that in H_m we have an interaction between different particle types the constraints are identical to the ones in the linear case for individual particles. In fact, H_m is indeed linear in $K^{(1)}$ and $K^{(2)}$ and the terms involving the products of $K^{(1)}$ and $K^{(2)}$ operators are Hermitian. Adding some genuinely bilinear combinations in the Hamiltonian is expected to generate a more intricate structure sheding light on interacting spins etc, but we leave this for future investigations as it goes beyond the simple comment we intended to make in this section regarding explicit realisations.

There are of course many more realisations we could present. We finish by mentioning the famous BCS-Hamiltonian [53], which can be expressed in terms of many copies of algebra (2.7) and plays a key role in the theory of superconductivity

$$H_{\rm BCS} = \sum_{n,\sigma} \varepsilon_n a_{n\sigma}^\dagger a_{n\sigma} - \sum_{n,n'} I_{n,n'} a_{n\downarrow}^\dagger a_{n\uparrow}^\dagger a_{n'\downarrow} a_{n'\uparrow}. \tag{5.18}$$

Here $a_{n\sigma}^\dagger$ and $a_{n\sigma}$ are creation and annihilation operators of electrons with spin σ in a state n, respectively. The ε_n are eigenvalues of the one-body Hamiltonian and the $I_{n,n'}$ are matrix elements of short range electron-electron interaction.

6. Conclusions

The central aim of this manuscript was to analyse systematically $\mathcal{PT}$-symmetric Hamiltonians of Lie algebraic type with regard to their quasi-Hermitian solvability properties. We have considered Hamiltonians of $sl_2(\mathbb{R})$-Lie algebraic type and for some specific cases we

constructed a similarity transformation together with isospectral Hermitian counterparts. We indicated the difficulty these types of Hamiltonians pose with regard to the outlined programme, mainly due to the feature that the Hermitian conjugation does not close within the set of $sl_2(\mathbb{R})$-generators. Nonetheless, for specific realisations of the algebra the outlined programme may be carried out explicitly.

Considering Hamiltonians of $su(1,1)$-Lie algebraic type instead circumvents these issues and we were able to construct systematically exact solutions for metric operators, which are of exponential form with arguments linear in the $su(1,1)$-generators. Our solutions fall into various subcases and are characterized by the constraints on the coupling constants in the model. In several cases we used the square root of the metric operator to construct the corresponding similarity transformation and its Hermitian counterparts. Alternatively we constructed the energy spectrum together with their corresponding eigenfunctions by means of generalized Bogoliubov transformations, which map the original Hamiltonians onto harmonic oscillator type Hamiltonians. The comparison between these two approaches exhibits agreement in some cases, but the overlap is not complete and we can obtain SQH-models which can not be mapped to a harmonic oscillator type Hamiltonian by means of generalized Bogoliubov transformations and vice versa. On one hand this is probably due to our restrictive ansatz for the operator η by demanding it to be Hermitian and in addition assuming it to be of exponential form with arguments linear in the $su(1,1)$-generators. On the other hand we could of course also make a more general ansatz for the "target Hamiltonian" in the generalized Bogoliubov transformation approach.

There are some obvious omissions and further problems resulting from our analysis. It would be desirable to complete the programme for more general operators η and different types of Bogoliubov transformations for the Hamiltonians of rank one Lie algebraic type. With regard to some concrete physical models, e.g. the Regge model or other types mentioned in [27], it is necessary to investigate Hamiltonians with a mixture between $su(1,1)$- and $sl_2(\mathbb{R})$-generators. Naturally systems related to higher rank algebras constitute an interesting generalisation.

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