City Research Online

City, University of London Institutional Repository

This is the accepted version of the paper.

This version of the publication may differ from the final published version.

Vector Nonlinear Schrödinger Equation on the half-line

V. Caudrelier and Q. C. Zhang

*Centre for Mathematical Science, City University London,
Northampton Square,London EC1V 0HB, UK.*

Abstract

We investigate the Manakov model or, more generally, the vector nonlinear Schrödinger equation on the half-line. Using a Bäcklund transformation method, two classes of integrable boundary conditions are derived: mixed Neumann/Dirichlet and Robin boundary conditions. Integrability is shown by constructing a generating function for the conserved quantities. We apply a nonlinear mirror image technique to construct the inverse scattering method with these boundary conditions. The important feature in the reconstruction formula for the fields is the symmetry property of the scattering data emerging from the presence of the boundary. Particular attention is paid to the discrete spectrum. An interesting phenomenon of transmission between the components of a vector soliton interacting with the boundary is demonstrated. This is specific to the vector nature of the model and is absent in the scalar case. For one-soliton solutions, we show that the boundary can be used to make certain components of the incoming soliton vanishingly small. This is reminiscent of the phenomenon of light polarization by reflection.

1 Introduction

In this paper, we consider the vector nonlinear Schrödinger equation (VNLS) on the half-line. Historically, the model on the line for a vector field with two components was introduced and solved by the inverse scattering method by Manakov [1]. The VNLS on the half-line can be seen as a generalization of the famous (scalar) one-dimensional nonlinear Schrödinger equation (NLS) along two directions: adding internal degrees of freedom and adding a boundary. It is worth mentioning that each direction is not just a trivial generalization exercise. Indeed, such considerations have been at the origin of major breakthroughs in the history of integrable systems. For instance, when the original Lieb-Liniger model [2] for spinless particles with contact interactions was generalized by Yang [3] to particles with spin, this gave rise to the now famous rational solution of the Yang-Baxter equation. Similarly, when studying integrable systems with boundaries, Sklyanin unraveled new algebraic structures controlled by the so-called reflection equation [4]. Concerning the NLS equation, these questions have been studied rather well at the quantum level. Vector generalizations on the line [5, 6] and on the half-line [7] have been investigated essentially from an algebraic point of view. At the classical level, a very general framework for dealing with NLS equations related to symmetric spaces has been presented in [8]. Although the VNLS equation is only a particular case of this general theory, it is still the object of active research: for instance, the full understanding of the factorization property of vector soliton collisions has only been achieved relatively recently (see for instance [9] and references therein). At the classical level again, the general question of generating boundary conditions of arbitrary order compatible with a general multicomponent nonlinear Schrödinger equation (expressed in terms of Jordan triple algebras) has been treated in [10]. However, to the best of the authors' knowledge (and surprise), the problem of formulating the inverse scattering method with integrable boundary conditions to tackle for instance the question of interactions of vector solitons with a boundary has not been addressed yet. This is in sharp contrast with the scalar case where a rich literature has appeared over the years since the early attempts of adapting the inverse scattering method to initial-boundary value problems [11] up to the very general method developed by Fokas (see e.g. the textbook [12]), in passing by the Bäcklund transformation method found in [13, 14, 15]. Recently, another idea [16] was revived in [17] and used to investigate soliton interactions in the presence of an integrable boundary by means of a (nonlinear) mirror image technique. As far as integrable boundary conditions are concerned, all these techniques are related and it seems that the central object is a special Bäcklund matrix that allows for the implementation of the mirror image technique. In the context of Fokas' method, this matrix evaluated at the location of the boundary is the matrix involved in the so-called linearizable boundary conditions.

The goal of this paper is to introduce and study the VNLS with an integrable boundary in order to lay the foundations for a deeper understanding of integrable vector field interactions with a boundary. Section 2 contains known results for the inverse scattering method of the VNLS on the line. This allows us to introduce our notations as well as to collect useful results for the rest of the paper. Section 3 represents the core of this paper. We first derive two classes of integrable boundary conditions using the Bäcklund matrix approach. Integrability is established by constructing an explicit generating function for the conserved quantities for the VNLS on the half-line. This is then used

to implement the mirror image technique that allows us to use the inverse scattering method results on the line to obtain the reconstruction formulae for the fields on the half-line. In section 4, we use our general results to investigate the behaviour of one soliton bouncing off a boundary. An interesting phenomenon of transmission between different modes of polarization is demonstrated. It is due to the interaction between the soliton polarization and what we can call the natural boundary polarization. This will be made more precise in the text. Our conclusion are gathered in section 5.

The vector nonlinear Schrödinger equation (VNLS) on the half line is defined as the following initial-boundary value problem for the n-component vector field $R(x,t)$

$$i\frac{\partial R}{\partial t} + \frac{\partial^2 R}{\partial x^2} - 2\lambda R R^\dagger R = 0\,, \qquad x,t \in [0,\infty)\,, \tag{1.1}$$

$$R(x,0) = R_0(x)\,, \quad R(0,t) = g_0(t)\,, \quad R_x(0,t) = g_1(t)\,, \tag{1.2}$$

where $R^\dagger(x,t)$ is the transpose conjugate of $R(x,t)$ and λ is the (real) coupling constant which can be normalised to $\lambda = \pm 1$. We assume that the functions R_0, g_0 and g_1 live in appropriate functional spaces so as to ensure that the calculations in this paper are meaningful (for details in the scalar case see for instance the "rigorous considerations" section in [18]). In particular, they are decaying functions at infinity. A powerful method has been developed by Fokas and various co-workers over the last two decades to tackle integrable nonlinear equations on the half-line (see e.g. [12] and references therein). The key ingredient is the simultaneous treatment of the two equations appearing in the Lax pair formulation of the nonlinear equation. We follow this principle in this paper.

2 Vector NLS on the line

In this section, we review the inverse scattering method for VNLS on the line. We follow chapter four of the textbook [19] and collect the main results and notations needed in the rest of the paper.

2.1 Lax pair

Given $R(x,t)$, define $Q(x,t)$ as the following $(n+1) \times (n+1)$ matrix-valued field

$$Q(x,t) = \begin{pmatrix} 0 & R(x,t) \\ \lambda R^\dagger(x,t) & 0 \end{pmatrix}\,. \tag{2.1}$$

Equation (1.1) is the compatibility condition of the following linear problem for the matrix-valued field $\Phi(x,t,k)$[1]

$$\Phi_x + ik[\Sigma_3, \Phi] = Q\,\Phi\,, \tag{2.2}$$

$$\Phi_t + 2ik^2[\Sigma_3, \Phi] = G\,\Phi\,, \tag{2.3}$$

where

$$\Sigma_3 = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad G = 2kQ - iQ_x\Sigma_3 - iQ^2\Sigma_3\,, \tag{2.4}$$

---

[1]From now on, we drop the x,t (and k) dependence for conciseness unless there is ambiguity.

I_n being the $n \times n$ identity matrix. Using standard arguments, one can show that $\det \Phi(x, t, k) = f(k)$ for all $x, t \in \mathbb{R}$, and all values of $k \in \mathbb{C}$ for which Φ is defined, where f can be determined for instance by fixing the asymptotic behaviour of Φ as $x \to \pm\infty$. Also, if Φ and Ψ are two solutions of (2.2,2.3) then

$$\Phi(x, t, k) = \Psi(x, t, k) \, e^{-i\phi(x,t,k)\Sigma_3} \, T(k) \, e^{i\phi(x,t,k)\Sigma_3} \,, \tag{2.5}$$

where T depends on k only and $\phi(x, t, k) = kx + 2k^2 t$. Finally, from

$$WQW^{-1} = -Q^\dagger \,, \tag{2.6}$$

where

$$W = \begin{pmatrix} -\lambda I_n & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{2.7}$$

one obtains that $W\Phi^\dagger(x, t, k^*)W^{-1}$ and $\Phi^{-1}(x, t, k)$ satisfy the same equations and are thus related as in (2.5).

2.2 Direct scattering problem

We define two fundamental (or Jost) solutions $M_\pm$ of (2.2,2.3) satisfying

$$\lim_{x \to \pm\infty} e^{i\phi(x,t,k)\Sigma_3} M_\pm(x, t, k) e^{-i\phi(x,t,k)\Sigma_3} = I_{n+1} \quad, \quad k \in \mathbb{R} \,. \tag{2.8}$$

They have the following Volterra integral equation representations

$$M_\pm(x, t, k) = I_{n+1} + \int_{\pm\infty}^{x} e^{-ik(x-y)\Sigma_3} Q(y, t) M_\pm(y, t, k) e^{ik(x-y)\Sigma_3} \, dy \,. \tag{2.9}$$

Their analyticity and boundedness properties then follow as in the scalar case. Splitting the matrices into left and right "column vectors" [2], one can indicate the domain of the complex k plane where they are bounded and analytic using the following notations

$$M_+(x, t, k) = (\overline{N}(x, t, k), N(x, t, k)), \quad M_-(x, t, k) = (M(x, t, k), \overline{M}(x, t, k)), \tag{2.10}$$

where $\overline{N}$, $\overline{M}$ (resp. N, M) are bounded analytic in the lower (resp. upper) half plane as functions of k. From (2.8), one finds that

$$\det M_\pm(x, t, k) = 1 \,. \tag{2.11}$$

Having defined the fundamental solutions, one can proceed with the definition of the scattering data. Recalling (2.5), we define the matrix $S(k)$ for $k \in \mathbb{R}$ as

$$e^{i\phi(x,t,k)\Sigma_3} M_-(x, t, k) e^{-i\phi(x,t,k)\Sigma_3} = e^{i\phi(x,t,k)\Sigma_3} M_+(x, t, k) e^{-i\phi(x,t,k)\Sigma_3} S(k) \,. \tag{2.12}$$

Several important properties follow from this definition. First, $\det S(k) = 1$. Then, $S(k)$ has the following integral representation

$$S(k) = I_{n+1} + \int_{-\infty}^{\infty} e^{iky\Sigma_3} Q(y, 0) M_-(y, 0, k) e^{-iky\Sigma_3} \, dy \,. \tag{2.13}$$

---

[2]Here, the left column vector is made of the first n left columns and the right one is made of the remaining column.

Splitting $S(k)$ into block matrices of natural sizes[3]

$$S(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \tag{2.14}$$

one gets that $a(k)$ can be analytically continued to $\operatorname{Im} k > 0$ while $\bar{a}(k)$ can be analytically continued to $\operatorname{Im} k < 0$. It is known that $\bar{a}(k^*) = (\det a(k))^*$. In the focusing case ($\lambda = -1$), the usual assumption is that $\det a(k)$ has a finite number of simple zeros located in the upper half-plane off the real axis. Denote them k_j, $\operatorname{Im} k_j > 0$, $j = 1, \ldots, J$. Consequently, $\bar{a}(k)$ has the same number J of simple zeros k_j^*, $j = 1, \ldots, J$ in the lower half-plane. These zeros play an important role in the inverse scattering problem described below. In particular, the residues of $b(k)a^{-1}(k)$ at $k = k_j$, called norming constants, are used to reconstruct the solitonic part of the vector field R.

2.3 Inverse scattering problem

Let us rewrite the relations (2.12) as

$$M(x,t,k)a^{-1}(k) = \overline{N}(x,t,k) + e^{2i\phi(x,t,k)}N(x,t,k)\rho(k), \tag{2.15}$$
$$\overline{M}(x,t,k)\bar{a}^{-1}(k) = e^{-2i\phi(x,t,k)}\overline{N}(x,t,k)\bar{\rho}(k) + N(x,t,k), \tag{2.16}$$

where $\rho(k) = b(k)a^{-1}(k)$ and $\bar{\rho}(k) = \bar{b}(k)\bar{a}^{-1}(k)$. Next, apply the Cauchy projector P^- (resp. P^+) to the first (resp. second) equality where

$$P^{\pm}(f)(k) = \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{f(\omega)}{\omega - (k \pm i0)}\,d\omega. \tag{2.17}$$

Using standard contour integration techniques and the asymptotic behaviour in k that can be obtained using integration by parts on the integral representations (2.9) and (2.13), one obtains

$$\overline{N}(x,t,k) = \begin{pmatrix} I_n \\ 0 \end{pmatrix} + \sum_{j=1}^{J}\frac{e^{2ik_jx+4ik_j^2t}}{k - k_j}N(x,t,k_j)\,C_j$$
$$+ \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{e^{2i\omega x+4i\omega^2 t}N(x,t,\omega)\rho(\omega)}{\omega - (k - i0)}\,d\omega, \tag{2.18}$$

$$N(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{J}\frac{e^{-2ik_j^*x-4ik_j^{*2}t}}{k - k_j^*}\overline{N}(x,t,k_j^*)\,\overline{C}_j$$
$$- \frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{e^{-2i\omega x-4i\omega^2 t}\overline{N}(x,t,\omega)\bar{\rho}(\omega)}{\omega - (k + i0)}\,d\omega, \tag{2.19}$$

where

$$C_j = Res(b(k)a^{-1}(k), k_j) \quad \text{and} \quad \overline{C}_j = Res(\bar{b}(k)\bar{a}^{-1}(k), k_j^*). \tag{2.20}$$

Note that (2.6) implies that $\overline{C}_j = \lambda C_j^{\dagger}$. Evaluating the first equation at $k = k_j^*$ and the second at $k = k_j$, one obtains a closed system of linear algebraic integral equations that

---

[3]For instance, a is $n \times n$ and $\bar{a}$ is a scalar.

can be solved in principle[4]. Below, we will see that this system of equations allows for closed-form formulas in the purely solitonic case *i.e.* when one sets $\rho(k) = 0 = \bar{\rho}(k)$ for $k \in \mathbb{R}$.

In the scalar case, the norming constants γ_ℓ are defined as the proportionality constants between the two column vector forming a Jost solution when the spectral parameter equals to a zero k_ℓ

$$M(x, t, k_\ell) = e^{2i\phi(x,t,k_\ell)} N(x, t, k_\ell) \gamma_\ell. \tag{2.21}$$

When working in the class of exponentially fast decreasing functions, there is a nice characterization of γ_ℓ in terms of $b(k)$ which can be analytically continued up to the strip $\{k \in \mathbb{C}; 0 \leq \operatorname{Im} k \leq K\}$ where K controls the decrease of the functions. With $K \geq \max\{\operatorname{Im} k_j; j = 1, \ldots, J\}$, we then have $\gamma_\ell = b(k_\ell)$. In the vector case, the norming constants can be defined as the vectors describing the linear dependence of the column of the Jost solution at the zeros k_j

$$M(x, t, k_j)\Gamma_j = e^{2i\phi(x,t,k_j)} N(x, t, k_j) C_j, \tag{2.22}$$

where Γ_j is an $n \times n$ matrix and C_j a row vector. Again, in the class of exponentially fast decreasing functions one has, without loss of generality,

$$\Gamma_j = A(k_j)^t \quad \text{and} \quad C_j = \frac{1}{\alpha'(k_j)} b(k_j) A(k_j)^t, \tag{2.23}$$

where $\alpha'(k) = \frac{d \det a(k)}{dk}$ and $A(k)^t$ is the transpose of the cofactor matrix of $a(k)$. Although not the most general, we will work with this convenient characterization to derive in particular proposition (3.2) below. We stress that the results would still be valid in a more general case, but this is enough for our purposes of studying soliton solutions.

Once N and $\overline{N}$ are known, one obtains $R(x, t)$ by the formula

$$R(x, t) = 2i\lambda \sum_{j=1}^{J} e^{-2ik_j^* x - 4ik_j^{*2} t} \overline{N}^{(up)}(x, t, k_j^*) C_j^\dagger + \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2i\omega x - 4i\omega^2 t} \overline{N}^{(up)}(x, t, \omega) \bar{\rho}(\omega) \, d\omega, \tag{2.24}$$

where $\overline{N}^{(up)}$ is the $n \times n$ upper block of $\overline{N}$. This is obtained by comparing the $O(1/k)$ term in the expansion obtained from (2.9) by integration by parts and the $O(1/k)$ term obtained from (2.19).

In the pure soliton case, (2.18, 2.19) yield the following coupled equations

$$\overline{N}(x, t, k_\ell^*) - \lambda \sum_{j,m=1}^{J} \frac{e^{2i(k_j - k_m^*)x + 4i(k_j^2 - k_m^{*2})t}}{(k_\ell^* - k_j)(k_j - k_m^*)} N(x, t, k_m^*) C_m^\dagger C_j. = I_n \quad, \quad \ell = 1, \ldots, J. \tag{2.25}$$

Introducing the matrix $\mu(x, t)$ whose entries are the following $n \times n$ matrices

$$\mu_{m\ell}(x, t) = \delta_{m\ell} I_n - \lambda \sum_{j=1}^{J} \frac{e^{2i(k_j - k_m^*)x + 4i(k_j^2 - k_m^{*2})t}}{(k_\ell^* - k_j)(k_j - k_m^*)} C_m^\dagger C_j, \tag{2.26}$$

---

[4]That this is the case is usually established by using the equivalent formulation in terms of a Riemann-Hilbert problem.

and the notation $\overline{N}(x,t,k_\ell^*) = \overline{N}_\ell$, equations (2.25) can be compactly rewritten as

$$\left(\overline{N}_1 \, \overline{N}_2 \, \ldots \, \overline{N}_J\right) \mu(x,t) = \left(I_n \, I_n \ldots I_n\right) . \tag{2.27}$$

One then obtains

$$R(x,t) = 2i\lambda \left(I_n \, I_n \ldots I_n\right) \mu^{-1}(x,t) \begin{pmatrix} C_1^\dagger e^{-2ik_1^* x - 4ik_1^{*2} t} \\ \vdots \\ C_J^\dagger e^{-2ik_J^* x - 4ik_J^{*2} t} \end{pmatrix} . \tag{2.28}$$

This is a central formula to understand the behaviour of solitons. In the next section, we show how to modify this formula to take into account the presence of the boundary. This is then used in Section 4 for numerical simulations.

3 Vector NLS on the half-line

Although the most general method for tackling initial boundary value problem exactly is the one developed by Fokas and co-workers, since we restrict our attention to a family of integrable boundary conditions (the vector generalisation of the Robin boundary conditions), we employ the mirror image technique as used in [17]. These two approaches give equivalent results for this class of boundary conditions which would be called linearizable in the terminology used by Fokas. The advantage of the mirror image technique is that it allows one to use the results on the line with minor (but important) modifications to get explicit solutions. One inconvenience though is that one has to prove that the boundary conditions are indeed satisfied by the field. This is because one uses the framework presented in the previous section for a modified field $P(x,t,k)$ instead of $Q(x,t)$ whose structure is chosen a priori. We note that, to the authors' knowledge, such a check has only been performed explicitly in [16] and only in the case of Dirichlet boundary conditions. Below, we present a similar proof for the mixed Neumann/Dirichlet boundary conditions in the vector case. We also show that the class of boundary conditions we consider is integrable in the sense that it allows for an infinite number of conserved quantities. This is done by exhibiting a generating function for such conserved quantities.

3.1 Deriving integrable boundary conditions

In order to start with an appropriate form for $P(x,t,k)$, we use the Bäcklund transformation method presented in [13] and further developed in [14, 15] for the scalar nonlinear Schrödinger equation to treat a class of integrable boundary conditions. Contrary to the scalar case, in the vector case it is important to study both the x-part and the t-part of the equations defining the Bäcklund transformation to completely characterize integrable boundary conditions. This is done below. This is all the more important as the t-part of the Bäcklund method provides the link between this method and Fokas' approach in the case of linearizable boundary conditions.

We introduce a Bäcklund matrix $L(x,t,k)$ relating the auxiliary problem (2.2,2.3) for Φ to the same auxiliary problem for $\widetilde{\Phi}$, with the potential Q replaced by a new potential $\widetilde{Q}$, by the equation

$$\widetilde{\Phi}(x,t,k) = L(x,t,k)\Phi(x,t,k) . \tag{3.1}$$

It is well-known that L, also known as a gauge transformation of the auxiliary problem, satisfies the following equations

$$L_x + ik[\Sigma_3, L] = \widetilde{Q}\, L - LQ\,, \tag{3.2}$$

$$L_t + 2ik^2[\Sigma_3, L] = \widetilde{G}\, L - LG\,, \tag{3.3}$$

and preserves the compatibility condition (or zero curvature condition). We look for a solution in the form

$$L(x, t, k) = kI_{n+1} + A(x, t)\,, \tag{3.4}$$

under the symmetry constraint $\widetilde{Q}(x, t) = Q(-x, t)$. Writing A in natural block form

$$\begin{pmatrix} A_1(x, t) & A_2(x, t) \\ A_3(x, t) & A_4(x, t) \end{pmatrix}\,, \tag{3.5}$$

and inserting in (3.2) yields the equations

$$2iA_2(x, t) = R(-x, t) - R(x, t)\,, \tag{3.6a}$$

$$-2iA_3(x, t) = \lambda\left(R^\dagger(-x, t) - R^\dagger(x, t)\right)\,, \tag{3.6b}$$

and

$$A_{1x}(x, t) = R(-x, t)A_3(x, t) - \lambda A_2(x, t)R^\dagger(x, t)\,, \tag{3.7a}$$

$$A_{2x}(x, t) = R(-x, t)A_4(x, t) - A_1(x, t)R(x, t)\,, \tag{3.7b}$$

$$A_{3x}(x, t) = \lambda\left[R^\dagger(-x, t)A_1(x, t) - A_4(x, t)R^\dagger(x, t)\right]\,, \tag{3.7c}$$

$$A_{4x}(x, t) = \lambda R^\dagger(-x, t)A_2(x, t) - A_3(x, t)R(x, t)\,. \tag{3.7d}$$

Note that

$$A_3(x, t) = \lambda A_2^\dagger(x, t)\,. \tag{3.8}$$

Combining (3.6) and (3.7b,3.7c) at $x = 0$ yields the following boundary conditions

$$R_x(0, t) = -\,i(A_4(0, t)I_n - A_1(0, t))R(0, t)\,, \tag{3.9a}$$

$$R_x^\dagger(0, t) = iR^\dagger(0, t)(A_1(0, t) - A_4(0, t)I_n)\,. \tag{3.9b}$$

To ensure the compatibility between (3.9a) and (3.9b), we impose

$$A_1(0, t) - A_4(0, t)I_n = -(A_1(0, t) - A_4(0, t)I_n)^\dagger\,, \tag{3.10}$$

showing that $H \equiv -i(A_1(0, t) - A_4(0, t)I_n)$ is a hermitian matrix. The boundary condition now reads

$$R_x(0, t) + HR(0, t) = 0\,. \tag{3.11}$$

We note that at this stage, we have boundary conditions that depend on time a priori. We remove this time dependence by requiring that $A_1(0, t)$ and $A_4(0, t)$ be time-independent. Then, H is time independent. We see that (3.11) is the vector generalisation of the usual Robin boundary condition in the scalar case $(r_x(0, t) + \alpha r(0, t) = 0,\ \alpha \in \mathbb{R})$. The fact that H is hermitian is the analog of α being real. Let us denote $A_4(0) = \beta$. What we have obtained so far reads

$$L(0, k) = kI_{n+1} + \begin{pmatrix} \beta I_n + iH & 0 \\ 0 & \beta \end{pmatrix}\,, \tag{3.12}$$

7

with L independent of t at $x = 0$. To complete the characterization of $L(0, k)$, insert (3.12) in (3.3) to get

$$\widetilde{G}(0, k)\, L(0, k) - L(0, k) G(0, k) = 0\,. \tag{3.13}$$

Noting that $\widetilde{G}(0, k) = \Sigma_3 G(0, -k) \Sigma_3$, this reads

$$G(0, -k)\, \Sigma_3 L(0, k) = \Sigma_3 L(0, k) G(0, k)\,. \tag{3.14}$$

It is now convenient to use the hermiticity of H that guarantees that H is diagonalisable by a unitary matrix V

$$H = VDV^\dagger, \qquad \text{where } D = \mathrm{diag}\{d_1, \ldots, d_n\}, \tag{3.15}$$

and the d_j's are all real numbers. Note that the transformation $R(x, t) \mapsto V^\dagger R(x, t) = R'(x, t)$ leaves the nonlinear Schrödinger equation invariant and in the new basis the boundary condition takes the simple, diagonal form

$$R'_x(0, t) + DR'(0, t) = 0\,. \tag{3.16}$$

This shows that, in the presence of a boundary described by H, the VNLS equation has a preferred polarization basis determined by the boundary. In the following, we work in this basis and drop the $'$. Then,

$$L(0, k) = kI_{n+1} + \begin{pmatrix} \beta I_n + iD & 0 \\ 0 & \beta \end{pmatrix}, \tag{3.17}$$

and inserting the relation $R_x(0, t) + DR(0, t) = 0$ in $G(0, k)$, equation (3.14) yields

$$(2i\beta I_n - D)DR(0, t) = 0\,, \tag{3.18a}$$
$$R^\dagger(0, t)(2i\beta I_n - D)D = 0\,, \tag{3.18b}$$
$$RR^\dagger(0, t)D = DRR^\dagger(0, t)\,. \tag{3.18c}$$

The compatibility of the first two equations imposes that β be purely imaginary: $\beta = i\alpha$, $\alpha \in \mathbb{R}$, unless $R(0, t) = 0$ (this Dirichlet boundary condition for all the components is formally obtained when all the d_j are infinite). Then, the first equation shows that either $d_j = 0$ or $d_j = -2\alpha$ or $R_j(0, t) = 0$. Finally, the last equation reads

$$d_j R_j R_k^*(0, t) = d_k R_j R_k^*(0, t)\,, \quad j, k = 1, \ldots, n\,. \tag{3.19}$$

In general, this means that $d_j = d_k \equiv d$ i.e. $D = dI_n$ is proportional to the identity matrix. The particular case $R_j(0, t) = 0$ for some j requires some attention. In this case, either $R_{jx}(0, t)$ is also zero and $d_j = d$ as before, or in general $R_{jx}(0, t) \neq 0$, meaning that $d_j = \infty$ is different from the common value d and we must have $d_j + 2\alpha = 0$. So this case occurs when formally $\alpha = -\infty$. To summarize the results, we have the following possible integrable boundary conditions

$$R_x(0, t) - 2\alpha R(0, t) = 0\,, \quad \alpha \in \mathbb{R}, \tag{3.20}$$

or a mixture or Neumann and Dirichlet boundary conditions

$$R_j(0, t) = 0\,, \quad j \in S\,, \tag{3.21}$$
$$R_{kx}(0, t) = 0\,, \quad k \in \{1, \ldots, n\} \setminus S, \tag{3.22}$$

where S is a subset of $\{1, \ldots, n\}$. In the rest of this paper, for convenience we will assume that S is a proper subset of $\{1, \ldots, n\}$. This avoids redundancy for the pure Dirichlet or Neumann condition which can be recovered from the Robin condition in the limits $\alpha \to \infty$ or $\alpha \to 0$ respectively.

In terms of $L(0, k)$ the previous results are more conveniently written by considering

$$\mathcal{L}(x, t, k) = \frac{1}{k + i\alpha} L(x, t, k) \,.$$

Note that $\mathcal{L}$ is completely equivalent to L since a Bäcklund matrix is always defined up to a function of k but it has the advantage of accommodating the $\alpha = -\infty$ case. Then, the previous two cases correspond to

$$\mathcal{L}(0, k) = \begin{pmatrix} \frac{k - i\alpha}{k + i\alpha} I_n & \\ & 1 \end{pmatrix} \quad \text{or} \quad \mathcal{L}(0, k) = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \\ & & & 1 \end{pmatrix}, \tag{3.23}$$

where $\sigma_j = -1$, $j \in S$ and $\sigma_j = 1$, $j \in \{1, \ldots, n\} \setminus S$.

We note that in [10], the previous mixed Neumann/Dirichlet boundary condition was not found. In section 4, we will see that this is the most interesting one as far as interactions between vector solitons and boundary are concerned.

It is now worthwhile to comment on relation (3.14). This is precisely the relation that is imposed in the Fokas' method to obtain the so-called linearizable boundary conditions (with the identification $\Sigma_3 L(0, k) \equiv N(k)$, see e.g. [12]). Thus, for each solution $N(k)$ of equation (3.14) that can be identified as the value at $x = 0$ of a Bäcklund matrix $L(x, t, k)$, we see that the so-called linearizable boundary conditions are integrable. In fact this is shown in section 3.2 where a generating function for the conserved quantities is constructed for the class of boundary conditions described by such an $N(k)$.

3.2 Integrability

The fact that our boundary conditions derive from a Bäcklund matrix ensures the existence of an infinite number of conserved quantities. This is what we mean by integrability in this context. Adapting a well-known argument for deriving a generating function of conserved quantities from the Lax pair to our situation, we show this explicitly below. Note that we expect integrability in the Liouville sense to hold as well once one equips the present system with a Hamiltonian structure. Indeed, Bäcklund transformations are known to have nice canonical properties (see e.g. [20]).

Proposition 3.1 *A generating function for the conserved quantities of the VNLS with integrable boundary conditions is given by*

$$i(I(k) - I^\dagger(k^*)) \,, \tag{3.24}$$

where

$$I(k) = tr \left[\int_0^\infty [R(x, t)(\Gamma(x, t, k) - \Gamma(x, t, -k))] \, dx \right], \tag{3.25}$$

and $\Gamma(x, t, k)$ satisfies the following Ricatti equation

$$\Gamma_x = 2ik\Gamma + \lambda R^\dagger - \Gamma R \Gamma \,. \tag{3.26}$$

Proof: The proof follows the one presented in [21] which we adapt to the present vector case with a boundary. Recall that we consider two copies of the auxiliary problem (2.2,2.3) related by (3.1). All the matrices involved in the argument, like Φ, G or L are split in natural blocks

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \quad , \quad Z = \Phi, G, L \,, \tag{3.27}$$

where Z_{11} is an $n \times n$ matrix, Z_{12} is an n-component column vector, etc. Define $\Gamma(x,t,k) = \Phi_{21}\Phi_{11}^{-1}(x,t,k)$ and $\widetilde{\Gamma}(x,t,k) = \widetilde{\Phi}_{21}\widetilde{\Phi}_{11}^{-1}(x,t,k)$. Then, (2.2) yields (3.26) for Γ and the same equation for $\widetilde{\Gamma}$ with R replaced by $\widetilde{R}$. Also, we can write

$$\Phi_{11x}\Phi_{11}^{-1} = R\Gamma \quad , \quad \Phi_{11t}\Phi_{11}^{-1} = G_{11} + G_{12}\Gamma \,, \tag{3.28}$$

so using $(\ln \Phi)_{xt} = (\ln \Phi)_{tx}$ we get $(R\Gamma)_t = (G_{11} + G_{12}\Gamma)_x$ where G_{ij} are the appropriate blocks of G. A similar relation holds for $\widetilde{\Gamma}$. Finally, from (3.1) we have

$$\begin{aligned} \widetilde{\Phi}_{11t} &= [(L_{11} + L_{12}\Gamma)\Phi_{11}]_t \\ &= [(L_{11} + L_{12}\Gamma)_t + (L_{11} + L_{12}\Gamma)(G_{11} + G_{12}\Gamma)](L_{11} + L_{12}\Gamma)^{-1}\widetilde{\Phi}_{11}, \quad (3.29) \end{aligned}$$

which we compare to $\widetilde{\Phi}_{11t} = (\widetilde{G}_{11} + \widetilde{G}_{12}\widetilde{\Gamma})\widetilde{\Phi}_{11}$ to get

$$\mathrm{tr}(\widetilde{G}_{11} + \widetilde{G}_{12}\widetilde{\Gamma}) = \mathrm{tr}(G_{11} + G_{12}\Gamma) + \mathrm{tr}\ln(L_{11} + L_{12}\Gamma)_t \,. \tag{3.30}$$

Swapping the roles of Φ and $\widetilde{\Phi}$ and introducing $\widetilde{L} = L^{-1}$ we can also obtain

$$\mathrm{tr}(G_{11} + G_{12}\Gamma) = \mathrm{tr}(\widetilde{G}_{11} + \widetilde{G}_{12}\widetilde{\Gamma}) + \mathrm{tr}\ln(\widetilde{L}_{11} + \widetilde{L}_{12}\widetilde{\Gamma})_t \,. \tag{3.31}$$

This allows us to obtain a more symmetric form of the final result. Putting everything together, we get the general result

$$\begin{aligned} &\partial_t \mathrm{tr}\left[\int_{-\infty}^{0} \widetilde{R}\widetilde{\Gamma}(x,t,k)\,dx + \int_{0}^{\infty} R\Gamma(x,t,k)\,dx\right] \\ &= \frac{1}{2}\partial_t \mathrm{tr}\left[\ln\left(L_{11}(0,k) + L_{12}(0,k)\Gamma(0,t,k)\right) - \ln\left(\widetilde{L}_{11}(0,k) + \widetilde{L}_{12}(0,k)\widetilde{\Gamma}(0,t,k)\right)\right].(3.32) \end{aligned}$$

Now, for the problem with boundary, the thing to note is that under the reduction $\widetilde{R}(x,t) = R(-x,t)$, we have

$$\widetilde{\Gamma}(x,t,k) = -\Gamma(-x,t,-k) \quad \text{and} \quad \widetilde{L}(x,t,k) = \frac{1}{k^2 + \alpha^2}\Sigma_3 L(-x,t,-k)\Sigma_3 \,. \tag{3.33}$$

Finally, at $x = 0$, $L(0,k)$ does not depend on t for the class of boundary conditions we have derived and $L_{12}(0,k) = 0$ so the right-hand-side in (3.32) vanishes. Therefore, we have shown

$$\partial_t I(k) = 0 \,. \tag{3.34}$$

The special form (3.24) is used to get real conserved quantities. ∎

10

In practice, the conserved quantities are determined recursively by inserting the following expansion

$$\Gamma(x, t, k) = \sum_{n=1}^{\infty} \frac{\Gamma_n(x, t)}{(2ik)^n} \,, \tag{3.35}$$

into the Ricatti equation to obtain

$$\Gamma_1 = -\lambda R^\dagger \;,\quad \Gamma_{n+1} = \Gamma_{nx} + \sum_{k=1}^{n-1} \Gamma_k R \Gamma_{n-k} \;,\quad n \geq 1 \,. \tag{3.36}$$

As expected from the presence of the boundary, we see that the conserved quantities corresponding to even powers of $1/k$ do not appear. In particular, the momentum (order $1/k^2$) is not conserved.

The connection between linearizable boundary conditions and Bäcklund transformations was noted in [22]. Here, in view of the previous result, this connection allows one to identify explicitly the infinite set of conserved quantities once a solution $\Sigma_3 L(0, k) \equiv N(k)$ of the "symmetry" relation (3.14) is known. Of course, this connection goes beyond the specific example of VNLS and holds for any integrable system characterized by a Lax pair (with possible complications for equations that are not invariant under $x \to -x$ such as the KdV equation).

3.3 Mirror image construction

Following [16, 17], we can now use the result of the previous section to introduce the following extended potential

$$P(x, t, k) = \theta(x)Q(x, t) + \theta(-x)\mathcal{B}(k)Q(-x, t)\mathcal{B}(-k) \,, \tag{3.37}$$

where θ is the Heaviside function and

$$\mathcal{B}(k) \equiv \begin{pmatrix} B(k) & 0 \\ 0 & -1 \end{pmatrix} = \Sigma_3 \, \mathcal{L}(0, k) \,. \tag{3.38}$$

Note that

$$B^\dagger(k^*)B(k) = I_n = B(-k)B(k) \;,\quad k \in \mathbb{C} \,. \tag{3.39}$$

We have the property

$$P(-x, t, -k) = -\mathcal{B}(-k)P(x, t, k)\mathcal{B}(k) \,. \tag{3.40}$$

The idea is to apply the inverse scattering method as explained in section 2 to P instead of Q. By construction, for $x > 0$, any solution found for P gives a solution Q of VNLS on the half-line. The delicate point is to check whether this solution does indeed satisfy the boundary conditions that we derived in the previous section and that we encoded in Fourier space in the matrix $B(k)$ obtained from $\mathcal{L}(0, k)$.

For conciseness, we keep the same notations as in the previous section but with Q replaced by P. Using (3.40), we obtain that $M_+(x, t, k)$ and $\mathcal{B}(k)M_-(-x, t, -k)\mathcal{B}(-k)$

satisfy the same Volterra integral equation. Noting that they have the same asymptotics as $x \to \infty$ we deduce

$$M_+(x,t,k) = \mathcal{B}(k)M_-(-x,t,-k)\mathcal{B}(-k)\,. \tag{3.41}$$

From this and (2.12), we obtain

$$S^{-1}(k) = \mathcal{B}(k)S(-k)\mathcal{B}(-k)\,. \tag{3.42}$$

Next, from (2.6), we have that $WM_\pm^\dagger(x,t,k^*)W^{-1} = M_\pm^{-1}(x,t,k)$ which in turn yields[5]

$$S^{-1}(k) = WS^\dagger(k^*)W^{-1}\,. \tag{3.43}$$

Combining the two previous equations yields the important result

$$WS^\dagger(k^*)W^{-1} = \mathcal{B}(k)S(-k)\mathcal{B}(-k)\,. \tag{3.44}$$

In components, this reads

$$a^\dagger(k^*) = B(k)a(-k)B(-k)\,, \quad \bar{a}^*(k^*) = \bar{a}(-k)\,, \tag{3.45}$$
$$\bar{b}^\dagger(k^*) = \lambda b(-k)B(-k)\,, \quad \lambda b^\dagger(k^*) = B(k)\bar{b}(-k)\,. \tag{3.46}$$

Note that due to the property $B^\dagger(k^*) = B(-k)$, the last two relations are consistent. The first relation implies that if k_j is a zero of $\det a(k)$ then $-k_j^*$ is also a zero. The same holds true for $\bar{a}(k)$. Therefore, we find that the main observation of [17] that the relevant zeros to formulate the inverse problem come in quartets $\{\pm k_j, \pm k_j^*\}$ is also valid in the vector case. The total number of zeros in each half-plane is even: $J = 2N$ and there are N zeros in each quadrant of the complex plane. For each $j = 1, \ldots, N$, it is enough to assume that k_j is in the first quadrant *i.e.* $\operatorname{Re} k_j \geq 0$ and $\operatorname{Im} k_j > 0$. The other three zeros are obtained by successive reflections with respect to the imaginary and real axes.

Associated to these zeros are norming constants which we denote C_j (for k_j) and $C_j{}'$ (for $-k_j^*$). From the symmetry (2.6), the norming constants associated to k_j^* (resp. $-k_j$) are then $\lambda C_j^\dagger$ (resp. $\lambda C_j'^\dagger$).

The subtle point of the method now is to reconstruct $R(x,t)$ from the previous scattering data corresponding to $P(x,t,k)$. This is achieved by establishing the analogs of (2.18), (2.19) and (2.24) and restricting to $x > 0$ where $P(x,t,k) = Q(x,t)$. To do so we must investigate the analytic properties of $M_\pm(x,t,k)$ as functions of k for $x > 0$. The argument follows that given in [16]. For $x > 0$, $P(x,t,k) = Q(x,t)$ so the analytic properties of M_+ are the same as in the infinite line case. Then, the analytic properties of M_- are obtained using the symmetry relation (3.41). In the mixed Neumann/Dirichlet case, no extra pole arises from the presence of $B(k)$ (which is in fact independent of k). In the Robin case, M gets an extra pole at $k = i\alpha$ if $\alpha > 0$ and $\overline{M}$ gets an extra pole at $k = -i\alpha$ if $\alpha > 0$ but are otherwise analytic in their respective half-plane. Then, $a(k)$ and $\bar{a}(k)$ share the same analytic properties as

---

[5]We note that strictly speaking, $S(k)$ is defined first for $k \in \mathbb{R}$. But as we have seen, its entries can be analytically continued to various parts of the complex plane. Eq (3.43) is then to be understood as relations for the entries of S valid wherever they make sense. Consequently, the same is true of eq (3.44).

M and $\overline{M}$. As a consequence, the fundamental quantities Ma^{-1} and $\overline{M}\bar{a}^{-1}$ have at worst a removable singularity. Therefore, the analysis of section 2.3 goes through as before, taking into account the additional symmetry properties of the scattering data discussed above. This is summarized in the following proposition which constitutes the main result of this work.

Proposition 3.2 *The reconstruction formulae for $Q(x,t)$, $x > 0$ are given by (2.18), (2.19) and (2.24) with $J = 2N$ and, without loss of generality, the substitution $k_{2j-1} \rightarrow k_j$, $j = 1, \ldots, N$ with corresponding norming constants C_j and $k_{2j} \rightarrow -k_j^*$, $j = 1, \ldots, N$ with corresponding norming constants C_j'. The scattering data involved in the formula satisfy the symmetry relations (3.45, 3.46) and*

$$\lambda C_j'^{\dagger} C_j = B(-k_j) \frac{A(k_j)^t}{\alpha'^2(k_j)} \quad , \quad j = 1, \ldots, N \,. \tag{3.47}$$

<u>Proof:</u> The only thing left to prove are the relations (3.47) between the norming constants C_j of the real solitons (those living on $x > 0$), and the norming constants C_j' of the mirror solitons (those living on $x < 0$). We use the following chain of relations obtained from (2.12) and (3.41) evaluated at $k = k_j$ and $k = -k_j$

$$M(x,t,k_j)A^t(k_j) = e^{2i\phi(x,t,k_j)} N(x,t,k_j) b(k_j) A^t(k_j) \,, \tag{3.48}$$

$$N(x,t,k_j) = \begin{pmatrix} -B(k_j) & \\ & 1 \end{pmatrix} \overline{M}(-x,t,-k_j) \,, \tag{3.49}$$

$$\overline{M}(-x,t,-k_j) = e^{-2i\phi(-x,t,-k_j)} \overline{N}(-x,t,-k_j) \bar{b}(-k_j) \,, \tag{3.50}$$

$$\overline{N}(-x,t,-k_j) = \begin{pmatrix} B(-k_j) & \\ & -1 \end{pmatrix} M(x,t,k_j) B(k_j) \,. \tag{3.51}$$

The compatibility of these relations yields $A^t(k_j) = -B(k_j)\bar{b}(-k_j)b(k_j)A^t(k_j)$ which is just (3.47) upon using $C_j = \frac{1}{\alpha'(k_j)} b(k_j) A(k_j)^t$, $\bar{C}_j' = \frac{\bar{b}(-k_j)}{\bar{a}'(-k_j)} = \lambda C_j'^{\dagger}$ and $\bar{a}'(-k_j) = -\alpha'(k_j)$. ∎

Note that following the same argument but for $k = -k_j^*$ and $k = k_j^*$, we arrive at the following similar relations

$$\lambda C_j^{\dagger} C_j' = B(k_j^*) \frac{A(-k_j^*)^t}{\alpha'^2(-k_j^*)} \quad , \quad j = 1, \ldots, N \,, \tag{3.52}$$

which are compatible with (3.47).

The symmetries of the scattering data ensure that Q satisfies the desired boundary conditions. We show this in general in the mixed Neumann/Dirichlet case.

Proposition 3.3 *Without loss of generality take $B(k) = diag(1, \ldots, 1, -1, \ldots, -1) \equiv B$ then the reconstruction formulae yield a potential R which satisfies*

$$R(-x,t) = BR(x,t) \,, \tag{3.53}$$

i.e. R satisfies mixed Neumann/Dirichlet boundary conditions.

<u>Proof:</u> In the case of fast decreasing solutions, the reconstruction formula (2.24) can be compactly rewritten as

$$R(x,t) = \frac{1}{\pi} \int_{\overline{C}} e^{-2i\omega x - 4i\omega^2 t} \overline{N}^{(up)}(x,t,\omega) \bar{\rho}(\omega) \, d\omega \,, \qquad (3.54)$$

where $\overline{C}$ is a contour in the lower half-plane from $-\infty$ to $+\infty$ that passes below all the zeros of $\bar{a}(k)$ and that is symmetric with respect to the imaginary axis. Repeating the arguments of section 2.3 from the relation

$$e^{i\phi(x,t)\Sigma_3} M_-(x,t,k) e^{-i\phi(x,t)\Sigma_3} S^{-1}(k) = e^{i\phi(x,t)\Sigma_3} M_+(x,t,k) e^{-i\phi(x,t)\Sigma_3} \,, \qquad (3.55)$$

instead of (2.12), one can derive an equivalent reconstruction formula

$$R(x,t) = -\frac{1}{\pi} \int_C e^{-2i\omega x - 4i\omega^2 t} M^{(up)}(x,t,\omega) \tau(\omega) \, d\omega \,, \qquad (3.56)$$

where M (and $\overline{M}$) satisfy coupled integral equations similar to (2.18,2.19) and $\tau(k) = d(k)c^{-1}(k)$ is obtained from the entries of $S^{-1}(k)$

$$S^{-1}(k) = \left(\begin{array}{cc} \bar{c}(k) & d(k) \\ \bar{d}(k) & c(k) \end{array} \right) \,. \qquad (3.57)$$

Now combining (3.43) and (3.44), we get $\tau(k) = -B\bar{\rho}(-k)$. Also, from (3.41), we get $\overline{N}^{(up)}(-x,t,-k) = BM^{(up)}(x,t,k)B$. Finally, performing the change of variable $\omega \to -\omega$ in (3.54) and using the previous relations yields the result. $\blacksquare$

To the authors' knowledge, a general proof that the field obtained with the mirror image technique obeys the desired boundary conditions is not known within this formalism. Even in the scalar case, an explicit argument has been given in [16] only in the case of Dirichlet boundary conditions. Arguments based on the linear limit of the nonlinear Schrödinger equation in the case of general boundary conditions have been given in [17]. It seems that the proper way to attack the problem is to use the general method developed by Fokas and co-workers in the case of linearizable boundary conditions. In view of the connection that we have made above between this method and the Bäcklund transformation approach that we have used here in the vector case, we see that a full treatment of the problem requires to formulate the vector version of Fokas method. Hopefully, this problem is still amenable for the class of linearizable boundary conditions that we consider here. This is beyond the scope of the present paper and is left for future investigation.

It is worth noting the appearance of the matrix $A^t(k_j)$ in the relations between the norming constants of the "real" solitons and those of the mirror image solitons. This is a new feature compared to the scalar case. Using the dressing method [23], it can be shown that these matrices can be expressed in terms of all the norming constants. Hence, the system (3.47) is a system of coupled matrix equations that one has to solve in order the completely characterize the soliton solutions with boundary. This is technically more challenging than in the scalar case where one could use directly the trace formula for $a(k)$ to solve the problem [17]. This is reminiscent of the difference between the scalar NLS and the Manakov model on the line where the presence of polarization vectors for

the solitons greatly complicates the analysis of the soliton interactions [9]. Here, the presence of the cofactor matrix captures the influence of the polarization vectors of the solitons on their interaction with the boundary.

A general efficient algorithm has not yet been found to solve these coupled equations and completely characterize the N-soliton solutions with a boundary. We hope to return to this question in a future work. In the next section, we show how this can be done for one soliton and use the result to investigate its interaction with the boundary.

4 One soliton reflection: transmission between modes

We consider the pure soliton case with $N = 1$ $i.e.$ we study the behaviour of one soliton bouncing off the boundary. The scattering data is k_1 and C_1. From this, we deduce the scattering data corresponding to the mirror soliton: $-k_1^*$ and C_1' determined by (3.52) (with $\lambda = -1$)

$$C_1^\dagger C_1' = -B(k_1^*)\frac{A(-k_1^*)^t}{\alpha'^2(-k_1^*)} \,. \tag{4.1}$$

We use the dressing method to compute $A(-k_1^*)^t$. Let us first recall this method to generate two arbitrary solitons from the vacuum solution and then we will apply it in the present special case. Let κ_1, κ_2, $\mathcal{C}_1$ and $\mathcal{C}_2$ be the poles and norming constants for the two solitons. The scattering coefficient $a(k)$ is then given by

$$a(k) = D_1(k)D_2(k) \,, \tag{4.2}$$

where

$$D_j(k) = I_n + \left(\frac{k - \kappa_j}{k - \kappa_j^*} - 1\right)\Pi_j \,, \tag{4.3}$$

and Π_j is a (one-dimensional) orthogonal projector obtained recursively by requiring $\mathcal{C}_j\, a(\kappa_j) = 0$, $j = 1, 2$. From this, one easily deduces

$$\frac{A(\kappa_2)^t}{\alpha'(\kappa_2)} = (\kappa_2 - \kappa_2^*)\Pi_2 D_1(\kappa_2)^{-1} \,, \tag{4.4}$$

with $\mathcal{C}_1\Pi_1 = \mathcal{C}_1$ and $\rho_2\Pi_2 = \rho_2$ where $\rho_2 = \mathcal{C}_2 D_1(\kappa_2)$.

Now, we insert this in (4.1) with the identification $\kappa_1 = k_1$, $\kappa_2 = -k_1^*$, $\mathcal{C}_1 = C_1$ and $\mathcal{C}_2 = C_1'$. Multiply on the right by $D_1(\kappa_2)$ and project on $\rho_2^\dagger$ to obtain

$$\frac{\rho_2^\dagger}{\rho_2\rho_2^\dagger} = -\frac{\alpha'(-k_1^*)}{k_1 - k_1^*}B(-k_1^*)C_1^\dagger \equiv V_1^\dagger \,. \tag{4.5}$$

Hence, we deduce the solution

$$C_1' = \frac{V_1}{V_1 V_1^\dagger}D_1(-k_1^*)^{-1} \,. \tag{4.6}$$

For practical calculations, we note finally that $\alpha'(-k_1^*)$ can be obtained from the well-known trace formula for $\det a(k)$ (see e.g. [19]) which gives, in our case,

$$\alpha'(-k_1^*) = \frac{k_1 + k_1^*}{2k_1^*(k_1 - k_1^*)} \,. \tag{4.7}$$

In the following, we present numerical results for the case of two components ($n = 2$) and for the mixed Neumann/Dirichlet case. The Robin boundary condition does not bring anything new in the vector case as compared to the scalar case in the sense that in any polarization basis, each component of R always satisfies a scalar Robin boundary condition. However, in the mixed Neumann/Dirichlet case, if the polarization basis is different from the preferred boundary basis *i.e.* if we restore the unitary V matrix in (3.15), then an interesting phenomenon of reflection-transmission between the modes appears: the amplitude of the soliton envelope $|R_j(x,t)|$ is different before and after its interaction with the boundary!

As an example, we take

$$V = \begin{pmatrix} \cos\theta e^{i\zeta} & \sin\theta e^{i\xi} \\ -\sin\theta e^{-i\xi} & \cos\theta e^{-i\zeta} \end{pmatrix}, \tag{4.8}$$

with $\theta, \xi, \zeta \in \mathbb{R}$ and consider the boundary matrix B_θ given by

$$B_\theta = V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V^{-1}. \tag{4.9}$$

The parameters θ, ζ, ξ measure the "deviation" from the natural boundary basis corresponding to $\theta = \zeta = \xi = 0$. Below are plots of $|R_1|$ and $|R_2|$ in x, t space for $k_1 = 1 + \frac{i}{2}$ and $C_1 = (2 \ 1)^t$.

First, for $\theta = \zeta = \xi = 0$ (Fig. 1 and 2) we see that, as expected, the first mode satisfies a Neumann boundary condition while the second mode satisfies a Dirichlet boundary condition. These plots are very similar to those of [17] in each case. Each mode behaves like a scalar solution seeing its own boundary condition.
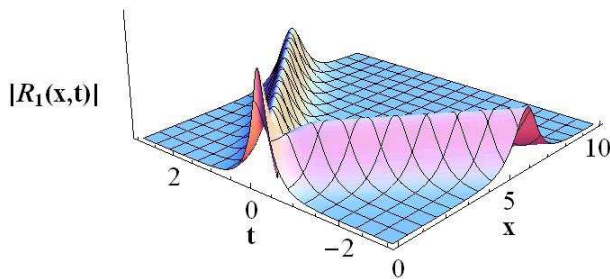

FIG. 1: Reflection of the first component of the soliton on the boundary with Neumann boundary condition. The amplitude before and after the reflection is the same.
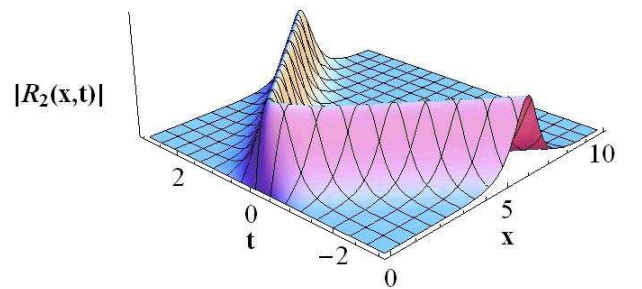
FIG. 2: Reflection of the second component of the soliton on the boundary with Dirichlet boundary condition. The amplitude before and after the reflection is the same.

Now, for $\theta = \frac{\pi}{6}$ and $\zeta = \xi = 0$ (Fig. 3 and 4), we clearly see that the amplitude of each mode before and after the interaction with the boundary is different. Both modes are reflected but part of mode 1 is transmitted to mode 2.

This shows that the boundary acts as some sort of polarization filter. We emphasize though that there is no loss in the transmission process in the sense that the quantity
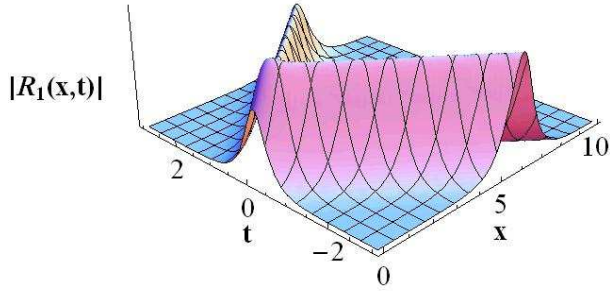
16

Fig. 3: Reflection of the first component of the soliton on the boundary. The outgoing mode has a smaller amplitude than the incoming one.
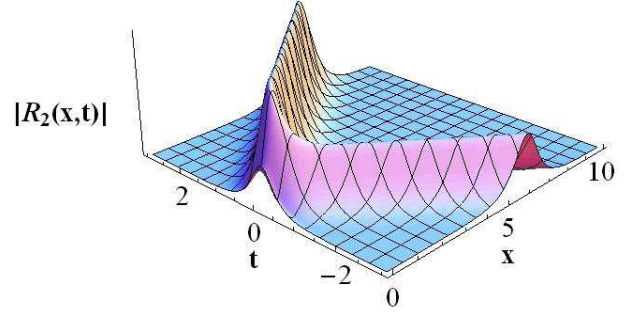
Fig. 4: Reflection of the second component of the soliton on the boundary. Part of the first mode has been transmitted to the outgoing mode here.

$|R_1|^2 + |R_2|^2$ is indeed a conserved density. Another way to see this is to investigate the same quantity in the reference frame of the incoming or outgoing soliton. Here the velocity of the incoming (outgoing) soliton is $-v$ (v) with $v = 4 \operatorname{Re} k_1 = 2$. We find indeed that

$$\lim_{t \to -\infty} (|R_1|^2 + |R_2|^2)(-vt + x_{in}, t) = \lim_{t \to \infty} (|R_1|^2 + |R_2|^2)(vt + x_{out}, t), \qquad (4.10)$$

where x_{in} (x_{out}) is the incoming (outgoing) soliton centre.

One can then wonder if one can use the boundary to control the amplitudes of the modes by changing the values of the boundary parameters. The following plot shows that this can be done. We keep $k_1 = 1 + \frac{i}{2}$ and $C_1 = (2 \ 1)^t$ and we vary θ from 0 to $\frac{\pi}{2}$ while fixing $\zeta = 1.11$ and $\xi = 0$. The incoming amplitudes are of course constant (lines) while the outgoing amplitudes vary (dashed curves). In our example, we see that the second mode (in blue) can be made vanishingly small (for $\theta \approx 1.15$). The total outgoing amplitude is constant though (black line) and equals the total incoming amplitude.
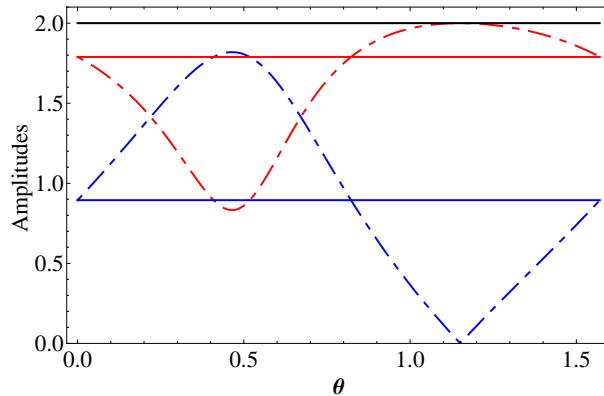

Fig. 5: Amplitudes in the soliton frame as functions of θ. The red line is the incoming $|R_1|$ amplitude. The blue line is incoming $|R_2|$. Dashed curves are the corresponding outgoing amplitudes. The black line is the total amplitude $\sqrt{|R_1|^2 + |R_2|^2}$.

17

5 Conclusions

The main goal of this paper was to establish the inverse scattering formulae for the VNLS model on the half-line with integrable boundary conditions. We had first to identify such integrable boundary conditions and this was done using the Bäcklund matrix approach. An important difference with the scalar case is the need to consider the time part of the Bäcklund transformation. Indeed, for the scalar case, this does not lead to new information and, in fact, it was not considered this way in the original papers [14, 15]. Only the space part was considered and then it was checked that the solution was compatible with time evolution. Here, we considered the time part directly and found that the solution to the space part has to be further reduced to two classes of possible integrable boundary conditions (which are then automatically compatible with the time evolution). On the technical side, this is chiefly due to the fact that in the scalar case, there is no difference between $RR^\dagger$ and $R^\dagger R$ while in the vector case, the former is a square matrix of size n while the latter is a scalar.

There seems to be a sharp contrast between vector and scalar cases in terms of which boundary conditions holding in the linear case go through to the interacting case. Indeed, in the scalar case, the Robin boundary condition $q_x(0,t) + dq(0,t) = 0$, easily established for the free Schrödinger equation survives the switching on of the interaction in the sense that it is compatible with the integrability of the nonlinear Schrödinger equation. But here in the vector case, only subfamilies of the general Robin condition $R_{jx}(0,t) + d_j R_j(0,t) = 0$, $j = 1, \ldots, n$ survive the requirement of integrability of the nonlinear equation when implemented in the form of the Bäcklund transformation method. Interestingly, this is not seen if one only looks at the x-part of the Bäcklund transformation method and the analysis of t-part also yields an important connection with the notion of linearizable boundary conditions in the Fokas method. In the Appendix, we examine in more details the linear and nonlinear systems.

Using the boundary conditions thus found, we implemented the vector version of the mirror image technique [17] to obtain the reconstruction formulae for the fields. Here again, a major difference with the scalar case is the appearance of coupled equations to determine the norming constants of the "virtual" solitons from the given scattering data characterizing the "real" solitons. We solved this problem explicitly in the case of one soliton. Using these results, we discovered an interesting phenomenon: the boundary can be tuned to alter the polarization of a soliton. This was demonstrated with numerical simulations.

The question of an efficient treatment of the general N-soliton problem in the presence of a boundary is an exciting open issue.

Acknowledgements

Appendix: Similarities and differences between linear and nonlinear Schrödinger equations with a boundary

We take some time to discuss the classes of integrable boundary conditions that we have found using the Bäcklund transformation method. It is well known that for the free vector Schrödinger equation on the half-line (corresponding to switching off the nonlinearity by setting $\lambda = 0$), the following general boundary conditions hold

$$(U - I_n)R(0, t) + i(U + I_n)R_x(0, t) = 0, \tag{5.1}$$

where U is some $n \times n$ unitary matrix. This is the same as (3.11) (and hence (3.16) in the boundary basis) where we allow formally some of the eigenvalues of H to be infinite if the corresponding eigenvalues of $U + I_n$ are zero. But with the nonlinear term, we found that only subclasses of these general boundary conditions survive if we want to maintain integrability. Therefore, it appears that the nonlinearity imposes severe restrictions. One can then wonder if such restrictions are due to the use of the Bäcklund transformation method and if they could be circumvented by a generalization of it. There is an indication that the restrictions are genuine or at least consistent with existing methods. Indeed, in the quantum version of the system considered here, the integrable boundary conditions are controlled by the reflection equation [4]. The solutions for the corresponding reflection matrix have been classified in [7] and the same reduction of possible boundary conditions appears due to the nonlinearity. Algebraically, this means that the R matrix of the Yangian of $gl(n)$ appears in the reflection equation instead of just the identity matrix in the free case.

To conclude this discussion, it is instructive to perform an analysis of the free case along the line of the Lax pair approach and compare with the nonlinear case. This sheds some more light on the differences between the two cases and on why integrability of the nonlinear case is likely to impose severe restrictions on the allowed boundary conditions. In doing so, we provide an alternative and more direct proof of proposition 3.1 in the mixed Neumann/Dirichlet case. In particular, this proof does not involve the Bäcklund matrix L.

Keeping the notations of section 2.1, a Lax pair formulation for the free vector Schrödinger equation

$$iQ_t - Q_{xx}\Sigma_3 = 0, \tag{5.2}$$

reads

$$\mu_x + ik[\Sigma_3, \mu] = Q, \tag{5.3}$$

$$\mu_t + 2ik^2[\Sigma_3, \mu] = 2kQ - iQ_x\Sigma_3. \tag{5.4}$$

Splitting as usual in natural blocks, the linear analog of Γ in proposition 3.1 is $\mu_{21} \equiv \Gamma^0$ which satisfies the following linear version of the Ricatti equation (3.26)

$$\Gamma^0_x = 2ik\Gamma^0 + R^\dagger, \tag{5.5}$$

which gives a series expansion $\Gamma^0 = \sum_{n=1}^{\infty} \dfrac{\Gamma^0_n}{(2ik)^n}$ with $\Gamma^0_n = -\partial_x^{n-1}R^\dagger$. We also have $(\Gamma^0 R)_t = i(\Gamma^0 R_x - \Gamma^0_x R)_x$ and hence, on the half-line, a generating function for the

conserved charges is

$$I^0(k) = \int_0^\infty \left(\Gamma^0(x,t,k) - \Gamma^0(x,t,-k) \right) R(x,t)\, dx\,, \tag{5.6}$$

with

$$\partial_t I^0(k) = i \left(\Gamma_x^0(0,t,k) - \Gamma_x^0(0,t,-k) \right) R(0,t) - i \left(\Gamma^0(0,t,k) - \Gamma^0(0,t,-k) \right) R_x(0,t)\,. \tag{5.7}$$

Collecting everything, the charges are

$$I_p^0 = \int_0^\infty \partial_x^{2p} R^\dagger R\, dx \quad,\quad p \ge 0\,, \tag{5.8}$$

and they are conserved iff

$$\left(\partial_x^{2p+1} R^\dagger R - \partial_x^{2p} R^\dagger \partial_x R \right)_{x=0} = 0 \quad,\quad p \ge 0\,. \tag{5.9}$$

Looking at $p = 0$, the usual charge I_0^0 is conserved when the current is zero at $x = 0$: $\left(R_x^\dagger R - R^\dagger R_x \right)_{x=0} = 0$. This is equivalent to (5.1) and this is how these boundary conditions arise in the first place (note that they also ensure that the energy is real). It is rather easy to adapt the Fourier transform method to incorporate these boundary conditions. The beauty of the linear case is that then, all the higher charges are automatically conserved (this is precisely the main difference with the nonlinear case). Indeed, the general solution to the free Schödinger equation with boundary conditions (5.1) can be written as

$$R(x,t) = \int_\infty^\infty \frac{dk}{2\pi} e^{ikx - ik^2 t} A(k)\,, \tag{5.10}$$

where A is a vector-valued function with the property $A(k) = B(-k)A(-k)$ where

$$B(k) = -\left[(U - I_n) + k(U + I_n) \right]^{-1} \left[(U - I_n) - k(U + I_n) \right]\,. \tag{5.11}$$

But then, we get more than just (5.1). In fact, the reconstructed field (5.10) satisfies the following mirror image symmetry

$$\left[(U - I_n)R + i(U + I_n)R_x \right](x,t) = -\left[(U - I_n)R + i(U + I_n)R_x \right](-x,t)\,, \tag{5.12}$$

of which (5.1) is a consequence. Hence,

$$\left[(U - I_n)\partial_x^{2p} R + i(U + I_n)\partial_x^{2p+1} R \right](x,t) = -\left[(U - I_n)\partial_x^{2p} R + i(U + I_n)\partial_x^{2p+1} R \right](-x,t)\,, \tag{5.13}$$

which immediately implies (5.9) as required.

Now for the nonlinear case, using the ingredients in the proof of proposition 3.1, we have

$$\partial_t \int_0^\infty (\Gamma(k) - \Gamma(-k)) R\, dx = \left[i \left(\Gamma_x(k) - \Gamma_x(-k) \right) R - i \left(\Gamma(k) - \Gamma(-k) \right) R_x \right.$$
$$\left. + i(\Gamma(k)R\Gamma(k)R - \Gamma(-k)R\Gamma(-k)R) \right]_{x=0}\,, \tag{5.14}$$

with

$$\Gamma_x = 2ik\Gamma + \lambda R^\dagger - \Gamma R\Gamma \,. \tag{5.15}$$

Comparing with (5.7) and (5.5), the nonlinearity shows up at two levels: first there is an extra term in the right hand side of (5.14) and second, Γ satisfies a nonlinear equation with the extra $\Gamma R\Gamma$ term compared to Γ^0. Despite this fact, an explicit calculation shows that for the first charge, everything is the same as in the linear case. So one is led to impose (5.1). A crucial difference now is that one cannot use the Fourier transform but has to use the inverse scattering method instead. It is quite plain that this is likely to impose further restrictions on the possible integrable boundary conditions. The proof of proposition 3.1 shows that the Bäcklund transformation method selects such integrable boundary conditions. Although it is not a general proof, a careful inspection of the first few higher conserved charges (already for $n = 2$) shows that there is little hope that more general boundary conditions than the ones we found ensure the conservation of all higher charges. In contrast, we now show that for the mixed Neumann/Dirichlet conditions, a nice argument similar to that of the linear case can be built to conclude that the left-hand side of (5.14) vanishes. This provides a direct check of proposition 3.1 and is based on the result of proposition 3.3 *i.e.* $R(-x) = BR(x)$. The latter fact implies the important relation mirror image relation

$$\Gamma(x, t, k) = -\Gamma(-x, t, -k)B \,, \tag{5.16}$$

which is in fact the nonlinear analog of $R^\dagger(-x) = R^\dagger(x)B$ together will all the higher derivatives of this relation. As a consequence, $\Gamma(0, t, k)R(0, t) = -\Gamma(0, t, -k)R(0, t)$, $\Gamma(0, t, k)R_x(0, t) = \Gamma(0, t, -k)R_x(0, t)$ and $\Gamma_x(0, t, k)R(0, t) = \Gamma_x(0, t, -k)R(0, t)$. Inserting in the right-hand side of (5.14) yields zero as required.

References

[1] S. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys. JETP **38**, (1974), 248.

[2] E. H. Lieb, W. Liniger, *Exact Analysis of an Interacting Bose Gas. I. The General Solution and the ground state*, Phys. Rev. **130** No. 4 (1963) 1605.

[3] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312.

[4] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A**21 (1988), 2375.

[5] S. Murakami, M. Wadati, *Connection between Yangian symmetry and the quantum inverse scattering method*, J. Phys. **A**29 (1996), 7903.

[6] M. Mintchev, É. Ragoucy, P. Sorba, Ph. Zaugg, *Yangian symmetry in the Non Linear Schrödinger hierarchy*, J. Phys. **A**32 (1999), 5885.

[7] M. Mintchev, É. Ragoucy, P. Sorba, *Spontaneous symmetry breaking in the gl(N)-NLS hierarchy on the half line*, J. Phys. **A 34** (2001), 8345.

[8] A. P. Fordy, P. P. Kulish, *Nonlinear Schrödinger equations and simple Lie algebras*, Comm. Math. Phys. **89**, (1983), 427.

[9] T. Tsuchida, *N-soliton collision in the Manakov model*, Prog. Theor. Phys. **111**, (2004), 151.

[10] I. T. Habibullin, S. I. Svinolupov, *Integrable boundary value problems for the multicomponent Schrödinger equations*, Physica **D 87**, (1995), 134.

[11] M. Ablowitz, H. Segur, *The inverse scattering transform: Semi-infinite interval* , J. Math. Phys. **16**, (1975), 1054.

[12] A. S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-SIAM (2008).

[13] I. T. Habibullin, *Bäcklund transformation and integrable boundary-initial value problems*, in Nonlinear world", vol. 1 (Kiev, 1989), 130-138, World Sci. Publ., River Edge, NJ, 1990; *Integrable initial-boundary value problems*, Theor. and Math. Phys. **86** (1991),28.

[14] P. N. Bibikov, V. O. Tarasov, *A boundary-value problem for the nonlinear Schrödinger equation*, Theor. Math. Phys. **79** (1989), 334.

[15] R. F. Bikbaev, V. O. Tarasov, *Initial-boundary value problem for the nonlinear Schrödinger equation*, J. Phys. **A24** (1991) 2507.

[16] A. S. Fokas, *An initial-boundary value problem for the nonlinear Schödinger equation*, Physica **D 35**, (1989), 167.

[17] G. Biondini, G. Hwang, *Solitons, boundary value problems and a nonlinear method of images*, J. Phys. **A42**, (2009), 205207.

[18] A. S. Fokas, A. Its, *The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation*, SIAM J. Math. Anal. **27**, (1996), 738.

[19] M. J. Ablowitz, B. Prinari, A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*, London Mathematical Society Lecture Notes Series 302, Cambridge University Press, 2003.

[20] E. K. Sklyanin, *Canonicity of Bäcklund transformation: r-matrix approach. II*, `solv-int/9903017`.

[21] V. Caudrelier, *On a systematic approach to defects in classical integrable field theories*, IJGMMP **5**, (2008), 1085.

[22] A. S. Fokas, *Linearizable initial boundary value problems for the sine-Gordon equation on the half-line*, Nonlinearity **17**, (2004), 1521.

[23] V. E. Zakharov, A. B. Shabat, *Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II*, Funct. Anal. Appl. **13**, (1979), 166.