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Discrete Torsion, Covering Groups and Quiver Diagrams

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Abstract: Without recourse to the sophisticated machinery of twisted group algebras, projective character tables and explicit values of 2-cocycles, we here present a simple algorithm to study the gauge theory data of D-brane probes on a generic orbifold G with discrete torsion turned on. We show in particular that the gauge theory can be obtained with the knowledge of no more than the *ordinary* character tables of G and its covering group G^*. Subsequently we present the quiver diagrams of certain illustrative examples of $SU(3)$-orbifolds which have non-trivial Schur Multipliers. The paper serves as a companion to our earlier work and aims to initiate a systematic and computationally convenient study of discrete torsion.

Keywords: D-branes on Orbifolds, Covering Groups, Projective Representations, Discrete Torsion, Quiver Diagrams.

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1. Introduction

Discrete torsion [2, 3] has become a meeting ground for many interesting sub-fields of string theory; its intimate relation with background B-fields and non-commutative geometry is one of its many salient features. In the context of D-brane probes on orbifolds with discrete torsion turned on, new classes of gauge theories may be fabricated and their (non-commutative) moduli spaces, investigated (see from [4] to [16]). Indeed, as it was pointed out in [4], projection on the matter spectrum in the gauge theory by an orbifold G with non-trivial discrete torsion is performed by the *projective representations* of G, rather than the mere linear (ordinary) represenations as in the case without.

In a previous paper [1], to which the present work shall be a companion, we offered a classification of the orbifolds with $\mathcal{N} = 0, 1, 2$ supersymmetry which permit the turning on of discrete torsion. We have pointed there that for the orbifold group G, the discriminant agent is the Abelian group known as the **Schur Multiplier** $M(G) := H^2(G, \mathbb{C}^*)$; only if $M(G)$ were non-trivial could G afford a projective representation and thereby discrete torsion.

In fact one can do more and for actual physical computations one needs to do more. The standard procedure of calculating the matter content and superpotential of the orbifold gauge theory as developed in [17] can, as demonstrated in [13], be directly generalised to the case with discrete torsion. Formulae given in terms of the ordinary characters have their immediate counterparts in terms of the projective characters, the *point d'appui* being that the crucial properties of ordinary characters, notably orthogonality, carry over without modification, to the projective case.

And thus our task would be done if we had a method of computing the projective characters. Upon first glance, this perhaps seems formidable: one seemingly is required to know the values of the cocycle representatives $\alpha(x, y)$ in $M(G)$ for all $x, y \in G$. In actuality, one can dispense with such a need. There exists a canonical method to arrive at the projective characters, namely by recourse to the **covering group** of G. We shall show in this writing the methodology standard in the mathematics literature [18, 20] by which one, once armed with the Schur Multiplier, arrives at the cover. Moreover, in light of the physics, we will show how, equipped with no more than the knowledge of the character table of G and that of its cover G^*, one obtains the matter content of the orbifold theory with discrete torsion.

The paper is organised as follows. Section 2 introduces the necessary mathematical background for our work. Due to the technicality of the details, we present a paragraph at the beginning of the section to summarise the useful facts; the reader may then freely skip the rest of Section 2 without any loss. In Section 3, we commence with an explicit example, viz., the ordinary dihedral group, to demonstrate the method to construct the covering group. Then we present all the covering groups for transitive and intransitive discrete subgroups of $SU(3)$. In Section 4, we use these covering groups to calculate the corresponding gauge theories (i.e., the quiver diagrams) for all exceptional subgroups of $SU(3)$ admitting discrete torsion as well as some examples for the Delta series. In particular we demonstrate the algorithm of extracting the quivers from the ordinary character tables of the group and its cover. As a by-product, in Section 5 we present a method to calculate the *cocycles* directly which will be useful for future reference. The advantage of our methods for the quivers and the cocycles is their simplicity and generality. Finally, in Section 6 we give some conclusions and further directions for research.

Nomenclature

Throughout this paper, unless otherwise specified, we shall adhere to the following conventions for notation:

ω_n	n-th root of unity;
G	a finite group of order $\|G\|$;
$[x, y]$	$:= xyx^{-1}y^{-1}$, the group commutator of x, y;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$\gcd(m, n)$	the greatest common divisor of m and n;
$Z(G)$	centre of G;
$G' := [G, G]$	the derived (commutator) group of G;
G^*	the covering group of G;
$A = M(G)$	the Schur Muliplier of G;
$\mathrm{char}(G)$	the ordinary (linear) character table of G, given as an $(r+1) \times r$ matrix with r the number of conjugacy classes and the extra row for the class numbers;
$Q_\alpha(G, \mathcal{R})$	the α-projective quiver for G associated to the chosen representation $\mathcal{R}$.

2. Mathematical Preliminaries

We first remind the reader of some properties of the the theory of projective representations; in what follows we adhere to the notation used in our previous work [1].

Due to the technicalities in the ensuing, the audience might be distracted upon the first reading. Thus as promised in the introduction, we here summarise the keypoints in the next fews paragraphs, so that the remainder of this section may be loosely perused without any loss.

Our aim of this work is to attempt to construct the gauge theory living on a D-brane probing an orbifold G when "discrete torsion" is turned on. To accomplish such a goal, we need to know the projective representations of the finite group G, which may not be immediately available. However, mathematicians have shown that there exists (for representations in $GL(\mathbb{C})$) a group G^* called the *covering group* of G, such that there is a one-to-one correspondence between the projective representations of G and the linear (ordinary) representations of G^*. Thus the method is clear: we simply need to find the covering group and then calculate the ordinary characters of its (linear) representations.

More specificaly, we first introduce the concept of the covering group in Definition 2.2. Then in Theorem 2.1, we introduce the necessary and sufficient conditions for G^* to be a covering group; these conditions are very important and we use them extensively during actual computations.

However, G^* for any given G is not unique and there exist non-isomorphic groups which all serve as covering groups. To deal with this, we introduce *isoclinism* and show that these non-isomorphic covering groups must be isoclinic to each other in Theorem 2.2. Subsequently, in Theorem 2.3, we give an upper-limit on the number of non-isomorphic covering groups of G. Finally in Thereom 2.4 we present the one-to-one correspondence of all projective representations of G and all linear representations of its covering group G^*.

Thus is the summary for this section. The uninterested reader may now freely proceed to Section 3.

2.1 The Covering Group

Recall that a **projective representation** of G over $\mathbb{C}$ is a mapping $\rho : G \to GL(V)$ such that $\rho(\mathbb{I}_G) = \mathbb{I}_V$ and $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for any elements $x, y \in G$. The function α, known as the *cocycle*, is a map $G \times G \to \mathbb{C}^*$ which is classified by $H^2(G, \mathbb{C}^*)$, the second $\mathbb{C}^*$-valued cohomology of G. This case of $\alpha = 1$ trivially is of course our familiar ordinary (non-projective) representation, which will be called **linear**.

The Abelian group $H^2(G, \mathbb{C}^*)$ is known as the **Schur Multiplier** of G and will be denoted by $M(G)$. Its triviality or otherwise is a discriminant of whether G admits projective representation. In a physical context, knowledge of $M(G)$ provides immediate information as to the possibility of turning on discrete torsion in the orbifold model under study. A classification of $M(G)$ for all discrete finite subgroups of $SU(3)$ and the exceptional subgroups of $SU(4)$ was given in the companion work [1].

The study of the projective representations of a given group G is greatly facilitated by introducing an auxilliary object G^*, the **covering group** of G, which "lifts projective representations to linear ones." Let us refresh our memory what this means. Let there be a **central extension** according to the exact sequence $1 \to A \to G^* \to G \to 1$ such that A is in the centre of G^*. Thus we have $G^*/A \cong G$. Now we say

DEFINITION 2.1 *A projective representation ρ of G* **lifts** *to a linear representation ρ^* of G^* if*
(i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and
(ii) there is a section[2] *$\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g))$, $\forall g \in G$.*

Likewise it *lifts projectively* if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map (not necessarily a homomorphism) $t : G \to \mathbb{C}^*$.

[2] i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.

DEFINITION 2.2 *G^* is called a* **covering group** *(or otherwise known as the* **representation group**, *Darstellungsgruppe) of G over $\mathbb{C}$ if the following are satisfied:*

(i) *$\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;*

(ii) $|A| = |M(G)| = |H^2(G, \mathbb{C}^*)|$.

The covering group will play a central rôle in our work; as we will show in a moment, *the matter content of an orbifold theory with group G having discrete torsion switched-on is encoded in the quiver diagram of G^*.*

For actual computational purposes, the following theorem, initially due to Schur, is of extreme importance:

THEOREM 2.1 ([18] p143) *$G^\star$ is a covering group of G over $\mathbb{C}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |M(G)|$.

In the above, $[G^\star, G^\star]$ is the **derived group** $G^{*'}$ of G^*. We remind ourselves that for a group H, $H' := [H, H]$ is the group generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$ for $x, y \in H$. We stress that conditions (ii) and (iii) are easily satisfied while (i) is the more stringent imposition.

The solution of the problem of finding covering groups is certainly *not* unique: G in general may have more than one covering groups (e.g., the quaternion and the dihedral group of order 8 are both covering groups of $\mathbb{Z}_2 \times \mathbb{Z}_2$). The problem of finding the necessary conditions which two groups G_1 and G_2 must satisfy in order for both to be covering groups of the same group G is a classical one.

The well-known solution starts with the following

DEFINITION 2.3 *Two groups G and H are said to be* **isoclinic** *if there exist two isomorphisms*

$$\alpha : G/Z(G) \stackrel{\cong}{\to} H/Z(H) \quad \text{and} \quad \beta : G' \stackrel{\cong}{\to} H'$$

such that $\alpha(x_1 Z(G)) = x_2 Z(H)$ and $\alpha(y_1 Z(G)) = y_2 Z(H) \Rightarrow \beta([x_1, y_1]) = [x_2, y_2]$,

where we have used the standard notation that $xZ(G)$ is a coset representative in $G/Z(G)$. We note in passing that every Abelian group is obviously isoclinic to the trivial group $\langle \mathbb{I} \rangle$.

We introduce this concept of isoclinism because of the following important Theorem of Hall:

THEOREM 2.2 ([18] p441) *Any two covering groups of a given finite group G are isoclinic.*

Knowing that the covering groups of G are not isomorphic to each other, but isoclinic, a natural question to ask is how many non-isomorphic covering groups can one have. Here a theorem due to Schur shall be useful:

THEOREM 2.3 ([18] p149) *For a finite group G, let*

$$G/G' = \mathbb{Z}_{e_1} \times ... \times \mathbb{Z}_{e_r}$$

and

$$M(G) = \mathbb{Z}_{f_1} \times ... \times \mathbb{Z}_{f_s}$$

be decompositions of these Abelian groups into cyclic factors. Then the number of non-isomorphic covering groups of G is less than or equal to

$$\prod_{1 \le i \le r, 1 \le j \le s} \gcd(e_i, f_j).$$

2.2 Projective Characters

With the preparatory remarks in the previous subsection, we now delve headlong into the heart of the matter. By virtue of the construction of the covering group G^* of G , we have the following 1-1 correpondence which will enable us to compute α-projective representations of G in terms of the linear representations of G^*:

THEOREM 2.4 [Theorema Egregium] ([18] p139; [19] p8) *Let G^* be the covering group of G and $\lambda : A \to \mathbb{C}^*$ a homomorphism. Then*

(i) *For every linear representation $L : G^* \to GL(V)$ of G^* such that $L(a) = \lambda(a)\mathbb{I}_V \ \forall a \in A$, there is an induced projective representation P on G defined by*

$$P(g) := L(r(g)), \ \forall \ g \in G,$$

with $r : G \to G^*$ the map that associates to each coset $g \in G \cong G^*/A$ a representative element[3] *in G^*; and vice versa,*

(ii) *Every α-projective representation for $\alpha \in M(G)$ lifts to an ordinary representation of G^*.*

An immediate consequence of the above is the fact that knowing the linear characters of G^* suffices to establish the projective characters of G for all α [20]. This should ease our initial fear in that *one does not need to know a priori the specific values of the cocycles $\alpha(x, y)$ for all $x, y \in G$ (a stupendous task indeed) in order to construct the α-projective character table for G.*

We shall leave the uses of this crucial observation to later discussions. For now, let us focus on some explicit computations of covering groups.

[3] i.e., $r(g)A \to g$ is the isomorphism $G^*/A \stackrel{\cong}{\to} G$.

3. Explicit Calculation of Covering Groups

To theory we must supplant examples and to abstraction, concreteness. We have prepared ourselves in the previous section the rudiments of the theory of covering groups; in the present section we will demonstrate these covers for the discrete finite subgroups of $SU(3)$. First we shall illustrate our techniques with the case of D_{2n}, the ordinary dihedral group, before tabulating the complete results.

3.1 The Covering Group of The Ordinary Dihedral Group

The presentation of the ordinary dihedral group of order $2n$ is standard (the notation is different from some of our earlier papers (e.g. [25]) where the following would be called D_n):

$$D_{2n} = \langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}^n = 1, \tilde{\beta}^2 = 1, \tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = \tilde{\alpha}^{-1} \rangle.$$

We recall from [1] that the Schur Multiplier for $G = D_{2n}$ is $\mathbb{Z}_2$ when n is even and trivial otherwise, thus we restrict ourselves only to the case of n even. We let $M(D_{2n})$ be $A = \mathbb{Z}_2$ generated by $\{a | a^2 = \mathbb{I}\}$. We let the covering group be $G^* = \langle \alpha, \beta, a \rangle$.

Now having defined the generators we proceed to constrain relations thereamong. Of course, $A \subset Z(G^*)$ immediately implies that $\alpha a = a\alpha$ and $\beta a = a\beta$. Moreover, α, β must map to $\tilde{\alpha}, \tilde{\beta}$ when we identify $G^\star/A \cong D_{2n}$ (by part (ii) of Theorem 2.1). This means that $\mathbb{I}_G$ must have a preimage in $A \subset G^*$, giving us: $\alpha^n \in A, \beta^2 \in A$ and $\beta\alpha\beta^{-1}\alpha \in A$ by virtue of the presentation of G. And hence we have 8 possibilities, each being a central extension of D_{2n} by A:

$$\begin{array}{l}
G_1^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_2^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_3^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_4^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_5^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_6^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_7^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_8^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle
\end{array} \tag{3.1}$$

Therefore, conditions (ii) and (iii) of Theorem 2.1 are satified. One must check (i) to distinguish the covering group among these 8 central extensions in (3.1). Now since A is actually the centre, it suffices to check whether $A \subset G_i^{*'} = [G_i^*, G_i^*]$.

We observe G_1^* to be precisely $D_{2n} \times \mathbb{Z}_2$, from which we have $G_1^{*'} = \mathbb{Z}_{n/2}$, generated by α^2. Because $A = \{\mathbb{I}, a\}$ clearly is not included in this $\mathbb{Z}_{n/2}$ we conclude that G_1^* is not the covering group. For G_2^*, we have $G_2^{*'} = \langle \alpha^2 a \rangle$, which means that when $n/2 =$ odd (recall that $n =$ even), $G_2^{*'}$ can contain a and hence $A \subset G_2^{*'}$, whereby making G_2^* a covering group. By the same token we find that $G_3^{*'} = \langle \alpha^2 \rangle$, $G_4^{*'} = \langle \alpha^2 a \rangle$, $G_5^{*'} = \langle \alpha^2 \rangle$, $G_6^{*'} = \langle \alpha^2 a \rangle$, and $G_7^{*'} = \langle \alpha^2 \rangle$. We summarise these results in the following table:

Group	$G^{*'}$	$Z(G^*)$	$G^*/Z(G^*)$	Covering Group?
G_1^*	$\mathbb{Z}_{n/2} = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
$G_2^*(n = 4k+2)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes
$G_2^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
G_3^*	$\mathbb{Z}_{n/2} = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
$G_4^*(n = 4k+2)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes
$G_4^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
G_5^*	$\mathbb{Z}_n = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes
$G_6^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_4 = \langle \alpha^{n/2} \rangle$	D_n	no
$G_6^*(n = 4k)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes
G_7^*	$\mathbb{Z}_n = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes
$G_8^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_4 = \langle \alpha^{n/2} \rangle$	D_n	no
$G_8^*(n = 4k)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	D_{2n}	yes

Whence we see that G_1^* and G_3^* are not covering groups, while for $n/2 =$ odd $G_{2,4}^*$ are covers, for $n/2 =$ even $G_{6,8}^*$ are covers as well and finally $G_{5,7}^*$ are always covers. Incidentally, G_7^* is actually the binary dihedral group and we know that it is indeed the (double) covering group from [1]. Of course in accordance with Theorem 2.2, these different covers must be isoclinic to each other. Checking against Definition 2.3, we see that for G^* being $G_{2,4}^*$ with $n = 4k+2$, $G_{6,8}^*$ with $n = 4k$ and $G_{5,7}^*$ for all even n, $G^{*'} \cong \mathbb{Z}_n$ and $G^*/Z(G^*) \cong D_{2n}$; furthermore the isomorphisms α and β in the Definition are easily seen to satisfy the prescribed conditions. Therefore all these groups are indeed isoclinic. We make one further remark, for both the cases of $n = 4k$ and $n = 4k+2$, we have found 4 non-isomorphic covering groups. Recall Theorem 2.3, here we have $f_1 = 2$ and $G/G' = \mathbb{Z}_2 \times \mathbb{Z}_2$ (note that n is even) and so $e_1 = e_2 = 2$, whence the upper limit is exactly $2 \times 2 = 4$ which is saturated here. This demonstrates that our method is general enough to find all possible covering groups.

3.2 Covering Groups for the Discrete Finite Subgroups of $SU(3)$

By methods entirely analogous to the one presented in the above subsection, we can arrive at the covering groups for the discrete finite groups of $SU(3)$ as tabulated in [1]. Let us list the results (of course in comparison with Table 3.2 in [1], those with trivial Schur Multipliers have no covering groups and will not be included here). Of course, as mentioned earlier, the covering group is not unique. The particular ones we have chosen in the following table are the same as generated by the computer package GAP using the Holt algorithm [29].

Intransitives

- $G = \mathbb{Z}_m \times \mathbb{Z}_n = \langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}^n = 1, \tilde{\beta}^m = 1, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha} \rangle;$

$$\begin{array}{l} M(G) = \mathbb{Z}_{p=\gcd(m,n)} = \langle a | a^p = \mathbb{I} \rangle; \\ G^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \beta a = a\beta, a^p = 1, \alpha^n = 1, \beta^m = 1, \alpha\beta = \beta\alpha a \rangle \end{array} \quad (3.2)$$

- $G = \langle \mathbb{Z}_{n=4k}, \widehat{D_{2m}} \rangle = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^m, \tilde{\beta}^{2m} = 1, \tilde{\beta}^m = \tilde{\gamma}^2, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle;$

$$\begin{cases} m \text{ even} & M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = 1 = b^2, ab = ba \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a, b | ab = ba, \alpha a = a\alpha, \alpha b = b\alpha, \beta a = a\beta, \beta b = b\beta, \\ & \quad \gamma a = a\gamma, \gamma b = b\gamma, a^2 = 1 = b^2, \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha b, \\ & \quad \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \\ m \text{ odd,} & M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha\beta = \beta\alpha, \\ & \quad \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \end{cases} \quad (3.3)$$

- $G = \langle \mathbb{Z}_{n=3k}, \widehat{E_6} \rangle$

$$\begin{array}{ll} k \text{ odd} & G \cong \mathbb{Z}_n \times \widehat{E_6} = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^n = 1, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2 \rangle; \\ & M(G) = \mathbb{Z}_3 = \langle a | a^3 = \mathbb{I} \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a | a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^n = 1, \\ & \quad \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2 \rangle \\ k = 2(2p+1) & G \cong \mathbb{Z}_{n/2} \times \widehat{E_6} \\ k = 4p & G \cong (\mathbb{Z}_n \times \widehat{E_6})/\mathbb{Z}_2 = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^3, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2 \rangle; \\ & M(G) = \mathbb{Z}_3 = \langle a | a^3 = \mathbb{I} \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a | a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^3, \\ & \quad \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2 \rangle \end{array} \quad (3.4)$$

- $$G = \langle \mathbb{Z}_{n=4k}, \widehat{E_7} \rangle = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^4, \tilde{\beta}^4 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2 \rangle;$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = \mathbb{I} \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^4,$$
$$\alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha, \beta^4 = \gamma^3 = (\beta\gamma)^2 \rangle \qquad (3.5)$$

- $G = \langle \mathbb{Z}_n, D_{2m} \rangle$

n odd, m even
$$G = \mathbb{Z}_n \times D_{2m} = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^n = 1, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\beta}^m = 1, \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle;$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha(\beta/\gamma) = (\beta/\gamma)\alpha, \alpha^n = 1, \beta^m = a, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

n even, m odd
$$G = \mathbb{Z}_n \times D_{2m}$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha a, \alpha^n = 1, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

m even, $n = 2(2l+1)$
$$G = \mathbb{Z}_{n/2} \times D_{2m}$$

$n = 4k, m = 2(2l+1)$
$$G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^{n/2} = \beta^{m/2}, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\beta}^m = 1, \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle;$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

$n = 4k, m = 4l$
$$G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2$$
$$M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = 1, b^2 = 1, ab = ba \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a, b | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha b, \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

(3.6)

where we have used the shorthand notation $(x/y/\ldots/z)$ to mean the relation to be applied to each of the elements $x, y, \ldots, z$.

Transitives

We first have the two infinite series.

- $G = \Delta(3n^2) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^n = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle;$

$$\left\{\begin{array}{ll} \gcd(n,3) = 1, n \text{ even} & M(G) = \mathbb{Z}_n = \langle a | a^n = 1\rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\ & \qquad a^n = \alpha^n a^{n/2} = \beta^n a^{n/2} = \gamma^3 = 1, \\ & \qquad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\ \gcd(n,3) = 1, n \text{ odd} & M(G) = \mathbb{Z}_n; \\ & G^* = \langle \alpha, \beta, \gamma, a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\ & \qquad a^n = \alpha^n = \beta^n = \gamma^3 = 1, \\ & \qquad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\ \gcd(n,3) \neq 1, n \text{ even} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3 = \langle a, b | a^n = 1, b^3 = 1\rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\ & \qquad ab = ba, a^n = b^3 = \gamma^3 = \alpha^n a^{n/2} b = 1, \\ & \qquad \beta^n a^{n/2} = b, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\ \gcd(n,3) \neq 1, n \text{ odd} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3; \\ & G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\ & \qquad a^n = b^3 = \gamma^3 = \alpha^n b = \beta^n b^{-1} = 1, \\ & \qquad ab = ba, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \end{array}\right. \tag{3.7}$$

- $G = \Delta(6n^2) = \langle \alpha, \beta, \gamma, \delta | \alpha^n = \beta^n = \gamma^3 = \delta^2 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma,$
$\alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle;$

$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1\rangle;$$

$$\left\{\begin{array}{ll} \gcd(n,4) = 4 & G^* = \langle \alpha, \beta, \gamma, \delta, a | \alpha^n = \beta^n = \gamma^3 = \delta^2 = a^2 = 1, \\ & \qquad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma, \\ & \qquad \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle; \\ \gcd(n,4) = 2 & G^* = \langle \alpha, \beta, \gamma, \delta, a | \alpha^n a = \beta^n a = \gamma^3 = \delta^2 = a^2 = 1, \\ & \qquad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma, \\ & \qquad \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle; \end{array}\right. \tag{3.8}$$

Next we present the three exceptionals that admit discrete torsion.

- $G = \Sigma(60) \cong A_5 = \langle \alpha, \beta | \alpha^5 = \beta^3 = (\alpha\beta^{-1})^3 = (\alpha^2\beta)^2 = 1$
$\alpha\beta\alpha\beta\alpha\beta = \alpha\gamma\alpha^{-1}\beta\alpha^2\beta\alpha^{-2}\beta = 1\rangle;$

$$\begin{array}{l} M(G) = \mathbb{Z}_2; \\ G^* = \langle \alpha, \beta, a | \alpha^5 = a, \beta^3 = a^2 = 1, (\alpha/\beta)a = a(\alpha/\beta) \\ \qquad (\alpha\beta^{-1})^3 = 1, (\alpha^2\beta)^2 = a\rangle; \end{array} \tag{3.9}$$

- $G = \Sigma(168) = \langle \alpha, \beta, \gamma | \gamma^2 = \beta^3 = \beta\gamma\beta\gamma = (\alpha\gamma)^4 = 1, \alpha^2\beta = \beta\alpha, \alpha^3\gamma\alpha^{-1}\beta = \gamma\alpha\gamma \rangle;$

$$\begin{aligned} & M(G) = \mathbb{Z}_2; \\ & G^* = \langle \alpha, \beta, \gamma, \delta | \delta^2 = \gamma^2\delta = \beta^3\delta = (\beta\alpha)^3 = (\alpha\gamma)^3 = 1, \\ & \qquad \beta\gamma\beta = \gamma, \alpha\delta = \delta\alpha, \beta^2\alpha^2\beta = \alpha, \beta^{-1}\alpha^{-1}\beta\gamma\alpha^{-1}\gamma = \gamma\alpha\beta \rangle; \end{aligned} \tag{3.10}$$

- $G = \Sigma(1080) = \langle \alpha, \beta, \gamma, \delta | \alpha^5 = \beta^2 = \gamma^2 = \delta^2 = (\alpha\beta)^2(\beta\gamma)^2 = (\beta\delta)^2 = 1,$
$(\alpha\gamma)^3 = (\alpha\delta)^3 = 1, \gamma\beta = \delta\gamma\delta, \alpha^2\gamma\beta\alpha^2 = \gamma\alpha^2\gamma \rangle;$

$$\begin{aligned} & M(G) = \mathbb{Z}_2; \\ & G^* = \langle \alpha, \beta, \gamma, \delta, \epsilon | \alpha^5 = \epsilon^2 = \gamma^2\epsilon^{-1} = \beta^2\epsilon^{-1} = \delta^2\epsilon^{-1} = (\alpha\delta)^3 = 1, \\ & \qquad \alpha^{-1}\epsilon\alpha = \beta^{-1}\epsilon\beta = \gamma^{-1}\epsilon\gamma = \delta^{-1}\epsilon\delta = \epsilon, \\ & \qquad (\alpha\beta)^2 = (\beta\gamma)^2 = (\beta\delta)^2 = \gamma\beta\delta\gamma\delta = (\alpha\gamma)^3 = \epsilon, \\ & \qquad \alpha^2\gamma\beta\alpha^2\gamma\alpha^{-2}\gamma = 1 \rangle; \end{aligned} \tag{3.11}$$

4. Covering Groups, Discrete Torsion and Quiver Diagrams

4.1 The Method

The introduction of the host of the above concepts is not without a cause. In this section we shall provide an **algorithm** which permits the construction of the quiver $Q_\alpha(G, \mathcal{R})$ of an orbifold theory with group G having discrete torsion α turned-on, and with a linear representation $\mathcal{R}$ of G acting on the transverse space.

Our method dispenses of the need of the knowledge of the cocycles $\alpha(x, y)$, which in general is a formidable task from the viewpoint of cohomology, but which the current literature may lead the reader to believe to be required for finding the projective representations. We shall demonstrate that the problem of finding these α-representations is reducible to the far more manageable duty of finding the covering group, constructing its character table (which is of course straightforward) and then applying the usual prodecure of extracting the quiver therefrom. One advantage of this method is that we immediately obtain the quiver for all cocycles (including the trivial cocycle which corresponds to having no discrete torsion at all) and in fact the values of $\alpha(x, y)$ (which we shall address in the next section) in a unified framework.

All the data we require are

(i) G and its (ordinary) character table char(G);

(ii) The covering group G^* of G and its (ordinary) character table char(G^*).

We first recall from [4] that turning on discrete torsion α in an orbifold projection amounts to using an α-projective representation[4] Γ_α of $g \in G$

$$\Gamma_\alpha(g) \cdot A \cdot \Gamma_\alpha^{-1}(g) = A, \qquad \Gamma_\alpha(g) \cdot \Phi \cdot \Gamma_\alpha^{-1}(g) = \mathcal{R}(g) \cdot \Phi \tag{4.1}$$

on the gauge field A and matter fields Φ.

The above equations have been phrased in a more axiomatic setting (in the language of [17]), by virtue of the fact that much of ordinary representation theory of finite group extends in direct analogy to the projective case, in [13]. *However, we hereby emphasize that with the aid of the linear representation of the covering group, one can perform orbifold projection with discrete torsion entirely in the setting of [17] without usage of the formulae in [13] generalised to twisted group algebras and modules.* In other words, if we use the matrix of the linear representation of G^* instead of that of the corresponding projective representation of G, we will arrive at the same gauge group and matter contents in the orbifold theory. This can be alternatively shown as follows.

When we lift the projective matrix representation of G into the linear one of G^*, every matrix $\rho(g)$ will map to $\rho(ga_i)$ for every $a_i \in A$. The crucial fact is that $\rho(ga_i) = \lambda_i \rho(g)$ where λ_i is simply a phase factor. Now in (4.1) (cf. Sections 4.2 and 5 for more details), $\Gamma_\alpha(g)$ and $\Gamma_\alpha^{-1}(g)$ always appear in pairs, when we replace them by $\Gamma(ga_i)$ and $\Gamma^{-1}(ga_i)$, the phase factor λ_i will cancel out and leave the result invariant. This shows that the two results, the one given by projective matrix representations of G and the other by linear matrix representations of G^*, will give identical answers in orbifold projections.

4.2 An Illustrative Example: $\Delta(3 \times 3^2)$

Without much further ado, an illustrative example of the group $\Delta(3 \times 3^2) \in SU(3)$ shall serve to enlighten the reader. We recall from (3.7) that this group of order 27 has presentation $\langle \alpha, \beta, \gamma | \alpha^3 = \beta^3 = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$ and its covering group of order 243 (since the Schur Multiplier is $\mathbb{Z}_3 \times \mathbb{Z}_3$) is $G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), a^3 = b^3 = \gamma^3 = \alpha^3 b = \beta^3 b^{-1} = 1, ab = ba, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$.

Next we require the two (ordinary) character tables. As pointed out in the Nomenclatures section, character tables are given as $(r+1) \times r$ matrices with r being the number of conjugacy classes (and equivalently the number of irreps), and the first row giving the conjugacy class numbers.

$$\text{char}(\Delta(3 \times 3^2)) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & \omega_3 & \omega_3 & \omega_3 & \bar{\omega}_3 & \bar{\omega}_3 & \bar{\omega}_3 \\ \hline 1 & 1 & 1 & 1 & 1 & \bar{\omega}_3 & \bar{\omega}_3 & \bar{\omega}_3 & \omega_3 & \omega_3 & \omega_3 \\ \hline 1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & 1 & \omega_3 & \bar{\omega}_3 & 1 & \omega_3 & \bar{\omega}_3 \\ \hline 1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & \omega_3 & \bar{\omega}_3 & 1 & \bar{\omega}_3 & 1 & \omega_3 \\ \hline 1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & \bar{\omega}_3 & 1 & \omega_3 & \omega_3 & \bar{\omega}_3 & 1 \\ \hline 1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & 1 & \bar{\omega}_3 & \omega_3 & 1 & \bar{\omega}_3 & \omega_3 \\ \hline 1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & \omega_3 & 1 & \bar{\omega}_3 & \bar{\omega}_3 & \omega_3 & 1 \\ \hline 1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & \bar{\omega}_3 & \omega_3 & 1 & \omega_3 & 1 & \bar{\omega}_3 \\ \hline 3 & 3\bar{\omega}_3 & 3\omega_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 3\omega_3 & 3\bar{\omega}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \tag{4.2}$$

[4]More rigorously, this statement holds only when the D-brane probe is pointlike in the orbifold directions. More generally, when D-brane probes extend along the orbifold directions, one may need to use ordinary as well as projective representations. For further details, please refer to [10] as well as [28].

$\text{char}(\Delta(3 \times 3^2)^*) =$

1	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
3	3	3	3	3	3	3	3	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	X	Y	Z	0	0	0	0	0	0	0	0	0	P	M	N
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	Z	X	Y	0	0	0	0	0	0	0	0	0	M	N	P
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	Y	Z	X	0	0	0	0	0	0	0	0	0	N	P	M
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	M	P	N	0	0	0	0	0	0	0	0	0	Z	Y	X
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	N	M	P	0	0	0	0	0	0	0	0	0	Y	X	Z
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	P	N	M	0	0	0	0	0	0	0	0	0	X	Z	Y
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	A	$-B$	C	$-A$	$-C$	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	$-B$	C	A	$-C$	B	$-A$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	C	A	$-B$	B	$-A$	$-C$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-C$	$-A$	B	0	0	0	0	0	0	$-B$	C	A	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-A$	B	$-C$	0	0	0	0	0	0	A	$-B$	C	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	B	$-C$	$-A$	0	0	0	0	0	0	C	A	$-B$	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	N	P	0	0	0	Z	X	Y	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	P	M	N	0	0	0	X	Y	Z	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	N	P	M	0	0	0	Y	Z	X	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-A$	$-C$	B	A	$-B$	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-C$	B	$-A$	$-B$	C	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	B	$-A$	$-C$	C	A	$-B$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	X	Z	Y	0	0	0	P	N	M	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Y	X	Z	0	0	0	N	M	P	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Z	Y	X	0	0	0	M	P	N	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	$-B$	A	C	0	0	0	0	0	0	$-C$	B	$-A$	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	A	C	$-B$	0	0	0	0	0	0	$-A$	$-C$	B	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	C	$-B$	A	0	0	0	0	0	0	B	$-A$	$-C$	0	0	0	0	0	0

(4.3)

with $A := -\omega_3 + \bar\omega_3, B := \omega_3 + 2\bar\omega_3, C := 2\omega_3 + \bar\omega_3; M := -\omega_9^2 - 2\bar\omega_9^4, N := \omega_9^2 + \bar\omega_9^4, P := -\omega_9^2 + \bar\omega_9^4; X := \omega_9^4 - \bar\omega_9^2, Y := \omega_9^4 + 2\bar\omega_9^2, Z := -2\omega_9^4 - \bar\omega_9^2$.

A comparative study of these two tables shall suffice to demonstrate the method. We have taken extreme pains to re-arrange the columns and rows of $\text{char}(G^*)$ for the sake of perspicuity; whence we immediately observe that $\text{char}(G)$ and $\text{char}(G^*)$ are unrelated but that the latter is organised in terms of "cohorts" [23] of the former. What this means is as follows: columns 1 through 9 of $\text{char}(G^*)$ have their first 11 rows (not counting the row of class numbers) identical to the first column of $\text{char}(G)$, so too is column 10 of $\text{char}(G^*)$ with column 2 of $\text{char}(G)$, *et cetera* with $\{11\} \to \{3\}$, $\{12, 13, 14\} \to \{4\}$, $\{15, 16, 17\} \to \{5\}$, $\{18, 19, 20\} \to \{6\}$, $\{21, 22, 23\} \to \{7\}$, $\{24, 25, 26\} \to \{8\}$, $\{27, 28, 29\} \to \{9\}$, $\{30, 31, 32\} \to \{10\}$, and $\{33, 34, 35\} \to \{11\}$; using the notation that $\{X\} \to \{Y\}$ for the first 11 rows of columns $\{X\} \subset \text{char}(G^*)$ are mapped to column $\{Y\} \subset \text{char}(G)$. These are the so-called "splitting conjugacy classes" in G^* which give the (linear) $\text{char}(G)$ [19]. In other words, (though the conjugacy class numbers may differ), up to repetition $\text{char}(G) \subset \text{char}(G^*)$. This of course is in the spirit of the technique of Frøbenius Induction of finding the character table of a group from that of its subgroup; for a discussion of this in the context of orbifolds, the reader is referred to [24]. Thus the first 11 rows of $\text{char}(G^*)$ corresponds exactly to the *linear irreps* of G. The rest of the rows we shall shortly observe to correspond to the projective representations.

To understand these above remarks, let $A := \mathbb{Z}_3 \times \mathbb{Z}_3$ so that $G^*/A \cong G$ as in the notation of Section 2. Now $A \subseteq Z(G^*)$, hence the matrix forms of all of its elements must be $\lambda \mathbb{I}_{d\times d}$, where d is the dimension of the irreducible representation and λ some phase factor. Indeed the first 9 columns of char(G^*) have conjugacy class number 1 and hence correspond to elements of this centre. Bearing this in mind, if we only tabulated the phases λ (by suppressing the factor $d = 1$ or 3 coming from $\mathbb{I}_{d\times d}$) of these first 9 columns, we arrive at the following table (removing the first row of conjugacy class numbers):

rows	$\mathbb{I}$	a	a^2	b	ab	a^2b	b^2	ab^2	a^2b^2
$2-12$	1	1	1	1	1	1	1	1	1
$13-15$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$
$16-18$	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3
$19-21$	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
$22-24$	1	ω_3	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	1	$\bar{\omega}_3$	1	ω_3
$25-27$	1	$\bar{\omega}_3$	ω_3	ω_3	1	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	1
$28-30$	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3
$31-33$	1	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	1	ω_3	ω_3	$\bar{\omega}_3$	1
$34-36$	1	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	ω_3	1	ω_3	1	$\bar{\omega}_3$

The astute reader would instantly recognise this to be the character table of $\mathbb{Z}_3 \times \mathbb{Z}_3 = A$ (and with foresight we have labelled the elements of the group in the above table). This certainly is to be expected: G^* can be written as cosets gA for $g \in G$, whence lifting the (projective) matrix representation $M(g)$ of g simply gives $\lambda M(g)$ for λ a *phase factor* correponding to the representation (or character as A is always Abelian) of A.

What is happening should be clear: all of this is merely Part (i) of Theorem 2.1 at work. The phases λ are precisely as described in the theorem. The trivial phase 1 gives rows $2 - 12$, or simply the ordinary representation of G while the remaining 8 non-trivial phases give, in groups of 3 rows from char(G^*), the projective representations of G. And to determine to which cocycle the projective representation belongs, we need and only need to determine the the 1-dimensional irreps of A. We shall show in Section 5 how to read out the actual cocycle values $\alpha(g, h)$ for $g, h \in G$ directly with the knowledge of A and G^* without char(G^*).

Enough said on the character tables. Let us proceed to analyse the quiver diagrams. Detailed discussions had already been presented in the case of the dihedral group in [1]. Let us recapitulate the key points. It is the group action on the Chan-Paton bundle that we choose to be projective, the space-time action inherited from $\mathcal{N} = 4$ R-symmetry remain ordinary. In other words, $\mathcal{R}$ from (4.1) must still be a linear representation.

Now we evoke an obvious though handy result: the tensor product of an α-projective representation with that of a β-representation gives an $\alpha\beta$-projective representation (cf. [18] p119),

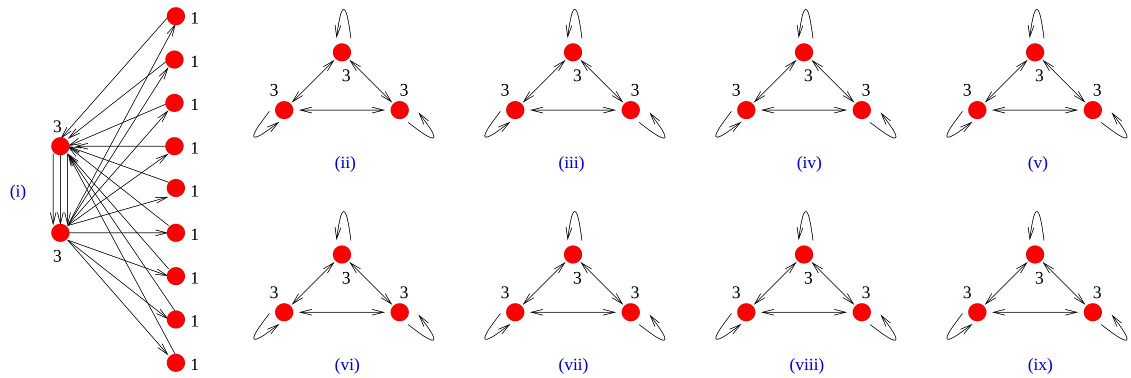

Figure 1: The Quiver Diagram for $\Delta(3\times3^2)^*$ (the Space Invaders Quiver): piece (i) corresponds to the usual quiver for $\Delta(3\times3^2)$ while the remaining 8 pieces (ii) to (ix) are for the cases of the 8 non-trivial discrete torsions (out of the $\mathbb{Z}_3\times\mathbb{Z}_3$) turned on.

i.e.,

$$\Gamma_\alpha(g)\otimes\Gamma_\beta(g)=\Gamma_{\alpha\beta}(g). \tag{4.4}$$

We recall that from (4.1) and in the language of [13, 17], the bi-fundamental matter content $a_{ij}^{\mathcal{R}}$ is given in terms of the irreducible representations R_i of G as

$$\mathcal{R}\otimes R_i=\bigoplus_j a_{ij}^{\mathcal{R}}R_j, \tag{4.5}$$

(with of course $\mathcal{R}$ linear and R_i projective representations). Because $\mathcal{R}$ is an $\alpha=1$ (linear) representation, (4.4) dictates that if R_i in (4.5) is a β-representation, then the righthand thereof must be written entirely in terms of β-representations R_j. In other words, the various projective representations corresponding to the different cocycles should not mix under (4.5). What this signifies for the matter matrix is that $a_{ij}^{\mathcal{R}}$ is block-diagonal and the quiver diagram $Q(G^*,\mathcal{R})$ for G^* *splits* into precisely $|A|$ pieces, one of which is the ordinary (linear) quiver for G and the rest, the various quivers each corresponding to a different value of the cocycle.

Thus motivated, let us present the quiver diagram for $\Delta(3\times3^2)^*$ in Figure 1. The splitting does indeed occur as desired, into precisely $|\mathbb{Z}_3\times\mathbb{Z}_3|=9$ pieces, with (i) being the usual $\Delta(3\times3^2)$ quiver (cf. [21, 22]) and the rest, the quivers corresponding to the 8 non-trivial projective representations.

4.3 The General Method

Having expounded upon the detailed example of $\Delta(3\times3^2)$ and witnessed the subtleties, we now present, in an algorithmic manner, the general method of computing the quiver diagram for an orbifold G with discrete torsion turned on:

1. Compute the character table char(G) of G;

2. Compute a covering group G^* of G and its character table char(G^*);

3. Judiciously re-order the rows and columns of char(G^*):

   - Columns must be arranged into cohorts of char(G), i.e., group the columns which contain a corresponding column in char(G) together;
   - Rows must be arranged so that modulo the dimension of the irreps, the columns with conjugacy class number 1 must contain the character table of the Schur Multiplier $A = M(G)$ (recall that $G^*/A \cong G$);
   - Thus char(G) is a sub-matrix (up to repetition) of char(G^*);

4. Compute the (ordinary) matter matrix $a_{ij}^{\mathcal{R}}$ and hence the quiver $Q(G^*, \mathcal{R})$ for a representation $\mathcal{R}$ which corresponds to a linear representation of G.

Now we have our final result:

THEOREM 4.5 *$Q(G^*, \mathcal{R})$ has $|M(G)|$ disconnected components (sub-quivers) in 1-1 correspondence with the quivers $Q_\alpha(G, \mathcal{R})$ of G for all possible cocycles (discrete torsions) $\alpha \in A = M(G)$. Symbolically,*

$$Q(G^*, \mathcal{R}) = \bigsqcup_{\alpha \in A} Q_\alpha(G, \mathcal{R}).$$

In particular, $Q(G^*, \mathcal{R})$ contains a piece for the trivial $\alpha = 1$ which is precisely the case without discrete torsion, viz., $Q(G, \mathcal{R})$.

This algorithm facilitates enormously the investigation of the matter spectrum of orbifold gauge theories with discrete torsion as the associated quivers can be found without any recourse to explicit evaluation of the cocycles and projective character tables. Another fine feature of this new understanding is that, not only the matter content, but also the superpotential can be directly calculated by the explicit formulae in [17] using the ordinary Clebsch-Gordan coefficients of G^*.

A remark is at hand. We have mentioned in Section 2 that the covering group G^* is not unique. How could we guarantee that the quivers obtained at the end of the day will be independent of the choice of the covering group? We appeal directly to the discussion in the concluding paragraph of Subsection 4.1, where we remarked that using the explicit form of (4.1), we see that the phase factor λ (being a $\mathbb{C}$-number) always cancels out. In other words, the linear representation of whichever G^* we use, when applied to orbifold projections (4.1) shall result in the same matrix form for the projective representations of G. Whence we conclude that the quiver $Q(G^*, \mathcal{R})$ obtained at the end will *ipso facto* be independent of the choice of the covering group G^*.

4.4 A Myriad of Examples

With the method at hand, we move on to the host of other subgroups of $SU(3)$ as tabulated in [1]. The character tables char(G) and char(G^*) will be left to the appendix lest the reader be too distracted. We present the cases of $\Sigma(60, 168, 1080)$, the exceptionals which admit nontrivial discrete torsion and some first members of the Delta series in Figure 2 to Figure 8.

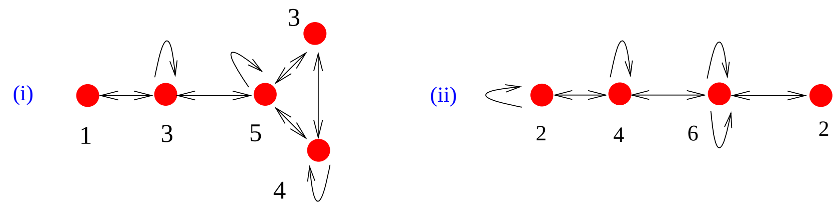

Figure 2: The quiver diagram of $\Sigma(60)^*$: piece (i) is the ordinary quiver of $\Sigma(60)$ and piece (ii) has discrete torsion turned on.

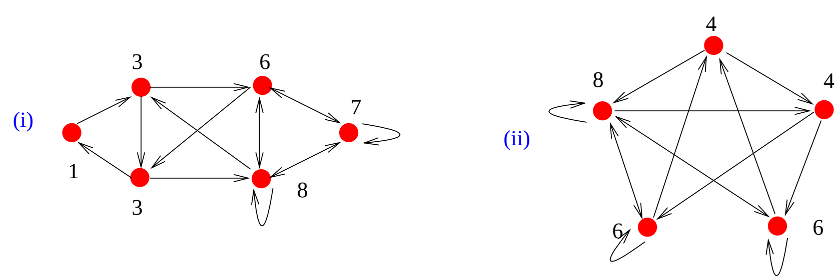

Figure 3: The quiver diagram of $\Sigma(168)$: piece (i) is the ordinary quiver of $\Sigma(168)$ and piece (ii) has discrete torsion turned on.

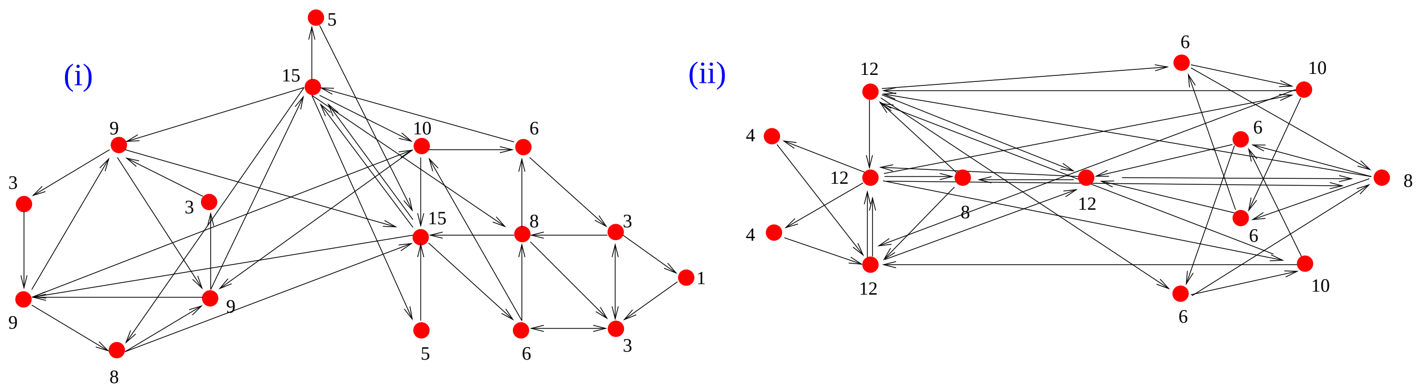

Figure 4: The quiver diagram of $\Sigma(1080)$: piece (i) is the ordinary quiver of $\Sigma(1080)$ and piece (ii) has discrete torsion turned on.

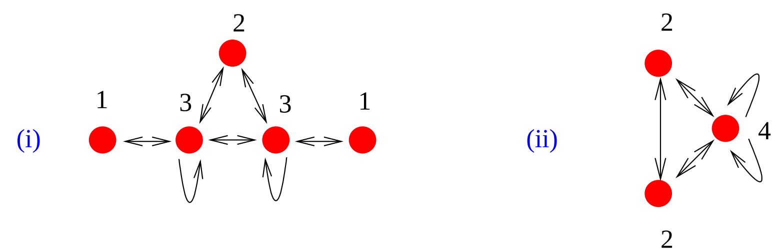

Figure 5: The quiver diagram of $\Delta(6 \times 2^2)$: piece (i) is the ordinary quiver of $\Delta(6 \times 2^2)$ and piece (ii) has discrete torsion turned on.

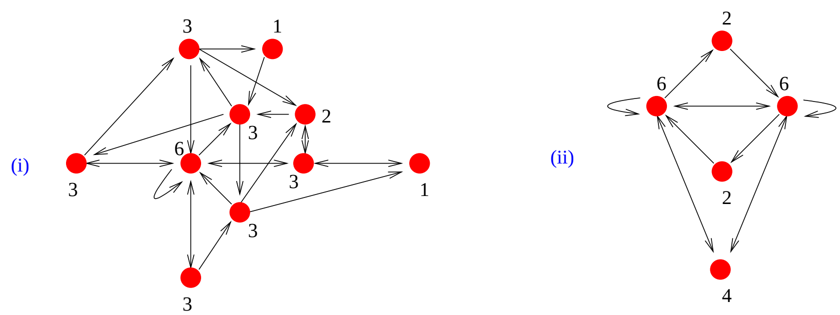

Figure 6: The quiver diagram of $\Delta(6 \times 4^2)$: piece (i) is the ordinary quiver of $\Delta(6 \times 4^2)$ and piece (ii) has discrete torsion turned on.

5. Finding the Cocycle Values

As advertised earlier, a useful by-product of the method is that we can actually find the values of the 2-cocycles from the covering group. Here we require even less information: only G^* and not even $\mathrm{char}(G^*)$ is needed.

Let us recall some facts from Subsection 4.2. The Schur multiplier is $A \subset Z(G^*)$, so every

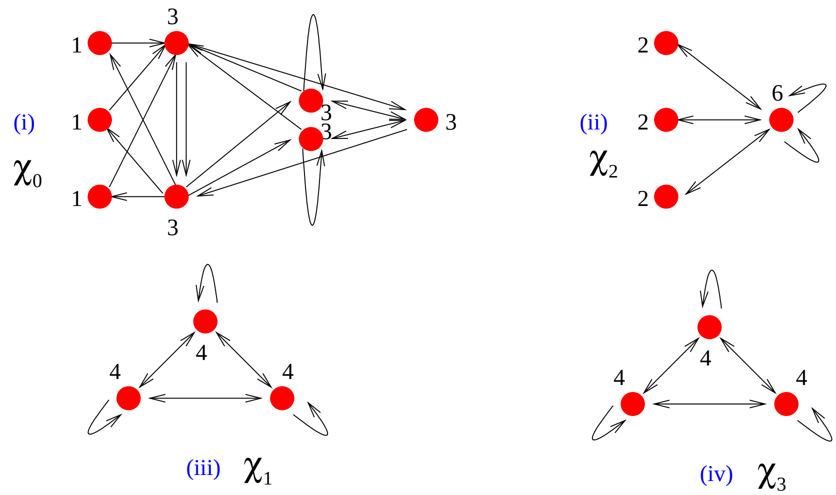

Figure 7: The quiver diagram of $\Delta(3\times 4^2)$: piece (i) is the ordinary quiver of $\Delta(3\times 4^2)$ and pieces (ii-iv) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_4$.

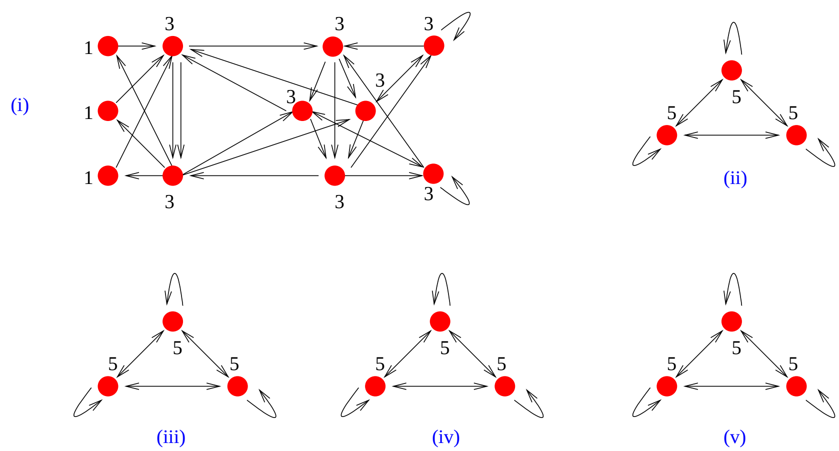

Figure 8: The quiver diagram of $\Delta(3\times 5^2)$: piece (i) is the ordinary quiver of $\Delta(3\times 4^2)$ and pieces (ii-v) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_5$.

element therein has its own conjugacy class in G^*. Hence for all linear representations of G^*, the character of $a_k \in A$ will have the form $d\chi_i(a_k)$ where d is the dimension of that particular irrep of G^* and $\chi_i(a_k)$ is the character of a_k in A in its $i-th$ 1-dimensional irrep (A is always Abelian and thus has only 1-dimensional irreps). This property has a very important consequence: merely reading out the factor $\chi_i(a_k)$ from char(G^*), we can determine which linear representations will give which *projective* representations of G. Indeed, two projective representations of G belong to the same cocycle *when and only when* the factor $\chi_i(a_k)$ is the same for every $a_k \in A$.

Next we recall how to construct the matrix forms of projective representations of G. $G^\star/A \equiv G$

implies that G^* can be decomposed into cosets $\bigcup\limits_{g\in G} gA$. Let $ga_i \in G^*$ correspond canonically to $\tilde{g} \in G$ for some fixed $a_i \in A$; then the matrix form of $\tilde{g}$ can be set to that of ga_i and furnishes the projective representation of $\tilde{g}$. Different choices of a_i will give different but projectively equivalent projective representations of G.

Note that if we have $\tilde{g}_i\tilde{g}_j = \tilde{g}_k$ in G, then in G^*, $g_ig_j = g_k a_{ij}^k$, or $(g_ia_i)(g_ja_j) = g_ka_k(a_{ij}^k a_i a_j a_k^{-1})$, but since (g_ia_i) is the projective matrix form for $\tilde{g}_i \in G$, this is exactly the definition of the cocycle from which we read:

$$\alpha(\tilde{g}_i, \tilde{g}_j) = \chi_p(a_{ij}^k a_i a_j a_k^{-1}), \tag{5.1}$$

where $\chi_p(a)$ is the p-th character of the linear representation of $a \in A$ defined above.

We can prove that (5.1) satisfies the 2-cocycle axioms (i) and (ii). Firstly notice that if $\tilde{g}_i = \mathbb{I} \in G$, we have $g_i = \mathbb{I} \in G^\star$; whence $a_{ij}^k = \delta_j^k \ \forall\ i$ and

$$(i) \quad \alpha(\mathbb{I}, \tilde{g}_j) = \chi_p(\delta_j^k a_j a_k^{-1}) = \chi_p(\mathbb{I}) = 1.$$

Secondly if we assume that $\tilde{g}_i\tilde{g}_j = \tilde{g}_q$, $\tilde{g}_q\tilde{g}_k = \tilde{g}_h$ and $\tilde{g}_j\tilde{g}_k = \tilde{g}_l$, we have $\alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) = \chi_p(a_{ij}^q a_i a_j a_q^{-1})\chi_p(a_{qk}^h a_q a_k a_h^{-1}) = \chi_p(a_{ij}^k a_{qk}^h a_i a_j a_k a_h^{-1})$ and $\alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k) = \chi_p(a_{jk}^l a_j a_k a_l^{-1})\chi_p(a_{il}^h a_i a_l a_h^{-1})$ $= \chi_p(a_{il}^h a_{jk}^l a_i a_j a_k a_h^{-1})$ However, because $(g_ig_j)g_k = g_q a_{ij}^q g_k = g_h a_{ij}^q a_{qk}^h = g_i(g_jg_k) = g_ig_l a_{jk}^l = g_h a_{il}^h a_{jk}^l$ we have $a_{ij}^k a_{qk}^h = a_{il}^h a_{jk}^l$, and so

$$(ii) \quad \alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) = \alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k).$$

Let us summarize the result. To read out the cocycle according to (5.1) we need only two pieces of information: the choices of the representative element in G^* (i.e., $a_i \in A$), and the definitions of G^* which allows us to calculate the $a_{ij}^k \in A$. We do not even need to calculate the character table of G^* to obtain the cocycle. Moreover, in a recent paper [27] the values of cocycles are being used to construct boundary states. We hope our method shall make this above construction easier.

6. Conclusions and Prospects

With the advent of discrete torsion in string theory, the hitherto novel subject of projective representations has breathed out its fragrance from mathematics into physics. However a short-coming has been immediate: the necessary tools for physical computations have so far been limited in the community due to the unavoidable fact that they, if present in the mathematical literature, are obfuscated under often too-technical theorems.

It has been the purpose of this writing, a companion to [1], to diminish the mystique of projective reprsentations in the context of constructing gauge theories on D-branes probing orbifolds with discrete torsion (non-trivial NS-NS B-fields) turned on. In particular we have deviced an algorithm (Subsection 4.3), culminating into Theorem 4.4, which computes the gauge theory data of the orbifold theory. The advantage of the method is its directness: without recourse to the sophistry of twisted group algebras and projective characters as had been suggested by some recent works [4, 13], all methods so-far known in the treatment of orbifolds (e.g. [17, 21]) are immediately generalisable.

We have shown that in computing the matter spectrum for an orbifold G with discrete torsion turned on, all that is required is the ordinary charater table $\text{char}(G^*)$ of the covering group G^* of G. This table, together with the available character table of G, immediately gives a quiver diagram which splits into $|M(G)|$ disjoint pieces ($M(G)$ is the Schur Multiplier of G), one of which is the ordinary quiver for G and the rest, are precisely the quivers for the various non-trivial discrete torsions.

A host of examples are then presented, demonstrating the systematic power of the algorithm. In particular we have tabulated the results for all the exceptional subgroups of $SU(3)$ as well as some first members of the Δ-series.

Directions for future research are self-evident. Brane setups for orbifolds with discrete torsion have yet to be established. We therefore need to investigate the groups satisfying BBM condition as defined in [25], such as the intransitives of the form $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times D$. Furthermore, we have given the presentation of the covering groups of series such as $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D, \mathbb{Z} \times E$ and $\Delta(3n^2), \Delta(6n^2)$. It will be interesting to find the analytic results of the possible quivers.

More important, as we have reduced the problem of orbifolds with discrete torsion to that of *linear* representations, we can instantly extend the methods of [17] to compute superpotentials and thence further to an extensive and systematic study of non-commutative moduli spaces in the spirit of [6]. So too do the families of toric varieties await us, methods utilised in [9, 26] eagerly anticipate their extension.

Acknowledgements

Ad Catharinae Sanctae Alexandriae et Ad Majorem Dei Gloriam...
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7. Appendix

We here present, for the reference of the reader, the (ordinary) character tables of the groups as well as the covering groups thereof, of the examples which we studied in Section 4.

$\Sigma(60)$

1	12	12	15	20
1	1	1	1	1
3	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	-1	0
3	$-\omega_5-\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	-1	0
4	-1	-1	0	1
5	0	0	1	-1

$\Sigma(60)^*$

1	1	12	12	12	12	30	20	20
1	1	1	1	1	1	1	1	1
3	3	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	$-\omega_5-\omega_5^{-1}$	-1	0	0
3	3	$-\omega_5-\omega_5^{-1}$	$-\omega_5-\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5^2-\omega_5^{-2}$	-1	0	0
4	4	-1	-1	-1	-1	0	1	1
5	5	0	0	0	0	1	-1	-1
2	-2	$-\omega_5^2-\omega_5^{-2}$	$\omega_5^2+\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	$\omega_5+\omega_5^{-1}$	0	-1	1
2	-2	$-\omega_5-\omega_5^{-1}$	$\omega_5+\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	$\omega_5^2+\omega_5^{-2}$	0	-1	1
4	-4	1	-1	1	-1	0	1	-1
6	-6	-1	1	-1	1	0	0	0

$\Sigma(168)$

1	21	42	56	24	24
1	1	1	1	1	1
3	-1	1	0	a	$\bar{a}$
3	-1	1	0	$\bar{a}$	a
6	2	0	0	-1	-1
7	-1	-1	1	0	0
8	0	0	-1	1	1

$\Sigma(168)^*$

1	1	42	42	42	56	56	24	24	24	24
1	1	1	1	1	1	1	1	1	1	1
3	3	-1	1	1	0	0	a	a	$\bar{a}$	$\bar{a}$
3	3	-1	1	1	0	0	$\bar{a}$	$\bar{a}$	a	a
6	6	2	0	0	0	0	-1	-1	-1	-1
7	7	-1	-1	-1	1	1	0	0	0	0
8	8	0	0	0	-1	-1	1	1	1	1
4	-4	0	0	0	1	-1	$-a$	a	$-\bar{a}$	$\bar{a}$
4	-4	0	0	0	1	-1	$-\bar{a}$	$\bar{a}$	$-a$	a
6	-6	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	-1	1	-1	1
6	-6	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	-1	1	-1	1
8	-8	0	0	0	-1	1	1	-1	1	-1

$$a := \frac{-1+\sqrt{7}i}{2}$$

$\Sigma(1080)$

1	1	1	45	45	45	72	72	72	72	72	72	90	90	90	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	X	Y	Z	W	$\bar{Z}$	$\bar{W}$	$\bar{A}$	A	1	0	0
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	Y	X	W	Z	$\bar{W}$	$\bar{Z}$	$\bar{A}$	A	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	Y	$\bar{Z}$	$\bar{W}$	Z	W	A	$\bar{A}$	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	X	$\bar{W}$	$\bar{Z}$	W	Z	A	$\bar{A}$	1	0	0
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	2	-1
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	2
6	$6\bar{A}$	$6A$	$2A$	$2\bar{A}$	2	1	1	$\bar{A}$	$\bar{A}$	A	A	0	0	0	0	0
6	$6A$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	A	A	$\bar{A}$	$\bar{A}$	0	0	0	0	0
8	8	8	0	0	0	X	Y	Y	X	Y	X	0	0	0	-1	-1
8	8	8	0	0	0	Y	X	X	Y	X	Y	0	0	0	-1	-1
9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	0	0
9	$9\bar{A}$	$9A$	A	$\bar{A}$	1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$\bar{A}$	A	1	0	0
9	$9A$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	A	$\bar{A}$	1	0	0
10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	1	1
15	$15\bar{A}$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	$-\bar{A}$	$-A$	-1	0	0
15	$15A$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	$-A$	$-\bar{A}$	-1	0	0

$A := \omega_3;$
$B := \omega_5;$
$C := \omega_{15};$
$X := -B-\bar{B};$
$Y := -B^2-\bar{B}^2;$
$Z := -C-C^4;$
$W := -\bar{C}^2-C^7;$

$\Sigma(1080)^*$

$D := B + \bar{B}, E := B^2 + \bar{B}^2, F := \bar{C} + \bar{C}^4, G := C^2 + \bar{C}^7, H := \omega_{24}, J := \bar{H}^7 - H^{11}, K := \bar{H}^5 - H$

1	1	1	1	1	1	90	90	90	72	72	72	72	72	72	72	72	72	72	72	72	90	90	90	90	90	90	120	120	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	X	X	Y	Y	Z	Z	W	W	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	Y	Y	X	X	W	W	Z	Z	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	X	Y	Y	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	Z	Z	W	W	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	Y	X	X	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	W	W	Z	Z	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	2	2	-1	-1
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	2	2
6	6	$6\bar{A}$	$6\bar{A}$	$6A$	$6A$	$2A$	$2\bar{A}$	2	1	1	1	1	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	A	A	A	A	0	0	0	0	0	0	0	0	0	0
6	6	$6A$	$6A$	$6\bar{A}$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	1	1	A	A	A	A	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	0	0	0	0	0	0	0	0	0	0
8	8	8	8	8	8	0	0	0	X	X	Y	Y	Y	Y	X	X	Y	Y	X	X	0	0	0	0	0	0	-1	-1	-1	-1
8	8	8	8	8	8	0	0	0	Y	Y	X	X	X	X	Y	Y	X	X	Y	Y	0	0	0	0	0	0	-1	-1	-1	-1
9	9	9	9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	0	0	0	0
9	9	$9\bar{A}$	$9\bar{A}$	$9A$	$9A$	A	$\bar{A}$	1	-1	-1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$-A$	$-A$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
9	9	$9A$	$9A$	$9\bar{A}$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	-1	-1	$-A$	$-A$	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
10	10	10	10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
15	15	$15\bar{A}$	$15\bar{A}$	$15A$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	-1	-1	0	0	0	0
15	15	$15A$	$15A$	$15\bar{A}$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	-1	-1	0	0	0	0
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	1	-1	-2	2
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	-2	2	1	-1
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	J	$-J$	K	$-K$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-J$	J	$-K$	K	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	$-K$	K	$-J$	J	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	K	$-K$	J	$-J$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
8	-8	-8	8	-8	8	0	0	0	D	X	E	Y	E	Y	D	X	E	Y	D	X	0	0	0	0	0	0	-1	1	-1	1
8	-8	-8	8	-8	8	0	0	0	E	Y	D	X	D	X	E	Y	D	X	E	Y	0	0	0	0	0	0	-1	1	-1	1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	1	-1	1	-1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	1	-1	1	-1
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	X	D	Y	E	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	Z	F	W	$\bar{G}$	0	0	0	0	0	0	0	0	0	0
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	Y	E	X	D	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	W	$\bar{G}$	Z	F	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	X	D	Y	E	Z	F	W	$\bar{G}$	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	Y	E	X	D	W	$\bar{G}$	Z	F	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	0	0	0	0	0	0	0	0	0	0

$\Delta(6 \times 2^2) =$

1	3	6	6	8
1	1	1	1	1
1	1	-1	-1	1
2	2	0	0	-1
3	-1	-1	1	0
3	-1	1	-1	0

$\Delta(6 \times 2^2)^* =$

1	1	6	6	6	12	8	8
1	1	1	1	1	1	1	1
1	1	1	-1	-1	-1	1	1
2	2	2	0	0	0	-1	-1
3	3	-1	-1	-1	1	0	0
3	3	-1	1	1	-1	0	0
2	-2	0	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	0	-1	1
2	-2	0	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	0	-1	1
4	-4	0	0	0	0	1	-1

$\Delta(6 \times 4^2) =$

1	3	3	3	6	12	12	12	12	32
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	-1	-1	-1	-1	1
2	2	2	2	2	0	0	0	0	-1
3	3	-1	-1	-1	-1	1	1	-1	0
3	3	-1	-1	-1	1	-1	-1	1	0
3	-1	$-1-2i$	$-1+2i$	1	-1	i	$-i$	1	0
3	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	i	1	0
3	-1	$-1-2i$	$-1+2i$	1	1	$-i$	i	-1	0
3	-1	$-1+2i$	$-1-2i$	1	1	i	$-i$	-1	0
6	-2	2	2	-2	0	0	0	0	0

$\Delta(6 \times 4^2)^* =$

1	1	3	3	6	6	12	24	12	12	12	12	24	32	32
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
2	2	2	2	2	2	2	0	0	0	0	0	0	-1	-1
3	3	3	3	-1	-1	-1	-1	1	1	1	1	-1	0	0
3	3	3	3	-1	-1	-1	1	-1	-1	-1	-1	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	-1	i	i	$-i$	$-i$	1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	$-i$	i	i	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	1	$-i$	$-i$	i	i	-1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	1	i	i	$-i$	$-i$	-1	0	0
6	6	-2	-2	2	2	-2	0	0	0	0	0	0	0	0
2	-2	-2	2	0	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	0	-1	1
2	-2	-2	2	0	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	0	-1	1
4	-4	-4	4	0	0	0	0	0	0	0	0	0	1	-1
6	-6	2	-2	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
6	-6	2	-2	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0

$\Delta(3 \times 4^2) =$

1	3	3	3	3	3	16	16
1	1	1	1	1	1	1	1
1	1	1	1	1	1	ω_3	$\bar{\omega}_3$
1	1	1	1	1	1	$\bar{\omega}_3$	ω_3
3	-1	-1	3	-1	-1	0	0
3	1	1	-1	$-1-2i$	$-1+2i$	0	0
3	1	1	-1	$-1+2i$	$-1-2i$	0	0
3	$-1-2i$	$-1+2i$	-1	1	1	0	0
3	$-1+2i$	$-1-2i$	-1	1	1	0	0

$\Delta(3 \times 4^2)^* =$

1	1	1	1	12	12	6	6	12	12	16	16	16	16	16	16	16	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	ω_3
3	3	3	3	-1	-1	3	3	-1	-1	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1-2i$	$-1+2i$	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1+2i$	$-1-2i$	0	0	0	0	0	0	0	0
3	3	3	3	$-1-2i$	$-1+2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
3	3	3	3	$-1+2i$	$-1-2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
2	-2	2	-2	0	0	2	-2	0	0	$-\omega_3$	ω_3	ω_3	$-\omega_3$	$-\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
2	-2	2	-2	0	0	2	-2	0	0	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\omega_3$	$-\omega_3$	ω_3	ω_3
2	-2	2	-2	0	0	2	-2	0	0	-1	1	1	-1	-1	-1	1	1
6	-6	6	-6	0	0	-2	2	0	0	0	0	0	0	0	0	0	0
4	$4i$	-4	$-4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	ω_3	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	1	$-i$	i	-1	1	-1	i	$-i$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	ω_3	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	1	i	$-i$	-1	1	-1	$-i$	i

$\Delta(3 \times 5^2) =$

1	3	3	3	3	3	3	3	3	25	25
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	E	$\bar{E}$
1	1	1	1	1	1	1	1	1	$\bar{E}$	E
3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0
3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0
3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0
3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0
3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0
3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0
3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0
3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0

$\Delta(3 \times 5^2)^* =$

1	1	1	1	1	15	15	15	15	15	15	15	15	25	25	25	25	25	25	25	25	25	25
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	E	E	E	E	E	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$
1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	E	E	E	E	E
3	3	3	3	3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0	0	0	0	0	0	0	0	0
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$

$$A := -\omega_5 - \bar{\omega}_5, B := -\omega_5^2 - \bar{\omega}_5^2, C := \bar{\omega}_5 - 2\bar{\omega}_5^2, D := 2\omega_5 + \bar{\omega}_5^2; E := \omega_3, F := \bar{\omega}_5, G := \omega_{15}.$$

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