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Non-Hermitian multi-particle systems from complex root spaces

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ABSTRACT: We provide a general construction procedure for antilinearly invariant complex root spaces. The proposed method is generic and may be applied to any Weyl group allowing to take any element of the group as a starting point for the construction. Worked out examples for several specific Weyl groups are presented, focusing especially on those cases for which no solutions were found previously. When applied in the defining relations of models based on root systems this usually leads to non-Hermitian models, which are nonetheless physically viable in a self-consistent sense as they are antilinearly invariant by construction. We discuss new types of Calogero models based on these complex roots. In addition we propose an alternative construction leading to q-deformed roots. We employ the latter type of roots to formulate a new version of affine Toda field theories based on non-simply laced roots systems. These models exhibit on the classical level a strong-weak duality in the coupling constant equivalent to a Lie algebraic duality, which is known for the quantum version of the undeformed case.

1. Introduction

More than fifty years ago Wigner [1] observed that operators left invariant under antilinear involutory transformations have real eigenvalues when in addition their eigenfunctions also possess this symmetry. Based on this fact one can regard such type of operators as being related to physical observables in classical, quantum mechanical and even quantum field theories. More recently [2, 3] this feature was exploited by taking $\mathcal{PT}$-symmetry, that is a simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$, as a concrete realisation of this antilinear involutory map. Especially when the operator is a quantum mechanical single particle Hamiltonian the symmetry is easily identified and many new physically meaningful and self-consistent models have been constructed. Also properties of older models could be explained more rigorously by exploiting it. In contrast, for multi-particle systems or field theories the deformations and the corresponding symmetries are less obvious and

may involve complicated transformations in the configuration space. Often the symmetry becomes only apparent after a suitable change of variables or even a full separation of variables has been carried out [4, 5]. Many interesting and even integrable multiparticle systems such as Calogero-Moser-Sutherland models [6] and also field theories such as Toda field theories [7, 8] are formulated generically in terms of root systems associated to Weyl or Coxeter groups. The dynamical variables or fields are in the dual space with respect to some standard inner product. Since these root spaces are naturally equipped with various symmetries due to the fact that by construction they are left invariant under the action of the entire Weyl group, it is by far easier and systematic to identify the antilinear symmetries in the root spaces rather than in the configuration space. Once identified they can be transformed to the latter.

This general idea was recently explored in [5, 9, 10], were some antilinear symmetric deformations were identified for several Weyl groups and the consequences were studied for some applications to modified Calogero models. It was shown that under the assumptions made in these papers the deformations with the desired properties did not exist for certain Weyl groups. One of the purposes of this manuscript is to fill this gap and provide solutions for the missing cases together with an explanation of why they do not exist based on the previous constructions.

The main steps of the construction proposed here is to select *any* element $\hat{\omega} \in \mathcal{W}$ of order two of the Weyl group, i.e. $\hat{\omega}$ is an involution $\hat{\omega}^2 = \mathbb{I}$. This element is then identified as the analogue of the parity operator $\mathcal{P}$. Subsequently $\hat{\omega}$, together with the root space it acts on, is deformed in an antilinear fashion. This means the root spaces have to be complex. One may also start from several elements and consequently construct deformations invariant under the same amount of different antilinear symmetries. Imposing further constraints on the number of the symmetries and the nature of the deformation, such as demanding it to be an isometry and possessing certain limiting behaviour, allows to determine it. Requiring maximal symmetry in all simple Weyl reflections is only possible for groups of rank 2. The explicit solutions for this scenario can be found for A_2, G_2 in [5] and B_2 in [11]. In [9] two possibilities have been investigated, to have a symmetry with respect to two $\mathcal{P}$-operators identified as two factors of the Coxeter element and one symmetry being the longest element of the Weyl group. For the specific example of the E_8-Weyl group the option of two symmetries giving rise to modified Coxeter transformations of order less than the standard Coxeter number was explored in [10]. It is mainly the latter construction which we generalise here, although the proposed procedure is completely generic.

We shall also propose a construction of complex root spaces based on q-deformations, which arose in the context of the study of the renormalisation of affine Toda field theories based on non-simply laced algebras [12, 13].

Clearly when applying these complex roots to define multi-particle systems of Calogero or Toda type models will be non-Hermitian. However, due to the build in antilinear symmetric invariance, the models are strong potential candidates for physically meaningful models.

Our manuscript is organised as follows: In section 2 we lay out the procedure of how to construct complex root spaces equipped with the desired property to be antilinearly

invariant. We exemplify this procedure in section 3 for spaces which possess two antilinear symmetries which are subfactors of the factors of the factorised Coxeter element. We provide solutions for several Weyl groups for which hitherto no solutions were found and for which it was even shown that solutions based on other assumptions do not exist. In section 4 we explore the possibility to take these two symmetries to be entirely arbitrary. In section 5 we reverse the construction and start with given deformations based on simple rotations in the configuration space and compute some of the corresponding root spaces. An alternative deformation method leading to q-deformed roots with no obvious antilinear symmetry is proposed in section 6. In section 7 and 8 we apply our constructions to propose new types of Calogero-Sutherland-Moser models and Toda field theories, respectively. We investigate some of the features of these new models. Our conclusions and an outlook to future investigations are presented in section 9.

2. Construction of antilinearly invariant complex rootspaces

We explain here the framework for a construction of complex extended antilinearly invariant root systems which we denote by $\tilde{\Delta}(\varepsilon)$. We present a generalisation of a method introduced and employed in [9, 10] with focus on obtaining solutions for cases which could not be found previously. The procedure consists of constructing two maps, which may be obtained in any order. In one step we extend the representation space Δ of the standard roots α from $\mathbb{R}^n$ to $\mathbb{R}^n \oplus i\mathbb{R}^n$. This means we are seeking a map

$$\delta: \ \Delta \to \tilde{\Delta}(\varepsilon), \qquad \alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha, \tag{2.1}$$

where $\alpha = \{\alpha_1, \ldots, \alpha_\ell\}$, $\Delta \subset \mathbb{R}^n$, $\tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus i\mathbb{R}^n$ and n is greater or equal to the rank ℓ of the Weyl group $\mathcal{W}$. The complex deformation matrix θ_ε introduced in (2.1) depends on the deformation parameter ε in such a way that $\lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}$. The deformation is constructed to facilitate the root space $\tilde{\Delta}$ with the crucial property for our purposes, namely to guarantee that it is left invariant under an antilinear involutory map

$$\varpi: \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon), \qquad \tilde{\alpha} \mapsto \omega \tilde{\alpha}. \tag{2.2}$$

This means the map in (2.2) satisfies $\varpi: \tilde{\alpha} = \mu_1 \alpha_1 + \mu_2 \alpha_2 \mapsto \mu_1^* \omega \alpha_1 + \mu_2^* \omega \alpha_2$ for μ_1, $\mu_2 \in \mathbb{C}$ and $\varpi \circ \varpi = \mathbb{I}$.

We assume next that ω decomposes into an element of the Weyl group $\hat{\omega} \in \mathcal{W}$ with $\hat{\omega}^2 = \mathbb{I}$ and a complex conjugation τ, $\omega = \tau \hat{\omega} = \hat{\omega} \tau$. The presence of τ ensures the antilinearity of ϖ. In some concrete applications it is understood that the maps $\hat{\omega}$ and τ correspond to analogues of the parity $\mathcal{P}$ and time reversal operator $\mathcal{T}$, respectively. Candidates for $\hat{\omega}$ previously explored are simple Weyl reflections σ_i [5], the two factors $\sigma_\pm$ of the Coxeter element [9], the longest element w_0 of the Weyl group [9] and some more general elements in $\mathcal{W}$ for the example of E_8 [10]. It is the latter construction which we focus on here and extend in a more general and systematic way.

Concretely we assume here that we have at least two different involutions ϖ of the type (2.2) at our disposal, say ϖ_i with $i = 1, 2, \ldots$ With our application in mind, namely to

construct physically viable self-consistent non-Hermitian multi-particle systems, one such map would in principle be sufficient. However, the presence of two maps leads immediately to some extremely useful constraints. We take the associated rules of correspondence to be of the form

$$\omega_i := \theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} = \tau \hat{\omega}_i, \qquad \text{for } i = 1, \ldots, \kappa \geq 2. \tag{2.3}$$

Then by

$$\omega_i \omega_j = \tau \hat{\omega}_i \tau \hat{\omega}_j = \tau^2 \hat{\omega}_i \hat{\omega}_j = \hat{\omega}_i \hat{\omega}_j = \theta_\varepsilon \hat{\omega}_i \hat{\omega}_j \theta_\varepsilon^{-1}, \tag{2.4}$$

it follows directly that the composition of any two of these elements of the Weyl group $\Omega_{ij} := \hat{\omega}_i \hat{\omega}_j$ commutes with the deformation matrix θ_ε

$$[\Omega_{ij}, \theta_\varepsilon] = 0. \tag{2.5}$$

Note that in general $\Omega_{ij} \neq \Omega_{ji}$. Examples previously considered [9] were for instance $\hat{\omega}_1 = \sigma_-$ and $\hat{\omega}_2 = \sigma_+$, such that $\Omega_{12} = \sigma$. Since by construction $\Omega_{ij} \in \mathcal{W}$ we can expand θ_ε in all elements $\breve{\omega}_i \in \mathcal{W}$ which commute with Ω_{ij}, i.e. $[\Omega_{ij}, \breve{\omega}_i] = 0$,

$$\theta_\varepsilon = \sum_k r_k(\varepsilon) \breve{\omega}_k \qquad \text{for } r_k(\varepsilon) \in \mathbb{C}, \tag{2.6}$$

and subsequently determine the coefficient functions $r_k(\varepsilon)$ from additional constraints. One further natural constraint is to demand that θ_ε is an isometry for the inner products on $\tilde{\Delta}(\varepsilon)$, i.e.

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j, \tag{2.7}$$

which means

$$\theta_\varepsilon^* = \theta_\varepsilon^{-1} \qquad \text{and} \qquad \det \theta_\varepsilon = \pm 1. \tag{2.8}$$

In summary, the task is to pick κ elements of the Weyl group $\hat{\omega}_i$, expand the the deformation matrix θ_ε in terms of the elements commuting with the products of these elements and finally determined the coefficient functions $r_k(\varepsilon)$ in these expansions for from the constraints

$$\theta_\varepsilon^* \hat{\omega}_i = \hat{\omega}_i \theta_\varepsilon, \quad [\hat{\omega}_i \hat{\omega}_j, \theta_\varepsilon] = 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}, \tag{2.9}$$

or possibly in reverse, that is for given θ_ε to identify meaningful involutions $\hat{\omega}_i$. It turn out that these constraints are quite restrictive and often allow to determine θ_ε with only very few free parameters left. In some situations it might not be desirable to preserve the inner products (2.7) after the deformation, in which case one may give up (2.8).

With our applications to physical models of Calogero or Toda type in mind, we may then easily construct a dual map $\delta^\star$ for δ

$$\delta^\star : \ \mathbb{R}^n \to \tilde{\Delta}^\star(\varepsilon) = \mathbb{R}^n \oplus i\mathbb{R}^n, \qquad x \mapsto \tilde{x} = \theta_\varepsilon^\star x, \tag{2.10}$$

i.e. this map acts on the coordinate space with $x = \{x_1, \ldots, x_n\}$ or possibly fields. Given θ_ε we construct $\theta_\varepsilon^\star$ by solving the ℓ equations

$$(\tilde{\alpha}_i \cdot x) = ((\theta_\varepsilon \alpha)_i \cdot x) = (\alpha_i \cdot \theta_\varepsilon^\star x) = (\alpha_i \cdot \tilde{x}), \quad \text{for } i = 1, \ldots, \ell, \tag{2.11}$$

involving the standard inner product. This means $(\theta^\star_\varepsilon)^{-1}\alpha_i = (\theta_\varepsilon\alpha)_i$. Note that in general $\theta^\star_\varepsilon \neq \theta^*_\varepsilon$. Naturally we can also identify an antilinear involutory map

$$\varpi^\star : \tilde{\Delta}^\star(\varepsilon) \to \tilde{\Delta}^\star(\varepsilon), \qquad \tilde{x} \mapsto \omega^\star\tilde{x}. \qquad (2.12)$$

corresponding to ϖ but acting in the dual space. Concretely we solve for this the $\kappa \times \ell$ relations

$$(\omega_i\tilde{\alpha})_j \cdot x = \alpha_j \cdot \omega^\star_i\tilde{x}, \quad \text{for } i = 1,\ldots\kappa; \, j = 1,\ldots,\ell, \qquad (2.13)$$

for $\omega^\star_i$ with given ω_i.

3. Deformation matrices from factorised modified Coxeter elements

As explained in the previous section, in principle the involution $\hat{\omega}_i$ could be *any* element in the Weyl group. We will now present a construction based on the selection of two specific, albeit still fairly generic, elements $\hat{\omega}_1 = \tilde{\sigma}_-$ and $\hat{\omega}_2 = \tilde{\sigma}_+$ defined as

$$\tilde{\sigma}_\pm := \prod_{i \in \tilde{V}_\pm} \sigma_i. \qquad (3.1)$$

The σ_i in (3.1) are simple Weyl reflections, acting as

$$\sigma_i(x) := x - 2\frac{x \cdot \alpha_i}{\alpha_i^2}\alpha_i, \qquad \text{with} \quad 1 \leq i \leq \ell \equiv \text{rank}\,\mathcal{W}. \qquad (3.2)$$

The sets $V_\pm$ are defined via the bi-colouration of the Dynkin diagram as explained in [9] and references therein. The difference towards the treatment in [9] is that the products in (3.1) do not have to extend over all possible elements in $V_\pm$, such that $\tilde{V}_\pm \subseteq V_\pm$. Denoting by $\sigma_\pm$ the factors of $\tilde{\sigma}$ when $\tilde{h} = h$ we may therefore express the reduced elements as $\tilde{\sigma}_\pm := \sigma_\pm \prod_{j \in \check{V}_\pm} \sigma_i$ for some values j, which follows by recalling $[\sigma_i, \sigma_j] = 0$ for $i, j \in V_+$ or $i, j \in V_-$ and $\sigma_i^2 = 1$. Thus $\check{V}_\pm$ is the complement of $\tilde{V}_\pm$ in $V_\pm$, that is $V_\pm = \check{V}_\pm \cup \tilde{V}_\pm$. This ensures that we have maintained the crucial involutory property $\tilde{\sigma}_\pm^2 = 1$.

This means the element Ω_{ij} in (2.5) can be viewed as a modified Coxeter element $\tilde{\sigma} := \tilde{\sigma}_-\tilde{\sigma}_+$ with property

$$\tilde{\sigma}^{\tilde{h}} = \mathbb{I}, \qquad \text{with } \tilde{h} \leq h. \qquad (3.3)$$

Therefore $\tilde{\sigma}$ equals a Coxeter element σ when the order $\tilde{h}$ becomes the Coxeter number h.

The reduced root space $\tilde{\Delta}$ is then constructed by acting with $\tilde{\sigma}$ on representatives $\tilde{\gamma}_i = c_i\tilde{\alpha}_i$ of a particular orbit $\tilde{\Omega}_i$ containing now $\tilde{h}$ instead of h roots

$$\tilde{\Omega}_i := \left\{\gamma_i, \tilde{\sigma}\gamma_i, \tilde{\sigma}^2\gamma_i, \ldots, \tilde{\sigma}^{\tilde{h}-1}\gamma_i\right\}. \qquad (3.4)$$

The corresponding entire root space containing $\ell \times \tilde{h}$ roots is the union of all orbits

$$\tilde{\Delta} = \bigcup_{i=1}^{\ell} \tilde{\Omega}_i. \qquad (3.5)$$

In analogy to the deformations defined in [9] we construct therefore the map ϖ as

$$\tilde{\sigma}^{\varepsilon}_{\pm} := \theta_{\varepsilon}\tilde{\sigma}_{\pm}\theta_{\varepsilon}^{-1} = \tilde{\sigma}_{\pm}\tau \tag{3.6}$$

where we assumed an additional property with θ_{ε} being the deformation matrix as introduced in (2.1). Defining the deformed reduced Coxeter element as $\tilde{\sigma}^{\varepsilon} := \tilde{\sigma}^{\varepsilon}_{-}\tilde{\sigma}^{\varepsilon}_{+}$ we use a similar line of reasoning as in the deduction of (2.5) to show that $[\tilde{\sigma},\theta_{\varepsilon}] = 0$. Therefore we make the following Ansatz for the deformation matrix

$$\theta_{\varepsilon} = \sum_{k=0}^{\tilde{h}-1}\mu_k(\varepsilon)\tilde{\sigma}^k, \qquad \text{with } \lim_{\varepsilon\to 0}\mu_k(\varepsilon) = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}, \ \mu_k(\varepsilon)\in\mathbb{C}. \tag{3.7}$$

The assumption for the coefficients $\mu_k(\varepsilon)$ ensures the appropriate limit $\lim_{\varepsilon\to 0}\theta_{\varepsilon} = \mathbb{I}$. Equation (3.6) yields the constraint $\theta^*_{\varepsilon}\tilde{\sigma}_{\pm} = \tilde{\sigma}_{\pm}\theta_{\varepsilon}$, from which we deduce with (3.7)

$$\theta_{\varepsilon} = \begin{cases} r_0(\varepsilon)\mathbb{I} + i\displaystyle\sum_{k=1}^{(\tilde{h}-1)/2} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \text{for } \tilde{h} \text{ odd,} \\[2ex] r_0(\varepsilon)\mathbb{I} + r_{\tilde{h}/2}(\varepsilon)\tilde{\sigma}^{\tilde{h}/2} + i\displaystyle\sum_{k=1}^{\tilde{h}/2-1} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \text{for } \tilde{h} \text{ even,} \end{cases} \tag{3.8}$$

where $\mu_0(\varepsilon) =: r_0(\varepsilon) \in \mathbb{R}$, $\mu_{\tilde{h}/2}(\varepsilon) =: r_{\tilde{h}/2}(\varepsilon) \in \mathbb{R}$ when $\tilde{h}$ is even. In addition we defined $\mu_k(\varepsilon) = ir_k(\varepsilon)$. Demanding next that θ_{ε} is an isometry, we invoke the constraint $\det\theta_{\varepsilon} = 1$. By means of the eigenvalue equations for $\tilde{\sigma}$

$$\tilde{\sigma}\tilde{v}_n = e^{2\pi i\tilde{s}_n/\tilde{h}}\tilde{v}_n \qquad \text{with } n = 1,\dots\ell, \tag{3.9}$$

we define a set of "modified exponents" $\tilde{s} = \{\tilde{s}_1,\dots,\tilde{s}_\ell\}$. Unlike as for the standard case, the eigenvalues may be degenerate in the modified scenario. In general, they take on the values

$$\tilde{s} = \left\{1^{\lambda_1}, 2^{\lambda_2},\dots,(\tilde{h}-1)^{\lambda_{\tilde{h}-1}}, \tilde{h}^{\lambda_{\tilde{h}}}\right\} \qquad \text{with } \sum_{k=1}^{\tilde{h}}\lambda_k = \ell, \tag{3.10}$$

with λ_i indicating the degeneracy of certain eigenvalues in (3.9). Due to the degeneracy there could be several solutions to (3.9) with different elements $\tilde{\sigma}^{(i)}$ for $i = 1,\dots m$ forming a similarity class

$$\Sigma_{\tilde{s}} = \left\{\tilde{\sigma}^{(1)}\tilde{\sigma}^{(2)},\dots,\tilde{\sigma}^{(m)}\right\}. \tag{3.11}$$

As in [9] we demand the preservation of the inner product between the original and deformed roots, which implies that $\det\theta_{\varepsilon} = 1$ and $\theta^*_{\varepsilon} = \theta_{\varepsilon}^{-1}$. Diagonalising (3.8) the constraint $\det\theta_{\varepsilon} = 1$ simply becomes

$$\begin{aligned} 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) - 2\sum_{k=1}^{(\tilde{h}-1)/2} r_k(\varepsilon)\sin\left(\frac{2\pi k}{\tilde{h}}\tilde{s}_n\right)\right] & \text{for } \tilde{h} \text{ odd,} \\ 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) + (-1)^{\tilde{s}_n}r_{\tilde{h}/2}(\varepsilon) - 2\sum_{k=1}^{\tilde{h}/2-1} r_k(\varepsilon)\sin\left(\frac{2\pi k}{\tilde{h}}\tilde{s}_n\right)\right] & \text{for } \tilde{h} \text{ even.} \end{aligned} \tag{3.12}$$

Solving these constraints for θ_ε allows us to construct the simple roots $\tilde{\alpha}_i$ and therefore the entire deformed reduced root space $\tilde{\Delta}(\varepsilon)$. Note that for simplicity we use the same notation for the unformed and deformed root space, distinguishing the latter always by the explicit mentioning of the deformation parameter ε. Hence we have

$$\tilde{\Omega}_i^\varepsilon = \theta_\varepsilon \tilde{\Omega}_i, \tag{3.13}$$

and therefore

$$\tilde{\Delta}(\varepsilon) = \bigcup_{i=1}^{\ell} \tilde{\Omega}_i^\varepsilon = \theta_\varepsilon \tilde{\Delta}. \tag{3.14}$$

This construction guarantees that the $\tilde{\sigma}_\pm^\varepsilon$ are indeed representations of the map ϖ in (2.2). Evidently it leaves the root space invariant

$$\tilde{\sigma}_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon \tilde{\sigma}_\pm \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon \tilde{\sigma}_\pm \tilde{\Delta} = \theta_\varepsilon \tilde{\Delta} = \tilde{\Delta}(\varepsilon). \tag{3.15}$$

For the latter property to hold we may also exclude some of the orbits $\tilde{\Omega}_i^\varepsilon$ in the union $\bigcup_{i=1}^{\ell}$, whenever they are mapped into themselves $\tilde{\sigma}_\pm^\varepsilon : \tilde{\Omega}_i^\varepsilon \to \tilde{\Omega}_i^\varepsilon$.

3.1 Antilinearly deformed A_ℓ root systems

When engaging into a case-by-case description in [9], we characterized different solutions group by group. Here we will take equation (3.12) as more fundamental and classify the solutions according to different values of the modified Coxeter number. In this manner different types of solutions to (3.12) are then characterized by different sets of modified exponents (3.10). This means we need to verify subsequently whether a corresponding $\tilde{\sigma}$ really exists.

For definiteness we fix our conventions and associate the colour values $c_i = 1$ or $c_i = -1$ when i is even or odd, respectively, to the vertices of the Dynkin diagram. We find various similarity classes $\Sigma_{\tilde{s}}$ characterized by different sets of modified exponents $\tilde{s}$.

3.1.1 The class with modified exponents $\{1, 2, 3, 4^{\ell-3}\}$ and $\tilde{\mathsf{h}}=4$

We find that the simplest similarity class Σ for which $x^4 = 1$ when $x \in \Sigma$ is

$$\Sigma_{\{1,2,3,4^{\ell-3}\}} = \left\{ \tilde{\sigma}^{(1)}, \dots, \tilde{\sigma}^{(\ell-2)} \right\}, \tag{3.16}$$

where the elements of that class are defined as

$$\tilde{\sigma}^{(i)} := (\sigma_{i+1}\sigma_i\sigma_{i+2})^{c_i} \qquad \text{for } i = 1, \dots, \ell-2. \tag{3.17}$$

It is clear that each element $\tilde{\sigma}^{(i)}$ in (3.17) has order 4, since it is formed from three consecutive elements on the Dynkin diagram and thus being isomorphic to the Coxeter element of A_3 when acting on the three corresponding roots.

Furthermore, by definition all elements of Σ have to be related by a similarity transformation. Indeed we find: *Two consecutive elements in $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are related as*

$$\varkappa_i \tilde{\sigma}^{(i)} = \tilde{\sigma}^{(i+1)} \varkappa_i \qquad \text{with } \varkappa_i := \sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_{i+3} \sigma_{i+1}. \tag{3.18}$$

Therefore all elements in Σ can be related to each other by an adjoint action simply by successive applications of (3.18).

Proof: Let us now prove the relation (3.18). The starting point is the identity

$$\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_{i-1}\sigma_i\sigma_{i+1}, \tag{3.19}$$

which follows by applying the left and right hand side to some arbitrary x using the definition of the simple Weyl reflection (3.2) consecutively. Normalising the length of the roots to be 2, we find in both cases

$$x - [(x \cdot \alpha_{i-1}) + (x \cdot \alpha_i) + (x \cdot \alpha_{i+1})]\,\alpha_{i-1} - [(x \cdot \alpha_i) + (x \cdot \alpha_{i+1})]\,(\alpha_i + \alpha_{i+1}). \tag{3.20}$$

Multiplying (3.19) from the left by $\prod_{k=1}^{i-2}\sigma_k$ and $\prod_{k=i+2}^{\ell}\sigma_k$ from the right and noting that for A_ℓ we have $[\sigma_i, \sigma_j] = 0$ for $|i - j| \geq 2$, it follows

$$\hat{\sigma}\sigma_i = \sigma_{i+1}\hat{\sigma}, \qquad \text{with } \hat{\sigma} := \prod_{k=1}^{\ell}\sigma_k. \tag{3.21}$$

The element $\hat{\sigma}$ is the standard Coxeter element. Multiplying next the identity (3.18) from the left by $\prod_{k=1}^{i-1}\sigma_k$ and $\sigma_{i+1}\prod_{k=i+4}^{\ell}\sigma_k$ from the right and recalling that $\sigma_i^2 = 1$ yields

$$\hat{\sigma}\,(\sigma_i\sigma_{i+2}\sigma_{i+1})^{c_i} = (\sigma_{i+1}\sigma_{i+3}\sigma_{i+2})^{c_i}\,\hat{\sigma}. \tag{3.22}$$

This relation is now easily established by commuting all three simple Weyl reflections through the Coxeter element using the identity (3.21), which in turn also proves (3.18). $\square$

Some special elements in Σ are related by the adjoint action of the Coxeter element σ. We find: *The first and the last element in $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are related as*

$$\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h-c_\ell}{2}} = \sigma^{\frac{h-c_\ell}{2}}\tilde{\sigma}^{(1)}, \tag{3.23}$$

Proof: We prove (3.23) by using the more elementary relations

$$\sigma_{\ell+1-i}\sigma^{\frac{h}{2}+\frac{c_i+c_ic_\ell}{4}} = \sigma^{\frac{h}{2}+\frac{c_i+c_ic_\ell}{4}}\sigma_i. \tag{3.24}$$

For even h we compute by a successive use of (3.24)

$$\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h}{2}} = \sigma_{\ell-2}\sigma_\ell\sigma_{\ell-1}\sigma^{\frac{h}{2}} = \sigma_{\ell-2}\sigma_\ell\sigma^{\frac{h}{2}}\sigma_2 = \sigma_{\ell-2}\sigma^{\frac{h}{2}}\sigma_1\sigma_2 = \sigma^{\frac{h}{2}}\sigma_3\sigma_1\sigma_2 = \sigma^{\frac{h}{2}}\tilde{\sigma}^{(1)}. \tag{3.25}$$

Similarly we compute for odd h

$$\begin{aligned}
\tilde{\sigma}^{(\ell-2)}\sigma^{\frac{h-1}{2}} &= \sigma_{\ell-1}\sigma_{\ell-2}\sigma_\ell\sigma^{\frac{h-1}{2}} = \sigma_{\ell-1}\sigma_{\ell-2}\sigma^{\frac{h-1}{2}}\sigma_1 = \sigma_{\ell-1}\sigma^{\frac{h-1}{2}}\sigma_3\sigma_1 \\
&= \sigma_{\ell-1}\sigma^{\frac{h+1}{2}}\sigma^{-1}\sigma_3\sigma_1 = \sigma^{\frac{h+1}{2}}\sigma_2\sigma^{-1}\sigma_3\sigma_1 = \sigma^{\frac{h-1}{2}}\sigma_-\sigma_+\sigma_2\sigma_+\sigma_-\sigma_3\sigma_1 \\
&= \sigma^{\frac{h-1}{2}}\sigma_3\sigma_1\sigma_2 = \sigma^{\frac{h-1}{2}}\tilde{\sigma}^{(1)}.
\end{aligned} \tag{3.26}$$

Thus we have established that the first element $\tilde{\sigma}^{(1)}$ in the similarity class Σ is related via the similarity transformation (3.23) to the last element $\tilde{\sigma}^{(\ell-2)}$ in this class. In comparison to one rank less the last element is the only additional one. For the other elements we can use the same argumentation but employing the Coxeter element for one rank less. $\square$

$(\tilde{\sigma}^{(1)})^j \backslash \alpha_i$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1)}$	$\mathbf{-1,2}$	$\mathbf{1,2,3}$	$\mathbf{-2,3}$	$2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}$	$\mathbf{-3}$	$\mathbf{-2}$	$\mathbf{-1}$	$1,2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}\tilde{\sigma}^{(1)}$	$\mathbf{2,3}$	$\mathbf{-1,2,3}$	$\mathbf{1,2}$	$3,4$	5	6	7	8

Table 1: The reduced A_8-root space $\tilde{\Delta}$ generated from the orbits of $\tilde{\sigma}^{(1)}$.

Expl.: A_8 We illustrate now the working of these formulae for a concrete example. We consider A_8 and generate the entire root space $\tilde{\Delta}$ as described in (3.13) from $\tilde{\sigma}^{(1)}$. The results are depicted in table 1.

For convenience we used the following conventions: For any non-simple root $\beta = \sum_i \mu_i \alpha_i$ we present only the non-vanishing coefficients μ_i in the table with the overall sign written in front, e.g. $\alpha_1 + \alpha_2 + \alpha_3$ is represented as $1, 2, 3$ and $-\alpha_1 - \alpha_2$ as $-1, 2$. We indicate the A_3 substructure in bold. Further examples for root spaces obtained from different elements in $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ are presented in appendix A.

Crucial to our construction is the invariance under the action of $\tilde{\sigma}^{(1)}_{\pm}$. Acting on the roots as depicted in table 1 with $\tilde{\sigma}^{(1)}_{\pm}$ we recover all the elements in table 1, albeit in a permuted way as indicated in table 2.

$\tilde{\sigma}^{(1)}_{-}(\tilde{\Delta})$	-1	$1,2,3$	-3	$3,4$	5	6	7	8
	$-2,3$	2	$-1,2$	$1,2,3,4$	5	6	7	8
	3	$-1,2,3$	-1	$2,3,4$	5	6	7	8
	$1,2$	-2	$2,3$	4	5	6	7	8

$\tilde{\sigma}^{(1)}_{+}(\tilde{\Delta})$	$1,2$	-2	$2,3$	4	5	6	7	8
	-1	$1,2,3$	-3	$3,4$	5	6	7	8
	$-2,3$	2	$-1,2$	$1,2,3,4$	5	6	7	8
	3	$-1,2,3$	1	$3,4$	5	6	7	8

Table 2: The invariance of the A_8-root space $\tilde{\Delta}$ generated from $\tilde{\sigma}^{(1)}$ under the action of $\tilde{\sigma}^{(1)}_{\pm}$.

3.1.2 The class with modified exponents $\{1, 2^2, 3, 4^{\ell-3}\}$ and $\tilde{h}=4$

Other classes become considerably more complicated. We present here only some examples to indicate this. For instance in the class

$$\Sigma_{\{1,2^2,3,4^{\ell-4}\}} = \left\{ \tilde{\sigma}^{(1,1,1)}, \tilde{\sigma}^{(1,1,2)}, \ldots, \tilde{\sigma}^{(2,1,\ell-4)} \right\}, \qquad (3.27)$$

we have to label the elements by three indices

$$\tilde{\sigma}^{(1,i,j)} := \sigma_i \sigma_{i+2} \sigma_{i+3+j} \sigma_{i+1} \qquad \text{and} \qquad \tilde{\sigma}^{(2,i,j)} := \sigma_i \sigma_{i+1+j} \sigma_{i+3+j} \sigma_{i+j+2} \qquad (3.28)$$

with $i = 1, \ldots, \ell - j - 3$ and $j = 1, \ldots, \ell - 4$. It is easy to convince oneself that these elements have order 4. In both types of labeling we have three consecutive elements and

one additional factor which commutes with all the other elements, that is σ_{i+3+j} in $\tilde{\sigma}^{(1,i,j)}$ and σ_i in $\tilde{\sigma}^{(2,i,j)}$, respectively. Thus by the same argument as in the previous class and the fact that $\sigma_i^2 = 1$ it follows that the order of all elements in (3.28) is 4.

Arguing along similar lines as for the class presented in the previous subsection, we can also show that all elements in $\Sigma_{\{1,2^2,3,4^{\ell-4}\}}$ are indeed related by a similarity transformation. We will not present this proof here.

3.1.3 The similarity class structure with $\tilde{h}=4$

It is clear that for higher ranks more and more possible sets of exponents characterising different classes may exist. Here we only indicate in table 3 the general structure but do not report a detailed construction of the elements of these classes and their interrelations as the argumentation goes along the same lines as in the two previous subsections. By inspection of the table we notice the onset of two new classes when we increase the rank by two, that is the number of classes increases by 2 for $\ell = 2n + 5$ for $n = 1, 2, \ldots$ We also observe that the number of classes for $\ell = 2n + 1$ and $\ell = 2n + 2$ is the same.

ℓ						
3	$\{1,2,3\}$					
4	$\{1,2,3,4\}$					
5	$\{1,2,3,4^2\}$	$\{1,2^2,3,4\}$				
6	$\{1,2,3,4^3\}$	$\{1,2^2,3,4^2\}$				
7	$\{1,2,3,4^4\}$	$\{1,2^2,3,4^3\}$	$\{1,2^3,3,4^2\}$	$\{1^2,2^2,3^2,4\}$		
8	$\{1,2,3,4^5\}$	$\{1,2^2,3,4^4\}$	$\{1,2^3,3,4^3\}$	$\{1^2,2^2,3^2,4^2\}$		
9	$\{1,2,3,4^6\}$	$\{1,2^2,3,4^5\}$	$\{1,2^3,3,4^4\}$	$\{1^2,2^2,3^2,4^3\}$	$\{1^2,2^3,3^2,4^2\}$	$\{1,2^4,3,4^3\}$
10	$\{1,2,3,4^7\}$	$\{1,2^2,3,4^6\}$	$\{1,2^3,3,4^5\}$	$\{1^2,2^2,3^2,4^4\}$	$\{1^2,2^3,3^2,4^3\}$	$\{1,2^4,3,4^4\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
ℓ	$\{1,2,3,4^{\ell-3}\}$	$\{1,2^2,3,4^{\ell-4}\}$	$\{1,2^3,3,4^{\ell-5}\}$	$\{1^2,2^2,3^2,4^{\ell-6}\}$	$\{1^2,2^3,3^2,4^{\ell-7}\}$	$\ldots$

Table 3: Similarity classes in A_ℓ with $\tilde{h} = 4$.

In addition we note that the number of factors in the elements of a similarity class increases by one in the table in each column from the left to the right, starting with three factors on the very left.

3.1.4 The class with modified exponents $\{1, 2, \ldots, 4n-1, 4n^{\ell-4n+1}\}$ and $\tilde{h}=4n$

Let us now generalize the previous considerations towards classes with larger amounts of eigenvalues, such that they are related to modified Coxeter numbers of higher powers. The class (3.16) acquires the more general form

$$\Sigma_{\{1,2,\ldots,4n-1,4n^{\ell-4n+1}\}} = \left\{\tilde{\sigma}^{(n,1)}, \ldots, \tilde{\sigma}^{(n,\ell+2-4n)}\right\}, \tag{3.29}$$

when $x^{4n} = 1$ for $x \in \Sigma$. In this case the elements of the class

$$\tilde{\sigma}^{(n,i)} := \left[\left(\prod_{k=2}^{n} \sigma_{i-1+4(k-1)}\sigma_{i+1+4(k-1)}\right)\sigma_{i+1}\left(\prod_{k=1}^{n} \sigma_{i+4(k-1)}\sigma_{i+2+4(k-1)}\right)\right]^{c_i}, \tag{3.30}$$

are characterised by two indices, n distinguishing the particular type of class and $i = 1, \ldots, \ell + 2 - 4n$ labeling the individual elements in that class. The case $n = 1$ reduces to our previous simpler example with $\tilde{\sigma}^{(n,i)} = \tilde{\sigma}^{(i)}$ as defined in (3.17). Evidently the element $\tilde{\sigma}^{(n,i)}$ contains the $4n - 1$ consecutive factors σ_i to σ_{i+4n-3} separated into odd and even indices. This means each element can be viewed as a Coxeter element for the A_{4n-1}-Weyl group and therefore the order of $\tilde{\sigma}^{(n,i)}$ is $\tilde{h} = 4n$.

In this case we will also establish that all elements in Σ are indeed related by a similarity transformation. Two consecutive elements in this class are related as

$$\varkappa_i^{(n)} \tilde{\sigma}^{(n,i)} = \tilde{\sigma}^{(n,i+1)} \varkappa_i^{(n)} \qquad \text{with } \varkappa_i^{(n)} := \prod_{k=1}^{4n} \sigma_{i+k-1} \prod_{k=1}^{2n-1} \sigma_{i+2k-1}, \qquad (3.31)$$

which in turn implies that all elements in Σ are related by a similarity transformation. The proof for this identity goes along the same line as the one for the particular case $n = 1$ of the identity (3.18).

3.1.5 The class with modified exponents $\left\{1, 2^2, \ldots, 4n - 1, 4n^{\ell - 4n}\right\}$ and $\tilde{\mathbf{h}} = \mathbf{4n}$

For higher order the similarity class (3.27) generalises to

$$\Sigma_{\{1,2^2,\ldots,4n-1,4n^{\ell-4n+1}\}} = \left\{\tilde{\sigma}^{(1,1,1,1)}, \tilde{\sigma}^{(1,2,1,1)}, \ldots\right\}, \qquad (3.32)$$

where we label its elements

$$\tilde{\sigma}^{(1,n,i,j)} := \prod_{k=1}^{n} \sigma_{i+4(k-1)} \sigma_{i+2+4(k-1)} \sigma_{i+j+(\tilde{h}-1)} \sigma_{i+1} \prod_{k=2}^{n} \sigma_{i-1+4(k-1)} \sigma_{i+1+4(k-1)}, \qquad (3.33)$$

$$\tilde{\sigma}^{(2,n,i,j)} := \sigma_{i+j+2} \prod_{k=1}^{n} \sigma_{i+j+4(k-1)} \sigma_{i+j+2+4(k-1)} \sigma_i \prod_{k=2}^{n} \sigma_{i+j+1+4(k-1)} \sigma_{i+j+3+4(k-1)}, \qquad (3.34)$$

now by four indices with $j = 1, \cdots, \ell - 4n$ and $i = 1, \cdots, \ell - j - (4n - 1)$. We recover the case discussed in the previous section for $n = 1$. Using similar arguments as before we can show that all elements in this class have order $\tilde{\mathbf{h}} = \mathbf{4n}$. For instance for the element $\sigma_{i+j+(\tilde{h}-1)}$ in (3.33) the subscript obeys $i + j + (\tilde{h} - 1) > i + \tilde{h}$, which means that the element may be commuted to the left. Taking then the $\tilde{h}$-th power of the entire expression we find

$$\left[(\sigma_{i+j+(\tilde{h}-1)})^{\tilde{h}} \left[\left(\prod_{k=1}^{n} \sigma_{i+4(k-1)} \sigma_{i+2+4(k-1)}\right) \sigma_{i+1} \left(\prod_{k=2}^{n} \sigma_{i-1+4(k-1)} \sigma_{i+1+4(k-1)}\right)\right]^{\tilde{h}}\right]. \qquad (3.35)$$

Since $\tilde{h}$ is even we have $(\sigma_{i+j+(\tilde{h}-1)})^{\tilde{h}} = 1$ and since the expression in the bracket is a reduced Coxeter element for $A_{\tilde{h}=4n}$ the expression in (3.35) equals 1, thus establishing the order of $\tilde{\sigma}^{(1,n,i,j)}$ to be $\tilde{h} = 4n$. Similar arguments can be used for $\tilde{\sigma}^{(2,n,i,j)}$ to prove that this element has the same order.

3.1.6 Antilinearly invariant complex root spaces

Based on the various classes constructed in the previous sections we may now compute the deformation matrix with the help of the Ansatz (3.8) subject to the mentioned constraints. In [9] we found some relatively simple solutions for $h = 4n$. We present now similar solutions for $\tilde{h} = 4n$. Taking in (3.12) all but three coefficients to be zero

$$r_i(\varepsilon) = 0 \qquad \text{for } i \neq 0, n, 2n, \tag{3.36}$$

the equation reduces with the help of (3.10) to

$$1 = (r_0 + r_{2n})^{2\sum_{k=1}^{n} \lambda_{2k}} \left[(r_0 - r_{2n})^2 - 4r_n^2 \right]^{\sum_{k=1}^{n} \lambda_{2k-1}}. \tag{3.37}$$

As can be seen directly, this equation is solved by

$$r_{2n} = 1 - r_0 \qquad \text{and} \qquad r_n = \sqrt{r_0(r_0 - 1)} =: \vartheta. \tag{3.38}$$

Thus the corresponding deformation matrix resulting from (3.8) reads

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + [1 - r_0(\varepsilon)]\,\tilde{\sigma}^{2n} + i\vartheta(\tilde{\sigma}^n - \tilde{\sigma}^{-n}). \tag{3.39}$$

All what is left now is to establish whether the set of modified exponents in (3.10) really exists for some concrete elements of $\tilde{\sigma} \in \mathcal{W}$ of order $\tilde{h} = 4n$ and possibley to specify the function $r_0(\varepsilon)$.

It is useful to consider a concrete example. For instance, the deformed roots resulting from $\tilde{\sigma}^{(3)}$ of the class $\Sigma_{\{1,2,3,4^{\ell-3}\}}$ for A_8 according to (3.39) are

$$
\begin{aligned}
&\tilde{\alpha}_1 = \alpha_1, \ \tilde{\alpha}_7 = \alpha_7, \ \tilde{\alpha}_8 = \alpha_8, \\
&\tilde{\alpha}_2 = \alpha_2 + (1 - r_0)\alpha_3 + (1 - r_0 + i\vartheta)\alpha_4 + (1 - r_0)\alpha_5, \\
&\tilde{\alpha}_3 = (r_0 - i\vartheta)\alpha_3 - 2i\vartheta\alpha_4 + (r_0 - i\vartheta - 1)\alpha_5, \\
&\tilde{\alpha}_4 = 2i\vartheta\alpha_3 + (2r_0 + 2i\vartheta - 1)\alpha_4 + 2i\vartheta\alpha_5, \\
&\tilde{\alpha}_5 = (r_0 - i\vartheta - 1)\alpha_3 - 2i\vartheta\alpha_4 + (r_0 - i\vartheta)\alpha_5, \\
&\tilde{\alpha}_6 = (1 - r_0)\alpha_3 + (1 - r_0 + i\vartheta)\alpha_4 + (1 - r_0)\alpha_5 + \alpha_6.
\end{aligned}
\tag{3.40}
$$

The θ_ε resulting from different elements in the same class have a similar form with the A_3-substructure displaced similarly as for the undeformed roots. We do not report these solutions here. Unlike as in (3.40) all eight roots are deformed when constructing θ_ε for instance from $\tilde{\sigma}^{(2,1)}$ as specified in (3.30)

$$
\theta_\varepsilon = \begin{pmatrix}
r_0 & 0 & i\vartheta & 2i\vartheta & i\vartheta & 0 & r_0 - 1 & 0 \\
0 & r_0 - i\vartheta & -2i\vartheta & -2i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 & 0 \\
i\vartheta & 2i\vartheta & r_0 + 2i\vartheta & 2i\vartheta & r_0 + 2i\vartheta - 1 & 2i\vartheta & i\vartheta & 0 \\
-2i\vartheta & -2i\vartheta & -2i\vartheta & 2r_0 - 2i\vartheta - 1 & -2i\vartheta & -2i\vartheta & -2i\vartheta & 0 \\
i\vartheta & 2i\vartheta & r_0 + 2i\vartheta - 1 & 2i\vartheta & r_0 + 2i\vartheta & 2i\vartheta & i\vartheta & 0 \\
0 & r_0 - i\vartheta - 1 & -2i\vartheta & -2i\vartheta & -2i\vartheta & r_0 - i\vartheta & 0 & 0 \\
r_0 - 1 & 0 & i\vartheta & 2i\vartheta & i\vartheta & 0 & r_0 & 0 \\
1 - r_0 & 1 - r_0 & 1 - r_0 & 1 - r_0 - i\vartheta & 1 - r_0 & 1 - r_0 & 1 - r_0 & 1
\end{pmatrix}.
\tag{3.41}
$$

The dual map $\delta^\star$ is obtained by solving (2.11) for the dual deformation matrix $\theta^\star_\varepsilon$ with the explicit form for θ_ε. Taking the latter to be given by (3.41) we compute for the standard $(\ell+1)$-dimensional representation of A_ℓ $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $i = 1, 2, \ldots, \ell$, $j = 1, 2, \ldots, \ell+1$

$$\theta^\star_\varepsilon = \begin{pmatrix} r_0 & 0 & 0 & i\vartheta & -i\vartheta & 0 & 0 & 1-r_0 & 0 \\ 0 & r_0 & -i\vartheta & 0 & 0 & i\vartheta & 1-r_0 & 0 & 0 \\ 0 & i\vartheta & r_0 & 0 & 0 & 1-r_0 & -i\vartheta & 0 & 0 \\ -i\vartheta & 0 & 0 & r_0 & 1-r_0 & 0 & 0 & i\vartheta & 0 \\ i\vartheta & 0 & 0 & 1-r_0 & r_0 & 0 & 0 & -i\vartheta & 0 \\ 0 & -i\vartheta & 1-r_0 & 0 & 0 & r_0 & i\vartheta & 0 & 0 \\ 0 & 1-r_0 & i\vartheta & 0 & 0 & -i\vartheta & r_0 & 0 & 0 \\ 1-r_0 & 0 & 0 & -i\vartheta & i\vartheta & 0 & 0 & r_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.42}$$

By construction the corresponding dual root space $\tilde{\Delta}^\star(\varepsilon)$ is invariant under the action of some antilinear maps $\varpi^\star$, obtained by solving (2.13). For antilinear symmetry $\omega_1 = \tau\sigma_2\sigma_4\sigma_6$ we compute the dual antilinear transformation to

$$\omega^\star_1 = \tau \begin{pmatrix} \nu^2 & 0 & 0 & -2i\vartheta\nu & 2i\vartheta\nu & 0 & 0 & \mu & 0 \\ 0 & -2i\vartheta\nu & \nu^2 & 0 & 0 & \mu & 2i\vartheta\nu & 0 & 0 \\ 0 & \nu^2 & 2i\vartheta\nu & 0 & 0 & -2i\vartheta\nu & \mu & 0 & 0 \\ -2i\vartheta\nu & 0 & 0 & \mu & \nu^2 & 0 & 0 & 2i\vartheta\nu & 0 \\ 2i\vartheta\nu & 0 & 0 & \nu^2 & \mu & 0 & 0 & -2i\vartheta\nu & 0 \\ 0 & \mu & -2i\vartheta\nu & 0 & 0 & 2i\vartheta\nu & \nu^2 & 0 & 0 \\ 0 & 2i\vartheta\nu & \mu & 0 & 0 & \nu^2 & -2i\vartheta\nu & 0 & 0 \\ \mu & 0 & 0 & 2i\vartheta\nu & -2i\vartheta\nu & 0 & 0 & \nu^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.43}$$

where we abbreviated $\nu := 2r_0 - 1$ and $\mu := 4(r_0 - r_0^2)$. Then the action on the deformed and original variables amounts with (3.43) simply to

$$\omega^\star_1 : \tilde{\Delta}^*(\varepsilon) \to \tilde{\Delta}^*(\varepsilon), \quad \tilde{x}_1 \mapsto \tilde{x}_1, \tilde{x}_2 \leftrightarrow \tilde{x}_3, \tilde{x}_4 \leftrightarrow \tilde{x}_5,, \tilde{x}_6 \leftrightarrow \tilde{x}_7, \tilde{x}_8 \mapsto \tilde{x}_8, \tilde{x}_9 \mapsto \tilde{x}_9, \tag{3.44}$$

$$x_1 \mapsto x_1, x_2 \leftrightarrow x_3, x_4 \leftrightarrow x_5,, x_6 \leftrightarrow x_7, x_8 \mapsto x_8, x_9 \mapsto x_9, i \mapsto -i.$$

A similar computation leads to the dual antilinear symmetry corresponding for $\omega_2 = \tau\sigma_1\sigma_3\sigma_5\sigma_7$.

Obviously these solutions only capture part of all possibilities as we may of course also consider the cases $\tilde{h} = 4n$ and since (3.39) is a restriction of the most general Ansatz (3.7). Some solutions filling these gaps were presented in [9]. Having been fairly detailed for the A_ℓ-Weyl group, we will only indicate some selected examples for reference for the other cases.

3.2 Antilinearly deformed B_ℓ root systems

The simplest class for $\tilde{h} = 4$ contains only one element comprised of two Weyl reflections

$$\Sigma_{\{1,3,4^{\ell-2}\}} = \{\tilde{\sigma} = \sigma_{\ell-1}\sigma_\ell\}. \tag{3.45}$$

The next class with $\tilde{h} = 4$ contains $2\ell - 6$ elements

$$\Sigma_{\{1,2,3,4^{\ell-3}\}} = \left\{ \tilde{\sigma}^{(1,1)}, \ldots, \tilde{\sigma}^{(1,\ell-3)}, \tilde{\sigma}^{(2,1)}, \ldots, \tilde{\sigma}^{(2,\ell-3)} \right\}, \tag{3.46}$$

build from a composition of three Weyl reflections

$$\tilde{\sigma}^{(1,i)} := \sigma_i \sigma_{i+2} \sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}^{(2,i)} := \sigma_\ell \sigma_{\ell-i-2} \sigma_{\ell-1} \quad \text{for} \quad i = 1, \cdots, \ell - 3. \tag{3.47}$$

In table 4 we indicate the different types of classes with increasing rank ℓ. We note that whenever the rank increases by one, a new type of class emerges with one additional Weyl reflection in the element $\tilde{\sigma}$.

ℓ							
3	$\{1,3,4\}$						
4	$\{1,3,4^2\}$	$\{1,2,3,4\}$					
5	$\{1,3,4^3\}$	$\{1,2,3,4^2\}$	$\{1,2^2,3,4\}$				
6	$\{1,3,4^4\}$	$\{1,2,3,4^3\}$	$\{1,2^2,3,4^2\}$	$\{1^2,2,3^2,4\}$			
7	$\{1,3,4^5\}$	$\{1,2,3,4^4\}$	$\{1,2^2,3,4^3\}$	$\{1^2,2,3^2,4^2\}$	$\{1,2^3,3,4^2\}$		
8	$\{1,3,4^6\}$	$\{1,2,3,4^5\}$	$\{1,2^2,3,4^4\}$	$\{1^2,2,3^2,4^3\}$	$\{1,2^3,3,4^3\}$	$\{1^2,2^2,3^2,4^2\}$	
9	$\{1,3,4^7\}$	$\{1,2,3,4^6\}$	$\{1,2^2,3,4^5\}$	$\{1^2,2,3^2,4^4\}$	$\{1,2^3,3,4^4\}$	$\{1^2,2^2,3^2,4^3\}$	
10	$\{1,3,4^8\}$	$\{1,2,3,4^7\}$	$\{1,2^2,3,4^6\}$	$\{1^2,2,3^2,4^5\}$	$\{1,2^3,3,4^5\}$	$\{1^2,2^2,3^2,4^4\}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$
ℓ	$\{1,3,4^{\ell-2}\}$	$\{1,2,3,4^{\ell-3}\}$	$\{1,2^2,3,4^{\ell-4}\}$	$\{1^2,2,3^2,4^{\ell-5}\}$	$\{1,2^3,3,4^{\ell-5}\}$	$\{1,2^4,3,4^{\ell-6}\}$	$\ldots$

Table 4: Similarity classes in B_ℓ with $\tilde{h} = 4$.

We also report explicitly one deformation matrix resulting from these classes for an example for which no solution exists with the assumptions made in [9]. Using for B_5 the same general Ansatz as for the A_ℓ-case in (3.39), we obtain for the antilinearly deformed symmetry of $\tilde{\sigma}^{(1,1)}$ the solution

$$\theta_\varepsilon = \begin{pmatrix} r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 & 0 \\ 2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 & 0 \\ r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 & 0 \\ 1 - r_0 & 1 - r_0 + i\vartheta & 1 - r_0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.48}$$

We compute the dual map $\delta^\star$ by solving (2.11) for the dual deformation matrix $\theta_\varepsilon^\star$ with the explicit form for θ_ε as in (3.48). For the standard root representation of the B_ℓ-roots $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $(\alpha_\ell)_j = \delta_{\ell j}$, $i = 1, 2, \ldots, \ell - 1$, $j = 1, 2, \ldots, \ell$ we obtain

$$\theta_\varepsilon^\star = \begin{pmatrix} r_0 & -i\vartheta & i\vartheta & 1 - r_0 & 0 \\ i\vartheta & r_0 & 1 - r_0 & -i\vartheta & 0 \\ -i\vartheta & 1 - r_0 & r_0 & i\vartheta & 0 \\ 1 - r_0 & i\vartheta & -i\vartheta & r_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.49}$$

By construction the corresponding root space $\tilde{\Delta}^\star(\varepsilon)$ is invariant under the action of some antilinear maps $\varpi^\star$, obtained by solving (2.13). For $\omega_1 = \tau\sigma_1\sigma_3$ we compute the dual antilinear transformation to

$$
\omega_1^\star = \tau \begin{pmatrix}
-2i\vartheta(2r_0 - 1) & (1 - 2r_0)^2 & -4(r_0 - 1)r_0 & 2i\vartheta(2r_0 - 1) & 0 \\
(1 - 2r_0)^2 & 2i\vartheta(2r_0 - 1) & -2i\vartheta(2r_0 - 1) & -4(r_0 - 1)r_0 & 0 \\
-4(r_0 - 1)r_0 & -2i\vartheta(2r_0 - 1) & 2i\vartheta(2r_0 - 1) & (1 - 2r_0)^2 & 0 \\
2i\vartheta(2r_0 - 1) & -4(r_0 - 1)r_0 & (1 - 2r_0)^2 & -2i\vartheta(2r_0 - 1) & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\tag{3.50}
$$

The action on the variables amounts with (3.50) simply to

$$
\omega_1^\star : \tilde{\Delta}^*(\varepsilon) \to \tilde{\Delta}^*(\varepsilon), \quad \tilde{x}_1 \leftrightarrow \tilde{x}_2, \tilde{x}_3 \leftrightarrow \tilde{x}_4, \tilde{x}_5 \mapsto \tilde{x}_5,
\tag{3.51}
$$

$$
x_1 \leftrightarrow x_2, x_3 \leftrightarrow x_4, x_5 \mapsto x_5, i \mapsto -i.
\tag{3.52}
$$

A similar computation leads to the dual antilinear symmetry corresponding to $\omega_2 = \tau\sigma_2$.

3.3 Antilinearly deformed C_ℓ root systems

A simple class for $\tilde{h} = 4$ with only one element $\tilde{\sigma} = \sigma_1\sigma_3\sigma_2$ is the case $\Sigma_{\{1,2,3,4^{\ell-3}\}}$. We present the deformation matrix for the C_4-case resulting from this element

$$
\theta_\varepsilon = \begin{pmatrix}
r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 \\
2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 \\
r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 \\
2(1 - r_0) & 2(1 - r_0) + 2i\vartheta & 2(1 - r_0) & 1
\end{pmatrix}.
\tag{3.53}
$$

Note that the classes for C_ℓ are the same as those for B_ℓ, albeit the deformation matrices are different due to the difference of the Weyl reflections.

3.4 Antilinearly deformed D_ℓ root systems

In this case a simple class for $\tilde{h} = 4$ contains $\ell - 1$ elements

$$
\Sigma_{\{1,3,4^{\ell-3}\}} = \left\{ \tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)}, \ldots, \tilde{\sigma}^{(\ell=2)}, \tilde{\sigma}^{(\ell)} \right\}.
\tag{3.54}
$$

with

$$
\tilde{\sigma}^{(i)} = \sigma_i\sigma_{i+2}\sigma_{i+1} \quad \text{and} \quad \tilde{\sigma}^{(\ell)} = \sigma_{\ell-3}\sigma_\ell\sigma_{\ell-2} \quad \text{for} \quad i = 1, \cdots, \ell - 2.
\tag{3.55}
$$

As an example for a deformation matrix for D_4 we present the one resulting from $\tilde{\sigma}^{(1)} = \sigma_1\sigma_3\sigma_2$

$$
\theta_\varepsilon = \begin{pmatrix}
r_0 - i\vartheta & -2i\vartheta & r_0 - i\vartheta - 1 & 0 \\
2i\vartheta & 2r_0 + 2i\vartheta - 1 & 2i\vartheta & 0 \\
r_0 - i\vartheta - 1 & -2i\vartheta & r_0 - i\vartheta & 0 \\
1 - r_0 - i\vartheta & 2 - 2r_0 & 1 - r_0 - i\vartheta & 1
\end{pmatrix}.
\tag{3.56}
$$

3.5 Antilinearly deformed E_{6+n} root systems

We may treat the exceptional algebras together using for the labeling the E_8-convention in [10] and removing vertices from the long end Dynkin diagram to obtain the E_7 and E_6-cases. A simple class for $\tilde{h} = 4$ contains $n + 5$ elements

$$\Sigma_{\{1,2,3,4^3\}} = \left\{ \sigma_1\sigma_3\sigma_4, \sigma_1\sigma_5\sigma_4, \sigma^{(2)}, \sigma^{(3)} \ldots, \sigma^{(n+4)} \right\}, \tag{3.57}$$

with $\sigma^{(i)} = \sigma_i\sigma_{i+2}\sigma_{i+1}$ for $i = 2, .., n + 4$. The deformation matrix for $\sigma^{(2)} = \sigma_3\sigma_2\sigma_4$ is computed to

$$\theta_\varepsilon = \begin{pmatrix} 1 & 1 - r_0 & -r_0 - i\vartheta + 1 & 1 - r_0 & 0 & 0 & \cdots \\ 0 & r_0 + i\vartheta & 2i\vartheta & r_0 + i\vartheta - 1 & 0 & 0 & \cdots \\ 0 & -2i\vartheta & 2r_0 - 2i\vartheta - 1 & -2i\vartheta & 0 & 0 & \cdots \\ 0 & r_0 + i\vartheta - 1 & 2i\vartheta & r_0 + i\vartheta & 0 & 0 & \cdots \\ 0 & 1 - r_0 & -r_0 - i\vartheta + 1 & 1 - r_0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.58}$$

A further class is $\Sigma_{\{1,2^2,3,4^{2+n}\}}$ with elements $\tilde{\sigma} = \sigma_1\sigma_4\sigma_2\sigma_5, \ldots$

3.6 Antilinearly deformed F_4 root systems

The simplest class for $\tilde{h} = 4$ contains only one element

$$\Sigma_{\{1,3,4^2\}} = \{\sigma_3\sigma_2\}. \tag{3.59}$$

The deformation matrix is computed to

$$\theta_\varepsilon = \begin{pmatrix} 1 & 2(1 - r_0) & 2(1 - r_0) - 2i\vartheta & 0 \\ 0 & 2r_0 + 2i\vartheta - 1 & 4i\vartheta & 0 \\ 0 & -2i\vartheta & 2r_0 - 2i\vartheta - 1 & 0 \\ 0 & 1 - r_0 + i\vartheta & 2(1 - r_0) & 1 \end{pmatrix}. \tag{3.60}$$

4. Deformation matrices from two arbitrary elements in $\mathcal{W}$

The procedure outlined in section 2 is entirely generic and may of course also be carried out by starting from any arbitrary elements in $\mathcal{W}$ different from σ_+ and σ_-. Due to the random choice we allow for the symmetries we have to consider now concrete cases. It is instructive to discuss some examples for which no nontrivial solutions were found previously.

Let us therefore consider B_3. As an abstract Coxeter group B_3 is fully characterized by three involutory generators $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{I}$ together with the three relations $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $\sigma_2\sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_3\sigma_2$. Choosing now in (2.3) the involutions different from the previous section as $\hat{\omega}_1 = \sigma_1$ and $\hat{\omega}_2 = \sigma_1\sigma_3$ yields $\Omega_{12} = \sigma_3$. Thus we have taken $\hat{\omega}_1$ and $\hat{\omega}_2$ both to be factors in σ_-. According to (2.6) we have to identify next

all elements in B_3 commuting with σ_3. Using the three relations and the three generators we find $\{\mathbb{I}, \sigma_1, \sigma_3, \sigma_1\sigma_3, \sigma_1\sigma_2\sigma_3\sigma_2\}$ leading to the Ansatz

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + r_1(\varepsilon)\sigma_1 + r_2(\varepsilon)\sigma_3 + r_3(\varepsilon)\sigma_1\sigma_3 + r_4(\varepsilon)\sigma_1\sigma_2\sigma_3\sigma_2. \tag{4.1}$$

which solves all the constraints (2.9) when

$$r_0 = \sqrt{1 - r_4^2} - r_2, \quad r_1 = -r_3 = 1 + r_2 - \frac{r_4}{2} - \frac{1}{2}\sqrt{1 - r_4^2}, \quad \lim_{\varepsilon \to 0} r_4 = 0, \quad r_4 \in i\mathbb{R}. \tag{4.2}$$

Upon substitution into the Ansatz (4.1) the deformation matrix takes on the form

$$\theta_\varepsilon = \begin{pmatrix} \sqrt{1 - r_4^2} + r_4 & 2r_4 & 2r_4 \\ -r_4 & \sqrt{1 - r_4^2} - r_4 & \sqrt{1 - r_4^2} - r_4 - 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Thus we have only one free function left. The same Ansatz (4.1) can be used for the choice $\hat{\omega}_1 = \sigma_2$ and $\hat{\omega}_2 = \sigma_2\sigma_3$, but in that case we are led to the trivial solution $\theta_\varepsilon = \mathbb{I}$.

With the help of (4.3) we may now also find the dual map $\delta^\star$ by solving (2.11) for the dual deformation matrix $\theta_\varepsilon^\star$. For the standard root representation of B_3 we obtain

$$\theta_\varepsilon^\star = \begin{pmatrix} \sqrt{1 - r_4^2} & -r_4 & 0 \\ r_4 & \sqrt{1 - r_4^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

The corresponding root space $\tilde{\Delta}^\star(\varepsilon)$ is then by construction invariant under the action of some antilinear maps $\varpi^\star$, obtained by solving (2.13). For $\omega_1 = \tau\sigma_1$ and $\omega_2 = \tau\sigma_1\sigma_3$ we compute

$$\omega_1^\star : \tilde{\Delta}^\star(\varepsilon) \to \tilde{\Delta}^\star(\varepsilon), \quad \tilde{x}_1 \leftrightarrow \tilde{x}_2, \tilde{x}_3 \mapsto \tilde{x}_3 \equiv x_1 \leftrightarrow x_2, x_3 \mapsto x_3, i \mapsto -i, \tag{4.5}$$

$$\omega_2^\star : \tilde{\Delta}^\star(\varepsilon) \to \tilde{\Delta}^\star(\varepsilon), \quad \tilde{x}_1 \leftrightarrow \tilde{x}_2, \tilde{x}_3 \mapsto -\tilde{x}_3 \equiv x_1 \leftrightarrow x_2, x_3 \mapsto -x_3, i \mapsto -i. \tag{4.6}$$

Taking $r_4 = \pm i \sinh \varepsilon$ we notice that $\theta_\varepsilon^\star$ becomes a rotation about the complex angle $\pm i\varepsilon$ for the variables x_1 and x_2 accompanied by a reflection in x_3 for the latter case.

5. Deformation matrices from rotations in the dual space

So far we have started with given antilinear involution ϖ_i and constructed the deformation map δ by solving the constraints (2.9) for a given Weyl group, i.e. given some ω_i we determined the deformation matrix θ_ε. Subsequently we constructed the corresponding maps $\delta^\star$ and $\varpi^\star$ acting in the dual spaces. We may also try to reverse the procedure and start with the dual space with given maps $\delta^\star$ and $\varpi^\star$ and determine the maps ϖ_i and δ thereafter. In view of the last section it is natural to assume the $\theta_\varepsilon^\star$ to be an element of the special orthogonal group. We define therefore the $(2n + 1) \times (2n + 1)$-matrix

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \quad \text{with } R = \begin{pmatrix} \cosh \varepsilon & i \sinh \varepsilon \\ -i \sinh \varepsilon & \cosh \varepsilon \end{pmatrix}, \tag{5.1}$$

and construct the deformation matrix θ_ε by solving (2.11). We note that this constraint might not admit any solutions for certain Weyl groups. In fact for the standard representation for A_ℓ it is easy to verify that indeed there exists no solution. However, for the special orthogonal Weyl groups $B_\ell \equiv SO(2\ell+1)$ and $D_\ell \equiv SO(2\ell)$ one can solve (2.11). Since previously we did not find solutions for odd ranks in the B-series in [9] based on the assumptions made in there, we present here some solutions for B_{2n+1}. Solving (2.13) for θ_ε using the standard representation for the B_ℓ-roots we compute the deformed roots to

$$\tilde{\alpha}_{2j-1} = \cosh\varepsilon\alpha_{2j-1} + i\sinh\varepsilon\left(\alpha_{2j-1} + 2\sum_{k=2j}^{\ell}\alpha_k\right) \qquad \text{for } j = 1,\dots,n, \qquad (5.2)$$

$$\tilde{\alpha}_{2j} = \cosh\varepsilon\alpha_{2j} - i\sinh\varepsilon\left(\sum_{k=2j}^{2j+2}\alpha_k + 2\sum_{k=2j+3}^{\ell}2\alpha_k\right) \quad \text{for } j = 1,\dots,n-1, \quad (5.3)$$

$$\tilde{\alpha}_{\ell-1} = \cosh\varepsilon(\alpha_{\ell-1} + \alpha_\ell) - \alpha_\ell - i\sinh\varepsilon\left(\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell\right), \qquad (5.4)$$

$$\tilde{\alpha}_\ell = \alpha_\ell. \qquad (5.5)$$

By construction we have satisfied the last three constraints in (2.9). Furthermore, we find that $\theta_\varepsilon^*\sigma_- = \sigma_-\theta_\varepsilon$ but $\theta_\varepsilon^*\sigma_+ \neq \sigma_+\theta_\varepsilon$ with $\sigma_- = \prod_{k=1}^{n+1}\sigma_{2k-1}$ and $\sigma_+ = \prod_{k=1}^{n}\sigma_{2k}$. Thus in this case $\tau\sigma_+$ does not constitute an antilinear symmetry which implies that $[\sigma,\theta_\varepsilon] \neq 0$. This is the reason why this solution has escaped the previous analysis. However, besides under the action of $\omega_- := \tau\sigma_- = \sigma_-^\varepsilon$ the root space $\tilde{\Delta}(\varepsilon)$ is left invariant under various other antilinear maps which consist of subfactors of σ_-. For B_3 we observed this in section 4 with $\sigma_- = \sigma_1\sigma_3$ and σ_3 being the additional symmetry. A generalization to B_{2n} is straightforward simply by starting in (5.1) with an $(2n)\times(2n)$-matrix of the form (5.1) without the entry 1.

Similarly as for B_{2n+1} we may also solve (2.11) for the D_{2n} Weyl group for which we demonstrated in [9] that no solution to the constraining equations (2.9) based on the Ansatz (3.8) could exist, that is for given invariance σ_-^ε and σ_+^ε. Starting with $\theta_\varepsilon^\star$ in the form (5.1) we construct the deformed roots with standard representation for the D_ℓ-roots $(\alpha_i)_j = \delta_{ij} - \delta_{(i+1)j}$, $(\alpha_\ell)_j = \delta_{j(\ell-1)} + \delta_{j\ell}$, $i = 1,2,\dots,\ell-1$, $j = 1,2,\dots,\ell$ as

$$\tilde{\alpha}_{\ell-(2j+1)} = \cosh\varepsilon\alpha_{\ell-(2j+1)} + i\sinh\varepsilon\left[\sum_{k=\ell-(2j+1)}^{\ell}\alpha_k + \sum_{\ell-2j}^{\ell-2}\alpha_k\right], \qquad (5.6)$$

$$\tilde{\alpha}_{\ell-(2j+2)} = \cosh\varepsilon\alpha_{\ell-(2j+2)} - i\sinh\varepsilon\left[\sum_{k=\ell-2j-3}^{\ell}\alpha_k + \sum_{\ell-2j}^{\ell-2}\alpha_k\right], \qquad (5.7)$$

$$\tilde{\alpha}_{\ell-2} = \cosh\varepsilon\alpha_{\ell-2} - i\sinh\varepsilon(\alpha_{\ell-3} + \alpha_{\ell-2} + \alpha_\ell), \qquad (5.8)$$

$$\tilde{\alpha}_{\ell-1} = \cosh\varepsilon\alpha_{\ell-1} + i\sinh\varepsilon\alpha_\ell, \qquad (5.9)$$

$$\tilde{\alpha}_\ell = \cosh\varepsilon\alpha_\ell - i\sinh\varepsilon\alpha_{\ell-1}. \qquad (5.10)$$

Similarly as for the B_{2n+1} case we find that $\theta_\varepsilon^*\sigma_- = \sigma_-\theta$ whereas $\theta_\varepsilon^*\sigma_+ \neq \sigma_+\theta$ with $\sigma_- = \prod_{k=1}^{n}\sigma_{2k-1}$ and $\sigma_+ = \prod_{k=1}^{n-2}\sigma_{2k}$. Again it is easy enough to generalize this to the the D_{2n+1} case.

For the standard $(n+1)$-dimensional representation of A_ℓ a rotation on a subspace of $\tilde{\Delta}^*(\varepsilon)$ for the first two coordinates and its conjugate momenta was suggested in [14, 15]. In that case, and for its generalisation (5.1), the corresponding deformation $\tilde{\Delta}(\varepsilon)$ can not be constructed since (2.11) admits no solution.

6. The construction of q-deformed roots

Mainly motivated by an applications to affine Toda field theories in mind we provide in this section a construction for q-deformed roots, meaning that we are seeking a map

$$\delta_q : \ \Delta \subset \mathbb{R}^n \to \Delta_q \subset \mathbb{R}^n[q], \qquad \alpha \mapsto \alpha_q = \Theta_q \alpha, \tag{6.1}$$

with $\mathbb{R}^n[q]$ denoting a polynomial ring in $q \in \mathbb{C}$. In this case the complex deformation matrix Θ_q depends on the deformation q in such a way that $\lim_{q \to 1} \Theta_q = \mathbb{I}$. Our construction is centered around a q-deformation of the Coxeter element in the factorised form already used in this manuscript $\sigma := \sigma_- \sigma_+$ as introduced in [12, 13]

$$\sigma_q := \sigma_-^q \, \tau_q \, \sigma_+^q \, \tau_q \ . \tag{6.2}$$

The deformations of the Coxeter factors $\sigma_\pm$ are defined as

$$\sigma_\pm^q := \prod_{i \in V_\pm} \sigma_i^q \ , \tag{6.3}$$

where the product is taken over q-deformed Weyl reflections, whose action on simple roots $\alpha_i \in \Delta$ is given as

$$\sigma_i^q(\alpha_j) := \alpha_j - (2\delta_{ij} - [I_{ji}]_q)\alpha_i \ . \tag{6.4}$$

We employed here one of the standard definition for a q-deformed integer[1]

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \ . \tag{6.5}$$

A further deformation in q results from the map τ_q also employed in (6.2)

$$\tau_q(\alpha_i) := q^{t_i}\alpha_i \ . \tag{6.6}$$

The integers t_i are the symmetrizers of the incidence matrix I, i.e. $I_{ij}t_j = I_{ji}t_i$. From these definitions it is evident the q-deformed Coxeter element is only different from the ordinary one when the associated Weyl group is related to non-simply laced algebras.

Since σ_q is defined by its action on the simple roots α it is natural to seek an operator $\mathcal{O}_q$ acting on elements $\alpha_q \in \Delta_q$ with the appropriate limit $\lim_{q \to 1} \mathcal{O}_q = \sigma$. Recalling that the order of σ is the Coxeter number h, i.e. $\sigma^h = 1$, whereas the order of σ_q is deformed $\sigma_q^h = q^{2H}$, it is obvious that the relation can not be a simple similarity transformation. Here H is the ℓ-th Coxeter number of the dual algebra, see e.g. [16] for more details. Therefore we make the Ansatz

$$\sigma_q \alpha = q^{2H/h}\Theta_q^{-1}\sigma\Theta_q \, \alpha \ = q^{2H/h}\Theta_q^{-1}\sigma \, \alpha_q. \tag{6.7}$$

---

[1]We will frequently use the identities $[1]_q = 1$, $[2]_q = q + q^{-1}$ and $[3]_q = 1 + q^2 + q^{-2}$.

and readily identify the operator $\mathcal{O}_q = q^{2H/h}\Theta_q^{-1}\sigma$. The relation (6.7) serves as the defining relation for the q-deformed simple roots $\alpha_q = \Theta_q \alpha$.

In analogy to the undeformed situation we introduce a q-deformed simple root dressed by a colour value as a separate quantity $(\gamma_q)_i := c_i (\alpha_q)_i$. This serves as a representant to introduce the q-deformed Coxeter orbits

$$(\Omega_q)_i := \left\{ (\gamma_q)_i, \sigma (\gamma_q)_i, \ldots, \sigma^{h-1} (\gamma_q)_i \right\}. \tag{6.8}$$

The entire q-deformed root system Δ_q is then spanned by the union of all ℓ q-deformed Coxeter orbits

$$\Delta_q := \bigcup_{i=1}^{\ell} (\Omega_q)_i. \tag{6.9}$$

At this stage it is not obvious under which type of symmetry Δ_q is left invariant.

6.1 The q-deformed root space for $\left(C_2^{(1)}, D_3^{(2)}\right)$

Let is now illustrate the working of the above formulae with a simple explicit example. The incidence matrix for C_2 is in this case defined as $I_{12} = 1$, $I_{21} = 2$, such that the symmetrizers are $t_1 = 1$ and $t_2 = 2$. The Coxeter numbers are $h = 4$ and $H = 6$. Therefore we obtain

$$\sigma_-^q = \begin{pmatrix} -1 & 0 \\ [2]_q & 1 \end{pmatrix}, \quad \sigma_+^q = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \tau_q = \begin{pmatrix} q & 1 \\ 0 & q^2 \end{pmatrix}, \quad \sigma_q = q^2 \begin{pmatrix} -1 & -q \\ [2]_q & 1 \end{pmatrix}. \tag{6.10}$$

Solving equation (6.7) then yields the deformed roots

$$(\alpha_q)_1 = r_1\alpha_1 + \frac{q}{1+q}(r_1 - r_2)\alpha_2, \tag{6.11}$$

$$(\alpha_q)_2 = \frac{r_2 + (r_2 - 2r_1)q^2}{q + q^2}\alpha_1 + r_2\alpha_2, \tag{6.12}$$

where r_1, r_2 depend on q with the limiting behaviour $\lim_{q\to 1} r_1 = 1$ and $\lim_{q\to 1} r_2 = 1$.

7. Non-Hermitian Calogero models

We can now formulate and investigate models on these complex root spaces. Thus we may consider new types of non-Hermitian generalisations of Calogero models

$$\mathcal{H}_{0,\varepsilon,q}(p,x) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_\alpha (\alpha \cdot x)^2 + \sum_\alpha \frac{g_\alpha}{(\alpha \cdot x)^2}, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q, \tag{7.1}$$

or the analogues of Calogero-Moser-Sutherland models when replacing the rational potential by a trigonometric or elliptic one. The model $\mathcal{H}_\varepsilon$ for the rational potential has been investigated previously [4, 5, 9, 10] and was found to have remarkable properties when compared with the standard undeformed models $\mathcal{H}_0$. As a result of the deformation into the complex domain the singularities in the potential are regularized. Therefore the models no longer have to be defined in separate disjointed regimes and continued by phase

factors corresponding to some selected statistics. As was shown in [9], in the $\mathcal{H}_\varepsilon$-models the anyonic phase factors are automatically present and the models can be defined on the entire domain of the configuration space. As a consequence the energy spectra of these models will also be different. Various ground state wavefunctions and those corresponding to exited states were computed in [9] and [5], respectively.

Since the Hamiltonians $\mathcal{H}_{\varepsilon,q}$ are not Hermitian the canonical variables p and x are non-observable in the standard Hilbert space. However, it is by now well understood how to reconcile this by constructing a well defined metric operator ρ [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. One seeks a linear, invertible, Hermitian and positive operator acting in the Hilbert space, such that $\mathcal{H}_{\varepsilon,q}$ becomes a self-adjoint operator with regard to this metric such that p and x become observable in this space. For this purpose one constructs a so-called Dyson map η, which maps the non-Hermitian Hamiltonian H adjointly to a Hermitian Hamiltonian h

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \text{ with } \rho = \eta^\dagger \eta. \tag{7.2}$$

Depending on the assumptions made on the metric such type of Hamiltonians are referred to with different terminology. When no assumption is made on the positivity of the ρ in (7.2), the relation on the right hand side constitutes the *pseudo-Hermiticity* condition, see e.g. [28, 29, 30], whenever the operator ρ is linear, invertible and Hermitian. In case the operator ρ is positive but not invertible this condition is usually referred to as *quasi-Hermiticity* [31, 32]. Different terminology is used at times with a less clear meaning.

In general we can not map the Hamiltonians $\mathcal{H}_{\varepsilon,q}$ to some Hermitian counterparts in a very obvious way, but in some case we can provide the explicit transformation η. We recall that the rotations in (5.1) on two variables can be realised by means of the angular momentum operators $L_{ij} = x_i p_j - x_j p_i$

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\}, \ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}. \tag{7.3}$$

Noting furthermore that

$$\mathcal{H}_0(\tilde{p}, \tilde{x}) = \mathcal{H}_\varepsilon(p, x), \tag{7.4}$$

we can find many explicit transformations of the type (7.2), which map these Hamiltonians to some isospectral Hermitian counterpart

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}. \tag{7.5}$$

For instance for the B_ℓ-models based on the deformations (5.1) the Dyson map is simply

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}. \tag{7.6}$$

In other cases based on special orthogonal groups the rotations involved might not commute. For instance, for the B_5-model based on the deformation (3.49) with $r_0 = \cosh^2 \varepsilon$ we find that

$$\tilde{x} = \theta_\varepsilon^\star x = R_{24}^{-1} R_{13} R_{34} R_{12}^{-1} x = \eta x \eta^{-1}, \qquad \text{with } \eta = \eta_{24}^{-1} \eta_{13} \eta_{34} \eta_{12}^{-1}. \tag{7.7}$$

When the deformation in the configuration space is not based on rotations such that inner products are not preserved it remains a challenge to find the corresponding Dyson maps and isospectral Hermitian couterparts. We also leave the investigation for the $\mathcal{H}_q(p, x)$-models for further investigations.

8. Non-Hermitian affine Toda theories

One of the main obstacles to overcome when passing from a classical description of a field theory to a full-fledged quantum field theory is renormalisation. In 1+1 space-time dimensions many miracles occur which allow to express a number of physical quantities in an exact, that is non-perturbative, manner. In particular it is possible to formulate classical Lagrangians which are in some sense exact from the quantum field theory point of view. The classical affine Toda field theory is a prototype for this kind of behaviour and has the remarkable property that its the classical mass ratios remain preserved in the quantum field theory after renormalisation, whenever the associated Lie algebra is simply laced [33, 34, 35, 36, 37, 38, 39, 40]. This property ceases to be valid when the algebra becomes non-simply laced [41, 42, 43, 44, 45, 46, 12, 13], in which case one has to consider a dual pair of affine Lie algebras [16] and the quantum mass ratios interpolate via an effective coupling constant between the values obtained from these two algebras. In the strong and weak limit of the coupling constant either of these two cases is obtained.

One may now pose the question whether it is also possible to formulate some naturally modified Lagrangians for non-simply laced algebras which already capture some exact features from the quantum level, such as preserving the classical mass ratios when renormalised. In addition we may study models in which the roots are elements of the antilinearly invariant space. In terms of simple roots we consider now the three different versions of affine Toda field theories defined by the Lagrangians

$$\mathcal{L}_{0,\varepsilon,q} := \frac{1}{2} \sum_{i=1}^{\ell} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{m^2}{\beta^2} \sum_{i=0}^{\ell} n_i e^{\beta \alpha_i \cdot \phi}, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q. \qquad (8.1)$$

The Lagrangian $\mathcal{L}_0$ corresponds to the standard version whereas $\mathcal{L}_{\varepsilon,q}$ are newly proposed models. The ℓ components of ϕ are real scalar fields, m an overall mass scale and the β is the coupling constant. The α's are simple roots with α_0 being the negative of the longest root, whose expansion in terms of simple roots in the relevant spaces $\alpha_0 = -\sum_{i=1}^{\ell} n_i \alpha_i$ is the defining relation for the integers n_i, often referred to as Kac labels. The $\mathcal{L}_0$ theories are known to fall roughly into two different classes characterised by β taken to be either real or or purely complex in which case the Yang-Baxter equation obeyed by the scattering matrix is either trivial or non-trivial, respectively. When $\beta \in i\mathbb{R}$ the theory is in general non-Hermitian, except for the A_2-case corresponding to the sine-Gordon model, but the classical mass spectra were still found to be real and stable with respect to small perturbations [47]. Here we conjecture that the $\mathcal{L}_{\varepsilon,q}$-models are also meaningful.

The classical mass matrix for the scalar fields is simply given by the quadratic term in the fields of the Lagrangian and is easily extracted from the formulation (8.1)

$$M_{ij}^2 = m^2 \sum_{a=0}^{\ell} n_a \alpha_a^i \alpha_a^j, \qquad \alpha_i \in \Delta, \tilde{\Delta}(\varepsilon), \Delta_q. \tag{8.2}$$

The mathematical fact that the overall length of the roots is a matter of convention is reflected in the physical property that the overall mass scale is not fixed. This is captured in the constant m.

8.1 The mass spectrum of $\left(C_2^{(1)}, D_3^{(2)}\right)$-$\mathcal{L}_q$

Taking the two q-deformed simple roots to be of the form (6.11), (6.12), noting that the Kac labels for C_2 are $n_1 = 2$, $n_2 = 1$ and using the non-standard representation for the undeformed C_2-roots $\alpha_1 = \{0, 1\}$, $\alpha_2 = \{1, -1\}$ we compute the mass matrix in (8.2). The virtue of this basis is that in the limit $q \to 1$ the mass matrix is diagonal. For $q \neq 1$ the direct evaluation leads to a nondiagonal matrix. However, imposing the additional constraint

$$r_2 = r_1 q \frac{3q^2 - 5q + 2 + (q+1)\sqrt{(16 - 7q)q - 8}}{2\left(2q^3 - q^2 + q - 1\right)}, \tag{8.3}$$

eliminates the off-diagonal elements. We obtain

$$M_{11}^2 = r_1^2 q^3 \frac{2q^3 + 8q^2 - 7q + \left(1 - 2q^2\right)\sqrt{16q - 7q^2 - 8}}{\left(1 - 2q^3 + q^2 - q\right)^2}, \tag{8.4}$$

$$M_{22}^2 = r_1^2 q \frac{11q^5 - 18q^4 + 19q^3 - 10q^2 + q + \left(q^4 + 2q^3 - 3q^2 + 2q - 1\right)\sqrt{16q - 7q^2 - 8}}{\left(2q^3 - q^2 + q - 1\right)^2},$$

with $m_1 = M_{11}$, $m_2 = M_{22}$ being the classical masses of the two scalar fields. As can be found in the above mentioned literature, the quantum mass ratios of the $\mathcal{L}_0$-theory are given by

$$\frac{m_1}{m_2} = \frac{\sin\left[\frac{1}{24}(6 - B)\pi\right]}{\cos\left(\frac{B\pi}{12}\right)}, \qquad \text{with } B = \frac{2H\beta^2}{H\beta^2 + 4\pi\ell h}, \tag{8.5}$$

where $B \in [0, 2]$ denotes the effective coupling constant. From (8.4), (8.5) and (8.5) we can therefore fix the deformation parameter such that the quantum mass ratios of $\mathcal{L}_0$ correspond to the classical mass ratios of $\mathcal{L}_q$. We find

$$q = \frac{1}{1 + \sqrt{3\left(\cos\frac{B\pi}{24} + \sin\frac{B\pi}{24}\right) + 2\sin\frac{B\pi}{12} - 3}}, \tag{8.6}$$

$$= 1 - \frac{1}{2}\sqrt{\frac{7\pi}{6}}\sqrt{B} + \frac{7\pi B}{24} - \frac{193\pi^{3/2}B^{3/2}}{192\sqrt{42}} + \frac{95\pi^2 B^2}{1152} + O\left(B^{5/2}\right). \tag{8.7}$$

Notice that deformation parameter $q(B)$ is a decreasing real valued function of B taking values between 1 and ≈ 0.435936. Consequently the coefficients in (6.11), (6.12) in front of the simple roots acquire complex when the effective coupling constant varies between 0 and 2.

The classical mass spectrum of $\mathcal{L}_q$ equals the quantum mass spectrum of $\mathcal{L}_0$.

9. Conclusions

We have provided two alternative general methods of construction for complex root systems. The first is based on using some selected elements of the Weyl group as analogues of the parity transformation, which are then extended such that the entire root space remains invariant under at least one antilinear symmetry. We have provided explicit solutions of different types for a large number of specific Weyl groups. Since the suggested method is very generic, i.e. allowing to start from any element in the Weyl group, it is useful to select a further principle providing some guidance. Starting from the factors of the Coxeter element serves for that purpose, but we have also seen that this is often too restrictive and for certain algebras it could be shown that no solutions exist in such a setting. However, we demonstrated that this can be overcome when starting from reduced versions of these factor. The drawback is then that this gives rise to a large number of possibilities. Nonetheless, as we demonstrated many of them lie in the same similarity class, which provides a certain ordering principle. When giving up even this guiding principle one can still find interesting solutions. The construction becomes even less restrictive if we also give up the demand of preserving the inner products. We have paid particular attention to the construction of the deformed variables in the dual space together with the corresponding antilinear symmetries. The second type of construction is based on deformations of the standard Coxeter element. The complex roots resulting from this procedure are not naturally invariant under an obvious symmetry.

For the deformations related to the special orthogonal groups we identified in some cases the corresponding rotations in the dual space. We also reversed the construction in some examples and identified the corresponding deformed roots when starting from certain rotations. It would be interesting to have a precise one-to-one relation between the deformed roots and deformed variables. We leave this as an open challenge.

Both constructions may be employed in the context of multi-particle systems. Here we indicated that all non-Hermitian Calogero-Moser-Sutherland models of the type $\mathcal{H}_\varepsilon(p, x)$ based on B_ℓ and D_ℓ Weyl groups may be mapped onto a Hermitian model via similarity transformations involving various combinations of the angular momentum operators. For the models based on other Weyl groups we expect this transformation to exist, but leave the explicit construction for future investigations. Further interesting open questions for future investigations are to find the explicit solutions including their modified spectra and to settle the questions of whether the deformed models are still integrable.

The second type of construction was employed explicitly to define a new type of non-Hermitian affine Toda theory. These models were found to have the interesting property that their classical mass ratios are identical in all orders of the coupling constant to the quantized and renomalised version of their undeformed couterparts. We leave the interesting problem of investigating more examples for different types of algebras for the future.

A. Appendix

In this appendix we provide more examples of reduced root spaces generated from different types of classes. We exhibit also the action of $\tilde\sigma_\pm$ on the simple roots from which one can easily infer the invariance of the entire root space. We use the same conventions as for the tables 2 and 3.

A.1 A_8-Root spaces based on the class $\Sigma_{\{1,2,3,4,\ell-3\}}$ and their invariance

$\tilde\sigma^{(i)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde\sigma^{(1)}$	$-\mathbf{1},\mathbf{2}$	$\mathbf{1},\mathbf{2},\mathbf{3}$	$-\mathbf{2},\mathbf{3}$	$2,3,4$	5	6	7	8
$\tilde\sigma^{(1)^2}$	$-\mathbf{3}$	$-\mathbf{2}$	$-\mathbf{1}$	$1,2,3,4$	5	6	7	8
$\tilde\sigma^{(1)^3}$	$\mathbf{2},\mathbf{3}$	$-\mathbf{1},\mathbf{2},\mathbf{3}$	$\mathbf{1},\mathbf{2}$	$3,4$	5	6	7	8
$\tilde\sigma^{(1)}_-$	-1	$1,2,3$	-3	$3,4$	5	6	7	8
$\tilde\sigma^{(1)}_+$	$1,2$	-2	$2,3$	4	5	6	7	8
$\tilde\sigma^{(2)}$	$1,2$	$\mathbf{3},\mathbf{4}$	$-\mathbf{2},\mathbf{3},\mathbf{4}$	$\mathbf{2},\mathbf{3}$	$4,5$	6	7	8
$\tilde\sigma^{(2)^2}$	$1,2,3,4$	$-\mathbf{4}$	$-\mathbf{3}$	$-\mathbf{2}$	5	6	7	8
$\tilde\sigma^{(2)^3}$	$1,2,3$	$-,\mathbf{2},\mathbf{3}$	$\mathbf{2},\mathbf{3},\mathbf{4}$	$-,\mathbf{3},\mathbf{4}$	$3,4,5$	6	7	8
$\tilde\sigma^{(2)}_-$	1	$2,3$	-3	$3,4$	5	6	7	8
$\tilde\sigma^{(2)}_+$	$1,2$	-2	$2,3,4$	-4	$4,5$	6	7	8
$\tilde\sigma^{(3)}$	1	$2,3,4$	$-\mathbf{3},\mathbf{4}$	$\mathbf{3},\mathbf{4},\mathbf{5}$	$-\mathbf{4},\mathbf{5}$	$4,5,6$	7	8
$\tilde\sigma^{(3)^2}$	1	$2,3,4,5$	$-\mathbf{5}$	$-\mathbf{4}$	$-\mathbf{3}$	$3,4,5,6$	7	8
$\tilde\sigma^{(3)^3}$	1	$2,3$	$\mathbf{4},\mathbf{5}$	$-\mathbf{3},\mathbf{4},\mathbf{5}$	$\mathbf{3},\mathbf{4}$	$5,6$	7	8
$\tilde\sigma^{(3)}_-$	1	$2,3$	-3	$3,4,5$	-5	$5,6$	7	8
$\tilde\sigma^{(3)}_+$	1	2	$3,4$	-4	$4,5$	6	7	8
$\tilde\sigma^{(4)}$	1	2	$3,4$	$\mathbf{5},\mathbf{6}$	$-\mathbf{4},\mathbf{5},\mathbf{6}$	$\mathbf{4},\mathbf{5}$	$6,7$	8
$\tilde\sigma^{(4)^2}$	1	2	$3,4,5,6$	$-\mathbf{6}$	$-\mathbf{5}$	$-\mathbf{4}$	$4,5,6,7$	8
$\tilde\sigma^{(4)^3}$	1	2	$3,4,5$	$-\mathbf{4},\mathbf{5}$	$\mathbf{4},\mathbf{5},\mathbf{6}$	$-\mathbf{5},\mathbf{6}$	$5,6,7$	8
$\tilde\sigma^{(4)}_-$	1	2	3	$4,5$	-5	$5,6$	7	8
$\tilde\sigma^{(4)}_+$	1	2	$3,4$	-4	$4,5,6$	-6	$6,7$	8
$\tilde\sigma^{(5)}$	1	2	3	$4,5,6$	$-\mathbf{5},\mathbf{6}$	$\mathbf{5},\mathbf{6},\mathbf{7}$	$-\mathbf{6},\mathbf{7}$	$6,7,8$
$\tilde\sigma^{(5)^2}$	1	2	3	$4,5,6,7$	$-\mathbf{7}$	$-\mathbf{6}$	$-\mathbf{5}$	$5,6,7,8$
$\tilde\sigma^{(5)^3}$	1	2	3	$4,5$	$\mathbf{6},\mathbf{7}$	$-\mathbf{5},\mathbf{6},\mathbf{7}$	$\mathbf{5},\mathbf{6}$	$7,8$
$\tilde\sigma^{(5)}_-$	1	2	3	$4,5$	-5	$5,6,7$	-7	$7,8$
$\tilde\sigma^{(5)}_+$	1	2	3	4	$5,6$	-6	$6,7$	8
$\tilde\sigma^{(6)}$	1	2	3	4	$5,6$	$\mathbf{7},\mathbf{8}$	$-\mathbf{6},\mathbf{7},\mathbf{8}$	$\mathbf{6},\mathbf{7}$
$\tilde\sigma^{(6)^2}$	1	2	3	4	$5,6,7,8$	$-\mathbf{8}$	$-\mathbf{7}$	$-\mathbf{6}$
$\tilde\sigma^{(6)^3}$	1	2	3	4	$5,6,7$	$-\mathbf{6},\mathbf{7}$	$\mathbf{6},\mathbf{7},\mathbf{8}$	$-\mathbf{7},\mathbf{8}$
$\tilde\sigma^{(6)}_-$	1	2	3	4	5	$6,7$	-7	$7,8$
$\tilde\sigma^{(6)}_+$	1	2	3	4	$5,6$	-6	$6,7,8$	-8

A.2 A_8-Root spaces based on the class $\Sigma_{\{1,2^2,3,4,\ell-4\}}$ and their invariance

$\tilde{\sigma}^{(i,j)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1,1)}$	$\mathbf{-1,2}$	$\mathbf{1,2,3}$	$\mathbf{-2,3}$	$2,3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}^{(1,1)^2}$	$\mathbf{-3}$	$\mathbf{-2}$	$\mathbf{-1}$	$1,2,3,4$	5	6	7	8
$\tilde{\sigma}^{(1,1)^3}$	$\mathbf{2,3}$	$\mathbf{-1,2,3}$	$\mathbf{1,2}$	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_-^{(1,1)}$	-1	$1,2,3$	-3	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_+^{(1,1)}$	$1,2$	-2	$2,3$	4	5	6	7	8
$\tilde{\sigma}^{(2,1)}$	$1,2$	$\mathbf{3,4}$	$\mathbf{-2,3,4}$	$\mathbf{2,3}$	$4,5,6$	-6	$6,7$	8
$\tilde{\sigma}^{(2,1)^2}$	$1,2,3,4$	$\mathbf{-4}$	$\mathbf{-3}$	$\mathbf{-2}$	$2,3,4,5$	6	7	8
$\tilde{\sigma}^{(2,1)^3}$	$1,2,3$	$\mathbf{-2,3}$	$\mathbf{2,3,4}$	$\mathbf{-3,4}$	$3,4,5,6$	-6	$6,7$	8
$\tilde{\sigma}_-^{(2,1)}$	1	$2,3$	-3	$3,4$	5	6	7	8
$\tilde{\sigma}_+^{(2,1)}$	$1,2$	-2	$2,3,4$	-4	$4,5,6$	-6	$6,7$	8
$\tilde{\sigma}^{(3,1)}$	1	$2,3,4$	$\mathbf{-3,4}$	$\mathbf{3,4,5}$	$\mathbf{-4,5}$	$4,5,6,7$	-7	$7,8$
$\tilde{\sigma}^{(3,1)^2}$	1	$2,3,4,5$	$\mathbf{-5}$	$\mathbf{-4}$	$\mathbf{-3}$	$3,4,5,6$	7	8
$\tilde{\sigma}^{(3,1)^3}$	1	$2,3$	$\mathbf{4,5}$	$\mathbf{-3,4,5}$	$\mathbf{3,4}$	$5,6,7$	-7	$7,8$
$\tilde{\sigma}_-^{(3,1)}$	1	$2,3$	-3	$3,4,5$	-5	$5,6,7$	-7	$7,8$
$\tilde{\sigma}_+^{(3,1)}$	1	2	$3,4$	-4	$4,5$	6	7	8
$\tilde{\sigma}^{(4,1)}$	1	2	$3,4$	$\mathbf{5,6}$	$\mathbf{-4,5,6}$	$\mathbf{4,5}$	$6,7,8$	-8
$\tilde{\sigma}^{(4,1)^2}$	1	2	$3,4,5,6$	$\mathbf{-6}$	$\mathbf{-5}$	$\mathbf{-4}$	$4,5,6,7$	8
$\tilde{\sigma}^{(4,1)^3}$	1	2	$3,4,5$	$\mathbf{-4,5}$	$\mathbf{4,5,6}$	$\mathbf{-5,6}$	$5,6,7,8$	-8
$\tilde{\sigma}_-^{(4,1)}$	1	2	3	$4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_+^{(4,1)}$	1	2	$3,4$	-4	$4,5,6$	-6	$6,7,8$	-8

A.3 A_8-Root spaces based on the class $\Sigma_{\{1,2^2,3,4,\ell-4\}}$ and their invariance

$\tilde{\sigma}^{(i,j)}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\tilde{\sigma}^{(1,2)}$	-1	$1,2,3,4$	$\mathbf{-3,4}$	$\mathbf{3,4,5}$	$\mathbf{-4,5}$	$4,5,6$	7	8
$\tilde{\sigma}^{(1,2)^2}$	1	$2,3,4,5$	$\mathbf{-5}$	$\mathbf{-4}$	$\mathbf{-3}$	$3,4,5,6$	7	8
$\tilde{\sigma}^{(1,2)^3}$	-1	$1,2,3$	$\mathbf{4,5}$	$\mathbf{-3,4,5}$	$\mathbf{3,4}$	$5,6$	7	8
$\tilde{\sigma}_-^{(1,2)}$	-1	$1,2,3$	-3	$3,4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_+^{(1,2)}$	1	2	$3,4$	-4	$4,5$	6	7	8
$\tilde{\sigma}^{(2,2)}$	$1,2$	-2	$2,3,4$	$\mathbf{5,6}$	$\mathbf{-4,5,6}$	$\mathbf{4,5}$	$6,7$	8
$\tilde{\sigma}^{(2,2)^2}$	1	2	$3,4,5,6$	$\mathbf{-6}$	$\mathbf{-5}$	$\mathbf{-4}$	$4,5,6,7$	8
$\tilde{\sigma}^{(2,2)^3}$	$1,2$	-2	$2,3,4,5$	$\mathbf{-4,5}$	$\mathbf{4,5,6}$	$\mathbf{-5,6}$	$5,6,7$	8
$\tilde{\sigma}_-^{(2,2)}$	1	2	3	$4,5$	-5	$5,6$	7	8
$\tilde{\sigma}_+^{(2,2)}$	$1,2$	-2	$2,3,4$	-4	$4,5,6$	-6	$6,7$	8

$\tilde{\sigma}^{(3,2)}$	1	2, 3	−3	3, 4, 5, 6	**−5, 6**	**5, 6, 7**	**−6, 7**	6, 7, 8
$\tilde{\sigma}^{(3,2)^2}$	1	2	3	4, 5, 6, 7	**−7**	**−6**	**−5**	5, 6, 7, 8
$\tilde{\sigma}^{(3,2)^3}$	1	2, 3	−3	3, 4, 5	**6, 7**	**−5, 6, 7**	**5, 6**	7, 8
$\tilde{\sigma}_{-}^{(3,2)}$	1	2, 3	−3	3, 4, 5	−5	5, 6, 7	−7	7, 8
$\tilde{\sigma}_{+}^{(3,2)}$	1	2	3	4	5, 6	−6	6, 7	8
$\tilde{\sigma}^{(4,2)}$	1	2	3, 4	−4	4, 5, 6	**7, 8**	**−6, 7, 8**	**6, 7**
$\tilde{\sigma}^{(4,2)^2}$	1	2	3	4	5, 6, 7, 8	**−8**	**−7**	**−6**
$\tilde{\sigma}^{(4,2)^3}$	1	2	3, 4	−4	4, 5, 6, 7	**−6, 7**	**6, 7, 8**	**−7, 8**
$\tilde{\sigma}_{-}^{(4,2)}$	1	2	3	4	5	6, 7	−7	7, 8
$\tilde{\sigma}_{+}^{(4,2)}$	1	2	3, 4	−4	4, 5, 6	−6	6, 7, 8	−8

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