$\mathcal{PT}$-symmetry and Integrability *

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Abstract

We briefly explain some simple arguments based on pseudo Hermiticity, supersymmetry and $\mathcal{PT}$-symmetry which explain the reality of the spectrum of some non-Hermitian Hamiltonians. Subsequently we employ $\mathcal{PT}$-symmetry as a guiding principle to construct deformations of some integrable systems, the Calogero-Moser-Sutherland model and the Korteweg deVries equation. Some properties of these models are discussed.

1 Introduction

Non-Hermitian Hamiltonians with complex eigenvalue spectrum have been studied almost since the formulation of quantum mechanics, most prominently as consistent descriptions of dissipative systems resulting for instance from channel coupling [1]. It is also known for a very long time that many interesting non-Hermitian Hamiltonians with *real* eigenvalue spectrum result naturally in various circumstances. For instance, it was argued more than thirty years ago that the lattice versions of Reggeon field theory [2]

$$\mathcal{H} = \sum\nolimits_{\vec{\imath}} \left[\Delta a_{\vec{\imath}}^{\dagger} a_{\vec{\imath}} + i g a_{\vec{\imath}}^{\dagger} (a_{\vec{\imath}} + a_{\vec{\imath}}^{\dagger}) a_{\vec{\imath}} + \tilde{g} \sum\nolimits_{\vec{\jmath}} (a_{\vec{\imath}+\vec{\jmath}}^{\dagger} - a_{\vec{\imath}}^{\dagger})(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right] \qquad (1)$$

with $a_{\vec{\imath}}^{\dagger}, a_{\vec{\imath}}$ being standard creation and annihilation operators and $\Delta, g, \tilde{g} \in \mathbb{R}$, possess a real eigenvalue spectrum[1] [3]. The reduction of the Hamiltonian in (1) to a single lattice site in zero transverse dimension [4] is very reminiscent of the so-called Swanson model [5], which results by replacing the interaction term with a simpler bilinear expression $g a^{\dagger} a^{\dagger} + \tilde{g} a a$. The latter model serves currently as a concrete popular solvable model to exemplify various general features related to the study of non-Hermitian Hamiltonians [5, 6, 7, 8, 9]. Affine Toda field theory with complex coupling constant is a very prominent class of

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[1] I am grateful to John Cardy for pointing this out.

field theoretical models, which are argued [10, 11] to be consistent despite their Hamiltonians being non-Hermitian. Besides the study of such explicit models related to non-Hermitian Hamiltonians, in particular their spectral properties, the question of how to formulate the corresponding quantum mechanical description consistently was first addressed in [12]. The useful insight of how to implement $\mathcal{PT}$-symmetry into this formulation has been obtained thereafter [13].

The current large interest in the subject of non-Hermitian Hamiltonian systems was initiated about nine years ago [14] by the surprising numerical observation that even the class of simple non-Hermitian Hamiltonians

$$\mathcal{H} = p^2 - g(iz)^N, \tag{2}$$

defined on a suitable domain, possesses a real positive and discrete eigenvalue spectrum for integers $N \geq 2$ with $g \in \mathbb{R}$. Supported by the numerous new results and insights (for some recent reviews see [15, 16, 17, 18]), which have been obtained since, the natural question arises of how to construct non-Hermitan Hamiltonians with real eigenvalue spectra in a more systematical way.

The question I would like to address in this talk is how this may be achieved, in particular by generalizing some integrable models.

2 Real spectra of non-Hermitian Hamiltonians

The activities in spectral theory usually focus on normal or self-adjoined operators in some Hilbert space. With regard to the remarks made in the introduction we shall first briefly review some arguments which may be used to explain the reality of the spectra of non-Hermitian Hamiltonians and thereafter employ them to construct new models, which depending on the argument used are guaranteed, or at least are likely, to have a real eigenvalue spectrum.

2.1 Pseudo-Hermiticity

Since a Hermitian operator, say $h = h^\dagger$, is guaranteed to have real eigenvalues, i.e. $h\phi = \varepsilon\phi$ with $\varepsilon \in \mathbb{R}$, one may trivially construct isospectral Hamiltonians by means of a similarity transformation $H = \eta^{-1} h \eta$, such that $H\Phi = \varepsilon\Phi$ with $\Phi = \eta^{-1}\phi$. When η is a Hermitian operator this implies that the conjugation of H is simply achieved by $H^\dagger = \eta^2 H \eta^{-2}$. Such type of Hamiltonians are denoted as pseudo Hermitian Hamiltonians [12, 19, 20, 21, 22]. One of the immediate virtues of the aforementioned relations is that η^2 can be used consistently as a metric operator.

Given a Hermitian Hamiltonian it is of course trivial to construct several isospectral non-Hermitian Hamiltonians in this manner simply by computing $\eta^{-1} h \eta \to H$ for some positive η. However, the interesting situations arise when given simple non-Hermitian Hamiltonians, such as for instance (1) and (2), possible together with the knowledge that they possess a positive real spectrum, and one tries to construct their Hermitian counterparts by seeking convenient

Hermitian operators η, such that $\eta H \eta^{-1} \rightarrow h = h^{\dagger}$. Unfortunately, this is only feasible in an exact manner in some very rare cases [23, 5, 6, 7, 8, 9] and mostly one has to rely on perturbation theory, see e.g. [24, 25, 26, 8, 27]. More awkward is the fact that when given exclusively the non-Hermitian Hamiltonian H, there might be several Hermitian Hamiltonians counterparts and the metric is therefore not even uniquely determined. One may select out a particular metric by specifying for instance at least one more observable [12] or the spectrum.

2.2 Supersymmetry

Another standard procedure, which produces isospectral Hamiltonians is to employ Darboux transformations or equivalently a supersymmetric quantum mechanical construction [28, 29]. For this one considers Hamiltonians $\mathcal{H}$, which can be decomposed into the form

$$\mathcal{H} = H_{+} \oplus H_{-} = Q\tilde{Q} \oplus \tilde{Q}Q \ . \tag{3}$$

As indicated in (3) one assumes that the two superpartner Hamiltonians $H_{\pm}$ factor into the two supercharges Q and $\tilde{Q}$, which intertwine the Hamiltonians $H_{\pm}$ as $QH_{-} = H_{+}Q$ and $\tilde{Q}H_{+} = H_{-}\tilde{Q}$. Evidently the two charges commute with the Hamiltonian $\mathcal{H}$, i.e. $[\mathcal{H}, Q] = [\mathcal{H}, \tilde{Q}] = 0$, and thus the $sl(1/1)$ algebra constitutes a symmetry of $\mathcal{H}$. As pointed out by various authors [30, 31, 32, 33, 34, 35, 36], one does not require the Hamiltonians $H_{\pm}$ to be Hermitian, such that we allow $H_{\pm}^{\dagger} \neq H_{\pm}$. The only constraints, which are natural to impose when one wishes to make contact with the pseudo-Hermitian treatment in the previous section, are that the individual factors of $H_{\pm}$ are conjugated as [34]

$$Q^{\dagger} = \eta_{-}^{2}\tilde{Q}\eta_{+}^{-2} \qquad \text{and} \qquad \tilde{Q}^{\dagger} = \eta_{+}^{2}Q\eta_{-}^{-2}, \tag{4}$$

where the operators $\eta_{\pm}$ are Hermitian $\eta_{\pm}^{\dagger} = \eta_{\pm}$. As an immediate consequence of (4), both Hamiltonians $H_{\pm}$ in (3) become pseudo-Hermitian and possess Hermitian counterparts $h_{\pm}^{\dagger} = h_{\pm}$

$$H_{\pm}^{\dagger} = \eta_{\pm}^{2}H_{\pm}\eta_{\pm}^{-2} \qquad \Leftrightarrow \qquad h_{\pm} = \eta_{\pm}H_{\pm}\eta_{\pm}^{-1}. \tag{5}$$

By construction all four Hamiltonians $h_{\pm}$, $H_{\pm}$ are therefore isospectral

$$H_{\pm}\Phi^{\pm} = \varepsilon\Phi^{\pm} \qquad \text{and} \qquad h_{\pm}\phi^{\pm} = \varepsilon\phi^{\pm} \tag{6}$$

and their corresponding wavefunctions are intimately related

$$\Phi^{+} = Q\Phi^{-} = \eta_{+}^{-1}\phi_{+} \quad \text{and} \quad \Phi^{-} = \tilde{Q}\Phi^{+} = \eta_{-}^{-1}\phi_{-}. \tag{7}$$

One may now characterize four qualitatively different cases depending on the properties of the Hermitian operators $\eta_{\pm}$ in (7), namely i) for generic $\eta_{\pm}$ we have isospectral quartets, ii) for generic η_{+} and $\eta_{-} = \mathbb{I}$ and iii) for generic

η_- and $\eta_+ = \mathbb{I}$ we find isospectral triplets and finally $iv)$ for $\eta_\pm = \mathbb{I}$ we have isospectral doublets. The interesting cases $ii)$ and $iii)$, which an contain Hermitian Hamiltonian, have been considered in [31].

Next one needs to specify the explicit representation for the supercharges in terms of the superpotential $W(x)$. Setting the parameter $\hbar^2/2m = 1$, the simplest choices are differential operators of first order

$$Q = \frac{d}{dx} + W \qquad \text{and} \qquad \tilde{Q} = -\frac{d}{dx} + W \tag{8}$$

such that the two superpartner Hamiltonians may be written as

$$H_\pm = -\Delta + W^2 \pm W' = -\Delta + V_\pm. \tag{9}$$

Alternative choices with higher order differential operators are discussed for instance in [37]. Assuming further that H_- possesses a discrete spectrum $H_-\Phi_n^- = \varepsilon_n \Phi_n^-$, one may adjust the energy scale such that $H_-\Phi_m^- = 0$ for some chosen m. In order to single out this groundstate wavefunction we denote it as $\psi_m := \Phi_m^- = c\exp[-\int W_m dx]$, $c \in \mathbb{C}$. Consequently the superpartner potentials may be expressed in terms of the groundstate wavefunctions and acquire the forms

$$W_m = -\frac{\psi_m'}{\psi_m}, \qquad V_-^m = \frac{\psi_m''}{\psi_m}, \qquad V_+^m = 2\left(\frac{\psi_m'}{\psi_m}\right)^2 - \frac{\psi_m''}{\psi_m}. \tag{10}$$

Therefore the Hamiltonians

$$H_\pm^m = -\Delta + V_\pm^m + E_m = -\Delta + W_m^2 \pm W_m' + E_m \tag{11}$$

are isospectral

$$H_\pm^m \Phi_n^\pm = E_n \Phi_n^\pm \quad \text{for } n > m. \tag{12}$$

In order to disentangle the Hermitian from the non-Hermitian case, we separate the superpotential into its real and imaginary part $W_m = w_m + i\hat{w}_m$ with $w_m = w_m^\dagger$, $\hat{w}_m = \hat{w}_m^\dagger$ and likewise for the groundstate energy $E_m = \varepsilon_m + i\hat{\varepsilon}_m$. With these notations we can re-write (11) as

$$H_\pm^m = -\Delta + w_m - \hat{w}_m \pm w_m' + \varepsilon_m + i(2w_m\hat{w}_m \pm \hat{w}_m' + \hat{\varepsilon}_m) \tag{13}$$

Clearly we encounter the situation $ii)$ or $iii)$ when

$$w_m = (\mp\hat{w}_m' - \hat{\varepsilon}_m)/2\hat{w}_m \qquad \text{or} \qquad \hat{w}_m = 0, \tag{14}$$

respectively.

When given a Hamiltonian, irrespective of being Hermitian or non-Hermitian, and at least one wavefunction, the exploitation of supersymmetry is a very constructive procedure to obtain isospectral Hamiltonians, which could also be Hermitian or non-Hermitian.

2.3 $\mathcal{PT}$-symmetry

A further very simple and transparent way to explain the reality of the spectrum of some non-Hermitian Hamiltonians results when we encounter unbroken $\mathcal{PT}$-symmetry, which in the recent context was first pointed out in [13]. It means that both the Hamiltonian *and* the wavefunction remain invariant under a simultaneous parity transformation $\mathcal{P} : x \to -x$ and time reversal $\mathcal{T} : t \to -t$, that is we require

$$[H, \mathcal{PT}] = 0 \quad \text{and} \quad \mathcal{PT}\Phi = \Phi, \tag{15}$$

where Φ is a square integrable eigenfunction on some domain of H. It is crucial to note that the $\mathcal{PT}$-operator is an anti-linear operator, i.e. it acts as $\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^* \mathcal{PT}\Phi + \mu^* \mathcal{PT}\Psi$ with $\lambda, \mu \in \mathbb{C}$ and Φ, Ψ being some eigenfunctions. An easy way to convince oneself of this property is to consider the standard canonical commutation relation $[x, p] = i$. Since $\mathcal{PT}: x \to -x, p \to p$, we require $\mathcal{PT}: i \to -i$ to keep this relation invariant. Utilizing now *both* relations in (15) and the anti-linear nature of the $\mathcal{PT}$-operator, a very simple argument leads to the reality of the spectrum

$$\varepsilon\Phi = H\Phi = H\mathcal{PT}\Phi = \mathcal{PT}H\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi. \tag{16}$$

Whereas the first relation in (15) is usually trivial to check, the second is in general difficult to access as one rarely knows all the wavefunctions. In case it does not hold one speaks of a broken $\mathcal{PT}$-symmetry and the eigenvalues come in complex conjugate pairs. All arguments in this subsection were essentially already known to Wigner in 1960 [38] relating to anti-linear operators in a completely generic form. Noting that the $\mathcal{PT}$-operator is an example of such an operator these ideas have been revitalized in a modified form and developed further in the recent context of the study of non-Hermitian Hamiltonians [13].

3 $\mathcal{PT}$-symmetry as a guiding principle to construct new models

If we now wish to construct new models with real eigenvalue spectra, we may in principle use any of the previous arguments. Clearly the exploitation of $\mathcal{PT}$-symmetry on the level of the Hamiltonian is the most direct and transparent way, as one can just read of this property immediately. Thereafter one can write down some new $\mathcal{PT}$-symmetric Hamiltonians by means of simple deformations, i.e. replacing for instance the potential $V(x)$ by $V(x)f(ix)$, $V(x)f(ixp)$, $V(x) + f(ix)$ or $V(x) + f(ixp)$, etc. with f being some arbitrary function. Clearly the Hamiltonians in (1) and (2) are of this type. Of course these new models are not guaranteed to have real spectra as the second property in (15) might be spoiled. Nonetheless, they have a high chance to describe non-dissipative physics and are potentially interesting.

3.1 $\mathcal{PT}$-symmetric extensions for multi-particle systems

Basu-Mallick and Kundu [39] were the first to write down some non-Hermitian extensions for some integrable many-particle systems, i.e. the rational A_ℓ-Calogero models [40]

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2}\sum\nolimits_i q_i^2 + \frac{g^2}{2}\sum\nolimits_{i\neq k}\frac{1}{(q_i - q_k)^2} + i\tilde{g}\sum\nolimits_{i\neq k}\frac{1}{(q_i - q_k)}p_i \qquad (17)$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$. There are some immediate questions one may pose [41] with regard to the properties of $\mathcal{H}_{BK}$: i) How can one formulate $\mathcal{H}_{BK}$ independently of the representation for the roots? ii) Can one generalize $\mathcal{H}_{BK}$ to other potentials apart from the rational one? iii) Can one generalize $\mathcal{H}_{BK}$ to other algebras or more precisely Coxeter groups? iv) Is it possible to include more coupling constants? and in particular v) Are the extensions still integrable? It turns out that the answer to all these questions become all quite simple when one realises that (17) corresponds in fact to the standard Calogero model simply shifted in the momenta. This means the similarity transformation η is simply the translation operator in p-space.

In order to see this and to answer the above questions we ignore the confining term in (17) by taking $\omega = 0$ and re-write the Hamiltonian as

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum\nolimits_{\alpha\in\Delta} g_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p, \qquad (18)$$

where Δ is now any root system invariant under Coxeter transformations, $\mu = 1/2\sum_{\alpha\in\Delta}\tilde{g}_\alpha f(\alpha\cdot q)\alpha$, $f(x) = 1/x$ and $V(x) = f^2(x)$. We have also introduced coupling constants $g_\alpha, \tilde{g}_\alpha$ for each individual root. The Hamiltonians $\mathcal{H}_\mu$ are meaningful for any representation of the roots and all Coxeter groups. For a specific choice of the representation for the roots, namely $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell$ with $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$ and the Coxeter group, i.e. A_ℓ, we recover the expression in (18). To establish the integrability of these models it is crucial to note the following not obvious property

$$\mu^2 = \alpha_s^2\tilde{g}_s^2\sum\nolimits_{\alpha\in\Delta_s} V(\alpha\cdot q) + \alpha_l^2\tilde{g}_l^2\sum\nolimits_{\alpha\in\Delta_l} V(\alpha\cdot q) \qquad (19)$$

where Δ_s, Δ_l denotes the short and long roots, respectively. For the details of the proof of this identity we refer to [41]. As a consequence (19), we may re-express $\mathcal{H}_\mu$ in form of the usual Calogero Hamiltonian with shifted momenta together with some redefinitions of the coupling constants

$$\mathcal{H}_\mu = \frac{1}{2}(p + i\mu)^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q), \quad \hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2\tilde{g}_s^2 & \text{for } \alpha\in\Delta_s \\ g_l^2 + \alpha_l^2\tilde{g}_l^2 & \text{for } \alpha\in\Delta_l \end{cases}. \qquad (20)$$

Therefore, upon the redefinition of the coupling constant, we may obtain $\mathcal{H}_\mu$ by a similarity transformation as $\mathcal{H}_\mu = \eta^{-1}h_{\mathrm{Cal}}\eta$ with $\eta = e^{-x\cdot\mu}$. The results of section 2.1 apply therefore and one may construct for instance the corresponding wavefunctions by $\Phi_\mu = \eta^{-1}\phi_{\text{Cal}}$. Similarly one can establish integrability with the help of a Lax pair with a shifted momentum. One may verify that

$$L = (p+i\mu)\cdot H + i\sum_{\alpha\in\Delta}\hat{g}_\alpha f(\alpha\cdot q)E_\alpha \quad \text{and} \quad M = m\cdot H + i\sum_{\alpha\in\Delta}\hat{g}_\alpha f'(\alpha\cdot q)E_\alpha \tag{21}$$

fulfill the Lax equation $\dot{L} = [L, M]$, upon the validity of the classical equation of motion resulting from (18), where the Lie algebraic commutation relations

$$[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha^i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha\cdot H, \quad [E_\alpha, E_\beta] = \varepsilon_{\alpha,\beta}E_{\alpha+\beta}.$$

are taken to be in the Cartan-Weyl basis, i.e. they are normalized as $\text{tr}(H_i H_j) = \delta_{ij}$, $\text{tr}(E_\alpha E_{-\alpha}) = 1$. The vector m can be expressed in terms of the structure constant $\varepsilon_{\alpha,\beta}$ and the potential in the usual fashion. We note that the Lax equation is $\mathcal{PT}$-symmetric as $\mathcal{PT}$:$L \to L, M \to -M$. Naturally the conserved charges $I_k = tr(L^k)/2$, notably the Hamiltonian I_2, have the same property.

Having established the integrability of the Calogero models one may address the question ii) and try to extend these considerations to other potentials. Allowing now $f(x) = 1/\sinh x$ and $f(x) = 1/\,\text{sn}\, x$, we obtain the hyperbolic and elliptic case with $V(x) = f^2(x)$. The integrability is guaranteed by means of the same Lax pairs (21). However, when expanding the square in (20) the resulting Hamiltonian is not quite of the form (18)

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p - \frac{1}{2}\mu^2, \tag{22}$$

because the identity (19) does not hold for the other potentials. This means the Hamiltonians in (22) constitute non-Hermitian *integrable* extensions for Calogero-Moser-Sutherland (CMS)-models for all crystallographic Coxeter groups, including, besides the rational, also trigonometric, hyperbolic and elliptic potentials. Dropping the last term would break the integrability for the non-rational potentials.

3.2 $\mathcal{PT}$-symmetric deformations of the Korteweg deVries equation

An even more popular integrable model than the CMS-model is one having the Korteweg-de Vries (KdV) equation [42] as equation of motion

$$u_t + uu_x + u_{xxx} = 0. \tag{23}$$

This equation is known to remain invariant under $x \to -x, t \to -t, u \to u$, i.e. it is $\mathcal{PT}$-symmetric. By the same recipe outlined above we may then carry out the following deformation $u_x \to -i(iu_x)^\varepsilon$ with $\varepsilon \in \mathbb{R}$, which was originally performed for the second term in [43] and for the third term in [44], leading to the equations

$$u_t - iu(iu_x)^\varepsilon + \ u_{xxx} = 0 \qquad\qquad \varepsilon\in\mathbb{R} \tag{24}$$

and

$$u_t + uu_x + i\varepsilon(\varepsilon-1)(iu_x)^{\varepsilon-2}\ u_{xx}^2 + \varepsilon(iu_x)^{\varepsilon-1}u_{xxx} = 0, \tag{25}$$

respectively. For model in (24) one can establish the following properties: the Galilean symmetry is broken, the model possess two conserved quantities in terms of infinite sums and exhibits steady state solutions. However, it is unclear how $\mathcal{PT}$-symmetric can be utilized further. Instead (25), despite being more complicated, has some simpler properties: it is Galilean invariant, possess three simple conserved charges, exhibits steady state solutions, $\mathcal{PT}$-symmetry can be utilized to explain the reality of the energy and it allows for a Hamiltonian formulation with non-Hermitian Hamiltonian density

$$\mathcal{H} = u^3 - \frac{1}{1+\varepsilon}(iu_x)^{\varepsilon+1} \qquad \varepsilon \in \mathbb{R}. \tag{26}$$

Analogues of various different types of solutions of the KdV-equation have been studied in [43, 44]. No soliton solutions have been found and it seems unlikely that the models are integrable.

4 Conclusions

We have demonstrated that $\mathcal{PT}$-symmetry serves as a very useful guiding principle to construct new interesting models, some of which even remain integrable. Being closely related to integrable models, these new models have appealing features and deserve further investigation. Naturally one may also reverse the setting and employ methods, which have been developed in the context of integrable to address questions which arise in the study of non-Hermitian Hamiltonians. For instance, one [45] may employ Bethe ansatz techniques to establish the reality of the spectrum for Hamiltonians of the type (2).

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