THE CITY UNIVERSITY
DEPARTMENT OF MATHEMATICS

THE DYNAMICS OF PENSION FUNDING

by
Daniel Dufresne

Thesis submitted for the
degree of Doctor of Philosophy

July 1986

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ACKNOWLEDGEMENTS

The research was conducted under the supervision of Dr. Steve Haberman. I wish to thank him for the help and encouragement (both discrete and continuous) he provided during these four years.

I also wish to thank Prof. Anatole Joffe (Un. of Montreal) whose help concerning the recondite subject of the convergence of semimartingales (Proposition 4.1) was essential.

For their valuable advice and comments, I am also grateful to Mr. Sidney Benjamin, Miss Barbara Cairns (both of Bacon & Woodrow), Dr. Mark Davis (Imperial College), Miss Monique Tremblay (Towers, Perrin, Forster & Crosby, Montreal) and Mr. David Wilkie (Watson & Sons). For her invaluable skills and patience, I wish to thank Mrs Francine Houle, who typed the thesis.

I am grateful to the Commonwealth Scholarship Commission, and to the Natural Sciences and Engineering Council Canada, whose sponsorship made this research possible.

DECLARATION - COPYING OF THIS THESIS

ABSTRACT

In the context of North American and British actuarial practice, a mathematical model is used to study the evolution over time of the fund levels (F) and contributions (C). First, actuarial cost methods (e.g. Unit Credit, Aggregate) are examined in the traditional "static" framework. Three points are studied: (1) comparison of the various methods, (2) inclusion of new entrants in the valuation basis, and (3) the rate at which $F(t)$ reaches its ultimate level, as $t \to \infty$. Next, the model is modified to include varying rates of return and of inflation. Two "methods of adjusting the normal cost" are considered: (1) the adjustment is equal to the unfunded liability divided by the present value of an annuity for a term of "m" years (Spread method); (2) each intervaluation loss is liquidated by a fixed number of payments over the following years (Amortization of Losses method). The core of the thesis has to do with random rates of return. In discrete time, these rates are supposed independent and identically distributed. Recursive equations are derived for the first and second moments of $F(t)$ and $C(t)$, under methods (1) and (2). In the case of the Spread method, an "optimal region" is specified for "m": it is shown that for $m \geq m^*$ the variances of both F and C are increasing functions of m. The optimal region is thus $1 \leq m \leq m^*$. The Spread method is also studied in continuous time, assuming rates of return to be a white noise process. A proof is given of the convergence of the discrete processes F^n (representing the fund when "n" valuations are performed every year) to a diffusion process F, as $n \to \infty$. Using the Itô calculus of diffusion processes, the first two moments of $F(t)$ and $C(t)$ are then shown to satisfy some particular

differential equations. The final chapter applies similar ideas to the calculation of the moments of annuities-certain, when rates of return are a white noise process.

CHAPTER 0
INTRODUCTION

0.1 SUBJECT-MATTER

This is a study of some aspects of pension funding, in the context of North American and British actuarial practice. The central idea is to view fund levels and contributions as "processes" taking place over time. In other words, the emphasis is put on determining the characteristics of the whole sequence of fund levels and contributions, rather than on analyzing them at isolated points in time.

Various methods of calculating the contributions are described, first in the traditional "static" setting, and, next, in the case of varying rates of return and of inflation. The major topic studied is the derivation of the moments of the fund levels and contributions, when rates of return are independent identically distributed (i.i.d.) random variables (discrete-time formulation) or a white noise process (continuous-time formulation).

0.2 METHODOLOGY

The analysis is carried out mathematically using a simplified model for the pension scheme and population. A few numerical illustrations are included.

Another way of treating the subject would be to use computer simulations. These can be applied to a wider range of situations, since they do not require as many simplifying assumptions as mathematical modelling usually does. Two comments are called for in order to explain, and perhaps justify, the approach adopted in this thesis.

(1) From a purely academic point of view, mathematical results are preferable to numerical ones. As concerns

pension funding, it is felt that more theoretical research is needed, since little has been done so far.

(2) In relation to the concrete problems encountered by actuaries, however, mathematical modelling cannot claim to be the final answer. The practical applications of the results presented here are restricted to (i) problems which are themselves simple enough to fit into the models studied, and (ii) obtaining approximate answers, for the more complex ones, without having to resort to full-scale simulations.

It is hoped that practitioners, even though they may find few practical uses for the results of this research, will at least benefit from the new insight it brings into pension funding.

0.3 OUTLINE

Each chapter begins with an introduction which describes its content and relates it to previously published work. The second section is about notation and assumptions (except for Chapter 5). Long proofs and numerical calculations are contained in the appendices. Summaries of Z-transform techniques and of the Itô calculus are also provided (Appendices 3.2 and 4.1). The text is fairly self-contained, except as regards weak convergence and Proposition 4.1 (Chapter 4).

Section 1.3 of Chapter 1 describes the funding methods traditionally used in North America and Britain (Aggregate, Unit Credit, etc.). The remainder of the chapter is devoted to analyzing some of these methods in a static environment, using a continuous-time model. The three major points studied are

(i) comparing the various methods;

(ii) the effects of including new entrants in the valuation basis; and

(iii) the rate at which the fund reaches its ultimate level.

The results extend those of Trowbridge (1952).

Chapter 1 is, to a large extent, necessary background for Chapters 2,3 and 4. Nevertheless, some of the results are mainly related to the underlying population of members, and so have little to do with the rest of the thesis (for example Sections 1.4.4 and 1.5.1).

In Chapter 2, the transition is made from the static model of Chapter 1, to the stochastic model assumed in Chapters 3 and 4. The major hypotheses are now

(i) varying rates of return and of inflation;

(ii) fixed actuarial assumptions; and

(iii) unindexed benefits.

Since economic assumptions are no longer in agreement with actuarial assumptions, the methods of Chapter 1 have to be supplemented with "methods of adjusting the normal cost". Two of these are described:

(1) The "Spread" method. The adjustment is equal to the unfunded liability (i.e. actuarial liability - actual fund) divided by the present value of an annuity for a fixed term.

(2) The "Amortization of Losses" method. Each intervaluation loss is liquidated by a fixed number of payments over the following years. The total contribution is then the normal cost plus the sum of those payments which are still in force.

Chapters 3 and 4 constitute the core of the thesis. They are primarily concerned to calculate the first and second moments of the fund and contribution, when rates of return are random. The model of Chapter 2 is kept largely unchanged, the only differences being

(i) there is no inflation on salaries (or, equivalently, benefits are fully indexed), and

(ii) rates of return form a white noise process.

Chapter 3 deals with the discrete-time formulation. Recursive equations are derived for the moments of the fund and contributions, under the two methods of adjusting the normal cost defined in Chapter 2. Asymptotic formulae and numerical illustrations are also provided.

In the case of the Spread method, an "optimal region" is specified for "m", the number of years over which the unfunded liability is spread. It is shown that for m greater than a particular value m^*, the variances of both the fund and the contribution are increasing functions of m. Thus the "optimal" values of m are $1 \leq m \leq m^*$.

Chapter 4 examines the continuous counterpart of the Spread method. Section 4.3 identifies which stochastic process F describes the evolution of the fund, when rates of return are a continuous-time white noise. First, a sequence of processes $\{F^n, n \geq 1\}$ is defined. Each F^n represents a fund subject to i.i.d. rates of return, as in Chapter 3, but with "n" valuations being performed every year. Then, the process F is found by taking the limit, as $n \to \infty$, of the sequence $\{F^n\}$. The convergence proof relies on recent results by Joffe and Metivier (1986) about the weak convergence of semimartingales.

In Section 4.4, differential equations are derived for the moments of the fund and contributions, with the help of the Itô calculus of diffusion processes.

Chapter 5 applies the ideas set forth in Chapters 3 and 4 to the calculation of the moments of annuities-certain, when rates of return are a white noise process. Some of the results of Boyle (1976) are reproduced, and further extended to continuous annuities.

0.4 NOTATION

Chapters 1 and 2

(i) All symbols corresponding to "actual" monetary amounts have a bar ("-") above them; the same symbols without a bar refer to "real-term" values (viewed at time 0). Example: $\bar{F}(t)$ is the actual fund built up at time t, while F(t) is the "deflated" counterpart

$$F(t) = \exp(-\beta t)\bar{F}(t),$$

β being the assumed rate of inflation on salaries.

(ii) The argument "s" or "t" is dropped when a function is constant over time; e.g. if B(x,t) is the same for all t, then it is written as B(x).

(iii) The argument standing for age (x or y usually) is dropped when a summation over all ages is performed. For instance

$$NC(t) = \sum_{x=a}^{r-1} NC(x,t).$$

(iv) The same symbols are often used in both the discrete-time and the continuous-time formulations. Any difference in meaning should be clear from the context.

<u>Annuities</u>

The symbols $\ddot{a}_{x:\overline{m}|}$, $\ddot{s}_{\overline{n}|}$, etc. have the usual meaning, except that, to simplify the notation

(i) the rate of interest is not shown when it is implicit from the context;

(ii) in Chapters 1 and 2, the <u>force</u> of interest (and not the <u>rate</u> of interest) is shown, when needed, as a superscript. For example,

$$\sum_{j=0}^{n-1} \exp(j \cdot \gamma)$$

is denoted by $\ddot{a}_{\overline{n}|}^{(\gamma)}$, instead of $\ddot{a}_{\overline{n}|}^{(e^{\gamma}-1)}$.

In Chapter 5, the random variables a(t), $\bar{a}(t)$, $\ddot{s}(t)$ and $\bar{s}(t)$ are the counterparts of $a_{\overline{t}|}$, $\bar{a}_{\overline{t}|}$, $\ddot{s}_{\overline{t}|}$ and $\bar{s}_{\overline{t}|}$, respectively, when rates of return are random.

0.5 LIST OF SYMBOLS

The sections in which the symbols are defined are indicated in brackets.

$\alpha, \alpha(n)$	Parameters representing the rate of convergence of $F(t)$ to $F(\infty)$ under the Aggregate method (1.4.1) or Aggregate with New Entrants method (1.4.3). Note that $\alpha(0) = \alpha$. In Chapters 2 and 4, α refers to the Spread method in general, with $\alpha = 1/\bar{a}_{\overline{m}\rceil} - \gamma$ (2.4.2.1).
β	Assumed rate of inflation on earnings
$\beta(t)$	Actual rate of inflation on salaries (2.2)
γ	Assumed net rate of return $= \eta - \beta$
$\gamma(t)$	Actual net rate of return (2.2)
γ_V	Valuation force of interest (4.2)
$\delta(\cdot)$	Dirac delta function
$\Delta\beta(t)$	$= \beta(t) - \beta$
$\Delta\gamma(t)$	$= \gamma(t) - \gamma$
$\Delta\eta(t)$	$= \eta(t) - \eta$
Δi	$= Ei(t) - i_V$
ς	Increase of pensions in payment (1.2.2)
η	Assumed nominal rate of return on assets
$\eta(t)$	Actual nominal rate of return (2.2)
μ_x	Force of mortality. See comment in 1.2.1
σ^2	$=$ Var $i(t)$ (Chapter 3). In continuous time (Chapters 4 and 5), σ^2 is also the variance of the instantaneous rates of return (4.2(v))
ω	End of life table
a	Entry age into scheme (1.2.2)
$\ddot{a}_{\overline{m}\rceil}, a_{\overline{m}\rceil}, \bar{a}_{\overline{m}\rceil}$	Present value of an m-year annuity-certain, with payments made at the beginning of the year, resp. the end of the year, continuously.

$\ddot{a}_x$	Present value, at age x, of a life annuity with payments made at the beginning of the year.
$\ddot{a}_{x:\overline{m}\rceil}$	Present value of an annuity payable until age x+m-1 or the annuitant's death, whichever occurs first, with payments made at the beginning of the year.
$_{m\|}\ddot{a}_x$	Present value of an annuity payable for life, with the first payment deferred m years. It is equal to $\exp(-\eta m)(\ell_{x+m}/\ell_x)\ddot{a}_{x+m}$. The life tables used before and after age x+m may differ if x+m=r.
$a(t), \bar{a}(t)$	Equivalents of $a_{\overline{t}\rceil}$ and $\bar{a}_{\overline{t}\rceil}$, when rates of return are random (5.3.2) and 5.5.2).
AAL	Active members actuarial liability (2.2)
AAN	Attained Age Normal cost method
ADJ(t)	Adjustment to normal cost
AL(x,t)	Actuarial liability w.r. to one member age x, at time t
b	In calculating benefits at retirement, fraction of salary which constitutes the pension, per year of service (1.2.2)
B(x,t)	Pension paid to one member age x at time t
B^*	Benefits paid when $\beta(t) = \beta$ for all t (in real terms) (2.2)
c	= sb(r-a) in Chapter 2
c(t)	Contribution, as fraction of payroll
c(n,t)	Contribution under Aggregate with New Entrants method, as a fraction of payroll (1.4.3). Note that $c(0,t) = {}^{AGG}c(t)$
C(t)	Overall contribution at time t

$C(n,t)$	Overall contribution under the Aggregate with New Entrants method
d, d_V	$= i/(1+i)$, resp. $i_V/(1+i_V)$
$e(y)$	In Chapter 1, rate of entry of new members age y (1.6)
$e(j)$	$= a_{\overline{m-1-j}\vert}/\ddot{a}_{\overline{m}\vert}$ in Chapter 3 (3.4)
$\mathring{e}_{x:\overline{m}\vert}$	$= \int_0^m \ell_{x+s}/\ell_x ds = \bar{a}^{(0)}_{x:\overline{m}\vert}$ (known as "partial life expectancy")
E	Expectation operator
EAN	Entry Age Normal cost method
ELT 13	English Life Table No. 13 (males)
$F(t)$	Fund at time t (before contributions and benefits are paid, in the discrete case)
$F(n,t)$	Fund at time t under the Aggregate with New Entrants method (1.4.3). Note that $F(0,t) = {}^{AGG}F(t)$
FIL	Frozen Initial Liability cost method
H_t	σ-field of events prior to (inclusive of) time t (3.2 and 4.2)
i	$= Ei(t)$ (3.2)
i_V	Valuation rate of interest (3.2)
$i(t)$	Earned rate of return during $(t-1,t)$ (3.2)
ILP	Individual Level Premium cost method
$\ell(t)$	Actuarial loss during $(t-1,t)$ (at exact time t in continuous time) (2.3.2.1 and 2.4.2.1)
ℓ_x	Survival function (1.2.1)
m^*	Boundary of "optimal region" (3.5.3)
n	Number of years over which new entrants are assumed to enter the scheme under the Aggregate with New Entrants method (Chapter 1). In Chapters 4 and 5, the superscript "n" refers to the situation where n valuations are performed every year.

$NC(x,t)$	Normal cost for one member age x at time t
$o(t)$	Let $-\infty \le a \le \infty$. Definition: $f(t) = o(t^n)$ as $t \to a$ if $\lim_{t \to a} \lvert f(t)/t^n \rvert = 0$.
$p(t)$	Payments made to liquidate $\ell(t)$ (2.3.3)
$P(t)$	Payment towards amortization of unfunded liability
PG	Pay-as-you-go
$PVB(t)$	Present value of future benefits of all current members (1.4.2)
$PVB(t,n)$	Present value of future benefits of current members *and* of new entrants coming into the scheme over the next n years (1.4.3). Note that $PVB(t,0) = PVB(t)$
PVB^*	$PVB(t)$ when $\beta(t) = \beta$ for all t (2.2)
$PVS(t)$	Present value of future salaries of all active members (1.4.2)
$PVS(t,n)$	Present value of future salaries of all currently active members and of new entrants coming into the scheme over the next n years (1.4.3). Note that $PVS(t,0) = PVS(t)$
q	$= e^{\gamma}(1- 1/\ddot{a}_{\overline{m}|})$ (2.3.2.1) or $(1+i)(1- 1/\ddot{a}_{\overline{m}|})$ (3.3)
r	In Chapter 1, retirement age. In Chapter 3, $r = (1+i)(NC-B+AL/\ddot{a}_{\overline{m}|})$
R	$= S \cdot PVB/PVS-B$ (1.4.1)
$RAL(t)$	Actuarial liability w.r. to retired members = present value of future benefits (2.2)
RAL^*	$RAL(t)$ when $\beta(t) = \beta$ for all t (2.2)
s	Salary of one active member at time 0
$\ddot{s}_{x:\overline{m}|}$	Value of m payments starting at age x, accumulated with interest and survivorship in service to the end of the m-th year

$\ddot{s}(t), \bar{s}(t)$	Equivalents of $\ddot{s}_{\overline{t}\vert}$ and $\bar{s}_{\overline{t}\vert}$, when rates of return are random (5.3.1 and 5.5.1)
$\ddot{s}_{\overline{t}\vert}, \bar{s}_{\overline{t}\vert}$	Accumulated value, at time t, of a t-year annuity-certain with payments made at the beginning of the year (resp. continuously)
$S(t)$	Payroll at time t
u	$= 1+i$
$u(t)$	$= 1+i(t)$
$U(t)$	Unamortized part of the initial unfunded liability (1.4.1)
UC	Unit Credit cost method
$UL(t)$	$= AL(t)-F(t) =$ unfunded liability
v	$= Ev(t)$
$v(t)$	$= 1/(1+i(t))$
W	Wiener process. See 4.2 and Appendix 4.1
$Z(t)$	Sum of forces of interest up to time t (5.6)

0.6 ABOUT "INFLATION" AND "REAL TERMS"

The only type of inflation considered in this text will be inflation on salaries.

The analysis of Chapters 1 to 4 refers to final-salary schemes and, accordingly, all monetary values relate, directly or indirectly, to the rate at which average salaries grow from year to year. As it is the evolution of these "actual" (or "nominal") monetary values which is studied, it is mathematically useful to work with "de-inflated" monetary values, which will be called "real-term" values. As is described in Sections 0.4, 2.3.1 and 2.4.1, this is done by dividing actual amounts at time t by the index number

$$\exp\left[\sum_{k=1}^{t} \beta(k)\right] \quad \text{or} \quad \exp\left[\int_0^t \beta(s)ds\right].$$

In consequence, "growth in real terms" will mean growth at a rate superior to the rate of increase of salaries. This explains the importance of the quantity

$$\begin{aligned} \gamma(t) &= \eta(t) - \beta(t) \\ &= \text{rate of return on assets} \\ &\quad - \text{rate of increase of salaries} \\ &= \text{"real rate of return".} \end{aligned}$$

The expressions "real terms" and "real rate of return" will thus have meanings slightly different from the ones they have in economics. In particular, *no* reference is made to price inflation or to the retail price index.

CHAPTER 1
ACTUARIAL COST METHODS:
CLASSICAL THEORY

1.1 INTRODUCTION

The main purpose of the first chapter is to give an account of actuarial cost methods, as background to the second chapter. This is done in Section 1.3. The description is chiefly North American in content and style, and follows (in decreasing order of importance) Trowbridge (1952), Winklevoss (1977) and Anderson (1985). British methods are briefly mentioned, following Colbran (1982) and, more importantly, Turner et al (1984).

Rather than providing an exhaustive study of actuarial cost methods, Section 1.3 endeavours to give an insight into the functioning of the most common ones. Gains and losses are not discussed (see Chapter 2), and only retirement benefits are considered; in any case, there is no "accepted" or "standard" way of handling benefits other than retirement when applying any one of the cost methods (in this respect, see Chapter 11 of Winklevoss (1977) and Chapter 4 of Anderson (1985)). Furthermore, the presentation specifically refers to defined benefit schemes, where benefits are linked to final salary.

The remainder of Chapter 1 is devoted to a certain number of theoretical problems connected with pension funding. Section 1.4 treats a particular family of cost methods of the aggregate type. Three points are studied. The first one is the question of identifying the limiting fund levels and contribution rates that this family of methods lead to, when they are repeatedly applied to a stationary population. The second one concerns the effect of introducing new entrants into the valuation basis; while almost unheard of in North America, new entrants assumptions are indeed a feature of the "Discontinuance" or "Control" methods used in Great Britain. The problem is treated mathematically, which of course means that several simplifying assumptions have to be made. The third topic studied is the influence of the salary distribution and of the rate of interest on the "rate of convergence" of $F(t)$ to its limit $F(\infty)$.

Section 1.5 is an attempt at comparing the different actuarial cost methods, with the help of mathematics instead of numerical simulations. There is no shortage of numerical comparisons in the literature; consider, among others, Colbran (1982), McGill (1979), McLeish (1983), and especially the notable mass of simulation results displayed in Winklevoss (1977). On the other hand, truly mathematical comparisons of actuarial cost methods are scarce: Trowbridge (1952) is probably the best example, while Winklevoss (1977) also does indicate some interesting relationships - yet without proving them. Section 1.5 contains proofs of generalized versions of some of the claims made by these two authors; as regards methods, however, this section is closer to what Picot (1976) and Hickman (1968) have done in related though slightly different contexts. A numerical illustration follows the mathematical analysis.

Section 1.6 discusses the validity of some of the results proved in 1.4 and 1.5, when one of the major assumptions - a single entry age into the scheme - is disposed of.

While in 1.3 the methods are described assuming yearly valuations and a "discrete" population, the rest of the chapter supposes continuously performed valuations and a continuous age distribution. This simplifies some of the proofs, and in my view makes the theory more elegant.

How salary scales are dealt with is explained in 1.2.1, and the model population and scheme are set forth in 1.2.2.

1.2 PRELIMINARIES

1.2.1 Promotional Salary Scales

In this chapter, no explicit reference will be made to salary scales. The reason is simple: as far as the completely deterministic situation is concerned, the salary scale and the survival function ℓ_x can be lumped together to form a new function ℓ_x^s, without causing any loss in generality.

Let me explain the last statement. Consider the present value of the retirement benefits an individual now aged a will receive starting at age r. Let s be his current salary and SS(•) the salary scale (excluding inflation). Also let η and β be the nominal rate of return and the rate of inflation on salaries, respectively. If the benefit formula is 100c% of final salary,

Present value of retirement benefits

$$= c \cdot s \cdot \exp[\beta(r-a)](SS(r)/SS(a))\,{}_{r-a|}\ddot{a}_a$$

$$= c \cdot s \cdot \exp[(\beta-\eta)(r-a)](SS(r)/SS(a))(\ell_r/\ell_a)\ddot{a}_r$$

$$= c \cdot s \cdot \exp[(\beta-\eta)(r-a)](\ell_r^s/\ell_a^s)\ddot{a}_r$$

$$\text{where} \quad \ell_x^s = \ell_x \cdot SS(x)$$

$$= c \cdot s \cdot \exp[\beta(r-a)]\ {}_{r-a|}\ddot{a}_a^s \ .$$

In the last expression ${}_{r-a|}\ddot{a}_a^s$ is computed using the "salary-survival" function ℓ_x^s (only in deferment, i.e. up to age r).

This is why one may find, in what follows, phrases like "if μ_x is smaller than zero", which usually do not make much sense, but do have a meaning if SS(x) increases quickly enough. As an example, Colbran (1982) uses for his numerical illustration a "flat" salary distribution (i.e. total salaries at age x are equal for all x). If one believes that this state of affairs is to persist in the future, then it could equivalently be said that $\mu_x^s = 0$ for all x. One could even envisage a situation where $\mu_x^s < 0$ at all ages, in other words that decrements are more than offset by salary increases.

The function μ_x^s will therefore not be assumed of any shape whatsoever. The superscript "s" will be dropped, for clarity, from μ_x^s and ℓ_x^s.

This kind of simplification would evidently not be possible if random deviations in mortality or withdrawals were considered.

Thus, to "lower μ_x" will mean any combination of

(i) decreasing the death rate

(ii) decreasing the withdrawal rate

(iii) making the salary scale *steeper* at age x.

1.2.2 Model Population and Scheme

The basic model population is the same as the one adopted by Trowbridge (1952): it is stationary from the start, and new entrants age a come into the scheme at rate ℓ_a per year; however, several comments will be made concerning the case where the initial population is not stationary.

The benefits are b times years in service times final salary, in the form of a straight life annuity. At any time t, all new entrants earn a salary equal to $s \cdot \exp(\beta t)$, increasing at rate β till they retire. Both η (the continuous rate of return on the fund's assets) and β are constants. In the discrete case, salaries are assumed to be paid in full at the beginning of the year. Benefits in payment increase at rate ζ, not necessarily equal to β.

Remark 1.1. In the discrete case there is a minor technical difficulty with the calculation of the benefit at retirement. A member having entered at age $a = 30$ on January 1, 1432 retired at $r = 65$ on January 1 1467, if still in service. As salaries are assumed to be paid at the beginning of the year, this member's final salary was paid on January 1466. Hence the pension would have been

$$b(r-a)\cdot\exp[\beta(r-a-1)]\cdot\text{initial salary.}$$

To make the formulas less cumbersome (and more similar to their continuous counterparts) I will suppose the pension to have been increased with an extra year's inflation, i.e. that

$$\text{pension} = b(r-a)\exp[\beta(r-a)]\cdot\text{initial salary.}$$

1.2.3 Terminology

Nomenclature is a problem in the pension world. Several different terminologies have been proposed for the actuarial cost methods, but I will retain the "old" one used by Trowbridge (1952), since it apparently still is the one most widely understood by practitioners.

1.3 DESCRIPTION OF TRADITIONAL METHODS

1.3.1 Unit Credit

1.3.1.1 Description

Trowbridge (1952) writes (p.22):

> Unit credit funding is based on the principle that the pension to be provided at retirement age will be divided into as many "units" as there are active membership years, with one unit assigned to each year. The normal cost as to any individual pension in any year becomes the cost to fully fund on a single premium basis the unit assigned to that year. The accrued liability at any time is the present value of all units of pension assigned to prior

years. Under this method of funding particularly the accrued liability is often referred to as the "past service" liability.

While there are theoretically no restrictions as to what these "units" should be, I will describe the version of the method which is most popular in practice. Here one unit is defined as the projected benefit at retirement divided by the number of active years before retirement (this is the so-called "Projected Unit Credit" method).

For each of the ℓ_a new members entering the scheme at time t, one unit is

$$\underbrace{\underbrace{s\cdot\exp(\beta t)}_{\substack{\text{earnings}\\ \text{at time t}}}\cdot\underbrace{\exp[\beta(r-a)]}_{\substack{\text{inflation on}\\ \text{earnings till}\\ \text{age r}}}\cdot b\cdot\underbrace{(r-a)}_{\substack{\text{years of}\\ \text{service at}\\ \text{age r}}}}_{\text{projected benefit}}\cdot\underbrace{\frac{1}{(r-a)}}_{\substack{\text{divide by}\\ \text{number of active}\\ \text{membership years}}}$$

$$= s\cdot b\cdot\exp[\beta(t+r-a)].$$

Hence the normal cost at time t with respect to each of these new entrants is

$$\bar{N}\bar{C}(a,t) = sb\exp[\beta(t+r-a)]\cdot{}_{r-a|}\ddot{a}_a.$$

The following year, the normal cost with respect to each member aged a+1 will be

$$\bar{N}\bar{C}(a+1,t+1) = sb\exp[\beta(t+r-a)]\cdot{}_{r-(a+1)|}\ddot{a}_{a+1}$$

$$= sb\cdot\exp[\beta((t+1)+r-(a+1))]\cdot{}_{r-(a+1)|}\ddot{a}_{a+1}.$$

It can be seen that for any x, $a \le x \le r-1$,

$$(1.1) \qquad \bar{NC}(x,t) = sb\ \exp[\beta(t+r-x)]\cdot {}_{r-x|}\ddot{a}_x .$$

The actuarial liability is the "mathematical reserve" (in the life insurance sense) built up by these normal costs, if all assumptions are exactly realized. Consider the same ℓ_a new entrants at time t. The accumulated value of the normal costs (taking survivorship in service into account) is, at time $t^1 = t+x-a$ (i.e. when the remaining members reach age x):

$$\sum_{k=0}^{x-a-1} \ell_{a+k} \bar{NC}(a+k,\ t+k)\ \exp[\eta(x-a-k)]$$

$$= s\cdot b\cdot \sum_{k=0}^{x-a-1} \ell_{a+k} \exp[\beta(t+r-a)]\cdot$$

$$ {}_{r-(a+k)|}\ddot{a}_{a+k} \exp[\eta(x-a-k)]$$

$$= s\cdot b\cdot \sum_{k=0}^{x-a-1} \ell_r\ \exp[\beta(t+r-a)]\ \exp[-\eta(r-x)]\ddot{a}_r$$

$$= s\cdot b\cdot \ell_x\ \exp[\beta(t^1+r-x)]\cdot(x-a)\cdot {}_{r-x|}\ddot{a}_x .$$

If one lets x=r, the preceding formula says that benefits are fully funded by the time of retirement. (This also shows that the prospective value of the actuarial liability - or "mathematical reserve" - is equal to the above calculated retrospective value.)

In practice one *defines* the actuarial liability for a member age x as

$$(1.2) \qquad \bar{AL}(x,t) = sb\ \exp[\beta(t+r-x)]\cdot(x-a)\cdot {}_{r-x|}\ddot{a}_x ,$$

since it is quite improbable that the normal costs would have been paid on the same basis in previous years as the one adopted at time t (like with a mathematical reserve in life insurance).

Compare $\bar{AL}(x,t)$ in (1.2) with $\bar{NC}(x,t)$ in (1.1). One notices that the actuarial liability is the amount required to fund the number of units allocated to date (x-a of them) on a single premium basis. That is to say

$$\bar{AL}(x,t)/\bar{NC}(x,t) = x-a, \quad a \le x \le r-1.$$

Moreover, rewriting (1.2) as

$$(1.3) \quad \bar{AL}(x,t) = \underbrace{s \cdot b \cdot \exp[\beta(t+r-x)](r-a)_{r-x|}\ddot{a}_x}_{\text{Present value of future benefits}} \cdot \frac{x-a}{r-a}$$

shows the actuarial liability at each age before retirement to be equal to a pro rata fraction $(x-a)/(r-a)$ of the present value of retirement benefits.

With regard to retired members, the actuarial liability is defined as the present value of benefits, and there is no normal cost.

In practice, the overall contribution is found by adding

(i) the sum of the individual normal costs $\bar{NC}(x,t)$, and

(ii) a payment towards the liquidation of the "unfunded liability".

The unfunded liability is the excess of the actuarial liability over the actual fund, i.e.

$$\bar{UL}(t) = \bar{AL}(t) - \bar{F}(t).$$

The way in which the unfunded liability is dealt with over the years depends on whether it arises from benefit improvements, gains or losses, etc. and on relevant legislation.

It is worth emphasizing that the Unit Credit method (like the following two, the Entry Age Normal and the Individual Level Premium) is an *individual* method. By this is meant that normal costs and actuarial liabilities are estimated separately for each member and then summed up, to yield, finally, an overall cost that is the sum of

(i) and (ii) above. Aggregate-type methods do not rely on individual costs and liabilities directly but solely on aggregate values (see Sections 1.3.4 to 1.3.6).

I will now show that the Unit Credit method "works" in the aggregate, i.e. that if actuarial assumptions are exactly realized and normal costs paid when due, then each member's benefits are funded by the time he reaches retirement age.

Assuming the model population, the derivation of Eq. (1.2) clearly implies that

$$\begin{aligned} &\exp(\eta)(\ell_x \bar{A}\bar{L}(x,t) + \ell_x \bar{N}\bar{C}(x,t)) \\ &= \ell_{x+1}\bar{A}\bar{L}(x+1,t+1), \quad a \le x \le r-1. \end{aligned}$$

A similar formula holds for retirees; if $x \ge r$

$$\begin{aligned} \bar{A}\bar{L}(x,t) &= \text{present value of future benefits} \\ &= \underbrace{\bar{B}(r,t+r-x)}_{\substack{\text{pension}\\\text{accrued at}\\\text{age } r}} \cdot \underbrace{\exp[\zeta(x-r)]}_{\substack{\text{post-retirement}\\\text{increase}}} \cdot \ddot{a}_x^{(\eta-\zeta)} \end{aligned}$$

and then

$$\begin{aligned} &\exp(\eta)\ell_x[\bar{A}\bar{L}(x,t) \\ &\quad -\text{benefits paid to a member age } x \text{ in year } t] \\ = \quad &\exp(\eta)\ell_x\{\bar{A}\bar{L}(x,t)-\bar{B}(r,t+r-x)\exp[\zeta(x-r)]\}) \\ = \quad &\ell_{x+1}\bar{B}[r,(t+1)+r-(x+1)]\exp[\zeta(x+1-r)]\cdot\ddot{a}_{x+1}^{(\eta-\zeta)} \\ = \quad &\ell_{x+1}\bar{A}\bar{L}(x+1,t+1). \end{aligned}$$

Therefore

$$\exp(\eta)(\bar{A}\bar{L}(t) + \bar{N}\bar{C}(t) - \bar{B}(t))$$

$$= \cdot\exp(\eta)\left(\sum_{x=a}^{\omega} \ell_x \bar{A}\bar{L}(x,t) + \sum_{x=a}^{r-1} \ell_x \bar{N}\bar{C}(x,t) - \sum_{x=r}^{\omega} \ell_x \bar{B}(x,t)\right)$$

$$= \sum_{x=a+1}^{\omega} \ell_x \bar{A}\bar{L}(x,t+1)$$

$$= \bar{A}\bar{L}(t+1), \quad \text{since} \quad \bar{A}\bar{L}(a,t+1) = 0.$$

This proves that when all assumptions work out in practice, paying the normal costs when due will provide for all retirement benefits and build up a fund that remains equal to the actuarial liability.

1.3.1.2 Ultimate Values

The model population has a constant rate of new entrants ℓ_a. Benefits, as a fraction of final salary, do not change over time. It is then clear from the description of the Unit Credit method that as soon as the initial unfunded liability has been amortized and the population become stationary (after at most $\omega-a$ years), the following formulae hold:

$$C(t) = \bar{C}(t)\cdot\exp(-\beta t)$$

$$= sb \sum_{x=a}^{r-1} \ell_x \exp[\beta(r-x)]\cdot {}_{r-x|}\ddot{a}_x$$

$$= sb\ \ell_r \ddot{a}_r \cdot a^{(\gamma)}_{\overline{r-a}|} ;$$

$$F(t) = \bar{F}(t) \exp(-\beta t)$$

$$= \bar{A}\bar{L}(t) \exp(-\beta t)$$

$$= sb\left\{ \sum_{x=a}^{r-1} \ell_x \exp[\beta(r-x)](x-a)\cdot {}_{r-x|}\ddot{a}_x \right.$$

$$\left. + \sum_{x=r}^{\omega} \ell_x \exp[(\beta-\zeta)(r-x)](r-a)\ddot{a}_x^{(\eta-\zeta)} \right\}.$$

1.3.2 Entry Age Normal

1.3.2.1 Description

This method, as its title implies, visualizes the normal cost for any given employee as the level payment (or level percentage of pay) necessary to fund the benefit over the working lifetime of such employee (Trowbridge (1952), p.23).

For a member age a at time t,

Present value of retirement benefits

$$= \underbrace{\underbrace{s\cdot\exp(\beta t)}_{\text{current earnings}} \cdot \underbrace{\exp[\beta(r-a)]}_{\text{increasing earnings to age } r}}_{\text{projected retirement benefit}} \cdot b(r-a)\cdot {}_{r-a|}\ddot{a}_a$$

and

Present value of future earnings

$$= s\cdot\exp(\beta t)\sum_{x=a}^{r-1} \underbrace{\exp[-\eta(x-a)]}_{\text{interest discount}} \cdot \underbrace{\exp[\beta(x-a)]}_{\text{increase in earnings}} \cdot \ell_x/\ell_a$$

$$= s\cdot\exp(\beta t)\cdot\ddot{a}^{(\gamma)}_{a:\overline{r-a}|},$$

so that the level fraction of earnings required to fund the benefits is

$$b(r-a)\cdot\exp[\beta(r-a)]\cdot{}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}.$$

Thus the normal cost for year $(t,t+1)$ is

$$\bar{N}\bar{C}(t) = \sum_{x=a}^{r-1} \ell_x\cdot sb(r-a)\exp[\beta(t+r-a)]\cdot{}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}.$$

Valued prospectively, the actuarial liability is

AL = Present value of future benefits w.r. to active and retired members

-Present value of future normal costs w.r. to active members.

In respect of retired members, the actuarial liability is the present value of the remaining pension payments; for an active member age x at time t, it is equal to

$$sb(r-a)\exp[\beta(t+r-x)]\cdot{}_{r-x|}\ddot{a}_x$$

$$-\ sb(r-a)\exp[\beta(t+r-a)]\cdot({}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|})\cdot\ddot{a}^{(\gamma)}_{x:\overline{r-x}|}\ .$$

This expression can be simplified, (dropping the factor $sb\exp(\beta t)(r-a)$ for the time being):

$$\exp[\beta(r-x)]{}_{r-x|}\ddot{a}_x - \exp[\beta(r-a)]({}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{x:\overline{r-a}|})\ddot{a}^{(\gamma)}_{x:\overline{r-x}|}$$

$$= \exp[\beta(r-x)]{}_{r-x|}\ddot{a}_x$$

$$\times\{1-\frac{[\exp[-\gamma(r-a)](\ell_r/\ell_a)\ddot{a}_r\ddot{a}^{(\gamma)}_{x:\overline{r-x}|}]}{[\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}\cdot\exp[-\gamma(r-x)](\ell_r/\ell_x)\ddot{a}_r]}\}$$

$$= \exp[\beta(r-x)]({}_{r-x|}\ddot{a}_x/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|})$$

$$\times[\ddot{a}^{(\gamma)}_{a:\overline{r-a}|} - \exp[-\gamma(x-a)](\ell_x/\ell_a)\ddot{a}^{(\gamma)}_{x:\overline{r-x}|}]$$

$$(1.4)\qquad = \exp[\beta(r-x)]{}_{r-x|}\ddot{a}_x\cdot\ddot{a}^{(\gamma)}_{a:\overline{x-a}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}\ .$$

Thus we define

$$\bar{A}\bar{L}(x,t) = sb(r-a)\exp[\beta(t+r-x)]{}_{r-x|}\ddot{a}_x\cdot\ddot{a}^{(\gamma)}_{a:\overline{x-a}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

if $x \le r-1$. It can be seen that

$$\bar{A}\bar{L}(x,t)/\bar{N}\bar{C}(x,t) = \ddot{s}^{(\gamma)}_{a:\overline{x-a}|}\ ,$$

and that

$$\bar{A}\bar{L}(x,t) = (\text{Present value of future benefits}) \times (\ddot{a}^{(\gamma)}_{a:\overline{x-a}|} / \ddot{a}^{(\gamma)}_{a:\overline{r-a}|}),$$

a formula similar to Eq. (1.3).

So, under both the Unit Credit and the Entry Age Normal methods, $AL(x,t)$ increases from 0 at age a to $B(r,t)\cdot\ddot{a}_r$ at age r.

It is also true, as in the case of the Unit Credit method, that when all assumptions are exactly realized

$$\exp(\eta)(\bar{A}\bar{L}(t) + \bar{N}\bar{C}(t) - \bar{B}(t)) = \bar{A}\bar{L}(t+1).$$

1.3.2.2 Ultimate Values

After the initial unfunded liability has been paid off *and* the population become stationary

$$F(t) = \exp(-\beta t)\bar{F}(t)$$

$$= \sum_{x=a}^{r-1} sb\ \ell_x \exp[\beta(r-x)](r-a)_{r-x|}\ddot{a}_x \ddot{a}^{(\gamma)}_{a:\overline{x-a}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

$$+ \sum_{x=r}^{\omega} sb\ \ell_x(r-a)\ \exp[(\zeta-\beta)(x-r)]\ddot{a}_x^{(\eta-\zeta)},$$

and

$$C(t) = \exp(-\beta t)\bar{N}\bar{C}(t)$$

$$= \sum_{x=a}^{r-1} s\ell_x\ \exp[\beta(r-a)]\ b(r-a)_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

$$= \sum_{x=a}^{r-1} s\ell_x\ \exp[-\gamma(r-a)]b(r-a)(\ell_r/\ell_a)\ddot{a}_r/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}.$$

$$(1.5) \qquad = sb(r-a)\ell_r\ddot{a}_r\exp[-\gamma(r-a)]\ddot{a}^{(0)}_{a:\overline{r-a}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

1.3.3 Individual Level Premium

1.3.3.1 Description

This method is closely related to the previous one. It "funds the benefits as to any individual from date of entry (or date plan is established, if later) to retirement date as a level amount (or as a level percentage of pay)" (Trowbridge (1952), p.24).

Let the scheme be set up at time 0 and consider a member age x who has, under the terms of the arrangement, x-a years of credited service. When using the Entry Age Normal methods, one computes

$$^{EAN}\overline{NC}(x,0) = sb(r-a)\exp[\beta(r-a)]_{r-a|}\ddot{a}_a / \ddot{a}^{(\gamma)}_{a:\overline{r-a}|} ;$$

and the "initial actuarial liability"

$$^{EAN}\overline{AL}(x,0) = sb(r-a)\exp[\beta(r-a)]_{r-x|}\ddot{a}_x \cdot \ddot{a}^{(\gamma)}_{a:\overline{x-a}|} / \ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

these $^{EAN}\overline{AL}(x,0)$ are summed up for all members with credited past service, and the outcome $^{EAN}\overline{AL}(0)$ (= initial unfunded liability) is amortized over a fixed number of years.

The rationale of the Individual Level Premium method is that in order to ensure that the fund will not go negative during the first few years (a possibility in the case of small schemes), each member's benefits are funded over his remaining years in service. Consequently, if normal costs are a level fraction of earnings,

$$^{ILP}\overline{NC}(y,t) = sb(r-a)\exp[\beta(t+r-x)]_{r-x|}\ddot{a}_x / \ddot{a}^{(\gamma)}_{x:\overline{r-x}|}$$

and

$$^{ILP}\overline{AL}(y,t) = sb(r-a)\exp[\beta(t+r-y)]_{r-y|}\ddot{a}_y \cdot \ddot{a}^{(\gamma)}_{x:\overline{y-x}|} / \ddot{a}^{(\gamma)}_{x:\overline{r-x}|}$$

$$t = y-x. \quad y \leq r-1.$$

The normal costs fully fund the member's benefits as he reaches age r, and no identifiable initial unfunded liability arises.

The Individual Level Premium method also differs from the previous two in the way gains and losses due to salary increases are dealt with.

To keep things simple, say a member j enters at age a at time t=0, with a salary $\bar{S}(j,0)$ and no credited service. Valuations are performed at times 0 and 1 on the same basis. Under both the Individual Level Premium and the Entry Age Normal methods, the normal cost at time 0 is

$$\bar{NC}(j,0) = \bar{S}(j,0)b(r-a)\exp[\beta(r-a)]\cdot {}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|} .$$

At time 1 the two methods differ:

<u>EAN</u>: ${}^{EAN}\bar{NC}(j,1)$ and ${}^{EAN}\bar{AL}(j,1)$ are estimated using current data, i.e.

$${}^{EAN}\bar{NC}(j,1) = \bar{S}(j,1)b(r-a)\exp[\beta(r-a)]\cdot {}_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$

and

$${}^{EAN}\bar{AL}(j,1) = \bar{S}(j,1)b(r-a)\exp[\beta(r-a-1)]\cdot {}_{r-a-1|}\ddot{a}_{a+1} \times \ddot{a}^{(\gamma)}_{a:\overline{1}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|} .$$

In practice $\bar{S}(j,1) \neq \exp(\beta)\bar{S}(j,0)$, with the result that a gain or loss arises from the unexpected increase in salary $\Delta\bar{S} = \bar{S}(j,1)-\exp(\beta)\bar{S}(j,0)$. These individual gains or losses are subsumed in the overall actuarial gain or loss, which is amortized in the fashion thought appropriate.

<u>ILP</u>: The difference here is that the method still requires the present value of normal costs (past and future) to equal the present value of benefits. Defining the normal cost as ${}^{EAN}\bar{NC}(j,1)$ (above) will not permit this since the projected benefit has changed. Instead,

$${}^{ILP}\bar{NC}(j,1) = \bar{S}(j,0)\exp(\beta)b(r-a)\exp[\beta(r-a)]_{r-a|}\ddot{a}_a/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}$$
$$+\Delta\bar{S}b(r-a)\exp[\beta(r-a-1)]_{r-a-1|}\ddot{a}_{a+1}/\ddot{a}^{(\gamma)}_{a+1:\overline{r-a-1}|}$$

The first term is what the normal cost would have been had all assumptions worked out, and the second one spreads the unexpected change in projected benefits over j's remaining years in service.

It follows that

$$
\begin{aligned}
{}^{ILP}\bar{\bar{AL}}(j,1) &= \text{Present value of updated projected benefits} \\
&\quad - \text{Present value of future updated normal costs} \\
&= \bar{S}(j,\emptyset)\exp(\beta)\exp[\beta(r-a-1)]_{r-a-1|}\ddot{a}_{a+1} \\
&\quad \times\ \ddot{a}^{(\gamma)}_{a:\overline{1}|}/\ddot{a}^{(\gamma)}_{a:\overline{r-a}|}
\end{aligned}
$$

just as though no unexpected change in salary had occurred. (This can be verified in the same way Eq. (1.4), Section 1.3.2.1, was derived.)

Updating normal costs in this fashion (each year) implies

$$
{}^{ILP}\bar{\bar{AL}}(j,t+1) = ({}^{ILP}\bar{\bar{AL}}(j,t) + {}^{ILP}\bar{\bar{NC}}(j,t))\ \exp(\eta)\ell_x/\ell_{x+1}
$$

whether or not the salary increase assumption is realized.

Finally (Anderson (1985), p.24):

> Thus we see that the individual level-premium method resembles entry-age-normal with entry age defined as the age of hire or age at the effective date (whichever is greater). The difference is that under ILP we take a normally large component of the actuarial gain - a component which is normally negative - out of the accrued liability and spread it into the future normal costs. Under entry-age-normal, this portion of the gain was simply amortized in the manner of other gains - i.e. over a period which may or may not have been longer than the future working lifetime of a particular individual.

1.3.3.2 Ultimate values

It is clear that they are the same as with the Entry Age Normal method, since after r-a years and in the absence of deviations from actuarial assumptions the two methods are identical.

1.3.4 Aggregate

> The principle behind the aggregate method is that of equating present value of unfunded future benefits to present value of future contributions, where the contribution per active life (or per dollar of salary) per year is assumed constant. It may seem at first thought that the resulting contributions should remain level from year to year for an initially stable population, since the very principle implies spreading the value of total benefits levelly over future life years.
>
> This supposition regarding the aggregate method is absolutely correct *provided* future new entrants are taken into account, both in valuing present value of future benefits and in calculating present value of future active life years. (Trowbridge (1952), p.26).

It will be shown in 1.4.3.2 that this last approach leads to partial funding only (or to no funding at all if the initial fund is nil).

In practice, though, the Aggregate method ignores future new entrants. The portion of covered payroll c(t) representing the cost of funding the benefits for the coming year is

$$c(t) = \frac{\text{(Present value of future benefits of all members)} - \text{(Fund at time (t))}}{\text{(Present value of future earnings of current members)}}$$

For the model population

$$c(t) = \frac{\sum_{x=a}^{r-1} sb(r-a)\exp[\beta(t+r-x)]_{r-x|}\ddot{a}_x + \sum_{x=r}^{\omega} \ell_x \bar{B}(x,t)\ddot{a}_x^{(\eta-\zeta)} - \bar{F}(t))}{\sum_{x=a}^{r-1} s\ell_x \exp(\beta t)\ddot{a}^{(\gamma)}_{x:\overline{r-x|}}}$$

and the overall cost for year $(t,t+1)$ is

$$\bar{C}(t) = c(t)\bar{S}(t) = c(t)\sum_{x=a}^{r-1} s\ell_x \exp(\beta t)$$

Derivation of ultimate fund values and contribution rates is deferred till Section 1.4, for all methods of the aggregate type.

1.3.5 Attained Age Normal

> Total benefits are divided into past service and future service benefits exactly as under unit credit funding, and [...] there is complete freedom as to the manner in which the past service liability shall be paid off. For future service benefits, however, the aggregate method is adopted. (Trowbridge (1952), p.28)

The overall contribution for year $(t,t+1)$ is

$$c(t)\cdot\bar{S}(t) + \bar{P}(t)$$

where

$$c(t) = \frac{\text{(Present value of future benefits of all members)} - \text{(Unamortized part of past service liability} + \text{Fund at time } t)}{\text{(Present value of future earnings of current members)}}$$

$\bar{P}(t)$ = Payment towards amortizing the initial (past service) unfunded liability.

After the unfunded liability has been completely paid off, the method is applied exactly like the Aggregate.

Remark 1.2. In practice, a new layer of unfunded liability is created every time

(i) a significant actuarial assumption is modified
(ii) benefits are updated.

For this reason it is very likely that the payments $\bar{P}(t)$ will be present for a period much longer than the initial amortization period.

1.3.6 Frozen Initial Liability

This method is applied in the same way as the Attained Age Normal, except that the initial unfunded liability is computed using the Entry Age Normal method; Remark 1.2 holds verbatim.

1.3.7 British Methods

According to Colbran (1982), the methods used most in the United Kingdom are the Aggregate, Attained Age Normal and Discontinuance Target. Only the last one of these (which is unknown in North America) has not been described so far. Turner et al. (1984) make this method a variation of the Unit Credit method, in which a control period (5, 10 or 20 years usually) is employed.

> Where a control period is used, the Standard Contribution Rate [normal cost rate], is found by dividing the present value of all benefits which will accrue in the control period (rather than simply in the year following the valuation date) by the present value of member's earnings in the control period. Thus the notional Standard Fund [actuarial liability] at the end of the control period is the present value of all benefits accrued at that date [...].
>
> In order to assume a stable age structure, in conjunction with the use of a control period an assumption is usually made that new entrants will replace current members as they leave service, die or retire. (Turner et al. (1984), p.19)

A control period can also be used to modify the Aggregate method. (The effect of bringing new entrants into the valuation basis, under the Aggregate method, is examined in Sections 1.4.3 and 1.4.4.)

1.3.8 Other Methods

The following three "funding methods" are unimportant in the context of funded schemes providing defined benefits; they are presented solely for the sake of completeness.

1.3.8.1 Initial Funding

Each new entrant's benefits are paid up in full at time of entry. The model population gives

$$\bar{C}(t) = \text{Cost at time } t$$

$$= \text{Present value of benefits of the } \ell_a \text{ members entering at time } t$$

$$= \ell_a \cdot sb(r-a)\exp[\beta(t+r-a)]_{r-a|}\ddot{a}_a$$

$$= \ell_r sb(r-a)\exp[\beta t-\gamma(r-a)]\ddot{a}_r.$$

One could define

$$\bar{AL}(x,t) = \text{Present value of benefits of a member age } x \text{ at time } t$$

$$= sb(r-a)\exp[\beta(t+r-x)]_{r-x|}\ddot{a}_x$$

$$= (\ell_r/\ell_x)sb(r-a)\exp(\beta t)\exp[-\gamma(r-x)]\ddot{a}_r$$

$$= \exp[\eta(x-a)]\bar{C}[t-(x-a)]/\ell_x$$

i.e. the fund on hand with respect to members age x ($\leq r$) is merely the contribution made x-a years ago, increased with interest.

1.3.8.2 Terminal Funding

Benefits are funded in their totality when members retire. The model population implies

$$\bar{C}(t) = \ell_r sb(r-a)\exp(\beta t)\ddot{a}_r,$$

and, if one insists on defining an actuarial liability,

$$\bar{AL}(x,t) = 0, \quad \forall x < r,$$

$$\bar{AL}(x,t) = sb(r-a)\exp[\beta t+(\varsigma-\beta)(x-r)]\ddot{a}_x^{(\eta-\varsigma)}, \quad x \geq r.$$

1.3.8.3 Pay-as-you-go

There is no funding: benefits are simply paid when due; with the model population

$$\bar{B}(t) = \sum_{x=r}^{\omega} \ell_x \bar{B}(x,t)$$

$$= \sum_{x=r}^{\omega} \ell_x \, sb(r-a)\exp[\beta(t+r-x)+\varsigma(x-r)]$$

$$= sb(r-a)\exp(\beta t)\ell_r \ddot{a}_r^{(\beta-\varsigma)}.$$

1.4 AGGREGATE METHODS

1.4.1 General Remarks

The rest of Chapter 1 is carried out in continuous time, taking the model population and scheme for granted. Furthermore, the analysis will be performed using "real-term" monetary values (see Section 0.6).

The Aggregate, Attained Age Normal, Frozen Initial Liability and Aggregate with New Entrants methods (the last one is described in 1.4.3 below) all operate in the following fashion:

(1.6) $\bar{F}'(t) = \eta\bar{F}(t) + c(t)\bar{S}(t) + \bar{P}(t) - \bar{B}(t)$

(1.7) $c(t) = [\bar{PVB}(t)-(\bar{F}(t)+\bar{U}(t))]/\bar{PVS}(t)$

where

$\bar{F}(t)$ = fund at time t;

$c(t)$ = contribution rate, as a fraction of payroll;

$\bar{PVB}(t)$, $\bar{PVS}(t)$ are positive functions, increasing at rate β, i.e. $\bar{PVB}(t) = \exp(\beta t)\cdot PVB$, $\bar{PVS}(t) = \exp(\beta t)\cdot PVS$;

$\bar{S}(t)$ = payroll;

$\bar{U}(t)$ = Unamortized part of the initial past service liability;

$\bar{P}(t)$ = rate at which $\bar{U}(t)$ is being paid off;

$\bar{B}(t)$ = rate at which benefits are paid = $B\cdot\exp(\beta t)$.

I will now derive the differential equation satisfied by the "de-inflated" fund $F(t) = \exp(-\beta t)\bar{F}(t)$. First substitute (1.7) into (1.6)

$$\begin{aligned}\bar{F}'(t) &= \eta\bar{F}(t) + \bar{S}(t)\bar{P}\bar{V}\bar{B}(t)/\bar{P}\bar{V}\bar{S}(t) - \bar{S}(t)\bar{F}(t)/\bar{P}\bar{V}\bar{S}(t)\\ &\quad -\bar{S}(t)\bar{U}(t)/\bar{P}\bar{V}\bar{S}(t) + \bar{P}(t) - \bar{B}(t)\\ &= (\eta-S/PVS)\bar{F}(t) + \exp(\beta t)\cdot S\cdot PVB/PVS\\ &\quad - \exp(\beta t)B + (\bar{P}(t)-S\cdot\bar{U}(t)/PVS).\end{aligned}$$

Then

$$\begin{aligned}F'(t) &= \exp(-\beta t)\bar{F}'(t) - \beta F(t)\\ &= (\gamma-S/PVS)F(t) + (S\cdot PVB/PVS - B)\\ &\quad + \exp(-\beta t)(\bar{P}(t)-S\cdot\bar{U}(t)/PVS).\end{aligned}$$

With the notation

$$\alpha = S/PVS - \gamma,$$

$$R = (S\cdot PVB/PVS - B),$$

and

$$h(t) = \exp(-\beta t)(S\cdot\bar{U}(t)/PVS - \bar{P}(t)),$$

we finally get

$$F'(t) = -\alpha F(t) + R - h(t), \tag{1.8}$$

which means that

$$F(t) = F(0)\exp(-\alpha t) + R\int_0^t \exp[-\alpha(t-s)]ds - \int_0^t \exp[-\alpha(t-s)]h(s)ds.$$

If

(i) $\alpha > 0$, and

(ii) $\lim_{t\to\infty}\int_0^t \exp[-\alpha(t-s)]h(s)ds$ exists and is denoted by H,

then $F(\infty)$ exists and is equal to

$$\lim_{t\to\infty} [F(0)\exp(-\alpha t)+R(1-\exp(-\alpha t))/\alpha -\int_0^t \exp[-\alpha(t-s)]h(s)ds]$$

$$= R/\alpha - H.$$

Assume $\alpha > 0$. Obviously $H = 0$ if $U(0)$ is completely paid off in a finite time, and then

$$(1.9) \quad F(\infty) = R/\alpha = (S\cdot PVB/PVS-B)/(S/PVS-\gamma).$$

Furthermore,

$$c(\infty) = \lim_{t\to\infty}(PVB-F(t)-\exp(-\beta t)\bar{U}(t))/PVS$$

$$= (PVB-F(\infty))/PVS$$

$$= [PVB(S/PVS-\gamma)-(S\cdot PVB/PVS-B)]/(S-\gamma PVS)$$

$$(1.10) \quad = (B-\gamma PVB)/(S-\gamma PVS).$$

1.4.2 Aggregate, Attained Age Normal and Frozen Initial Liability

Under these three methods

PVB = Present value of benefits of all current members at time 0

$$(1.11) = \int_a^r sb\ell_x \exp[\beta(r-x)](r-a)_{r-x|}\bar{a}_x dx$$

$$+ \int_r^\omega sb\,\ell_x \exp[(\varsigma-\beta)(x-r)](r-a)\bar{a}_x^{(\eta-\varsigma)} dx \;;$$

PVS = Present value of future earnings of active members at time 0

$$(1.12) = \int_a^r s\ell_x\, \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\, dx.$$

I first show that $\alpha = S/PVS-\gamma > 0$, which implies that the three methods lead to the same ultimate contribution rate and fund level (from Eq. (1.10)). Clearly $\alpha > 0$ if $\gamma \le 0$. If $\gamma > 0$, then

$$
\begin{aligned}
PVS &= \text{present value of future earnings of current members} \\
&< \text{present value of future earnings of all current } \textit{and} \text{ future members} \\
&= \int_0^\infty \exp(-\eta t)\bar{S}(t)\,dt \\
&= S/\gamma\ ;
\end{aligned}
$$

consequently α is still strictly greater than 0.

Next, the ultimate contribution under the three methods mentioned above will be shown to be identical to the ultimate contribution under the Entry Age Normal method. Recall that the latter is

$$(1.13)\qquad {}^{EAN}C(\infty) = sb(r-a)\ell_r\bar{a}_r\exp[-\gamma(r-a)]\bar{a}^{(0)}_{a:\overline{r-a|}}\ /\bar{a}^{(\gamma)}_{a:\overline{r-a|}}$$

(this is the continuous-time counterpart of Eq. (1.5), Section 1.3.2.2). I will now prove that the expression in Eq. (1.10) boils down to (1.13), when Eqs. (1.11) and 1.12) are substitited into it.

$$
\begin{aligned}
(1)\quad & \int_a^r \ell_x\exp[\beta(r-x)]{}_{r-x|}\bar{a}_x dx \\
&= \int_a^r \ell_r\bar{a}_r^{(\eta-\zeta)}\exp[-\gamma(r-x)]dx \\
&= \ell_r\bar{a}_r^{(\eta-\zeta)}\bar{a}^{(\gamma)}_{\overline{r-a|}}
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & \int_r^\omega \ell_x\exp[(\zeta-\beta)(x-r)]\bar{a}_x^{(\eta-\zeta)}dx \\
&= \int_r^\omega \ell_x\exp[(\zeta-\beta)(x-r)]\int_x^\omega \exp[(\zeta-\eta)(u-x)(\ell_u/\ell_x)]\,dudx \\
&= \int_r^\omega\int_r^u \ell_u\exp[(\zeta-\beta)(x-r)+(\zeta-\eta)(u-x)]\,dxdu
\end{aligned}
$$

(changing the order of integration)

$$= \int_r^\omega \ell_u\ \exp[(-(\zeta-\beta)r+(\zeta-\eta)u]\ \int_r^u \exp[(\eta-\beta)x]dxdu$$

$$= \int_r^\omega \{\ell_u\{\exp[(\zeta-\beta)(u-r)]-\exp[(\zeta-\eta)(u-r)]\}/\gamma\}du$$

$$= \ell_r(\bar{a}_r^{(\beta-\zeta)}-\bar{a}_r^{(\eta-\zeta)})/\gamma .$$

(3) $$B = sb(r-a)\int_r^\omega \ell_x \exp[(\zeta-\beta)(x-r)]\,dx$$

$$= sb(r-a)\ell_r\bar{a}_r^{(\beta-\zeta)}$$

(4) From (1), (2) and (3) (see Eq. (1.11)),

$$B-\gamma PVB = sb(r-a)\cdot\{\ell_r\bar{a}_r^{(\beta-\zeta)}$$

$$-[\ell_r\bar{a}_r^{(\eta-\zeta)}(1-\exp[-\gamma(r-a)])+\ell_r\bar{a}_r^{(\beta-\zeta)}-\ell_r\bar{a}_r^{(\eta-\zeta)}]\}$$

(1.14) $$= sb(r-a)\exp[-\gamma(r-a)]\ell_r\bar{a}_r^{(\eta-\zeta)}$$

(5) $$\int_a^r \ell_x\ \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\,dx$$

$$= \int_a^r \ell_x \int_x^r \exp[-\gamma(u-x)]\ell_u/\ell_x\,du dx$$

$$= \int_a^r \ell_u \exp(-\gamma u) \int_a^u \exp(\gamma x)\,dx du$$

(changing the order of integration)

(1.15) $$= \ell_a(\bar{a}^{(\emptyset)}_{a:\overline{r-a}|} - \bar{a}^{(\gamma)}_{a:\overline{r-a}|})/\gamma .$$

(6) $$S = s\int_a^r \ell_x\,dx$$

$$= s\ell_a\cdot\bar{a}^{(\emptyset)}_{a:\overline{r-a}|} .$$

(7) From (5) and (6),

$$S-\gamma PVS = s\ell_a\cdot\bar{a}^{(\gamma)}_{a:\overline{r-a}|} .$$

(8) Finally, (4), (6) and (7) imply

$$AGG_{C(\infty)} = sb(r-a)\ell_r\bar{a}_r^{(\eta-\zeta)}\exp[-\gamma(r-a)]\bar{a}^{(\emptyset)}_{a:\overline{r-a}|}/\bar{a}^{(\gamma)}_{a:\overline{r-a}|}$$

$$= EAN_{NC} .$$

Appendix 1.1 shows that the same result holds when the population is just asymptotically stationary. The conclusions are enunciated as

<u>Proposition 1.1</u>. Assume

(i) there is only one entry age;

(ii) the population is asymptotically stationary; and

(iii) the initial unfunded liability, if any, is liquidated within a finite time.

Then

$$AGG_{C(\infty)} = AAN_{C(\infty)} = FIL_{C(\infty)} = EAN_{NC},$$

and

$$AGG_{F(\infty)} = AAN_{F(\infty)} = FIL_{F(\infty)} = EAN_{AL} .$$

<u>Remark 1.3</u>. Prop. 1.1 is essentially what is proved in Demonstration II of Trowbridge (1952), pp. 41-43, except that Trowbridge works in discrete time, and does not explicitly refer to the case where the rate interest is smaller than or equal to 0. Moreover, the case of an initially immature population is only illustrated with a numerical example, no proof of convergence being provided.

<u>Remark 1.4</u>. Steps (1) to (8) show the identity of ultimate contributions whatever γ may be (smaller than, equal to, or greater than 0). When $\gamma \neq 0$, the equation of equilibrium

$$0 = \gamma F(\infty) + C(\infty) - B \tag{1.16}$$

proves that the same identity holds for fund levels. However, when $\gamma = 0$ Eq. (1.16) only says that $C(\infty) = B$, and does *not* imply the identity of fund levels. This minor inconvenience is easily overcome using the following continuity argument.

On the one hand the limiting fund under the aggregate methods is (Section 1.4.1)

$$AGG_{F(\infty)} = (S \cdot PVB/PVS - B)/(S/PVS - \gamma). \tag{1.9}$$

Because of the definitions of B, S, PVB and PVS, this is a continuous function of γ. On the other hand,

$$EAN_{F(\infty)} = EAN_{AL}$$

which is again a continuous function of γ. Consequently

$$AGG_{F(\infty)} = EAN_{F(\infty)}$$

has to hold when $\gamma = 0$, since two continuous functions, identical everywhere except at one point, have to be equal at that point as well.

Remark 1.5. Eq. (1.10) (Section 1.4.1) for the ultimate contribution has an interesting interpretation when $\gamma > 0$. Rewrite it as

(1.17) $\qquad C(\infty) = S\cdot(B/\gamma-PVB)/(S/\gamma-PVS)$.

(i) $\qquad B/\gamma$ = Present value of all benefits to be paid out of the fund

(ii) $\qquad S/\gamma$ = Present value of all salaries to be earned by current and future members.

Hence the ultimate contribution is a fraction of payroll equal to the ratio of

(i) the present value of benefits of all future members, excluding current ones, to

(ii) the present value of all future members' earnings, again excluding current ones.

1.4.3 Aggregate with New Entrants

Imagine a modification of the Aggregate method, under which every valuation includes new entrants coming into the scheme at rate ℓ_a per year, over the next n years.

More precisely, define

$\bar{P}\bar{V}\bar{B}(t,n)$ = Present value of benefits of (a) current members *and* (b) new entrants coming into the scheme over the next n years, at time t;

$\bar{P}\bar{V}\bar{S}(t,n)$ = Present value of earnings of (a) currently active members, *and* (b) new entrants coming into the scheme over the next n years, at time t.

Then define the overall contribution at time t as a fraction of covered payroll equal to

$$c(n,t) = (\bar{P}\bar{V}\bar{B}(t,n)-\bar{F}(t))/\bar{P}\bar{V}\bar{S}(t,n).$$

Clearly, the model population produces values of $\bar{P}\bar{V}\bar{B}(t,n)$ and $\bar{P}\bar{V}\bar{S}(t,n)$ that increase at rate β, i.e.

$$\bar{P}\bar{V}\bar{B}(t,n) \quad = \quad \exp(\beta t)PVB(n),$$

and

$$\bar{P}\bar{V}\bar{S}(t,n) \quad = \quad \exp(\beta t)PVS(n),$$

where

$$(1.18) \qquad PVB(n) = \int_a^r s\ell_x \exp[\beta(r-x)]b(r-a)_{r-x|}\bar{a}_x dx + \int_r^{\omega} \ell_x B(x)\bar{a}_x^{(\eta-\zeta)}dx + \int_0^n sb\ \exp(-\gamma s)\ell_a \exp[\beta(r-a)](r-a)_{r-a|}\bar{a}_a ds$$

and

$$(1.19) \qquad PVS(n) \quad = \quad \int_a^r s\ell_x \bar{a}^{(\gamma)}_{x:\overline{r-x|}}\, dx + \int_0^n s\exp(-\gamma s)\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a|}}\ \ ds.$$

<u>Remark 1.6</u>. The third integral in (1.18) represents the present value of benefits of new entrants coming into the scheme at rate ℓ_a per year, over the next n years; the second integral in (1.19) is the present value of their earnings.

1.4.3.1 $\quad n < \infty$

Eq. (1.10) represents $c(n,\infty)$ (with PVB(n) and PVS(n) replacing PVB and PVS, respectively) *if*

$$\alpha(n) \quad = \quad S/PVS(n) - \gamma \quad > \quad 0.$$

This is the case in general, for (refer to steps (5) to (8) of Section 1.4.2)

$$(1.20) \quad S-\gamma PVS(n) = s\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a|}} - \gamma\bar{a}^{(\gamma)}_{\overline{n|}} \cdot s\ell_a \cdot \bar{a}^{(\gamma)}_{a:\overline{r-a|}}$$

$$= \exp(-\gamma n)(S-\gamma PVS(0)) \quad > \quad 0.$$

Therefore the ultimate contribution is

$$C(n,\infty) \quad = \quad S(B-\gamma PVB(n))/(S-\gamma PVS(n)).$$

The numerator of this expression can be simplified (refer to steps (1) to (4) of Section 1.4.2):

$$B-\gamma PVB(n) = B-\gamma PVB - \gamma\cdot\bar{a}^{(\gamma)}_{\overline{n}|}\ sb(r-a)\ell_r \exp[-\gamma(r-a)]\bar{a}_r$$
$$= \exp(-\gamma n)(B-\gamma PVB(0)).$$

Finally,

$$(1.21)\qquad C(n,\infty) = S(B-\gamma PVB(n))/(S-\gamma PVS(n))$$
$$= S(B-\gamma PVB(0))/(S-\gamma PVS(0))$$
$$= EAN_{NC}.$$

<u>Proposition 1.2</u>. Under the hypotheses of Prop. 1.1,

$$C(n,\infty) = AGG_{C(\infty)} = EAN_{NC};$$
$$F(n,\infty) = AGG_{F(\infty)} = EAN_{AL}.$$

<u>Remark 1.7</u>. It is not very difficult to see that Proposition 1.2 also holds when new entrants come into the scheme at a (varying) rate $n(s)$ s years into the future, ovided that salaries are fully projected to retirement and that $\alpha(n) > 0$. Here $PVB(n)$ and $PVS(n)$ are defined as before, except that the third integral in Eq. (1.18) becomes

$$\int_0^\infty \exp(-\gamma s)n(s)\ell_a s\exp[\beta(r-a)]b(r-a)_{r-a|}\bar{a}_a\ ds$$

and the second one in Eq. (1.19)

$$\int_0^\infty s\ \exp(-\gamma s)n(s)\ \ell_a\ \bar{a}^{(\gamma)}_{a:\overline{r-a}|}\,ds.$$

It follows easily that

$$B-\gamma PVB(n) = (B-\gamma PVB(0))(1-\ \gamma\int_0^\infty n(s)\exp(-\gamma s)ds)$$

and

$$S-\gamma PVS(n) = (S-\gamma PVS(0))(1-\gamma\int_0^\infty n(s)\exp(-\gamma s)ds)$$

which imply that $C(n,\infty) = EAN_{NC}$ as before.

Remark 1.8. Remark 1.5 (Section 1.4.2) still applies: the ultimate contribution $C(n,\infty)$ is a fraction of total payroll equal to the ratio of

(i) the present value of all future benefits, excluding members present in the valuation, to

(ii) the present value of future salaries, excluding members present in the valuation.

1.4.3.2 $n = \infty$ and $\gamma > 0$

All future new entrants are taken into account, and thus

$$\begin{aligned} \alpha(\infty) &= S/PVS(\infty) - \gamma \\ &= S/(S/\gamma) - \gamma \\ &= 0 \\ R(\infty) &= S \cdot PVB(\infty)/PVS(\infty) - B \\ &= 0, \end{aligned}$$

and Eq. (1.8) of Section 1.4.1 becomes

$$(1.22) \qquad F'(t) = 0, \quad t \geq 0,$$

implying that $F(t) \equiv F(0)$, i.e. $\bar{F}(t) = \exp(\beta t)F(0)$. In words including all future new entrants in the valuation amounts to no funding at all (in real terms). In particular, if $F(0) = 0$, the method is equivalent to Pay-as-you-go.

Remark 1.9. Eq. (1.22) took for granted that no initial unfunded liability was present at $t=0$; but the result would be the same had there been one; after its amortization, the fund would stop growing in real terms.

1.4.4 The Parameter α

Recall that

$$(1.8) \qquad F'(t) = -\alpha F(t) + R - h(t).$$

Assume $\alpha > 0$. After the initial unfunded liability has been taken care of,

$$F'(t) = \alpha(R/\alpha - F(t))$$
$$= \alpha(F(\infty) - F(t));$$

in consequence, α entirely determines the rate at which any difference $F(\infty)-F(t)$ is reduced over time. This is the justification for the remainder of Section 1.4: an analysis of the dependence of α upon the real rate of return γ, on the survival function ℓ_x, and on the inclusion of new entrants in the valuation basis.

1.4.4.1 No New Entrants

1.4.4.1.1 Effect of γ

Proposition 1.3. α is a decreasing (resp. strictly decreasing) function of γ if $g(x) = \dot{e}_{x:\overline{r-x}|}$ is a decreasing (resp. strictly decreasing) function of x.

One may intuitively believe that $\dot{e}_{x:\overline{r-x}|}$ always is a decreasing function of x, but this is not true; in particular, the inclusion of withdrawals and of a promotional salary scale can make $g(x)$ increase over part of the interval $[a,r]$.

The proof of Prop. 1.3 is deferred till after Prop. 1.4 and Lemma 1.1.

Proposition 1.4. It is sufficient in order for $g(x) = \dot{e}_{x:\overline{r-x}|}$ to be strictly decreasing that either

(i) $\mu_x \leq 0 \quad \forall x \in [a,r]$, or

(ii) μ_x be non-decreasing for $x \in [a,r]$.

Proof. $(d/dx)\dot{e}_{x:\overline{r-x}|} = -1+\mu_x \dot{e}_{x:\overline{r-x}|}$

(i) Obviously, $\mu_x \leq 0 \Rightarrow (d/dx)\dot{e}_{x:\overline{r-x}|} < 0$

(ii) If μ_x is increasing, then for any x such that $\mu_x > 0$

$$\mu_x \dot{e}_{x:\overline{r-x}|} = \mu_x \int_0^{r-x} \exp(-\int_0^u \mu_{x+s} ds)du$$
$$\leq \mu_x \int_0^{r-x} \exp(-\int_0^u \mu_x ds)du$$
$$= 1- \exp[-\mu_x(r-x)]$$

$$\Rightarrow \qquad (d/dx)\mathring{e}_{x:\overline{r-x}|} = -1 + \mu_x \mathring{e}_{x:\overline{r-x}|}$$
$$\leq -\exp[-\mu_x(r-x)] < 0. \ \square$$

<u>Lemma 1.1.</u> (Apostol (1974), p.177)

Assume that m(x) is increasing, and that

$$\int_a^b f\,dm, \quad \int_a^b g\,dm \quad \text{and} \quad \int_a^b fg\,dm$$

each exist in the Riemann sense. Then

$$\int_a^b dm \cdot \int_a^b fg\,dm \leq \int_a^b f\,dm \cdot \int_a^b g\,dm$$

if f is increasing and g decreasing. The reverse inequality holds if f and g are both increasing (or both decreasing).

<u>Proof</u>. Let f be increasing, g decreasing; the other cases are similar. Then

$$\frac{1}{2}\int_a^b\int_a^b (f(y)-f(x))(g(y)-g(x))dm(y)dm(x)$$
$$= \int_a^b dm \cdot \int_a^b fg\,dm - \int_a^b f\,dm \cdot \int_a^b g\,dm$$

(straightforward expansion of the left hand side of the equation). Since

$$(f(y)-f(x))(g(y)-g(x)) \leq 0$$

for all x,y, the result follows. □

Note that if there exist two subintervals of positive m-measure I_f, I_g [a,b] such that f is strictly increasing in I_f and g strictly decreasing in I_g, then "≤" can be replaced with "<" in the lemma (similar modifications apply for the other cases).

<u>Proof of Prop. 1.3</u>. I now show that $d\alpha/d\gamma \leq 0$ (or < 0) under the conditions stated.

From step (7) of the proof of Prop. 1.1 (Section 1.4.2),

$$(1.23) \qquad \alpha = S/PVS - \gamma = \ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a}|} \Big/ \int_a^r \ell_x \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\,dx.$$

Transform the denominator as follows:

$$\int_a^r \ell_x \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\,dx$$

$$= \int_a^r \ell_x \int_x^r \exp[-\gamma(u-x)](\ell_u/\ell_x)\,du\,dx$$

$$= \int_\emptyset^{r-a} \ell_{a+x} \int_\emptyset^{r-a-x} \exp(-\gamma u)(\ell_{a+u+x}/\ell_{a+x})\,du\,dx$$

$$= \int_\emptyset^{r-a} \int_\emptyset^{r-a-u} \ell_{u+a+x}\,dx\,\exp(-\gamma u)\,du$$

(changing the order of integration)

$$= \int_a^r \int_\emptyset^{r-u} \ell_{u+x}\,dx\,\exp[-\gamma(u-a)]\,du$$

$$= \int_a^r g(u)\,\ell_u\,\exp[-\gamma(u-a)]\,du$$

where

$$g(u) = \ell_u^{-1} \int_\emptyset^{r-u} \ell_{u+x}\,dx = \bar{a}^{(\emptyset)}_{u:\overline{r-u}|} = \dot{e}_{u:\overline{r-u}|}\ .$$

Consequently,

$$(d\alpha/d\gamma) = \left(\int_a^r \ell_x \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\,dx\right)^{-2}$$

$$\times\Big\{-\int_a^r (x-a)\ell_x \exp[-\gamma(x-a)]\,dx$$

$$\cdot\int_a^r g(x)\ \ell_x \exp[-\gamma(x-a)]\,dx$$

$$+\int_a^r \ell_x \exp[-\gamma(x-a)]\,dx$$

$$\cdot\int_a^r (x-a)\ell_x g(x)\exp[-\gamma(x-a)]\,dx\Big\}.$$

Let $m(x) = \int_a^x \ell_u \exp[-\gamma(u-a)]\,du.$ $m(x)$ is strictly increasing, which means that the sign of $d\alpha/d\gamma$ is the same as that of

$$\{\ \} = \int_a^r dm \int_a^r (x-a)g(x)\,dm - \int_a^r (x-a)\,dm \int_a^r g(x)\,dm.$$

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According to Lemma 1.1, { } ≤ 0 (resp. < 0) if $g(x) = \bar{a}^{(0)}_{x:\overline{r-x}|} = \mathring{e}_{x:\overline{r-x}|}$ is decreasing (resp. strictly decreasing). □

1.4.4.1.2 Effect of Pre-Retirement Decrements

For the sake of tractability assume $\mu_x \equiv \mu$ for all $x \le r$. When $\gamma \neq 0$,

$$S = s\int_a^r \ell_x \, dx$$

$$= s\, \ell_a \int_a^r \exp(-\mu x)dx$$

$$= s\ell_a \; \bar{a}^{(\mu)}_{\overline{r-a}|} \;;$$

$$PVS = s \int_a^r \ell_x \bar{a}^{(\gamma)}_{x:\overline{r-x}|} \, dx$$

$$= s\ell_a \; (\bar{a}^{(\mu)}_{\overline{r-a}|} - \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|})/\gamma.$$

As before,

$$\alpha = \ell_a \; \bar{a}^{(\gamma)}_{a:\overline{r-a}|} / \int_a^r \ell_x \; \bar{a}^{(\gamma)}_{x:\overline{r-x}|} dx$$

$$= \gamma \cdot \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|} / (\bar{a}^{(\mu)}_{\overline{r-a}|} - \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|})$$

$$= \gamma / (\bar{a}^{(\mu)}_{\overline{r-a}|} / \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|} - 1).$$

Hence

$$(d\alpha/d\mu) = -\gamma(\bar{a}^{(\mu)}_{\overline{r-a}|} / \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|} - 1)^{-2} \cdot (d/d\mu)(\bar{a}^{(\mu)}_{\overline{r-a}|} / \bar{a}^{(\mu+\gamma)}_{\overline{r-a}|}).$$

$$\frac{d}{d\mu}\left(\frac{\bar{a}^{(\mu)}_{\overline{r-a}|}}{\bar{a}^{(\mu+\gamma)}_{\overline{r-a}|}}\right) = (\bar{a}^{(\mu+\gamma)}_{\overline{r-a}|})^{-2}\left[\left(\frac{d}{d\mu}\,\bar{a}^{(\mu)}_{\overline{r-a}|}\right)\bar{a}^{(\mu+\gamma)}_{\overline{r-a}|} - \left(\frac{d}{d\mu}\,\bar{a}^{(\mu+\gamma)}_{\overline{r-a}|}\right)\bar{a}^{(\mu)}_{\overline{r-a}|}\right].$$

$$[\;] = -\int x \exp(-\mu x)dx \int \exp[-(\mu+\gamma)x]dx$$

$$+ \int x \exp[-(\mu+\gamma)x]dx \int \exp(-\mu x)dx$$

$$= \int x \exp(-\gamma x)dm \int dm - \int x dm \int \exp(-\gamma x)dm$$

where $m(x) = \int_0^x \exp(-\mu s)ds.$

If $\gamma > 0$, Lemma 1.1 implies [ ] < 0 and $d\alpha/d\mu > 0$. If $\gamma < 0$, then Lemma 1.1 implies [ ] > 0 and, still, $d\alpha/d\mu > 0$.

Remark 1.10. As to the case $\gamma = 0$, one can only assert that $d\alpha/d\mu \geq 0$. ($d\alpha/d\mu > 0$, $\forall\gamma \neq 0$, and is continuous w.r. to $\gamma \Rightarrow d\alpha/d\mu \geq 0$ at $\gamma = 0$.)

Though the calculations above do not prove that in general any increase in μ_x increases α, they make this claim plausible. It results that of two populations, the one with the higher μ_x's will produce fund values converging faster to the ultimate fund value, other things being equal.

1.4.4.1.3 Asymptotic values of α

There may be some theoretical interest in knowing the limiting behaviour of α as $\gamma \to \pm\infty$. Firstly (from step (7) of the Prop. 1.1)

$$\lim_{\gamma\to\infty} \alpha = \lim_{\gamma\to\infty} \gamma \cdot s\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a}|} / (S - s\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a}|})$$

$$= s\ell_a / S$$

$$= 1/\bar{a}^{(0)}_{a:\overline{r-a}|}$$

since $\gamma\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a}|} \to \ell_a$ as $\gamma \to \infty$. Secondly α is asymptotically equal to $-\gamma$ as $\gamma \to -\infty$, because

$$\lim_{\gamma\to-\infty} (S/PVS-\gamma)/\gamma = \lim_{\gamma\to-\infty} S/(\gamma \cdot PVS) - 1$$

$$= -1.$$

1.4.4.2 Aggregate with New Entrants

As explained at the beginning of Section 1.4.3, assume that new entrants coming into the scheme at rate ℓ_a per year for the next n years, $n < \infty$, are included in the valuation basis. Recall that $\alpha(0)$, PVB(0) and PVS(0) are the same as α, PVB and PVS respectively, and correspond to the usual Aggregate method.

1.4.4.2.1 Effect of n

Obviously PVS(n) increases with n, and so $\alpha(n) = S/PVS(n)-\gamma$ decreases as n increases. In other words, introducing more new entrants into the valuation basis slows down convergence of F(t) to its limit, whatever η, β or μ_x may be.

1.4.4.2.2 Effect of γ

$$PVS(n) = PVS + s\ell_a \bar{a}^{(\gamma)}_{a:\overline{r-a}|} \cdot \bar{a}^{(\gamma)}_{\overline{n}|}$$

$$= PVS + \alpha PVS \cdot \bar{a}^{(\gamma)}_{\overline{n}|}$$

(from step (7) of Section 1.4.2)

$$= PVS(1+ \alpha\bar{a}^{(\gamma)}_{\overline{n}|})$$

$$\Rightarrow \quad \alpha(n) = S/PVS(n)-\gamma$$

$$= (S-\gamma PVS(n))/PVS(n)$$

$$= \exp(-\gamma n)\alpha/(1+\alpha\bar{a}^{(\gamma)}_{\overline{n}|}) \tag{1.24}$$

(from (1.20), Section 1.4.3.1).

(This is another way of showing $\alpha(n) > 0$ when $n < \infty$).

<u>Proposition 1.5</u>. If $d\alpha/d\gamma \le 0$, then $d\alpha(n)/d\gamma < 0$.

<u>Proof</u>. $$\frac{d\alpha(n)}{d\gamma} = \frac{\partial\alpha(n)}{\partial\gamma} + \frac{\partial\alpha(n)}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial\gamma}.$$

(i) Fix α. From Eq. (1.24),

$$\frac{\partial\alpha(n)}{\partial\gamma} =$$

$$(1+\alpha\bar{a}^{(\gamma)}_{\overline{n}|})^{-2}[-n\exp(-\gamma n)\alpha(1+\alpha\bar{a}^{(\gamma)}_{\overline{n}|})-\exp(-\gamma n)\alpha^2(\partial\bar{a}^{(\gamma)}_{\overline{n}|}/\partial\gamma)]$$

$$[\ ] = -\alpha\exp(-\gamma n)[n+\alpha(n\bar{a}^{(\gamma)}_{\overline{n}|} + (\partial\bar{a}^{(\gamma)}_{\overline{n}|}/\partial\gamma))]$$

$$< 0,$$

since

$$
\begin{aligned}
n\bar{a}_{\overline{n}|}^{(\gamma)} + \partial \bar{a}_{\overline{n}|}^{(\gamma)}/\partial\gamma &= n\bar{a}_{\overline{n}|}^{(\gamma)} - \int_0^n x\exp(-\gamma x)dx \\
&= \int_0^n (n-x)\exp(-\gamma x)dx \\
&> 0.
\end{aligned}
$$

Accordingly, $\partial\alpha(n)/\partial\gamma < 0$.

(ii) $$\partial\alpha(n)/\partial\alpha$$

$$
= \left(1+\alpha\bar{a}_{\overline{n}|}^{(\gamma)}\right)^{-2}\left[\exp(-\gamma n)\left(1+\alpha\bar{a}_{\overline{n}|}^{(\gamma)}\right)-\alpha\exp(-\gamma n)\bar{a}_{\overline{n}|}^{(\gamma)}\right]
$$

$$
\begin{aligned}
[\ ] &= \exp(-\gamma n) + \alpha\exp(-\gamma n)\bar{a}_{\overline{n}|}^{(\gamma)} - \alpha\exp(-\gamma n)\bar{a}_{\overline{n}|}^{(\gamma)} \\
&= \exp(-\gamma n) > 0.
\end{aligned}
$$

Therefore $\partial\alpha(n)/\partial\gamma > 0$.

Finally

$$
\begin{aligned}
\partial\alpha(n)/\partial\gamma &= \underbrace{\partial\alpha(n)/\partial\gamma}_{<0} + \underbrace{(\partial\alpha(n)/\partial\alpha)}_{>0}\cdot\underbrace{(\partial\alpha/\partial\gamma)}_{\leq 0} \\
&< 0. \quad \square
\end{aligned}
$$

<u>Proposition 1.6</u>. $\alpha(n)$ is strictly decreasing w.r to γ if $g(x) = \mathring{e}_{x:\overline{r-x}|}$ is a decreasing function of x.

1.4.4.2.3 Effect of Pre-retirement Decrements

Let $\mu_x \equiv \mu$, as in 1.4.4.1.2. It is easy to see that $d\alpha/d\mu > 0$ (resp. ≥ 0) implies $d\alpha(n)/d\mu > 0$ (resp. ≥ 0) (differentiate each side of Eq. (1.24)).

1.4.4.2.4 Asymptotic Values of $\alpha(n)$

If $\gamma > 0$, then $\alpha(n) \to 0$ as $n \to \infty$, from Eq. (1.24). If $\gamma \leq 0$, then $PVS(n) \to \infty$ and $\alpha(n) \to -\gamma$. Consequently

$$\lim_{n\to\infty} \alpha(n) = \max(-\gamma, 0).$$

From 1.4.4.1.3, α has a finite limit as $\gamma \to \infty$. Hence (again from Eq. (1.24)), $\alpha(n) \to 0$ as $\gamma \to \infty$, for any $n > 0$.

As when $n = 0$, it is easy to see that $\alpha(n)/\gamma$ tends to -1 as $\gamma \to -\infty$.

1.4.4.3 Numerical values of $\alpha(n)$

Table 1.1 shows numerical values of $\alpha(n)$, computed on different bases. The real rate of return is varied from -.10 to +.10, the column $\gamma = \infty$ being supplied to illustrate how little $\alpha(n)$ varies when γ becomes very large.

Row (1) tells that if the survival function ℓ_x is that of ELT 13, and if $\gamma = .01$, then

$$F'(t) = .0565(F(\infty)-F(t));$$

in other words at any time t the method steers the fund level towards $F(\infty)$ at a rate equal to the difference $F(\infty)-F(t)$ multiplied by $\alpha = .0565$.

It is evident that in all cases $\alpha(n)$ decreases as γ increases, whatever n or the population may be.
Scenarios (1), (2) and (3), on the one hand, and (5) and (6), on the other, illustrate how α depends on the age distribution. In (3), $\mu_x \equiv -.004$, which yields a survival curve ℓ_x which is very similar to the ELT13 ℓ_x curve, but this time slanting upwards.

Extending the valuation basis to include new entrants has a much greater influence on α, as scenarios (1), (4) and (5) exemplify (see Section 1.5.2 for further comments on this.)

Population l_x , $x < 65$	n	γ									
		$-.10$	$-.05$	$-.02$	$-.01$	0	$.01$	$.02$	$.05$	$.10$	∞
(1) ELT 13	0	.1133	.0810	.0669	.0630	.0596	.0565	.0538	.0475	.0411	.0307
(2) $\mu_x = 0$	0	.1122	.0791	.0646	.0607	.0571	.0540	.0512	.0447	.0383	.0286
(3) $\mu_x = -.004$	0	.1118	.0782	.0635	.0595	.0558	.0526	.0498	.0431	.0365	.0266
(4) ELT 13	10	.1045	.0651	.0469	.0419	.0373	.0333	.0296	.0210	.0120	0
(5) ELT 13	20	.1016	.0582	.0377	.0321	.0272	.0229	.0191	.0109	.0041	0
(6) $\mu_x = 0$	20	.1015	.0578	.0372	.0316	.0267	.0223	.0186	.0105	.0039	0

TABLE 1.1 $a(n)$ $(a = 30,\ r = 65)$

1.5 COMPARISON OF METHODS

1.5.1 In the Limit

In Section 1.4 it is proved that when there is only one entry age, the Entry Age Normal and all the aggregate methods lead to the same ultimate situation, except when all future new entrants are part of the valuation basis. (In which case there is no funding at all - in real terms - and the method is tantamount to Pay-as-you-go.) Three methods therefore need to be compared: Unit Credit, Entry Age Normal, and Pay-as-you-go.

The Unit Credit method reputedly leads to higher contributions than the Entry Age Normal, after a scheme has matured (see for example Trowbridge (1952) and p.96 of Winklevoss (1977)). This is indeed the case under the "classical" assumptions of $\gamma > 0$ and the function ℓ_x decreasing with x. It will be seen presently that more generally this is not always the case, though it appears unlikely that practitioners would ever encounter the reverse situation.

Proposition 1.7. Assume that there is only one entry age into the scheme, and that the population is stationary (or only asymptotically so).

If $\gamma > 0$ (resp. =, <), then

$$PG_{C(\infty)} > EAN_{C(\infty)} \qquad \text{(resp. =, <)}$$

and

$$PG_{C(\infty)} > UC_{C(\infty)} \qquad \text{(resp. =, <)}$$

Proof. From 1.3.1.2 and 1.3.2.2,

$$\begin{aligned} UC_{C(\infty)} &= UC_{NC} \\ &= sb\ \ell_r \bar{a}_r^{(\eta-\zeta)}\ \bar{a}_{\overline{r-a}|}^{(\gamma)}, \end{aligned} \tag{1.25}$$

$$\begin{aligned} EAN_{C(\infty)} &= EAN_{NC} \\ &= sb\ell_r \bar{a}_r^{(\eta-\zeta)}(r-a)\exp[-\gamma(r-a)]\bar{a}_{a:\overline{r-a}|}^{(0)} / \bar{a}_{a:\overline{r-a}|}^{(\gamma)}. \end{aligned} \tag{1.26}$$

The Pay-as-you-go contribution is

$$
\begin{aligned}
{}^{PG}C(\infty) &= \int_r^\omega sb(r-a)\ell_x \exp[(\zeta-\beta)(x-r)]dx \\
&= sb(r-a)\ell_r \bar{a}_r^{(\beta-\zeta)}.\} \qquad (1.27)
\end{aligned}
$$

Dividing by $sb(r-a)\ell_r$, the ultimate rates are proportional to

(1.28) UC: $\bar{a}_r^{(\eta-\zeta)}\bar{a}_{\overline{r-a}|}^{(\gamma)} /(r-a)$

(1.29) EAN: $\bar{a}_r^{(\eta-\zeta)}\exp[-\gamma(r-a)]\bar{a}_{a:\overline{r-a}|}^{(0)} /\bar{a}_{a:\overline{r-a}|}^{(\gamma)}$

PG: $\bar{a}_r^{(\beta-\zeta)}$.

The result follows trivially. □

This agrees with intuition: if the fund earns a positive real return on top of salary increases, then Pay-as-you-go is more expensive, in the long run, than any kind of prefunding. The opposite happens in the case of a negative real return.

Now leave Pay-as-you-go aside, and compare Unit Credit with Entry Age Normal.

<u>Proposition 1.8</u>. The assumptions are the same as for Prop. 1.7. Denote

$$f(x) = \exp(-\gamma x)\ell_{x+a}/\ell_a, \quad x \in [0,r-a].$$

If $f(x)$ is strictly decreasing, then

(1i) ${}^{UC}F(\infty) < {}^{EAN}F(\infty)$;

(1ii) $\gamma > 0 \Rightarrow {}^{UC}C(\infty) > {}^{EAN}C(\infty)$;

(1iii) $\gamma < 0 \Rightarrow {}^{UC}C(\infty) < {}^{EAN}C(\infty)$.

If $f(x)$ is strictly increasing, then

(2i) ${}^{UC}F(\infty) > {}^{EAN}F(\infty)$

(2ii) $\gamma > 0 \Rightarrow {}^{UC}C(\infty) < {}^{EAN}C(\infty)$;

(2iii) $\gamma < 0 \Rightarrow {}^{UC}C(\infty) > {}^{EAN}C(\infty)$.

<u>Proof</u>. Multiplying (1.28) and (1.29) by

$$\exp[\gamma(r-a)](r-a)\ \bar{a}^{(\gamma)}_{a:\overline{r-a}|}\ /\bar{a}^{(\eta-\zeta)}_{r},$$

one obtains ultimate rates proportional to

$$[UC] = \bar{a}^{(\gamma)}_{a:\overline{r-a}|}\ \cdot \bar{s}^{(\gamma)}_{\overline{r-a}|}$$

$$= \int_0^{r-a} \exp(-\gamma x)\ell_{x+a}/\ell_a\ dx \int_0^{r-a} \exp(\gamma x)dx,$$

$$[EAN] = (r-a)\ \bar{a}^{(0)}_{a:\overline{r-a}|}$$

$$= \int_0^{r-a} 1dx \int_0^{r-a} \ell_{x+a}/\ell_a\ dx.$$

Let $f(x) = \exp(-\gamma x)\ell_{x+a}/\ell_a$, $g(x) = \exp(\gamma x)$ and $m(x) = x$ in Lemma 1.1 (Section 1.4.4), to find (1ii), (1iii), (2ii) and (2iii). When $\gamma \neq 0$, (1i) and (2i) are then consequences of the equation of equilibrium

(1.30) $$0 = \gamma F(\infty) + C(\infty) - B.$$

When $\gamma = 0$, Eq. (1.30) does not tell anything about ${}^{UC}F(\infty)$ or ${}^{EAN}F(\infty)$. However, from Sections 1.3.1.2 and 1.3.2.2, we know that if $\gamma = 0$,

$$^{UC}F(\infty) = {}^{UC}AL$$

$$= sb(r-a)\ell_r\bar{a}^{(\eta-\zeta)}_r \int_a^r (x-a)/(r-a)\ dx$$

$$+ sb(r-a)\int_r^{\omega} \ell_x\ \exp[(\zeta-\beta)(x-r)]\bar{a}^{(\eta-\zeta)}_x\ dx$$

and

$$^{EAN}F(\infty) = {}^{EAN}AL$$

$$= sb(r-a)\ell_r\bar{a}^{(\eta-\zeta)}_r \int_a^r \bar{a}^{(0)}_{a:\overline{x-a}|}\ /\bar{a}^{(0)}_{a:\overline{r-a}|}\ dx$$

$$+sb(r-a)\int_r^{\omega} \ell_x\exp[(\zeta-\beta)(x-r)]\bar{a}^{(\eta-\zeta)}_x\ dx.$$

Say $f(x) = \ell_{x+a}/\ell_a$ is strictly decreasing; then it is not very difficult to see that $(0 < x < r-a)$:

$$\bar{a}^{(0)}_{a:\overline{x-a}|} / \bar{a}^{(0)}_{a:\overline{r-a}|} = \int_0^{x-a} f(u)du / \int_0^{r-a} f(u)du$$

$$> \int_0^{x-a} du / \int_0^{r-a} du$$

$$= (x-a)/(r-a)$$

$$\Rightarrow \qquad {}^{UC}F(\infty) < {}^{EAN}F(\infty).$$

The case of a strictly increasing $f(x)$ is similar; finally, (1i) and (2i) still hold when $\gamma = 0$. □

Remark 1.11. The argument used to prove the case $\gamma = 0$ in the above also works when $\gamma \neq 0$. Hence it is an alternative way of proving the whole of Prop. 1.8.

1.5.1.1 Numerical Example

Table 1.2 shows numerical values of ultimate costs and funds, as percentages of payroll, and illustrates many of the claims made in the previous section. The assumptions are

Entry age	$a = 30$ (only)
Retirement age	$r = 65$
Benefits	1% of final salary per year of service
Post-retirement mortality	ELT 13
Return on fund	$\eta = .05$

The other assumptions are indicated in the table. The rate of increase of earnings (β) is varied from .03 to .07 in scenarios (1) to (5), with no increase of benefits in payment.

Because there is only one entry age, the Entry Age Normal columns also correspond to any of the aggregate methods (of course excluding the variant that takes all future new entrants into account).

Section 1.5

Except for scenarios (3) and (8) the Entry Age Normal method produces a lower contribution rate than the Unit Credit. The real rate of return is nil in scenario (3), which means that any method ultimately leads to a contribution rate equal to the Pay-as-you-go rate,

Situation (6) is the same as (2), the only difference being that in (6) benefits are fully indexed. Under the Entry Age and Unit Credit methods, ultimate costs are multiplied by

$$k_1 = \bar{a}_{65}^{(.01)}/\bar{a}_{65}^{(.05)} = 1.337$$

while the Pay-as-you-go cost is multiplied by

$$k_2 = \bar{a}_{65}^{(0)}/\bar{a}_{65}^{(.04)} = 1.355.$$

It may be of interest to note that fund levels do *not* grow in size by either factor k_1 or k_2, but rather by

$$^{UC}k_3 = \frac{k_2 \cdot {}^{PG}C(\infty) - k_1 \cdot {}^{UC}C(\infty)}{{}^{PG}C(\infty) - {}^{EAN}C(\infty)}$$
$$= 1.424.$$

and

$$^{EAN}k_3 = \frac{k_2 \cdot {}^{PG}C(\infty) - k_1 \cdot {}^{EAN}C(\infty)}{{}^{PG}C(\infty) - {}^{EAN}C(\infty)}$$
$$= 1.418.$$

(${}^{UC}C(\infty)$ and ${}^{EAN}C(\infty)$ refer to scenario (2); these formulae are derived from the equation of equilibriun (1.30), Section 1.5.1.)

Scenario (7) is also similar to (2), but this time the salary distribution is "flat", i.e. the ℓ_x curve is constant. Finally, scenario (8) is an illustration of claim (2ii) of Prop. 1.8. Here the ℓ_x curve rises steeply enough for $f(x) = \exp(-\gamma x) \cdot \ell_{x+a}/\ell_a$ to be strictly increasing over the whole range (0,35), and as predicted, the Entry Age Normal contribution is higher than the one produced by Unit Credit funding.

Scen.	Infl. Sal.	Net Ret.	Pre-Ret.	Pens. Incr.	Pay-as-you-go	Unit Credit		Entry Age Normal		$f(x)$	Prop. 1.8
	β	γ	μ_x	ζ	C/S	C/S	F/S	C/S	F/S		
(1)	.03	.02	ELT 13	0	7.61	4.76	142.2	4.51	154.8	$\downarrow$	1ii
(2)	.04	.01	"	0	7.09	5.59	149.9	5.49	159.4	$\downarrow$	1ii
(3)	.05	0	"	0	6.62	6.62	159.7	6.62	164.5	$\downarrow$	-
(4)	.06	-.01	"	0	6.21	7.93	172.2	7.91	170.0	$\uparrow$	2iii
(5)	.07	-.02	"	0	5.83	9.59	187.8	9.35	175.7	$\uparrow$	2iii
(6)	.04	.01	"	.04	9.60	7.47	213.4	7.34	226.1	$\downarrow$	1ii
(7)	.04	.01	$\mu_x = 0$	0	8.99	7.09	190.1	7.01	197.3	$\downarrow$	1ii
(8)	.04	.01	$\mu_x = -.02$	0	12.50	9.86	264.4	9.96	254.3	$\uparrow$	2ii

TABLE 1.2 Ultimate Costs and Funds (% of payroll)

1.5.2 Transient Behaviour: Numerical Example

Table 1.3 compares how contribution rates and fund levels vary over time, under the Unit Credit, Entry Age Normal, Aggregate and Aggregate with New Entrants methods. The assumptions are

Decrements	ELT 13 (pre- and post-retirement)
Return on fund	$\eta = .05$
Inflation on salaries	$\beta = .04$
Net return	$\gamma = \eta - \beta = .01$
Benefits	Pension equal to 1% of final salary per year of service; no post-retirement increases
Initial fund	nil
Amortization period	20 years (for methods that specify an initial unfunded liability)
New entrants assumption (Aggregate with New Entrants method)	Full replacement of members leaving the scheme over the next 20 years

The scheme is assumed mature from the start, i.e. both the group of active members and the group of retired members are stationary. This implies a very high initial unfunded liability; one would probably not meet this situation in practice, but it is easier to work out, and simply exaggerates the characteristics of the different methods.

Figs 1.1 and 1.2 are graphs of overall contribution rates (including amortization of unfunded liability, if any) and fund levels achieved.

The Entry Age Normal and Unit Credit methods yield very similar results, with these particular assumptions. The aggregate methods spread the initial unfunded liability far into the future, in comparison with methods that identify a separate past service liability. Here

$\alpha(0) = .0565$ and $\alpha(20) = .0229$ (see Table 1.1, rows (1) and (5)). Figures 1.1 and 1.2 illustrate what was meant by "rate of convergence" of the fund level to its limiting value. Both the Aggregate and the Aggregate with New Entrants methods lead to the same situation in the limit, but the inclusion of new entrants slows down convergence significantly. For example, the fund reaches half its ultimate level in

$$\log 2/\alpha(0) = 12.3 \text{ years}$$

under the usual Aggregate method, and in

$$\log 2/\alpha(20) = 30.3 \text{ years}$$

under the Aggregate with New Entrants. Spreading past service liabilities over the active years of present and future members produces contribution rates that are more level than under any of the other methods considered, but, as can be seen in this example, it may also mean costs which remain relatively high for a long period of time.

Remark 1.12. In the case of the Entry Age Normal and Unit Credit methods, the unfunded liability is paid off with level payments (not a level fraction of payroll), following North-American practice.

t	Pay-as-you-go	Unit Credit		Entry Age Normal		Aggregate No N.E.		Aggregate N.E. 20 Years	
	C/S	C/S	F/S	C/S	F/S	C/S	F/S	C/S	F/S
0	7.08	17.4	0	18.1	0	16.1	0	10.7	0
2	"	16.5	20.0	17.1	21.3	15.0	17.0	10.5	7.1
4	"	15.7	38.6	16.2	41.1	14.0	32.3	10.3	13.9
6	"	14.9	56.0	15.4	59.5	13.0	45.9	10.1	20.4
8	"	14.2	72.2	14.6	76.8	12.2	58.0	9.9	26.6
10	"	13.5	87.3	13.9	92.9	11.5	68.8	9.7	32.6
20	"	10.9*	149.9	11.2*	159.4	8.9	108.0	8.8	58.5
30	"	5.6	"	5.5	"	7.4	130.2	8.1	79.1
40	"	"	"	"	"	6.6	142.8	7.6	95.5
50	"	"	"	"	"	6.1	150.0	7.2	108.6
∞	"	"	"	"	"	5.5	159.4	5.5	159.4

*Dropping to the ultimate level at $t = 20$

TABLE 1.3 Transient Behaviour of Costs and Funds (% of payroll)

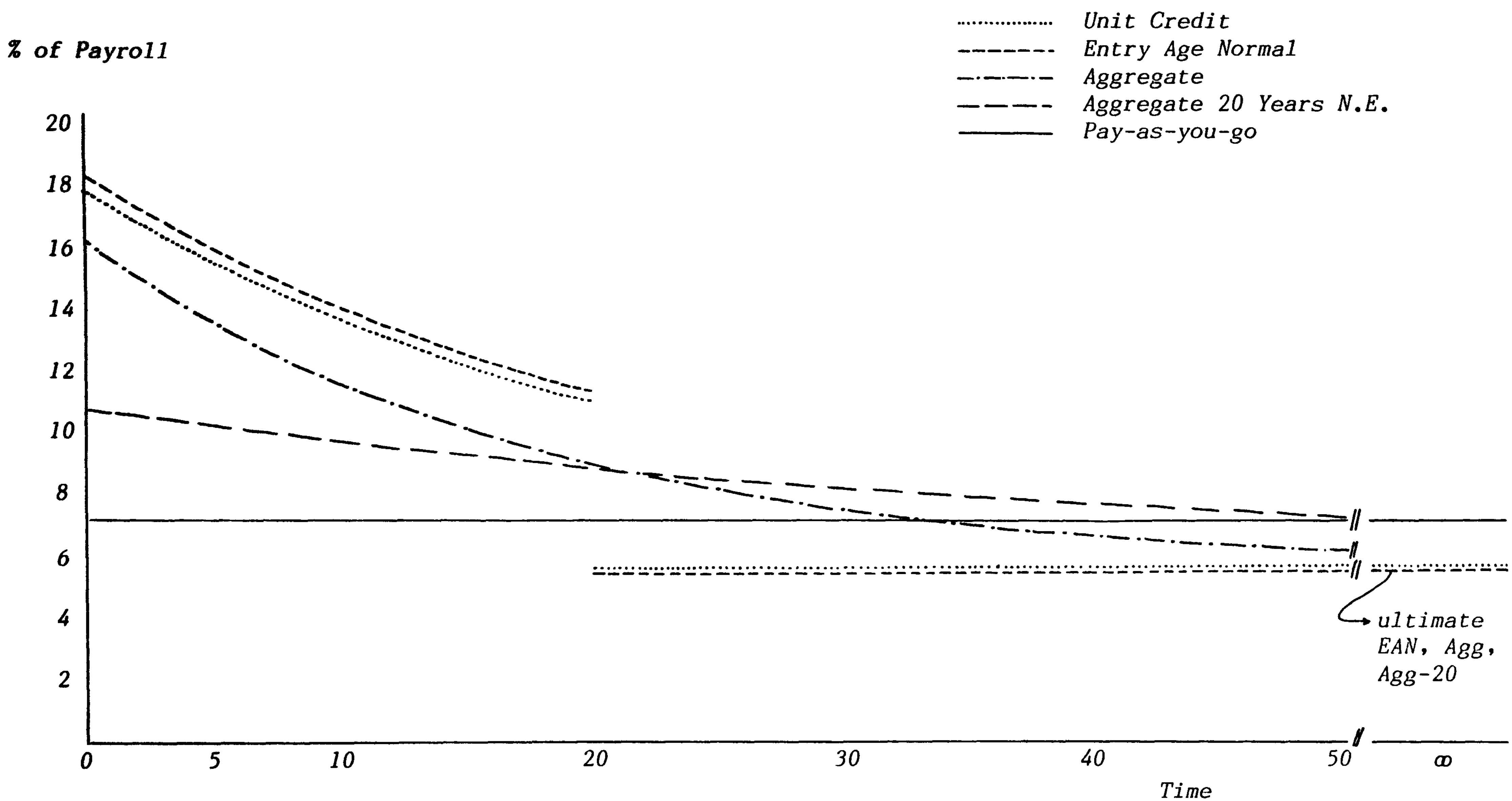

Figure 1.1 Overall Costs over Time (% of payroll)

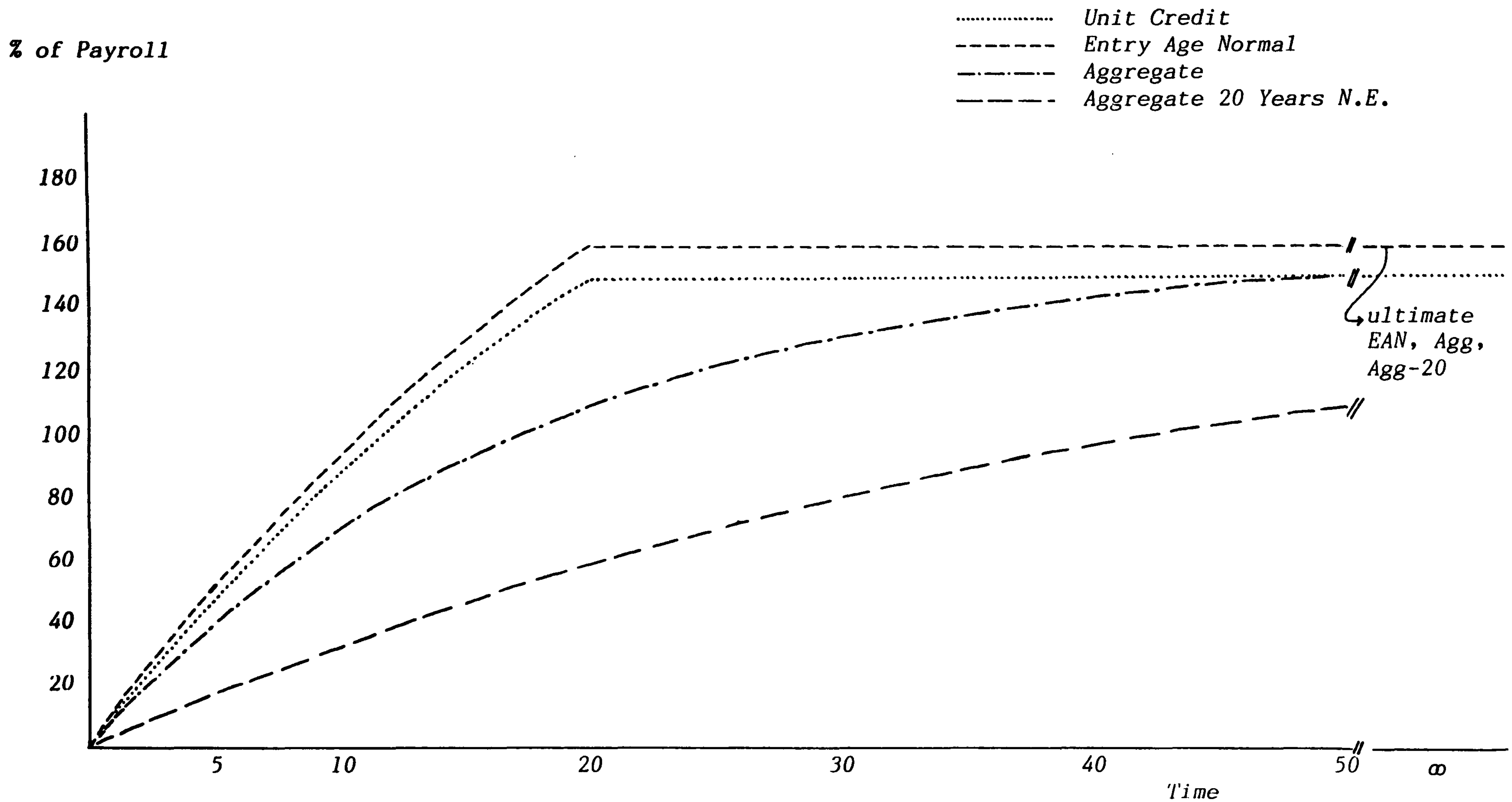

Figure 1.2 Fund Levels over Time (% of payroll)

1.6 MORE THAN ONE ENTRY AGE

Here are a few comments about the more general case of members entering at different ages, instead of at a single one.

Let a now stand for the earliest entry age, and let new entrants come into the scheme at rate $e(y)\cdot\ell_y$ if age y, $a \leq y < r$. Each method is applied in exactly the same fashion as before, taking for granted the fact that members having entered at age y receive a pension equal to $b(r-y)$ times final salary. Assume that whatever the entry age, the members are subject to the same decrements and salary scale; in this section, the ℓ_x function refers only to the service table, and not to the actual population as well.

Prop. 1.8 (Section 1.5.1) plainly remains valid in the multi-entry age case, since the Unit Credit and Entry Age Normal are individual methods: one only has to think of the fund as the aggregation of many smaller funds, each corresponding to the subpopulation of members entering at one particular age "y".

As to whether aggregate methods are still equivalent - in the limit - to the Entry Age Normal method, the situation is not as clear since it is not additive. The derivation of ultimate values is entirely the same as in Section 1.4.2; for instance, consider the Aggregate method: define the real-term constants S, PVB, PVS, and B as previously, i.e.

$$S = \text{total earnings}$$

$$= \int_a^r S(y)e(y)dy,$$

$e(y)S(y)$ = total earnings of members having, entered the scheme at age y

$$S(y) = s\int_y^r \ell_x dx = s\ell_y \cdot \bar{a}^{(0)}_{y:\overline{r-y}|} ;$$

PVB = present value of future benefits of all members

$$= \int_a^r PVB(y)e(y)dy;$$

$$PVB(y) = sb(r-y)\int_y^r \ell_x \exp[\beta(r-x)]_{r-x|}\bar{a}_x dx$$

$$+sb(r-y)\int_r^\omega \ell_x \exp[(\zeta-\beta)(x-r)]\ \bar{a}^{(\eta-\zeta)}_x dx;$$

PVS = present value of future salaries of current members

$$= \int_a^r PVS(y)e(y)dy,$$

$$PVS(y) = s\int_y^r \ell_x\ \bar{a}^{(\gamma)}_{x:\overline{r-x}|}\, dx;$$

B = benefit outgo

$$= \int_r^\omega B(y)e(y)dy,$$

$$B(y) = sb(r-y)\int_r^\omega \ell_x \exp[(\zeta-\beta)(x-r)]dx.$$

($e(y)dy$ will be shortened to $dE(y)$ below.)

As before $\alpha = S/PVS-\gamma > 0$ and

$$AGG_{C(\infty)} = \lim_{t\to\infty} \exp(-\beta t)\bar{S}(t)c(t)$$

$$= S\cdot c(\infty)$$

$$= S(B-\gamma PVB)/(S-\gamma PVS) \quad \text{(Section 1.4.1)}$$

$$= \frac{\int_a^r S(y)dE(y) \cdot \int_a^r [B(y)-\gamma PVB(y)]dE(y)}{\int_a^r [S(y)-\gamma PVS(y)]dE(y)}$$

$$= sb\frac{\int_a^r \ell_y \bar{a}^{(0)}_{y:\overline{r-y}|}\, dE(y) \; \int_a^r (r-y)\exp[-\gamma(r-y)]\ell_r \bar{a}_r dE(y)}{\int_a^r \ell_y \; \bar{a}^{(\gamma)}_{y:\overline{r-y}|} \; dE(y)}$$

(Section 1.4.2).

Comparing this expression with

$${}^{EAN}C(\infty) \;=\; sb\int_a^r (r-y)\exp[-\gamma(r-y)]\ell_r \bar{a}_r \bar{a}^{(0)}_{y:\overline{r-y}|} / \bar{a}^{(\gamma)}_{y:\overline{r-y}|}\, dE(y)$$

we see that in general ${}^{AGG}C(\infty) \neq {}^{EAN}C(\infty)$. When $\gamma = 0$, the contributions are equal, but the funds built up are not, for (see Section 1.4.1)

$$\begin{aligned} {}^{AGG}F(\infty) &= PVB - B \cdot PVS/S \\ &= \int_a^r PVB(y)dE(y) - [\int_a^r B(y)dE(y) \int_a^r PVS(y)dE(y) \\ &\qquad \div \int_a^r S(y)dE(y)] \end{aligned}$$

and

$$\begin{aligned} {}^{EAN}F(\infty) &= \int_a^r PVB(y) - B(y)PVS(y)/S(y)dE(y) \\ &= \int_a^r PVB(y)dE(y) - \int_a^r B(y)PVS(y)/S(y)dE(y). \end{aligned}$$

Not much can be said in general about ultimate costs and funds when new entrants are taken into account. Rate of convergence is evidently reduced, but otherwise the limiting contribution rates and fund levels are not necessarily equal to those shown above for the Aggregate method.

APPENDIX 1.1

CONVERGENCE TO A STATIONARY POPULATION

The setting is the same as in Section 1.4.2, except that the population is only asymptotically stationary. It will be shown that Prop. 1.1 still holds, i.e. that the ultimate (real-term) costs and funds under the Aggregate, Attained Age Normal, Frozen Initial Liability and Entry Age Normal methods are equal to EAN_{NC} and EAN_{AL}, respectively, provided the initial unfunded liability, if any, is liquidated in a finite time.

Eqs. (1.6) and (1.7) are unchanged, but

$$PVB(t) = \exp(-\beta t)\overline{PVB}(t),$$

$$PVS(t) = \exp(-\beta t)\overline{PVS}(t),$$

$$S(t) = \exp(-\beta t)\bar{S}(t)$$

and

$$B(t) = \exp(-\beta t)(\bar{B}(t)$$

are no longer constants. Then

$$F'(t) = -\alpha(t)F(t) + R(t) - h(t)$$

where

$$\alpha(t) = S(t)/PVS(t) - \gamma$$

$$R(t) = S(t)PVB(t)/PVS(t)-B(t)$$

and

$$h(t) = -\exp(-\beta t)(\bar{P}(t)-\bar{S}(t)\bar{U}(t)/\overline{PVS}(t)).$$

From the assumptions $h(t) = 0$ for t larger than some t_0, $\alpha(t) \to \alpha > 0$, and $R(t) \to R$ as $t \to \infty$.

The problem can be formulated as follows: consider the differential equations

$$F'(t) = -\alpha(t)F(t) + R(t), \quad (t > t_0)$$

$$G'(t) = -\alpha G(t) + R, \quad G(0) = F(0).$$

Clearly $G(\infty) = R/\alpha$; I will show that $F(\infty) = R/\alpha$.

Let $D(t) = F(t)-G(t)$. It is sufficient to show that $D(t) \to 0$ as $t \to \infty$.

$$D'(t) = -\alpha(t)F(t) + \alpha G(t) + R(t) - R$$
$$= -\alpha(t)D(t) + \underbrace{(\alpha-\alpha(t))G(t)+R(t)-R}_{Z(t)}$$
$$= -\alpha(t)D(t) + Z(t).$$

Observe that $Z(t) \to 0$ as $t \to \infty$.

$\varepsilon > 0$. Let t_1 be such that $t > t_1$ implies $\alpha(t) > \bar{\alpha} > 0$ and $|Z(t)| < \varepsilon$. Then for any $t > t_1$,

$$D(t) = D(t_1)\exp(-\int_{t_1}^{t} \alpha(s)ds) + \int_{t_1}^{t} \exp(-\int_{s}^{t} \alpha(u)du)Z(s)\, ds$$

$$\Rightarrow \quad |D(t)| \le |D(t_1)| \exp[-(t-t_1)\bar{\alpha}] + \varepsilon(1-\exp[-\bar{\alpha}(t-t_1)])/\bar{\alpha}$$

$$\Rightarrow \quad \limsup_{t\to\infty} |D(t)| \le \varepsilon/\bar{\alpha}. \quad \square$$

Remark 1.13. Obviously the same argument completes the proof of Prop. 1.2 as well; that is, it shows that

$$C(n,\infty) = EAN_{NC}$$

and

$$F(n,\infty) = EAN_{AL}$$

when the population is only asymptotically stationary, and $n < \infty$.

APPENDIX 1.2

DETAILS ABOUT THE NUMERICAL EXAMPLES

The values of $\bar{a}_{30:\overline{35}|}$ and $\bar{a}_{65}$ required were computed directly from the English Life Table No. 13; it is not clear how good the approximation used,

$$\bar{a}_x^{(\gamma)} = \ddot{a}_x - .5 - (\mu_x+\gamma)/12,$$

becomes when γ is large or negative.

Table 1.1. First $\alpha(0)$ was calculated using (steps (5) and (7) of Section 1.4.2)

$$\alpha(0) = S/PVS-\gamma = \gamma/(\bar{a}_{30:\overline{35}|}^{(0)}/\bar{a}_{30:\overline{35}|}^{(\gamma)}-1),$$

and then $\alpha(n)$ resulted from

$$\alpha(n) = \exp(-\gamma n)\cdot\alpha(0)/(1+\alpha(0)\cdot\bar{a}_{\overline{n}|}^{(\gamma)})$$

(at $\gamma = 0$ interpolation was required).

Table 1.2. The limiting contribution rates were found from Eq. (1.25), (1.26) and (1.27) (Section 1.5.1), dividing by $.01\cdot s\cdot \ell_{30}\cdot\bar{a}_{30:\overline{35}|}^{(0)}$ to obtain percentages of payroll. Ultimate fund levels then follow from

$$0 = \gamma\cdot F(\infty) + C(\infty) - B$$

(interpolation required at $\gamma = 0$).

Table 1.3 and Figs. 1.1 and 1.2. Under the Unit Credit and Entry Age Normal methods, the contribution is the normal cost plus the amortization payment. Since the population is mature from the start, and the initial fund nil, the initial unfunded liability is $F(\infty)$. Thus, for $t < 20$,

$$\text{overall contribution rate at time } t$$
$$= \text{ultimate rate} + (F(\infty)/\bar{a}_{\overline{20}|}^{(.05)})/\text{payroll};$$

as a fraction of payroll,

$$C(t)/S = C(\infty)/S + \exp(-\beta t)(F(\infty)/S)\bar{a}_{\overline{20}|}^{(.05)}.$$

The fund reaches its ultimate level at t=20. For $t < 20$,

$$F'(t) = \gamma F(t) + NC - B + \exp(-\beta t)F(\infty)/\bar{a}_{\overline{20}|}^{(.05)}$$

$$\Rightarrow \quad F(t) = \int_0^t \exp[\gamma(t-s)][NC-B + \exp(-\beta s)F(\infty)/\bar{a}_{\overline{20}|}^{(.05)}]ds$$

$$= F(\infty)\int_0^t -\gamma \exp[\gamma(t-s)]+\exp[-\beta s+\gamma(t-s)]/\bar{a}_{\overline{20}|}^{(.05)}\, ds$$

$$(\text{since } NC - B = -\gamma F(\infty))$$

$$= F(\infty)(1-\exp(\gamma t)+\exp(\gamma t)\bar{a}_{\overline{t}|}^{(.05)}/\bar{a}_{\overline{20}|}^{(.05)}).$$

Aggregate methods (n=0,20): firstly

$$F(n,t) = R_n\int_0^t \exp[-\alpha_n(t-s)]ds$$

$$= (R_n/\alpha_n)[1-\exp(-\alpha_n t)]$$

$$= F(n,\infty)(1-\exp(-\alpha_n t))$$

$$= {}^{EAN}AL(1-\exp(-\alpha_n t)).$$

Secondly

$$C(n,t) = S(PVB(n)-F(t))/PVS(n)$$

$$= S(PVB(n)-F(n,\infty))/PVS(n)$$

$$+S(F(n,\infty)-F(n,t))/PVS(n)$$

$$= {}^{EAN}NC + (S/PVS(n))\cdot\exp(-\alpha_n t)\cdot F(n,\infty)$$

$$= {}^{EAN}NC + (\alpha_n+\gamma)\exp(-\alpha_n t)\cdot {}^{EAN}AL .$$

CHAPTER 2

VARYING RATES OF RETURN AND OF INFLATION

2.1 INTRODUCTION

In the first chapter, actuarial assumptions were always borne out by experience. Because they never are in the real world, actuaries have devised methods of adjusting contributions for deviations from these assumptions. This chapter considers rates of return ($\eta(t)$) and of increase on salaries ($\beta(t)$) which differ from the assumed rates η and β. All other assumptions (e.g., mortality, withdrawals) are supposed to be consistently realized.

The chapter has two purposes. Firstly, it describes two methods of taking deviations into account, and derives formulas which are essential to Chapters 3 and 4. Secondly, it includes a brief comparison of the two methods, in the case of a single deviation from acturial assumptions. This comparison is further translated into the language of control theory.

The "Spread" method is the first one studied. The normal cost is adjusted by an amount equal to the overall unfunded liability divided by the present value of an annuity for a fixed term. It is shown that aggregate cost methods have a built-in method of adjusting contributions, and that it is mathematically equivalent to the Spread method.

The other method considered I have termed "Amortization of Losses". At each valuation date, an "actuarial loss" is estimated, corresponding to the time elapsed since the last valuation only. Each intervaluation loss is liquidated in full by a series of level payments, over a fixed number of years. At any one valuation date, the adjustment to the normal cost is the sum of those payments which are still in force.

The Amortization of Losses method, or variations of it, has been widely used in Canada. The Spread method appears to be more popular in the United Kingdom.

Several simplifying assumptions are necessary to keep the formulae at a bearable level of complexity. Perhaps the most significant of these is that surpluses and deficiencies (or, alternatively, gains and losses) receive the same treatment under either of the methods outlined above. The other assumptions are set out in 2.2.

Section 2.3 examines the discrete-time situation. The same results are formulated in continuous time in 2.4.

Even though the concept of "actuarial loss" pervades it, this chapter has little to do with the literature on gain and loss analysis. I would go as far as to affirm that the sole idea required from this subject is (Street (1977), p. 407):

> Pension plan gains may be described as the excess of the expected over the actual unfunded accrued liability at the end of the period to be analyzed.

Three aspects of Chapter 2 are original:

(i) The explicit formulas of 2.3.1 and 2.4.1 for the dependence of the acturial liability on past inflation rates;

(ii) Sections 2.3.3 and 2.4.3, dealing with the Amortization of Losses method; and

(iii) Remarks 2.4 and 2.6 which interpret the two methods of adjusting contributions as "negative feedback controls".

This chapter raises many more questions than are answered in Chapters 3 and 4. After all, these are only concerned with random rates of return. Chapter 2 could be the starting point of further research; for example, as to the effects of varying rates of inflation on contributions and fund levels, when benefits are not indexed.

2.2 MODELLING ASSUMPTIONS

(i) The population is stationary.

(ii) For aggregate cost methods, it is required that there be only one entry age into the scheme.

(iii) Actuarial assumptions (η, β, ℓ_x, etc.) are fixed. Only rates of return ($\eta(t)$) and of inflation ($\beta(t)$) differ from the assumptions.

(iv) Benefits are not indexed.

(v) As in Chapter 1, benefits are a fixed fraction of final earnings. The constant sb(r-a) will be replaced by "c", for simplicity.

(vi) In discrete time, $\eta(t)$ and $\beta(t)$ are the actual rates experienced during $(t-1,t)$. η and β are the rates assumed in the valuation. The net rate of return during $(t-1,t)$ is $\gamma(t) = \eta(t)-\beta(t)$. I will also denote

$$\Delta\eta(t) = \eta(t) - \eta$$

$$\Delta\beta(t) = \beta(t) - \beta$$

$$\Delta\gamma(t) = \gamma(t)-\gamma = \Delta\eta(t) - \Delta\beta(t).$$

(vii) In continuous time $\eta(t)$ and $\beta(t)$ are instantaneous rates at time t. The other symbols have the same meaning as in (vi) above.

(viii) Like in Chapter 1, a bar ("$\bar{\ }$") above a symbol refers to a nominal quantity, while the same symbol without a bar corresponds to a real-term quantity.

(ix) AAL and RAL will refer to the part of the actuarial liability (AL) attributed to active and retired members, respectively.

(x) B^* will mean the level of the benefits paid, if there have never been any deviations $\Delta\beta(\cdot)$ from the assumed rate β. Same comment for RAL^* and PVB^*.

(xi) An "individual cost method" will be any cost method which produces an actuarial liability and a normal cost, and such that

(2.1) 1. $$AAL = e^{\gamma}(AAL+NC) - \ell_r AL(r)$$
(discrete time)

(2.2) $$0 = \gamma AAL + NC - \ell_r AL(r)$$
(continuous time)

2. RAL(t) is the present value of benefits of retired members.

The Unit Credit and Entry Age Normal are examples of individual cost methods. This family of methods is characterized in Cooper and Hickman (1967).

(xii) By aggregate cost method I will mean any of the methods studied in Section 1.4, excepting the one which includes all future new entrants $(n = \infty)$.

Finally, it should be emphasized that all the equations derived in this chapter relate to "real-term" values (see Section 0.6). Sections 2.3.1 and 2.4.1 derived basic real-term relationships, in discrete and continuous time, respectively.

2.3 DISCRETE TIME

2.3.1 Real-Term Variables

Salaries increase by a factor $e^{\beta(t)}$ during the year $(t-1,t)$, and thus the overall payroll at time t is proportional to

$$\exp\left(\sum_{k=1}^{t} \beta(k)\right). \tag{2.3}$$

Accordingly, (2.3) is the index number used to convert nominal amounts into real-term amounts, i.e.

(real-term variable at time t)

$= $ (nominal-term variable at time t) $\times \exp\left(-\sum_{k=1}^{t} \beta(k)\right)$.

Because benefits are a fraction of final salary, both the normal cost and the actuarial liability with respect to active members (AAL) are constant in real terms. On the contrary, the fact that benefits in payment are not indexed makes $B(t)$ and $RAL(t)$ functions of $\beta(k)$, $k \le t$. We find

$$\begin{aligned}
B(r,t) &= c, \\
B(r+1,t) &= B(r,t-1)\exp(-\beta(t)) \\
&= c\cdot\exp(-\beta(t)), \\
B(r+2,t) &= B(r+1,t-1)\exp(-\beta(t)) \\
&= c\cdot\exp(-\beta(t-1)-\beta(t)), \\
&\cdots \\
B(x,t) &= c\cdot\exp\left(-\sum_{k=t-x+r+1}^{t} \beta(k)\right).
\end{aligned}$$

I will now show that

$$AL(t+1) = e^{\gamma}(AAL+NC) + \exp(\eta-\beta(t+1))(RAL(t)-B(t)). \tag{2.4}$$

When $x \ge r$, $AL(x,t) = B(x,t)\ddot{a}_x^{(\eta)}$, and so

$$\ell_{x+1}AL(x+1,t+1) = c\cdot\exp\left(-\sum_{k=t-x+r+1}^{t+1} \beta(k)\right)\ell_{x+1}\ddot{a}_{x+1}^{(\eta)}$$

$$= c \cdot \exp\left(-\sum_{k=t-x+r+1}^{t} \beta(k)\right)\exp(-\beta(t+1))$$

$$\times\ e^{\eta}(\ell_x \ddot{a}_x^{(\eta)} - \ell_x)$$

$$= \exp(\eta-\beta(t+1))(\ell_x AL(x,t)-\ell_x B(x,t)).$$

Summing for all $x \geq r$,

$$RAL(t+1) - \ell_r AL(r) = \exp(\eta-\beta(t+1))(RAL(t)-B(t)).$$

Equation (2.4) is obtained by adding this formula to Eq. (2.1) (Section 2.2).

Finally, as concerns the fund and contributions, from dividing both sides of

$$\bar{F}(t+1) = \exp(\eta(t+1))(\bar{F}(t)+\bar{C}(t)-\bar{B}(t))$$

by (2.3), we deduce

$$F(t+1) = \exp(\gamma(t+1))(F(t)+C(t)-B(t)) \tag{2.5}$$

2.3.2 Spread Method

2.3.2.1 Individual Cost Methods

The adjustment to the normal cost is equal to the overall unfunded liability, divided by the present value of an annuity for a term of "m" years:

$$\begin{aligned} C(t) &= NC + ADJ(t) \\ &= NC + UL(t)/\ddot{a}_{\overline{m|}}. \end{aligned} \tag{2.6}$$

From Eqs. (2.4), (2.5) and (2.6),

$$\begin{aligned} UL(t+1) &= AL(t+1) - F(t+1) \\ &= e^{\gamma}(AAL+NC) + \exp(\eta-\beta(t+1))(RAL(t)-B(t)) \\ &\quad -\exp(\gamma(t+1))(F(t)+NC-B(t)+UL(t)/\ddot{a}_{\overline{m|}}) \end{aligned}$$

$$
\begin{aligned}
&= \exp(\gamma(t+1))[(AAL+NC+RAL(t)-B(t)) \\
&\qquad\qquad -(F(t)+NC-B(t)+UL(t)/\ddot{a}_{\overline{m|}})] \\
&\quad +[\exp(\gamma)-\exp(\gamma(t+1))](AAL+NC) \\
&\quad +[\exp(\eta-\beta(t+1))-\exp(\gamma(t+1))](RAL(t)-B(t)) \\
&= \exp(\gamma(t+1))(1-1/\ddot{a}_{\overline{m|}})UL(t) \\
&\quad +e^{\gamma}\{[1-\exp(\Delta\gamma(t+1))](AAL+NC) \\
&\qquad +[\exp(-\Delta\beta(t+1))-\exp(\Delta\gamma(t+1))] \\
&\qquad\quad \times (RAL(t)-B(t))\}
\end{aligned}
$$

$$
(2.7) \qquad = q\cdot UL(t) + \ell(t+1),
$$

where

$$
(2.8) \qquad q = e^{\gamma}(1-1/\ddot{a}_{\overline{m|}})
$$

and

$$
\begin{aligned}
(2.9) \quad \ell(t+1) &= (\exp(\Delta\gamma(t+1))-1)qUL(t) \\
&\quad +e^{\gamma}\{[1-\exp(\Delta\gamma(t+1))](AAL+NC) \\
&\quad +[\exp(-\Delta\beta(t+1))-\exp(\Delta\gamma(t+1))](RAL(t)-B(t))\}.
\end{aligned}
$$

$\ell(t+1)$ represents the (actuarial) loss, with respect to inflation on earnings and return on assets, incurred during the period $(t, t+1)$. If actuarial assumptions are realized during that period, that is to say, if $\beta(t+1) = \beta$ and $\eta(t+1) = \eta$, then $\ell(t+1) = 0$. The loss is measured at the end of the year, which explains the factor e^{γ} on the right hand side of (2.9).

$q\cdot UL(t)$ is what $UL(t+1)$ would be, if all actuarial assumptions had been correct. The first term,

$$
(\exp(\Delta\gamma(t+1))-1)qUL(t),
$$

therefore represents the loss on the unfunded liability itself, caused by the net return discrepancy $\Delta\gamma(t)$. The second term,

$$
e^{\gamma}[1-\exp(\Delta\gamma(t+1))](AAL+NC),
$$

is the loss on the active members' actuarial liability and normal cost, again attributed to $\Delta\gamma(t)$. If $\gamma(t+1) > \gamma$, a gain arises. The third term,

$$e^{\gamma}[1-\exp(\Delta\eta(t+1))]\exp(-\Delta\beta(t+1))(RAL(t)-B(t)),$$

is the loss on the retired members' actuarial liability and benefits; it results from the nominal return discrepancy $\Delta\eta(t)$. A gain is experienced if $\eta(t+1) > \eta$.

Notice the difference between the second and third term of $\mathcal{E}(t+1)$. On the one hand, active members' liabilities and normal costs all increase at the same rate as the payroll. Hence only the *net* rate of return deviation $\Delta\gamma(t)$ is of importance. On the other hand, benefits are unindexed, and so the loss on the retired members' actuarial liability depends on the variation of the *nominal* rate of return, $\Delta\eta(t+1)$.

2.3.2.2 Aggregate Cost Methods

Consider the Aggregate method (see 1.3.4 and 1.4.2):

$$\bar{C}(t) = \bar{S}(t)(\bar{P}\bar{V}\bar{B}(t)-\bar{F}(t))/\bar{P}\bar{V}\bar{S}(t).$$

Multiply by $\exp(-\sum_{k=1}^{t} \beta(k))$ to get the real-term contribution

$$C(t) = S(PVB(t)-F(t))/PVS.$$

PVB(t) and PVS are defined as in 1.4.2, except that PVB(t) does not turn out to be a constant, because benefits are not indexed.

The discrete-time version of Prop. 1.1 (Section 1.4.2) tells us that

$$EAN_{NC} = S(PVB^{*}-{}^{EAN}AL)/PVS.$$

Furthermore, it is easy to see that

$$PVB(t) - PVB^* = RAL(t) - RAL^*$$
$$= {}^{EAN}AL(t) - {}^{EAN}AL^*.$$

In consequence,

$$C(t) = S(PVB^* - {}^{EAN}AL^*)/PVS$$
$$+S(PVB(t)-PVB^*+{}^{EAN}AL^*-F(t))/PVS$$
$$= {}^{EAN}NC + S({}^{EAN}AL(t)-F(t))/PVS$$

(2.10) $$= {}^{EAN}NC + {}^{EAN}UL(t)\cdot S/PVS.$$

Thus Eqs. (2.7), (2.8) and (2.9) still hold, with $\ddot{a}_{\overline{m|}}$ replaced by PVS/S. The unfunded liability, estimated on the basis of the Entry Age Normal method, is spread over "m" years, m being such that

(2.11) $$\ddot{a}_{\overline{m|}} = PVS/S.$$

That is, m is a kind of "salary-weighted" average of remaining years of service. The numerical example in Section 3.5 further illustrates this point.

<u>Remark 2.1</u>. Aggregate with New Entrants method ($n < \infty$).

Let the valuation basis include new entrants coming into the scheme over the next "n" years. In view of the discrete-time version of Prop. 1.2 (Section 1.4.3.1), it is clear that Eqs. (2.10) and (2.11) become

$$C(n,t) = {}^{EAN}NC + {}^{EAN}UL(t)\cdot S/PVS(n)$$

and

$$\ddot{a}_{\overline{m'|}} = PVS(n)/S.$$

Consequently,

$${}^{EAN}UL(t+1) = q'\,{}^{EAN}UL(t) + \ell(t+1)$$

where $\ell(t+1)$ is defined as in (2.9), but

$$\begin{aligned} q' &= e^{\gamma}(1 - S/PVS(n)) \\ &= e^{\gamma}(1 - 1/\ddot{a}_{\overline{m'|}}). \end{aligned}$$

2.3.3 Amortization of Losses Method

Under this method of adjusting the normal cost, each intervaluation loss $\ell(t)$ is amortized over m years. I first find an expression for $\ell(t)$. Denote the overall adjustment by ADJ(t), i.e.

$$C(t) = NC + ADJ(t).$$

From Eqs. (2.4) and (2.5), we find

$$(2.12) \qquad UL(t+1) = e^{\gamma}(UL(t)-ADJ(t)) + \ell(t+1)$$

where $\ell(t+1)$ has a definition very similar to Eq. (2.9):

$$\begin{aligned} (2.13) \quad \ell(t+1) = {} & (\exp(\Delta\gamma(t+1))-1)e^{\gamma}(UL(t)-ADJ(t)) \\ & +e^{\gamma}\{[1-\exp(\Delta\gamma(t+1))](AAL+NC) \\ & +[\exp(-\Delta\beta(t+1))-\exp(\Delta\gamma(t+1))](RAL(t)-B(t))\}. \end{aligned}$$

The only difference between the two definitions of $\ell(t+1)$ lies in their first term. It is explained by the fact that the expected unfunded liability at time t+1 (i.e. supposing all actuarial assumptions to have worked out during (t,t+1)) is now

$$e^{\gamma}(UL(t)-ADJ(t)).$$

The loss on the unfunded liability differs accordingly.

Each loss $\ell(s)$ is liquidated by m payments

$$p(s) = \ell(s)/\ddot{a}^{(\gamma)}_{\overline{m|}}$$

to be paid at times s,s+1,...,s+m-1. Thus the overall adjustment becomes

$$\begin{aligned} ADJ(t) &= \text{sum of } p(s)\text{'s in force at time t} \\ &= \sum_{k=0}^{m-1} p(t-k) \end{aligned}$$

$$(2.14) \qquad = \sum_{k=0}^{m-1} \ell(t-k)/\ddot{a}_{\overline{m}|} .$$

I shall now derive an expression for UL(t), in terms of the $\ell(s)$'s only. This expression, together with Eq. (2.14), will be fundamental in calculating the moments of F(t) and C(t) in Section 3.4.

Imagine the scheme to be set up at time 0. Then $\ell(s) = 0$, $s < 0$, and $\ell(0) = UL(0)$. From Eqs. (2.12) and (2.14),

$$UL(1) = e^{\gamma}(UL(0)-ADJ(0)) + \ell(1)$$

$$(2.15) \qquad = e^{\gamma}(1-1/\ddot{a}_{\overline{m}|})\ell(0) + \ell(1)$$

$$(2.16) \qquad = (\ddot{a}_{\overline{m-1}|}/\ddot{a}_{\overline{m}|})\ell(0) + \ell(1);$$

$$\begin{aligned} UL(2) &= e^{\gamma}(UL(1)-ADJ(1)) + \ell(2) \\ &= e^{\gamma}[(\ddot{a}_{\overline{m-1}|}/\ddot{a}_{\overline{m}|})\ell(0) + \ell(1) \\ &\qquad -\ell(0)/\ddot{a}_{\overline{m}|} - \ell(1)/\ddot{a}_{\overline{m}|}] + \ell(2) \\ &= (\ddot{a}_{\overline{m-2}|}/\ddot{a}_{\overline{m}|})\ell(0) + (\ddot{a}_{\overline{m-1}|}/\ddot{a}_{\overline{m}|})\ell(1)+\ell(2); \end{aligned}$$

$$\cdots$$

$$(2.17) \qquad UL(t) = \sum_{k=0}^{m-1} \ell(t-k)\ddot{a}_{\overline{m-k}|}/\ddot{a}_{\overline{m}|}$$

$$(2.18) \qquad = \sum_{k=0}^{m-1} p(t-k)\ddot{a}_{\overline{m-k}|} .$$

In words, UL(t) is the present value of those payments p(s) which have yet to be made, in order to liquidate each of the losses having arisen over the past m years.

Remark 2.2. It is implicit in the derivation of Eq. (2.17) that the rate used to calculate $\ddot{a}_{\overline{m}|}$ is γ (this is why (2.16) can be deduced from (2.15)). In practice, however, valuations are not done on a "real-term" basis, and so $\ddot{a}_{\overline{m}|}$ is calculated at rate η. The formulae obtained in that case are similar, but more complex.

2.3.4 Response to a Single Loss

The two methods of adjusting the normal cost will be compared in Chapter 3, when $\Delta\beta(t) \equiv 0$ and $\{\eta(t)\}_{t\geq 1}$ is a sequence of i.i.d. random variables. In this section, the methods are compared in the simplest possible case, that of a single loss $\ell(0)$.

Remark 2.3. In the parlance of control theory, this section studies the response of the systems to an "impulse" input (= unique disturbance). Chapter 3 studies the response of the system to a random (uncorrelated) input.

Suppose a unique loss $\ell(0)$ and $\ell(s) = 0$, $\forall s \neq 0$. With the Spread method, one obtains (from Eq. 2.7))

$$UL(1) = q\cdot\ell(0),$$

$$UL(2) = q^2\cdot\ell(0),$$

$$\cdots$$

$$UL(t) = q^t\cdot\ell(0). \tag{2.19}$$

It follows that

$$C(t) = NC + q^t\ell(0)/\ddot{a}_{\overline{m}|}. \tag{2.20}$$

Therefore, the unfunded liability converges geometrically to 0, and the contributions to NC, *if* $q < 1$. This is the case in general if the rate used to calculate $\ddot{a}_{\overline{m}|}$ is equal to the assumed net rate of return γ, for

$$e^{\gamma}(1- 1/\ddot{a}_{\overline{m}|}^{(\gamma)}) = \ddot{a}_{\overline{m-1}|}^{(\gamma)}/\ddot{a}_{\overline{m}|}^{(\gamma)} < 1.$$

When the Amortization of Losses method is used, Eqs. (2.14) and (2.17) tell us that

$$
(2.21)\qquad
\begin{aligned}
UL(t) &= \ell(0)\ddot{a}_{\overline{m-t}|}/\ddot{a}_{\overline{m}|} \\
C(t) &= NC + \ell(0)/\ddot{a}_{\overline{m}|}
\end{aligned}
\quad t < m
\qquad
\begin{aligned}
UL(t) &= 0 \\
C(t) &= NC.
\end{aligned}
\quad t \geq m.
$$

This merely confirms that the method actually does liquidate losses in exactly m years. Notice that, on the other hand, the Spread method never gets rid of $\ell(0)$ completely.

Remark 2.4. Eqs. (2.6), (2.7), (2.12), (2.14) and (2.17) imply that

(i) under the Spread method

$$
(2.22)\qquad
\begin{aligned}
UL(t+1) &= UL(t) - (1-q)UL(t) + \ell(t+1) \\
ADJ(t) &= UL(t)/\ddot{a}_{\overline{m}|} ;
\end{aligned}
$$

(ii) under the Amortization of Losses method,

$$
(2.23)\qquad
\begin{aligned}
UL(t+1) &= UL(t) - \sum_{k=0}^{m-1} \exp[-\gamma(m-1-k)]\ell(t-k)/\ddot{a}_{\overline{m}|} + \ell(t+1) \\
ADJ(t) &= \sum_{k=0}^{m-1} \ell(t-k)/\ddot{a}_{\overline{m}|} .
\end{aligned}
$$

Now think of UL(t) as the state of the "system", of $\ell(t)$ as the disturbance experienced, and of ADJ(t) as the control applied to the system. It can be seen that

(i) the Spread method applied a *proportional* negative feedback control, while

(ii) the Amortization of Losses method is a kind of *integral* negative feedback control.

(The feedback is "negative", for ADJ(t) is subtracted from UL(t); for more about the different types of controls, see Burghes and Graham (1980), pp. 101-102.)

2.4 CONTINUOUS TIME

Below is the continuous time translation of the results of the previous section. Several details and interpretations have not been repeated.

2.4.1 Real-Term Variables

Total payroll is proportional to

$$\exp\left(\int_0^t \beta(s)ds\right).$$

Thus, in what follows,

(real-term variable at time t)

$=$ (nominal-term variable at time t) $\exp\left(-\int_0^t \beta(s)ds\right)$.

If $x \geq r$,

$$B(x,t) = c\cdot\exp\left(-\int_{t-x+r}^{t} \beta(s)ds\right)$$

and

$$AL(x,t) = B(x,t)\cdot\bar{a}_x^{(\eta)}.$$

I will now show that

(2.24) $\quad AL'(t) = \gamma(t)AL(t)-\Delta\gamma(t)AAL-\Delta\eta(t)RAL(t)+NC-B(t).$

(i) Say $x \geq r$.

$$\frac{\partial}{\partial x}(\ell_x AL(x,t)) = c\cdot\ell_x\bar{a}_x^{(\eta)}\frac{\partial}{\partial x}\exp\left(-\int_{t-x+r}^{t} \beta(s)ds\right)$$

$$+c\cdot\exp\left(-\int_{t-x+r}^{t} \beta(s)ds\right)\frac{\partial}{\partial x}\left(\ell_x\bar{a}_x^{(\eta)}\right).$$

From

$$\frac{\partial}{\partial x}\left(\ell_x\bar{a}_x^{(\eta)}\right) = -\ell_x + \eta\ell_x\bar{a}_x^{(\eta)},$$

we get

$$\frac{\partial}{\partial x}(\ell_x AL(x,t)) = (\eta-\beta(t-x+r))\ell_x AL(x,t)-\ell_x B(x,t).$$

(ii) $$\frac{\partial}{\partial t}(\ell_x AL(x,t)) = (\beta(t-x+r)-\beta(t))\ell_x AL(x,t)$$

$$= (\eta-\beta(t))\ell_x AL(x,t) - \frac{\partial}{\partial x}(\ell_x AL(x,t)) - \ell_x B(x,t).$$

(iii) $$\frac{\partial}{\partial t}RAL(t) = \int_r^\omega \frac{\partial}{\partial t}(\ell_x AL(x,t))dx$$

$$= \int_r^\omega (\eta-\beta(t))\ \ell_x AL(x,t) - \frac{\partial}{\partial x}(\ell_x AL(x,t)) - \ell_x B(x,t)dx$$

(from (ii))

$$= (\eta-\beta(t))RAL(t) + \ell_r AL(r) - B(t).$$

Adding this to Eq. (2.2), we find

$$AL'(t) = \frac{d}{dt}(AAL + RAL(t))$$

$$= \gamma AAL + NC - \ell_r AL(r)$$

$$+(\eta-\beta(t))RAL(t) + \ell_r AL(r) - B(t),$$

which proves (2.24).

Finally, as regards contributions and fund levels,

$$\bar{F}'(t) = \eta(t)\bar{F}(t) + \bar{C}(t) - \bar{B}(t),$$

which implies

$$\frac{d}{dt}F(t) = \frac{d}{dt}(\bar{F}(t)\exp(-\int_0^t \beta(s)ds))$$

$$= \bar{F}'(t)\exp(-\int_0^t \beta(s)ds) - \beta(t)F(t)$$

(2.25) $$= \gamma(t)F(t) + C(t) - B(t).$$

2.4.2 Spread Method

2.4.2.1 Individual Cost Methods

Assume

$$C(t) \quad = \quad NC + UL(t)/\bar{a}_{\overline{m}|} . \tag{2.26}$$

From Eqs. (2.24) and (2.25)

$$UL'(t) = AL'(t) - F'(t)$$

$$= \gamma(t)UL(t) - UL(t)/\bar{a}_{\overline{m}|} - \Delta\gamma(t)AAL - \Delta\eta(t)RAL(t)$$

$$= -\alpha UL(t) + \Delta\gamma(t)UL(t) - \Delta\gamma(t)AAL - \Delta\eta(t)RAL(t), \tag{2.27}$$

where $\alpha = 1/\bar{a}_{\overline{m}|} - \gamma$.

After defining

$$\ell(t) \quad = \quad \Delta\gamma(t)UL(t) - \Delta\gamma(t)AAL - \Delta\eta(t)RAL(t), \tag{2.28}$$

Eq. (2.27) becomes

$$UL'(t) \quad = \quad -\alpha UL(t) + \ell(t). \tag{2.29}$$

Eq. (2.28) is the continuous-time equivalent of Eq. (2.9) (Section 2.3.2.1), and has a similar interpretation. $\ell(t)$ could be named the "instantaneous loss" at time t.

(i) $\Delta\gamma(t)UL(t)$ is the net return loss on UL(t) itself;

(ii) $-\Delta\gamma(t)AAL$ is the net return loss on active members' liabilities; and

(iii) $-\Delta\eta(t)RAL(t)$ is the nominal return loss on retired members' liabilities.

The difference between (ii) and (iii) can again be imputed to the fact that benefits are not indexed.

2.4.2.2 Aggregate Cost Methods

Consider the Aggregate method:

$$C(t) \quad = \quad S(PVB(t)-F(t))/PVS.$$

We infer from Prop. 1.1 that

$$(2.30) \qquad C(t) = {}^{EAN}NC + ({}^{EAN}AL(t) - F(t)) \cdot S/PVS.$$

The method is equivalent to the Entry Age Normal, when the unfunded liability is spread over m years, m being such that

$$(2.31) \qquad \bar{a}_{\overline{m}|} = PVS/S.$$

Because of this, Eqs. (2.27), (2.28) and (2.29) remain valid, with $\alpha = S/PVS - \gamma$.

<u>Remark 2.5</u>. Aggregate with New Entrants method ($n < \infty$). From Prop. 1.2 we can write

$$C(n,t) = {}^{EAN}NC + {}^{EAN}UL(t) \cdot S/PVS(n)$$

and

$$\bar{a}_{\overline{m'}|} = PVS(n)/S.$$

Thence

$$UL'(t) = -\alpha(n)UL(t) + \ell(t).$$

We see that the size of α (or $\alpha(n)$) determines how close $F(t)$ will stay from $AL(t)$. Thus the results of Section 1.4.4 are also relevant when actuarial assumptions η and β are not realized.

2.4.3 <u>Amortization of Losses Method</u>

If $C(t) = NC + ADJ(t)$, then Eqs. (2.24) and (2.25) yield

$$(2.32) \qquad UL'(t) = \gamma UL(t) - ADJ(t) + \ell(t)$$

where

$$(2.33) \qquad \ell(t) = \Delta\gamma(t)UL(t) - \Delta\gamma(t)AAL - \Delta\eta(t)RAL(t).$$

Let m be the number of years over which losses are to be amortized. Then

$$(2.34) \qquad ADJ(t) = \int_{t-m}^{t} \ell(s)/\bar{a}^{(\gamma)}_{\overline{m}|}\, ds.$$

It is not unreasonable to suspect that the continuous counterpart of Eq. (2.17) is

$$(2.35) \qquad UL(t) = \int_{t-m}^{t} \ell(s)\, \bar{a}_{\overline{m-t+s|}} / \bar{a}_{\overline{m|}}\, ds.$$

This can be verified by substituting the right hand side of Eq. (2.35) into Eq. (2.32). The details are in Appendix 2.1.

2.4.4 Response to a Single Loss

Suppose a unique loss $\ell(0) \neq 0$, and $\ell(s) = 0 \ \forall s \neq 0$. In continuous time, this is expressed as

$$\ell(s) = \ell(0)\delta(s)$$

where $\delta(\cdot)$ is the Dirac delta function.

(i) Spread method:

$$UL'(t) = -\alpha UL(t) + \ell(0)\delta(t)$$

$$\Rightarrow \quad \begin{aligned} UL(t) &= \ell(0)e^{-\alpha t} \\ C(t) &= NC + \ell(0)e^{-\alpha t}/\bar{a}_{\overline{m|}}\,. \end{aligned} \qquad t \geq 0$$

Notice the way the effect of the loss dies out exponentially, for both the fund and the contribution. Of course this assumes that $\alpha > 0$; this is the case if $\bar{a}_{\overline{m|}}$ iscalculated at rate γ, for in general

$$1/\bar{a}_{\overline{m|}} - \gamma > 0, \quad \forall \gamma \in \mathbb{R}.$$

See also Sections 1.4.2 and 1.4.3.1 concerning the α's produced by aggregate cost methods.

(ii) Amortization of Losses method: from Eqs. (2.34) and (2.35)

$$\begin{aligned} UL(t) &= \ell(0)\bar{a}_{\overline{m-t|}} / \bar{a}_{\overline{m|}} \\ ADJ(t) &= \ell(0)/\bar{a}_{\overline{m|}} \end{aligned} \qquad t < m$$

and

$$UL(t) = ADJ(t) = 0 \qquad t \geq m.$$

<u>Remark 2.6</u>. From Eqs. (2.26), (2.29), (2.32), (2.34) and (2.35), we obtain

(i) Spread method:

$$UL'(t) = -\alpha UL(t) + \ell(t)$$
$$ADJ(t) = UL(t)/\bar{a}_{\overline{m|}} .$$

(ii) Amortization of Losses method:

$$UL'(t) = -\int_{t-m}^{t} \{\exp[-\gamma(m-t+s)]/\bar{a}_{\overline{m|}}\}\ell(s)ds + \ell(t)$$
$$ADJ(t) = \int_{t-m}^{t} \ell(s)/\bar{a}_{\overline{m|}}\, ds.$$

Again we see that

(i) the Spread method amounts to a *proportional* negative feedback, and

(ii) the Amortization of Losses method is a kind of *integral* negative feedback control.

APPENDIX 2.1

PROOF OF EQ. (2.35) FOR UL(t)

Define $\ell(t)$ as in Eq. (2.33) for $t \geq 0$, and $\ell(t) = 0$ for $t < 0$. Also define

$$Z(t) = \int_{t-m}^{t} \ell(s)\, \bar{a}_{\overline{m-t+s}|} / \bar{a}_{\overline{m}|}\, ds$$

for $t > 0$, $Z(0) = UL(0)$. I first show that

$$Z'(t) = \gamma Z'(t) - ADJ(t) + \ell(t).$$

Leibniz's rule yields

$$Z'(t) = \frac{d}{dt} \int_{t-m}^{t} \ell(s)\, \bar{a}_{\overline{m-t+s}|} / \bar{a}_{\overline{m}|}\, ds$$

$$= \ell(t)\bar{a}_{\overline{m}|} / \bar{a}_{\overline{m}|} - \ell(t-m)\bar{a}_{\overline{0}|} / \bar{a}_{\overline{m}|}$$

$$+ \int_{t-m}^{t} \ell(s)\left(\frac{d}{dt}\, \bar{a}_{\overline{m-t+s}|}\right) / \bar{a}_{\overline{m}|}\, ds.$$

Now

$$\frac{d}{dt}\, \bar{a}_{\overline{m-t+s}|} = \frac{d}{dt} \int_{0}^{m-t+s} e^{-\gamma u} du$$

$$= -\exp[-\gamma(m-t+s)],$$

which implies

$$Z'(t) = \ell(t) - \int_{t-m}^{t} \ell(s) \exp[-\gamma(m-t+s)] / \bar{a}_{\overline{m}|}\, ds$$

$$= \ell(t) + \gamma \int_{t-m}^{t} \ell(s)\bar{a}_{\overline{m-t+s}|} / \bar{a}_{\overline{m}|}\, ds$$

$$- \int_{t-m}^{t} \ell(s) / \bar{a}_{\overline{m}|}\, ds$$

or

(2.36) $$Z'(t) = \ell(t) + \gamma Z(t) - ADJ(t).$$

From Eqs. (2.32) and (2.36), we get

$$(UL(t)-Z(t))' = \gamma(UL(t) - Z(t)).$$

This means $UL(t)-Z(t) = e^{\gamma t}(UL(0)-Z(0)) = 0$, i.e. $UL(t) = Z(t)\ \forall t \geq 0$.

CHAPTER 3
RANDOM RATES OF RETURN: DISCRETE TIME

3.1 INTRODUCTION

In this chapter, the first and second moments of $F(t)$ and $C(t)$ are calculated, under the assumption that rates of return are independent identically distributed (i.i.d.) random variables. (The i.i.d. hypothesis is reviewed in Section 5.2.) The two methods of adjusting the normal cost described in Chapter 2 are examined, in a discrete-time framework.

This is the central chapter of the thesis, because

(i) both methods of adjusting the normal cost are considered (while the continuous-time analysis of the next chapter is only concerned with the Spread method);

(ii) it includes the discussion of the "optimal region" (Section 3.5.3); and

(iii) partly due to (i) and (ii), and also because it is set in discrete time, this chapter is the one which comes nearest to practical actuarial problems.

As in Chapter 2, it is supposed that surpluses and deficiencies (or gains and losses) receive the same treatment when adjusting the normal cost. The other assumptions are very similar to those of Chapter 2, except that only varying rates of return are taken into account. These assumptions are briefly restated in Section 3.2, along with some new notation.

The moments of F and C are derived in 3.3, in the case of the Spread method, and in 3.4, in the case of the Amortization of Losses method.

Section 3.5 is a numerical example. 3.5.2 places the two methods side by side, and comments on the variance of $F(\infty)$ and $C(\infty)$ produced by different values of "m". In 3.5.3, an "optimal region" for m is specified, in the case of the Spread method.

The results of this chapter are original (parts of 3.3 and 3.5 are included in Dufresne (1986)).

3.2 ASSUMPTIONS AND NOTATION

(i) The population is stationary.

(ii) All acturial assumptions are consistently borne out by experience, except for investment returns.

(iii) There is *no* inflation on salaries.

Alternatively, one may imagine that benefits in payment increase at the same rate as salaries, and that only real-term variables are considered (see Section 2.3.1).

(iv) Valuation assumptions are fixed, including the rate of interest i_V.

(v) The actually earned rates of return $\{i(t)\}_{t\geq 1}$ are i.i.d. random variables, with $i(t) > -1$ w.p. 1 and $\mathrm{Var}\, i(t) = \sigma^2 < \infty$. $i(t)$ is the rate earned during $(t-1,t)$.

(vi) An "individual cost method" will mean the same thing as it did in Chapter 2. I will use the fact that under these methods

$$AL = (1+i_V)(AL+NC-B). \tag{3.1}$$

This is easily deduced from assumptions (iii) above and (xi) of Section 2.2.

(vii) $\mathrm{Prob}(F(0) = F_0) = 1$ for some $F_0 \in \mathbb{R}$.

From these assumptions,

$$F(t+1) = (1+i(t+1))(F(t)+C(t)-B). \tag{3.2}$$

Define

i_V = valuation rate of interest

$i = Ei(t)$

$d = i/(1+i)$

$d_V = i_V/(1+i_V)$

$u(t) = 1+i(t)$

$u = Eu(t) = 1+i$

$\sigma^2 = \text{Var } i(t) = \text{Var } u(t)$

and

H_t = {σ-algebra of events prior to (inclusive of) time t}.

It follows from Eq. (3.2) that $i(t)$, $F(t)$ and $C(t)$ are each H_t-measurable.

I will repeatedly use the identity

$$EX = EE(X|H),$$

which presumes H to be a sub-σ-field of the σ-field on which X is defined.

The z-transform (see Appendix 3.2) of any sequence $\{x(t)\}$ will be denoted by $\tilde{x}(z)$.

<u>Note</u>. The analysis is conducted with real-term values, as in Section 2.3. The only exceptions are Sections 3.3.3 and 3.3.4, where it is shown that, as far as the Spread method is concerned, similar results hold for nominal monetary values.

<u>Remark 3.1</u>. The "exponential" rates $e^{\gamma(t)}$ had their purpose in Chapter 2, in showing the similarity between the discrete and continuous time situations. In this chapter, however, I use

$$u(t) = e^{\gamma(t)}$$

and

$$i(t) = e^{\gamma(t)} - 1,$$

to simplify the formulae and their interpretation.

3.3 SPREAD METHOD

Consider any individual cost method (see Section 3.2) and suppose that

$$C(t) = NC + UL(t)/\ddot{a}_{\overline{m|}} \tag{3.3}$$

where $\ddot{a}_{\overline{m|}}$ is evaluated at rate i_V, and $2 \leq m < \infty$. This implies

$$\begin{aligned} F(t+1) &= u(t+1)(F(t)+C(t)-B) \\ &= u(t+1)[F(t)+NC+(AL-F(t))/\ddot{a}_{\overline{m|}} - B] \\ &= (u(t+1)/u)(qF(t)+r) \qquad (3.4) \end{aligned}$$

where

$$u(t+1) = 1 + i(t+1),$$

$$u = Eu(t+1),$$

$$q = u(1 - 1/\ddot{a}_{\overline{m|}}) \tag{3.5}$$

and

$$r = u(NC - B + AL/\ddot{a}_{\overline{m|}}). \tag{3.6}$$

Remark 3.2. Because of the i.i.d. assumption imposed on $\{i(t)\}$, Eq. (3.4) shows that $F(t)$ is a Markov process. This is also true of $C(t)$ (Eq. (3.3)).

3.3.1 First Moments

Eq. (3.4) implies

$$\begin{aligned} EF(t+1) &= EE(F(t+1)|H_t) \\ &= qEF(t) + r, \qquad (3.7) \end{aligned}$$

which in turn means that

$$EF(t) = q^t F_0 + r(1-q^t)/(1-q), \quad t \geq 0. \tag{3.8}$$

3.3.1.1 $Ei(t) = i_V$.

In this case,

$$\begin{aligned} 0 < q &= (1+i_V)(1 - 1/\ddot{a}_{\overline{m|}}) \\ &= \ddot{a}_{\overline{m-1|}} / \ddot{a}_{\overline{m|}} < 1. \qquad (3.9) \end{aligned}$$

<u>Proposition 3.1</u>. Let $m \geq 1$. If $Ei(t) = i_V$, then under the Spread method

$$\text{(3.10)} \qquad EF(\infty) = \lim_t EF(t) = AL$$

$$\text{(3.11)} \qquad EC(\infty) = \lim_t EC(t) = NC.$$

<u>Proof</u>. If $m \geq 2$, then from Eqs. (3.8) and (3.9),

$$\text{(3.12)} \qquad \lim_t EF(t) = r/(1-q).$$

This limit is AL, for Eqs. (3.1), (3.5) and (3.6) yield

$$AL = (1+i)(AL+NC-B)$$

$$\Rightarrow \qquad B-NC = dAL$$

$$\Rightarrow \qquad r/(1-q) = \frac{AL(1+i)(1/\ddot{a}_{\overline{m}|} - d)}{[1-(1+i)(1- 1/\ddot{a}_{\overline{m}|})]}$$

$$= AL.$$

Eq. (3.11) follows from (3.3).

The case $m = 1$ is dealt with in Remark 3.3 below. □

3.3.1.2 $Ei(t) \neq i_V$.

If i_V differs from $Ei(t)$, then Prop. 3.1 does not hold. All that can be said is that if

$$q = (1+Ei(t))(1-1/\ddot{a}_{\overline{m}|}) < 1$$

then

$$EF(\infty) = r/(1-q)$$

and

$$EC(\infty) = NC + (AL-EF(\infty))/\ddot{a}_{\overline{m}|}.$$

<u>Remark 3.3</u>. When $m = 1$,

$$C(t) = NC + UL(t),$$

$$F(t+1) = \dot{u}(t+1)(AL+NC-B)$$

$$= u(t+1) \cdot AL/(1+i_V),$$

which imply

$$EF(t+1) = [u/(1+i_V)]\cdot AL \qquad t \geq 0$$

$$EC(t) = NC + AL - EF(t).$$

If $Ei(t) = i_V$, then $EF(t) = AL$, and $EC(t) = NC$ for all $t \geq 1$.

3.3.2 Second Moments

I will use Eq. (3.4) and the identity (proved in Appendix 3.1)

(3.13) $$VarF(t+1) = EVar(F(t+1)|H_t) + VarE(F(t+1)|H_t)$$

to show that

(3.14) $$VarF(t+1) = k\cdot VarF(t) + s(EF(t+1))^2$$

where $k = q^2(1+\sigma^2u^{-2})$ and $s = \sigma^2u^{-2}$. Firstly,

$$Var(F(t+1)|H_t) = Var(u(t+1)/u)\cdot(qF(t)+r)^2$$
$$= s(qF(t)+r)^2$$
$$\Rightarrow \quad EVar(F(t+1)|H_t) = sE(qF(t)+r)^2$$
$$= sE[q(F(t)-EF(t))+qEF(t)+r]^2$$

(3.15) $$= sq^2VarF(t)+s(EF(t+1))^2$$

from Eq. (3.7).

$$VarE(F(t+1)|H_t) = Var(qF(t)+r)$$

(3.16) $$= q^2VarF(t).$$

Finally, Eq. (3.14) is the sum of Eqs. (3.15) and (3.16).

Denote $M(t) = EF(t)$, and $V(t) = VarF(t)$. Eq. (3.14) becomes

(3.17) $$V(t+1) = kV(t) + sM(t+1)^2.$$

Assumption (vii) (Section 3.2) says that $VarF(0) = 0$, and so

$$V(1) = sM(1)^2,$$

$$V(2) = ksM(1)^2 + sM(2)^2$$

$$\ldots$$

(3.18) $$V(t) = s \sum_{j=1}^{t} k^{t-j} M(j)^2, \quad t \geq 1.$$

Prop. 3.2 identifies the limits of VarF(t) and VarC(t), as $t \to \infty$, and Prop. 3.3 is about covariances.

Proposition 3.2. Let $2 \leq m \leq \infty$. If $k = q^2(1+\sigma^2 u^{-2}) < 1$, then

(3.19) $$\mathrm{VarF}(\infty) = \sigma^2 u^{-2}(EF(\infty))^2/(1-k),$$

(3.20) $$\mathrm{VarC}(\infty) = [\mathrm{VarF}(\infty)]/(\ddot{a}_{\overline{m|}})^2.$$

If $k \geq 1$, then both $\mathrm{VarF}(\infty)$ and $\mathrm{VarC}(\infty)$ are infinite.

Proof. First note the following properties of limits inferior and superior. Say $f(t), g(t) \geq 0$. Then

$$\liminf_t (f+g)(t) \geq \liminf_t f(t) + \liminf_t g(t)$$

and

$$\limsup_t (f+g)(t) \leq \limsup_t f(t) + \limsup_t g(t).$$

$k < 1$ implies $q < 1$, and thereby $M(\infty)^2 < \infty$. Then, from Eq (3.18),

$$V(t) \leq \sum_{j=1}^{t} k^{t-j}(\sup_t M(t)^2)$$

$$\leq (1-k)^{-1} \sup_t M(t)^2 < \infty,$$

which means that $\sup_t V(t) < \infty$.

Hence we may take limits inferior on both sides of Eq. (3.17) to obtain

$$\liminf_t V(t) \geq sM(\infty)^2/(1-k).$$

Taking limits superior, we also find

$$\limsup_t V(t) \le sM(\infty)^2/(1-k).$$

These prove (3.19). Eq. (3.20) is a consequence of Eq. (3.3).

<u>$k \ge 1$</u>: $M(\infty)$ can never be 0. This is because

$$\begin{aligned} r &= u(NC-B+AL/\ddot{a}_{\overline{m|}}) \\ &= uAL(1/\ddot{a}_{\overline{m|}} - d_V) > 0 \end{aligned}$$

(cf. Eqs. (3.7) and (3.8)). If $k \ge 1$, Eq. (3.18) then implies $\liminf_t V(t) = \infty$. □

<u>Proposition 3.3</u>. Let $h \ge 0$.

$$\begin{aligned} \mathrm{Cov}(F(t),F(t+h)) &= q^h \mathrm{Var}F(t) \\ \mathrm{Cov}(C(t),C(t+h)) &= q^h \mathrm{Var}C(t) \\ \mathrm{Cov}(F(t),C(t+h)) &= -q^h[\mathrm{Var}F(t)]/\ddot{a}_{\overline{m|}}. \end{aligned} \tag{3.21}$$

If $k < 1$, then, as $t \to \infty$,

$$\begin{aligned} \text{Correlation } (F(t),F(t+h)) &\to q^h \\ \text{Correlation } (C(t),C(t+h)) &\to q^h \\ \text{Correlation } (F(t),C(t+h)) &\to -q^h \end{aligned} \tag{3.22}$$

<u>Proof</u>. Define $F^*(t) = F(t) - EF(t)$. Then (Eqs. (3.4) and (3.7)),

$$\begin{aligned} & F^*(t+1) = (u(t+1)/u)(qF(t)+r) - qEF(t) - r \\ \Rightarrow\quad & E(F^*(t+1)|H_t) = qF^*(t) \\ \Rightarrow\quad & \mathrm{Cov}(F(t+1),F(t)) = EqF^*(t)^2 \\ & \qquad = q\mathrm{Var}F(t). \end{aligned}$$

Hence, if $j \ge 0$,

$$\begin{aligned} & E(F^*(t+j+1)|H_{t+j}) = qF^*(t+j) \\ \Rightarrow\quad & \mathrm{Cov}(F(t+j+1),F(t)) = EF^*(t+j+1)F^*(t) \\ & \qquad = EE(F^*(t+j+1)F^*(t)|H_{t+j}) \end{aligned}$$

$$= EqF^*(t+j)F^*(t)$$

$$= qCov(F(t+j),F(t)).$$

This implies

$$Cov(F(t+h),F(t)) = q^h VarF(t), \quad h \geq 0.$$

The other covariances easily follow.

As to correlation coefficients, for example,

$$\text{Correlation } (F(t+h),F(t))$$

$$= \; Cov(F(t+h),F(t))/[VarF(t+h)\cdot VarF(t)]^{1/2}$$

$$\to \quad q^h, \quad \text{if} \quad k < 1. \; \square$$

Remark 3.4. The case $m=1$ (cf. Remark 3.3). From $F(t+1) = u(t+1)AL/(1+i_V)$, we get

$$VarF(t) = VarC(t) = \sigma^2 AL^2/(1+i_V)^2$$

and

$$Cov(F(t),F(t+h)) = Cov(C(t),C(t+h))$$

$$= 0, \quad \text{for all} \quad h \neq 0.$$

3.3.3 Non-Stationary Population

Recursive relationships similar to Eqs. (3.7) and (3.14) also apply when some of the assumptions are discarded. Suppose now that the population is no longer stationary, that salaries grow with inflation - constant or not, but *not* random - and that the valuation interest rate i_V is not necessarily equal to $i = Ei(t)$. From

$$F(t+1) = u(t+1)[F(t)+NC(t)-B(t)+(AL(t)-F(t))/\ddot{a}_{\overline{m}|}]$$

we easily get

$$EF(t+1) = qEF(t)+r(t)$$

and

$$VarF(t+1) = kVarF(t) + s(EF(t+1))^2,$$

where q,k and s are defined as before, and

$$r(t) = u(NC(t)-B(t)+AL(t)/\ddot{a}_{\overline{m}|}).$$

3.3.4 Aggregate Cost Methods

Given the same setting as in 3.3.3, if

$$C(t) = S(t)(PVB(t)-F(t))/PVS(t),$$

then

$$EF(t+1) = q(t)EF(t) + r(t)$$

and

$$VarF(t+1) = k(t)VarF(t) + s(EF(t+1))^2,$$

where

$$q(t) = u(1-S(t)/PVS(t)),$$

$$r(t) = u(S(t)PVB(t)/PVS(t)-B(t)),$$

$$k(t) = q(t)^2(1+\sigma^2u^{-2}).$$

Now suppose a single entry age, and reinstate the assumptions of Section 3.2. From Eq. (2.10),

$$C(t) = {}^{EAN}NC + {}^{EAN}UL(t)/\ddot{a}_{\overline{m|}}$$

where $\ddot{a}_{\overline{m|}} = PVS/S$. We see that Propositions 3.1 and 3.2 once more apply:

- if $i_V = Ei(t)$, then

$$\lim_t EF(t) = {}^{EAN}AL,$$

$$\lim_t EC(t) = {}^{EAN}NC;$$

- if $k = u^2(1-S/PVS)^2(1+\sigma^2u^{-2}) < 1$, then

$$\lim_t VarF(t) = sEF(\infty)^2/(1-k).$$

$$\lim_t VarC(t) = [VarF(\infty)]S^2/PVS^2.$$

Note: The results of this section hold for either the Aggregate or Aggregate with New Entrants methods; see Section 2.3.2.2.

3.4 AMORTIZATION OF LOSSES METHOD

Recall that in this case (see Eqs. (2.14) and (2.17) of Section 2.3.3)

$$(3.23)\qquad C(t) = NC + ADJ(t)$$

$$(3.24)\qquad ADJ(t) = \sum_{j=0}^{m-1} \ell(t-j)/\ddot{a}_{\overline{m}|}$$

$$(3.25)\qquad UL(t) = \sum_{j=0}^{m-1} \ell(t-j)\ddot{a}_{\overline{m-j}|}/\ddot{a}_{\overline{m}|}.$$

Now turn to Eq. (2.13) of the same section. Taking into account the assumptions and different notation of Section 3.2, this equation is first rewritten as

$$\begin{aligned}\ell(t+1) &= (i(t+1)-i_V)(UL(t)-ADJ(t)) \\ &\quad +(i_V-i(t+1))(AAL+NC+RAL-B) \\ &= (i(t+1)-i_V)[UL(t)-ADJ(t)-(1+i_V)^{-1}AL],\end{aligned}$$

since

$$\begin{aligned}AAL+NC+RAL-B &= AL+NC-B \\ &= (1+i_V)^{-1}AL.\end{aligned}$$

Next, subtract (3.24) from (3.25) to derive

$$UL(t)-ADJ(t) = \sum_{j=0}^{m-2} e(j)\ell(t-j)$$

where

$$\begin{aligned}e(j) &= (\ddot{a}_{\overline{m-j}|}-1)/\ddot{a}_{\overline{m}|} \\ (3.26)\qquad &= a_{\overline{m-1-j}|}/\ddot{a}_{\overline{m}|},\ \emptyset \le j \le m-2.\end{aligned}$$

Finally, Eq. (2.13) is restated as

$$(3.27)\qquad \ell(t+1) = (i(t+1)-i_V)(\sum_{j=0}^{m-2} e(j)\ell(t-j)-A)$$

in which $A = (1+i_V)^{-1}AL$. Also define

$$\Delta i = Ei(t)-i_V.$$

Throughout the rest of Section 3.4, the following strategy will be adopted:

(i) calculate the moments of the $\ell(t)$'s first, using Eq. (3.27); and then

(ii) use Eqs. (3.24) and (3.25), together with the results of (i), to calculate the moments of $C(t)$ and $F(t)$.

The last comment is about "initial conditions": Eq. (3.27) shows that $\ell(1)$ depends on $\ell(0)$, $\ell(-1), \ldots, \ell(-m+2)$. These are the initial conditions. In the sequel they are presumed known and non-random.

<u>Remark 3.5</u>. It will be assumed that $2 \le m < \infty$. When $m = 1$, the Amortization of Losses and Spread methods are indistinguishable; see Remarks 3.3 and 3.4.

<u>Remark 3.6</u>. Eq. (3.27) clearly shows that neither $F(t)$ nor $C(t)$ is a Markov process, for $\ell(t)$ explicitly depends on "the past", i.e. on $\ell(t-1), \ell(t-2), \ldots, \ell(t-m+1)$.

3.4.1 <u>First Moments</u>

3.4.1.1 $Ei(t) = i_v$.

Let $t \ge 0$. From Eq. (3.27),

$$E\ell(t+1) = EE(\ell(t+1) | H_t)$$

$$= E(i(t+1)-i_v)E\left(\sum_{j=0}^{m-2} e(j)\ell(t-j)-A\right)$$

$$= 0. \qquad (3.28)$$

This makes sense: if the valuation rate of interest is correct "on average", then on average $\ell(t)$ is 0.

Consequently,

$$\left.\begin{aligned} EC(t) &= NC + \sum_{j=t}^{m-2} \ell(t-j)/\ddot{a}_{\overline{m|}} \\ EF(t) &= AL - \sum_{j=t}^{m-2} \ell(t-j)\ddot{a}_{\overline{m-j|}} / \ddot{a}_{\overline{m|}} \end{aligned}\right\} \quad t \le m-2$$

$$\left.\begin{aligned} EC(t) &= NC \\ EF(t) &= AL. \end{aligned}\right\} \quad t > m-2$$

The initial conditions have an effect on the first moments of $C(t)$ and $F(t)$ when $t \le m-2$, and none afterwards.

Matters are slightly more convoluted when $Ei(t) \neq i_V$.

3.4.1.2 $Ei(t) \neq i_V$.

From Eq. (3.27).

$$E(\ell(t+1)|H_t) = \Delta i\Big(\sum_{j=0}^{m-2} e(j)\ell(t-j) - A\Big)$$

$$\Rightarrow \quad E\ell(t+1) = EE(\ell(t+1)|H_t)$$

(3.29)
$$= \Delta i\Big(\sum_{j=0}^{m-2} e(j)E\ell(t-j) - A\Big).$$

Define $M(t) = E\ell(t)$, and

$$\check{M}(z) = \sum_{t\geq 0} z^{-t}M(t).$$

(z-transforms are briefly explained in Appendix 3.2.)

Eq. (3.29) implies

(3.30)
$$\sum_{t\geq 0} z^{-t}M(t+1)$$

$$= \Delta i \sum_{t\geq 0} z^{-t} \sum_{j\geq 0} e(j)M(t-j) - \Delta i\cdot A\cdot \sum_{t\geq 0} z^{-t}.$$

The left hand side of this equation is $z\check{M}(z)-zM(0)$, while on the right hand side we have

$$\sum_{j\geq 0} e(j)z^{-j} \sum_{t\geq 0} z^{-t+j}M(t-j)$$

(3.31)
$$= \sum_{j\geq 0} e(j)z^{-j} \sum_{s\geq 0} z^{-s}M(s) + \sum_{j\geq 0} e(j)z^{-j} \sum_{s=-j}^{-1} z^{-s}\ell(s)$$

$$= \tilde{e}(z)\check{M}(z) + \check{Q}(z), \text{ say.}$$

$\check{Q}(z)$ reflects the initial conditions. Since $\sum_{t\geq 0} z^{-t} = 1/(1-z^{-1})$ Eq. (3.30) becomes

$$z\check{M}(z) - z\ell(0) = \Delta i\tilde{e}(z)\check{M}(z) + \Delta i\check{Q}(z) - \Delta iA/(1-z^{-1})$$

(3.32)
$$\Rightarrow \quad \check{M}(z) = (\ell(0)+\Delta iz^{-1}\check{Q}(z))/(1-\Delta iz^{-1}\tilde{e}(z))$$
$$-\Delta iAz^{-1}(1-z^{-1})^{-1}/(1-\Delta iz^{-1}\tilde{e}(z)).$$

From $\tilde{M}(z)$, I shall now find an expression for $M(t)$, taking for granted that initial conditions are nil ($\ell(s) = 0$, $-m+2 \le s \le 0$). First,

$$(1-\Delta i z^{-1}\tilde{e}(z))^{-1} = 1+\Delta i z^{-1}\tilde{e}(z)+\Delta i^2 z^{-2}\tilde{e}(z)^2+\ldots$$

corresponds to

$$1_{\{t=0\}} + \Delta i e(t-1) + \Delta i^2 e^{(2)}(t-2) + \ldots$$

where $e^{(n)}$ is the n-th convolution of $e(\cdot)$, i.e.

$$e^{(n+1)}(t) = \sum_{j\ge 0} e(j)e^{(n)}(t-j), \quad e^{(1)}(t) = e(t).$$

The first term on the right hand side of (3.32) vanishes and thus (using basic properties of z-transforms, see Appendix 3.2)

$$E\ell(t) = M(t)$$

$$(3.33) \qquad = -\Delta i A[1_{\{t\ge 1\}} + \Delta i \sum_{j=0}^{t-2} e(j) + \Delta i^2 \sum_{j=0}^{t-3} e^{(2)}(j)$$

$$+\ldots+ \Delta i^{t-1} \sum_{j=0}^{0} e^{(t-1)}(j)].$$

Once $E\ell(s)$, $0 \le s \le t$, have been calculated - either from Eq. (3.29) or from Eq. (3.33) - the expectations of $F(t)$ and $C(t)$ are found from Eqs. (3.24) and (3.25):

$$EF(t) = AL - EUL(t)$$

$$(3.34) \qquad = AL - \sum_{j=0}^{m-1} E\ell(t-j)\, \ddot{a}_{\overline{m-j|}} / \ddot{a}_{\overline{m|}},$$

$$EC(t) = NC + EADJ(t)$$

$$(3.35) \qquad = NC + \sum_{j=0}^{m-1} E\ell(t-j)/\ddot{a}_{\overline{m|}}.$$

Proposition 3.4 tells what $EF(t)$ and $EC(t)$ become in the limit (the proof is in Appendix 3.3).

<u>Proposition 3.4</u>. If $|\Delta i| < (\Sigma e(j))^{-1}$, then $E\ell(t)$, $EC(t)$ and $EF(t)$ have finite limits as $t \to \infty$, and

(3.36) $$E\ell(\infty) = -\Delta i(1+i_V)^{-1}AL/(1-\Delta i\Sigma e(j))$$

(3.37) $$EC(\infty) = NC + m\ddot{a}_{\overline{m|}}^{-1}E\ell(\infty)$$

(3.38) $$EF(\infty) = AL - \left(\sum_{j=0}^{m-1} \ddot{a}_{\overline{m-j|}}/\ddot{a}_{\overline{m|}}\right)E\ell(\infty)$$

(3.39) $$= AL[1 + \Delta i(1+i_V)^{-1}]/(1-\Delta i\Sigma e(j)).$$

3.4.2 <u>Second Moments</u>

Only the case $\Delta i = 0$ (i.e. $Ei(t) = i_V$) is considered. Two facts are essential:

(i) $E\ell(t) = 0 \quad \forall t \geq 1$ (Section 3.4.1.1).

(ii) $\{\ell(t)\}_{t\geq 1}$ is an uncorrelated sequence, for, if $1 \leq s \leq t$,

$$\begin{aligned} Cov(\ell(s),\ell(t+1)) &= E\ell(s)\ell(t+1) \\ &= EE(\ell(s)\ell(t+1)|H_t) \\ &= E(i(t+1)-i_V)E\ell(s)\left(\sum_{j=0}^{m-2} e(j)\ell(t-j)-A\right) \\ &\qquad \text{(see Eq. (3.27))} \\ &= 0. \end{aligned}$$

(<u>Note</u>: though the $\ell(t)$'s are uncorrelated, they are certainly *not* independent.)

<u>Remark 3.7</u>. It may now be explained why the condition $\Delta i = 0$ is imposed here, while it wasn't needed in the case of the Spread method (Section 3.3.2).

When $\Delta i \neq 0$, properties (i) and (ii) above hold no more. Therefore, all the simplifications brought about by the fact that the $\ell(t)$'s are uncorrelated (see below) are not permitted.

Accordingly, calculating variances becomes a much more arduous task - or so it seems at this point in time. The partial results so far obtained do not merit inclusion here.

From Eq. (3.27),

$$\begin{aligned} \mathrm{Var}\ell(t+1) &= EE(\ell(t+1)^2 | H_t) \\ &= \sigma^2 E\Big(\sum_{j=0}^{m-1} e(j)\ell(t-j) - A\Big)^2 \\ &= \sigma^2 E\Big[\sum_{j=0}^{m-2\wedge t-1} e(j)\ell(t-j) \\ &\quad + \Big(\sum_{j=t}^{m-2} e(j)\ell(t-j) - A\Big)\Big]^2 \\ &= \sigma^2 \sum_{j=0}^{m-2\wedge t-1} e(j)^2 \mathrm{Var}\ell(t-j) \\ &\quad + \sigma^2\Big(\sum_{j=t}^{m-2} e(j)\ell(t-j) - A\Big)^2. \end{aligned}$$

I used the fact that

$$A(t) = \sum_{j=t}^{m-2} e(j)\ell(t-j) - A$$

is not random (it only depends on the initial conditions $\ell(s)$, $s \le 0$). Thus

$$\mathrm{Var}\ell(t+1) = \sigma^2 \sum_{j=0}^{m-2} e(j)^2 \mathrm{Var}\ell(t-j) + \sigma^2 A(t)^2. \tag{3.40}$$

Define $V(t) = \mathrm{Var}\ell(t)$, and take z-transforms on both sides of Eq. (3.40) to get

$$z\breve{V}(z) = \sigma^2 \breve{e}_2(z)\breve{V}(z) + \sigma^2 \breve{A}_2(z)$$

or

$$\breve{V}(z) = \sigma^2 z^{-1} \breve{A}_2(z) / (1 - \sigma^2 z^{-1} \breve{e}_2(z)) \tag{3.41}$$

where

$$\breve{e}_2(z) = \sum_{j \ge 0} e(j)^2 z^{-j}$$

and

$$\tilde{A}_2(z) = \sum_{j\geq 0} A(j)^2 z^{-j}.$$

Using Eq. (3.41) it is possible to write down a somewhat explicit expression for Var $\ell(t)$:

$$\tilde{V}(z) = \sigma^2 z^{-1}\tilde{A}_2(z)(1+\sigma^2 z^{-1}\tilde{e}_2(z)+\sigma^4 z^{-2}\tilde{e}_2(z)^2+\ldots)$$
$$= \sigma^2 z^{-1}\tilde{A}_2(z)+\sigma^4 z^{-2}\tilde{A}_2(z)\tilde{e}_2(z)+\sigma^6 z^{-3}\tilde{A}_2(z)\tilde{e}_2(z)^2+\ldots$$

which means that

$$V(t) = \sigma^2 A(t-1)^2 + \sigma^4 \sum_{j=0}^{t-2} e(j)^2 A(t-2-j)^2 \tag{3.42}$$
$$+ \sigma^6 \sum_{j=0}^{t-3} e_2^{(2)}(j)A(t-3-j) + \ldots +$$
$$+ \sigma^{2t} \sum_{j=0}^{0} e_2^{(t-1)}(j)A(-j)^2.$$

($e_2^{(n)}$ is the n-th convolution of $e(\cdot)^2$.)

VarF(t) and VarC(t) can be expressed in terms of Var$\ell(s)$, $t-m+1 \leq s \leq t$. From Eqs. (3.23), (3.24) and (3.25), keeping in mind that the $\ell(s)$'s are uncorrelated.

$$\mathrm{VarC}(t) = \ddot{a}_{\overline{m}|}^{-2} \sum_{j=0}^{m-1} \mathrm{Var}\ell(t-j) \tag{3.43}$$

$$\mathrm{VarF}(t) = \ddot{a}_{\overline{m}|}^{-2} \sum_{j=0}^{m-1} \ddot{a}_{\overline{m-j}|}^{2}\, \mathrm{Var}\ell(t-j). \tag{3.44}$$

The next proposition is concerned with the limits of VarC(t) and VarF(t), when $t \to \infty$. Its proof is in Appendix 3.3.

<u>Proposition 3.5.</u> Assume $Ei(t) = i_V$. If

$$\sigma^2 < (\Sigma\, e(j)^2)^{-1},$$

then Var$\ell(t)$, VarC(t) and VarF(t) have the following limits as $t \to \infty$:

$$(3.45)\qquad \mathrm{Var}\ell(\infty) = \sigma^2(1+i_V)^{-2}AL^2/(1-\sigma^2\Sigma e(j)^2)$$

$$(3.46)\qquad \mathrm{Var}C(\infty) = \ddot{a}_{\overline{m}|}^{-2}\, m\mathrm{Var}\ell(\infty)$$

$$(3.47)\qquad \mathrm{Var}F(\infty) = \ddot{a}_{\overline{m}|}^{-2} \sum_{j=0}^{m-1} (\ddot{a}_{\overline{m-j}|})^2\, \mathrm{Var}\ell(\infty).$$

If

$$\sigma^2 \geq (\Sigma\, e(j)^2)^{-1},$$

then all those limits are equal to ∞.

Covariances can also be calculated:

Proposition 3.6. Assume $Ei(t) = i_V$, and let $0 \leq h < m$

$$\mathrm{Cov}(F(t),F(t+h)) = \mathrm{Cov}(UL(t),\ UL(t+h))$$
$$= \ddot{a}_{\overline{m}|}^{-2} \sum_{j=0}^{m-h-1} \mathrm{Var}\ell(t-j)\ddot{a}_{\overline{m-j}|}\, \ddot{a}_{\overline{m-h-j}|}$$

and

$$\mathrm{Cov}(C(t),C(t+h)) = \ddot{a}_{\overline{m}|}^{-2} \sum_{j=0}^{m-h-1} \mathrm{Var}\ell(t-j).$$

If $\sigma^2 < (\Sigma\, e(j)^2)^{-1}$, then

$$\lim_t \text{Correlation } (F(t),F(t+h)) = \frac{\sum_{j=0}^{m-h-1} \ddot{a}_{\overline{m-j}|}\, \ddot{a}_{\overline{m-h-j}|}}{\sum_{j=0}^{m-1} \ddot{a}_{\overline{m-j}|}^2}$$

and

$$\lim_t \text{Correlation } (C(t),C(t+h)) = 1-h/m.$$

All covariances and correlations vanish when $h \geq m$.

The contrast with the Spread method is striking (see Prop. 3.3). On the one hand, under the Spread method, $(F(t),C(t))$ and $(F(t+h),C(t+h))$ are correlated for any h. In the limit, the correlation is q^h, and q may be quite high (e.g. if $Ei(t) = i_V$, then $q = \ddot{a}_{\overline{m-1}|}/\ddot{a}_{\overline{m}|}$).

On the other hand, under the Amortization of Losses method, the correlation between $(F(t),C(t))$ and $(F(t+h),C(t+h))$ diminishes rapidly as h increases, and vanishes for any $h \geq m$.

3.5 COMPARISON OF METHODS: NUMERICAL EXAMPLE

The purpose of this section is to illustrate and complement the results of 3.3 and 3.4. In 3.5.2 the two methods of adjusting the normal cost are compared, using as criteria the variances of F and C. The "trade-off" observed between VarF and VarC is further analyzed in 3.5.3.

3.5.1 Assumptions

The illustrations of Sections 3.5.2 and 3.5.3 are based on the following assumptions.

Population	English Life Table No. 13 (males), stationary
Entry Age	30 (only)
Retirement Age	65
No salary scale, no inflation on salaries	
Benefits	Straight life annuity (2/3 of salary)
Funding methods	Entry Age Normal and Aggregate
Valuation interest rate	$i_V = .01$
Actuarial liability	EAN_{AL} = 451% of payroll[1]
Normal cost	EAN_{NC} = 14.5% of payroll[1]
Actual rates of return	$\{i(t)\}_{t\geq 1}$, i.i.d., with $Ei(t) = i_V = .01$.

[1] The calculations are in Appendix 3.6.

Because $Ei(t) = i_v$,

$$\lim_{t} EF(t) = EAN_{AL}$$

$$\lim_{t} EC(t) = EAN_{NC}$$

in all cases. (Including the Aggregate method - see Section 3.3.4. Given this particular population and interest rate, the value of m satisfying

$$\ddot{a}_{\overline{m}|} = PVS/S$$

is slightly less than 17.)

The tables and figure show the "relative standard deviations"

$$\lim_{t\to\infty} [VarF(t)]^{1/2}/EF(t)$$

and

$$\lim_{t\to\infty} [VarC(t)]^{1/2}/EC(t).$$

That is to say, the standard deviations of $F(\infty)$ and $C(\infty)$ are expressed as percentages of their respective expected values.

3.5.2 The Trade-off Between VarF and VarC

Table 3.1 contains the results produced by the Spread method, and Table 3.2 those produced by the Amortization of Losses method. The standard deviation of the earned rates of return, σ, takes the values 2.5%, 5% and 10%.

Comments:

1. Comparing the figures resulting from identical values of m, we see that
   (i) under the Amortization of Losses method, greater emphasis is laid on security of benefits (i.e. VarF is smaller) than under the Spread method.
   (ii) however, contributions have a smaller variance under the Spread method and are thus more "stable" than with the other method.

2. It is seen that, for $\sigma \leq 10\%$, the standard deviations of $F(\infty)$ and $C(\infty)$ are nearly linear in σ. This linearity gradually disappears, though, as σ and m become larger.

3. Within the range of σ and m chosen, no single value of m is "better" than the others. As m is varied, there is a trade-off between VarF and VarC, e.g. increasing m reduces VarC, but increases VarF.

4. This trade-off is a direct outcome of Eqs. (3.19), (3.20), (3.46) and (3.47) - see Props. 3.2 and 3.5. However, the following asymptotic formulas give a more intuitive understanding of the way VarF and VarC vary with m. They are valid when $i=0$ and $\sigma^2 m \to 0$ as $m \to \infty$. See Appendix 3.4 for their derivation.

   Spread method:

$$\mathrm{VarF}(\infty) \sim \sigma^2 \frac{m}{2} AL^2, \tag{3.48}$$

$$\mathrm{VarC}(\infty) \sim \sigma^2 \frac{1}{2m} AL^2. \tag{3.49}$$

   Amortization of Losses method:

$$\mathrm{VarF}(\infty) \sim \sigma^2 \frac{m}{3} AL^2, \tag{3.50}$$

$$\mathrm{VarC}(\infty) \sim \sigma^2 \frac{1}{m} AL^2. \tag{3.51}$$

In words: when i is close to 0, the standard deviation of F (resp. of C) is roughly proportional to $\sqrt{m}$ (resp. to $1/\sqrt{m}$). For instance, in Tables 3.1 and 3.2, moving from m=5 to m=20 approximately doubles the standard deviation of $F(\infty)$, and halves the standard deviation of $C(\infty)$.

m	$\sigma = .025$		$\sigma = .05$		$\sigma = .10$	
	$[\mathrm{Var}\ F(\infty)]^{\frac{1}{2}}_{AL}$	$[\mathrm{Var}\ C(\infty)]^{\frac{1}{2}}_{NC}$	$[\mathrm{Var}\ F(\infty)]^{\frac{1}{2}}_{AL}$	$[\mathrm{Var}\ C(\infty)]^{\frac{1}{2}}_{NC}$	$[\mathrm{Var}\ F(\infty)]^{\frac{1}{2}}_{AL}$	$[\mathrm{Var}\ C(\infty)]^{\frac{1}{2}}_{NC}$
1	2.5 %	77.0 %	5.0 %	154.0 %	9.9 %	307.8 %
5	4.2	26.4	8.3	52.9	16.8	106.5
10	5.8	18.9	11.7	37.9	23.7	77.1
20	8.3	14.2	16.8	28.7	35.0	59.8
40	12.4	11.6	25.3	23.8	56.2	52.6
Aggregate ($m \doteq 17$)	7.6	15.2	15.3	30.6	31.6	63.2

TABLE 3.1 Relative Standard Deviations of $F(\infty)$ and $C(\infty)$ under the Spread Method ($Ei(t) = .01$, $\sigma = [\mathrm{Var}\ i(t)]^{\frac{1}{2}}$)

m	$\sigma = .025$		$\sigma = .05$		$\sigma = .10$	
	$\frac{[Var\ F(\infty)]^{\frac{1}{2}}}{AL}$	$\frac{[Var\ C(\infty)]^{\frac{1}{2}}}{NC}$	$\frac{[Var\ F(\infty)]^{\frac{1}{2}}}{AL}$	$\frac{[Var\ C(\infty)]^{\frac{1}{2}}}{NC}$	$\frac{[Var\ F(\infty)]^{\frac{1}{2}}}{AL}$	$\frac{[Var\ C(\infty)]^{\frac{1}{2}}}{NC}$
1	2.5 %	77.0 %	5.0 %	154.0 %	9.9 %	307.8 %
5	3.7	35.1	7.4	70.3	14.8	141.3
10	4.9	25.5	9.9	51.1	19.9	103.2
20	6.8	18.9	13.7	38.1	28.0	78.1
40	9.7	14.7	19.6	29.9	41.6	63.3

TABLE 3.2 Relative Standard Deviations of $F(\infty)$ and $C(\infty)$ under the Amortization of Losses Method ($Ei(t) = .01$, $\sigma = [Var\ i(t)]^{\frac{1}{2}}$)

3.5.3 The Optimal Region $1 \leq m \leq m^*$

The object of this section is to refine the observations made in the last section. Most of what follows is taken from Dufresne (1986); only the Spread method is considered. The assumptions are unchanged (Section 3.5.1).

Table 3.3 and Figures 3.1 show the standard deviations of $F(\infty)$ and $C(\infty)$ when $i = .01$ and $\sigma = .05$. But this time a wider range of m's is taken into account.

m	$\frac{\sqrt{VarF(\infty)}}{AL}$	$\frac{\sqrt{VarC(\infty)}}{NC}$
1	5.0%	154.0%
5	8.3	52.9
10	11.7	37.9
20	16.8	28.7
40	25.3	23.8
60 $(=m^*)$	33.4	22.9
80	41.9	23.5
100	51.4	25.1

TABLE 3.3

Relative standard deviations of $F(\infty)$ and $C(\infty)$
under the Spread method
$(i = .01, \sigma = .05)$.

The trade-off alluded to previously does take place, but only up to $m^* = 60$; beyond this point, augmenting m causes both VarV and VarC to increase. With a view to minimizing variances, any $m > 60$ should therefore be rejected, for clearly some $m < 60$ would reduce both VarF and VarC. For this reason, I will call the range $1 \leq m \leq m^*$ the "optimal region".

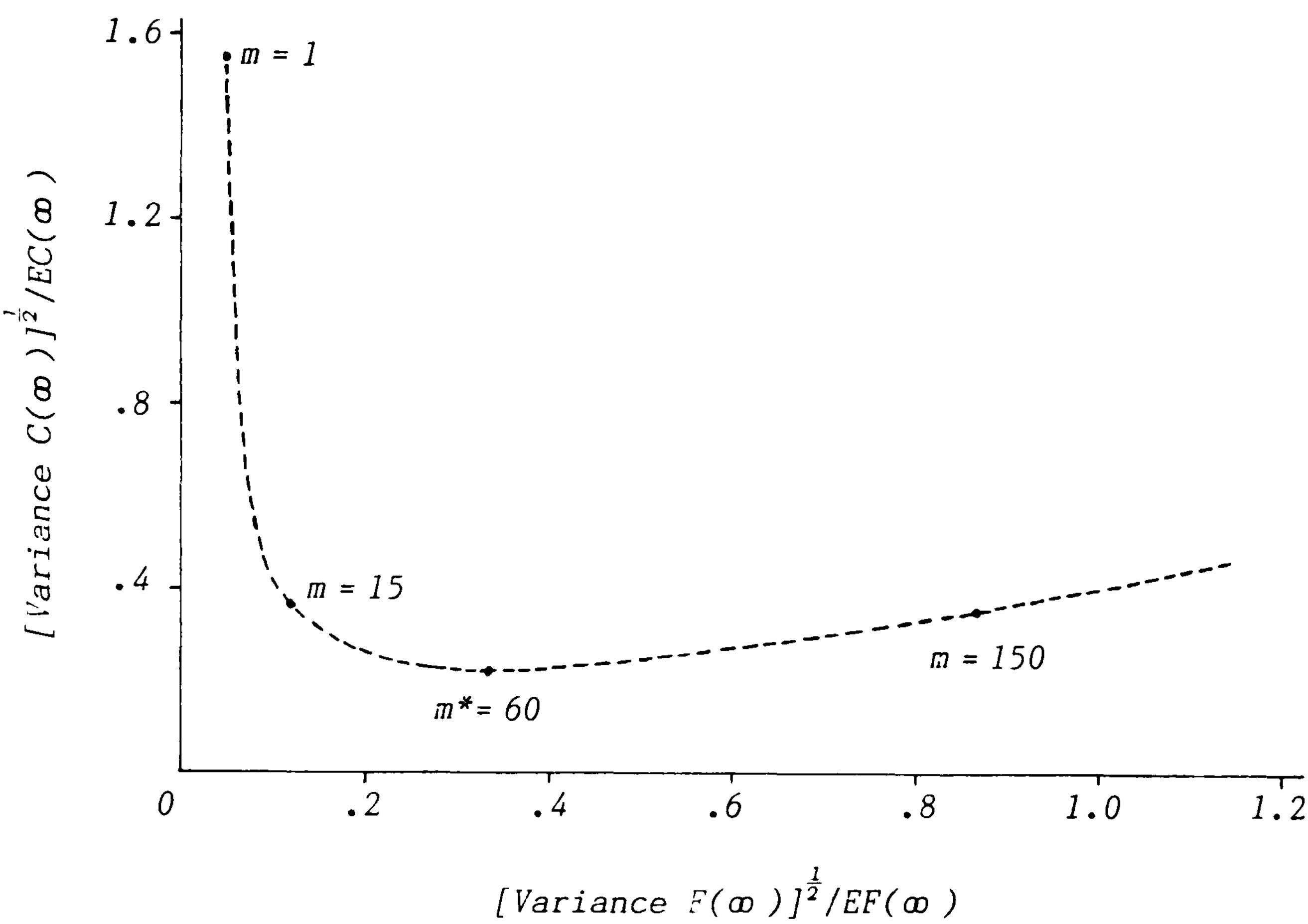

FIGURE 3.1

Relative standard deviation of $F(\infty)$ and $C(\infty)$

($i = .01$, $\sigma = .05$, cf. Table 3.3)

Proposition 3.7 ascertains under which conditions m^* exists and, when it does, gives an explicit formula for it. The proof is in Appendix 3.5.

Proposition 3.7. Assume $Ei(t) = i = i_V$, and define

$$y = (1+i)^2 + \sigma^2.$$

(i) If $y > 1$, then both $VarF(\infty)$ and $VarC(\infty)$ become infinite for some finite m, and there exists m^* such that

(1) for $m \leq m^*$ $VarF(\infty)$ increases and $VarC(\infty)$ decreases with m increasing;

(2) for $m \geq m^*$ both $VarF(\infty)$ and $VarC(\infty)$ increase with m increasing.

Moreover, if $v = (1+i_V)^{-1}$,

- when $i \neq 0$,
$$m^* = -\log[(vy-1)/(y-1)]/\log(1+i);$$
- when $i = 0$,
$$m^* = 1+ 1/\sigma^2.$$

(ii) If $y = 1$, $VarC(\infty) \to 0$ and $VarF(\infty) \to \infty$ as $m \to \infty$, although $VarF(\infty)$ stays finite for all m.

(iii) If $y < 1$, $VarC(\infty) \to 0$ and $VarF(\infty)$ has a finite limit as $m \to \infty$.

In (ii) and (iii), $VarF(\infty)$ increases and $VarC(\infty)$ decreases as m increases, for $2 \leq m < \infty$.

σ \ i	-.01	0	.01	.03	.05
.05	-	401	60	23	14
.10	-	101	42	20	13
.15	158	45	28	16	11
.20	41	26	19	13	10
.25	22	17	14	10	8

TABLE 3.4

m^* as a function of i_V and σ (nearest integer)

Table 3.4 contains numerical values of m^*, as a function of i and σ. It should be borne in mind that "i" is an average *real* rate of return, when interpreting the figures.

Remark 3.8. The example in Table 3.3 and Figure 3.1 found m^* to be 60, which has no practical consequence, since deficiencies or surpluses are not currently spread over periods of 50 years or more.

However, Prop. 3.7 may have some practical importance, if m^* turns out to be smaller. For instance, Table 3.4 tells us that if $i = .03$ and $\sigma = .20$, the optimal region shrinks to $1 \leq m \leq 13$.

APPENDIX 3.1

PROOF OF $\mathrm{Var}X = E\mathrm{Var}(X|H)+\mathrm{Var}E(X|H)$

Let H G be two σ-fields, and X a random variable defined on G. Then

$$
\begin{aligned}
\mathrm{Var}X &= EX^2 - (EX)^2 \\
&= EE(X^2|H) - [EE(X|H)]^2 \\
&= EE(X^2|H) - E(E(X|H))^2 \\
&\quad + E(E(X|H))^2 - [EE(X|H)]^2 \\
&= E\mathrm{Var}(X|H) + \mathrm{Var}E(X|H),
\end{aligned}
$$

since

$$\mathrm{Var}(X|H) = E(X^2|H) - [E(X|H)]^2.$$

APPENDIX 3.2

Z-TRANSFORMS

Z-transforms are the discrete time counterpart of Laplace transforms.

Definition. $\tilde{x}(z) = Z[x(t)] = \sum_{t \geq 0} z^{-t} x(t).$

Properties.

1. Translation $(h > 0)$

(a) $Z[x(t+h)] = z^h Z[x(t)] - \sum_{j=0}^{h-1} x(j) z^{h-j}.$

In particular,

$$Z[x(t+1)] = zZ[x(t)] - z \cdot x(0).$$

(b) If $x(t) = 0$ for all $t < 0$ then

$$Z[x(t-h)] = z^{-h} Z[x(t)].$$

2. Convolutions

Assume $x(t) = y(t) = 0$ for all $t < 0$. Define the convolution of $x(\cdot)$ and $y(\cdot)$ as

$$(x*y)(t) = \sum_{j=0}^{t} x(t-j) y(j).$$

Then

$$Z[(x*y)(t)] = Z[x(t)] \cdot Z[y(t)].$$

3. Summation

This is a special case of Property 2. If

$$x(t) = \sum_{j=0}^{t} y(t),$$

then

$$Z[x(t)] = Z[1] \cdot Z[y(t)]$$
$$= (1-z^{-1})^{-1} Z[y(t)].$$

Z-transforms are explained in greater detail in Bishop (1975), Gupta (1966) and Lifermann (1975).

Note. Here is an alternative definition of the Z-transform, in terms of the backward operator B from time series analysis. If $Bx(t) = x(t-1)$, then B^{-1}, the inverse of B, is the forward operator $B^{-1}x(t) = x(t+1)$. Thus

$$Z[x(t)] = \sum_{t \geq 0} z^{-t}B^{-t}x(0)$$

$$= (1-z^{-1}B^{-1})^{-1}x(0).$$

Upon defining the operator $\Psi(B)$ as

$$\Psi(B) = (1-z^{-1}B^{-1})^{-1}$$

we get

$$Z[x(t)] = \Psi(B)x(0).$$

APPENDIX 3.3

PROOFS OF PROPOSITIONS 3.4 AND 3.5

Lemma 3.1. Let

$$x(t) = \sum_{j=0}^{t} y^{(n)}(j) \tag{3.52}$$

where $y^{(n)}(\cdot)$ is the n-th convolution of $y(\cdot)$. Assume further that there exists $k \in \mathbb{N}$ such that

$$y(j) = 0, \quad j > k.$$

Then

$$x(t) = \Big(\sum_{j\geq 0} y(j)\Big)^n, \quad t \geq n\cdot k.$$

Proof. Taking z-transforms on both sides of (3.52), we obtain

$$\check{x}(z) = [\tilde{y}(z)]^n(1-z^{-1})^{-1}. \tag{3.53}$$

Now

$$y^{(1)}(j) = 0 \qquad\qquad j > k$$

$$y^2(j) = \sum_{i=0}^{j} y(i)y(j-i) = 0, \qquad j > 2k$$

etc., implying that $x(t)$ is constant for $t \geq nk$, which in turn means that the coefficients of $\check{x}(z)$ are identical for $t \geq n\cdot k$.

Note that in general

$$\sum_{j\geq 0} (x(j+1)-x(j))z^{-j} = (z-1)\check{x}(z) - zx(0),$$

which in the case at hand can be rewritten as

$$(z-1)\check{x}(z) = \sum_{j=0}^{nk-1} z^{-j}(x(j+1)-x(j)) + zx(0).$$

First take the limit as $z \to 1$ on the left hand side to get (from Eq. 3.53)

$$\lim_{z\to 1} (z-1)\tilde{x}(z) = \lim_{z\to 1} z\,[\tilde{y}(z)]^n$$

$$= [\tilde{y}(1)]^n$$

$$= (\sum_{j\geq 0} y(j))^n.$$

On the right hand side the limit is

$$\sum_{j=0}^{nk-1} (x(j+1)-x(j))+x(0) = x(nk). \quad \square$$

Proof of Proposition 3.4.

(i) First assume zero initial conditions and refer to Eq. (3.33).

(1) Say $0 < \Delta i < (\Sigma e(j))^{-1}$. Define

$$y(j) = \Delta i \cdot e(j).$$

Then $0 < \tilde{y}(1) = \Sigma y(j) < 1$.

To show that $E\ell(t)$ has a finite limit, it is sufficient to show that

$$g(t) = 1 + \sum_{j=0}^{t-1} y(j) + \sum_{j=0}^{t-2} y^{(2)}(j) + \ldots$$

has a finite limit. From Lemma 3.1,

$$0 \leq g(t) \leq 1+\tilde{y}(1)+\tilde{y}(1)^2+\ldots = (1-\tilde{y}(1))^{-1} < \infty.$$

Furthermore, $g(t)$ increases with t. Thus $g(t)$ does converge in $\mathbb{R}$.

(2) Say $-(\Sigma e(j))^{-1} < \Delta i < 0$. Define

$$y(j) = -\Delta i e(j).$$

Then $0 < \tilde{y}(1) = \Sigma y(j) < 1$. I will show that

$$h(t) = 1 - \sum_{j=0}^{t-1} y(j) + \sum_{j=0}^{t-2} y^{(2)}(j) - \ldots$$

has a finite limit as $t \to \infty$.

If $n > 0$,

$$|h(t+n)-h(t)| = \left| \sum_{j=t}^{t+n-1} y(j) - \sum_{j=t-1}^{t+n-2} y^{(2)}(j) + \sum_{j=t-2}^{t+n-3} y^{(3)}(j) - \ldots \right|$$

$$\leq \sum_{j=t}^{t+n-1} y(j) + \sum_{j=t-1}^{t+n-2} y^{(2)}(j) + \sum_{j=t-2}^{t+n-3} y^{(3)}(j) + \ldots$$

$$= |g(t+n)-g(t)|.$$

$\{g(t)\}_{t\geq 1}$ is a Cauchy sequence (from Step (1)). Hence $\{h(t)\}_{t\geq 0}$ is also a Cauchy sequence, and converges in $\mathbb{R}$.

(3) Steps (1) and (2) above permit the taking of limits on each side of Eq. (3.29), which yields $E\ell(\infty)$, Eq. (3.36). Eq. (3.37) then results from Eqs. (3.23) and (3.24).

As to $EF(\infty)$, Eq. (3.25) implies

$$EF(\infty) = AL\text{-}EUL(\infty) = AL - E\ell(\infty) \sum_{j=0}^{m-1} \ddot{a}_{\overline{m-j}|} / \ddot{a}_{\overline{m}|}$$

$$= AL[1-\Delta i \Sigma e(j) + \Delta i(1+i_V)^{-1} \Sigma \ddot{a}_{\overline{m-j}|} / \ddot{a}_{\overline{m}|}]/(1-\Delta_i \Sigma e(j))$$

from which Eq. (3.39) follows, since

$$(1+i_V)^{-1} \sum_{j=0}^{m-1} \ddot{a}_{\overline{m-j}|} / \ddot{a}_{\overline{m}|} - \sum_{j=0}^{m-2} a_{\overline{m-1-j}|} / \ddot{a}_{\overline{m}|}$$

$$= \ddot{a}_{\overline{m}|}^{-1} \sum_{j=0}^{m-1} (a_{\overline{m-j}|} - a_{\overline{m-1-j}|})$$

$$= \ddot{a}_{\overline{m}|}^{-1} \sum_{j=0}^{m-1} (1+i_V)^{-m+j}$$

$$= (1+i_V)^{-1}.$$

(ii) Now assume arbitrary initial conditions. The first term of Eq. (3.32)

$$\tilde{R}(z) = (\ell(0)+\Delta i z^{-1}\tilde{Q}(z))/(1-\Delta i z^{-1}\tilde{e}(z))$$
$$= \tilde{P}(z)/(1-\Delta i z^{-1}\tilde{e}(z))$$

corresponds to the effects of the initial conditions on $E\ell(t)$. I will show that if $|\Delta i| < (\Sigma e(j))^{-1}$, then $R(t) \to 0$ as $t \to \infty$.

$\tilde{P}(z)$ is a polynomial of finite order in z^{-1},

$$\tilde{P}(z) = \sum_{j=0}^{n} z^{-j}P(j).$$

Now

$$\tilde{R}(z) = \tilde{P}(z)+\tilde{P}(z)\Delta i z^{-1}\tilde{e}(z)+\tilde{P}(z)\Delta i^2 z^{-2}\tilde{e}(z)^2+\ldots$$

which implies

$$R(t) = P(t)+\Delta i(P*e)(t-1)+\Delta i^2(P*e^{(2)})(t-2)+\ldots \tag{3.54}$$

where (define $\|P\| = \sup_t |P(t)|$)

$$|(P*e^{(n)})(t)| = |\sum_j P(j)e^{(n)}(t-j)|$$
$$\le \|P\| \sum_j e^{(n)}(j)$$
$$= \|P\| (\Sigma e(j))^n$$

(from Lemma 3.1).

Reasoning as in the proof of Lemma 3.1, it can be seen that

$$(P*e^{(n)})(t) = 0, \quad \text{for all } t > (n+1)(m-2).$$

In view of expression (3.54), we infer the existence of $\{t_n\}_{n\ge 1}$ such that

$$t \ge t_n \Rightarrow |R(t)| \le \sum_{j\ge n} \|P\| |\Delta i|^j (\Sigma e(j))^j$$
$$= \|P\| |\Delta i \Sigma e(j)|^n/(1-|\Delta i|\Sigma e(j))$$
$$\to 0 \text{ as } n \to \infty. \ \square$$

Proof of Proposition 3.5.

The situation is simpler than with Prop. 3.4, because $\mathrm{Var}\ell(t)$ is always non-negative.

First assume $\sigma^2 < (\Sigma e(j)^2)^{-1}$ and consider Eq. (3.42). Denote $\|A\| = \sup_t |A(t)|$.

Lemma 3.1 implies that for any $n \geq 1$,

$$\sum_{j=0}^{t-n} e_2^{(n-1)}(j)A(t-n-j)^2 \leq \sum_{j\geq 0} e_2^{(n-1)}(j)\|A\|^2$$
$$= [\tilde{e}_2(1)]^{n-1}\|A\|^2.$$

Thus,

$$\sup_t V(t) \leq \sigma^2\|A\|^2 + \sigma^4\tilde{e}_2(1)\|A\|^2 + \sigma^6\tilde{e}_2(1)^2\|A\|^2 + \ldots$$
$$= \sigma^2\|A\|^2/(1-\sigma^2\tilde{e}_2(1)) < \infty.$$

Hence we may take limits inferior and superior on each side of Eq. (3.40), to obtain

$$\liminf_t V(t) \geq \sigma^2 A^2/(1-\sigma^2\tilde{e}_2(1))$$

and

$$\limsup_t V(t) \leq \sigma^2 A^2/(1-\sigma^2\tilde{e}_2(1)).$$

These two inequalities account for Eq. (3.45). Eqs. (3.46) and (3.47) then follow from Eqs. (3.24) and (3.25).

Now suppose $\sigma_0^2 \geq (\Sigma e(j)^2)^{-1}$.

Eq. (3.40) clearly shows that, for fixed t, $\mathrm{Var}\ell(t)$ is a strictly increasing function of σ^2. Thus

$$\liminf_t \mathrm{Var}\ell(t)\Big|_{\sigma^2=\sigma_0^2} \geq \lim_{\sigma^2 \uparrow (\Sigma e(j)^2)^{-1}} \mathrm{Var}\ell(\infty).$$

From Eq. (3.45), the right hand side is infinite.

APPENDIX 3.4

ASYMPTOTIC RELATIONSHIPS FOR $\mathrm{VarF}(\infty)$ AND $\mathrm{VarC}(\infty)$

Suppose $i_V = i = 0$ and

(3.55) $$\sigma^2 \cdot m \to 0 \quad \text{as} \quad m \to \infty.$$

(i) Spread method

From Prop (3.2) (Section 3.3.2)

(3.56) $$\mathrm{VarF}(\infty) = \sigma^2 AL^2/(1-k)$$

(3.57) $$\mathrm{VarC}(\infty) = [\mathrm{VarF}(\infty)]/m^2.$$

Notice that

$$k < 1 \iff \sigma^2 m < m(q^{-2}-1).$$

Since

$$\begin{aligned} m(q^{-2}-1) &= m[(1-1/m)^{-2}-1] \\ &= m(m^2/(m-1)^2-1) \\ &= (2m^2-m)/(m-1)^2 \\ &\to 2 \quad \text{as} \quad m \to \infty, \end{aligned}$$

condition (3.55) ensures that $k < 1$ holds for all $m > N$, for some $N < \infty$.

To prove Eq. (3.48), I need to show that

$$\mathrm{VarF}(\infty)/[\sigma^2 mAL^2/2] \to 1$$

as $m \to \infty$. From Eq. (3.56), this is equivalent to showing that $m(1-k) \to 2$. We have

$$\begin{aligned} m(1-k) &= m[1-(1-1/m)^2] - \sigma^2 mq^2 \\ &= 2 - 1/m - \sigma^2 mq^2 \\ &\to 2 \quad \text{as} \quad m \to \infty \end{aligned}$$

from (3.55)

Formula (3.49) is a consequence of (3.57).

(ii) Amortization of Losses method

Appendix 3.4

Set $i_V = 0$ in Prop. 3.5 (Section 3.4.2) to obtain

$$(3.58) \qquad \mathrm{Var}\,\ell(\infty) = \sigma^2 AL^2/\{1 - \sigma^2 \sum_{j=0}^{m-2} [(m-1-j)/m]^2\}$$

$$(3.59) \qquad \mathrm{Var}F(\infty) = \sum_{j=0}^{m-1} [(m-j)/m]^2 \mathrm{Var}\,\ell(\infty)$$

$$(3.60) \qquad \mathrm{Var}C(\infty) = \mathrm{Var}\,\ell(\infty)/m.$$

First,

$$m^{-2} \sum_{j=0}^{m-2} (m-1-j)^2 = m^{-2} \sum_{j=1}^{m-1} j^2$$

$$= m^{-2}(m-1)m(2m-1)/6$$

$$\sim m/3 \quad \text{as} \quad m \to \infty.$$

(See Spiegel (1971), p. 98, for the summation formula used above.)

This, together with Eq. (3.58), proves that

$$(3.61) \qquad \mathrm{Var}\,\ell(\infty) \sim \sigma^2 AL^2 \quad \text{as} \quad m \to \infty.$$

Now turn to $\mathrm{Var}F(\infty)$ (Eq. (3.59)). The same summation formula shows that

$$\sum_{j=0}^{m-1} [(m-j)/m]^2 \sim m/3$$

which takes care of (3.50).

Finally, Eqs. (3.60) and (3.61) clearly imply (3.51).

Remark 3.9. As approximations for $\mathrm{Var}F(\infty)$ and $\mathrm{Var}C(\infty)$, formulas (3.48) to (3.51) are sometimes valuable, even when $i_V \neq 0$. For example, if $i_V = .01$, $\sigma = .05$ and $m = 10$,

(i) Spread method: formula (3.48) yields

$$[\mathrm{Var}F(\infty)]^{1/2}/AL \doteq 11.2\%$$

while the exact number is 11.7% (Table 3.1);

(ii) Amortization of Losses: formula (3.50) yields

$$[\mathrm{Var}F(\infty)]^{1/2}/AL \doteq 9.1\%$$

while the exact number is 9.9% (Table 3.2).

APPENDIX 3.5

PROOF OF PROPOSITION 3.7

Define $\upsilon = (1+i_V)^{-1}$. From Prop. 3.2, we need to look at the behaviour of

$$\text{for VarF}(\infty)\text{:} \quad 1/(1-k)$$

$$\text{and for VarC}(\infty)\text{:} \quad 1/[\ddot{a}^2_{\overline{m|}}(1-k)],$$

over the range $2 \le m < \infty$. Recall that

$$k = (1+i)^2(1-1/\ddot{a}_{\overline{m|}})^2(1+\sigma^2\upsilon^2)$$

where $\ddot{a}_{\overline{m|}}$ is evaluated at rate $i = i_V$.

(i) If $y = (1+i)^2+\sigma^2 > 1$, then

$$k = (\ddot{a}_{\overline{m-1|}}/\ddot{a}_{\overline{m|}})^2(1+\sigma^2\upsilon^2)$$

converges to $1+\sigma^2\upsilon^2 > 1$ if $i \ge 0$, and to $(1+i)^2+\sigma^2 > 1$ if $i < 0$. Thus both VarF(∞) and VarC(∞) reach infinity for some finite m.

(ii) If $y=1$, then $1-k \downarrow 0+$ as $m \to \infty$, and thus Var F(∞) tends to infinity, without ever reaching it. As to VarC(∞), note that in this case $i_V < 0$ and

$$1+\sigma^2\upsilon^2 = \upsilon^2[(1+i)^2+\sigma^2] = \upsilon^2.$$

Thus

$$\ddot{a}^2_{\overline{m|}}(1-k) = \ddot{a}^2_{\overline{m|}} - \upsilon^2\ddot{a}^2_{\overline{m-1|}},$$

which tends to infinity as $m \to \infty$. VarC(∞) therefore converges to 0.

(iii) If $y < 1$, it is easy to see that

$$\lim_m k = y < 1,$$

which means that VarF(∞) has a finite limit.

<u>Formula for m^*</u>

Let $y > 1$, and define

$$F(m) = 1/(1-k)$$

$$C(m) = 1/[\ddot{a}^2_{\overline{m|}}(1-k)].$$

Think of these as functions of a continuous variable m.

Geometrically, m^* is the m for which VarC(∞) is a minimum, as a function of VarF(∞). But VarF(∞) is a strictly increasing function of m, and thus $dF/dm > 0$ for any $m \geq 1$.

Because of this, the points where $dC/dF = 0$ are the same as those where $dC/dm = 0$, since

$$dC/dF = (dC/dm)/(dF/dm).$$

Now

$$(3.62) \qquad \frac{dC}{dm} = -[\ddot{a}^2_{\overline{m|}}(1-k)]^{-2}\frac{d}{dm}[\ddot{a}^2_{\overline{m|}} - \ddot{a}^2_{\overline{m-1|}}(1+s)]$$

where $s = \sigma^2 v^2$.

(i) <u>$i \neq 0$:</u> dC/dm vanishes if and only if

$$(\log(1+i)/d)2v^m\ddot{a}_{\overline{m|}} = (\log(1+i)/d)2v^{m-1}\ddot{a}_{\overline{m-1|}}(1+s)$$

$$\Longleftrightarrow \qquad 1-v^m = (1+i)(1-v^{m-1})(1+s)$$

$$\Longleftrightarrow \qquad v^m = (vy-1)/(y-1)$$

$$\Longleftrightarrow \qquad m = -\log[(vy-1)/(y-1)]/\log(1+i).$$

(ii) <u>$i = 0$:</u> From Eq. (3.62), dC/dm vanishes if and only if

$$\frac{d}{dm}[m^2-(m-1)^2y] = 0$$

$$\Longleftrightarrow \qquad m = 1+1/\sigma^2.$$

APPENDIX 3.6

AL AND NC (SECTION 3.5.1)

The following values are needed (based on ELT13):

$\ell_{30} = 95993$ $\qquad$ $\ell_{65} = 70426$

$\ddot{a}^{(0)}_{30:\overline{35}|} = 32.707$ $\qquad$ $\ddot{a}^{(.01)}_{30:\overline{35}|} = 27.942$

$\ddot{a}^{(0)}_{65} = 12.686$ $\qquad$ $\ddot{a}^{(.01)}_{65} = 11.739.$

Recall that $S = s \cdot \ell_a \cdot \ddot{a}^{(0)}_{a:\overline{r-a}|}$. From Sections 1.3.2.2 and 1.3.8.3,

$$\begin{aligned}
NC/S &= (2/3)(1.01)^{-35}(\ell_{65}/\ell_{30})\ddot{a}^{(.01)}_{65}/\ddot{a}^{(.01)}_{30:\overline{35}|} \\
&= .1451 \text{ or } \underline{14.5\%} \\
B/S &= (2/3)(\ell_{65}/\ell_{30})\ddot{a}^{(0)}_{65}/\ddot{a}^{(0)}_{30:\overline{35}|} \\
&= .1897 \\
\Rightarrow \quad AL/S &= (B/S - NC/S)/d \\
&= 4.509 \text{ or } \underline{451\%}.
\end{aligned}$$

CHAPTER 4

RANDOM RATES OF RETURN: CONTINUOUS TIME

4.1 INTRODUCTION

Like the preceding one, this chapter is concerned to calculate the moments of $F(t)$ and $C(t)$, but now in a continuous-time setting. Only the Spread method is considered. The Amortization of Losses method is mentioned briefly in Paragraph 4 of Section 4.3.5.

In 4.2, some comments are made about notation and assumptions. The latter are much the same as those of Chapter 3.

Actual rates of return are represented as continuous-time white noise; this is analogous to the i.i.d. assumption of the last chapter. (The merits of the white noise model are discussed in 5.2.)

Consequently, $F(t)$ now satisfies a particular stochastic differential equation (SDE). As regards specifying this SDE, two lines of action are possible. The first one is directly to write down the equation, using intuitive arguments only. In the case at hand, it appears that this approach lacks rigour. Therefore, the other possibility has been chosen, namely first to imagine valuations to be performed "n" times per year, with i.i.d. returns like in Chapter 3; and, next, to identify the limiting stochastic process, when $n \to \infty$. The outcome of this analysis is contained in Section 4.3, and Appendices 4.2 and 4.3.

The first and second moments of $F(t)$ and $C(t)$ are calculated in 4.4. Their derivation rests on elementary properties of Itô stochastic differential equations, which are outlined in Appendix 4.1.

This Section 4.4 is a continuous-time version of Section 3.3. Propositions 4.4 to 4.7 correspond to Propositions 3.1, 3.2, 3.3 and 3.7.

The results of Chapter 4 are original. Proposition 4.1 is essentially based on the paper by Joffe and Métivier (1986).

The model presented in this chapter does require a higher degree of mathematical sophistication than the one used in Chapter 3. However, it is believed that the following problems may be studied more easily within a continuous-time framework than with a discrete one:

- higher moments of $F(t)$ and $C(t)$;
- other methods of adjusting the normal cost;
- probability densities of $F(t)$ and $C(t)$;
- hitting times, e.g. the time it takes the fund to move from one level F_0 to some other level F_1.

This judgment is based on the extensiveness of the theory of SDE's and on the great number of applications it has found in engineering, economics, finance, etc.

As concerns actuarial science, diffusion processes have been applied to ruin theory (cf. Beekman and Fuelling (1977), Emmanuel et al. (1975), Iglehart (1969), Ruohonen (1980)).

Note: "Diffusion processes" and "solutions of stochastic differential equations" are virtually the same class of processes. See Chapter 9 of Arnold (1974).

4.2 ASSUMPTIONS AND NOTATION

For Section 4.3, the assumptions are identical to those described in 3.2. The only difference lies in the notation: the superscript "n" added to a symbol makes reference to the situation where n valuations are performed every year (at times 0, 1/n, 2/n, etc.). The absence of a superscript alludes to the limit as $n \to \infty$ (e.g. $F^n \to F$ as $n \to \infty$).

For example, B^n is the amount of the benefits paid at time k/n, $k=0,1,2,\ldots$ This means that B^n is of the order of B^1/n. Thus it is natural to require the limiting "instantaneous" rate of benefit outgo to be

$$B = \lim_n nB^n$$

(see Props. 4.2 and 4.3). The same comment applies for NC^n and S^n.

On the contrary, AL^n is of the same order as AL^1. Accordingly, Props. 4.2 and 4.3 suppose that

$$AL = \lim_n AL^n.$$

(Same comment for PVB^n and PVS^n.)

In Section 4.4, hypotheses (i), (ii) and (iii) of 3.2 remain unchanged. The other ones become:

(iv) Valuation assumptions are fixed, including the valuation force of interest γ_V.

(v) First define $W(t)$ as the Wiener process (see Appendix 4.1). Then the actual (instantaneous) rates of return are

$$\gamma(t) = \gamma + \sigma dW(t)/dt. \tag{4.1}$$

$dW(t)/dt$ is what is known as "white noise". It is *not* a stochastic process in the usual sense, since $W(t)$ is nowhere differentiable w.p.1. But expression (4.1) is a convenient abuse of notation. It is also convenient (though not strictly accurate) to say that

γ is the mean (instantaneous) rate of return

and that

σ^2 is the variance of the rates of return.

(vi) "Individual cost methods" have been defined in Chapter 2.

From assumption (xi) of Section 2.2, we deduce

$$0 = \gamma_V \cdot AL + NC - B. \tag{4.2}$$

(vii) $\quad \text{Prob}(F(0)=F_0) = 1$ for some $F_0 \in \mathbb{R}$.

Let

$$H_t \;=\; \sigma\text{-field of events prior to time } t$$
$$\text{(i.e. generated by } \{W(s),\ s \le t\})$$

In all the cases considered in 4.4, it will turn out that $F(t)$ and $C(t)$ are H_t-measurable, for every $t \geq 0$.

Note. The analysis is conducted with real-term values, as in Section 2.4. The only exceptions are Sections 4.4.3 and 4.4.4, which show that similar results hold for nominal monetary values.

4.3 CONVERGENCE TO A DIFFUSION

4.3.1 The Problem

Imagine that contributions and benefits are paid n times a year. Changing the time-scale, Eq. (3.2) of Section 3.2 is rewritten as

$$(4.3)\quad F^n\left(\frac{k+1}{n}\right) \;=\; \left(1+ i^n(k+1)\right)\left(F^n\left(\frac{k}{n}\right)+ C^n\left(\frac{k}{n}\right)- B^n\right),$$
$$k = 0,1,2,\ldots$$

Each sequence $\{i^n(k),\ k \geq 1\}$ is still supposed i.i.d.. (A small inconsistency should be noted: $i^n(k+1)$ is the rate earned during the period $\left(\frac{k}{n}, \frac{k+1}{n}\right)$.)

In order to prove convergence, it is essential to know how $i^n(\cdot)$ is defined. Two different ways of doing so will be described, in Sections 4.3.3 and 4.3.4, respectively.

Section 4.3.2 says a few words about the particular type of convergence which will concern us, and also states the general result from which Propositions 4.2 and 4.3 will follow.

4.3.2 General Convergence Result

Proposition 4.1 is about the "weak" convergence of a sequence of stochastic processes of a certain type. Weak convergence can be said to be a generalization of convergence in distribution. The latter concept does not suit the present situation, since an infinite number of random variables are involved at once. The monograph by Billinsley (1968) is by now a classical reference on weak convergence. Its introduction and first chapter explain the preceding ideas in great detail.

The proof of Prop. 4.1 is very technical. It can be found in Appendix 4.2.

Proposition 4.1. For each $n \geq 1$, let $\{h^n(k), k \geq 1\}$ be an i.i.d. sequence of random variables, with $Eh^n(k) = 0$ and $\mathrm{Var}\,h^n(k) = 1$. Further assume that $\{h^n(1)^2, n \geq 1\}$ is uniformly integrable. Also, let

$$u^n(x) = d^n x + e^n, \quad v^n(x) = f^n x + g^n,$$

(4.4)

$$u(x) = dx + e, \quad v(x) = fx + g,$$

with $d^n \to d$, $e^n \to e$, $f^n \to f$ and $g^n \to g$ as $n \to \infty$. Define the processes X^n by

$$(4.5) \qquad X^n\left(\frac{k+1}{n}\right)$$

$$= X^n\left(\frac{k}{n}\right) + \frac{1}{n} u^n\left(X^n\left(\frac{k}{n}\right)\right) + \frac{1}{\sqrt{n}} v^n\left(X^n\left(\frac{k}{n}\right)\right) h^n(k+1)$$

$k=0,1,2,\ldots$

$$(4.6) \qquad X^n(t) = X^n([nt]/n), \quad t \geq 0.$$

$$(4.7) \qquad X^n(0) = X_0^n \in \mathbb{R} \text{ w.p.1.} \quad X_0^n \to X_0 \in \mathbb{R}.$$

Then $\{X^n, n \geq 1\}$ converges weakly to the process X, which is the unique solution of the Itô SDE

$$(4.8) \qquad dZ(t) = u(Z(t))dt + v(Z(t))dW(t)$$
$$Z(0) = X_0 \quad \text{w.p.1.}$$

The observations below should clarify the meaning of this proposition.

Definition. A family $\{X_i,\ i \in I\}$ of random variables is said to be uniformly integrable if

$$\sup_I \int_{|X_i|>a} |X_i| dP \to 0$$

as $a \to \infty$ (Billingsley (1968), p.32).

Eq. (4.6) is a technical requirement, which turns the realization of X^n (as defined by Eq. (4.5)) into right-continuous functions (note: $[z]$ is the greatest integer $\leq z$).

(4.7) indicates that the initial values $X^n(0)$ are not random, and converge to some finite X_0.

That the processes X^n, as defined by (4.5), should converge to the solution of SDE (4.8) is on the whole not very surprising. Prop.4.1 has only one distinctive feature: it asks for very little in connection with $\{h^n(\cdot),\ n\geq 1\}$. There is no condition on moments higher than the second; it is only supposed that $\{h^n(1)^2,\ n\geq 1\}$ is uniformly integrable.

4.3.3 Subdividing $i^1(\cdot)$

Let us now return to the question of defining $\{i^n(k), k\geq 1\}$ for every n.

Required here is a distribution for $i^n(\cdot)$, such that "n" independent random variables having this distribution, say $(i^n(1),\ldots,i^n(n))$, will in some way be equivalent to $i^1(1)$. This is not immediately obvious, for the "noise" (or "randomness") introduced into the system is not additive, but multiplicative.

In what follows the distribution of $i^1(\cdot)$ - the "initial" distribution - will be left unspecified. From it, the distribution of $i^n(\cdot)$ will be defined, using a linear transformation. Two ways of accomplishing this will be considered. The first one is directly concerned with the "discrete" rate $i^1(\cdot)$, whereas the other one involves the "instantaneous" rate $\gamma^1(\cdot) = \log(1+i^1(\cdot))$.

The first possibility is to ask for both

$$E \prod_{k=1}^{n} (1+i^n(k)) = E(1+i^1(1))$$

and

$$\operatorname{Var} \prod_{k=1}^{n} (1+i^n(k)) = \operatorname{Var} (1+i^1(1)).$$

The i.i.d. assumption then implies

$$i^n = Ei^n(\cdot) = (1+Ei^1(\cdot))^{1/n} - 1 \tag{4.9}$$

and

$$\operatorname{Var} i^n(\cdot) = [E(1+i^1(\cdot))^2]^{1/n} - (1+Ei^1(\cdot))^{2/n}. \tag{4.10}$$

Therefore, in order to let $i^n(\cdot)$ have a distribution of the same form as that of $i^1(\cdot)$, it is defined as

$$i^n(\cdot) \overset{\text{dist}}{=} (1+Ei^1(\cdot))^{1/n} - 1 + [i^1(\cdot) - Ei^1(\cdot)](\operatorname{Var} i^n(\cdot)/\operatorname{Var} i^1(\cdot))^{1/2} \tag{4.11}$$

Finally, define the normalized variables

$$h^n(k) = (i^n(k) - i^n)/(\operatorname{Var} i^n(\cdot))^{1/2}. \tag{4.12}$$

The definition of $i^n(\cdot)$ can now be used to transform Eq. (4.3), in a way which will allow the application of Prop. 4.1.

(a) In the case of individual cost methods

$$C^n(t) = NC^n + (AL^n - F^n(t))/\ddot{a}^{(i^n)}_{\overline{m \cdot n|}} . \tag{4.13}$$

Observe that

(i) the unfunded liability is spread over m·n periods of 1/n year = m years;

(ii) the annuity is evaluated at rate $i^n = Ei^n(\cdot)$.

In view of Eq. (4.13), Eq. (4.3) becomes

(4.14) $$F^n\left(\frac{k+1}{n}\right) = (1+i^n(k+1))(q^nF^n(\frac{k}{n}) + r^n)$$

where

$$q^n = 1 - 1/\ddot{a}_{\overline{m\cdot n|}}$$

and

$$r^n = NC^n + AL^n/\ddot{a}_{\overline{m\cdot n|}} - B^n.$$

This expression is in turn transformed as follows

$$\begin{aligned}
F^n\left(\frac{k+1}{n}\right) &= (1+i^n)(q^nF^n(\frac{k}{n}) + r^n) \\
&\quad +(i^n(k+1)-i^n)(q^nF^n(\frac{k}{n})+ r^n) \\
&= F^n(\frac{k}{n}) + [(1+i^n)q^n-1]F^n(\frac{k}{n}) + (1+i^n)r^n \\
&\quad + (i^n(k+1)-i^n)(q^nF^n(\frac{k}{n}) + r^n) \\
&= F^n(\frac{k}{n})+ n^{-1}\{n[(1+i^n)q^n-1]F^n(\frac{k}{n}) + n(1+i^n)r^n\} \\
&\quad + n^{-1/2}\{(nVar\, i^n(\cdot))^{1/2}(q^nF^n(\frac{k}{n})+r^n)\}h^n(k+1)
\end{aligned}$$

(4.15) $$= F^n(\frac{k}{n}) + \frac{1}{n} u^n(F^n(\frac{k}{n})) + \frac{1}{\sqrt{n}} v^n(F^n(\frac{k}{n}))h^n(k+1).$$

In Eq. (4.15),

$$u^n(x) = n[(1+i^n)q^n-1]x + n(1+i^n)r^n,$$

$$v^n(x) = (n\mathrm{Var}\, i^n(\cdot))^{1/2}(q^nx+r^n).$$

(b) In the case of aggregate cost methods,

$$C^n(t) = S^n(PVB^n-F^n(t))/PVS^n.$$

Eq. (4.14) results once more, if q^n and r^n are redefined as

$$q^n = 1 - S^n/PVS^n,$$

$$r^n = S^n\cdot PVB^n/PVS^n - B^n.$$

Eq. (4.15) is thus unchanged, as well as the definitions of u^n and v^n above.

Before Prop. 4.1 can be applied, it remains to

(i) show that $\{h^n(1)^2,\ n\geq 1\}$ is uniformly integrable, and

(ii) determine the limits $u(x)$ and $v(x)$.

The details are in Appendix 4.3.

<u>Proposition 4.2.</u> For every $n \geq 1$, let $\{i^n(k), k \geq 1\}$ be an i.i.d. sequence, with $i^n(\cdot)$ given by Eq. (4.11).

(a) Individual cost methods.

Suppose that $AL^n \to AL$ and $n(NC^n-B^n) \to NC-B$ as $n \to \infty$. If

$$C^n(t) = NC^n + (AL^n-F^n(t))/\ddot{a}^{(i^n)}_{\overline{m\cdot n}|},$$

then the processes F^n converge weakly to the process F satisfying the Itô SDE

$$(4.16)\qquad \begin{aligned} dF(t) &= (\gamma F(t)+C(t)-B)dt + \sigma F(t)dW(t),\\ F(0) &= \lim_n F^n(0),\\ C(t) &= NC + (AL-F(t))/\bar{a}^{(\gamma)}_{\overline{m}|}, \end{aligned}$$

$$(4.17) \qquad \gamma = \log(1 + Ei^1(\cdot))$$

$$(4.18) \qquad \sigma^2 = \log\{[E(1+i^1(\cdot))^2]/[E(1+i^1(\cdot))]^2\}.$$

(b) Aggregate cost method

Suppose that $nS^n \to S$, $PVB^n \to PVB$, $PVS^n \to PVS$ and $nB^n \to B$ as $n \to \infty$. If

$$C^n(t) = S^n(PVB^n - F^n(t))VS^n,$$

then the processes F^n converge weakly to the process F satisfying (4.16), with

$$(4.19) \qquad C(t) = S(PVB-F(t))/P.$$

γ and σ^2 are still given by (4.17) and (4.18).

4.3.4 Subdividing $\gamma^1(1)$

Another possible way of defining $i^n(\cdot)$ consists in fractioning the instantaneous rate $\gamma^1(\cdot)$. Denote

$$\gamma^n(\cdot) = \log(1 + i^n(\cdot)).$$

We now require

$$E \sum_{k=1}^{n} \gamma^n(k) = E\gamma^1(1)$$

and

$$\mathrm{Var} \sum_{k=1}^{n} \gamma^n(k) = \mathrm{Var}\, \gamma^1(1).$$

The appropriate linear transformation of the distribution of $\gamma^1(\cdot)$ is

$$(4.20) \qquad \gamma^n(\cdot) \overset{\text{dist}}{=} \frac{1}{n} E\gamma^1(\cdot) + \frac{1}{\sqrt{n}}(\gamma^1(\cdot) - E\gamma^1(\cdot)).$$

We may then revert to discrete rates and define

$$(4.21) \qquad i^n(k) = \exp[\gamma^n(k)] - 1.$$

<u>Proposition 4.3</u>. For every $n \geq 1$, let $\{i^n(k), k \geq 1\}$ be an i.i.d. sequence, with $i^n(\cdot)$ given by Eq. (4.21). Assume furthermore that $\mathrm{Var}\gamma^1(\cdot) < \infty$.

The conclusions of Prop. 4.2 remain unchanged, <u>except that</u> Eqs. (4.17) and (4.18) are replaced with

$$(4.22) \qquad \gamma = E\gamma^1(\cdot) + \frac{1}{2}\,\mathrm{Var}\gamma^1(\cdot)$$

$$(4.23) \qquad \sigma^2 = \mathrm{Var}\gamma^1(\cdot).$$

(Proof in Appendix 4.3).

4.3.5 Comments

1. Comparison of Props 4.2 and 4.3

An examination of Eqs. (4.17) and (4.18), on the one hand, and Eqs.(4.22) and (4.23), on the other, shows that to a certain extent the limiting pcess F does depend on the way $i^n(\cdot)$ is defined (given the same initial distribution $i^1(\cdot)$).

For the purpose of assessing how different the two pairs (γ,σ^2) may be, it is helpful to make use of the "cumulant generating function", defined as

$$k(t) = \log E\exp(tX)$$

for a random variable X. When it exists in a neighbourhood of $t = 0$, $k(t)$ has the expansion

$$(4.24) \qquad k(t) = \sum_{j\geq 1} t^j \frac{k^j}{j!}$$

where

$$k_1 = EX,$$
$$k_2 = E(X-EX^2) = \mathrm{Var}X,$$
$$k_3 = E(X-EX)^3 ,$$
$$k_4 = E(X-EX)^4 - 3(\mathrm{Var}X)^2,$$

etc. For more about cumulants, the reader is referred to Cramer (1946), pp. 185-187.

Denote the two ways of defining $i^n(\cdot)$, i.e. subdividing $i^1(\cdot)$ and subdividing $\gamma^1(\cdot)$, by the subscripts "a" and "b", respectively. Also, consider the cumulant generating function of $\gamma^1(\cdot)$.

Props. 4.2 and 4.3 mean that, firstly,

$$\begin{aligned}\gamma_a &= \log E(i^1(\cdot)+1)\\ &= \log E \exp(\gamma^1))\\ &= k(1)\\ &= E\gamma^1(\cdot) + \frac{1}{2}\,\mathrm{Var}\ \gamma^1(\cdot) + \ldots\end{aligned}$$

while

$$\gamma_b = E\gamma^1(\cdot) + \frac{1}{2}\,\mathrm{Var}\ \gamma^1(\cdot).$$

Secondly,

$$\begin{aligned}\sigma_a^2 &= \log\{[E(1+i^1(\cdot))^2]/[E(1+i^1(\cdot))]^2\}\\ &= k(2) - 2k(1).\end{aligned}$$

Expression (4.24) tells us that

$$k(2t)-2k(t) = t^2\mathrm{Var}\gamma^1(\cdot) + o(t^2)$$

and so

$$\sigma_a^2 = \mathrm{Var}\ \gamma^1(\cdot) + \ldots$$

may not be very different from

$$\sigma_b^2 = \mathrm{Var}\ \gamma^1(\cdot).$$

In conclusion, if the cumulants of $\gamma^1(\cdot)$ of third and higher order are negligible, then (γ_a,σ_a^2) and (γ_b,σ_b^2) are likely to be very close.

One particular initial distribution is of special interest. Suppose $\gamma^1(\cdot)$ is a normal random variable. Then

$$\begin{aligned}k(t) &= \log E \exp(t\gamma^1(\cdot))\\ &= t\cdot E\gamma^1(\cdot) + \frac{t^2}{2}\,\mathrm{Var}\ \gamma^1(\cdot),\end{aligned}$$

and so $(\gamma_a,\sigma_a^2) = (\gamma_b,\sigma_b^2)$.

2. Eq. (4.16) can be rewritten as

$$dF(t) = (-\alpha F(t)+r)dt + \sigma F(t)dW(t)$$

with

$$\alpha = 1/\bar{a}^{(\gamma)}_{\overline{m}|} - \gamma, \qquad r = NC + AL/\bar{a}^{(\gamma)}_{\overline{m}|} - B$$

for individual cost methods, or

$$\alpha = S/PVS - \gamma, \quad r = S\cdot PVB/PVS - B$$

for aggregate cost methods.

The parameters α and r are defined here in the same way they were in Sections 1.4.1, 2.4.2.1 and 2.4.2.2. This shows that the SDE (4.16) is a simple modification of the ordinary differential equations which hold in the deterministic case. The added term $\sigma F(t)dW(t)$ reflects the randomness introduced into the system.

3. Like each of the F^n's, F is a Markov process. $F(t)$ is also continuous w.p.1.

4. Amortization of Losses method

Here are the partial results so far obtained in connection with this method.

If valuations are performed n times a year, and if intervaluation losses are amortized over m years (= m·n periods of 1/n year), then Eqs. (3.23) and (3.24) of Section 3.4 become

$$C^n(t) = NC^n + \sum_{k=0}^{m\cdot n-1} \ell^n(t-\frac{k}{n})/\ddot{a}^{(i^n)}_{\overline{m\cdot n}|} .$$

At present, it is surmised that the weak limit of the sequence $\{F^n\}$ thus produced satisfies the Itô stochastic differential equation

$$dF(t) = [\gamma F(t)+NC-B-(\bar{a}^{(\gamma)}_{\overline{m}|})^{-1} \int_{t-m}^{t} \sigma F(u)dW(u)]dt + \sigma F(t)dW(t), \tag{4.25}$$

where γ and σ^2 are the same as in Props. 4.2 or 4.3 depending on how $i^n(\cdot)$ is defined.

Eq. (4.25) can be formally obtained from Eqs. (2.32) to (2.34) (Section 2.4.3), upon letting $\Delta\beta(t) \equiv 0$ and $\Delta\gamma(t) = \sigma dW(t)/dt$. Attempts to provide a rigorous proof have so far been inconclusive.

5. One point that was left in the dark when defining the sequences $\{i^n(k),\ k \geq 1\}$ is whether $i^n(\cdot) > 1$ w.p. 1. This is not required in the proofs of Props. 4.2 and 4.3, but is needed if F^n is to make sense as an accumulating fund.

The result is obvious with the second way of defining $i^n(\cdot)$ (Section 4.3.4). So assume $i^n(\cdot)$ to be given by Eq. (4.11) of Section 4.3.3, with $i^1(\cdot) > -1$ w.p. 1 and $\mathrm{Var}\ i^1(\cdot) > 0$. Let

$$\begin{aligned} a &= [E(1+i(\cdot))^2]^{1/n} \\ b &= [E(1+i^1(\cdot))]^{1/n} \end{aligned}$$

Observe that $a > b^2$ and

$$\begin{aligned} (a^n-b^{2n})/(a-b^2) &= a^{n-1} + a^{n-2}b^2 + \ldots + b^{2n-2} \\ &> n \cdot b^{2n-2} \end{aligned}$$

Then, from Eqs. (4.9), (4.10) and (4.11),

$$\begin{aligned} 1+i^n(\cdot) &= b + (1+i^1(\cdot)-b^n)[(a-b^2)/(a^n-b^{2n})]^{1/2} \\ &> b - b^n[(a-b^2)/(a^n-b^{2n})]^{1/2} \\ &> b - b^n/(n \cdot b^{2n-2})^{1/2} \\ &= b(1-n^{1/2}) > 0 \quad \text{for all } n \geq 2. \end{aligned}$$

4.4 MOMENTS OF F(t) AND C(t)

From now on it is assumed that $F(t)$ satisfies the Itô SDE

$$dF(t) = (\gamma F(t)+C(t)-B)dt + \sigma F(t)dW(t) \tag{4.26}$$

with

$$C(t) = NC + (AL-F(t))/\bar{a}^{(\gamma)}_{\overline{m|}}, \tag{4.27}$$

in the case of individual cost methods, or

$$C(t) = S(PVB-F(t))/PVS \tag{4.28}$$

in the case of aggregate cost methods. It is also supposed that

$$F(0) = F_0 \in \mathbb{R} \quad \text{w.p. } 1.$$

It can be shown that $F(t)$ has finite moments of any order (see Paragraph 6 of Appendix 4.1). These moments are continuous functions of t. It is thus permissible to change the order of $E(\cdot)$ and $\int$, i.e. it is always true that

$$E\int_0^t F(s)^k ds = \int_0^t EF(s)^k ds, \quad t < \infty, \ k \geq 0.$$

4.4.1 First Moments

In the case of individual cost methods, Eq. (4.26) becomes

$$(4.29) \qquad dF(t) = (-\alpha F(t)+r)dt + \sigma F(t)dW(t)$$

where

$$\alpha = 1/\bar{a}_{\overline{m}|} - \gamma, \qquad r = NC + AL/\bar{a}_{\overline{m}|} - B.$$

Assume that $\bar{a}_{\overline{m}|}$ is calculated at rate γ_v, and that $0 < m < \infty$.

Rewrite (4.29) as an integral equation:

$$(4.30) \quad F(t) = F(0) + \int_0^t (-\alpha F(s)+r)ds + \int_0^t \sigma F(s)dW(s).$$

Since

$$E\int_0^t F(s)dW(s) = 0$$

(see Paragraphs 3 and 6 of Appendix 4.1), we obtain

$$EF(t) = F(0) + \int_0^t (-\alpha EF(s)+r)ds,$$

which is equivalent to the ordinary differential equation

$$(4.31) \qquad \frac{d}{dt} EF(t) = -\alpha EF(t) + r.$$

Define $M(t) = EF(t)$:

$$M'(t) = -\alpha M(t) + r, \quad M(0) = F(0)$$

$$\Rightarrow \quad \frac{d}{dt}(M(t)e^{\alpha t}) = re^{\alpha t}$$

$$\Rightarrow \quad M(t)e^{\alpha t} - F(0) = \int_0^t re^{\alpha s}ds$$

$$\Rightarrow \quad M(t) = F(0)e^{-\alpha t} + \int_0^t re^{-\alpha(t-s)}ds$$

$$= F(0)e^{-\alpha t} + (1-e^{-\alpha t})r/\alpha. \tag{4.32}$$

4.4.1.1 $\gamma = \gamma_V$

We get

$$\alpha = \gamma_V/[1-\exp(-\gamma_V \cdot m)] - \gamma_V > 0.$$

Proposition 4.4. If $\gamma = \gamma_V$ and $0 < m < \infty$, then

$$EF(\infty) = \lim_t EF(t) = AL \tag{4.33}$$

$$EC(\infty) = \lim_t EC(t) = NC. \tag{4.34}$$

Proof. Since $\alpha > 0$, Eq. (4.32) implies

$$\begin{aligned} EF(\infty) &= r/\alpha \\ &= (NC - B + AL/\bar{a}_{\overline{m}|})/\alpha \\ &= AL(1/\bar{a}_{\overline{m}|} - \gamma)/\alpha \\ &\quad \text{from Eq. (4.2) (Section 4.2)} \\ &= AL. \end{aligned}$$

Clearly $EC(\infty) = NC + (AL-EF(\infty))/\bar{a}_{\overline{m}|} = NC$. □

4.4.1.2 $\gamma \neq \gamma_V$

If γ_V differs from the mean rate of return γ, then Prop. 4.4 does not hold. We can only say that if $\alpha > 0$, then

$$\lim_t EF(t) = r/\alpha$$

and

$$\lim_{t} EC(t) = NC + (AL-EF(\infty))/\bar{a}_{\overline{m}|} .$$

4.4.2 Second Moments

I will first use Itô's formula (Paragraph 5 of Appendix 4.1) to show that $V(t) = \mathrm{Var}\, F(t)$ satisfies the following differential equation:

$$(4.35) \qquad V'(t) = (-2\alpha+\sigma^2)V(t) + \sigma^2(EF(t))^2$$
$$V(0) = 0.$$

Let $g(t,x) = (x-M(t))^2$, $M(t) = EF(t)$. We have

$$g_t'(t,x) = -2(x-M(t))M'(t)$$
$$= -2(x-M(t))(-\alpha M(t)+r)$$
$$g_x'(t,x) = 2(x-M(t))$$
$$g_{xx}''(t,x) = 2.$$

$M(t)$ is continuous, and so the above partial derivatives are also continuous. Itô's formula may thus be applied:

$$\begin{aligned} dg(t,F(t)) &= [g_t'(t,F(t))+g_x'(t,F(t))(-\alpha F(t)+r) \\ &\quad + \frac{1}{2} g_{xx}''(t,F(t))\sigma^2 F(t)^2]dt \\ &\quad + g_x'(t,F(t))\sigma F(t)dW(t) \\ &= [-2(F(t)-M(t))(-\alpha M(t)+r) \\ &\quad + 2(F(t)-M(t))(-\alpha F(t)+r) + \sigma^2 F(t)^2]dt \\ &\quad + 2(F(t)-M(t))\sigma F(t)dW(t) \\ &= [-2\alpha(F(t)-M(t))^2 + \sigma^2 F(t)^2]dt \\ &\quad + 2\sigma(F(t)-M(t))F(t)dW(t). \end{aligned}$$

Next, proceed as with $EF(t)$: first rewrite the preceding differential equation in integral form. Then, noticing that

$$E \int_0^t \sigma(F(s)-M(s))F(s)dW(s) = 0,$$

revert to the differential formulation to obtain Eq. (4.35)

$$\begin{aligned} V'(t) &= \frac{d}{dt} Eg(t,F(t)) \\ &= -2\alpha E(F(t)-M(t))^2 + \sigma^2 EF(t)^2 \\ &= (-2\alpha+\sigma^2)V(t) + \sigma^2 M(t)^2 \end{aligned}$$

since $EF(t)^2 = VarF(t) + (EF(t))^2$.

An explicit expression for $VarF(t)$ can be derived in the same manner as for $EF(t)$, this time using the integrating factor $\exp[(2\alpha-\sigma^2)t]$. Since $V(0) = 0$, we find

(4.36) $$V(t) = \sigma^2 \int_0^t \exp[(-2\alpha+\sigma^2)(t-s)](EF(s))^2 ds.$$

Proposition 4.5.

Let $0 < m < \infty$. If $\alpha > \frac{1}{2}\sigma^2$, then

(4.37) $$Var\ F(\infty) = \sigma^2(EF(\infty))^2/(2\alpha-\sigma^2)$$

(4.38) $$Var\ C(\infty) = [Var\ F(\infty)]/(\bar{a}_{\overline{m}|})^2.$$

If $\alpha \leq \frac{1}{2}\sigma^2$, then both $Var\ F(\infty)$ and $Var\ C(\infty)$ are infinite.

Proof. $\alpha > \frac{1}{2}\sigma^2$. $EF(\infty) = M(\infty)$ is finite since $\alpha > 0$. Let $U(0) = 0$ and

$$U'(t) = (-2\alpha+\sigma^2)U(t) + \sigma^2 M(\infty)^2.$$

It is easy to see that $U(\infty) = \sigma^2 M(\infty)^2/(2\alpha-\sigma^2)$. Define

$$D(t) = V(t) - U(t).$$

Then

$$\begin{aligned} D'(t) &= (-2\alpha+\sigma^2)V(t) + (2\alpha-\sigma^2)U(t) + \sigma^2 M(t)^2 - \sigma^2 M(\infty)^2 \\ &= (-2\alpha+\sigma^2)D(t) + Z(t) \end{aligned}$$

where $Z(t) = \sigma^2[M(t)^2-M(\infty)^2] \to 0$ as $t \to \infty$.

Section 4.4

The proof that $D(t) \to 0$ as $t \to \infty$ is contained in Appendix 1.1 (simply substitute $2\alpha-\sigma^2$ for $\alpha(t)$ in the expression for $D'(t)$).

This proves Eq. (4.37). Eq. (4.38) follows from the definition of $C(t)$.

$\alpha \le \frac{1}{2}\sigma^2$. $EF(\infty)$ cannot be 0. This is because

$$\begin{aligned} r &= NC - B + AL/\bar{a}_{\overline{m}|} \\ &= AL(1/\bar{a}_{\overline{m}|} - \gamma_v) > 0 \end{aligned}$$

(see Eqs. (4.31) and (4.32)).

So there exists t_0 such that

$$(EF(s))^2 \ge b > 0, \quad s > t_0.$$

Eq. (4.36) implies $(t > t_0)$.

$$V(t) \ge \sigma^2 \int_{t_0}^{t} b\, ds \to \infty \text{ as } t \to \infty. \ \Box$$

<u>Proposition 4.6</u>. Let $u \ge 0$.

$$\begin{aligned} \mathrm{Cov}(F(t),F(t+u)) &= e^{-\alpha u}\mathrm{Var}\ F(t) \\ \mathrm{Cov}(C(t),C(t+u)) &= e^{-\alpha u}\mathrm{Var}\ C(t) \\ \mathrm{Cov}(F(f),C(t+u)) &= -e^{-\alpha u}\{\mathrm{Var}\ F(t)\}/\bar{a}_{\overline{m}|}. \end{aligned}$$

If $\alpha > \frac{1}{2}\sigma^2$, then, as $t \to \infty$,

$$\begin{aligned} &\text{Correlation } (F(t),F(t+u)) \to e^{-\alpha u} \\ &\text{Correlation } (C(t),C(t+u)) \to e^{-\alpha u} \\ &\text{Correlation } (F(t),C(t+u)) \to -e^{-\alpha u} \end{aligned}$$

<u>Proof</u>. Fix $t \ge 0$. Let $F^*(s) = F(s)-EF(s)$ and

$$\begin{aligned} g(u) &= \mathrm{Cov}(F(t),F(t+u)) \\ &= EF^*(t)\ F^*(t+u). \end{aligned}$$

We have

$$dF(s) = (-\alpha F(s)+r)ds + \sigma F(s)dW(s)$$

and

$$dEF(s) = (-\alpha EF(s)+r)ds$$

which imply

$$dF^*(s) = -\alpha F^*(s)ds + \sigma F(s)dW(s).$$

Thus

$$F^*(t+u) = F^*(t) + \int_t^{t+u} (-\alpha F^*(s))ds + \int_t^{t+u} \sigma F(s)dW(s).$$

Multiplying by $F^*(t)$ and taking expectations, we get

$$EF^*(t)F^*(t+u) = \text{Var}F(t) - \int_0^u \alpha EF^*(t)F^*(t+v)dv$$

since

$$E(F^*(t) \int_t^{t+u} F(s)dW(s)|H_t) = 0.$$

Consequently,

$$g(u) = g(0) - \alpha\int_0^u g(V)dV$$

or

$$g'(u) = -\alpha g(u), \qquad g(0) = \text{Var } F(t)$$

$$\Rightarrow \quad g(u) = \text{Cov}(F(t),F(t+u)) = e^{-\alpha u}\text{Var } F(t).$$

The other formulas follow easily. □

4.4.3 Non-Stationary Population

Now suppose that

(i) the population is not stationary;

(ii) salaries grow with inflation (independent of the process $W(t)$); and

(iii) γ is not necessarily equal to γ_U.

Then

$$dF(t) = [\gamma F(t)+NC(t)-B(t)+[AL(t)-F(t)]/\bar{a}_{\overline{m}|}]dt$$
$$+\sigma F(t)dW(t).$$

Provided $B(t)$, $NC(t)$ and $AL(t)$ are continuous functions of t, it can be shown (in the same fashion as in Section 4.4.2) that

$$\frac{d}{dt} EF(t) = -\alpha EF(t) + r(t)$$

and

$$\frac{d}{dt}\,\text{VarF}(t) = (-2\alpha+\sigma^2)\text{VarF}(t) + \sigma^2(\text{EF}(t))^2$$

where α is defined as before, and

$$r(t) = \text{NC}(t) - \text{B}(t) + \text{AL}(t)/\bar{a}_{\overline{m|}}\,.$$

4.4.4 Aggregate Cost Methods

Retain (i), (ii) and (iii) of 4.4.3, and suppose that S(t), PVB(t) and PVS(t) are continuous functions of t. Then

$$\frac{d}{dt}\,\text{EF}(t) = -\alpha(t)\text{EF}(t) + r(t)$$

and

$$\frac{d}{dt}\,\text{Var F}(t) = (-2\alpha(t)+\sigma^2)\text{Var F}(t) + \sigma^2(\text{EF}(t))^2$$

where

$$\alpha(t) = \text{S}(t)/\text{PVS}(t) - \gamma$$

$$r(t) = \text{S}(t)\text{PVB}(t)/\text{PVS}(t) - \text{B}(t).$$

If the assumptions of Section 4.2 are reinstated, and if, moreover, there is only one entry age into the scheme, then

$$\text{C}(t) = {}^{\text{EAN}}\text{NC} + ({}^{\text{EAN}}\text{AL} - \text{F}(t))\text{S/PVS}$$

(Eq. (2.30) of Section 2.4.2.2). In consequence, Props 4.4 and 4.5 apply once more:

- if $\gamma = \gamma_v$, then

$$\lim_t \text{EF}(t) = {}^{\text{EAN}}\text{AL},$$

$$\lim_t \text{EC}(t) = {}^{\text{EAN}}\text{NC};$$

- if $\alpha = \text{S/PVS}-\gamma > \frac{1}{2}\sigma^2$, then

$$\lim_t \text{VarF}(t) = \sigma^2(\text{EF}(\infty))^2/(2\alpha-\sigma^2),$$

$$\lim_t \text{VarC}(t) = [\text{VarF}(\infty)]\text{S}^2/\text{PVS}^2.$$

<u>Note</u>. The same formulas hold if PVS is replaced by PVS(n), $n < \infty$; see Section 2.4.2.2.

4.4.5 <u>The Optimal Region</u>

Prop. 3.7 has the following continuous-time version:

<u>Proposition 4.7</u>. Assume $\gamma = \gamma_V$.

(i) If $\gamma > -\frac{1}{2}\sigma^2$, then both VarF($\infty$) and VarC($\infty$) become infinite for some finite m, and there exists m^* such that

(1) for $m \le m^*$, VarF(∞) increases and VarC(∞) decreases with m;

(2) for $m \ge m^*$, both VarF(∞) and VarC(∞) increase with m.

Moreover,

- when $\gamma \neq 0$,

$$m^* = -\log[1-\gamma/(2\gamma+\sigma^2)]/\gamma;$$

- when $\gamma = 0$

$$m^* = 1/\sigma^2.$$

(ii) If $\gamma = -\frac{1}{2}\sigma^2$, VarC($\infty$) $\to 0$ and VarF(∞) $\to \infty$ as $m \to \infty$, although VarF(∞) stays finite for all $m < \infty$.

(iii) If $\gamma < -\frac{1}{2}\sigma^2$, VarC($\infty$) $\to 0$ and VarF(∞) has a finite limit as $m \to \infty$.

In (ii) and (iii), VarF(∞) increases and VarC(∞) decreases as m increases, for $0 < m < \infty$.

The proof uses the same arguments as for Prop. 3.7 and the algebra is in fact simpler. It is therefore omitted.

<u>Comments</u>

1. Notice that the determining factor is now $2\gamma+\sigma^2$, while it was $(1+i)^2+\sigma^2$ in Prop.3.7. This is not surprising, for letting $i = e^\gamma-1$,

$$(1+i)^2+\sigma^2 = e^{2\gamma}+\sigma^2$$
$$\simeq 1+2\gamma+\sigma^2$$

and thus $(1+i)^2+\sigma^2 \gtrless 1$ are essentially the same conditions as $2\gamma+\sigma^2 \gtrless 0$.

2. The formulae for m^* can also be shown to be approximately equivalent. In discrete time, m^* is (if $i \neq 0$)

$$-\log[(vy-1)/(y-1)]/\log(1+i).$$

Letting $1+i = e^\gamma$ omce more, gives, firstly,

$$vy-1 = (1+i)+v\sigma^2-1$$
$$\simeq \gamma+\sigma^2,$$

and secondly,

$$y-1 \simeq 2\gamma+\sigma^2$$

and thus

$$(vy-1)/(y-1) \simeq 1-\gamma/(2\gamma+\sigma^2).$$

3. The continuous-time expressions for m^* make it easier to show that it is continuous at $\gamma = 0$. Since

$$\log(1-x) = -(x + x^2/2 + x^3/3 + \ldots)$$

we get

$$-\log[1-\gamma/(2\gamma+\sigma^2)]/\gamma = 1/(2\gamma+\sigma^2) + (\text{terms in } \gamma,\gamma^2,\ldots)$$
$$\to 1/\sigma^2 \quad \text{as} \quad \gamma \to 0.$$

This ensures that m^* does not have any odd behaviour at $\gamma = 0$. The same can be said concerning $i=0$ in discrete time.

APPENDIX 4.1

STOCHASTIC DIFFERENTIAL EQUATIONS

1. The Wiener Process
2. White Noise
3. Itô Stochastic Integrals
4. Definition of SDE
5. Itô's Formula
6. Existence of Moments.

This short account of SDE's is intended for people with no background in the subject. It supplies the minimum amount of knowledge needed to understand Section 4.4.

For a thorough treatment of the theory of SDE's, the reader is referred to Arnold (1974), and Gihman and Skorohod (1972). Kallianpur (1980) gives a more up-to-date presentation, using martingale theory.

1. THE WIENER PROCESS

$W(t)$ is a Wiener process if it is a homogeneous Gaussian process, with (i) independent increments, (ii) $W(0) = 0$ w.p. 1 and (iii) $EW(t) = 0$ and $\mathrm{Var}W(t) = t$. It follows that $W(t+h)-W(t)$ is a normal random variable, of mean 0 and variance h.

It is possible to choose a version of $W(t)$ which is continuous w.p. 1.

The Wiener process is one instance of "diffusion processes". These processes have the disconcerting property of being both continuous and nowhere differentiable. Their paths are also of unbounded variation (w.p.1). This explains why stochastic integrals (= integrals with respect to diffusion processes) have to be defined differently from the traditional Stieltjes integrals.

<u>Note</u>: The Wiener process is also known as "Standard Brownian Motion".

2. WHITE NOISE

Having just asserted that $W(t)$ is not differentiable anywhere, I now introduce the derivative of $W(t)$, $dW(t)/dt$, *alias* "white noise".

White noise does have a (mathematical) existence, when seen as a generalized function (or "distribution"). It is, in a certain sense, the continuous-time equivalent of a sequence of i.i.d. normal random variables. Though it is convenient in giving an intuitive idea of the behaviour of diffusion processes, it should be remembered that $dW(t)/dt$ does not in general obey the usual rules of the calculus.

3. STOCHASTIC INTEGRALS

Say $T < \infty$ and $\mathcal{H}_2[0,T]$ = {suitably measurable random functions such that $\int_0^T f(t)^2 dt < \infty$ w.p.1}. The Itô integral is defined in two steps.

1. Consider a process $G(t)$, such that every realization $G(t,w)$ of $G(t)$ is w.p.1 a step function. In other words, $G(t)$ is constant over the intervals $[0,t_1)$, $[t_1,t_2)$, etc., $0 < t_1 < t_2 <\ldots< t_n = T$.

The Itô integral of G (with respect to W) is defined as

$$(4.39)\quad \int_0^T G(t)dW(t) = \sum_{k=1}^{n} G(t_{i-1})[W(t_i)-W(t_{i-1})].$$

2. It can be shown that for any $G \in \mathcal{H}_2[0,T]$ there exists a sequence $\{G_n,\ n \geq 1\}$ of step functions in $\mathcal{H}_2[0,T]$, such that G_n converges to G in the following sense:

$$\lim_{n\to\infty} \int_0^T |G(s)-G_n(s)|^2 ds = 0 \quad \text{w.p.1.}$$

The Itô integral of G is defined as the quadratic mean limit of the integrals of G_n, i.e.

$$\int_0^T G dW = A$$

where A is the random variable such that

$$\lim_n E[A - \int_0^T G_n dW]^2 = 0$$

Here are two basic properties of Itô integrals.

(a) Linearity: if $a, b \in \mathbb{R}$ then

$$\int (aG_1 + bG_2) dW = a\int G_1 dW + b\int G_2 dW.$$

(b) Say $\{H_t, t \geq 0\}$ is the filtration attached to W. (Intuitively, the σ-field H_t represents the information known at time t.)

Suppose $G \in \mathcal{H}_2[0,T]$, and also

$$\int_0^T EG(t)^2 < \infty.$$

Then

$$E\int_0^T G dW = 0$$

and

$$E(\int_0^T G dW)^2 = \int_0^T EG(s)^2 ds.$$

These remain valid when conditioning on the information known at time $a < T$, that is,

$$E(\int_a^b G dW | H_a) = 0$$

and

$$E[(\int_a^b G dW)^2 | H_a] = \int_a^b E[G(s)^2 | H_a] ds,$$

for any $0 \leq a < b \leq T$.

<u>Note</u>: The stochastic integral described above is termed "Itô" because it is not the only possible way of constructing a stochastic integral. For example, the integral in (4.39) has different properties if $\frac{1}{2}[G(t_{i-1})+G(t_i)]$ replaces $G(t_{i-1})$ on the right hand side of the equation. This difference carries over to the limit $\int G = \lim_n \int G_n$ (step 2). The integral so obtained is known as the "Stratonovich" stochastic integral. Stochastic differential equations of the Stratonovich type have also been studied, yielding a theory slightly different from the theory of Itô SDE's. The interested reader is referred to Chapter 10 of Arnold (1974), or to Schuss (1980).

4. ITO STOCHASTIC DIFFERENTIAL EQUATIONS

A stochastic differential equation is an expression of the form

$$(4.40)\qquad dY(t) = b(t,Y(t))dt + \sigma(t,Y(t))dW(t),$$
$$Y(0) = c \text{ w.p.1}, \quad 0 \le t \le T < \infty.$$

Rewrite (4.40) as

$$(4.41)\quad Y(t) = c + \int_0^t b(s,Y(s))ds + \int_0^t \sigma(s,Y(s))dW(s).$$

$X(t)$ is said to be a solution of the Itô SDE (4.40) if $X(t)$ satisfies (4.41) for every t, assuming that

$$\int_0^t \sigma(s,Y(s))dW(s)$$

is taken in the Itô sense.

<u>Theorem</u> (Existence and uniqueness of solutions). If

(i) c is independent of $W(t)$ for $t \ge 0$;

(ii) $b(t,x)$ and $\sigma(t,x)$ are suitably measurable;

(iii) there exists k such that

$$|b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)| \le k|x-y|$$

and

$$b(t,x)^2+\sigma(t,x)^2 \leq k^2(1+x^2)$$

for all $(x,t) \in \mathbb{R}\times[0,T]$;

then Eq. (4.40) has a unique solution $X(t)$ with $X(0) = c$, and which is continuous w.p. 1. (Arnold (1974), p.105).

The equations considered in this text are all linear, that is $b(t,x)$ and $\sigma(t,x)$ are each of the form $Z(t)x+z(t)$, where Z and z are continuous functions of t only. Therefore they always satisfy the requirements of the theorem above, for any $T < \infty$.

5. ITO'S FORMULA

Assume

$$dX(t) = b(t)dt + \sigma(t)dW(t),$$

and let $g(t,x)$ be a function with partial derivatives $g_t'(t,x)$, $g_x'(t,x)$ and $g_{xx}''(t,x)$ that are continuous everywhere in $[0,T]\times\mathbb{R}$.

Then Itô's formula says that the process

$$Y(t) = g(t,X(t))$$

satisfies the SDE

$$dY(t) = (g_t'(t,X(t)) + g_x'(t,X(t))b(t) + \frac{1}{2}g_{xx}''(t,X(t))\sigma(t)^2)dt$$
$$+ g_x'(t,X(t))\sigma(t)dW(t).$$

(Arnold (1974), p.92).

6. EXISTENCE OF MOMENTS

Consider the linear SDE

$$\text{(4.42)} \qquad dX(t) = (A(t)X(t)+a(t))dt + (B(t)X(t)+b(t))dW(t)$$
$$X(0) = c$$

where A, a, B and b are continuous functions of t.

Theorem. The solution of (4.42) has for all $0 \leq t \leq T < \infty$ a p-th-order moment if and only if $E|c|^p < \infty$. In particular

Appendix 4.1

$$\frac{d}{dt} EX(t) = A(t)EX(t) + a(t)$$

$$EX(0) = Ec.$$

(Arnold (1974), pp. 138-139).

The SDE's studied in this text all satisfy the conditions of this theorem, and, moreover, $X(0) = c$ is always a constant. And so $E|F(t)|^p$ and $E|C(t)|^p$ are finite for all $0 \leq t,p < \infty$.

APPENDIX 4.2

PROOF OF PROPOSITION 4.1

The greater part of the proof consists in verifying the definitions and conditions set out on pp. 43-49 of Joffe and Métivier (1986) (designated by "J-M" in what follows). Then Theorem 3.3.1 (p. 49 therein) can be applied.

Define

$D = \{x:\mathbb{R}^+ \to \mathbb{R} \mid x(t+) = x(t)$ and $x(t-)$ exists for all $t \geq 0\}$.

1. THE STOCHASTIC BASES $(\Omega^n, \mathcal{A}^n, \{H_t^n, t \geq 0\}, P^n)$ OF X^n.

I will assume $\Omega^n = D$ and $\mathcal{A}^n = \mathcal{B}(D)$ for all n. ($\mathcal{B}(D)$ is the Borel σ-field on D. It rests on the Skorohod topology of D, discussed on pp. 31-32 of J-M; see also Chapter 3 of Billingsley (1968).)

The filtration $\{H_t^n, t \geq 0\}$ results from letting H_t^n be generated by $\{X^n(s), s \leq t\}$. This implies that X^n is H^n-adapted, and also that $h^n(k)$ is $H^n_{k/n}$-measurable.

2. DEFINITION OF L^n

Let

$$C = \{\phi:\mathbb{R} \to \mathbb{R} \mid \phi'' \text{ is uniformly bounded}\},$$

$$A^n(t) = \frac{1}{n}[nt]$$

and

$$L^n(\phi,x) = nE_{h^n}\{\phi[x + \frac{1}{n}u^n(x) + \frac{1}{\sqrt{n}} v^n(x)\cdot h] - \phi(x)\}$$

where u^n and v^n are the linear functions defined in Prop. 4.1. The expectation is taken w.r. to the distribution of $h^n(\cdot)$.

3. Conditions (D.1) to (D.3) pose no serious problem.

4. LOCAL COEFFICIENTS.

Let $\phi_1(x) = x$ and $\phi_2(x) = x^2$. Then

$$b^n(x) = L^n(\phi_1,x) = u^n(x)$$

and

$$a^n(x) = L^n(\phi_2,x) - 2xb^n(x)$$
$$= \frac{1}{n}u^n(x) + v^n(x)^2.$$

5. TIGHTNESS OF THE SEQUENCE $\{P^n\}$.

Since u^n and v^n are linear functions which converge to u and v, it is clear that there exists $k < \infty$ such that

$$b^n(x)^2+a^n(x) = (1+\frac{1}{n})u^n(x)^2 + v^n(x)^2$$
$$\leq k(1+x^2)$$

for all $n \geq 1$ and $x \in \mathbb{R}$. This takes care of condition (H1) (i). (H1)(ii), (H2) and (H3) are obvious.

6. WEAK CONVERGENCE OF $\{P^n\}$.

Let $C_\emptyset = C$ and define

$$L(\phi,x) = u(x)\phi'(x) + \frac{1}{2}v(x)^2\phi''(x).$$

(H2') and (H5) trivially hold. It remains to verify condition (H4), namely that for all $t < \infty$

$$(4.43) \qquad \lim_{n\to\infty} \int_\emptyset^t E^n|L^n(\phi,X_{s-})-L(\phi,X_{s-})|dA^n(s) = \emptyset.$$

(Here the expectation operator E^n corresponds to P^n.)

Two important facts:

(i) $E_{h^n}h = \emptyset$, $E_{h^n}h^2 = 1$, $E_{h^n}|h| \leq 1$.

(ii) From Lemma 3.2.2 of J-M (p.46), for any $t < \infty$ there exists a constant $Q_t < \infty$ such that

$$E^n \sup_{s\leq t} X_s^2 \leq Q_t \quad \text{for all} \quad n \geq 1.$$

7. PROOF OF (H4)

Define $\|\phi''\| = \sup_x |\phi''(x)|$. From Taylor's Theorem, there exists z (between x and $x+\frac{1}{n}u^n(x)+\frac{1}{\sqrt{n}}v^n(x)h$) such that

$$
\begin{aligned}
&L^n(\phi,x) - L(\phi,x) \\
&= nE_{h^n}\{\phi[x+\tfrac{1}{n}u^n(x)+\tfrac{1}{\sqrt{n}}v^n(x)h] - \phi(x)\} \\
&\qquad -u(x)\phi'(x) - \tfrac{1}{2}v(x)^2\phi''(x) \\
&= E_{h^n}[u^n(x)+h\sqrt{n}\,v^n(x)]\phi'(x) - u(x)\phi'(x) \\
&\quad + \tfrac{1}{2}E_{h^n}[n(\frac{u^n(x)}{n}+\frac{h}{\sqrt{n}}v^n(x))^2\phi''(z)] \\
&\quad - \tfrac{1}{2}v(x)^2\phi''(x) \\
&= (u^n(x)-u(x))\phi'(x) \\
&\quad + \tfrac{1}{2}E_{h^n}[\frac{u^n(x)^2}{n}+\frac{2h}{\sqrt{n}}u^n(x)v^n(x)]\phi''(z) \\
&\quad + \tfrac{1}{2}E_{h^n}\{h^2[v^n(x)^2\phi''(z) - v(x)^2\phi''(x)]\} \\
&= f_1(n,x) + \tfrac{1}{2}f_2(n,x) + \tfrac{1}{2}f_3(n,x).
\end{aligned}
$$

(i) u^n and u are linear in x, and

$$|\phi'(x)| \le |\phi'(0)| + |x|\|\phi''\|.$$

Hence there exists a sequence $\{C_n, n \ge 1\}$ with $C_n \to 0$ such that

$$
\begin{aligned}
|u^n(x)-u(x)||\phi'(x)| &= |(d^n-d)x+e^n-e||\phi'(x)| \\
&\le C_n(1+x^2)
\end{aligned}
$$

and consequently,

$$\int_0^t E^n|f_1(n,X_{s-})|dA^n(s) \le C_n(t+1)(1+E^n \sup_{s\le t} X_s^2)$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

(ii) There exists a constant C such that

$$|f_2(n,x)| \le \frac{C}{\sqrt{n}}(1+x^2)\|\phi''\|$$

and so

$$\int_0^t E^n|f_2(n,X_{s-})|dA^n(s) \to 0 \text{ as } n \to \infty.$$

(iii) Fix $x \in \mathbb{R}$ and let

$$R(n,x,h) = h^2[v^n(x)^2\phi''(z) - v(x)^2\phi''(x)],$$
$$Y_n = |R(n,x,h^n)| \le C'\cdot(h^n)^2, \quad C' \text{ a constant.}$$

Since (1) $\{(h^n)^2\}$ are uniformly integrable and (2) $z = x + \frac{1}{n}u^n(x) + \frac{1}{\sqrt{n}}v^n(x)h \to x$ as $n \to \infty$, the random variables $\{Y_n\}$ are also uniformly integrable, and, moreover, $Y_n \to 0$ in probability.

Therefore

$$\begin{aligned} |f_3(n,x)| &= |E_{h^n}R(n,x,h)| \\ &\le E_{h^n}|R(n,x,h)| \\ &= EY_n \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

from Theorem 9.4C, p.165, of Loève (1977) (or else from Theorem 21, p.36, of Dellacherie and Meyer (1975)).

(iv) Next

$$|f_3(n,X_s)| \le C''(1+X_s^2) \le C''(1+\sup_{s\le t} X_s^2)$$

and so (from (iii) above)

$$E^n|f_3(n,X_s)| \to 0 \quad \text{as} \quad n \to \infty, \tag{4.44}$$

using the Dominated Convergence Theorem.

(v) Finally

$$E^n|f_3(n,X_s)| \leq C''(1+E^n \sup_{s\leq t} X_s^2) \leq C''(1+Q_t) = K_1 < \infty$$

for all n and s, which allows another application of the Dominated Convergence Theorem:

$$\int_0^t E^n|f_3(n,X_{s-})|dA^n(s)$$

$$\leq \int_0^t E^n|f_3(n,X_{s-})|ds + \left|\int_0^t E^n|f_3(n,X_{s-})|(dA^n(s)-ds)\right|$$

$$\leq \int_0^t E^n|f_3(n,X_{s-})|ds + \left|\int_0^t K_1(dA^n(s)-ds)\right|$$

$$\to 0 \quad \text{as} \quad n\to\infty, \qquad \text{from (4.44).}$$

This completes the proof of condition (H4).

8. From Theorem 3.3.1 of J-M, the weak limits (if any) of $\{X^n\}$ are solutions of the martingale problem (L,C,μ_0) where the measure μ_0 is concentrated at $X_0 \in \mathbb{R}$.

It's a standard result of the theory of Itô SDE's that

$$dX(t) = u(X(t))dt + v(X(t))dW(t), \quad X(0) = X_0 \quad \text{w.p. } 1$$

has a unique solution (see Paragraph 4 of Appendix 4.1). This solution is therefore a solution of the martingale problem (L,C,μ_0).

To show that the martingale problem has a unique solution, it is sufficient to check that the moments of $X(t)$, under the limit measure P, are unique.

Let $\phi(x) = x^k$, $k=1,2,\ldots$ The equation

$$E\phi(X(t)) = E\phi(X_0) + E\int_0^t [(dX(s)+e)\phi'(X(s)) + \frac{1}{2}(fX(s)+g)^2\phi''(X(s))ds$$

implies that the moments of $X(t)$ are recursively determined, and therefore unique. (This is the argument used in the proof of Theorem 4.2.2, pp. 53-55 of J-M.)

APPENDIX 4.3

PROOFS OF PROPOSITIONS 4.2 AND 4.3

PROOF OF PROP. 4.2

That $\{h^n(1)^2,\ n\geq 1\}$ is uniformly integrable follows trivially from Eqs. (4.11) and (4.12):

$$h^n(1) = (i^n(1)-i^n)/(\mathrm{Var}\ i^n(\cdot))^{1/2}$$
$$\overset{\text{dist}}{=} (i^1(1)-i^1)/(\mathrm{Var}\ i^1(\cdot))^{1/2}, \quad n\geq 1.$$

(The $h^n(1)$'s all have the same square-integrable distribution.)

(a) Individual cost methods

We have (see Eq. (4.15))

$$u^n(x) = n[(1+i^n)q^n-1]x + n(1+i^n)r^n$$

and

$$v^n(x) = (n\mathrm{Var}\ i^n(\cdot))^{1/2}(q^n x+r^n)$$

where

$$q^n = 1- 1/\ddot{a}_{\overline{m\cdot n|}}$$

and

$$r^n = NC^n + AL^n/\ddot{a}_{\overline{m\cdot n|}} - B^n.$$

I will show that u^n and v^n converge respectively to

(4.45) $\qquad u(x) = -\alpha x+r \quad \text{and} \quad v(x) = \sigma x,$

with

$$\alpha = 1/\bar{a}^{(\gamma)}_{\overline{m|}} -\gamma$$
$$r = NC + AL/\bar{a}^{(\gamma)}_{\overline{m|}} - B,$$
$$\gamma = \log(1+i^1)$$

and

$$\sigma^2 = \log\{[E(1+i^1(\cdot))^2]/[E(1+i^1(\cdot))]^2\}.$$

(i) $\qquad n[(1+i^n)q^n-1] = n[q^n-1+i^nq^n]$
$$= n(q^n-1)+ni^nq^n.$$

Let $\gamma = \log(1+i^1)$. From Eq. (4.9)

Appendix 4.3

$$
\begin{aligned}
ni^n &= n[(1+i^1)^{1/n} - 1] \\
&= n[\exp(\gamma/n) - 1] \\
&= n(\gamma/n + \gamma^2/2n^2 + \ldots) \\
&\to \gamma \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This implies

$$
\begin{aligned}
n(q^n - 1) &= -n/\ddot{a}^{(i^n)}_{\overline{m\cdot n}|} \\
&= -ni^n/\{(1+i^n)[1-\exp(-\gamma\cdot m)]\} \\
&\to -\gamma/[1-\exp(-\gamma m)] \quad = \quad -1/\bar{a}^{(\gamma)}_{\overline{m}|} .
\end{aligned}
$$

This also implies that $q^n \to 1$, and so

$$
n[(1+i^n)q^n - 1] \quad \to \quad -1/\bar{a}_{\overline{m}|} + \gamma \quad = -\alpha .
$$

(ii) $n(1+i^n)r^n$. From the assumptions of Prop. 4.2, and the fact that $n/\ddot{a}_{\overline{mn}|} \to 1/\bar{a}^{(\gamma)}_{\overline{m}|}$, we get

$$
\begin{aligned}
n(1+i^n)r^n &= (1+i^n)(nNC^n + AL^n n/\ddot{a}_{\overline{m\cdot n}|} - nB^n) \\
&\to NC + AL/\bar{a}^{(\gamma)}_{\overline{m}|} - B.
\end{aligned}
$$

(iii) $(n\mathrm{Var}\ i^n(\cdot))^{1/2}(q^n x + r^n)$. From (i) and (ii), $q^n \to 1$ and $r^n \to 0$.

As to the limit of $n\mathrm{Var}\ i^n(\cdot)$, first note that for any positive constants a and b

$$
\begin{aligned}
n(a^{1/n} - b^{1/n}) &= n[\exp(\tfrac{1}{n}\log a) - \exp(\tfrac{1}{n}\log b)] \\
&\to \log a - \log b \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thence, from Eq. (4.10)

$$
n\mathrm{Var}\ i^n(\cdot) \quad \to \quad \log[E(1+i^1(\cdot))^2] - 2\log(1+i^1).
$$

Eqs. (4.45) are established, and Prop. 4.1 can be applied, yielding part (a) of Prop. 4.2.

(b) Aggregate cost methods

Eq. (4.15) holds as well, but with

$$
q^n \quad = \quad 1 - S^n/PVS^n
$$

and

$$
r^n \quad = \quad S^n \cdot PVB^n/PVS^n - B^n.
$$

Appendix 4.3

(i) $$n(q^n-1) = -nS^n/PVS^n \to -S/PVS.$$

(ii) $$n(1+i^n)r^n = (1+i^n)n(S^n PVB^n/PVS^n - B^n)$$
$$\to S\cdot PVB/PVS - B.$$

We conclude that $u^n \to u$ and $v^n \to v$, where

$$u(x) = -\alpha x+r, \quad v(x) = \sigma x,$$
$$\alpha = S/PVS-\gamma, \quad r = S\cdot PVB/PVS-B. \ \square$$

The proof of Prop. 4.3 necessitates the two following lemmas.

<u>Lemma 4.1</u>. Let $y \in \mathbb{R}$.

(a) $$[(e^{ty}-1)/t]^2 \leq y^2+2e^{2y}, \qquad 0 < t \leq 1/2.$$

(b) $$|(e^{ty}-1-ty)/t^2| \leq \frac{1}{2} y^2 + e^{2y}, \quad 0 < t \leq 1.$$

<u>Proof</u> (a) If $g(t) = e^{ty}$, then

$$g(t) = g(0) + tg'(z(t)), \quad 0 < z(t) < t$$

or

$$e^{ty} = 1 + ty\exp[z(t)y]$$

$$\Rightarrow \quad [(e^{ty}-1)/t]^2 = y^2\exp(2z(t)y).$$

If $y \geq 0$, then $y^2 \leq 2\ (1+ y + y^2/2 + \ldots) = 2e^y$, which implies

$$y^2\exp(2z(t)y) \leq y^2e^y \leq 2e^{2y}.$$

If $y < 0$, $\exp(2z(t)y) \leq 1$, and

$$y^2\exp(2z(t)y) \leq y^2.$$

(b) The Taylor expansion

$$e^{ty} = 1 + ty + \frac{t^2}{2} y^2 \exp(z(t)y), \quad 0 < z(t) < 1$$

implies (discerning between the cases $y \geq 0$ and $y < 0$ as in (a))

$$|(e^{ty}-1-ty)/t^2| = \frac{1}{2} y^2\exp(z(t)y) \leq \frac{1}{2} y^2 + e^{2y} . \ \square$$

<u>Lemma 4.2</u>. Let X be a real random variable, and define $f(t) = Ee^{tX}$, $t \geq 0$. Assume

(i) $EX = 0$

(ii) $EX^2 < \infty$

(iii) $Ee^{2X} < \infty$.

Then

$$(4.46) \qquad f(t) = 1 + \frac{t^2}{2} EX^2 + o(t^2) \quad \text{as} \quad t \downarrow 0.$$

<u>Proof</u>. If

$$g_t(x) = [(e^{tx}-1-tx)/t^2]$$

then (Lemma 4.1) $|g_t(x)| \leq \frac{1}{2} x^2 + e^{2x}$. From assumptions (ii) and (iii), together with the Dominated Convergence Theorem

$$\lim_{t \downarrow 0} Eg_t(X) = E \lim_{t \downarrow 0} g_t(X) = \frac{1}{2} X^2 ,$$

which proves (4.46) (since $EX = 0$). □

<u>PROOF OF PROP. 4.3</u>

Let

$$(4.47) \qquad h^n(k) = (i^n(k)-i^n)/\mathrm{Var}\ i^n(k))^{1/2}$$

with $i^n(\cdot)$ as in Eq. (4.21).

Eq. (4.15) results once more, with the same expressions for $u^n(x)$ and $v^n(x)$.

Let

$$f(t) = E \exp[t(\gamma^1(1)-E\gamma^1(1))].$$

Since $E \exp[2\gamma^1(\cdot)] = E(1+i^1(\cdot))^2 < \infty$ and $\mathrm{Var}\ \gamma^1(\cdot) < \infty$, the assumptions of Lemma 4.2 are satisfied, and

$$(4.48) \qquad f(t) = 1 + \frac{t^2}{2} \mathrm{Var}\ \gamma^1(\cdot) + o(t^2).$$

Appendix 4.3

It is convenient to first calculate the limits of ni^n and $n\mathrm{Var}\ i^n(\cdot)$, before proving that $\{h^n(1)^2,\ n \geq 1\}$ is uniformly integrable.

1. ni^n. We have (Eq. (4.20))

$$\begin{aligned} 1+i^n &= E\exp\{E\gamma^1(1)/n + [\gamma^1(1)-E\gamma^1(1)]/\sqrt{n}\} \\ &= \exp(\gamma^1/n)f(1/\sqrt{n}) \qquad (\gamma^1 = E\gamma^1(\cdot)). \end{aligned}$$

Thus

$$\begin{aligned} ni^n &= n[\exp(\gamma^1/n)\ f(1/\sqrt{n}) -1] \\ &= n\{\exp(\gamma^1/n)[1+ \frac{1}{2n} \mathrm{Var}\ \gamma^1(\cdot) + o(n^{-1})] -1\} \\ &= n[\exp(\gamma^1/n) -1] \\ &\quad + n\exp(\gamma^1/n)[\frac{1}{2n} \mathrm{Var}\ \gamma^1(\cdot) + o(n^{-1})] \\ &\to \gamma^1 + \frac{1}{2} \mathrm{Var}\ \gamma^1(\cdot) \quad \text{as} \quad n \to \infty. \end{aligned}$$

2. $n\ \mathrm{Var}\ i^n(\cdot)$.

$$\begin{aligned} n\mathrm{Var}\ i^n(\cdot) &= n\mathrm{Var}(1+i^n(\cdot)) \\ &= n\mathrm{Var}\ \exp(\gamma^n(\cdot)) \\ &= nE\exp(2\gamma^n(\cdot)) - n[E\exp(\gamma^n(\cdot))]^2 \\ &= n\exp(2\gamma^1/n)[f(2/\sqrt{n}) - f(1/\sqrt{n})^2]. \end{aligned}$$

From Eq. (4.46)

$$\begin{aligned} f(2/\sqrt{n}) &= 1+ 2[\mathrm{Var}\ \gamma^1(\cdot)]/n + o(n^{-1}) \\ f(1/\sqrt{n})^2 &= 1 + [\mathrm{Var}\ \gamma^1(\cdot)]/n + o(n^{-1}) \end{aligned}$$

and so

$$n\mathrm{Var}\ i^n(\cdot) \to \mathrm{Var}\ \gamma^1(\cdot).$$

3. $\{h^n(1)^2,\ n \geq 1\}$ is uniformly integrable.

(i) From the definition of $h^n(1)$ (Eq. (4.47)),

$$h^n(1)^2 \leq 2ni^n(1)^2/[n\mathrm{Var}\ i^n(\cdot)] + 2n(i^n)^2/[n\mathrm{Var}\ i^n(\cdot)].$$

From steps 1 and 2 above, the second term on the right hand side converges to 0. Moreover $\{1/[n\mathrm{Var}\ i^n(\cdot)]\}$ has a finite limit, and is thus uniformly bounded. Hence it is sufficient to show that $\{ni^n(1)^2,\ n\geq 1\}$ is uniformly integrable.

To simplify the equations, define

$$\Delta\gamma^1(1) = \gamma^1(1) - E\gamma^1(1).$$

(ii) We have

$$i^n(1)^2 = (\exp[\gamma^n(1)]-1)^2$$

$$\le 2(\exp[\gamma^n(1)] - \exp(\gamma^1/n))^2 + 2(\exp(\gamma^1/n)-1)^2$$

$$\Rightarrow \quad ni^n(1)^2 \le 2\exp(2\gamma^1/n)\cdot\left[\frac{\exp[\Delta\gamma^1(1)/\sqrt{n}] - 1}{1/\sqrt{n}}\right]^2$$

$$+ 2\left[\frac{\exp(\gamma^1/n)-1}{1/\sqrt{n}}\right]^2$$

$$\le 2\exp(2\gamma^1/n)(\Delta\gamma^1(1)^2 + 2\exp[2\Delta\gamma^1(1)])$$

$$+ (\gamma^1)^2 + 2e^{2\gamma^1} \qquad (n \ge 4)$$

from Part (a) of Lemma 4.1 (let $t = 1/\sqrt{n}$).

Finally, the supremum over n of the right hand side of the previous inequality is an integrable random variable, since Prop. 4.3 assumes that $Ei^1(\cdot)^2$ and $\mathrm{Var}\ \gamma^1(1)$ are finite. It follows that $\{ni^n(1)^2,\ n \ge 1\}$ is uniformly integrable.

4. Given the limits of ni^n and $n\mathrm{Var}\ i^n(\cdot)$ calculated in steps 1 and 2, the functions $u^n(\cdot)$ and $v^n(\cdot)$ converge to the same limits as in Props. 4.2, except for the fact that γ and σ^2 are now

$$\gamma = \lim_n ni^n = E\gamma^1(\cdot) + \frac{1}{2}\mathrm{Var}\ \gamma^1(\cdot)$$

and

$$\sigma^2 = \mathrm{Var}\ \gamma^1(\cdot). \quad \square$$

CHAPTER 5

MOMENTS OF ANNUITIES CERTAIN

5.1 INTRODUCTION

The techniques developed in Chapters 3 and 4 can also be used to calculate the moments of annuities-certain. This is what Chapter 5 proposes to show.

Section 5.2 examines the literature on the subject, especially as regards the interest rate processes the different authors have considered. This in turn leads to a brief discussion of the i.i.d. assumption of Chapters 3 and 4.

The moments of discrete annuities-certain are derived in Section 5.3. Their continuous-time counterparts are dealt with in 5.5, after showing convergence to diffusion processes in 5.4.

Boyle (1976) remarks that

> In the case of deterministic rates there is a neat reciprocal relationship between accumulating and discounting. With stochastic interest rates this relationship no longer holds (p. 695)

In Sections 5.3 to 5.5, it will indeed be seen that, when calculating first moments, the accumulating and discounting rates are different. The discrepancy between the two rates is quantified in Section 5.6, in the case of i.i.d. rates of return.

Section 5.7 is a short comment on moments higher than the second.

The results of Section 5.3 are those of Boyle (1976). Their continuous-time versions could also be deduced from the more general formulae of Panjer and Bellhouse (1980). The purpose of these sections is to indicate another way of obtaining the moments of annuities-certain, based on recursive or differential equations.

5.2 THE INTEREST RATE PROCESS

Let $\{i(t),\ t\geq 1\}$ be a sequence of random rates of interest. Notice the similarity between

$$(5.1)\qquad F(t+1) = (1+i(t+1))F(t)+C(t)-B(t))$$

and

$$(5.2)\qquad \ddot{s}(t+1) = (1+i(t+1))(\ddot{s}(t)+1).$$

$\ddot{s}(t)$ is the accumulated value, at time t, of payments of 1 unit made at times $\emptyset,1,\ldots,t-1$. In general, B(t) is not constant, and, furthermore, C(t) depends on i(s), $s \leq t$. In consequence, it is not always true that the statistical properties of F(t) can be studied along the same lines as those of $\ddot{s}(t)$.

Nevertheless, Chapter 3 has shown that in some cases Eq. (5.1) can be rewritten as

$$F(t+1) = [(1+i(t+1))/(1+i)](q(t)F(t)+r(t))$$

where q and r are independent of $\{i(t)\}$. As Section 5.3 will demonstrate, the moments of $\ddot{s}(t)$ can be calculated in exactly the same way as those of F(t), when $\{i(t)\}$ is an i.i.d. sequence.

Thus, when asking ourselves whether the analysis of Chapters 3 and 4 can be broadened to include more general processes $\{i(t)\}$, it is only natural to investigate what is already known about the simpler case of $\ddot{s}(t)$. This is why this discussion of the interest rate process is included in the chapter on annuities-certain.

Pollard (1971) assumed that the force of interest is a particular autoregressive process of order two:

$$\gamma(t)-\gamma_{\emptyset} = 2k(\gamma(t-1)-\gamma_{\emptyset}) + k(\gamma(t-2)-\gamma_{\emptyset}) - e(t)$$

where $\emptyset \leq k < 1$ and $\{e(t),\ t \geq 1]$ is an i.i.d. sequence of normal random variables (with $Ee(t) = \emptyset$). If

$$v(t) = 1/(1+i(t)) = \exp(-\gamma(t))$$

is the discount factor during $(t-1,t)$, then

$$\sum_{j=1}^{k} v(j) = v_{\emptyset}^{k}\exp[Z(k)]$$

where

$$Z(k) = \sum_{j=1}^{k} [\gamma_{\emptyset}-\gamma(j)]$$

is a normal random variable. Hence the first two moments of $Z(k)$ determine the distribution of $\sum_{j=1}^{k} v(j)$. By summing over k, they also determine the distribution of the discounted annuity certain

$$a(t) = \sum_{k=1}^{t} \sum_{j=1}^{k} v(j).$$

The author provides summation formulae for the moments of $Z(k)$, and approximations for the first and second moments of $a(t)$.

Wilke (1976) supposes $\{v(t),\ t\geq 1\}$ to be a sequence of i.i.d. normal random variables. This amounts to setting $k=\emptyset$ in Pollard's model. He points out the following recursive relationship for $E\ddot{s}(t)^n$ (attributed to R.E. Beard):

$$(5.3)\quad E\ddot{s}(t+1)^n = [E\exp(n\cdot\gamma(t+1))]\cdot[1+\sum_{j=1}^{n} \binom{n}{j}E\ddot{s}(t)^j]$$

for $1 \leq n \leq 4$. It may be noted that this is essentially the approach adopted to calculate the first and second moments of $F(t)$ in Section 3.3 (Spread method).

(Using the conditioning technique of Chapter 3, it is obvious that the same formulae holds for any $n \geq 1$. See Section 5.7.)

Boyle (1976) assumes the returns $\{i(t)\}$ to be i.i.d.. By summing the cross product moments of

$$\sum_{j=1}^{k} v(j) \quad \text{and} \quad \sum_{j=1}^{k} (1+i(j)),$$

the first three moments of $a(t)$ and $\ddot{s}(t)$ are derived. They are expressed as simple combinations of the usual annuities-certain functions, evaluated at different rates of interest.

Waters (1978) supposes $\{\gamma(t),\ t \geq 1\}$ to be an i.i.d. normal sequence. He calculates the first four moments of $a(t)$ but leaves them in summation form.

Panjer and Bellhouse (1980) first consider a general stationary process $\{\gamma(t),\ t \geq 1\}$. After defining

$$Z(t) = \sum_{k=1}^{t} \gamma(k),$$

they show how the first and second moments of $a(t)$ can be expressed in terms of the moment generating function of $Z(t)$. Next, they specify this moment generating function in the case of autoregressive processes of order one and two. The same analysis is carried out for a continuous-time process $\{\gamma(t),\ t \in \mathbb{R}^+\}$.

A second paper, Bellhouse and Panjer (1981), extends these results to conditional autoregressive processes (i.e. which depend on current rates of interest).

These two papers generalize the results of all the ones previously mentioned. The formulae are compact and intelligible, though the actual computation of moments is apparently no trivial matter. There is a fairly high price to pay for the extra "realism" of autoregressive processes, in comparison with the simplicity of Boyle's formula.

Wescott (1981) closely follows Pollard (1971), assuming the more general second order autoregressive process

$$y(t)-y_{\emptyset} = a(y(t-1)-y_{\emptyset}) - b(y(t-2)-y_{\emptyset}) + e(t),$$

where $4b > a^2$ and $\{e(t),\ t \geq 1\}$ is again an i.i.d. sequence of normal random variables. The first four moments of $\ddot{s}(t)$ are elicited, using the method suggested by Pollard, that is expressing $E\ddot{s}(t)^k$ as summations of the cross product moments

$$E \sum_{j=1}^{s} \exp[y(j)-y_{\emptyset}] \sum_{j=1}^{r} \exp[y(j)-y_{\emptyset}].$$

In the context of ruin theory, Schnieper (1983) considers a process $\{X(t),\ t \geq 1\}$ of i.i.d. cash flows independent of the process of discount factors $\{v(t),\ t \leq 1\}$. The latter are supposed to form a Markov chain. Since he is mostly concerned with ultimate ruin, the author is more interested in the first two moments of

$$\lim_{t\to\infty} \tilde{a}(t) = \sum_{k=1}^{\infty} X(k) \prod_{j=1}^{k} v(j).$$

But his formulae also enable one to calculate the same moments for finite t. This could be done using the "discounted transition matrices" that are specified in the paper.

I now leave annuities-certain aside, and briefly turn to the "term structure of interest rates". In financial economics, it has been attempted, based on theoretical considerations, to determine which stochastic processes should describe interest rates. One example is Vasicek (1977), who ends up suggesting the Ornstein-Uhlenbeck process. Boyle (1978) applied this model to a problem of immunization.

Beekman (1973) and Beekman and Fuelling (1977) have also used the Ornstein-Uhlenbeck process to represent "investment deviations", as part of a general collective risk model.

<u>Concluding remarks</u>. There is statistical evidence that autoregressive processes of order one or two describe historical interest rates better than i.i.d. sequences (Wilkie (1978), Panjer and Bellhouse (1980)). Hence the model of Chapters 3 and 4 is not as realistic as it could be. Further research is needed in order to improve the model in this respect.

However, the parallel with annuities-certain seems to indicate that replacing white noise with an autoregressive or Markov process will make the model much less tractable.

This suggests that the white noise model should also be taken further. As Chapters 3 and 4 have shown, the white noise assumption often leads to results which are both general and explicit. This may not be the case with more realistic interest rate processes.

5.3 DISCRETE TIME

Assume $\{i(t),\ t \geq 1\}$ is a. i.i.d. sequence, and define

$$i = Ei(t), \quad u(t) = 1+i(t),$$

$$u = Eu(t) = 1+i$$

$$v(t) = 1/(1+i(t)), \qquad v = Ev(t),$$

$$u_2 = Eu(t)^2, \qquad v_2 = Ev(t)^2.$$

Notice that $v \neq 1/u$ (see Section 5.6). As before H_t is the σ-field generated by $\{i(1),\ldots,i(t)\}$.

5.3.1 Accumulated Values

Let $\ddot{s}(0) = 0$ and

$$\ddot{s}(t+1) = u(t+1)(\ddot{s}(t)+1). \tag{5.4}$$

Clearly

$$E\ddot{s}(t+1) = u(E\ddot{s}(t)+1)$$

and so

$$E\ddot{s}(t) = \sum_{j=0}^{t-1} u^{t-j}$$

$$= (u^{t}-1)u/(u-1)$$

(5.5) $$= \ddot{s}_{\overline{t}|}^{(i)}.$$

It will be shown that

(5.6) $$\text{Var } \ddot{s}(t) = \frac{u_2+u}{u_2-u}\, \ddot{s}_{\overline{t}|}^{(u_2-1)} - \frac{2u_2}{u_2-u}\, \ddot{s}_{\overline{t}|}^{(i)} - (\ddot{s}_{\overline{t}|}^{(i)})^2.$$

(This is Eq. (2.19), p. 698, of Boyle (1976).)

In view of Eq. (5.5), this is the same as showing that

(5.7) $$(u_2-u)E\ddot{s}(t)^2 = (u_2+u)\ddot{s}_{\overline{t}|}^{(u_2-1)} - 2u_2\ddot{s}_{\overline{t}|}^{(i)}.$$

$$= A(t) .$$

Observe that (5.7) holds for t=1, as

$$E\ddot{s}(1)^2 = u_2$$

and

$$A(1) = (u_2+u)u_2 - 2u_2u$$

$$= (u_2-u)u_2.$$

Using the usual conditioning techniques, we find

$$E\ddot{s}(t+1)^2 = EE(\ddot{s}(t+1)^2|H_t)$$

$$= u_2E\ddot{s}(t)^2 + 2u_2E\ddot{s}(t) + u_2,$$

and so

$$(u_2-u)E\ddot{s}(t+1)^2$$

$$= u_2[(u_2-u)E\ddot{s}(t)^2] + 2u_2(u_2-u)\ddot{s}_{\overline{t}|}^{(i)} + u_2(u_2-u).$$

It remains to show that A(t) (i.e. the right hand side of (5.7)) satisfies

$$(5.8)\quad A(t+1) = u_2A(t) + 2u_2(u_2-u)\ddot{s}_{\overline{t|}}^{(i)} + u_2(u_2-u).$$

From the definition of $A(t)$,

$$A(t+1) = (u_2+u)(u_2\ddot{s}_{\overline{t|}}^{(u_2-1)}+u_2) - 2u_2(u\ddot{s}_{\overline{t|}}^{(i)}+u)$$

$$= u_2(u_2+u)\ddot{s}_{\overline{t|}}^{(u_2-1)} -2u_2u\ddot{s}_{\overline{t|}}^{(i)} +u_2(u_2-u)$$

$$= u_2[(u_2+u)\ddot{s}_{\overline{t|}}^{(u_2-1)} -2u_2\ddot{s}_{\overline{t|}}^{(i)}]$$

$$+2u_2(u_2-u)\ddot{s}_{\overline{t|}}^{(i)} + u_2(u_2-u)$$

which proves (5.8), and completes the proof of Eq. (5.6).

5.3.2 Discounted Values

When the rate of interest is constant,

$$a_{\overline{t+1|}} = v(a_{\overline{t|}} + 1).$$

When it is random, we may define

$$a(t+1) = v(t+1)(a(t)+1), \quad a(0) = 0.$$

Then

$$Ea(t+1) = v(Ea(t)+1)$$

and

$$Ea(t) = (1-v^t)v/(1-v)$$
$$= a_{\overline{t|}}^{(j)}$$

where

$$j = 1/v - 1$$
$$= [1/Ev(t)]-1.$$

As in Section 3.5.1, it can be proved that

$$(5.9)\quad \text{Var } a(t+1) = \frac{v_2+v}{v_2-v}\, a_{\overline{t|}}^{(j_2)} - \frac{2v_2}{v_2-v}\, a_{\overline{t|}}^{(j)} - (a_{\overline{t|}}^{(j)})^2$$

where j is as before and

$$j_2 = 1/v_2 - 1$$
$$= [1/Ev(t)^2] - 1$$

(Eq. (5.9) is equivalent to Eq. (2.22), p. 699, of Boyle (1976).)

5.4 CONVERGENCE TO DIFFUSIONS

Imagine payments of 1/n unit made at times, 0, 1/n, 1/n, etc. and a sequence $\{i^n(k),\ k \geq 1\}$ of i.i.d. rates of return. $i^n(k)$ is the return during the period $(\frac{k-1}{n}, \frac{k}{n})$. The accumulating process $\ddot{s}^n$ can be defined recursively as follows:

$$(5.10) \qquad \ddot{s}^n(\frac{k+1}{n}) = (1+i^n(k+1))(\ddot{s}^n(\frac{k}{n}) + \frac{1}{n}),$$
$$\ddot{s}^n(0) = 0.$$

Prop. 5.1 specifies the weak limit of $\{\ddot{s}^n,\ n \geq 1\}$, when the sequence $\{i^n(k),\ k \geq 1\}$ are defined as in Chapter 4.

Proposition 5.1. Let $Ei^1(k)^2 < \infty$. The processes $\ddot{s}^n$ converge weakly to a diffusion $\bar{s}$ satisfying the Itô SDE

$$(5.11) \qquad d\bar{s}(t) = (\gamma\bar{s}(t)+1)dt + \sigma\bar{s}(t)dW(t)$$
$$\bar{s}(0) = 0.$$

γ and σ^2 are as in Prop. 4.2 or Prop. 4.3, depending on whether $i^n(\cdot)$ is defined by Eq. (4.11) or Eq. (4.21).

(The second way of defining $i^n(\cdot)$ requires the further assumption that

$$\text{Var} \log(1+i^1(\cdot)) < \infty\ ).$$

Proof. Rewrite (5.10) as

$$\ddot{s}^n\left(\frac{k+1}{n}\right) = \ddot{s}^n\left(\frac{k}{n}\right) + i^n\ddot{s}^n\left(\frac{k}{n}\right) + (1+i^n)\cdot\frac{1}{n}$$

$$+ \frac{i^n(k+1)-i^n}{\sqrt{\mathrm{Var}\ i^n(\cdot)}} \cdot \sqrt{\mathrm{Var}\ i^n(\cdot)}\ \left(\ddot{s}^n\left(\frac{k}{n}\right) + \frac{1}{n}\right)$$

$$= \ddot{s}^n\left(\frac{k}{n}\right) + \frac{1}{n}\, U^n\left(\ddot{s}^n\left(\frac{k}{n}\right)\right) + \frac{1}{\sqrt{n}}\, V^n\left(\ddot{s}^n\left(\frac{k}{n}\right)\right)h^n(k+1)$$

where

$$U^n(x) = ni^n x + 1 + i^n$$

$$V^n = \sqrt{n\mathrm{Var}\ i^n(\cdot)}\ \left(x + \frac{1}{n}\right)$$

$$h^n(k+1) = \frac{i^n(k+1)-i^n}{\sqrt{\mathrm{Var}\ i^n(\cdot)}}\ .$$

In Appendix 4.3 it was shown that $ni^n \to \gamma$ and $n\mathrm{Var}\ i^n \to \sigma^2$, and so U^n and V^n have the limits

$$U(x) = \gamma x+1, \quad V(x) = \sigma x.$$

Weak convergence is a consequence of Prop. 4.1. □

The discounting processes a^n can similarly be defined by

(5.12) $$a^n\left(\frac{k+1}{n}\right) = v^n(k+1)\left(a^n\left(\frac{k}{n}\right) + \frac{1}{n}\right)$$

where $v^n(k) = 1/(1+i^n(k))$.

<u>Proposition 5.2</u>. Let $Ev^1(t)^2 < \infty$.

(a) If $i^n(\cdot)$ is given by Eq. (4.11), assume $Ei^1(t)^2 < \infty$.

(b) If $i^n(\cdot)$ is given by Eq. (4.21), assume $E\gamma^1(t)^2 < \infty$.

The processes a^n converge weakly to the diffusion $\bar{a}$ satisfying

(5.13) $$d\bar{a}(t) = [-(\gamma-\sigma^2)\bar{a}(t)+1]dt + \sigma\bar{a}(t)dW(t)$$
$$\bar{a}(0) = 0,$$

where γ and σ^2 are as in Prop. 4.2 (case (a)) or Prop. 4.3 (case (b)).

The proof is in Appendix 5.1.

5.5 CONTINUOUS TIME

5.5.1 Accumulated Values

In accordance with Prop. 5.1, consider the process $\bar{s}$ satisfying

$$\text{(5.14)} \qquad d\bar{s}(t) = (\gamma\bar{s}(t)+1)dt + \sigma\bar{s}(t)dW(t)$$
$$\bar{s}(0) = 0.$$

It is immediately seen that

$$\frac{d}{dt}E\bar{s}(t) = \gamma E\bar{s}(t) + 1$$

and so

$$E\bar{s}(t) = \int_0^t e^{\gamma(t-r)}dr$$

$$= (e^{t\gamma}-1)/\gamma$$

$$\text{(5.15)} \qquad = \bar{s}_{\overline{t}|}^{(\gamma)}.$$

Let $g(x) = x^2$ and use Itô's formula (Appendix 4.1) to obtain

$$d\bar{s}(t)^2 = [g'_x(\bar{s}(t))(\gamma\bar{s}(t)+1) + \frac{1}{2}g''_{xx}(\bar{s}(t))\sigma^2\bar{s}(t)^2]dt$$
$$+ (\ldots)dW(t)$$
$$= [2\bar{s}(t)(\gamma\bar{s}(t)+1) + \sigma^2\bar{s}(t)^2]dt + (\ldots)dW(t)$$
$$= [(2\gamma+\sigma^2)\bar{s}(t)^2 + 2\bar{s}(t)]dt + (\ldots)dW(t).$$

Define $k = 2\gamma+\sigma^2$. We get

$$\frac{d}{dt}E\bar{s}(t)^2 = k\cdot E\bar{s}(t)^2 + 2E\bar{s}(t)$$

which implies, using Eq. (5.15),

$$E\bar{s}(t)^2 = 2\int_0^t e^{k(t-u)}E\bar{s}(u)du$$

$$= 2\int_0^t e^{ku}E\bar{s}(t-u)du$$

$$= 2\int_0^t e^{ku}\int_0^{t-u} e^{\gamma(t-u-v)}dv\ du$$

$$= 2\int_0^t e^{\gamma(t-v)}\int_0^{t-v} e^{(k-\gamma)u}\ du\ dv$$

(changing the order or integration)

$$= \frac{2}{k-\gamma}\int_0^t e^{\gamma(t-v)}(e^{(k-\gamma)(t-v)}-1)\ dv$$

$$= \frac{2}{k-\gamma}\int_0^t e^{k(t-v)} - e^{\gamma(t-v)}\ dv$$

$$= \frac{2}{k-\gamma}\ (\bar{s}_{\overline{t|}}^{(k)} - \bar{s}_{\overline{t|}}^{(\gamma)}).$$

Therefore

$$(5.16)\quad \mathrm{Var}\ \bar{s}(t) = \frac{2}{\gamma+\sigma^2}\ (\bar{s}_{\overline{t|}}^{(2\gamma+\sigma^2)} - \bar{s}_{\overline{t|}}^{(\gamma)}) - (\bar{s}_{\overline{t|}}^{(\gamma)})^2.$$

5.5.2 Discounted Values

Consider the process

$$(5.17)\qquad d\bar{a}(t) = (-\gamma_1\bar{a}(t)+1)dt + \sigma\bar{a}(t)dW(t)$$

$$\bar{a}(0) = 0$$

where $\gamma_1 = \gamma-\sigma^2$ (see Prop. 5.2).

Eq. (5.17) has the same form as Eq. (5.14), and so (from Eqs. (5.15) and (5.16))

$$E\bar{a}(t) = \bar{s}_{\overline{t|}}^{(-\gamma_1)}$$

$$= \bar{a}_{\overline{t|}}^{(\gamma-\sigma^2)}$$

$$\operatorname{Var} \bar{a}(t) = \frac{2}{-\gamma_1+\sigma^2}\left(\bar{s}_{\overline{t|}}^{(-2\gamma_1+\sigma^2)} - \bar{s}_{\overline{t|}}^{(-\gamma_1)}\right) - \left(\bar{s}_{\overline{t|}}^{(-\gamma_1)}\right)^2$$

$$= \frac{2}{2\sigma^2-\gamma}\left(\bar{a}_{\overline{t|}}^{(2\gamma-3\sigma^2)} - \bar{a}_{\overline{t|}}^{(\gamma-\sigma^2)}\right) - \left(\bar{a}_{\overline{t|}}^{(\gamma-\sigma^2)}\right)^2$$

since $\gamma_1 = \gamma-\sigma^2$, and, for any r,

$$\bar{s}_{\overline{t|}}^{(-r)} = (1-e^{-rt})/r$$

$$= \bar{a}_{\overline{t|}}^{(r)}.$$

The lack of symmetry between the formulae for the moments of $\bar{s}(t)$, on the one hand, and those for the moments of $\bar{a}(t)$, on the other, is explained in Section 5.6.

5.6 ACCUMULATING AND DISCOUNTING RATES

In Section 5.3, we have seen that, in discrete time,

$$Es(t) = s_{\overline{t|}}^{(i)} \quad \text{and} \quad Ea(t) = a_{\overline{t|}}^{(j)}$$

where

$$i = Ei(\cdot) \quad \text{and} \quad j = E\left(\frac{1}{1+i(\cdot)}\right)^{-1} - 1.$$

I first show that $j < i$ (of course assuming $\operatorname{Var} i(t) > 0$). Let

$$f(x) = 1/(1+x), \quad x > -1.$$

Consider the second-order Taylor series for $f(x)$, centered at x_0:

$$\frac{1}{1+x} = \frac{1}{1+x_0} + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(z).$$

Letting $x = i(\cdot)$, $x_0 = Ei(\cdot)$, and using the fact that f'' is always strictly positive, we find

$$E\frac{1}{1+i(\cdot)} = \frac{1}{1+Ei(\cdot)} + E(i(\cdot)-E(\cdot))f'(Ei(\cdot))$$

$$+ \frac{1}{2}E[(i(\cdot)-Ei(\cdot))^2 f''(z)]$$

$$> \frac{1}{1+Ei(\cdot)}$$

$$\Rightarrow \quad j = (E\frac{1}{1+i(\cdot)})^{-1} -1 < 1+Ei(\cdot) -1 = i.$$

Next, it will be shown that $i-j = [\mathrm{Var}\ i(\cdot)]/(1+i)$.

First assume $Ei(\cdot) = i = 0$. Then, from Lemma 5.1 (Appendix 5.1),

$$E\frac{1}{1+i(\cdot)} = 1 - E\frac{i(\cdot)}{1+i(\cdot)}$$

$$\simeq 1 + \mathrm{Var}\ i(\cdot)$$

$$\Rightarrow [E(\frac{1}{1+i(\cdot)})]^{-1} \simeq \frac{1}{1+\mathrm{Var}\ i(\cdot)} \simeq 1 - \mathrm{Var}\ i(\cdot).$$

If $i \neq 1$, then, letting $u = 1+i$ and $\Delta i(\cdot) = i(\cdot)-i$,

$$[E(\frac{1}{1+i(\cdot)})]^{-1} = u[E(\frac{u}{1+i(\cdot)})]^{-1}$$

$$= u[E(\frac{1}{1+\Delta i(\cdot)/u})]^{-1}$$

$$\simeq u(1-\mathrm{Var}[\Delta i(\cdot)/u])$$

$$= 1+i-[\mathrm{Var}\ i(\cdot)]/(1+i).$$

Finally,

$$i-j = 1+i- [E(\frac{1}{1+i(\cdot)})]^{-1} \simeq [\mathrm{Var}\ i(\cdot)]/(1+i).$$

<u>Example</u>. Say $i(\cdot)$ is $U[0,a]$, that is, $i(\cdot)$ has a density function equal to $1/a$ in the interval $[0,a]$ and equal to 0 elsewhere. Then

$$Ei(\cdot) = a/2$$

and

$$\mathrm{Var}\ i(\cdot) = (1/a)\int_0^a x^2 dx - a^2/4$$

$$= a^2/12.$$

Furthermore

$$E\,\frac{1}{1+i(\cdot)} = \frac{1}{a}\int_0^a \frac{dx}{1+x} = \frac{1}{a}\log(1+a)$$

and

$$j = \frac{a}{\log\ (1+a)} - 1.$$

(i) If $a = .20$, then $i = .10$, $j = .096962989\ldots$ and

$$i-j = .0030370104\ldots$$

while

$$[\mathrm{Var}\ i(\cdot)]/(1+i) = a^2/[12(1+\ a/2)]$$

$$= .00\overline{30}.$$

(ii) In the more extreme case where $a = .50$,

$$i = .25, \quad j = .2331517312\ldots$$

$$i-j = .0168482688\ldots$$

The approximation for $i-j$ is

$$a^2/[12(1+\ a/2)] = .01\bar{6}.$$

(The relative error is 1.1 % .)

The convergence results of Section 5.4 show that in continuous time the difference between the accumulating and discounting rates is $\gamma-\gamma_1 = \gamma-(\gamma-\sigma^2) = \sigma^2$. This can be seen to agree with the approximation given above for the discrete rates:

$$i^n-j^n \simeq [\mathrm{Var}\ i^n(\cdot)]/(1+i^n)$$

$$\Rightarrow \qquad n(i^n-j^n) \simeq n[\mathrm{Var}\ i^n(\cdot)]/(1+i^n)$$

$$\to \sigma^2$$

since $ni^n \to \gamma$ and $n\mathrm{Var}\ i^n(\cdot) \to \sigma^2$.

A question related to the difference between γ and γ_1 is the asymmetry of the formulae for the moments of $\bar{s}(t)$ and $\bar{a}(t)$ (Section 5.5). This will be accounted for by showing that the force of interest "processes" underlying the accumulating and discounting processes are

$$(5.18) \qquad A(t) = \gamma_0 + \sigma dW(t)/dt$$

and

$$(5.19) \qquad D(t) = -\gamma_0 + \sigma dW(t)/dt$$

respectively, where $\gamma_0 = \gamma - \frac{1}{2}\sigma^2$.

First, the stochastic process representing "the sum of the forces of interest up to time t" will be specified. Define (in the setting of Section 5.4)

$$Z^n(t) = \sum_{j=1}^{[nt]} \gamma^n(j), \qquad \gamma^n(j) = \log[1+i^n(j)]$$

$$Z^n(0) = 0.$$

The weak limit of the processes Z^n is determined as follows. If

$$Y^n(t) = \exp Z^n(t)$$

then

$$Y^n\left(\frac{k+1}{n}\right) = [1 + i^n(k+1)]Y^n\left(\frac{k}{n}\right)$$

which amounts to

$$Y^n\left(\frac{k+1}{n}\right) = Y^n\left(\frac{k}{n}\right) + \frac{1}{n} U^n\left(Y^n\left(\frac{k}{n}\right)\right) + \frac{1}{\sqrt{n}} V^n\left(Y^n\left(\frac{k}{n}\right)\right)h^n(k+1)$$

where

$$U^n(x) = ni^n x \to \gamma x = U(x)$$

$$V^n(x) = \sqrt{n\mathrm{Var}\ i^n(\cdot)}\ x \to \sigma x = V(x).$$

Prop. 4.1 tells us that Y^n converges weakly to the solution of the SDE

$$dY = \gamma Y dt + \sigma Y dW.$$

Thus $Z^n = \log Y^n$ converges weakly to $Z = \log Y$, since $h(x) = \log x$ is continuous (for a justification of this assertion, see Billingsley (1968), p.29).

Therefore, let $g(x) = \log x$ and use Itô's formula to obtain

$$dZ = [\gamma Y \cdot \frac{1}{Y} + \frac{1}{2}\sigma^2 Y^2 \cdot (-\frac{1}{Y^2})]dt + \sigma y \cdot \frac{1}{Y} dW$$

$$= (\gamma - \frac{1}{2}\sigma^2)dt + \sigma dW$$

$$\Rightarrow \quad Z(t) = \int_0^t (\gamma - \frac{1}{2}\sigma^2)dt + \sigma\int_0^t dW(t)$$

$$= \gamma_0 \cdot t + \sigma W(t).$$

That is, the sum of the forces of interest up to time t is, in the limit, a Brownian Motion with mean $\gamma_0 t$ and variance $\sigma^2 t$. This justifies (5.18); (5.19) is handled similarly.

The equations of Sections 5.4 can be restated as

$$d\bar{s}(t) = [(\gamma_0 + \frac{1}{2}\sigma^2)\bar{s}(t)+1]dt + \sigma\bar{s}(t)dW(t)$$

$$d\bar{a}(t) = [-(\gamma_0 - \frac{1}{2}\sigma^2)\bar{a}(t)+1]dt + \sigma\bar{a}(t)dW(t).$$

Finally, the formulae for the moments of $\bar{s}(t)$ and $\bar{a}(t)$ become entirely symmetrical when expressed in terms of $\gamma_0 \pm \frac{1}{2}\sigma^2$, instead of γ and γ_1.

This exemplifies the observation, previously made in Appendix 4.1, that $dW(t)/dt$ does not obey the usual rules of the calculus. If we formally substitute $A(t)$ for $\gamma(t)$ in the ordinary differential equation

$$d\bar{s}(t) = [\gamma(t)\bar{s}(t)+1]dt$$

we obtain

$$(5.20)\qquad d\bar{s}(t) = (\gamma_0\bar{s}(t)+1)dt + \sigma\bar{s}(t)dW(t),$$

which is not the correct Itô equation satisfied by $\bar{s}$.

However, it can be shown that if (5.20) is interpreted as a Stratonovich stochastic differential equation, then it is equivalent to the correct Itô equation (see Section 10.2 of Arnold (1974), especially pp. 169-171).

Notice that there was no need for the above remarks to be made in Chapter 4, since the discussion was not concerned with discounting, but only with the ccumulated values of the fund.

5.7 HIGHER MOMENTS

Recursive (or differential) equations can also be written down for $Ea(t)^k$ and $E\ddot{s}(t)^k$ (or $E\bar{a}(t)^k$ and $E\bar{s}(t)^k$), when $k \geq 3$ (as pointed out in Section 5.2).

First consider $E\ddot{s}(t)^k$. If $u_k = E(1+i(\cdot))^k$,

$$\begin{aligned} E\ddot{s}(t+1)^k &= EE(\ddot{s}(t+1)^k|H_t) \\ &= E(1+i(t+1))^k\cdot E(\ddot{s}(t)+1)^k \\ &= u_k[1 + \sum_{j=1}^{k} \binom{k}{j}E\ddot{s}(t)^j]. \end{aligned}$$

Similarly, if $v_k = E(1+i(\cdot))^{-k}$, then

$$(5.21)\qquad Ea(t+1)^k = v_k[1 + \sum_{j=1}^{k} \binom{k}{j} Ea(t)^j].$$

The differential equation satisfied by $E\bar{s}(t)^k$ results from Itô's formula, with $g(x) = x^k$:

$$\begin{aligned} d[\bar{s}(t)^k] = {} & [k\bar{s}(t)^{k-1}(\gamma\bar{s}(t)+ \tfrac{1}{2}k(k-1)\bar{s}(t)^{k-2}\cdot\sigma^2\bar{s}(t)^2]dt \\ & + (\ldots)\, dW(t) \end{aligned}$$

$$\Rightarrow \quad \frac{d}{dt} E\bar{s}(t)^k = k[\gamma + \frac{1}{2}(k-1)\sigma^2]E\bar{s}(t)^k + kE\bar{s}(t)^{k-1}.$$

In the case of discounted values, we obtain

$$(5.22) \quad \frac{d}{dt} E\bar{a}(t)^k = k[-\gamma_1 + \frac{1}{2}(k-1)\sigma^2]E\bar{a}(t)^k + kE\bar{a}(t)^{k-1}.$$

One simple application of these formulae is the calculation of the limits of the moments, when $t \to \infty$. Take $E\bar{a}(\infty)^k$, for example. It is finite if and only if $\gamma_1 > \frac{1}{2}(k-1)\sigma^2$ (from Eq. (5.22)), in which case

$$E\bar{a}(\infty)^k = \frac{1}{\gamma_1 - \frac{1}{2}(k-1)\sigma^2} \cdot E\bar{a}(\infty)^{k-1}.$$

In discrete time,

$$Ea(\infty)^k = [1 + \sum_{j=1}^{k-1} \binom{k}{j} Ea(\infty)^j] \cdot v_k/(1-v_k),$$

if $v_k < 1$.

APPENDIX 5.1

PROOF OF PROPOSITION 5.2

Define

$$\begin{aligned} v^n &= Ev^n(\cdot) \\ &= E[1/(1+i^n(\cdot))] \end{aligned}$$

and restate Eq. (5.12) as

$$\begin{aligned} a^n(\tfrac{k+1}{n}) &= a^n(\tfrac{k}{n}) + (v^n-1)a^n(\tfrac{k}{n}) + v^n/n \\ &+ \frac{v^n(k+1)-v^n}{\sqrt{\mathrm{Var}\ v^n(\cdot)}}[\sqrt{\mathrm{Var}\ v^n(\cdot)}\ (a^n(\tfrac{k}{n}) + \tfrac{1}{n})] \\ &= a^n(\tfrac{k}{n}) + \tfrac{1}{n}\ U^n(a^n(\tfrac{k}{n})) + \tfrac{1}{\sqrt{n}}\ V^n(a^n(\tfrac{k}{n}))h^n(k+1) \end{aligned}$$

where

$$U^n(x) = n(v^n-1)x + v^n$$

$$V^n(x) = \sqrt{n\mathrm{Var}\ v^n(\cdot)}\ (x+\tfrac{1}{n}),$$

$$h^n(k+1) = (v^n(k+1)-v^n)/\sqrt{\mathrm{Var}\ v^n(\cdot)}. \tag{5.23}$$

It will be shown that

$$U^n(x) \rightarrow U(x) = -(\gamma-\sigma^2)x+1,$$

$$V^n(x) \rightarrow V(x) = \sigma x.$$

γ and σ^2 are as in Prop. 4.2 or 4.3, depending on whether $i^n(\cdot)$ is defined by Eq. (4.11) or (4.21), respectively. Weak convergence will then follow.

(a) Subdividing $i^1(.)$.

Lemma 5.1. If

(i) $\qquad EX = 0$

(ii) $\qquad EX^2 < \infty$

(iii) $$E(1+X)^{-2} < \infty$$

then

$$E\frac{X}{1+tX} = -tEX^2 + o(t) \quad \text{as} \quad t \downarrow 0.$$

Proof. Let $t \le 1$. From Taylor's Theorem, if $f(t) = 1/(1+tx)$

$$1/(1+tx) = 1 - tx/(1+z(t)x)^2, \quad 0 \le z(t) \le t$$

$$\Rightarrow \quad x/(1+tx) = x - tx^2/(1+z(t)x)^2.$$

Let

$$\begin{aligned} g_t(x) &= [x/(1+tx)-x]t^{-1} \\ &= -x^2/[1+z(t)x]^2. \end{aligned}$$

Since

$$\begin{aligned} |g_t(X)| &\le X^2 1_{\{X\ge 0\}} + [X^2/(1+X)^2]1_{\{X<0\}} \\ &\le X^2 1_{\{X\ge 0\}} + 2[1+ 1/(1+X)^2]1_{\{X<0\}}, \end{aligned}$$

the Dominated Convergence Theorem implies

$$\begin{aligned} \lim_{t\downarrow 0} Eg_t(X) &= E \lim_{t\downarrow 0} g_t(X) \\ &= -EX^2. \quad \square \end{aligned}$$

Refer to Section 4.3.3. We have (Eqs. (4.9) and (4.11)

$$i^n(1) = i^n + \Delta i^1(1)\cdot t_n$$

with

$$\Delta i^1(1) = i^1(1) - i^1,$$

$$t_n = (\text{Var } i^n(1)/\text{Var } i^1(1))^{1/2}.$$

The proof of Prop. 4.2 (Appendix 4.3) tells us that

$$ni^n \to \gamma, \quad n\text{Var } i^n(\cdot) \to \sigma^2$$

(and so (1) $nt_n^2 \to \sigma^2/\text{Var } i^1(1)$, (2) $t_n \downarrow 0$.)

(i) $n\text{Var } v^n(\cdot)$. Notice that

$$\mathrm{Var}\ v^n(\cdot) = \mathrm{Var}[i^n(1)/(1+i^n(i))].$$

Define

$$X_n = \frac{i^n(1)\sqrt{n}}{1+i^n(1)}$$

$$= \frac{i^n\sqrt{n} + t_n\sqrt{n}\ \Delta i^1(1)}{1+i^n+ t_n\Delta i^1(1)}$$

There exists constants k_1, k_2 and $n_\emptyset$ such that

$$(5.24) \qquad X_n^2 \le (k_1 + k_2[\Delta i^1(1)]^2)1_{\{\Delta i^1(1)\ge\emptyset\}}$$

$$+ \frac{k_1 + k_2[\Delta i^1(1)]^2}{(1+ i^1(1))^2} 1_{\{\Delta i^1(1) < \emptyset\}}, \quad n\ge n_\emptyset.$$

Hence, we can apply the Dominated Convergence Theorem, to conclude that

$$\lim_n n\mathrm{Var}\ v^n(\cdot) = \lim_n \mathrm{Var}\ X_n$$

$$= (\lim_n nt_n^2)\cdot\mathrm{Var}\ \Delta i^1(1)$$

$$= \sigma^2$$

(ii) $n(v^n-1)$.

$$n(v^n-1) = nE([1/(1+i^n(1))]-1)$$

$$= -nE[i^n(1)/(1+i^n(1))]$$

$$= -ni^nE[1/(1+i^n(1))]$$

$$-E[nt_n\Delta i^i(1)/(1+i^n+t_n\Delta i^1(1))].$$

Firstly, $i^n(1) \to \emptyset$ w.p. 1, and so, using the same type of argument as in (1), we get

$$\lim_n ni^n\ E[1/(1+i^n(1))] = \gamma.$$

Secondly, from Lemma 5.1,

$$\lim_n E \frac{n\Delta i^1(1)\cdot t_n}{1+i^n+t_n\Delta i^1(1)} = \lim_n nt_n[-t_n \mathrm{Var}\ \Delta i^1(1) + o(t_n)]$$

$$= -\lim_n nt_n^2 \cdot \mathrm{Var}\ i^1(\cdot)$$

$$= -\sigma^2.$$

Finally, $n(\upsilon^n-1) \to -(\gamma-\sigma^2)$. This also implies $\upsilon^n \to 1$.

(iii) It only remains to show that $\{h^n(1)^2,\ n\geq 1\}$ is uniformly integrable. Since (1) $\upsilon^n \to 1$ and (2) $n\mathrm{Var}\ \upsilon^n$ has a strictly positive limit, it is sufficient to show that $(ni^n(1)^2/(1+i^n(1))^2,\ n\geq 1\}$ is uniformly integrable. This results from inequality (5.24) of step (i).

(b) Subdividing $\gamma^1(\cdot)$

Lemma 5.2. Let $f(t) = Ee^{tX}$, $t \leq 0$. If

(i) $EX = 0$

(ii) $EX^2 < \infty$

(iii) $Ee^{-2X} < \infty$

then

$$f(t) = 1 + \frac{t^2}{2} EX^2 + o(t^2), \text{ as } t \uparrow 0.$$

Proof. Similar to that of Lemma 4.2 (Appendix 4.3).□

Refer to Section 4.3.4. Recall that $\gamma = \gamma^1 + \frac{1}{2}\mathrm{Var}\ \gamma^1(\cdot)$ and $\sigma^2 = \mathrm{Var}\ \gamma^1(\cdot)$. Let $f(t) = E\exp(t\Delta\gamma^1(1))$, $t \leq 0$.

(i) $n\mathrm{Var}\ \upsilon^n(\cdot)$. From Lemma 5.2,

$$n\mathrm{Var}\ \upsilon^n(\cdot) = n\{E\exp[-2\gamma^n(\cdot)] - (E\exp[-\gamma^n(\cdot)])^2\}$$

$$= n\exp(-2\gamma^1/n)[f(-2/\sqrt{n})-f(-1/\sqrt{n})^2]$$

$$\to \quad \mathrm{Var}\ \gamma^1(\cdot) \quad = \quad \sigma^2,$$

as in the proof of Prop. 4.3 (Appendix 4.3).

(ii) $n(v^n-1)$.

$$\begin{aligned} n(v^n-1) &= nE(\exp[-\gamma^n(\cdot)]-1) \\ &= n[\exp(-\gamma^1/n)f(1-/\sqrt{n})-1] \\ &= n[\exp(-\gamma^1/n)-1] \\ &\quad +n\exp(-\gamma^1/n)[\frac{1}{2n}\ \mathrm{Var}\ \gamma^1(\cdot) + o(n^{-1})] \\ &\to -\gamma^1 + \frac{1}{2}\ \mathrm{Var}\ \gamma^1(\cdot) \\ &= -(\gamma-\sigma^2). \end{aligned}$$

(iii) $\{h^n(1)^2,\ n \geq 1\}$ is uniformly integrable.

From Eq. (5.23),

$$h^n(1)^2 \leq n(v^n(1)-1)^2/(n\mathrm{Var}\ v^n(\cdot)) + n(1-v^n)^2/(n\mathrm{Var}\ v^n(\cdot)).$$

The second term on the right hand side goes to 0 as $n \to \infty$ since $n(1-v^n)$ and $n\cdot\mathrm{Var}\ v^n(1)$ have finite limits. It is therefore sufficient to show that the sequence

$$n(v^n(1)-1)^2 \quad = \quad n(\exp[-\gamma^n(1)]-1)^2$$

is uniformly integrable. This is done in the same fashion as in step 3 of the proof of Prop. 4.3 (Appendix 4.3).

CONCLUSION

There are many other problems to be studied in the dynamics of pension funding. Here are a few ideas for further research.

1. Perhaps the most obvious way of improving the results of Chapters 3 and 4 is to consider more realistic interest rate processes. As was pointed out in Section 5.2, two possibilities are autoregressive and Markov processes. It would be especially interesting to see what becomes of the "optimal region" under different assumptions.

2. Nevertheless, the white noise model should probably not be forgotten totally, since it is the most tractable (this was discussed in Sections 4.1 and 5.2). As far as the theory of pension funding is concerned, the lack of realism of the model is compensated by the explicitness of the results obtained. The continuous-time formulation is particularly promising. The continuous-time formulation is particularly promising. For example under the Spread method, the process F is a diffusion and so its transition probabilities can be found from the so-called backward and forward partial differential equations. Should this approach fail (as these equations are by no means trivial to solve), the density functions of F(t) and C(t) may be estimated from their moments, which can be calculated explicitly (see Section 5.7).

3. The equations of Chapter 2 could serve to study how fluctuating inflation rates affect the evolution of the fund levels and contributions, when benefits are not totally indexed. As the equations show, this problem is mathematically more complex than the one of fluctuating

interest rates. This is because $B(t)$ and $AL(t)$ now depend on the inflation rates experienced over the last ω-r years (see for example the expression for $B(x,t)$ in Section 2.3.1).

Going still further, another possible development would be of considering inflation on salaries and investment earnings which are both stochastic.

4. Another problem is the determination of "security loadings" against unfavourable experience. For example, say we are given some model for the future behaviour of assets growth, inflation, mortality, etc. Then, for appropriate values of k, T and ε, what is the (minimum) increase in contributions needed to ensure that

$$\text{Prob}\{F(t) \geq k \cdot AL(t), \text{ for all } 0 \leq t \leq T\} \geq 1-\varepsilon \text{ ?}$$

This type of question may become more important in the future, in view of the current trend towards "realistic" actuarial assumptions (as opposed to "conservative" assumptions, which give an implicit, undetermined amount of "safety" to the valuation basis).

5. Finally, there is the hypothesis of fixed acturial assumptions that is quite unrealistic. One would like actuarial assumptions to take recent experience into account. The problems which arise are that

(1) a dependance on the past is introduced and

(2) NC, PVB etc. are now functions of these varying assumptions.

One possibility is to linearize for small changes in the assumptions. This was the approach adopted by S. Benjamin (1983), who assumed the valuation interest rate to be the average of the previous "k" earned rates of return on the assets.

REFERENCES

The following abbreviations are used:

JIA	Journal of the Institute of Actuaries
JIASS	Journal of the Institute of Actuaries' Student Society
JRI	Journal of Risk and Insurance
PCAPP	Proceedings of the Conference of Actuaries in Public Practice
PCAS	Proceedings of the Casualty Actuarial Society
SAJ	Scandinavian Actuarial Journal
TFA	Transactions of the Faculty of Actuaries
TSA	Transactions of the Society of Actuaries

Adams, W.R. (1967). The effect of interest on pension contributions. *TSA 19*: 170-183.

Allison, G.D. and Winklevoss, H.E. (1975). The interrelationship among inflation rates, salary rates, interest rates and pension costs. *TSA 27*: 197-210.

Anderson, A.W. (1985). *Pension Mathematics for Actuaries*. Society of Actuaries, Itasca, Illinois.

Apostol, T.M. (1974). *Mathematical Analysis*, Second Edition, Addison-Wesley, Reading, Mass.

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications*. Wiley, New York.

Astrom, K.J. (1970). *Introduction to Stochastic Control Theory*. Academic Press, New York.

Balzer, L.A. (1982). Control of insurance systems with delayed profit/loss sharing feedback and persisting unpredicted claims. *JIA 109*: 285-311.

References

Balzer, L.A. and Benjamin, S. (1980). Dynamic response of insurance systems with delayed profit/loss sharing feedback to isolated unpredicted claims. *JIA 107*: 513-528.

Beekman, J.A. (1973). A new collective risk model. *TSA 25*: 573-589.

Beekman, J.A. and Fuelling, C.P. (1977). Refined distributions for a multi-risk stochastic process *SAJ*: 175-183.

Bellhouse, D.R. and Panjer, H.H. (1981). Stochastic modelling of interest rates with applications to life contingencies - Part II. *JRI* 48: 628-637.

Benjamin, B. et al. (1985). *Pensions: The Problems of Today and Tomorrow.* Institute of Actuaries, London.

Benjamin, S. (1983). Seminar on "Control of pension funding systems". Presented at the City University, 17 March 1983.

Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

Bishop, A.B. (1975). *Introduction to Discrete Linear Controls: Theory and Applications.* Academic Press, New York.

Bizley, M.T.L. (1954). The effect on pension fund contributions of a change in the rate of interest. *JIASS 10*: 47-51.

Bowers, N.L., Hickman, J.C. and Nesbitt, C.J. (1976). Introduction to the dynamics of pension funding. *TSA 28*: 177-203.

Bowers, N.L., Hickman, J.C. and Nesbitt, C.J. (1979). The dynamics of pension funding: contribution theory. *TSA 31*: 93-119.

Bowers, N.L., Hickman, J.C. and Nesbitt, C.J. (1982). Notes on the dynamics of pension funding. *Insurance: Mathematics and Economics 1*: 261-270.

Boyle, P.P. (1976). Rates of return as random variables. *JRI* 43: 693-711.

Boyle, P.P. (1978). Immunization under stochastic models of the term structure. *JIA 105*: 177-187.

Burghes, D. and Graham, A. (1980). *Introduction to Control Theory, Including Optimal Control.* Ellis Horwood, Chichester.

Colbran, R.B. (1982). Valuation of final-salary pension schemes. *JIA 109*: 359-385.

Consael, R. and Lambrecht, J. (1954). *Theorie mathematique des assurances* (Lecture notes). Universite Libre de Bruxelles, Bruxelles.

Cooper, S.L. and Hickman, J.C. (1967). A family of accrued benefit cost methods. *TSA 19*: 53-59.

Cramer, H. (1946). *Mathematical Methods of Statistics.* Princeton University Press, Princeton.

Dellacherie, C. and Meyer, P.-A. (1975). *Probabilites et potentiel.* Hermann, Paris.

Dufresne, D. (1986). Pension funding and random rates of return. To appear in *Proceedings of the NATO Advanced Study Institute of Insurance and Risk Theory.*

Elkin, J.M. (1958). A method of allocating acturial gains and losses in a pension fund. *PCAPP 7*: 192-198.

Emmanuel, D.C. et al. (1975). A diffusion approximation for the ruin function of a risk process with compounding assets. *SAJ*: 240-247.

Ezra, D. (1979). *Understanding Pension Fund Finance and Investments.* Pagurian Press, Toronto.

Financial Executive Institute Canada (1981). Report on Survey of Pension Plans in Canada.

Fleischer, D.R. (1975). The forecast valuation method for pension plans. *TSA 27*: 93-125.

Gihman, I.I. and Skorohod, A.V. (1972). *Stochastic Differential Equations.* Springer-Verlag, New York.

Gupta, S.C. (1966). *Transforms and State Variable Methods in Linear Systems.* Wiley, New York.

Hickman, J.C. (1968). Funding theories for social insurance. *PCAS* 55: 303-311.

Iglehart, D.L. (1969). Diffusion approximations in collective risk theory. *Journal of Applied Probability* 6: 285-292.

Joffe, A. and Metivier, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Advances in Applied Probability 18*: 20-65.

Kallianpur, G. (1980). *Stochastic Filtering Theory.* Springer-Verlag, New York.

Liferman, J. (1975). *Les systemes discrets.* Masson, Paris.

Loeve, M. (1977). *Probability Theory I.* Fourth Edition. Springer-Verlag, New York.

Lyon, M. (1977). Controlled funding methods for group pension schemes. *JIASS 16*: 130-146.

McGill, D.M. (1979). *Fundamentals of Private Pensions.* Fourth Edition. Irwin, Homewood, Illinois.

McLeish, D.J.D. (1983). A financial framework for pension funds. *TFA* 38: 267-314.

Neveu, J. (1970). *Bases mathematiques du calcul des probabilites.* Masson, Paris.

Panjer, H.H. and Bellhouse, D.R. (1980). Stochastic modelling of interest rates with applications to life contingencies. *JRI 47*: 91-110.

Picot, J. (1976). Le fonctionnement d'un regime de retraite dans l'inflation. *Proceedings of the 20th International Congress of Actuaries* 3: 303-317.

Pollard, A.A. and Pollard, J.H. (1969). A stochastic approach to actuarial functions. *JIA* 95: 79-113.

Pollard, J.H. (1971). On fluctuating interest rates. *Bulletin de l'Association Royale des Actuaires Belges* 66: 68-94.

Pollard, J.H. (1973). *Mathematical Models for the Growth of Human Population.* Cambridge University Press, Cambridge.

Ruohonen, M. (1980). On the probability of ruin of risk processes approximated by a diffusion process. *SAJ*: 113-120.

Schnieper, R. (1983). Risk processes with stochastic discounting. *Bulletin of the Association of Swiss Actuaries 83*: 203-217.

Schuss, Z. (1980). *Theory and Applications of Stochastic Differential Equations.* Wiley, New York.

Seal, H.L. (1952). "Acceptable" funding methods for self-insured pension funds. *PCAPP 2*: 17-44.

Shapiro, A.F. (1983). Modified cost methods for small pension plans. *TSA 35*: 11-30.

Skorohod, A.V. (1956). Limit theorems for stochastic processes. *Theory of Prob. Appl. 1*: 261-290.

Soong, T.T. (1973). *Random Differential Equations in Science and Engineering.* Academic Press, New York.

Spiegel, M.R. (1971). *Schaum's Outline of Theory and Problems of Calculus of Finite Differences and Difference Equations.* McGraw Hill, New York.

Stewart, C.M. (1983). Pension problems and their solution. *JIA 110*: 289-313.

Street, C.C. (1977). Another look at group pension plan gain and loss. *TSA 29: 399-421.*

Taylor, J.R. (1967). The generalized family of aggregate actuarial cost methods for pension funding. TSA 19: 1-12.

Taylor, R.H. (1952). The probability distribution of life annuity reserves and its application to a pension system. *PCAPP 2*: 100-150.

Trowbridge, C.L. (1952). Fundamentals of pension funding. *TSA* 4: 17-43.

Trowbridge, C.L. (1963). The unfunded present value family of pension funding methods. *TSA* *15*: 151-169.

Turner, M.J. et al. (1984). *Codification of Pension Funding Methods.* Pension Standards Joint Committee of the Institute and Faculty of Actuaries.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* 5: 177-188.

Waters, H.R. (1978). The moments and distributions of actuarial functions. *JIA 105*: 61-75.

Westcott, D.A. (1981). Moments of compound interest functions under fluctuating interest rates. *SAJ*: 237-244.

Wilkie, A.D. (1976). The rate of interest as a stochastic process - Theory and applications. *Proceedings of the 20th International Congress of Actuaries 1*: 325-338.

Wilkie, A.D. (1978). Maturity (and other) guarantees under unit-linked policies. *TFA* 36: 27-41.

Winklevoss, H.E. (1977). *Pension Mathematics: With Numerical Illustrations.* Irwin, Homewood, Illinois.