Metrics and isospectral partners for the most generic cubic $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian

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Abstract: We investigate properties of the most general $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian of cubic order in the annihilation and creation operators as a ten parameter family. For various choices of the parameters we systematically construct an exact expression for a metric operator and an isospectral Hermitian counterpart in the same similarity class by exploiting the isomorphism between operator and Moyal products. We elaborate on the subtleties of this approach. For special choices of the ten parameters the Hamiltonian reduces to various models previously studied, such as to the complex cubic potential, the so-called Swanson Hamiltonian or the transformed version of the from below unbounded quartic $-x^4$-potential. In addition, it also reduces to various models not considered in the present context, namely the single site lattice Reggeon model and a transformed version of the massive sextic $\pm x^6$-potential, which plays an important role as a toy model to identify theories with vanishing cosmological constant.

1. Introduction

Non-Hermitian Hamiltonians are usually interpreted as effective Hamiltonians associated with dissipative systems when they possess a complex eigenvalue spectrum. However, from time to time also non-Hermitian Hamiltonians whose spectra were believed to be *real* have emerged sporadically in the literature, e.g. the lattice version of Reggeon field theory [1, 2]. Restricting this model to a single site leads to a potential very similar to the complex cubic potential $V = ix^3$. Somewhat later it was found [3] for the latter model that it possess a real spectrum on the real line. More recently the surprising discovery was made [4] that in fact the entire infinite family of non-Hermitian Hamiltonians involving the complex potentials $V^n = z^2(iz)^n$ for $n \geq 0$ possess a real spectrum, when its domain is appropriately continued to the complex plane.

Thereafter it was understood [4, 5] that the reality of the spectra can be explained by an unbroken $\mathcal{PT}$-symmetry, that is invariance of the Hamiltonian and its eigenfunctions under a simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$. In case only the Hamiltonian is $\mathcal{PT}$-symmetric the eigenvalues occur in complex conjugate pairs. In fact, the $\mathcal{PT}$-operator is a specific example of an anti-linear operator for which such spectral properties have been established in a generic manner a long time ago by Wigner [6]. However, in practical terms one is usually not in a position to know all eigenfunctions for a given non-Hermitian Hamiltoinian and therefore one has to resort to other methods to establish the reality of the spectrum. Since Hermitian Hamiltonians are guaranteed to have real spectra, one obvious method is to search for Hermitian counterparts in the same similarity class as the non-Hermitian one. This means one seeks similarity transformations η of the form

$$h = \eta H \eta^{-1} = h^\dagger = \eta^{-1} H^\dagger \eta \quad \Leftrightarrow \quad H^\dagger = \eta^2 H \eta^{-2}, \qquad \text{for } \eta = \eta^\dagger. \tag{1.1}$$

Non-Hermitian Hamiltonians H respecting the property (1.1) are referred to as pseudo-Hermitian [7]. Besides these spectral properties it is also understood how to formulate a consistent quantum mechanical description for such non-Hermitian Hamiltonian systems [8, 9, 5] by demanding the η^2-operator to be Hermitian and positive-definite, such that it can be interpreted as a metric to define the η-inner product. A special case of this is the $\mathcal{CPT}$-inner product [5], which results by taking $\eta^2 = \mathcal{CP}$ with $\mathcal{C} = \sum |\phi_n\rangle \langle\phi_n|$. For some recent reviews on pseudo Hermitian Hamiltonians see [10, 11, 12, 13, 14].

Since the metric-operator η^2 is of central importance many attempts have been made to construct it when given only a non-Hermitian Hamiltonian. However, so far one has only succeeded to compute exact expressions for the metric and isospectral partners in very few cases. Of course when the entire spectrum is known this task is straightforward, even though one might not always succeed to carry out the sum over all eigenfunctions. However, this is a very special setting as even in the most simple cases one usually does not have all the eigenfunctions at ones disposal and one has to resort to more pragmatic techniques, such as for instance perturbation theory [15, 16, 17, 18]. Rather than solving equations for operators, the entire problem simplifies considerably if one converts it into differential equations using Moyal products [19, 20, 21] or other types of techniques [22]. Here we wish to pursue the former method for the most generic $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian of cubic order in the creation and annihilation operators.

We refer models for which the metric can be constructed exactly as *solvable pseudo-Hermitian* (SPH) systems.

Our manuscript is organised as follows: In section 2 we introduce the model we wish to investigate in this manuscript, formulating it in terms of creation and annihilation operators and equivalently in terms of space and momentum operators. We comment on the reduction of the model to models previously studied. In section 3 we discuss in detail the method we are going to employ to solve the equations (1.1), namely to exploit the isomorphism between products of operator valued functions and Moyal products of scalar functions. In section 4 we construct systematically various exact solutions for the metric operator and the Hermitian counterpart to H. As special cases of these general considerations we focus in section 5 and 6 on the single site lattice Reggeon model and the massive $\pm x^6$-potential. In section 7 we provide a simple proof of the reality for the ix^{2n+1}-potentials and some of its generalizations. We state our conclusions in section 8.

2. A master Hamiltonian of cubic order

The subject of our investigation is the most general $\mathcal{PT}$-symmetric Hamiltonian, which is maximally cubic in creation and annihilation operators $a^\dagger, a$, respectively,

$$H_{\rm c} = \lambda_1 a^\dagger a + \lambda_2 a^\dagger a^\dagger + \lambda_3 aa + \lambda_4 + i(\lambda_5 a^\dagger + \lambda_6 a + \lambda_7 a^\dagger a^\dagger a^\dagger + \lambda_8 a^\dagger a^\dagger a + \lambda_9 a^\dagger aa + \lambda_{10} aaa). \quad (2.1)$$

The Hamiltonian $H_{\rm c}$ is a ten-parameter family with $\lambda_i \in \mathbb{R}$. It is clear that this Hamiltonian is $\mathcal{PT}$-symmetric by employing the usual identification $a = (\omega\hat{x} + i\hat{p})/\sqrt{2\omega}$ and $a^\dagger = (\omega\hat{x} - i\hat{p})/\sqrt{2\omega}$ with the operators in x-space $\hat{x}$ and $\hat{p} = -i\partial_x$. The effect of a simultaneous parity transformation $\mathcal{P} : \hat{x} \to -\hat{x}$ and time reversal $\mathcal{T} : t \to -t$, $i \to -i$ on the creation and annihilation operators is $\mathcal{PT} : a \to -a$, $a^\dagger \to -a^\dagger$. Without loss of generality we may set the parameter ω to one in the following as it is simply an overall energy scale.

In terms of the operators $\hat{x}$ and $\hat{p}$ the separation into a Hermitian and non-Hermitian part is somewhat more transparent and we may introduce in addition a coupling constant $g \in \mathbb{R}$ in order to be able to treat the imaginary part as perturbation of a Hermitian operator. In terms of these operators the most general expression is, as to be expected, yet again a ten-parameter family

$$H_{\rm c} = \alpha_1\hat{p}^3 + \alpha_2\hat{p}^2 + \alpha_3\frac{\{\hat{p},\hat{x}^2\}}{2} + \alpha_4\hat{p} + \alpha_5\hat{x}^2 + \alpha_6 + ig\left[\alpha_7\frac{\{\hat{p}^2,\hat{x}\}}{2} + \alpha_8\frac{\{\hat{p},\hat{x}\}}{2} + \alpha_9\hat{x}^3 + \alpha_{10}\hat{x}\right]. \quad (2.2)$$

model\constants	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
massive ix-potential	0	1	0	0	m^2	0	0	0	0	1
massive ix^3-potential	0	1	0	0	m^2	0	0	0	1	0
Swanson model	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	0	1	0	0
lattice Reggeon	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	1	0	1	-2
$\Delta a^\dagger a + iga^\dagger\left(a^\dagger\text{-}a\right)a$	g	$\frac{\Delta}{2}$	g	$-2g$	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	0	0	0	0
$H_{(5.11)}$	g	$\frac{\Delta}{2}$	g	$-2g$	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	1	0	1	-2
$H_{(5.14)}$	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	-$\frac{\Delta}{2}$	1	0	0	-2
$\frac{\hat{p}_z^2}{2}\text{-}\frac{g}{32}\hat{z}^4$	0	$\frac{1}{2}$	0	$\frac{1}{4}-\frac{1}{2g}$	$\frac{g}{2}$	-$\frac{g}{2}$	$\frac{1}{2g}$	0	0	-1
$\frac{\hat{p}_z^2}{2}+\lambda_1\hat{z}^6+\lambda_2\hat{z}^2$	0	$\frac{1}{2}$	0	$\frac{1}{4}-\frac{1}{2g}$	$192\lambda_1$	κ_1	$\frac{1}{2g}$	0	$\frac{64\lambda_1}{g}$	$\frac{\kappa_2}{g}$

Table 1: Special reductions of the Hamiltonian $H_{\rm c}$. The map $z(x)$ is defined in equation (6.5) and $g, \alpha, \Delta, m \in \mathbb{R}$ are coupling constants of the models. We abbreviated $\kappa_1 = -4(16\lambda_1 + \lambda_2)$ and $\kappa_2 = -4(48\lambda_1 + \lambda_2)$.

We have symmetrized in $H_{\rm c}$ terms which contain $\hat{p}$ and $\hat{x}$ by introducing anticommutators, i.e. $\{A, B\} = AB + BA$. This allows us to separate off conveniently the real and imaginary parts of $H_{\rm c}$ by defining

$$H_{\rm c}(\hat{x},\hat{p}) = h_0(\hat{x},\hat{p}) + igh_1(\hat{x},\hat{p}), \quad (2.3)$$

with $h_0^\dagger = h_0$ and $h_1^\dagger = h_1$. In addition, the symmetrized version (2.2) will lead to very simple expressions when we convert products of operator valued functions into expressions involving scalar functions multiplied via Moyal products. For our definition of the Moyal product it implies that the parameters α do not need to be re-defined. Depending on the context, one (2.1) or the other (2.2) formulation is more advantageous. Whereas the usage of creation and annihilation operators is more proned for an algebraic generalization, see e.g. [23], the formulation in terms of operators $\hat{x}$ and $\hat{p}$ is more suitable for a treatment with Moyal brackets. The relation between the two versions is easily computed from the aforementioned identifications between the a,$a^\dagger$ and $\hat{x}$,$\hat{p}$ via the relations $\alpha = M\lambda$ and $\lambda = M^{-1}\alpha$ with M being a 10×10-matrix. Below we will impose some constraints on the coefficients α and it is therefore useful to have an explicit expression for M at our disposal in order to see how these constraints affect the expression for the Hamiltonian in (2.1). We compute the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & -\frac{3}{2\sqrt{2}g} \\ 0 & -\frac{1}{g} & \frac{1}{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} \\ 0 & 0 & 0 & 0 & \frac{1}{g\sqrt{2}} & \frac{1}{g\sqrt{2}} & 0 & -\frac{1}{g\sqrt{2}} & -\frac{1}{g\sqrt{2}} & 0 \end{pmatrix} \tag{2.4}$$

and the inverse

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{g}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{g}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & \frac{g}{\sqrt{2}} \\ -\frac{3}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & \frac{g}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{g}{2\sqrt{2}} & 0 \\ \frac{3}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & 0 \\ -\frac{3}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{g}{2\sqrt{2}} & 0 \end{pmatrix}, \tag{2.5}$$

which also exists for $g \neq 0$ since $\det M = -g^{-4}$.

The Hamiltonian H_c encompasses many models and for specific choices of some of the α_i it reduces to various well studied examples, such as the simple massive ix-potential [24] or its massless version, the so-called Swanson Hamiltonian [25, 26, 27, 21, 28], the complex cubic potential together with his massive version [4] and also the transformed version of the $-\hat{x}^4$-potential [27]. As we will show below, in addition it includes several interesting new models, such as the single site lattice version of Reggeon field theory [29], which is a thirty year old model but has not been considered in the current context and the transformed version of the $\pm x^6$- potential, which serves as a toy model to identify theories with vanishing cosmological constant [30]. The latter models have not been solved so far with regard to their metric operators and isospectral partners. Besides these models, H_c also includes many new models not considered so far, some of which are even SPH.

To enable easy reference we summarize the various choices in table 1.

Most SPH-models which have been constructed so far are rather trivial, such as the massive ix-potential or the so-called Swanson Hamiltonian. The latter model can be obtained simply from the standard harmonic oscillator by means of a Bogolyubov transformation and a subsequent similarity transformation, which is bilinear in a and $a^\dagger$. Beyond these maximally quadratic models, the complex cubic potential was the first model which has been studied in more detail. Unfortunately so far it can only be treated perturbatively. The transformed version of the from below unbounded -z^4-potential is the first SPH-model containing at least one cubic term. Here we enlarge this class of models. As a special case we shall also investigate the single site lattice version of Reggeon field theory [29] in more detail. Before treating these specific models let us investigate first the Hamiltonian H_c in a very generic manner.

Our objective is to solve equation (1.1) and find an exact expression for the positive-definite metric operator η^2, subsequently to solve for the similarity transformation and construct Hermitian isospectral partner Hamiltonians.

3. Pseudo-Hermitian Hamiltonians from Moyal products

3.1 Generalities

Taking solely a non-Hermitian Hamiltonian as a starting point, there is of course not a one-to-one correspondence to one specific Hermitian Hamiltonian counterpart. The conjugation relation in (1.1) admits obviously a whole family of solutions. In order to construct these solutions we will not use commutation relations involving operators, but instead we will exploit the isomorphism between operator valued function in $\hat{x}$ and $\hat{p}$ and scalar functions multiplied by Moyal products in monomial of scalars x and p. We associate to two arbitrary operator valued functions $F(\hat{x},\hat{p})$ and $G(\hat{x},\hat{p})$ two scalar functions $F(x,p)$, $G(x,p) \in \mathcal{S}$ such that

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) \cong F(x,p)\star G(x,p), \tag{3.1}$$

where $\mathcal{S}$ is the space of complex valued integrable functions. Here we use the following standard definition of the Moyal product $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, see e.g. [31, 32, 21],

$$\begin{aligned} F(x,p) \star G(x,p) &= F(x,p) e^{\frac{i}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} G(x,p) \\ &= \sum_{s=0}^{\infty} \frac{(-i/2)^s}{s!} \sum_{t=0}^{s} (-1)^t \binom{s}{t} \partial_x^t \partial_p^{s-t} F(x,p) \partial_x^{s-t} \partial_p^t G(x,p). \end{aligned} \tag{3.2}$$

The Moyal product is a distributive and associative map obeying the same Hermiticity properties as the operator valued functions on the right hand side of (3.1), that is $(F \star$

$G)^* = G^* \star F^*$. Following standard arguments we provide now explicit representations for the $F(\hat{x},\hat{p})$ and $F(x,p)$. We may formally Fourier expand an arbitrary operator valued functions $F(\hat{x},\hat{p})$ and scalar functions $F(x,p)$ as

$$F(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} dsdt f(s,t) e^{i(s\hat{x}+t\hat{p})} \quad \text{and} \quad F(x,p) = \int_{-\infty}^{\infty} dsdt f(s,t) e^{i(sx+tp)}, \tag{3.3}$$

respectively. In terms of this representation the multiplication of two operator valued functions yields

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} dsdtds'dt' f(s,t) f(s',t') e^{\frac{i}{2}(ts'-t's)} e^{i(s+s')\hat{x}+i(t+t')\hat{p}}, \tag{3.4}$$

which follows using the identities $e^{i(s\hat{x}+t\hat{p})} = e^{is\hat{x}/2} e^{it\hat{p}} e^{is\hat{x}/2}$ and $e^{is\hat{x}/2} e^{it\hat{p}} = e^{it\hat{p}} e^{is\hat{x}/2} e^{ist}$. Is now straightforward to verify that the definition of the Moyal product (3.2) guarantees that the isomorphism (3.1) holds, since $F(x,p)\star G(x,p)$ yields formally the same expression as (3.4) with $\hat{x},\hat{p}$ replaced by x,p.

The Hermiticity property is important for our purposes. We find that

$$F^\dagger(\hat{x},\hat{p}) = F(\hat{x},\hat{p}) \cong F^*(x,p) = F(x,p). \tag{3.5}$$

This is easily seen by computing $F^\dagger(\hat{x},\hat{p})$ using the representation (3.3). Then this function is Hermitian if and only if the kernel satisfies $f^*(s,t) = f(-s,-t)$, which in turn implies that $F(x,p)$ is real. Positive definiteness of an operator valued function $F(\hat{x},\hat{p})$ is guaranteed if the logarithm of the operator is Hermitian, that is we need to ensure that $\log F(x,p)$ is real. Furthermore, it is easy to see that $F(\hat{x},\hat{p})$ is $\mathcal{PT}$-symmetric if and only if $f^*(s,t) = f(s,-t)$.

As an instructive example we consider $F(x,p) = x^m p^n$ for which we compute the corresponding kernel as $f(s,t) = i^{n+m}\delta^{(m)}(s)\delta^{(n)}(t)$. From this it is easy to see that $(ix)^m p^n$ is $\mathcal{PT}$-symmetric, since $i^m f(s,t)$ satisfies $[i^m f(s,t)]^* = i^m f(s,-t)$.

In the present context of studying non-Hermitian Hamiltonians this technique of exploiting the isomorphism between Moyal products and operator products has been exploited by Scholtz and Geyer [19, 20], who reproduced some previously known results and also in [21], where new solutions were constructed. In [19, 20] a more asymmetrical definition than (3.2) of the Moyal product was employed, i.e. $F(x,p) * G(x,p) = F(x,p) e^{i\overleftarrow{\partial}_x \overrightarrow{\partial}_p} G(x,p)$. In comparison with (3.2) this definition leads to some rather unappealing properties: i) the loss of the useful and natural Hermiticity relation, i.e. $(F * G)^* \neq G^* * F^*$, ii) the right hand side of the isomorphism in (3.5) is replaced by the less transparent expression $F^*(x,p) = e^{-i\partial_x\partial_p} F(x,p)$ and iii) in [21] it was shown that the definition $*$ leads to more complicated differential equations than the definition $\star$. The representation for the operator valued functions $F(\hat{x},\hat{p})$, which satisfies the properties resulting from the definition $*$ differs from (3.3) by replacing $e^{i(s\hat{x}+t\hat{p})} \to e^{is\hat{x}} e^{it\hat{p}}$ in the Fourier expansion.

3.2 Construction of the metric operator and isospectral partners

We briefly recapitulate the main steps of the procedure [19, 20, 21] of how to find for a given non-Hermitian Hamiltonian H a metric operator $\eta^2(\hat{x},\hat{p})$, a similarity transformation

$\eta(\hat{x},\hat{p})$ and an Hermitian counterpart $h(\hat{x},\hat{p})$ using Moyal products. First of all we need to solve the right hand side of the isomorphism

$$H^\dagger(\hat{x},\hat{p})\eta^2(\hat{x},\hat{p}) = \eta^2(\hat{x},\hat{p})H(\hat{x},\hat{p}) \cong H^\dagger(x,p)\star\eta^2(x,p) = \eta^2(x,p)\star H(x,p) \tag{3.6}$$

for the "scalar metric function" $\eta^2(x,p)$. Taking as a starting point the non-Hermitian Hamiltonian $H(\hat{x},\hat{p})$, we have to transform this expression into a scalar function $H(x,p)$ by replacing all occurring operator products with Moyal products. We can use this expression to evaluate the right hand side of the isomorphism of (3.6), which is a differential equation for $\eta^2(x,p)$ whose order is governed by the highest powers of x and p in $H(x,p)$. Subsequently we may replace the function $\eta^2(x,p)$ by the metric operator $\eta^2(\hat{x},\hat{p})$ using the isomorphism (3.1) now in reverse from the right to the left. Thereafter we solve the differential equation $\eta^2(x,p) = \eta(x,p)\star\eta(x,p)$ for $\eta(x,p)$. Inverting this expression we obtain $\eta^{-1}(x,p)$, such that we are equipped to compute directly the scalar function associated to the Hermitian counterpart by evaluating

$$h(x,p) = \eta(x,p)\star H(x,p)\star\eta^{-1}(x,p). \tag{3.7}$$

Finally we have to convert the function $\eta(x,p)$ into the operator valued function $\eta(\hat{x},\hat{p})$ and the "Hermitian scalar function" $h(x,p)$ into the Hamiltonian counterpart $h(\hat{x},\hat{p})$, once more by solving (3.1) from the right to the left.

So far we did not comment on whether the metric is a meaningful Hermitian and positive operator. According to the isomorphism (3.5) we simply have to verify that $\eta^2(x,p)$, $\eta(x,p)$ and $h(x,p)$ are real functions in order to establish that the corresponding operator valued functions $\eta^2(\hat{x},\hat{p})$, $\eta(\hat{x},\hat{p})$ and $h(\hat{x},\hat{p})$ are Hermitian. We may establish positive definiteness of these operators by verifying that their logarithms are real.

3.3 Ambiguities in the solution

Obviously when having a non-Hermitian Hamiltonian as the sole starting point there is not a unique Hermitian counterpart in the same similarity class associated to the adjoint action of one unique operator η. Consequently also the metric operator η^2 is not unique. The latter was pointed out for instance in [20] and exemplified in detail for the concrete example of the so-called Swanson Hamiltonian in [28]. In fact, it is trivial to see that any two non-equivalent metric operators, say η^2 and $\hat{\eta}^2$, can be used to construct a non-unitary symmetry operator $S := \eta^{-2}\hat{\eta}^2 \neq S^\dagger = \hat{\eta}^2\eta^{-2}$ for the non-Hermitian Hamiltonian H

$$H^\dagger = \eta^2 H\eta^{-2} = \hat{\eta}^2 H\hat{\eta}^{-2} \qquad \Leftrightarrow \qquad [S,H] = [S^\dagger, H^\dagger] = 0. \tag{3.8}$$

We may solve (3.8) and express one metric in terms of the other as

$$\hat{\eta}^2 = \left(H^\dagger\right)^n \eta^2 H^n \qquad \text{for } n \in \mathbb{N}. \tag{3.9}$$

Thus we encounter here an infinite amount of new solutions. Likewise this ambiguity can be related to the non-equivalent Hermitian counterparts

$$h = \eta H\eta^{-1}, \hat{h} = \hat{\eta}H\hat{\eta}^{-1} \qquad \Leftrightarrow \qquad [s,h] = [\hat{s},\hat{h}] = 0, \tag{3.10}$$

with symmetry operators $s = \eta\hat{\eta}^{-2}\eta$ and $\hat{s} = \hat{\eta}^{-1}\eta^{2}\hat{\eta}^{-1}$. When $\eta^{\dagger} = \eta$ and $\hat{\eta}^{\dagger} = \hat{\eta}$ we obviously also have $s^{\dagger} = s$ and $\hat{s}^{\dagger} = \hat{s}$. The expression for the symmetry operator s for h was also identified in [33].

There are various ways to select a unique solution. One possibility [8] is to specify one more observable in the non-Hermitian system. However, this argument is very impractical as one does not know a priori which variables constitute observables.

4. SPH-models of cubic order

Let us study $H_{\rm c}(\hat{x},\hat{p})$ by converting it first into a scalar function $H_{\rm c}(x,p)$. Most terms are non problematic and we can simply substitute $\hat{x} \to x, \hat{p} \to p$, but according to our definition of the Moyal bracket (3.2) we have to replace $\hat{p}^2\hat{x} \to p^2{\star}x = p^2x - ip$, $\hat{p}\hat{x}^2 \to p{\star}x^2 = px^2 - ix$, $\hat{p}\hat{x} \to p \star x = px - i/2$ etc. Replacing all operator products in this way we convert the Hamiltonian $H_{\rm c}(\hat{x},\hat{p})$ in (2.2) into the scalar function

$$H_{\rm c}(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_7 p^2 x + \alpha_8 px + \alpha_9 x^3 + \alpha_{10} x\,). \quad (4.1)$$

Substituting (4.1) into the right hand side of the isomorphism into (3.6) yields the third order differential equation

$$\left(\alpha_3 px\partial_p + \alpha_5 x\partial_p + \frac{\alpha_3}{8}\partial_x\partial_p^2 + \frac{\alpha_1}{8}\partial_x^3 - \alpha_2 p\partial_x - \frac{3}{2}\alpha_1 p^2\partial_x - \frac{\alpha_3}{2}x^2\partial_x - \frac{\alpha_4}{2}\partial_x\right)\eta^2 \quad (4.2)$$
$$= g\left(\alpha_9 x^3 + \alpha_{10}x + \alpha_8 px + \alpha_7 p^2 x + \frac{\alpha_7}{2}p\partial_x\partial_p + \frac{\alpha_8}{4}\partial_x\partial_p - \frac{\alpha_7}{4}x\partial_x^2 - \frac{3}{4}\alpha_9 x\partial_p^2\right)\eta^2$$

for the "metric scalar function" $\eta^2(x,p)$. There are various simplifications one can make at this stage. First of all we could assume that either $\hat{x}$ or $\hat{p}$ is an observable in the non-Hermitian system, such that $\eta^2(x,p)$ does not depend on p or x, respectively. As pointed out before it is not clear at this stage if any of these choices is consistent. However, any particular choice p or x will be vindicated if (4.2) can be solved subsequently for $\eta^2(p)$ or $\eta^2(x)$, respectively. Here we will assume that $\eta^2(x,p)$ admits a perturbative expansion. Making a very generic exponential $\mathcal{PT}$-symmetric ansatz, which is real and cubic in its argument for $\eta^2(x,p) = \exp g(q_1p^3 + q_2px^2 + q_3p^2 + q_4x^2 + q_5p)$, we construct systematically all exact solutions of this form. Substituting the ansatz into the differential equation (4.2) and reading off the coefficients in front of each monomial in x and p yields at each order in g ten equations. by solving these equations we find five qualitatively different types of exact solutions characterized by vanishing coefficients α_i and some additional constraints. We will now present these solutions.

4.1 Non-vanishing $\hat{p}\hat{x}^2$-term

4.1.1 Constraints 1

We consider the full Hamiltonian $H_{\rm c}(x,p)$ in (4.1) and impose as the only constraint that the px^2-term does not vanish, i.e. $\alpha_3 \neq 0$. For this situation we can solve the differential equation (4.2) exactly to all orders in perturbation theory for

$$H_{\rm c}(x,p) = h_0(x,p) + ig(\frac{\alpha_1\alpha_9}{\alpha_3}p^2x + \frac{\alpha_2\alpha_9 - \alpha_5\alpha_7}{\alpha_3}px + \alpha_9 x^3 + \frac{\alpha_4\alpha_9 - \alpha_5\alpha_8}{\alpha_3}x\,), \quad (4.3)$$

where we imposed the additional constraints

$$\alpha_1\alpha_9 = \alpha_3\alpha_7, \qquad \alpha_2\alpha_9 = \alpha_5\alpha_7 + \alpha_3\alpha_8 \qquad \text{and} \qquad \alpha_4\alpha_9 = \alpha_5\alpha_8 + \alpha_3\alpha_{10}. \tag{4.4}$$

In (4.3) we have replaced the constants α_7, α_8 and α_{10} using (4.4). The solution of the differential equation is the metric scalar function

$$\eta^2(x,p) = e^{-g\left(\frac{\alpha_7}{\alpha_3}p^2 + \frac{\alpha_8}{\alpha_3}p + \frac{\alpha_9}{\alpha_3}x^2\right)}. \tag{4.5}$$

Since $\eta^2(x,p)$ is real it follows from (3.5) that the corresponding metric operator is Hermitian. Next we solve $\eta(x,p) \star \eta(x,p) = \eta^2(x,p)$ for $\eta(x,p)$. Up to order g^2 we find

$$\begin{aligned}\eta(x,p) &= 1 - g\frac{\alpha_7 p^2 + \alpha_8 p + x^2\alpha_9}{2\alpha_3} + g^2\left(\frac{\alpha_9\left(\alpha_7 + 2\alpha_7 p^2 x^2 + 2\alpha_8 p x^2\right) + \alpha_9^2 x^4}{8\alpha_3^2}\right.\\ &\quad \left. + \frac{(p\alpha_7 + \alpha_8)^2 p^2}{8\alpha_3^2}\right) + \mathcal{O}(g^3).\end{aligned} \tag{4.6}$$

The corresponding Hermitian counterpart corresponding to this solution is computed by means of (3.7) to

$$\begin{aligned}h_c(x,p) &= \alpha_3 p x^2 + \alpha_5 x^2 + \alpha_6 + \frac{\alpha_3\alpha_7}{\alpha_9}p^3 + \frac{(\alpha_5\alpha_7 + \alpha_3\alpha_8)}{\alpha_9}p^2 + \frac{(\alpha_5\alpha_8 + \alpha_3\alpha_{10})}{\alpha_9}p\\ &\quad -g^2\frac{(2\,\alpha_7 p + \alpha_8)\,\left(p\,(\alpha_7 p + \alpha_8) + \alpha_9 x^2 + \alpha_{10}\right)}{4\,\alpha_3} + \mathcal{O}(g^4).\end{aligned} \tag{4.7}$$

Notice that since we demanded α_3 to be non-vanishing these solutions can not be reduced to any of the well studied models presented in table 1, but represent new types of solutions. We may simplify the above Hamiltonians by setting various αs to zero.

Demanding for instance that $\hat{x}$ is an observable in the non-Hermitian system we are forced by (4.5) to set $\alpha_7 = \alpha_8 = 0$ and by (4.4) also $\alpha_1 = \alpha_2 = 0$. The Hamiltonian in (4.3) then simplifies to

$$H_c(x,p) = \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3}x\,). \tag{4.8}$$

Since $\eta^2(x,p)$ only depends on x in this case, we can compute exactly $\eta(x,p) = e^{-g\frac{\alpha_9}{2\alpha_3}x^2}$. The Hermitian counterpart results to

$$h_c(x,p) = h_0(x,p) = \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6. \tag{4.9}$$

If we require on the other hand that $\hat{p}$ is an observable, we have to choose $\alpha_9 \to 0$. However, in that case the constraints (4.4) imply that the non-Hermitian part of the Hamiltonian (4.3) vanishes, i.e. we obtain the trivial case $H_c(x,p) = h_0(x,p)$.

4.1.2 Constraints 2

In the construction of the previous solution some coefficients had to satisfy a quadratic equations in the parameters to guarantee the vanishing of the perturbative expansion. The other solution for this equation leads to the constraints $\alpha_1 = \alpha_7 = 0$, such that the non-Hermitian Hamiltonian simplifies. If we now impose the additional constraints

$$\alpha_3\alpha_{10} = \alpha_4\alpha_9 \qquad \text{and} \qquad \alpha_3\alpha_8 = 2\alpha_2\alpha_9, \tag{4.10}$$

we can solve the differential equation (4.2) exactly. For

$$H_c(x,p) = \alpha_2 p^2 + \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\frac{2\alpha_2\alpha_9}{\alpha_3}px + \alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3}x) \tag{4.11}$$

we compute the exact scalar metric function to

$$\eta^2(x,p) = e^{-g\frac{\alpha_9}{\alpha_3}x^2}. \tag{4.12}$$

Clearly $\eta^2(\hat{x},\hat{p})$ is a Hermitian and positive definite operator, which follows from the facts that $\eta^2(x,p)$ and $\log\eta^2(x,p)$ are real, respectively. Notice the fact that the Hamiltonian (4.11) does not follow as a specialization of (4.3), since the constraints (4.10) do not result as a particular case of (4.4). The Hermitian Hamiltonian counterpart corresponding to (4.11) is computed with $\eta^2(x,p) = e^{-g\frac{\alpha_9}{2\alpha_3}x^2}$ by means of (3.7) to

$$h_c(x,p) = \alpha_2 p^2 + \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_2\,\alpha_9^2}{\alpha_3{}^2}x^2. \tag{4.13}$$

Once again we may simplify the above Hamiltonians by setting various αs to zero or other special values, except for the case $\alpha_9 \to 0$ for which the constraints (4.10) reduce the non-Hermitian part of the Hamiltonian (4.11) to zero.

Thus this case requires a separate consideration:

4.2 Non-vanishing $\hat{p}\hat{x}^2$-term and vanishing $\hat{x}^3$-term

Let us therefore embark on the treatment of the complementary case to the previous subsection, namely $\alpha_3 \neq 0$ and $\alpha_9 = 0$. For these constraints we can solve the differential equation (4.2) exactly for the Hamiltonian

$$H_c(x,p) = h_0(x,p) + ig(\alpha_7 p^2 x + \alpha_8 px + \frac{\alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7)}{\alpha_3^2}x), \tag{4.14}$$

when we impose one additional constraint

$$\alpha_{10}\alpha_3^2 = \alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7). \tag{4.15}$$

The "metric scalar function" results to

$$\eta^2(x,p) = \eta^2(p) = e^{g\left(\frac{\alpha_7}{2\alpha_3}p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{\alpha_3{}^2}p\right)}. \tag{4.16}$$

Once again $\eta^2(\hat{x},\hat{p})$ is a Hermitian and positive definite operator, which follows again from the facts that $\eta^2(x,p)$ and $\log\eta^2(x,p)$ are real. Since $\eta^2(x,p)$ only depends on p, we can simply take the square root to compute $\eta(p)$. Then the corresponding Hermitian counterpart is computed by means of (3.7) to

$$h_c = h_0 + g^2 \left(\frac{{\alpha_7}^2}{4\alpha_3} p^3 + \frac{2\alpha_3\alpha_7\alpha_8 - \alpha_5\alpha_7^2}{4{\alpha_3}^2} p^2 + \frac{{\alpha_3}^2{\alpha_8}^2 - {\alpha_5}^2{\alpha_7}^2}{4{\alpha_3}^3} p + \frac{\alpha_5 \left(\alpha_5\alpha_7 - \alpha_3\alpha_8\right)^2}{4{\alpha_3}^4} \right) \tag{4.17}$$

In fact we can implement the constraint (4.15) directly in the solution. The function

$$\eta^2(p) = (p\,\alpha_3 + \alpha_5)^{\frac{g\left({\alpha_5}^2\,\alpha_7 - \alpha_3\,\alpha_5\,\alpha_8 + {\alpha_3}^2\,\alpha_{10}\right)}{{\alpha_3}^3}} e^{g\left(\frac{\alpha_7}{2\alpha_3}p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{{\alpha_3}^2}p\right)} \tag{4.18}$$

solves (4.2) for the generic Hamiltonian (4.1) with the only constraint that $\alpha_3 \neq 0$ and $\alpha_9 = 0$. In this case the corresponding Hermitian counterpart is computed to

$$h_c(x,p) = h_0 + g^2 \frac{\left(p^2\,\alpha_7 + p\,\alpha_8 + \alpha_{10}\right)^2}{4\,\left(p\,\alpha_3 + \alpha_5\right)}. \tag{4.19}$$

Implementing the constraint (4.15), the Hamiltonian (4.19) reduces to the one in (4.17). Similarly as the model of the previous subsection, these solutions can not be reduced to any of the well studied models presented in table 1, since α_3 is assumed to be non-vanishing.

4.3 Vanishing $\hat{p}\hat{x}^2$-term and non-vanishing $\hat{x}^2$-term

Next we consider the complementary case to the previous two section, that is we take $\alpha_3 = 0$ in (2.2). For this set up we can only find an exact solution when we demand in addition that $\alpha_5 \neq 0$ and $\alpha_9 = 0$. For the non-Hermitian Hamiltonian

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_7 p^2 x + \alpha_8 p x + \alpha_{10} x\,), \tag{4.20}$$

we can solve the differential equation (4.2) exactly by

$$\eta^2(x,p) = e^{g\,\left(\frac{\alpha_7}{3\,\alpha_5}p^3 + \frac{\alpha_8}{2\alpha_5}p^2 + \frac{\alpha_{10}}{\alpha_5}p\right)}. \tag{4.21}$$

Once again $\eta^2(x,p)$ only depends on p and we can simply take the square root to compute $\eta(p)$. Using (3.7) the corresponding Hermitian Hamiltonian is subsequently computed to

$$h_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\left(p^2\alpha_7 + p\alpha_8 + \alpha_{10}\right)^2}{4\alpha_5}. \tag{4.22}$$

Obviously these solutions can be reduced to various cases presented in table 1, notably the transformed $-z^4$-potential and the Swanson Hamiltonian.

4.4 Vanishing $\hat{p}\hat{x}^2$-term and non-vanishing $\hat{p}$-term or non-vanishing $\hat{p}^2$-term

Finally we consider the complementary case of the previous section by taking $\alpha_3 = 0$ and allowing α_5 to acquire any value. To be able to find an exact solution we need to impose the additional constraints

$$\alpha_1 = \alpha_7 = \alpha_9 = 0, \qquad \text{and} \qquad \alpha_4\alpha_8 = 2\alpha_2\alpha_{10}, \tag{4.23}$$

i.e. we consider the non-Hermitian Hamiltonian

$$H_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig\left(\alpha_8 px + \alpha_{10} x\right), \tag{4.24}$$

for which we can solve equation (4.2) by

$$\eta^2(x,p) = e^{-g\alpha_{10}/\alpha_4 x^2} \qquad \text{for } \alpha_4 \neq 0, \tag{4.25}$$

$$\eta^2(x,p) = e^{-g\alpha_8/2\alpha_2 x^2} \qquad \text{for } \alpha_2 \neq 0. \tag{4.26}$$

As $\eta^2(x,p)$ only depends on x, we can take the square root to compute $\eta(x)$ and subsequently evaluate the corresponding Hermitian counterpart using (3.7)

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_2 \alpha_{10}^2}{\alpha_4^2} x^2 \qquad \text{for } \alpha_4 \neq 0, \tag{4.27}$$

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_8^2}{4\alpha_2} x^2 \qquad \text{for } \alpha_2 \neq 0. \tag{4.28}$$

The Hamiltonian in (4.24) can be reduced to the Swanson Hamiltonian. Notice that when we impose $\alpha_1 = \alpha_4 = \alpha_7 = \alpha_{10} = 0$ for the Hamiltonian in (4.20) and $\alpha_4 = \alpha_{10} = 0$ for the Hamiltonian in (4.24), they become both identical to the Swanson Hamiltonian. The corresponding solutions for the metric operators reduce to $\hat{\eta}^2(x,p) = e^{g\,\alpha_8/2\alpha_5 p^2}$ and $\eta^2(x,p) = e^{-g\alpha_8/2\alpha_2 x^2}$, respectively, which are the well known non-equivalent solutions for the Swanson Hamiltonian, see e.g. [28]. This means according to (3.8) we can identify a symmetry operator for the Swanson Hamiltonian as

$$S(\hat{x},\hat{p}) = e^{-g\,\frac{\alpha_8}{2\alpha_5}\hat{p}^2} e^{-g\frac{\alpha_8}{2\alpha_2}\hat{x}^2}. \tag{4.29}$$

Notice that $S(x,p)\star H(x,p) = H(x,p)\star S(x,p)$ is a more difficult equation to solve in this example than (3.6), since S is not of a simple exponential form as η^2. In fact this is what we expect. For instance supposing that the symmetry is some group of Lie type, a typical group element, when Gauβ decomposed, would be a product of three exponentials.

model\const	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
$H_{(4.3)}$	α_1	α_2	$\neq 0$	α_4	α_5	α_6	$\frac{\alpha_1\alpha_9}{\alpha_3}$	$\frac{\alpha_2\alpha_9-\alpha_5\alpha_7}{\alpha_3}$	α_9	$\frac{\alpha_4\alpha_9-\alpha_5\alpha_8}{\alpha_3}$
$H_{(4.11)}$	0	α_2	$\neq 0$	α_4	α_5	α_6	0	$\frac{2\alpha_2\alpha_9}{\alpha_3}$	α_9	$\frac{\alpha_4\alpha_9}{\alpha_3}$
$H_{(4.14)}$	α_1	α_2	$\neq 0$	α_4	α_5	α_6	α_7	α_8	0	$\frac{\alpha_5(\alpha_3\alpha_8-\alpha_5\alpha_7)}{\alpha_3^2}$
$H_{(4.20)}$	α_1	α_2	0	α_4	$\neq 0$	α_6	α_7	α_8	0	α_{10}
$H_{(4.24)}$	0	α_2	0	$\neq 0$	α_5	α_6	0	$\frac{2\alpha_2\alpha_{10}}{\alpha_4}$	0	α_{10}
$H_{(4.24)}$	0	$\neq 0$	0	α_4	α_5	α_6	0	α_8	0	$\frac{\alpha_8\alpha_4}{2\alpha_2}$

Table 2: SPH-models of cubic order.

Obviously using the relation $\lambda = M^{-1}\alpha$ we can also convert our solutions into expressions using creation and annihilation operators.

5. Lattice version of Reggeon field theory

Having studied the Hamiltonian (2.1), (2.2) in a very generic manner let us return to our original motivation and focus on some special cases, which have hitherto not been dealt with in the literature. It has been argued for more than thirty years that the lattice versions of Reggeon field theory (LR) [1]

$$H_{\text{LR}} = \sum_{\vec{\imath}} \left[\Delta a_{\vec{\imath}}^\dagger a_{\vec{\imath}} + i\tilde{g} a_{\vec{\imath}}^\dagger (a_{\vec{\imath}} + a_{\vec{\imath}}^\dagger) a_{\vec{\imath}} + \hat{g} \sum_{\vec{\jmath}} (a_{\vec{\imath}+\vec{\jmath}}^\dagger - a_{\vec{\imath}}^\dagger)(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right] \tag{5.1}$$

with $a_{\vec{\imath}}^\dagger, a_{\vec{\imath}}$ being standard creation and annihilation operators and $\Delta, \hat{g}, \tilde{g} \in \mathbb{R}$, possess a real eigenvalue spectrum [?]. This assertion was made despite the fact that the Hamiltonian in (5.1) is non-Hermitian.

5.1 The single site lattice Reggeon model

Many features of lattice models can be understood with a finite, possibly small, number of sites. Thus reducing the Hamiltonian in (5.1) to a single lattice site yields the harmonic oscillator perturbed by a cubic perturbation in a and $a^\dagger$, such that unlike for the so-called Swanson Hamiltonian its properties can not be understood in direct analogy to the harmonic oscillator. The resulting model, which we refer to as single site lattice Reggeon model (SSLR) reads

$$H_{\text{SSLR}}(a, a^\dagger) = \Delta a^\dagger a + i\tilde{g} a^\dagger \left(a + a^\dagger\right) a. \tag{5.2}$$

Obviously H_{SSLR} is a special case of the generic cubic $\mathcal{PT}$-symmetric Hamiltonians (2.1). One may find already in the old literature, e.g. [34], that the Hermitian conjugation of H as specified in (1.1) can be obtained by an adjoint action with the parity operator

$$\mathcal{P} = e^{\frac{i\pi}{2} a^\dagger a} = \eta^2. \tag{5.3}$$

This is easily seen by noting that $\mathcal{P}$ acts on the creation and annihilation operators as

$$\mathcal{P} a \mathcal{P} = -a \quad \text{and} \quad \mathcal{P} a^\dagger \mathcal{P} = -a^\dagger. \tag{5.4}$$

However, since the corresponding $\eta = \sqrt{\mathcal{P}}$ is not a Hermitian operator, we can not construct a Hermitian counterpart as specified in (1.1) from this solution for η^2. Another operator bilinear in a and $a^\dagger$, which has the same effect as $\mathcal{P}$ in (5.4) when acting adjointly on a,$a^\dagger$, is

$$\hat{\eta}^2 = e^{\frac{i\pi}{2}(aa - a^\dagger a^\dagger)}. \tag{5.5}$$

Taking now the square root of (5.5) yields a Hermitian operator $\hat{\eta}$ and serves therefore potentially for our purposes. Noting that this $\hat{\eta}$ acts as

$$\hat{\eta} a \hat{\eta}^{-1} = i a^\dagger \quad \text{and} \quad \hat{\eta} a^\dagger \hat{\eta}^{-1} = i a, \tag{5.6}$$

such that a corresponding Hermitian counterpart is trivially obtained as

$$h = \hat{\eta} H_{\text{SSLR}} \hat{\eta}^{-1} = -\Delta a a^\dagger + \tilde{g} a \left(a + a^\dagger\right) a^\dagger. \tag{5.7}$$

Clearly the spectrum of this Hamiltonian is not bounded from below for $\Delta > 0$, resulting essentially from the fact that $\hat{\eta}^2$ in not a legitimate metric operator as it is not positive definite. Nonetheless, in the context of Reggeon field theory there is a considerable interest in the regime $\Delta < 0$, such that the above argument contributes to an old discussion.

Ignoring whether the metric is positive-definite or Hermitain we have two alternative solutions to equation (1.1) involving H_{SSLR} and therefore we have simple examples for the symmetry operators $S = \mathcal{P}\hat{\eta}^2$ and $s = \hat{\eta}\mathcal{P}\hat{\eta}$ in (3.8) and (3.10), respectively.

Let us now try to find a more meaningful metric operator by using Moyal brackets. To commence, we have to convert the version (5.2) of the Hamiltonian into one which depends on $\hat{x}$ and $\hat{p}$ instead of a and $a^\dagger$. Using the aforementioned relations yields

$$H_{\text{SSLR}}(\hat{x}, \hat{p}) = \frac{\Delta}{2}(\hat{p}^2 + \hat{x}^2 - 1) + i\frac{\tilde{g}}{\sqrt{2}}(\hat{x}^3 + \hat{p}^2\hat{x} - 2\hat{x} + i\hat{p}) . \tag{5.8}$$

When ignoring the last three terms, this Hamiltonian becomes the massive version of the complex cubic potential of Bender and Boettcher [4]. To proceed with our analysis we have to change $H(\hat{x}, \hat{p})$ from a function depending on operators to a scalar function. In most terms in (5.8) we can simply replace operators by scalars using $\hat{x} \to x$, $\hat{p} \to p$, but care needs to be taken with the term $\hat{p}^2\hat{x} \to p^2 \star x = p^2x - ip$. The resulting scalar function is

$$H_{\text{SSLR}}(x, p) = a^\dagger \star a + i\tilde{g}a^\dagger \star \left(a + a^\dagger\right) \star a = \frac{1}{2}(x^2 + p^2 - 1) + ig(x^3 + p^2x - 2x), \tag{5.9}$$

where we have also scaled $\tilde{g} \to g\sqrt{2}$ and set $\Delta = 1$. For these values of the constants the differential equation (4.2) reduces to

$$(2x\partial_p - 2p\partial_x)\, \eta^2(x, p) = g\left(4x^3 - 8x + 4p^2x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2\right)\eta^2(x, p). \tag{5.10}$$

Due to its close resemblance to the cubic potential we do not expect this equation to be exactly solvable and therefore resort to perturbation theory.

5.2 SPH-models related to the single site lattice Reggeon model

However, there are various models closely related to H_{SSLR}, which fit into the scheme of the previous section and are SPH. For instance, we can identify the exact solution of section 4.1.1. by matching $H_c(x, p)$ in (4.3) with H_{SSLR}

$$H_{(4.3)}(a, a^\dagger, \Delta, g, \lambda) = H_{\text{SSLR}}(a, a^\dagger, \Delta/2, g) + H_{\text{SSLR}}(-a, a^\dagger, -\Delta/2, -\lambda). \tag{5.11}$$

We find that all the constraints for the ten parameters α_i in (4.4) are satisfied for this combination. An exact solution for the scalar metric function can then be identified as

$$\eta^2(x, p) = e^{-g/\lambda(x^2+p^2)}. \tag{5.12}$$

The Hamiltonian in (5.11) exhibits an interesting strong-weak symmetry

$$H_{(4.3)}(a, a^\dagger, \Delta, g, \lambda) = H_{(4.3)}(-a, a^\dagger, -\Delta, -\lambda, -g). \tag{5.13}$$

A further example for a SPH-model related to $H_{\rm SSLR}$ is

$$H_{(4.20)}(\hat{x},\hat{p},\Delta,g,\lambda) = H_{\rm SSLR}(\hat{x},\hat{p},\Delta,g) - i\tilde{g}\hat{x}^3, \tag{5.14}$$

which is identical to $H_{\rm c}(x,p)$ in (4.20) for certain values of the ten parameters α_i. Reading off those parameters yields as exact solution

$$\eta^2(\hat{x},\hat{p}) = e^{-\sqrt{2}g/\Delta(\hat{p}^3/3-2\hat{p}^2)}, \tag{5.15}$$

for the metric operator according to (4.21). In fact, this model can be matched with the transformed version of the $-z^4$-potential, for which the exact metric operator was constructed in [27].

5.3 Perturbative solution

In order to solve the differential equation (5.10) perturbatively we make now the ansatz

$$\eta^2(x,p) = 2\sum_{n=0}^{\infty} g^n c_n(x,p) \tag{5.16}$$

and the equation (5.10) is changed into a recursive equation for the coefficients c_n

$$(2x\partial_p - 2p\partial_x)\, c_n(x,p) = \left(4x^3 - 8x + 4p^2x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2\right) c_{n-1}(x,p). \tag{5.17}$$

We may solve this successively order by order. Using the fact that $\lim_{g\to 0}\eta^2(x,p) = 0$ the initial condition is taken to be $c_0(x,p) = 1$. We then obtain recursively order by order

$$\begin{aligned}
c_1(x,p) &= p^3 - 2p + px^2, \\
c_2(x,p) &= p^6 - 4p^4 + p^2 + x^2 - 4p^2x^2 + 2p^4x^2 + p^2x^4, \\
c_3(x,p) &= \frac{2}{3}p^9 - 4p^7 - 5p^5 + 24p^3 - 4p + 8px^2 - 6p^3x^2 - 8p^5x^2 + 2p^7x^2 - px^4 + \frac{2}{3}p^3x^6 \\
&\quad -4p^3x^4 + 2p^5x^4, \\
c_4(x,p) &= \frac{1}{3}p^{12} - \frac{8}{3}p^{10} - 12p^8 + 76p^6 - 5p^4 - 72p^2 + 24x^2 - 18p^2x^2 + 104p^4x^2 \\
&\quad -28p^6x^2 - 8p^8x^2 - 13x^4 + 28p^2x^4 - 20p^4x^4 - 8p^6x^4 + 2p^8x^4 - 4p^2x^6 \\
&\quad +\frac{4}{3}p^{10}x^2 - \frac{8}{3}p^4x^6 + \frac{4}{3}p^6x^6 + \frac{1}{3}p^4x^8.
\end{aligned} \tag{5.18}$$

Apart from the usual ambiguities, which were discussed in section 3.3, there are new uncertainties entering through this solution procedure. Apart from the different choice for the integration constants, it is evident that on the left hand side of (5.17) we can always add to $c_n(x,p)$ any function of the Hermitian part of $H_{\rm SSLR}(x,p)$, i.e. $c_n(x,p) \to c_n(x,p) + \frac{1}{2}(x^2+p^2-1)$ is also a solution of the left hand side of (5.17). We fix this ambiguity by demanding $\eta^2(x,p,g) \star \eta^2(x,p,-g) = 1$ for reasons explained in [17].

Next we solve the differential equation $\eta(x,p) \star \eta(x,p) = \eta^2(x,p)$ for $\eta(x,p)$ by making the ansatz

$$\eta(x,p) = 1 + \sum_{n=1}^{\infty} g^n q_n(x,p). \tag{5.19}$$

We find order by order

$$
\begin{aligned}
q_1(x,p) &= c_1(x,p), \qquad q_2(x,p) = c_2(x,p)/2, \qquad (5.20)\\
q_3(x,p) &= \frac{1}{6}p^9 - p^7 - \frac{17}{4}p^5 + 16p^3 - 3p - \frac{15}{2}p^3x^2 + \frac{1}{2}p^7x^2 + \frac{p^5x^4}{2} + \frac{p^3x^6}{6} - \frac{13}{4}px^4 \\
&\quad -p^3x^4 + 12px^2 - 2p^5x^2, \\
q_4(x,p) &= -35p^2 + \frac{11}{8}p^4 + \frac{51}{2}p^6 - \frac{9}{2}p^8 - \frac{1}{3}p^{10} + \frac{1}{24}p^{12} - \frac{25p^2x^2}{4} + \frac{39}{2}p^2x^4 + \frac{1}{24}p^4x^8 \\
&\quad + \frac{1}{6}p^{10}x^2 - \frac{61}{8}x^4 - \frac{23}{2}p^4x^4 - \frac{25}{2}p^6x^2 + \frac{p^8x^4}{4} - \frac{7}{2}p^2x^6 - \frac{1}{3}p^4x^6 + \frac{1}{6}p^6x^6 \\
&\quad +45p^4x^2 - p^8x^2 - p^6x^4 + 13x^2.
\end{aligned}
$$

We are now in the position to compute the Hermitian counterpart to $H_{\mathrm{SSLR}}(x,p)$ by means of (3.7)

$$
\begin{aligned}
h_{\mathrm{SSLR}}(x,p) &= \frac{1}{2}(x^2+p^2-1) + g^2\left(\frac{3}{2}p^4 - 4p^2 + 1 - 4x^2 + 3p^2x^2 + \frac{3}{2}x^4\right) \qquad (5.21)\\
&-g^4\left(\frac{17}{2}p^6 - 34p^4 + 4p^2 + 8 + 4x^2 - 48p^2x^2 + \frac{41}{2}p^4x^2 - 14x^4 + \frac{31}{2}p^2x^4 + \frac{7}{2}x^6\right) + \mathcal{O}(g^6)
\end{aligned}
$$

Finally we recast our solution again in terms of creation and annihilation operators. Up to order g^2 the square root of the metric becomes

$$
\eta = 1 + i\sqrt{2}ga^\dagger(a^\dagger - a)a + g^2a^\dagger\left[a^\dagger(2a^\dagger a - a^\dagger a^\dagger - aa + 5)a - 2a^\dagger a^\dagger - 2aa + 2\right]a \qquad (5.22)
$$

and the Hermitian counterpart to the non-Hermitian Hamiltonian H_{SSLR} acquires the form

$$
\begin{aligned}
h_{\mathrm{SSLR}} &= a^\dagger a + g^2a^\dagger(6a^\dagger a + 4)a + g^4\left[a^\dagger a^\dagger(10a^\dagger a^\dagger + 10aa - 48a^\dagger a)aa\right. \qquad (5.23)\\
&\qquad \left. + a^\dagger(20a^\dagger a^\dagger + 20aa - 120a^\dagger a)a - 32a^\dagger a\right] + \mathcal{O}(\hat{g}^6).
\end{aligned}
$$

As for all previously constructed perturbative solutions, it would be highly desirable to investigate in more detail the convergence properties of these solutions.

6. Potentials leading to zero cosmological constants and the SSLR-model

For the purpose of identifying vacuum solutions with zero cosmological constant 't Hooft and Nobbenhuis proposed in [30] an interesting complex space-time symmetry transformation between de-Sitter and anti-de-Sitter space

$$
\mathrm{dS} \to \mathrm{adS}: \; x^\mu \to ix^\mu \;\equiv\; x \to ix, p \to -ip\;. \qquad (6.1)
$$

Since this transformation relates vacuum solutions with positive cosmological constant to those with negative cosmological constant, it can only be a symmetry for the vacuum if the cosmological constant is vanishing. In order to match this with a quantum mechanical Hamiltonian one demands that the map dS→adS sends H to $-H$, such that the vacuum state is the only invariant state of theory. This means any Hamiltonian of the form

$$
H_{\mathrm{dS}}(\hat{x},\hat{p}) = \sum\nolimits_j \hat{x}^{n_j}\hat{p}^{m_j}f_j(\hat{x},\hat{p}) \quad \text{with} \begin{cases} n_j + m_j = 4k_j & \text{for } m_j \text{ odd} \\ n_j + m_j = 2k_j & \text{for } m_j \text{ even, } k_j \text{ odd,} \end{cases} \qquad (6.2)
$$

where the $f_j(\hat{x}, \hat{p}) = f_j(i\hat{x}, -i\hat{p})$ are arbitrary functions, is respecting this symmetry. A simple example for such a Hamiltonian was proposed in this context by Jackiw, see [30],

$$H_J(\hat{z}, \Omega, \lambda_1, \lambda_2) = \frac{\Omega}{2}\hat{p}_z^2 + \lambda_1 \hat{z}^6 + \lambda_2 \hat{z}^2. \tag{6.3}$$

Clearly dS→adS maps $H_J \to -H_J$. In fact, his Hamiltonian involving a sextic potential was investigated before in [35, 36, 37]. Notice that H_J is also $\mathcal{PT}$-symmetric. Furthermore, as pointed out in [35, 30] for $H_J(\hat{z}, 1, 2, -3)$ the groundstate wavefunction acquires a very simple form $\psi_0 = \exp(-\hat{z}^4/2)$ and H_J factorizes, such that it can be interpreted as the bosonic part of a supersymmetric pair of Hamiltonians. Moreover, this model is quasi-exactly solvable, meaning that a finite portion of the corresponding eigensystem has been constructed Bender:1996at. As discussed in [35, 4] one can continue the Schrödinger equation away from the real axis. Assuming an exponential fall off at infinity for H_J one may choose any parameterization which remains asymptotically inside the two wedges

$$\mathcal{W}_L = \left\{ \theta \left| -\frac{7}{8}\pi < \theta < -\frac{5}{8}\pi \right. \right\} \qquad \text{and} \qquad \mathcal{W}_R = \left\{ \theta \left| -\frac{3}{8}\pi < \theta < -\frac{1}{8}\pi \right. \right\}. \tag{6.4}$$

In fact, we can employ the same transformation as the one which was used successful for the $-z^4$-potential in [27]

$$z(x) = -2i\sqrt{1 + ix}. \tag{6.5}$$

For large positive x we find $z \sim e^{-i\pi/4} \in \mathcal{W}_R$ and likewise for large negative x we find $z \sim e^{-i\pi 3/4} \in \mathcal{W}_L$.
We then find

$$H_4(\hat{x}, \hat{p}_x, g) = H_4\left[\hat{z}(x), \hat{p}_z, g\right] = \frac{\hat{p}_{z(x)}^2}{2} - \frac{g}{32}\hat{z}(x)^4 = \frac{\hat{p}_x^2}{2} + \frac{\hat{p}_x}{4} + \frac{g}{2}\hat{x}^2 - \frac{g}{2} + \frac{i}{2}\hat{x}\hat{p}_x^2 - ig\hat{x}. \tag{6.6}$$

This allows us to interpret the Hamiltonian $H_J(z, \Omega, \lambda_1, \lambda_2)$ as a perturbation of the exactly solvable model $H_4(\hat{z}, \hat{p}_z, g)$, since

$$H_J\left[\hat{z}(x), 1, g/384, g/8\right] = H_4(\hat{x}, \hat{p}_x, g) + \frac{ig}{6}x^3 - \frac{g}{6}. \tag{6.7}$$

We can also relate the special case to an exactly solvable model

$$H_J\left[\hat{z}(x), 1, 2, -3\right] = \frac{\hat{p}_x^2}{2} - \frac{\hat{p}_x}{4} + 384\hat{x}^2 - 116 + i\left(\frac{\{\hat{x}, \hat{p}_x^2\}}{4} + 372\hat{x} + 128\hat{x}^3\right) \tag{6.8}$$

$$= H_{(4.20)}(\hat{x}, \hat{p}) + i128\hat{x}^3 \tag{6.9}$$

It would be very interesting to investigate also the possibility to have a $\hat{p}$-dependence in the potential $H_{\mathrm{dS}}(\hat{x}, \hat{p})$, but unfortunately this always leads to equations with order greater than three and is therefore beyond our generic treatment.

7. A simple reality proof for the spectrum of $p^2 + z^2(iz)^{2m+1}$

Considerable efforts have been made to prove the reality of the spectrum for the family of Hamiltonians $H_n = p^2 + \hat{z}^2(i\hat{z})^n$ for $n \geq 0$, see for instance [38, 39]. Unfortunately most of the proofs are rather cumbersome and not particularly transparent. The simplest way to establish the reality of the spectrum for a non-Hermitain Hamiltonian is to construct exactly the similarity transformation, which relates it to a Hermitian counterpart in the same similarity class. So far this could only be achieved for the case $n = 2$ in a remarkably simple manner [27]. Here we present a trivial argument, which establishes the reality for a subclass of the massive version of H_n with n being odd. Considering

$$\begin{aligned} H_m &= \frac{\Delta}{2}(\hat{p}^2 + \hat{x}^2) + gx^2(i\hat{x})^{2m-1}, \qquad \Delta, g \in \mathbb{R}, m \in \mathbb{N}, \\ &= \Delta a^\dagger a - \frac{ig(-1)^m}{2^{m+1/2}}\left(a + a^\dagger\right)^{2m+1} \end{aligned} \tag{7.1}$$

it is trivial to see that the metric operator $\hat{\eta} = e^{\frac{i\pi}{4}(aa - a^\dagger a^\dagger)}$ introduced in (5.5) transforms H_m adjointly into a Hermitian Hamiltonian

$$\begin{aligned} h_m &= \hat{\eta} H^m \hat{\eta}^{-1} = -\Delta a a^\dagger + \frac{g}{2^{m+1/2}}\left(a + a^\dagger\right)^{2m+1} = h_m^\dagger \\ &= -\frac{\Delta}{2}(\hat{p}^2 + \hat{x}^2) - g\hat{x}^{2m+1}. \end{aligned} \tag{7.2}$$

We can immediately apply the same argumentation to generalizations of the single site lattice Reggeon Hamiltonian. The non-Hermitian Hamiltonian

$$H_{\text{SSLR}}^m = \Delta a^\dagger a + \frac{ig}{2} a^\dagger \left(a + a^\dagger\right)^{2m+1} a \tag{7.3}$$

is transformed to the Hermitian Hamiltonian

$$h_{\text{SSLR}}^m = \hat{\eta} H_{\text{SSLR}}^m \hat{\eta}^{-1} = -\Delta a a^\dagger - \frac{g}{2^{m+1/2}} a^\dagger \left(a + a^\dagger\right)^{2m+1} a. \tag{7.4}$$

For the reasons mentioned in section 5 the operator $\hat{\eta}^2$ is not a proper positive-definite metric, but for the purpose of establishing the reality of the spectrum that is not important.

8. Conclusions

We have systematically constructed all exact solutions for the metric operator which is of exponential form with $\mathcal{PT}$-symmetric real and cubic argument and adjointly complex conjugates the most generic Hamiltonian of cubic order (2.1), (2.2). Our solutions are characterized by various constraints on the ten parameters in the model. Several of the SPH-models may be reduced to previously studied models, but some correspond to entirely new examples for SPH-models. We used the metric to construct the corresponding similarity transformation and its Hermitian counterparts.

We have demonstrated that exploiting the isomorphism between operator and Moyal products allows to convert the operator identities into manageable differential equations.

Even when no obvious exact solution exists, perturbation theory can be carried out on the level of the differential equation to almost any desired order. An important open question, which will be left for future investigation is concerning the convergence of the perturbative series. There are obvious limitations for the method as it works only well for potentials of polynomial form, as otherwise the differential equations will be of infinite order. In addition even for the differential equations of finite order not all solutions of all possible have been obtained as one has to make various assumptions. For instance the solution (4.18) would be missed for an ansatz for the metric of a purely exponential form. Therfore it would be very interesting to compare the method used in this manuscript with alternative techniques, such as the one proposed in [22].

The special cases considered, namely the single site lattice Reggeon model (5.2) as well as the sextic potential (6.3) are both not SPH within our framework. However, both of them may be understood as quasi exactly solvable models perturbed by some complex cubic potential.

There are some obvious generalizations of the presented analysis, such as for instance the study of generic quartic, quintic etc Hamiltonians.

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