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Unhiggsing the del Pezzo

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Abstract: We develop an unhiggsing procedure for finding the D-brane probe world volume gauge theory for blowups of geometries whose gauge theory data are known. As specific applications we unhiggs the well-studied theories for the cone over the third del Pezzo surface. We arrive at what we call pseudo del Pezzos and these will constitute a first step toward the understanding of higher, non toric del Pezzos. Moreover, our methods and results give further support for toric duality as well as obtaining superpotentials from global symmetry considerations.

Keywords: (Un)Higgsing, del Pezzo, D-brane probes, blowups, Toric Duality, Superpotentials.

Contents

1. Introduction

D-brane probes to singularities have by now become an important tool in understanding the compactification of string theory on Calabi-Yau manifolds. Indeed the resolution of the singularities [1, 2, 3] to smooth Calabi-Yau's by the sub-stringy scale dynamics of the world-volume gauge theories is of great interest to the physicist and mathematician alike.

With the help of the myriad of combinatorial techniques of toric geometry, notably the systematic partial resolution by blowups of Abelian orbifolds, a particular class of non-compact, singular Calabi-Yau threefolds have been extensively investigated[1] [1, 2, 4, 5]. These are the so-called toric singularities. Well-studied cases include Abelian orbifolds and the famous conifold. Though the construction of the world-volume gauge theories for arbitrary singularities which model the Calabi-Yau remains an open question [13, 14, 20], progress has been made in this subclass.

The method of extracting the world-volume gauge theory of the D-brane probing transversely to such toric singularities has been formalised and conveniently algorithmised in [6]. An interesting by-product of the so-called Inverse Algorithm is the phenomenon of Toric Duality where a systematic method has been created to construct classes of vastly different gauge theories having the same (toric) moduli space in the infra-red [6, 7, 8, 9]. A subset of the gauge theories that share the same toric moduli space, the *toric phases*, have the interesting property of laying in the conformal window. Recently, activities from three different perspectives have hinged on the conjecture that toric duality is generalised Seiberg's $\mathcal{N} = 1$ duality [8, 15, 23].

Prime examples of toric duality and the Inverse Algorithm have been the cones over del Pezzo surfaces. These surfaces sit as compact 4-cycles (divisors) in the Calabi-Yau and have been a long-time player in the field of String Theory. There are in total 10 of such surfaces, namely $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and $B_k := \mathbb{P}^2$ blown up at $k = 1, \ldots 8$ points. Research of these surfaces in string theory has been diverse and has ranged over directions from mirror symmetry [11] to mysterious dualities [12].

The first 5 members of the series, namely the cones over $F_0 := \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ (which gives the resolution of the famous orbifold $\mathbb{C}^3/\mathbb{Z}_3$) as well as $dP1, 2, 3$ (the cones of the first 3 del Pezzo surfaces) are toric and have been scrutinised in the context of D-brane probes by [4, 6, 7, 8, 15, 23, 9, 10, 16], especially since the advent of the Inverse Algorithm. The remaining members, $dP4, \ldots, 8$, are non-toric and the first venture into this *terra incognita* has been [16], wherein the quiver diagrams have been constructed.

Indeed, of late four techniques have been in circulation, towards the full understanding of probing toric singularities: (1) direct field theory techniques wherein the acquisition of vevs to

[1] Recently, new phenomenological constructions have been developed by wrapping D6-branes on compact, intersecting 3-cycles of Calabi-Yau manifolds [26, 27].

spacetime fields is considered [1, 2, 24, 4, 15], (2) brane configurations such as diamonds [31, 8] and (p, q)-web techniques [18, 16, 19, 10], (3) geometric engineering wherein exceptional collections of coherent sheafs over the divisors provide the gauge theory data and certain geometric transitions provide large N dualities [22, 23, 32], as well as (4) the Inverse Algorithm [6, 7, 8, 9], which is computationally very convenient and methodical. All these complementary techniques have thusfar supported each other perfectly, as in particular exemplified in the detailed study of the above five toric varieties.

However, to have a better understanding of the D-brane probe theory, we need to proceed beyond toric varieties. In this paper, we develop a systematic method, the so called **unhiggsing** mechanism[2], to deal with this problem. The basic idea is the following. Given a singularity Y, it is relatively simple to calculate the quiver diagram (matter content) by the aforementioned geometric methods. The difficult part is to find the corresponding superpotential (for example, by calculating the mapping among the collections of coherent sheafs). Now if we know the quiver and superpotential of a singularity X which is the blow down of Y, we can use the unhiggsing mechanism to get the superpotential of Y more easily[3].

This above method is of course perfectly adapted to our needs: we have the quivers of the higher dPk's from [16], we know that each dPk is the $\mathbb{P}^1$-blowup of $dP(k-1)$ and we have the full theories for $dP0, 1, 2, 3$ from [6]. Inspired by this philosophy and armed with this technique, we attempt at finding the corresponding superpotential of the non-toric $dP4$ and $dP5$ singularities, with quiver diagrams given in [16]. The results turn out to be toric. In other words, the moduli space of these gauge theories, unhiggsed from the known $dP3$ theory, defined by the subsequent superpotentials and quiver diagrams, are in fact toric varieties. We will see that these toric moduli spaces are not generic, smooth $dP4$ and $dP5$, but degenerate cases with non-isolated singularities. These singularities we shall call **pseudo del Pezzos**. These surfaces, which we denote as PB_k, over which the $PdPk$ are affine cones, bear close semblance to the del Pezzo surfaces B_k as they are also $\mathbb{P}^2$ blown up at points .

Although we do not reach our initial aim, the method itself is very useful and can be applied to hosts of examples in order to construct new classes of D-brane gauge theories. We will discuss more about this issue in the conclusion.

Furthermore, continuing along the path of [9, 10], we shall use elegant symmetries inherited from the very geometry (and indeed from the closed string sector), to arrive at the superpotentials for these theories [4]. Once again, we shall find that such symmetry considerations are powerful

[2]During the preparation of this manuscript, YHH has learnt from M. Wijnholt that the latter's collaboration group is also working in this direction and has reached similar results.

[3]There are some subtle points in this inverse process which we will discuss later.

[4]The issue of multiplicity symmetry, raised in [9], has also been considered in [21].

enough to uniquely determine the superpotential, the calculation of which is often a daunting task, either for the Inverse Algorithm, or for the composition of Ext's in the derived category of coherent sheafs.

The organization of the paper is as follows. In Section 2, we refresh the readers' memory on the four toric phases of the $dP3$ theories, known to the literature. Then, in Section 3, we present the other ingredient and explain the (un)higgsing mechanism in relation to geometric blow (down) ups. Thus prepared, we unhiggs the $dP3$ theories to obtain the $dP4$ gauge theory in Section 4, and check the consistency by higgsing back to $dP3$ in Section 5. As a hind-sight, in Section 6, we shall see that we have in fact obtained the $PdP4$ theory and discuss some of the geometric properties thereof. Continuing in this vein, we obtain the $PdP5$ theory in Section 7. As an additional confirmation to the unhiggsing method, we also use global symmetry arguments to check our superpotentials in Section 8. Finally, we conclude in Section 9.

Nomenclature

Unless otherwise stated, we shall throughout the paper adhere to the notation that dPk means the affine cone over the k-th del Pezzo surface B_k, i.e., $\mathbb{P}^2$ blown up at k generic points. When these blowup points are *not* generic, i.e., 3 or more may be colinear, or 6 or more may lie on a single conic, we shall call the resulting surface the non-generic (or Pseudo) del Pezzo, denoted as PB_k; some of these may actually be toric as we shall see. The affine cones over these surfaces we shall call $PdPk$.

Often we shall append a Roman numeral subscript as in dPk_I; this means the I-th (toric-dual) phase of the theory for dPk. And so likewise for $PdPk$.

In the quiver theory, the arrow X_{ij} corresponds to the bifundamental field from node i to j.

2. The Four Phases of $dP3$

The starting point for the unhiggsing process that we will use to generate the theories associated to higher del Pezzos is $dP3$. There are four toric phases corresponding to $dP3$ [8, 9, 10, 15]. To refresh the reader's memory, let us clarify what we mean by a **toric phase**, as inspired by the Toric Duality discussions in [6, 8]: we call any gauge theory where the quiver has the rank of all nodes equal to N (for simplicity, most times we set $N = 1$) as well as only monomial F-terms, i.e., suitable for the Forward Algorithm of [6, 5]. Indeed this is not a necessary condition for the moduli space to be toric. We can have phases without all the ranks of the nodes equal, and still obtaining a toric moduli space when calculating it in terms of gauge invariant operators.

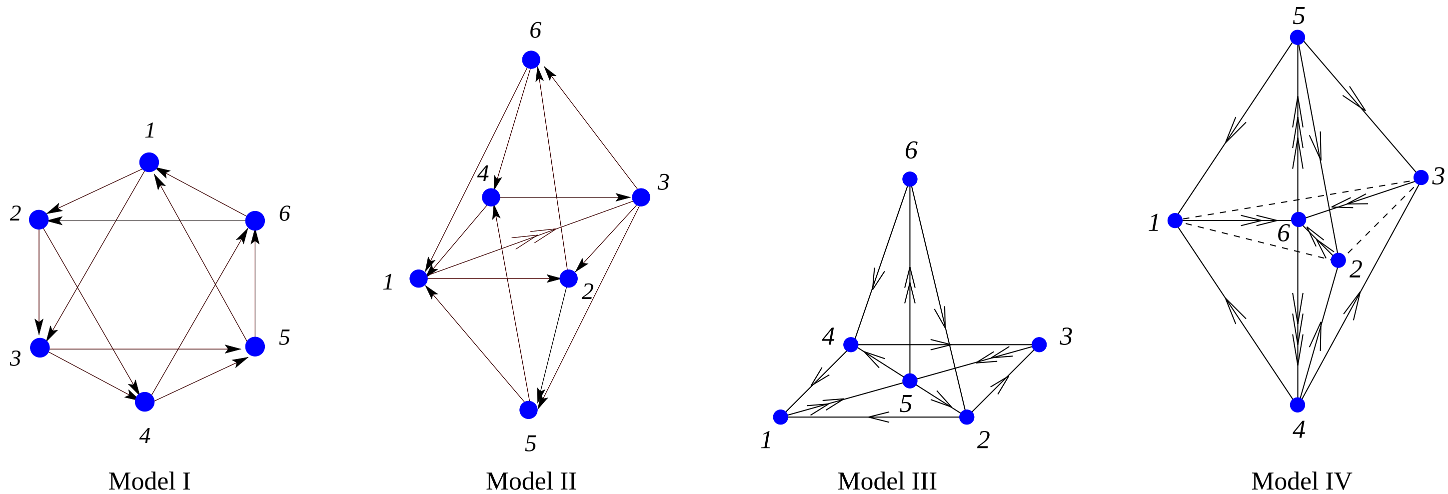


Figure 1: Quiver diagrams of the four phases of $dP3$.

Now, let us recall the $dP3$ quivers in Figure 1, where we have used the versions presented in [9], which make global symmetries explicit.

The superpotentials for these theories are

$$
\begin{aligned}
W_I &= X_{12}X_{23}X_{34}X_{45}X_{56}X_{61} - [X_{23}X_{35}X_{56}X_{62} + X_{13}X_{34}X_{46}X_{61} + X_{12}X_{24}X_{45}X_{51}] \\
&\quad +[X_{13}X_{35}X_{51} + X_{24}X_{46}X_{62}]
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
W_{II} &= [X_{12}X_{26}X_{61} - X_{12}X_{25}X_{51} + X_{36}X_{64}X_{43} - X_{35}X_{54}X_{43}] \\
&\quad +[-X_{61}X_{13}X_{36} + X_{51}Y_{13}X_{35}] + [-X_{26}X_{64}X_{41}Y_{13}X_{32} + X_{25}X_{54}X_{41}X_{13}X_{32}]
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
W_{III} &= [X_{41}X_{15}X_{54} - X_{54}X_{43}X_{35} + Y_{35}X_{52}X_{23} - X_{52}X_{21}Y_{15}] \\
&\quad +[-X_{41}Y_{15}X_{56}X_{64} + X_{64}X_{43}Y_{35}Y_{56} - X_{23}X_{35}X_{56}X_{62} + X_{62}X_{21}X_{15}Y_{56}]
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
W_{IV} &= [X_{41}X_{16}X_{64} + X_{43}X_{36}Y_{64} + X_{42}X_{26}Z_{64}] - [X_{41}Y_{16}Y_{64} + X_{43}Y_{36}Z_{64} + X_{42}Y_{26}X_{64}] \\
&\quad + [X_{51}Y_{16}X_{65} + X_{53}Y_{36}Y_{65} + X_{52}Y_{26}Z_{65}] - [X_{51}X_{16}Y_{65} + X_{53}X_{36}Z_{65} + X_{52}X_{26}X_{65}].
\end{aligned} \tag{2.4}
$$

From this data, we shall use the technique of “unhiggsing” to attempt to arrive the theories for the higher del Pezzos.

3. Blowing Up and Down versus Unhiggsing and Higgsing

Now we need our second ingredient and discuss the geometric origin of the (un)higgsing method. The philosophy is straight-forward and standard to the literature:

the blow-up of a point, replacing it by a compact 2-cycle, is translated to an unhiggsing of the field theory on the D-brane. Conversely, blowing down a 2-cycle corresponds to the higgsing of turning on a VEV for a bifundamental field that breaks two $U(1)$ factors down to a single one.

In terms of fractional branes, the higgsing process corresponds to the combination of the fractional branes of the higgsed gauge groups into bound states as discussed in [19].

Let us now discuss the connection between the higgsing and the partial resolution methods [15, 6, 4, 5]. When Fayet-Iliopoulos (FI) terms acquire generic values the singularity is completely resolved. On the other hand, when the FI terms lie on some non-generic cones, we obtain a partial resolution corresponding to a non-trivial (singular) geometry. This technique was exploited in [1, 2, 24, 4, 6] to obtain theories for various toric varieties starting from abelian orbifolds. To illustrate, let us consider the resolution of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ down to the Suspended Pinched Point (SPP). The quiver for $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is given in Figure 2(a) (which we quote from [24, 6]), while its superpotential is

$$
\begin{aligned}
W = {} & X_{13}Y_{34}Z_{41} - X_{13}Z_{32}Y_{21} + X_{31}Y_{12}Z_{23} - X_{31}Z_{14}Y_{43} \\
& + X_{24}Y_{43}Z_{32} - X_{24}Z_{41}Y_{12} + X_{42}Y_{21}Z_{14} - X_{42}Z_{23}Y_{34}.
\end{aligned} \tag{3.1}
$$

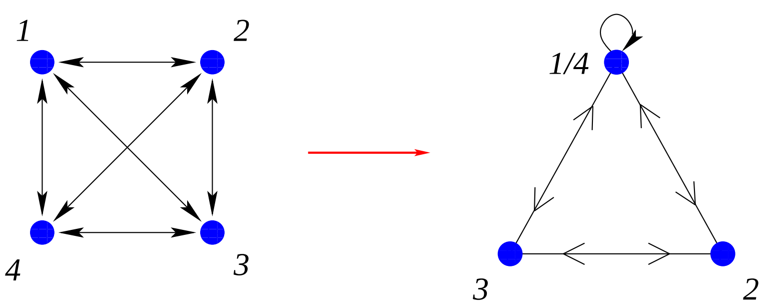


Figure 2: (a) The quiver for the parent orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$; (b) The quiver for the SPP, a partial resolution from higgsing the parent.

The SPP is obtained by constraining the four FI terms to be [24, 6]

$$
\zeta_2 = 0 \qquad \zeta_3 = 0 \quad \zeta_1 + \zeta_4 = 0 \qquad \zeta_1 \neq 0. \tag{3.2}
$$

This corresponds to higgsing $U(1)_{(1)} \times U(1)_{(4)}$ to a single $U(1)$. We can do it by giving a non-zero VEV to Z_{14} (the alternative of giving a VEV to Z_{41} is equivalent by symmetry). Let us set $\langle Z_{14} \rangle = 1$. During the higgsing process, mass terms are generated for X_{31}, Y_{43}, X_{42} and Y_{21}, so they have to be integrated out. Calling nodes $1(4) \to 1$, we get

$$
W = X_{21}Y_{12}Z_{23}Z_{32} - Z_{32}Z_{23}Y_{31}X_{13} + X_{13}Y_{31}Z_{11} - X_{21}Z_{11}Y_{12} \tag{3.3}
$$

and the quiver in Figure 2(b), which is exactly that for the SPP.

More explicitly, let us consider the D-terms of nodes 1 and 4. If we give only one field Z_{14} a nonzero VEV, to satisfy D-terms for these two nodes, both ζ_1 and ζ_4 can not be zero, but $\zeta_1 + \zeta_4 = 0$ because of the opposite sign of field Z_{14} in these two D-terms. This establishes the relationship between FI-parameters and fields which acquire nonzero VEV.

Therefore, we have shown in a simple example how the linear relations among FI parameters associated to a blow-down such as (3.2) straightforwardly determine a higgsing in the gauge theory. The methodology is of course easily generalised and the reverse of the procedure, viz., the unhiggsing is much in the same spirit and will be detailed in the next section. We remark that such relation between (un)higgsing and blowing (up) down is very conveniently visualised in the (p, q)-web picture [18, 19].

4. Unhiggsing From $dP3$ to $dP4$

Thus our ingredients are complete. With the full theories for $dP3$, the quivers for the higher (non-toric) del Pezzo's given in [16], as well as the preparatory étude on the SPP in the previous section, let us proceed.

The quiver diagram of $dP4$ given in [16] is redrawn here as Model I in Figure 3. The other models we shall obtain later. For this phase of $dP4$, we have a total of 15 fields. When we higgs down to $dP3$ in the manner of Section 3 therefore, we can reach at most three of total four phases, viz. $dP3_I$ with 12 fields as well as $dP3_{II}$ and $dP3_{III}$ with 14 fields. For $dP3_{IV}$ there are 18 fields so it obviously can not be higgsed from this phase of $dP4$. Let us analyze this process in more detail.

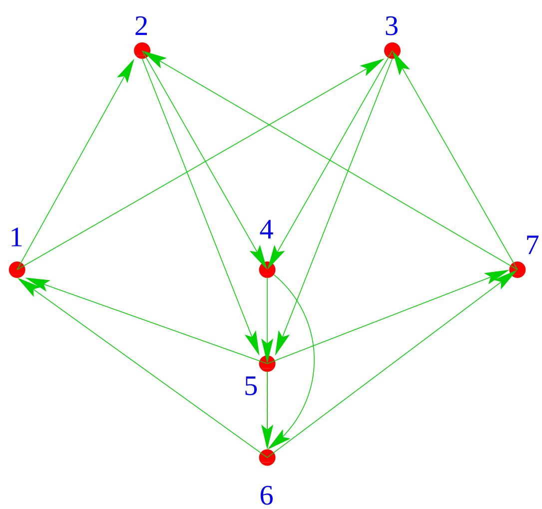


Figure 3: The quiver diagram for $dP4_I$, redrawn from the results in [16]. In this paper this model is referred to as Model I, a $U(1)^7$ theory with 15 bifundamentals.

4.1 Higgsing $dP4_I$

First, notice that there is an explicit symmetry of the quiver for $dP4_I$, namely the reflection about the 456-axis. This means that node-pair $(2, 3)$ as well as the pair $(1, 7)$ are equivalent. Now let us see whether $dP4_I$ can be higgsed down to $dP3_I$; the latter seems a natural choice because it, as with $dP4_I$, is the only model without multiple arrows between any two nodes. However, we can not find the reflection symmetry exhibited in $dP4_I$, i.e., we can not find such equivalence between pairs of nodes in $dP3_I$. This tells us that when we higgs down, such symmetry could be broken and in fact will be so.

Second, notice that for node 5 in $dP4$, we have three arrows coming in and three going out while there are only two incoming and two outgoing arrows for any node in $dP3_I$. This means that we must integrate out one incoming and one outgoing arrow at node 5 when we higgs down; these two fields must acquire mass when we higgs. In other words, there must be a cubic term in the superpotential that involve these two fields and another field to which we will give nonzero vacuum expectation value (VEV).

To set some notations, we shall label fields in $dP4$ by ϕ and those in $dP3$ by X. Moreover, in the quiver diagrams, the daughter of the higgsing will have its nodes indexed by numbers with "[ ]" around them and node $a/b[c]$ would thus mean node c in the daughter, obtained from higgsing nodes a and b in the parent.

Combining the above two observations, we see that the one field which is integrated out must be ϕ_{25}. Indeed we can make this choice due to the symmetry between nodes 2 and 3. Now to get the cubic term which includes ϕ_{25}, we have only two choices: $\phi_{25}\phi_{57}\phi_{72}$ and $\phi_{25}\phi_{51}\phi_{12}$ as they are the only closed loops in the quiver diagram (i.e. gauge invariant operators) involving node 5. Again, since node 7 is symmetric to node 1, these two choices are equivalent to each other. So without loss of generality we take $\phi_{25}\phi_{51}\phi_{12}$.

We should also condense field ϕ_{12} from $\phi_{25}\phi_{51}\phi_{12}$. The condensation process is shown in part (a) of Figure 4 wherein we give the corresponding nodes of $dP3$ in brackets for comparison. We have drawn dashed lines for the field that gets a non-zero VEV and those that become massive and are integrated out.

So far we have used the quiver alone, the next step is to start from the superpotential of $dP3_I$ to attempt to reach that for $dP4_I$ which is thusfar unknown in the literature. From the superpotential for $dP3_I$,

$$W_{dP3_I} = X_{12}X_{23}X_{34}X_{45}X_{56}X_{61} - [X_{23}X_{35}X_{56}X_{62} + X_{13}X_{34}X_{46}X_{61} + X_{12}X_{24}X_{45}X_{51}] + [X_{13}X_{35}X_{51} + X_{24}X_{46}X_{62}]$$

we replace by the corresponding fields in $dP4$ (in the way that is suggested by Figure 4.a) to get

$$W_{dP4_I} = \phi_{34}\phi_{45}\phi_{56}\phi_{67}\phi_{72}\phi_{12}^{-1}\phi_{13} - [\phi_{45}\phi_{57}\phi_{72}\phi_{24} + \phi_{35}\phi_{56}\phi_{61}\phi_{13} + \phi_{73}\phi_{34}\phi_{46}\phi_{67}] + [\phi_{73}\phi_{35}\phi_{57} + \phi_{24}\phi_{46}\phi_{61}\phi_{12}]$$

In W_{dP4_I} to close the loops we needed to replace X_{46} by $\phi_{61}\phi_{12}$. The crucial step is that the term $X_{12}X_{23}X_{34}X_{45}X_{56}X_{61}$ must be replaced by $\phi_{34}\phi_{45}\phi_{56}\phi_{67}\phi_{72}\phi_{12}^{-1}\phi_{13}$ where we have put in ϕ_{12}^{-1} to show that this term is the result of integrating out massive fields. In other words, this term should come from the replacement of ϕ_{25} or ϕ_{51} by their equations of motion.

If it came from the replacement of ϕ_{25}, we should have the term $\phi_{25}\phi_{56}\phi_{67}\phi_{72}$ which upon substitution of ϕ_{25} from $\phi_{12}\phi_{25} = \phi_{13}\phi_{34}\phi_{45}$ gives the required $\phi_{34}\phi_{45}\phi_{56}\phi_{67}\phi_{72}\phi_{12}^{-1}\phi_{13}$. Thus we get the final superpotential as

$$W_{dP4_I} = \phi_{24}\phi_{46}\phi_{61}\phi_{12} + \phi_{73}\phi_{35}\phi_{57} - \phi_{73}\phi_{34}\phi_{46}\phi_{67} - \phi_{45}\phi_{57}\phi_{72}\phi_{24} - \phi_{35}\phi_{56}\phi_{61}\phi_{13}$$

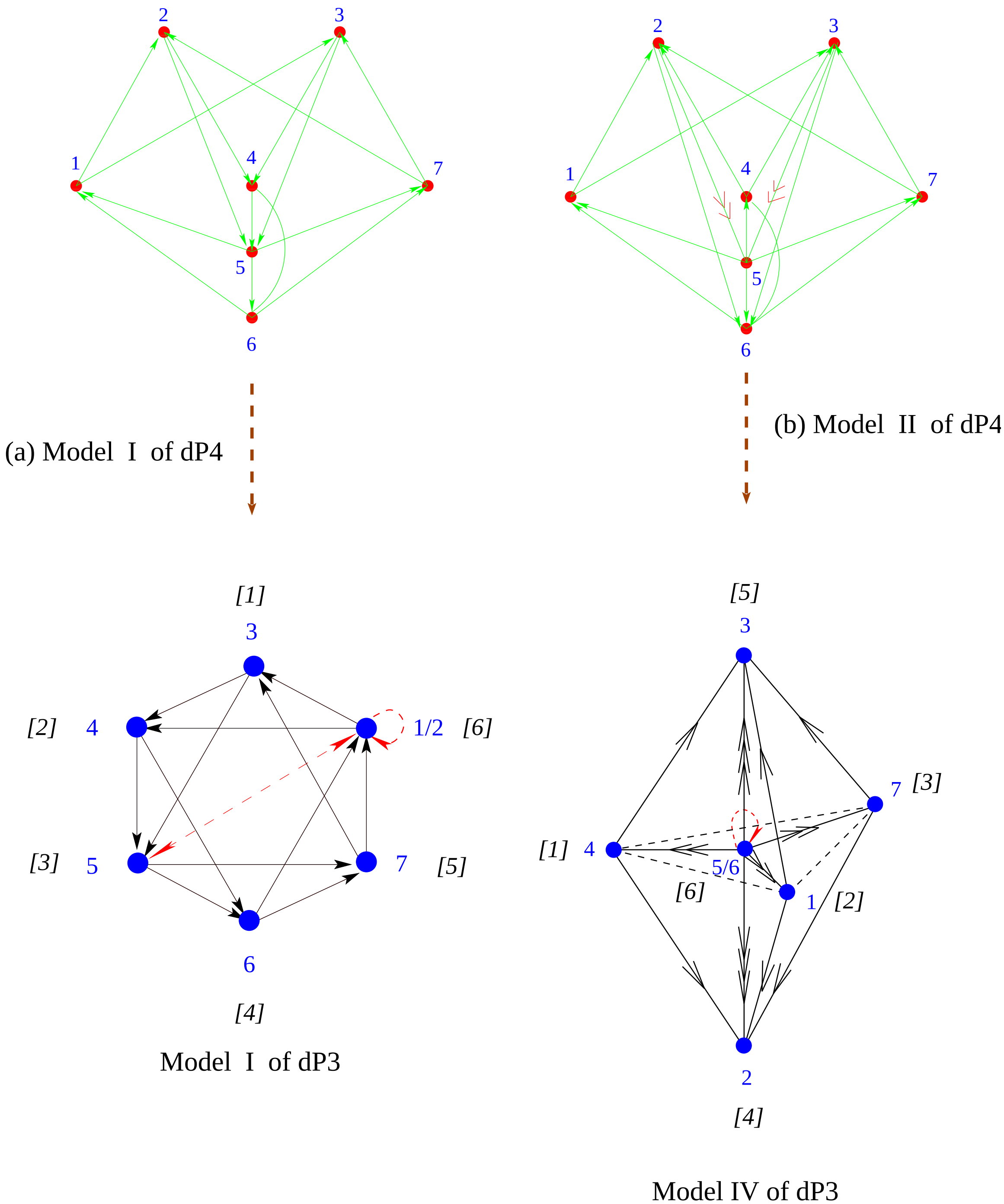


Figure 4: The higgsing of $dP4$ down to $dP3$. (a) The quiver diagram after we condense ϕ_{12} from $dP4_I$ to $dP3_I$; (b) similarly we obtain $dP3_{IV}$ from $dP4_I$ by turning on a VEV for ϕ_{56}.

$$+\phi_{25}\phi_{56}\phi_{67}\phi_{72} - \phi_{51}(\phi_{12}\phi_{25} - \phi_{13}\phi_{34}\phi_{45}). \tag{4.1}$$

If on the other hand we were to do the replacement of ϕ_{51}, we should have the term $\phi_{51}\phi_{13}\phi_{34}\phi_{45}$, with the EOM $\phi_{51}\phi_{12} = \phi_{56}\phi_{67}\phi_{72}$. From this we would have the superpotential

$$\begin{aligned} W_{dP4_I} &= \phi_{24}\phi_{46}\phi_{61}\phi_{12} + \phi_{73}\phi_{35}\phi_{57} - \phi_{73}\phi_{34}\phi_{46}\phi_{67} - \phi_{45}\phi_{57}\phi_{72}\phi_{24} - \phi_{35}\phi_{56}\phi_{61}\phi_{13} \\ &\quad +\phi_{51}\phi_{13}\phi_{34}\phi_{45} - \phi_{25}(\phi_{51}\phi_{12} - \phi_{56}\phi_{67}\phi_{72}). \end{aligned}$$

This is the same as (4.1). We have therefore obtained the superpotential for $dP4_I$.

4.2 The Various Phases of $dP4$

Having obtained one phase of the $dP4$ theory, it is natural to seek other phases related thereto by Seiberg duality. In this section, we shall look for the duality transformations which stay within the toric phase. We shall also find the closure of this set of dual theories.

For $dP4_I$, we can rewrite (4.1) as

$$\begin{aligned} W_{dP4_I} &= -[\phi_{51}\phi_{12}\phi_{25} - \phi_{57}\phi_{73}\phi_{35}] + [\phi_{12}\phi_{24}\phi_{46}\phi_{61} - \phi_{73}\phi_{34}\phi_{46}\phi_{67}] \\ &\quad + [\phi_{34}\phi_{45}\phi_{51}\phi_{13} - \phi_{24}\phi_{45}\phi_{57}\phi_{72}] + [\phi_{25}\phi_{56}\phi_{67}\phi_{72} - \phi_{35}\phi_{56}\phi_{61}\phi_{13}] \end{aligned} \tag{4.2}$$

where to show explicitly the symmetry between the pair $(1, 2)$ and $(7, 3)$ we have redefined the fields and grouped them properly. We remind the reader that this can be higgsed down to model I of $dP3$.

Now let us discuss the symmetries of this model in the spirit of [9]. First, from the quiver diagram in Figure 3 we see following symmetry: (1) nodes $1 \leftrightarrow 7$; (2) nodes $2 \leftrightarrow 3$; (3) nodes $4 \leftrightarrow 6$, $1 \leftrightarrow 3$, $2 \leftrightarrow 7$ as well as reversing the directions of all arrows.

However, the superpotential we found in (4.2) does not preserve all these symmetries. It is easy to see that only the following symmetries are preserved: (1) simultaneous exchange of nodes $1 \leftrightarrow 7$ and $2 \leftrightarrow 3$ (we have shown this symmetry explicitly by the brackets in (4.2)); (2) exchange of nodes $4 \leftrightarrow 6$, $1 \leftrightarrow 3$, $2 \leftrightarrow 7$ and reversal of the directions of all arrows.

These observations of symmetries are very important and will reduce much computation in tracing through the tree of generalised Seiberg dualities. For example, we see immediately that dualising on node 4 will give the same theory as on 6. Similarly, dualising on any of $1, 2, 3, 7$ will also produce the same theory.

Now starting from $dP4_I$ we can dualise either node $4, 6$ to give us a new model which we shall call $dP4_{II}$. The superpotential is after integrating out, given by

$$\begin{aligned} W_{dP4_{II}} &= -[\phi_{51}\phi_{12}\phi_{25} - \phi_{57}\phi_{73}\phi_{35}] + [\phi_{12}\widetilde{\phi}_{26}\phi_{61} - \phi_{73}\widetilde{\phi}_{36}\phi_{67}] \\ &\quad + [\widetilde{\phi}_{35}\phi_{51}\phi_{13} - \widetilde{\phi}_{25}\phi_{57}\phi_{72}] + [\phi_{25}\phi_{56}\phi_{67}\phi_{72} - \phi_{35}\phi_{56}\phi_{61}\phi_{13}] \\ &\quad + [\widetilde{\phi}_{25}\phi_{54}\phi_{42} - \widetilde{\phi}_{35}\phi_{54}\phi_{43}] - [\widetilde{\phi}_{26}\phi_{64}\phi_{42} - \widetilde{\phi}_{36}\phi_{64}\phi_{43}], \end{aligned} \tag{4.3}$$

where the $\widetilde{\phi}$ are dual meson fields and the last row comes from the added meson interaction of the form $Mq\tilde{q}$.

Now let us discuss the symmetries of $dP4_{II}$, which from the quiver we see as (1) $2 \leftrightarrow 3$ and (2) the permutations of nodes $(1,4,7)$. Again, the superpotential preserves only the symmetry of exchanging $1 \leftrightarrow 7$ and $2 \leftrightarrow 3$ at the same time. This is also explicitly shown in (4.3) by grouping the appropriate terms in brackets. The symmetry indicates that dualising nodes $1,7$ will give the same theory. It is also worth to mention that although ϕ_{35} and $\widetilde{\phi}_{35}$ are doubly degenerate in the quiver diagram, the superpotential breaks this degeneracy explicitly. The same conclusion holds for fields ϕ_{25} and $\widetilde{\phi}_{25}$.

Finally, we have nodes $1,2,3,7$ to choose from in dualising $dP4_I$. Let us without loss of generality choose to dualise node 1; we reach yet another model, which we call $dP4_{III}$. Comparing with the quiver of $dP4_I$, we notice that they are almost the same except one thing: there is a bi-directional arrow between nodes $3,5$. This difference is very important and non-trivial. For all del Pezzo surfaces we have encountered before, they are always chiral. This property of the del Pezzo surfaces was also pointed out in [16]. In fact, the rules given in [8, 10] about Seiberg duality can not be directly applied to such cases. It is certainly worth to investigate this and generalise the Seiberg duality rules. In any event we seem to have a puzzle here as the the del Pezzo surfaces admit only uni-directional arrows [16]. We shall address this puzzle in Section 6. For now let us present the superpotential:

$$\begin{aligned} W_{dP4_{III}} = & [\phi_{62}\phi_{24}\phi_{46} - \phi_{62}\phi_{21}\phi_{16}] + [\phi_{34}\phi_{45}\phi_{53} - \phi_{31}\phi_{15}\phi_{53}] \\ & + \phi_{57}\phi_{73}\phi_{35} - \phi_{35}\phi_{56}\phi_{63} + \phi_{63}\phi_{31}\phi_{16} \\ & - \phi_{73}\phi_{34}\phi_{46}\phi_{67} - \phi_{24}\phi_{45}\phi_{57}\phi_{72} + \phi_{21}\phi_{15}\phi_{56}\phi_{67}\phi_{72}. \end{aligned} \tag{4.4}$$

The quiver of this model has only an explicit $\mathbb{Z}_2$ symmetry $1 \leftrightarrow 4$.

These three models are the only toric phases of $dP4$ under Seiberg duality and we summarise them in Figure 5.

5. Higgsing from $dP4$ to $dP3$

In the previous sections we have studied how to obtain one of the phases of $dP4$ by unhiggsing $dP3_I$, and then calculated all the three toric phases of $dP4$ that are closed under Seiberg dualities. Now we will show how it is possible to get all the four toric $dP3$ phases by higgsing the $dP4$ models. One can conversely adopt the unhiggsing perspective, and think about the result we here present as possible ways of going from $dP3$ to $dP4$ by suitable unhiggsings. Once again one could take the (p,q) perspective [19] to visualise more easily.

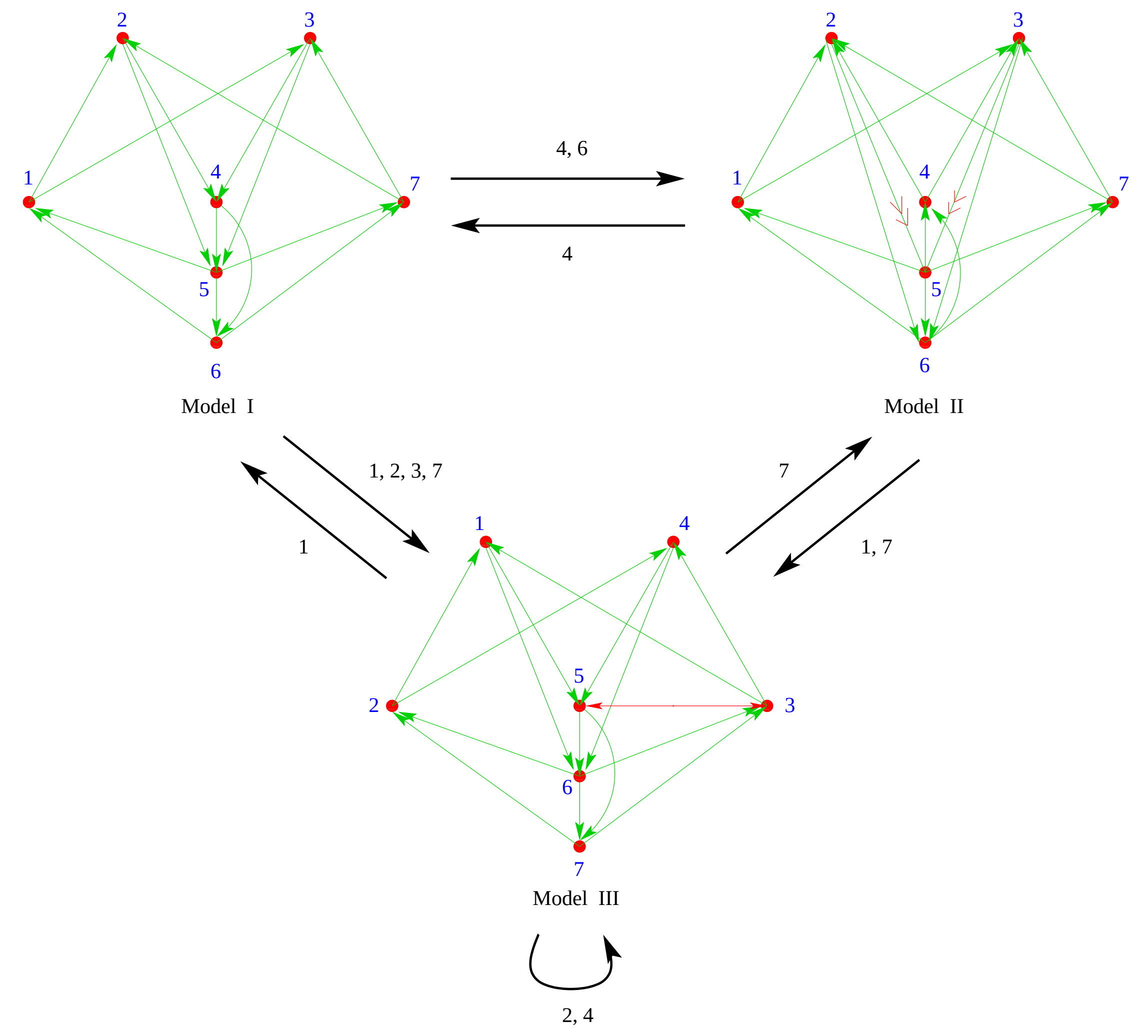


Figure 5: The quivers for the three toric Seiberg dual phases of $dP4$. The nodes upon which one dualises to transform between them are shown next to the arrows.

5.1 Phase I of $dP3$

The discussion in Section 4 showed how one obtains $dP3_I$ from $dP4_I$ and vice versa. Here we show how to accomplish the same using $dP4_{III}$ as our starting point.

Let us turn on a non-zero VEV for ϕ_{31} in $dP4_{III}$. This expectation value for a charged bi-fundamental field breaks $U(1)_{(1)} \times U(1)_{(3)}$ down to the $U(1)_{[5]}$ subgroup, thus leading to a theory with $U(1)^6$ gauge group. The subsequent quiver diagram is shown in Figure 6. Looking at the superpotential (4.4) we see that the cubic terms containing ϕ_{31} give rise to masses for $\tilde{\phi}_{53}$, ϕ_{15}, $\tilde{\phi}_{26}$ and ϕ_{16}. When looking at the IR limit of the gauge theory, these massive fields have to be integrated out using their equations of motion. The result, is a $U(1)^6$ gauge theory with 12 fields and superpotential given by

$$W = \tilde{\phi}_{62}\phi_{24}\phi_{46} + \phi_{57}\phi_{73}\phi_{35} + \phi_{21}\phi_{34}\phi_{45}\phi_{56}\phi_{67}\phi_{72}$$
$$-\phi_{35}\phi_{56}\tilde{\phi}_{62}\phi_{21} - \phi_{73}\phi_{34}\phi_{36}\phi_{67} - \phi_{24}\phi_{45}\phi_{57}\phi_{72},$$

which, after the following renaming of the gauge groups $(1/3, 2, 4, 5, 6, 7) \to (5, 4, 6, 1, 2, 3)$ and calling the fields X and setting $\langle\phi_{31}\rangle = 1$ becomes

$$W = X_{24}X_{46}X_{62} + X_{13}X_{35}X_{51} + X_{45}X_{56}X_{61}X_{12}X_{23}X_{34}$$
$$-X_{51}X_{12}X_{24}X_{45} - X_{35}X_{56}X_{62}X_{23} - X_{46}X_{61}X_{13}X_{34}.$$

We recognise this exactly as the superpotential, and part (b) of Figure 6, the quiver, for phase I of $dP3$, as is required.

5.2 Phase II of $dP3$

Referring to Figure 7, let us start from model II of $dP4$ and give a VEV to ϕ_{12}. In this case, only the $U(1)_{[4]}$ in $U(1)_{(1)} \times U(1)_{(2)}$ survives. Mass terms are generated for ϕ_{25}, ϕ_{51}, $\tilde{\phi}_{26}$ and ϕ_{61}. After integrating them out, we have a $U(1)^6$ theory with 14 fields and the following superpotential

$$W = \tilde{\phi}_{36}\phi_{64}\phi_{43} + \tilde{\phi}_{25}\phi_{54}\phi_{42} - \phi_{73}\tilde{\phi}_{36}\phi_{67} - \tilde{\phi}_{25}\phi_{57}\phi_{72} + \phi_{57}\phi_{73}\phi_{35}$$
$$-\tilde{\phi}_{35}\phi_{54}\phi_{43} + \tilde{\phi}_{35}\phi_{56}\phi_{67}\phi_{72}\phi_{13} - \phi_{35}\phi_{56}\phi_{64}\phi_{42}\phi_{13}.$$

Renaming the gauge groups $(1/2, 3, 4, 5, 6, 7) \to (4, 1, 5, 3, 2, 6)$ as well as the fields

$$\phi_{35} \to X_{13}, \quad \tilde{\phi}_{35} \to Y_{13}, \quad \tilde{\phi}_{36} \to X_{12}, \quad \phi_{43} \to X_{51},$$

we get

$$W = X_{12}X_{25}X_{51} + X_{43}X_{35}X_{54} - X_{61}X_{12}X_{26} - X_{43}X_{36}X_{64} + X_{36}X_{61}X_{13}$$
$$-Y_{13}X_{35}X_{51} + Y_{13}X_{32}X_{26}X_{64}X_{41} - X_{13}X_{32}X_{25}X_{54}X_{41}.$$

This is precisely, upto an overall minus sign, the superpotential for $dP3_{II}$. Likewise, the quiver of the $dP3$ model is reproduced exactly, as shown in Figure 7.

We can get the model II of $dP3$ also from phase III of $dP4$, whose superpotential is given by

$$W = \phi_{62}\phi_{24}\phi_{46} - \phi_{62}\phi_{21}\phi_{16} + \phi_{34}\phi_{45}\phi_{53} - \phi_{31}\phi_{15}\phi_{53} + \phi_{57}\phi_{73}\phi_{35} - \phi_{35}\phi_{56}\phi_{63}$$
$$+\phi_{63}\phi_{31}\phi_{16} - \phi_{73}\phi_{34}\phi_{46}\phi_{67} - \phi_{24}\phi_{45}\phi_{57}\phi_{72} + \phi_{21}\phi_{15}\phi_{56}\phi_{67}\phi_{72}.$$

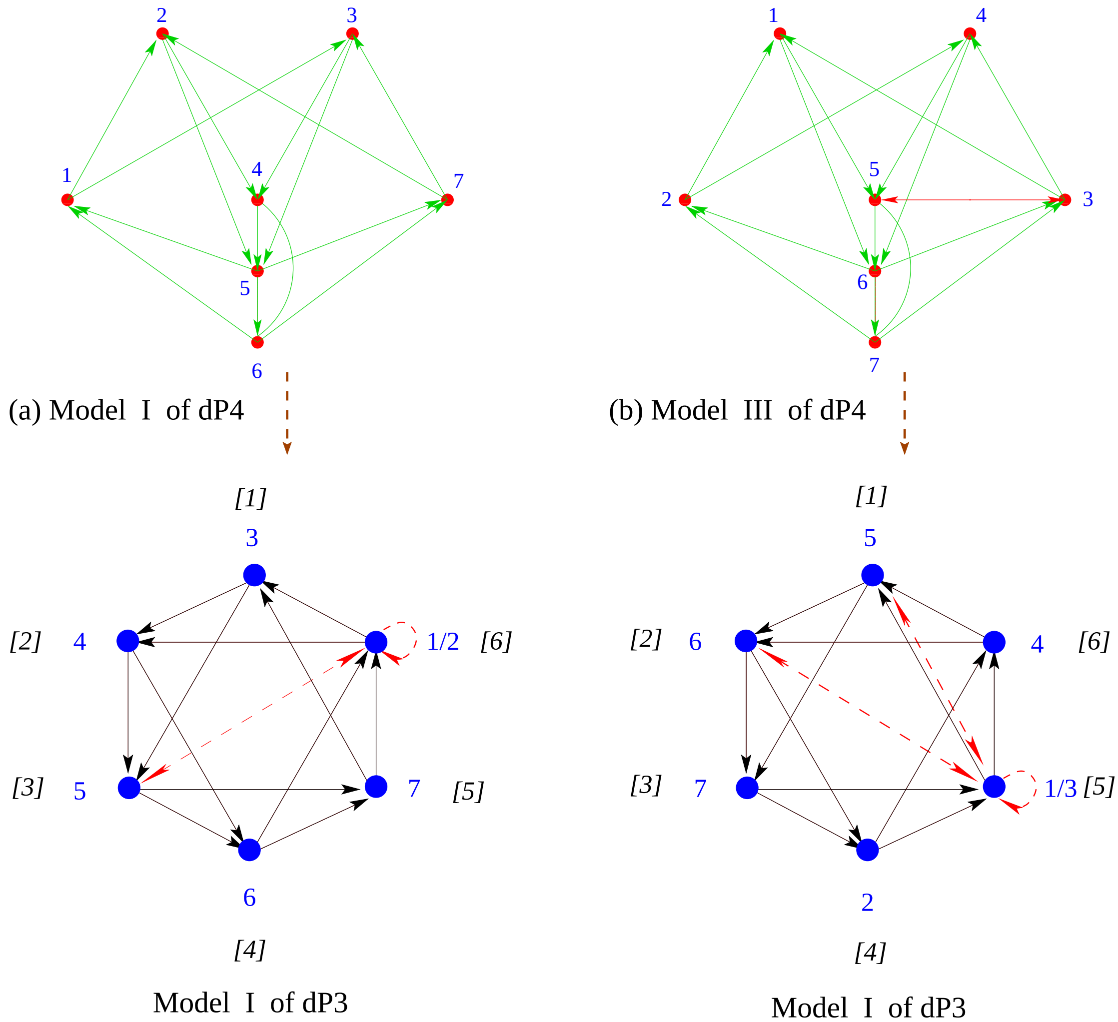


Figure 6: (a) Higgsing the field ϕ_{12} of $dP4_I$ to obtain $dP3_I$. (b) Higgsing the field ϕ_{31} of $dP4_{III}$ to also reach $dP3_I$. We have used the dashed lines to indicate the fields to be integrated out and nodes in bracket to indicate the corresponding nodes in model I of $dP3$.

Setting $\langle\phi_{73}\rangle = 1$, $U(1)_{(7)} \times U(1)_{(3)}$ is broken down to $U(1)_{[3]}$. During the higgsing, ϕ_{35} and ϕ_{57} become massive, with equations of motion

$$\phi_{35} = \phi_{24}\phi_{45}\phi_{72}$$

$$\phi_{57} = \phi_{56}\phi_{63}.$$

Finally, renaming nodes $(1, 2, 3/7, 4, 5, 6) \to (5, 2, 3, 6, 4, 1)$, and calling the two fields connecting nodes 1 and 3 in the final theory

$$\phi_{67} \to \tilde{X}_{13}, \qquad \phi_{63} \to Y_{13},$$

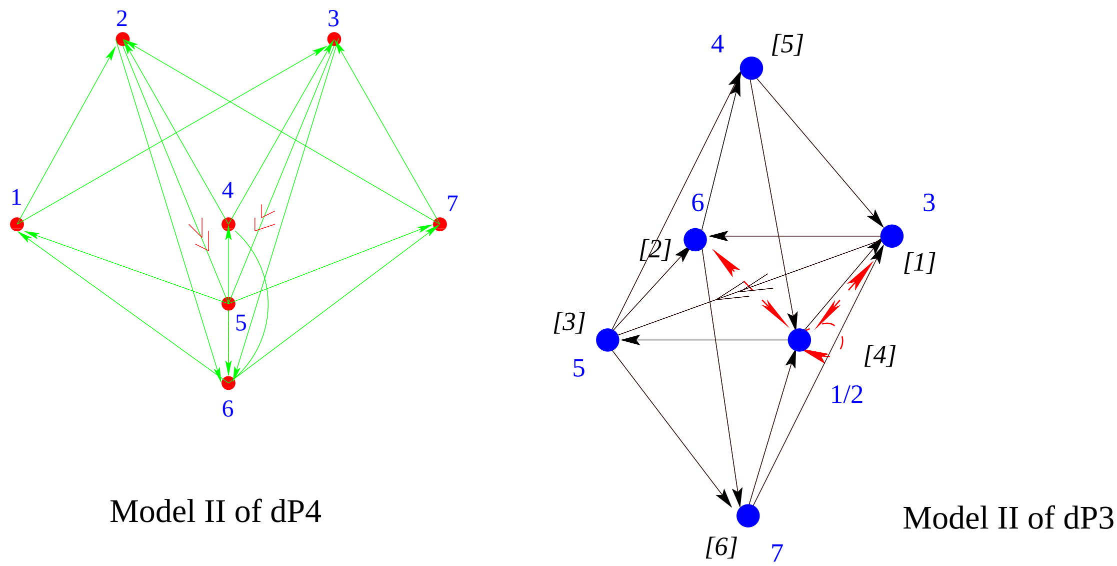


Figure 7: Higgsing the field ϕ_{12} of $dP4_{II}$ to reach $dP3_{II}$. We have used the dashed lines to indicate the fields to be integrated out and nodes in bracket to indicate the corresponding nodes in $dP3_{II}$.

we get

$$W = X_{12}X_{26}X_{61} - X_{12}X_{25}X_{51} + X_{36}X_{64}X_{43} - X_{35}X_{54}X_{43} + Y_{13}X_{35}X_{51} \\ -X_{36}X_{61}X_{13} - X_{26}X_{64}X_{41}Y_{13}X_{32} + X_{25}X_{54}X_{41}X_{13}X_{32},$$

which is again the superpotential for phase II of $dP3$.

5.3 Phase III of $dP3$

This time, we can start from any of the models I, II and III of $dP4$ to reach model III of $dP3$. First we start from $dP4_{III}$ and turn on a VEV for ϕ_{56}. The fields ϕ_{35} and $\tilde{\phi}_{63}$ will become massive. Then, in the IR we have a $U(1)^6$ theory with 14 fields. Taking $\langle\phi_{56}\rangle = 1$, the superpotential is

$$W = -\phi_{31}\phi_{15}\tilde{\phi}_{53} + \phi_{34}\phi_{45}\tilde{\phi}_{53} - \tilde{\phi}_{62}\phi_{21}\phi_{16} + \tilde{\phi}_{62}\phi_{24}\phi_{46} - \phi_{24}\phi_{45}\phi_{57}\phi_{72} \\ +\phi_{57}\phi_{73}\phi_{31}\phi_{16} - \phi_{73}\phi_{34}\phi_{46}\phi_{67} + \phi_{21}\phi_{15}\phi_{67}\phi_{72}$$

Let us rename the $U(1)$ gauge factors as $(1, 2, 3, 4, 5/6, 7) \to (1, 4, 2, 3, 5, 6)$ and call the fields X, except

$$\phi_{16} \to X_{15},\ \phi_{15} \to Y_{15},$$
$$\phi_{67} \to X_{56},\ \phi_{57} \to Y_{56},$$
$$\phi_{46} \to X_{35},\ \phi_{45} \to Y_{35}.$$

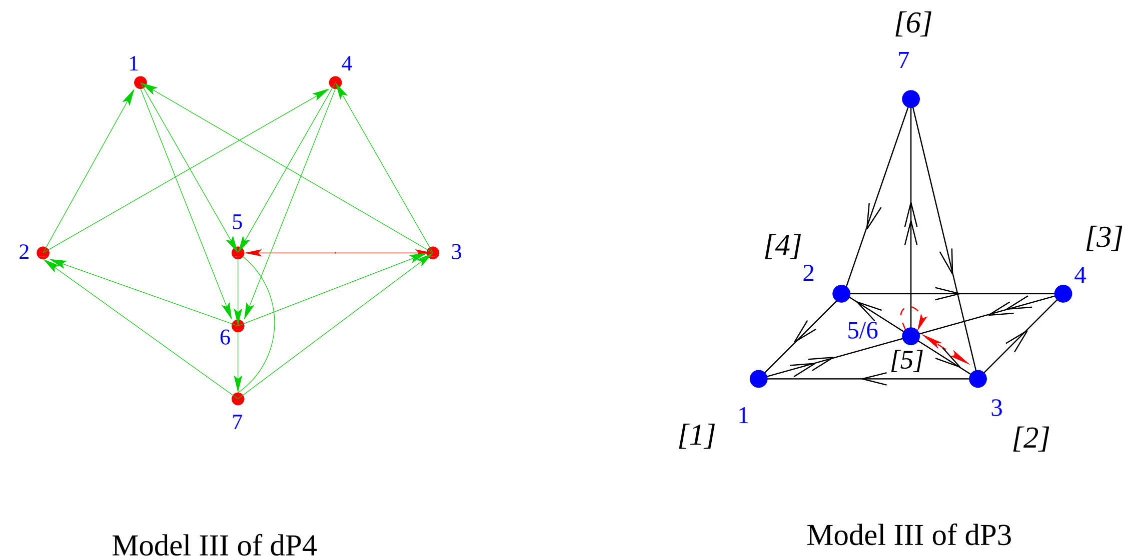


Figure 8: Higgsing the field ϕ_{56} of $dP4_{III}$ to reach $dP3_{III}$. We have used the dashed lines to indicate the fields to be integrated out and nodes in bracket to indicate the corresponding nodes in $dP3_{III}$.

Then, after redefining $X_{41} \to -X_{41}$ and $X_{43} \to -X_{43}$, the superpotential becomes

$$W = -X_{21}Y_{15}X_{52} + X_{23}Y_{35}X_{52} + X_{54}X_{41}X_{15} - X_{54}X_{43}X_{35} + X_{43}Y_{35}Y_{56}X_{64} \\ +X_{21}X_{15}Y_{56}X_{62} - X_{62}X_{23}X_{35}X_{56} - X_{41}Y_{15}X_{56}X_{64},$$

which is the superpotential for phase III of $dP3$. The quiver of this model is also correct, as drawn in Figure 8.

Next we start from from phase I of $dP4$, whose superpotential is

$$W = \phi_{73}\phi_{35}\phi_{57} + \phi_{51}\phi_{12}\phi_{25} + \phi_{24}\phi_{46}\phi_{61}\phi_{12} - \phi_{73}\phi_{34}\phi_{46}\phi_{67} \\ -\phi_{45}\phi_{57}\phi_{72}\phi_{24} - \phi_{35}\phi_{56}\phi_{61}\phi_{13} + \phi_{25}\phi_{56}\phi_{67}\phi_{72} - \phi_{51}\phi_{13}\phi_{34}\phi_{45}.$$

We turn on $\langle\phi_{56}\rangle = 1$. In this case, no mass terms are generated. After renaming nodes $(1, 2, 3, 4, 5/6, 7) \to (1, 4, 2, 6, 5, 3)$, calling

$$\phi_{51} \to X_{51},\ \phi_{61} \to Y_{51},\ \phi_{67} \to X_{53}$$
$$\phi_{57} \to Y_{53},\ \phi_{46} \to X_{65},\ \phi_{45} \to Y_{65},$$

and changing the signs $X_{46} \to -X_{46}$ and $Y_{65} \to -Y_{65}$, the superpotential becomes

$$W = X_{32}X_{25}Y_{53} + X_{51}X_{14}X_{45} - X_{25}Y_{51}X_{12} + X_{45}X_{53}X_{34} - X_{46}X_{65}Y_{51}X_{14} \\ -X_{32}X_{26}X_{65}X_{53} - Y_{65}Y_{53}X_{34}X_{46} + X_{51}X_{12}X_{26}Y_{65}.$$

We recognise this to be the superpotential for Phase III of $dP3$ after charge conjugation.

Finally we start from phase II of $dP4$ with superpotential

$$
\begin{aligned}
W = & -\phi_{51}\phi_{12}\phi_{25} + \phi_{57}\phi_{73}\phi_{35} + \phi_{12}\tilde{\phi}_{26}\phi_{61} - \phi_{73}\tilde{\phi}_{36}\phi_{67} + \tilde{\phi}_{35}\phi_{51}\phi_{13} - \tilde{\phi}_{25}\phi_{57}\phi_{72} \\
& + \tilde{\phi}_{25}\phi_{54}\phi_{42} - \phi_{35}\phi_{54}\phi_{43} - \tilde{\phi}_{26}\phi_{64}\phi_{42} + \tilde{\phi}_{36}\phi_{64}\phi_{43} + \phi_{25}\phi_{56}\phi_{67}\phi_{72} - \phi_{35}\phi_{56}\phi_{61}\phi_{13}.
\end{aligned}
$$

Setting $\langle\phi_{64}\rangle = 1$, $U(1)_{(6)} \times U(1)_{(4)}$ breaks to the $U(1)_{[3]}$ and mass terms are generated for $\tilde{\phi}_{26}$, ϕ_{42}, $\tilde{\phi}_{36}$ and ϕ_{43}, with equations of motion

$$
\begin{aligned}
\tilde{\phi}_{26} &= \tilde{\phi}_{25}\phi_{54} \qquad & \phi_{42} &= \phi_{12}\phi_{61} \\
\tilde{\phi}_{36} &= \tilde{\phi}_{35}\phi_{54} \qquad & \phi_{43} &= \phi_{73}\phi_{67} \ .
\end{aligned}
$$

Relabelling nodes $(1,2,3,4/6,5,7) \to (4,1,3,6,5,2)$, changing $X_{15} \to -X_{15}$ and calling

$$
\begin{aligned}
&\phi_{25} \to X_{15},\ \tilde{\phi}_{25} \to Y_{15} \\
&\phi_{35} \to Y_{35},\ \tilde{\phi}_{35} \to X_{35} \\
&\phi_{54} \to X_{56},\ \phi_{56} \to Y_{56},
\end{aligned}
$$

we get

$$
\begin{aligned}
W = & X_{54}X_{41}X_{15} + X_{52}X_{23}Y_{35} + X_{35}X_{54}X_{43} - Y_{15}X_{52}X_{21} - X_{15}Y_{56}X_{62}X_{21} \\
& -Y_{35}Y_{56}X_{64}X_{43} - X_{23}X_{35}X_{56}X_{62} + Y_{15}X_{56}X_{64}X_{41},
\end{aligned}
$$

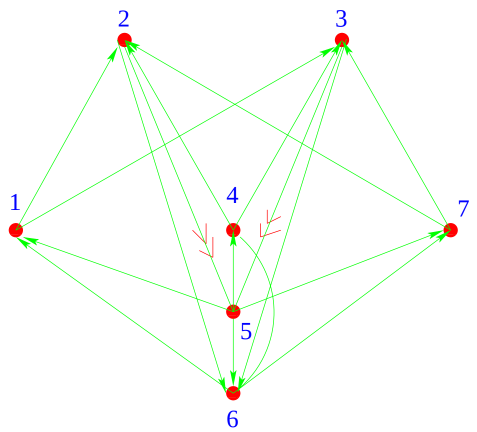
and once again obtain the theory for $dP3_{III}$.

5.4 Phase IV of $dP3$

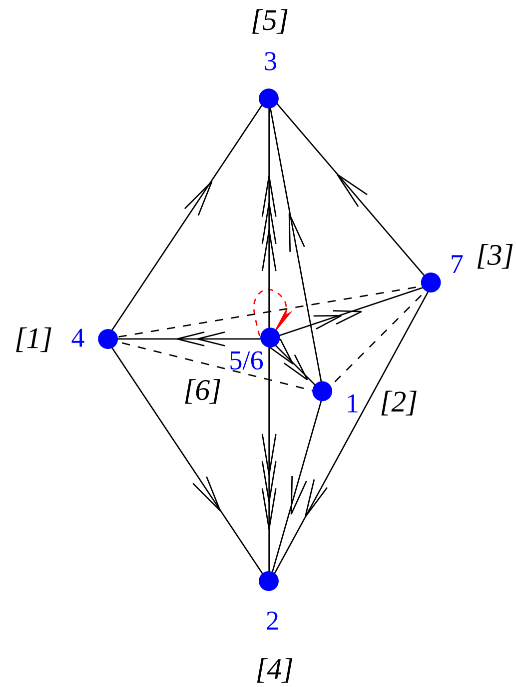
After the above detailed demonstrations, we will be brief in this part. In this case, we start from the model II of $dP4$ and give the nonzero VEV to ϕ_{56}. It is easy to see the quiver will be that of model IV of $dP3$, as drawn in Figure 9. Renaming nodes $(1,2,3,4,5/6,7) \to (2,4,5,1,6,3)$ and making the following replacements

$$
\begin{array}{lll}
\phi_{51} \to X_{62}, & \phi_{61} \to Y_{62}, & \\
\phi_{25} \to Z_{46}, & \widetilde{\phi}_{25} \to Y_{46}, & \widetilde{\phi}_{26} \to X_{46}, \\
\phi_{57} \to X_{63}, & \phi_{67} \to Y_{63}, & \\
\phi_{35} \to Z_{56}, & \widetilde{\phi}_{35} \to X_{56}, & \widetilde{\phi}_{36} \to Y_{56}, \\
\phi_{54} \to Y_{61}, & \phi_{64} \to X_{61}. &
\end{array}
$$

we get the correct superpotential, upto an overall minus sign and the charge conjugation of all fields suggested by the condensed quiver.



Model II of dP4



Model IV of dP3

Figure 9: Higgsing the field ϕ_{56} from $dP4_{II}$ to give $dP3_{IV}$. We have used the dashed lines to indicate the fields to be integrated out and nodes in bracket to indicate the corresponding nodes in model IV of $dP3$.

6. PdP4: del Pezzo Three Blownup at a Non-Generic Point

We have obtained, via the unhiggsing method, three toric phases of a new theory from the four phases of the cone over del Pezzo 3. In the previous sections, because we have used the quivers obtained from (p,q)-web techniques in, [16], we have assumed that we have arrived at the theory for $dP4$. Is this indeed so? The purpose of this section is to show that we are not quite right, even though we did arrive at a new theory which is $dP3$ blownup at one point.

Let us begin with the model $dP4_{III}$ obtained from unhiggsing. We recall the matter content and superpotential here:

$$\begin{aligned} W_{III} = & [\phi_{62}\phi_{24}\phi_{46} - \phi_{62}\phi_{21}\phi_{16}] + [\phi_{34}\phi_{45}\phi_{53} - \phi_{31}\phi_{15}\phi_{53}] \\ & + \phi_{57}\phi_{73}\phi_{35} - \phi_{35}\phi_{56}\phi_{63} + \phi_{63}\phi_{31}\phi_{16} \\ & - \phi_{73}\phi_{34}\phi_{46}\phi_{67} - \phi_{24}\phi_{45}\phi_{57}\phi_{72} + \phi_{21}\phi_{15}\phi_{56}\phi_{67}\phi_{72} \end{aligned}$$

and

$$
d_{III} = \begin{array}{c}
\begin{matrix} \phi_{15} & \phi_{16} & \phi_{21} & \phi_{24} & \phi_{31} & \phi_{34} & \phi_{35} & \phi_{45} & \phi_{46} & \phi_{53} & \phi_{56} & \phi_{57} & \phi_{62} & \phi_{63} & \phi_{67} & \phi_{72} & \phi_{73} \end{matrix} \\
\begin{pmatrix}
-1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1
\end{pmatrix}
\end{array}.
$$

We can obtain all the 17 F-terms from W_{III}:

$$
\begin{array}{l}
\phi_{21}\phi_{56}\phi_{67}\phi_{72} = \phi_{31}\phi_{53}, \quad \phi_{31}\phi_{63} = \phi_{21}\phi_{62}, \quad \phi_{15}\phi_{56}\phi_{67}\phi_{72} = \phi_{16}\phi_{62}, \quad \phi_{46}\phi_{62} = \phi_{45}\phi_{57}\phi_{72} \\
\phi_{16}\phi_{63} = \phi_{15}\phi_{53}, \quad \phi_{45}\phi_{53} = \phi_{46}\phi_{67}\phi_{73}, \quad \phi_{57}\phi_{73} = \phi_{56}\phi_{63}, \quad \phi_{34}\phi_{53} = \phi_{24}\phi_{57}\phi_{72}, \\
\phi_{24}\phi_{62} = \phi_{34}\phi_{67}\phi_{73}, \quad \phi_{34}\phi_{45} = \phi_{15}\phi_{31}, \quad \phi_{15}\phi_{21}\phi_{67}\phi_{72} = \phi_{35}\phi_{63}, \quad \phi_{35}\phi_{73} = \phi_{24}\phi_{45}\phi_{72}, \\
\phi_{24}\phi_{46} = \phi_{16}\phi_{21}, \quad \phi_{16}\phi_{31} = \phi_{35}\phi_{56}, \quad \phi_{15}\phi_{21}\phi_{56}\phi_{72} = \phi_{34}\phi_{46}\phi_{73}, \quad \phi_{15}\phi_{21}\phi_{56}\phi_{67} = \phi_{24}\phi_{45}\phi_{57}, \\
\phi_{35}\phi_{57} = \phi_{34}\phi_{46}\phi_{67}.
\end{array}
$$

These are all monomial relations! These F-terms thus generate a toric ideal. This is suggestive that our moduli space is actually toric and thus cannot be the cone over the generic del Pezzo 4. Let us perform the Forward Algorithm of [6] to check.

From the F-terms we can actually express the solution space in terms of the K-matrix prescribing a cone:

$$
K^T = \begin{array}{cc}
 & \begin{matrix} \phi_{15} & \phi_{16} & \phi_{21} & \phi_{24} & \phi_{31} & \phi_{34} & \phi_{35} & \phi_{45} & \phi_{46} & \phi_{53} & \phi_{56} & \phi_{57} & \phi_{62} & \phi_{63} & \phi_{67} & \phi_{72} & \phi_{73} \end{matrix} \\
\begin{matrix} \phi_{15} \\ \phi_{21} \\ \phi_{34} \\ \phi_{45} \\ \phi_{56} \\ \phi_{57} \\ \phi_{67} \\ \phi_{72} \\ \phi_{73} \end{matrix} &
\begin{pmatrix}
1 & 2 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array},
$$

where we express the 17 variables in terms of 9 as we read from the above vertically: $\phi_{j=1,\ldots,17} = \prod_{i=1}^{9} \phi_i^{K^T_{ij}}$.

Equipped with the d and K matrices we can now easily perform the Forward Algorithm to obtain the moduli space as a toric variety. The answer is:

$$
G_t = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
-1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1
\end{pmatrix}.
$$

We immediately see that after a permutation s and an $SL(3;\mathbb{Z})$ transformation, which certainly do not effect the moduli space, we can bring the above G_t to a familiar form:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix} \stackrel{s}{\Rightarrow} \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 & -1 \end{pmatrix}$$

$$\stackrel{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}}{\Longrightarrow} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 & 2 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}$$

We recognise the embedding of this toric diagram into our familiar orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ in Figure 10. We have explicitly labelled the multiplicity of the GLSM fields (homogeneous coordinates) and see that it is perfectly congruent with the observations in [9, 21] about the emergence of the Pascal's triangle.

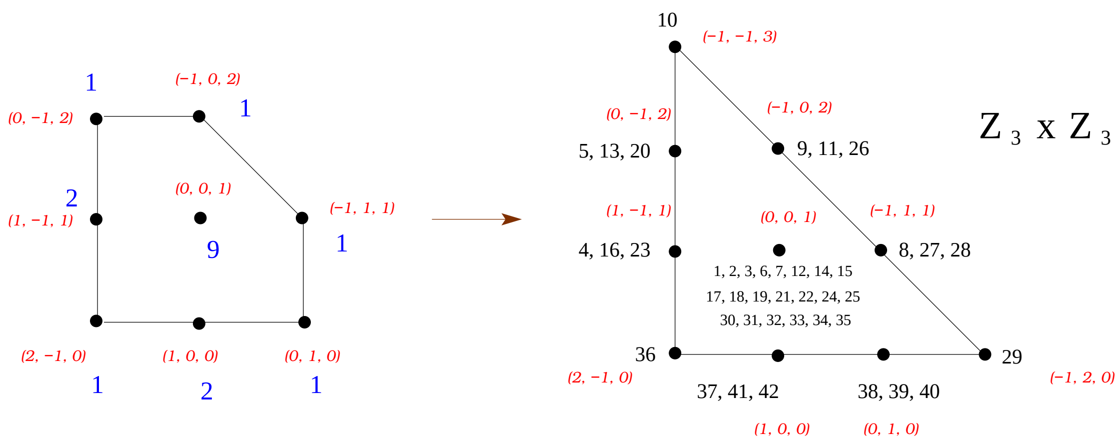


Figure 10: The embedding of the moduli space for the model III obtained from unhiggsing $dP3$, into the toric diagram of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. We have labelled all coordinates explicitly. In the left the numbers (in blue) are the multiplicities corresponding to the nodes (q.v. [9, 21]) and in the right, the numbers are the GLSM fields commonly used.

What we have for the moduli space is therefore a toric variety which is a blowup of $dP3$. According to [17], such a cone is no longer over an ample surface. Therefore whatever theory we have obtained, is not that of the generic del Pezzo 4 theory, because all del Pezzo surfaces are ample; to this point we shall return in the next subsection.

Certainly, unhiggsing the $dP3$ corresponds to blowing it up at a point and the so-called $dP4$ theories in the previous sections are indeed $dP3$ blowup at a point and hence the cone over $\mathbb{P}^2$ blowup at 4 points. We thus conclude that the theories we have obtained in the previous sections must be the cone over $\mathbb{P}^2$ blownup at 4 **non-generic** points. We shall henceforth call this variety the **Pseudo dP4**, or PdP4.

Here is an important fact: whereas $\mathbb{P}^2$ blownup at **generic** points are the del Pezzo surfaces, as we shall see below, blowing up at *non-generic* points no longer gives us del Pezzo surfaces in the strict sense. Indeed as remarked above, our $dP4$ is really a toric variety while the generic del Pezzo k for $k \geq 4$ certainly is not. Recently such non-generic del Pezzos have risen in the context of [30].

6.1 Some Properties of the Moduli Space

We here have a toric variety whose toric diagram is given in Figure 10; let us determine some of its geometrical properties in light of the discussion above that it should be a $PdP4$.

Let us study the compact surface as a projective variety because we know the properties of the del Pezzo surfaces well; our $dP4$ is simply an affine cone thereover. In other words we shall study the so-called Pseudo del Pezzo surface $PB4$ in comparison to the true $B4$.

First, given the toric diagram, one could immediately find the characteristic classes using combinatorics [28]. With the convenient help of the package Macaulay [33], we immediately arrive at the Betti numbers: $b^0 = b^4 = 1, b^1 = b^3 = 0$ and $b^2 = 5$. Indeed the middle-dimensional homology of $\mathbb{P}^2$ blown up at 4 points should consist of the hyperplane $\mathbb{P}^1$ class as well as 4 exceptional divisors of the blowup. Thus we pass our preliminary test of homology.

Next let us study the explicit embedding as projective varieties. We know, using the method of fat points[5] in $\mathbb{P}^2$ [34] that the generic del Pezzo 4 surface can be embedded as the smooth intersection of 5 quadrics in $\mathbb{P}^5$ (q.v. [36]). The affine cone over it, would have an isolated singular point at the conical apex (say, at the origin) and it is this point upon which we place our D-brane probe. A non-generic one however, say 3 co-linear points being blown-up, may have more complicated presentation. Moreover, the precise positions of the blowups determine the complex structure moduli space of the B_4, whereupon singularities may arise as one varies these positions and causes the Jacobian matrix to be non-maximal rank. In these cases the affine cone $dP4$ would have singularities at more than the point at the origin. To these we refer as non-generic, or pseudo $dP4$'s.

From the toric diagram in Figure 10, we can instantly determine the projective embedding by finding the relations of the homogeneous coordinate ring [35]. We find that we obtain the intersection of 5 quadrics in $\mathbb{P}^5$; and indeed upon computing the Jacobian matrix of the variety we find non-trivial singular loci. Therefore our toric diagram corresponds to a non-generic $dP4$, or $PdP4$.

[5]YHH would like to thank Hal Schenck of Texas AMU for extensive discussions on this point.

Let us re-iterate to our reader that a surface given by a toric diagram of the form Figure 10 does not have ample anti-canonical class (and hence not del Pezzo). Standard results from toric geometry (e.g., [28]) dictates that a Cartier divisor D on a complete toric variety X is ample iff its support function is strictly convex. Combinatorially this translates to the following: Let X be a complete toric variety with fan $\Sigma = \{\sigma_a\}$ with each cone σ generated by $\{v_i\}$ as $\sigma = \sum_i \mathbb{R}_{\geq 0} v_i$. A divisor D can be written as $\sum_{i=1}^r a_i D_i$ with D_i corresponding to v_i, then

THEOREM 6.1 *D is ample iff for each cone σ there exists an integer vector m_σ such that $\langle m_\sigma, v_i \rangle = -a_i$ for all i and such that $m_\sigma \neq m_\tau$ for different cones σ and τ.*

The anticanonical class is of course given by $K = -\sum_{i=1}^r D_i$ with all $a_i = -1$. We can thus easily proceed with the check for ampleness. The surface we have at hand has the fan as given in Figure 10: $\Sigma = \{\sigma_{i=1\ldots7}\}$ with σ_i generated by $\{v_i, v_{(i+1) \bmod 7}\}$ where $\{v_1, v_2, \ldots v_7\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$. The list of support function m_σ can be easily computed as $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$. Due to the repetition therein we conclude that $-K$ is indeed not ample and our surface is not del Pezzo.

In fact all the toric diagrams which satisfy the criteria of the above theorem are classified in dimension 2 [29] and are precisely the del Pezzo polytopes; Figure 10 is certainly not a member of the classification.

6.2 Confirmations from the Inverse Algorithm

Having assertained that the moduli space is actually toric with the explicit toric diagram and embedding given in Figure 10, we can naturally use the conjecture that toric duality is Seiberg duality [8, 15] to see whether we indeed obtain the above phases. We will use the algorithm of the multiplicity symmetry introduced in [9].

We have 3 models which we must obtain. Starting from the 42 GLSM fields of $\mathbb{Z}_3 \times \mathbb{Z}_3$ in Figure 10, we obtain a total of 216 theories which fall into various isoclasses. If we keep, for example, the fields { 4, 5, 6, 7, 8, 9, 12, 14, 15, 18, 21, 22, 23, 30, 36, 37, 38, 42 }, we obtain the theory with 17 fields, precisely the model III addressed above. If, on the other hand, we kept the fields { 4, 5, 6, 7, 8, 11, 12, 17, 18, 19, 21, 22, 23, 24, 25, 30, 32, 36, 37, 38, 42 }, then the resulting theory is the model II with 19 fields.

These consistency checks are very re-assuring: even though the moduli space we obtained is not that of the cone over the generic del Pezzo four, it is a perfectly well-defined toric Calabi-Yau variety. Most importantly, toric duality from the Inverse Algorithm indeed reproduces the Seiberg dual theories obtained from field theoretic analyses using unhiggsing.

However we have yet to obtain the model I with 15 fields. This poses a hitherto unencountered problem. The Inverse Algorithm does not give us any theories with 15 fields. What seems to be wrong? Let us attempt to find the moduli space of Model I using the Forward Algorithm. From the superpotential

$$W = -X_{12}\,X_{25}\,X_{51} + X_{13}\,X_{34}\,X_{45}\,X_{51} + X_{12}\,X_{24}\,X_{46}\,X_{61} - X_{13}\,X_{35}\,X_{56}\,X_{61} -$$
$$X_{24}\,X_{45}\,X_{57}\,X_{72} + X_{25}\,X_{56}\,X_{67}\,X_{72} + X_{35}\,X_{57}\,X_{73} - X_{34}\,X_{46}\,X_{67}\,X_{73},$$

we can solve for the 15 F-terms as

$$\begin{aligned}\{X_{12}, X_{73}, X_{35}, X_{57}, X_{25}, X_{51}\} = \pm\{&\sqrt{\frac{X_{13}\,X_{34}\,X_{45}\,X_{56}\,X_{67}\,X_{72}}{X_{24}\,X_{46}\,X_{61}}}, \sqrt{\frac{X_{13}\,X_{24}\,X_{45}\,X_{56}\,X_{61}\,X_{72}}{X_{34}\,X_{46}\,X_{67}}}, \sqrt{\frac{X_{24}\,X_{34}\,X_{45}\,X_{46}\,X_{67}\,X_{72}}{X_{13}\,X_{56}\,X_{61}}},\\ &\sqrt{\frac{X_{13}\,X_{34}\,X_{46}\,X_{56}\,X_{61}\,X_{67}}{X_{24}\,X_{45}\,X_{72}}}, \sqrt{\frac{X_{13}\,X_{24}\,X_{34}\,X_{45}\,X_{46}\,X_{61}}{X_{56}\,X_{67}\,X_{72}}}, \sqrt{\frac{X_{24}\,X_{46}\,X_{56}\,X_{61}\,X_{67}\,X_{72}}{X_{13}\,X_{34}\,X_{45}}}\}.\end{aligned} \tag{6.1}$$

These are not monomial relations! In fact no attempt of the solution space (the so-called space of commuting matrices [4]) of these F-terms could give purely monomial relations. In other words, we cannot generate a K-matrix which corresponds to an integral polyhedral cone. The Forward Algorithm thus already fails to be valid.

This is rather surprising. We have checked in Subsection 6.1 that the moduli space is toric and in particular, the toric $PdP4$. Furthermore we have checked above that we indeed consistently obtain models $dP4_{II}$ and $dP4_{III}$. Indeed we must be able to obtain this remaining model of $dP4_I$ from partial resolutions. Yet, the Forward Algorithm (and thus necessarily the Inverse Algorithm) already does not seem to succeed to generate a cone, and hence a toric description.

The situation however, is easily remedied. The F-terms in (6.1) generate a cone over $\mathbb{Q}$ instead of our usual circumstance of $\mathbb{Z}$. It corresponds to a K-matrix with $\frac{1}{2}$ entries due to the square root exponent; we simply reconvert our basis and work in a large integral cone by multiplying it by 2:

$$\begin{pmatrix}
\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\
-\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}\\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2}\\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\
-\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2}\\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2}\\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2}
\end{pmatrix} \stackrel{\times 2}{\Longrightarrow} K^T = \begin{pmatrix}
1 & 2 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1\\
-1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1\\
1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1\\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & -1\\
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1\\
-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1\\
1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & -1\\
1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 1
\end{pmatrix}$$

Now application of the standard Forward Algorithm on this integral matrix K and the incidence matrix for the quiver in Figure 3 readily gives us (after an appropriate unimodular transformation that does not change the geometry) the correct toric diagram in the left of Figure 10.

Therefore with the caveat of needing to convert a rational cone to an integral one, upon which both the Forward and Inverse Algorithms depend, we have shown that the remaining case of $dP4_I$

also gives the same toric variety. Therefore all 3 Seiberg dual phases for $dP4$ give the same moduli space, as expected. More importantly, the moduli space is toric, an affine variety which is a cone over $\mathbb{P}^2$ blown up at 4 colinear points. Thus in our notation, the models $dP4_{I,II,III}$ should really be called $PdP4_{I,II,III}$ and to this convention we shall henceforth adhere.

7. Unhiggsing $PdP4$ Once Again to $PdP5$

Having obtained the toric, non-generic, $PdP4$, it is natural to ask whether this pattern should continue. In other words, could we unhiggs/blowup this non-generic $dP4$ to something else that is perhaps also toric, and in particular, $PdP5$?

We shall see that this indeed is the case in this section, whereby confirming out unhiggsing procedure as well as the Inverse Algorithm. We find that there are in fact four toric phases which are related to each other by Seiberg duality. Without much ado let us present the results below.

7.1 Model $PdP5_I$

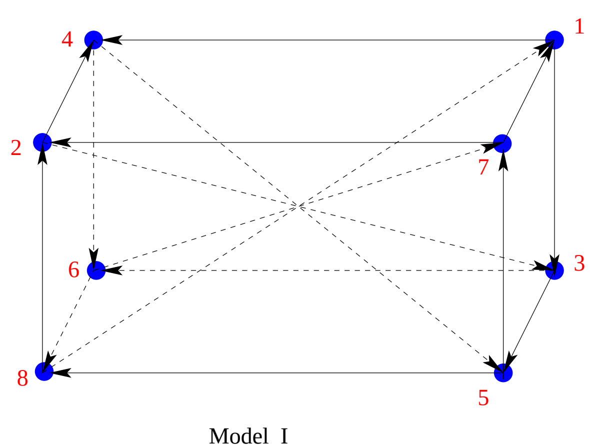


Figure 11: Model I of the $PdP5$ theory, unhiggsed from $PdP4_I$.

Now we unhiggs the above $PdP4$ to the $dP5$ given in [16]. Indeed it will turn out that it is not really $dP5$ either and we will use the notation $PdP5$ for pseudo-$dP5$. Comparing the quiver of model I of $PdP5$ in [16] with the quiver of $PdP4_I$, we see that giving the field ϕ_{68} nonzero VEV is the way to higgs $PdP5$ to $PdP4$.

Since the $PdP5_I$ has 16 fields while $PdP4_I$ has 15, there is no mass term generated in the higgsing process and the unhiggsing is straight-forward. We just need to lift the superpotential of $PdP4_I$ directly. With a little of algebra we reach the superpotential (7.1):

$$\begin{aligned} W_I = {} & \phi_{13}\phi_{35}\phi_{58}\phi_{81} + \phi_{14}\phi_{46}\phi_{68}\phi_{81} + \phi_{35}\phi_{57}\phi_{72}\phi_{23} - \phi_{46}\phi_{67}\phi_{72}\phi_{24} \\ & + \phi_{67}\phi_{71}\phi_{13}\phi_{36} - \phi_{57}\phi_{71}\phi_{14}\phi_{45} + \phi_{58}\phi_{82}\phi_{24}\phi_{45} - \phi_{68}\phi_{82}\phi_{23}\phi_{36}. \end{aligned} \tag{7.1}$$

Let us analyze the symmetry of the quiver. First there is a cyclic $\mathbb{Z}_4$ symmetry around the horizontal axis. Second, there is a $\mathbb{Z}_2$ symmetry which exchanges (1357) and (2864) and reverses the arrows (i.e., charge conjugation). The superpotential preserves both symmetries. It is easy to see that by redefining the signs of fields $\phi_{58}, \phi_{13}, \phi_{46}, \phi_{14}, \phi_{23}$ we can regroup the superpotential as

$$\begin{aligned} W_I = &[(1358) - (3572) + (5714) - (7136)] + \\ &[(3682) - (5824) + (7246) - (1468)], \end{aligned}$$

where the four terms in brackets are related to each other by $\mathbb{Z}_4$ and two brackets are related to each other by $\mathbb{Z}_2$.

Due to the abundance of such symmetries, Seiberg dualising any of the nodes will give the same result. We will call, without loss of generality, the result from dualising on node 8, $PdP5_{II}$, upon which we shall in the ensuing subsections continue to dualise to obtain models $PdP5_{III,IV}$.

7.2 Model $PdP5_{II}$

Without loss of generality, let us dualise $PdP5_I$ on node 8. Since there are no cubic terms in (7.1), no mass terms are generated. The resulting model has 20 fields, with no bi-directional arrows. It is then possible to call all fields ϕ_{ij}, and the superpotential is

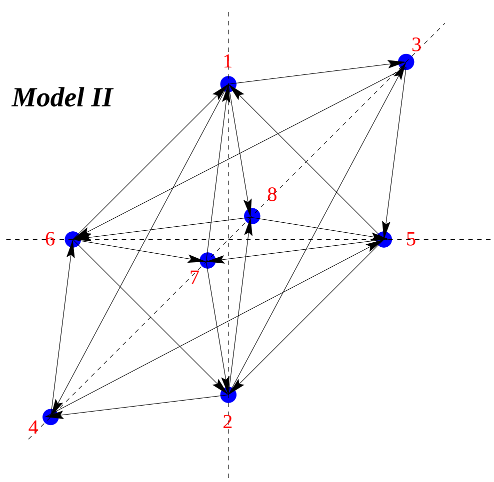


Figure 12: The quiver for the theory $PdP5_{II}$. This is a nice 3D representation, with nodes 7 and 8 located at the centre.

$$\begin{aligned} W = &\phi_{13}\phi_{35}\phi_{51} + \phi_{14}\phi_{46}\phi_{61} + \phi_{52}\phi_{24}\phi_{45} - \\ &\phi_{62}\phi_{23}\phi_{36} - \phi_{51}\phi_{18}\phi_{85} - \phi_{61}\phi_{18}\phi_{86} - \\ &\phi_{52}\phi_{28}\phi_{85} + \phi_{62}\phi_{28}\phi_{86} + \phi_{35}\phi_{57}\phi_{72}\phi_{23} - \\ &\phi_{46}\phi_{67}\phi_{72}\phi_{24} + \phi_{67}\phi_{71}\phi_{13}\phi_{36} - \phi_{57}\phi_{71}\phi_{14}\phi_{45} \end{aligned} \tag{7.2}$$

The symmetries are $\mathbb{Z}_2 : (1,5) \leftrightarrow (2,6)$ and $\mathbb{Z}_2 : (1,3) \leftrightarrow (2,4)$. Using these we can group the superpotential in the orbits of the global symmetries as

$$\begin{aligned} W_{II} = &[(7246) + (7145) - (7136) - (7235)] \\ &+[(513) + (623) - (524) - (614)] \\ &+[(528) + (618) - (518) - (628)]. \end{aligned}$$

Geometrically, we let (12) be the z-axis, (34), the y-axis, (56), the x-axis and $(7,8)$, around the origin as in Figure 12; Then symmetries are just the rotation with x, y, z axis.

From these symmetries, we see that to get new Seiberg dual phase, we have only two choices: dualise node 3 or node 7. Starting from node 3 we obtain a new theory: $PdP5_{III}$ and starting from node 7 we obtain $PdP5_{IV}$. To these we now turn.

7.3 Model $PdP5_{III}$

Recall from the above that dualising $PdP5_{II}$ on node 3 we obtain the quiver in Figure 13 with the dual superpotential

$$
\begin{aligned}
W = M_{15}\phi_{51} + \phi_{14}\phi_{46}\phi_{61} + \phi_{25}\phi_{24}\phi_{45} - \phi_{62}M_{26} - \phi_{51}\phi_{18}\phi_{85} - \phi_{61}\phi_{18}\phi_{86} \\
-\phi_{52}\phi_{28}\phi_{85} + \phi_{62}\phi_{28}\phi_{86} + M_{25}\phi_{57}\phi_{72} - \phi_{46}\phi_{67}\phi_{72}\phi_{24} + \phi_{67}\phi_{71}M_{16} \\
-\phi_{57}\phi_{71}\phi_{14}\phi_{45} - M_{15}\tilde{\phi}_{53}\tilde{\phi}_{31} - M_{16}\tilde{\phi}_{63}\tilde{\phi}_{31} - M_{25}\tilde{\phi}_{53}\tilde{\phi}_{32} + M_{26}\tilde{\phi}_{63}\tilde{\phi}_{32}.
\end{aligned}
$$

We see that M_{51}, ϕ_{15}, M_{62} and ϕ_{26} become massive, leading to a theory with 20 fields. Integrating them out using their equations of motion (and calling all fields ϕ_{ij}) we finally have

$$
\begin{aligned}
W = \phi_{14}\phi_{46}\phi_{61} + \phi_{52}\phi_{24}\phi_{45} - \phi_{61}\phi_{18}\phi_{86} - \phi_{52}\phi_{28}\phi_{85} + \phi_{25}\phi_{57}\phi_{72} + \phi_{67}\phi_{71}\phi_{17} \\
-\phi_{16}\phi_{63}\phi_{31} - \phi_{25}\phi_{53}\phi_{32} - \phi_{53}\phi_{31}\phi_{18}\phi_{85} + \phi_{63}\phi_{32}\phi_{28}\phi_{86} - \phi_{46}\phi_{67}\phi_{72}\phi_{24} - \phi_{57}\phi_{71}\phi_{14}\phi_{15}
\end{aligned}
\tag{7.3}
$$

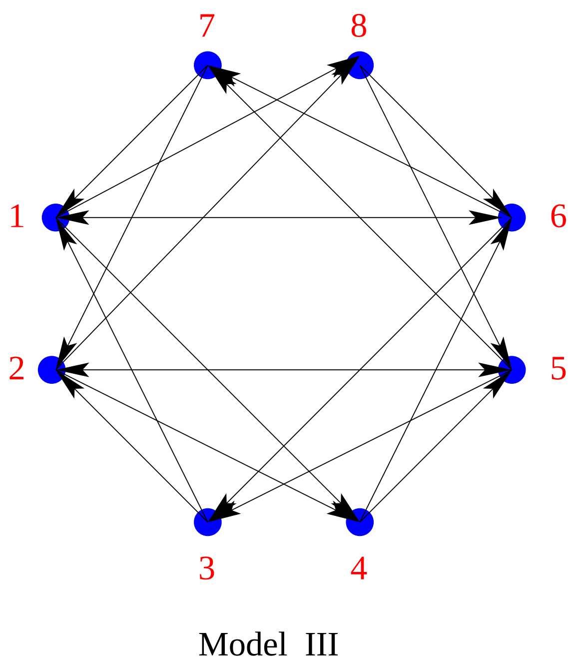


Figure 13: The theory for $PdP5_{III}$. Notice the bidirectional arrows (16) and (25).

From the quiver in Figure 13 we see that the theory has these symmetries: $\mathbb{Z}_2^{(1)}$: $(15) \leftrightarrow (26)$, $\mathbb{Z}_2^{2)}$: $(34) \leftrightarrow (78)$ and $\mathbb{Z}_2^{(3)}$: $(1237) \leftrightarrow (6548)$. Grouping terms together with respect to this symmetry we get the superpotential

$$
\begin{aligned}
W_{III} = [(7145) + (7246) - (3185) - \\
(3286)] + [(528) + (618) - (524) - (614)] \\
+[(163) + (253) - (167) - (257)],
\end{aligned}
$$

where every bracket is invariant under $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{2)}$ while the first bracket is invariant under $\mathbb{Z}_2^{(3)}$ and last two brackets are related by $\mathbb{Z}_2^{(3)}$. From the symmetry, we see that no new phase can be reached by Seiberg duality that still remains toric.

7.4 Model $PdP5_{IV}$

Recall that we have a final model which comes from $PdP5_{II}$ after dualising on node 7. Now this node does not appear in any cubic term of (7.2), thus there are no massive fields. This phase has 24 fields, with the quiver shown in Figure 14 and the superpotential is

$$
\begin{aligned}
W = \phi_{13}\phi_{35}\phi_{51} + \phi_{14}\phi_{46}\phi_{61} + \phi_{52}\phi_{24}\phi_{45} - \phi_{62}\phi_{23}\phi_{36} - \phi_{51}\phi_{18}\phi_{85} - \phi_{61}\phi_{18}\phi_{86} - \phi_{52}\phi_{28}\phi_{85} - \phi_{62}\phi_{28}\phi_{86} \\
+\phi_{35}\tilde{\phi}_{52}\phi_{23} - \phi_{46}\tilde{\phi}_{62}\phi_{24} + \tilde{\phi}_{61}\phi_{13}\phi_{36} - \tilde{\phi}_{51}\phi_{14}\phi_{45} + \tilde{\phi}_{51}\phi_{17}\phi_{75} - \tilde{\phi}_{52}\phi_{27}\phi_{75} - \tilde{\phi}_{61}\phi_{17}\phi_{76} + \tilde{\phi}_{62}\phi_{27}\phi_{76},
\end{aligned}
$$

where we have indicated the Seiberg mesons with tildes.

The theory has the symmetry: $\mathbb{Z}_2^{(1)}: (15) \leftrightarrow (26)$, $\mathbb{Z}_2^{(2)}: (73) \leftrightarrow (84)$ and changing tildes to non-tildes, $\mathbb{Z}_2^{(3)}: (78) \leftrightarrow (43)$ and $\mathbb{Z}_2^{(4)}: (173) \leftrightarrow (284)$ (here the tildes are not changed to non-tilde).

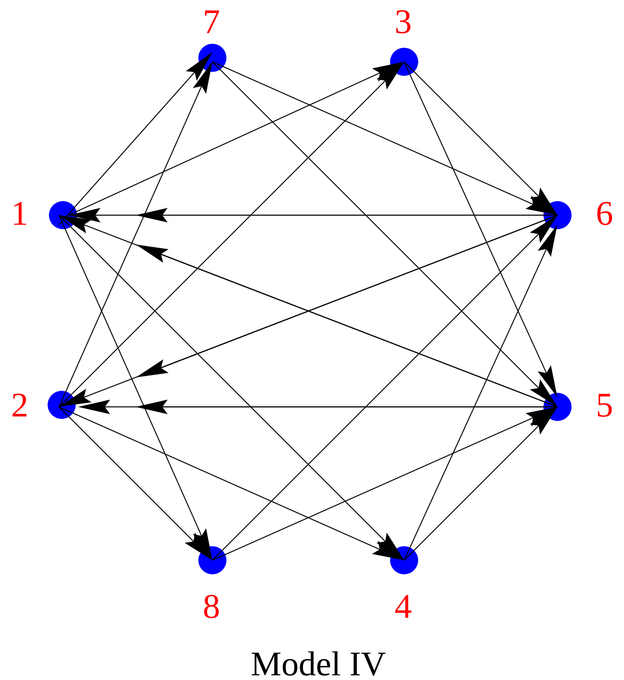


Figure 14: The quiver diagram for model $PdP5_{IV}$.

The superpotential can be accordingly grouped as

$$
\begin{aligned}
W_{IV} = \{- &[(\widetilde{627}) + (\widetilde{517}) + (628) + (518)] + \\
&[(\widetilde{624}) + (\widetilde{514}) + (623) + (513)]\} + \\
&\{[(\widetilde{617}) + (\widetilde{527}) + (618) + (528)] - \\
&[(614) + (524) + (\widetilde{613}) + (\widetilde{523})]\}.
\end{aligned}
$$

Here every bracket is grouped by $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(2)}$; the first and second brackets as well as the third and the fourth are each grouped by $\mathbb{Z}_2^{(3)}$. Moreover these two pairs of brackets ([1][2]) and ([3][4]) are related to each other by $\mathbb{Z}_2^{(4)}$. From the symmetry, we see that nodes $3, 4, 7, 8$ are equivalent to each other, so Seiberg duality can not give new phase that is toric.

In conclusion then, by unhiggsing the $PdP4$ theory we have obtained a toric phase for a blownup of thereof, which we have called $PdP5$. By applying Seiberg duality, we have found all toric phases of this theory and there are 4 of these: $PdP5_{I,II,III,IV}$. We summarize them in Figure 15.

7.5 PdP5 and the Orbifolded Conifold

In the vein of thought of Section 6, let us investigate whether continuing this method of blowing up/unhiggsing would give rise to yet another toric moduli space, in particular the cone over the so-called pseudo del Pezzo 5.

Let us study model $dP5_I$; the others are related thereto by Seiberg duality and hence have the same moduli space. From the quiver from Figure 11 and the superpotential in (7.1), we can readily proceed with the Forward Algorithm of [1, 2, 5, 4, 6]. The final moduli space we obtain is summarised in the G_t matrix for the toric diagram:

$$
G_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -2 \end{pmatrix}.
$$

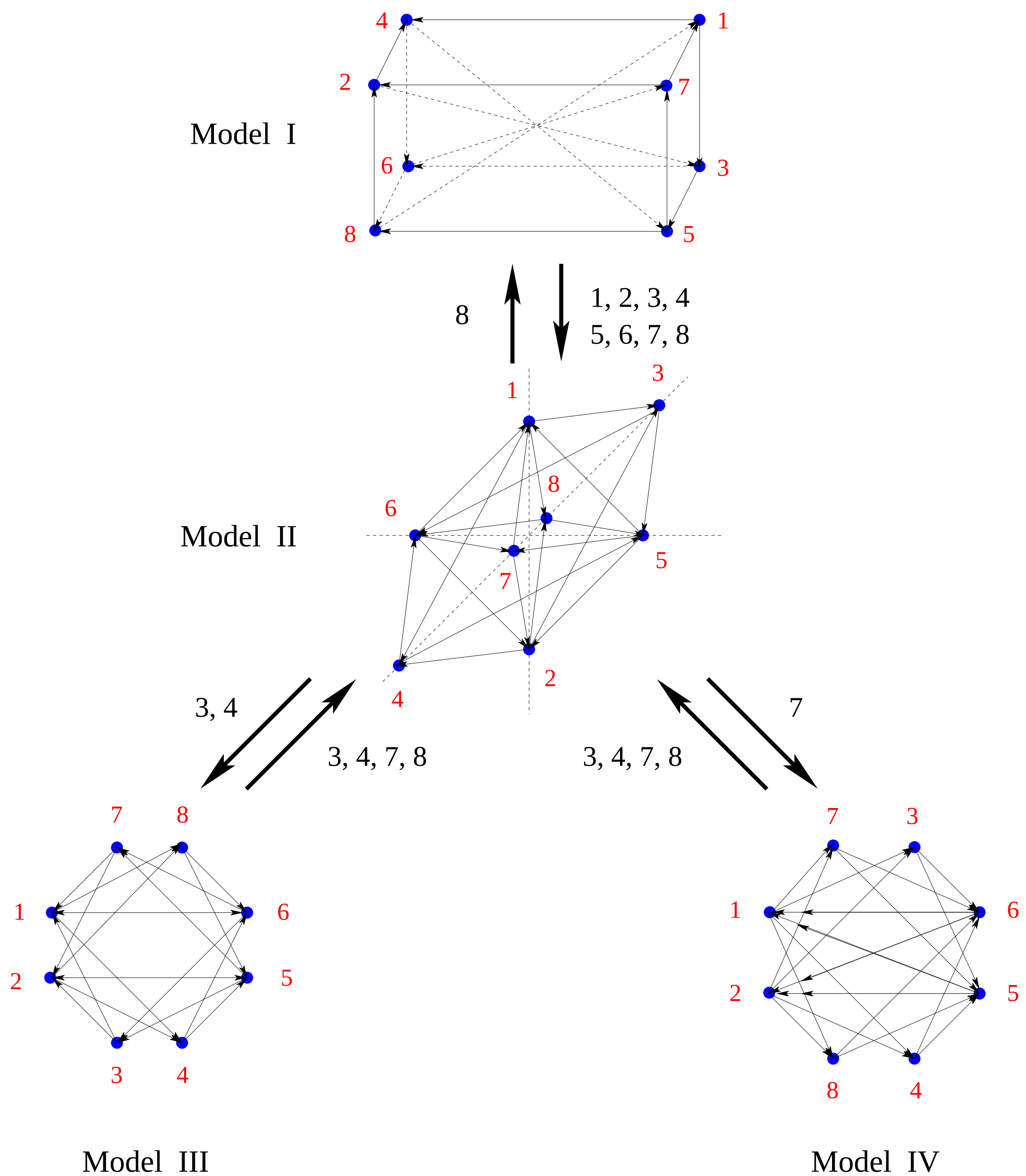


Figure 15: The quivers for the four Seiberg dual phases of $dP5$ which all have rank one nodes. The nodes upon which one dualises to transform between them are shown next to the arrows.

This corresponds to the toric diagram as shown in Figure 16 (we have performed the usual $SL(3;\mathbb{Z})$ transformation so as to make the presentation compatible with the standard $\mathbb{Z}_3 \times \mathbb{Z}_3$ toric diagram in [4, 6]).

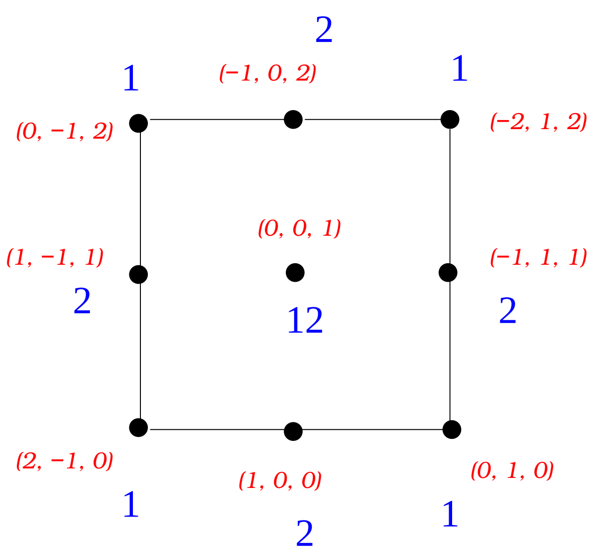


Figure 16: Toric diagram for the theory corresponding to $dP3$ blownup twice, or what have called the toric $PdP5$. We have shown the coordinates in red and the multiplicity of the GLSM fields, in blue.

The toric diagram is indeed expected, consistent with the node-addition of Section 6. Incidentally, the multiplicities of the GLSM fields (shown in blue next to the nodes) are still consistent with the observations of [9, 21]. What might be a surprise to the reader is that this is a well-known toric diagram (q.v. e.g. [31, 32, 25]) and the superpotential has been known from the brane diamond techniques.

Figure 16 is the orbifolded conifold, with the affine equation

$$xy = z^2 \quad uv = z^2.$$

Therefore the surface over which our Calabi-Yau is an affine cone is a compact divisor (4-cycle) in the orbifolded conifold $C_{2,2}$.

Along the lines of Subsection 6.1, let us again check the geometry we have obtained. As before, we focus on the compact projective surface over which our moduli space is an affine cone. Computing the homology of the toric surface corresponding to Figure 16, we obtain $b^0 = b^4 = 1, b^1 = b^3 = 0$ and $b^2 = 6$. This is indeed the homology of $\mathbb{P}^2$ blown up at 5 points.

We proceed to check the embedding equations. We recall that $\mathbb{P}^2$ blown up at 5 generic points is the well-known del Pezzo surface of degree 4, as the intersection of 2 quadrics in $\mathbb{P}^4$. If we have say, 5 non-generic points however, we could again use [34] to find that such a surface is given by 2 quadrics in $\mathbb{P}^4$, but with non-trivial singular loci. On the other hand, the homogeneous coordinate ring of the toric variety in Figure 16 gives us precisely such an embedding into $\mathbb{P}^4$. Thus we have shown that our moduli space is indeed a toric variety, the non-generic $PdP5$. Once again, we see that checking against Theorem 6.1, the anticanonical divisor is not ample and our surface is not del Pezzo.

Therefore with the current technology of the inherently toric method of (p, q) webs which provided us the quivers, the unhiggsing procedure stays within the toric realm. The unhiggsing can bring us from $\mathbb{P}^2$ to del Pezzo 3, and continue so to the surfaces corresponding to $\mathbb{P}^2$ blownup at 4 and 5 special points, which we have rather cavalierly called the non-generic or pseudo del Pezzos. We summarise our results for the unhiggsing/blowups in Figure 17.

8. Quiver Symmetries and the Superpotential

In this section, we will try, in the spirit of [9, 10], to use symmetry arguments to fix the superpotentials for the theories which we have called $PdP4_{I,II}$ in Section 4. The situation is more complex here than the cases discussed in [9] because $PdP4_{III}$ does not have any explicit symmetry. More precisely, the quiver has some symmetry but the superpotential breaks it. We will show here, by certain consistency arguments, that we can sometimes determine how the superpotential breaks the quiver symmetry.

8.1 Symmetries of $PdP4_I$

Let us start from model I. We recall from Figure 3 that there is an explicit $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(2)}$ ($1 \leftrightarrow 7$ and $2 \leftrightarrow 3$) quiver symmetry; here we list the orbits of loops (i.e., possible terms in the superpotential) under this group[6]:

1. $\{(125), (725), (135), (735)\}$
2. $\{(1256), (7256), (1356), (7356)\}$
3. $\{(1245), (7245), (1345), (7345)\}$
4. $\{(1246), (7246), (1346), (7346)\}$
5. $\{(12456), (72456), (13456), (73456)\}$.

Now there is yet another group $\mathbb{Z}_2^{(3)}$ defined by the simultaneous action of $(4, 1, 2) \leftrightarrow (6, 7, 3)$ plus charge conjugation. Under this third $\mathbb{Z}_2$, orbits (1), (4) and (5) are self-dual while orbit (2) maps to (3). Therefore under this full $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ quiver symmetry we have four orbits: (1), (4), (5) and (2-3). We can easily count the number of times the fields appear in these orbits to be respectively 12, 16, 20 and 32. Now in Section 6, we have assertained that $PdP4_I$ is toric, thus since it has 15 fields, a total of $15 \times 2 = 30$ fields must appear in the superpotential [6, 9]. This is incompatible with the orbit counting above. Therefore *the quiver symmetry must be broken.* But how?

First we assume only one $\mathbb{Z}_2$ is broken. We have the following cases: (A) $\mathbb{Z}_2^{(3)}$ is broken, giving us 5 orbits with number of fields $12, 16, 16, 16$ and 20. Again it does not work; (B) $\mathbb{Z}_2^{(2)}$ is broken. Now although $2 \leftrightarrow 3$ is broken, by $\mathbb{Z}_2^{(3)}$ we still have the loop $3 \leftrightarrow 1 \leftrightarrow 7 \leftrightarrow 2$, so we still have

[6] In fact, there is another set of gauge invariant operators $(1257346), (7251346), (1357246), (7351246)$ which contains the seven nodes. It is easy to concluded that this orbit has to be excluded, so we will neglect it in the following discussion.

orbits with field numbers $12, 16, 32, 20$ and it still can not be; (C) $\mathbb{Z}_2^{(1)}$ is broken, giving us the same situation as case (B). These cases tell us that we must break a combination of $\mathbb{Z}_2$'s and leave the diagonal term invariant. It is obvious that $\mathbb{Z}_2^{(3)}$ can not combine with $\mathbb{Z}_2^{(1)}$ or $\mathbb{Z}_2^{(2)}$. This leave us with the only choice (D) breaking the combination $\mathbb{Z}_2^{(3)}$ and $\mathbb{Z}_2^{(4)}$, defined by the action $(1,2) \leftrightarrow (7,3)$. This will turn out to be the right choice.

Now let us write down the orbits of loops under the symmetry $\mathbb{Z}_2^{(3)} \times \mathbb{Z}_2^{(4)}$:

- (Ia). $\{(125), (735)\}$
- (Ib). $\{(135), (725)\}$
- (Ic). $\{(1246), (7346)\}$
- (Id). $\{(7246), (1346)\}$
- (Ie). $\{(1256), (7356), (7345), (1245)\}$
- (If). $\{(7256), (1356), (7245), (1345)\}$
- (Ig). $\{(12456), (73456)\}$
- (Ih). $\{(72456), (13456)\}$

The number of fields in the orbits are respectively $6, 6, 8, 8, 16, 16, 10$ and 10. There are these ways to get the number 30: $6 + 8 + 16, 6 + 6 + 8 + 10$.

For the choice of $6 + 6 + 8 + 10$, From orbits (Ia) and (Ib), we get $(125) - (135) + (735) - (725)$ where we have chosen the sign properly. If we choose the orbit (Ic) we get $(125) - (135) + (735) - (725) - (1246) - (7346)$ where the minus sign of last two terms is determined by the positive sign of $(125), (735)$. However, we find that field ϕ_{46} shows up twice with the same sign and contradicts the toric condition [9]. The same argument shows that the orbit (Id) is not the correct choice either. This tells us that we should choose the other combination $6 + 8 + 16$.

For this combination of $6 + 8 + 16$, there are two orbits with 16 fields. However, since they are different by only relabelling $2 \leftrightarrow 3$, we can choose without loss of generality, for example, the orbit (If). Starting from this orbit we write down

$$-\phi_{13}\phi_{35}\phi_{56}\phi_{61} + \phi_{72}\phi_{25}\phi_{56}\phi_{67} - \phi_{72}\phi_{24}\phi_{45}\phi_{57} + \phi_{13}\phi_{34}\phi_{45}\phi_{51}.$$

Now we need to determine the other orbits. Since ϕ_{13} has appeared twice in orbit (If) already, we must choose orbit (Ic) and (Ia). Putting every thing together we get

$$\begin{aligned} W_I &= [-\phi_{13}\phi_{35}\phi_{56}\phi_{61} + \phi_{72}\phi_{25}\phi_{56}\phi_{67} - \phi_{72}\phi_{24}\phi_{45}\phi_{57} + \phi_{13}\phi_{34}\phi_{45}\phi_{51}] \\ &\quad +[\phi_{35}\phi_{57}\phi_{73} - \phi_{51}\phi_{25}\phi_{12}] + [\phi_{12}\phi_{24}\phi_{46}\phi_{61} - \phi_{73}\phi_{34}\phi_{46}\phi_{67}] \end{aligned}$$

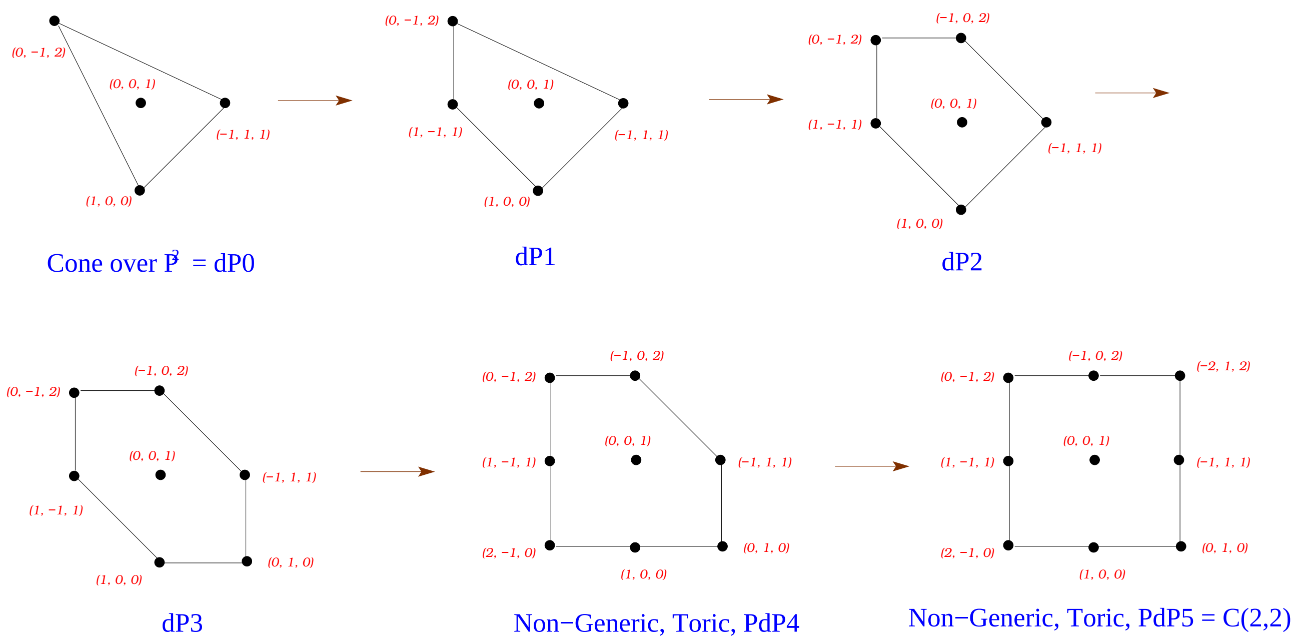


Figure 17: The sequence of generic $\mathbb{P}^1$ blowups from $dP0 = \mathcal{O}_{\mathbb{P}^2}(-3)$ to $dP3$. The last blownups from $dP3$ give non-generic, toric, $PdP4$ and $PdP5$. We have drawn the toric diagrams in a way such that it is obvious that each blowup corresponds to an addition of a node.

which is exactly the superpotential derived by the unhiggsing in (4.1).

8.2 Symmetries of $PdP4_{II}$

Now let us discuss model II with 19 fields. The quiver has the symmetry $\mathbb{Z}_2^{(1)} \times S_3$ where $\mathbb{Z}_2 : 2 \leftrightarrow 3$ and S_3 is the symmetric group on the 3 nodes $(1, 4, 7)$. As before, we write down the orbits as

1. $\{(125), (135)\} + \{(425), (435)\} + \{(725), (735)\}$
2. $\{(1256), (1356)\} + \{(4256), (4356)\} + \{(7256), (7356)\}$
3. $\{(125436), (135426)\} + \{(125736), (135726)\} + \{(425136), (435126)\} + \{(425736), (435726)\} + \{(725436), (735426)\} + \{(725136), (735126)\}$
4. $\{(126), (136)\} + \{(426), (436)\} + \{(726), (736)\}$
5. $\{(126735), (136725)\} + \{(126435), (136425)\} + \{(426735), (436725)\} + \{(426135), (436125)\} + \{(726135), (736125)\} + \{(726435), (736425)\}$,

where we have divided the action of $\mathbb{Z}_2^{(1)}$ and S_3. Notice that the number of fields in the orbits are $18, 24, 72, 18, 72$, it is impossible to get the 38 fields needed in the superpotential. Again, the quiver symmetry must be broken by the superpotential. Let us analyse how the symmetry is broken.

First we consider the case that only one symmetry is broken: (A) $\mathbb{Z}_2^{(1)}$ is broken and we get orbits with number of fields $9, 12, 36$, which can not in any way combine to get 38; (B) S_3 is broken to the cyclic subgroup $\mathbb{Z}_3$ so that only orbits (3) and (5) are broken to two parts and we get the numbers $18, 24, 36$ which again can not give 38; (C) S_3 is broken to the subgroup $\mathbb{Z}_2^{(2)}$ which we can take to be the action that exchanges nodes $1 \leftrightarrow 7$. In this case, every orbit is broken and we get numbers $12, 6, 16, 8, 24$. We have five solutions $24+8+6$, $16+16+6$, $16+8+8+6$, $12+12+8+6$, $12+8+6+6+6$ which give 38.

Now we will try to show that these five solutions can give at most one consistent result:

- The $12+8+6+6+6$ case: Only orbits (1) and (3) can be broken to provide the number 6. It is easy to see that fields ϕ_{42}, ϕ_{43} show up three times at least, so it is not the correct choice.

- The $12+12+8+6$ case: number 8 can come only from $\{(4256), (4356)\}$ and number 6 can come from $\{(425), (435)\}$ or $\{(426), (436)\}$. From the field ϕ_{64} we can choose only orbit $\{(425), (435)\}$ for the number 8. Number 12 can comes from (1)$\{(125), (135)\}$ +$\{((725), (735)\}$; (2) $\{(126), (136)\} + \{(726), (736)\}$. Since we need two 12, every orbit shows up once and only once by considering fields ϕ_{51}, ϕ_{61}. From these arguments, we can write down the superpotential uniquely as

$$
\begin{aligned}
W = & [\phi_{42}\phi_{25}\phi_{56}\phi_{64} - \phi_{43}\phi_{35}\phi_{56}\phi_{64}] - [\phi_{42}\widetilde{\phi_{25}}\phi_{54} - \phi_{43}\widetilde{\phi_{35}}\phi_{54}] \\
& +[-\phi_{12}\phi_{25}\phi_{51} + \phi_{13}\phi_{35}\phi_{51} + \phi_{72}\widetilde{\phi_{25}}\phi_{57} - \phi_{72}\widetilde{\phi_{35}}\phi_{57}] \\
& +[\phi_{12}\phi_{26}\phi_{61} - \phi_{13}\phi_{36}\phi_{61} - \phi_{72}\phi_{26}\phi_{67} + \phi_{73}\phi_{36}\phi_{67}]
\end{aligned}
$$

  This is a perfect legitimate toric superpotential, but is not the one found by Seiberg duality from Model I. To see why it is not correct choice heuristically, notice the term $[\phi_{42}\phi_{25}\phi_{56}\phi_{64} - \phi_{43}\phi_{35}\phi_{56}\phi_{64}]$ where field ϕ_{56} couples to ϕ_{64} two times. In [9] we observed that in toric models fields try to couple different field if it is possible. This may indicate why this is not the right choice.

- The $16+8+8+6$ case: since number 8 can come only from $\{(4256), (4356)\}$, by repeating two times we get that the field ϕ_{64} shows up four times, so it is again ruled out.

- The $16+16+6$ case: number 16 comes only from $\{(1256), (1356)\} + \{((7256), (7356)\}$. Repeating two times will give field ϕ_{56} appearing four times, so it is not allowed either.

- The $24+8+6$ case:
  number 24 comes from (1) $\{(125436), (135426)\}$+$\{(725436), (735426)\}$; (2)$\{(125736), (135726)\}$+$\{(725136), (735126)\}$; (3) $\{(425136), (435126)\}$+$\{(425736), (435726)\}$; (4) $\{(126735), (136725)\}$+

$\{(726135),(736125)\}$; (5) $\{(426735),(436725)\}+\{(426135),(436125)\}$; (6) $\{(126435),(136425)\}+\{(726435),(736425)\}$. As we have showed that 8+6 can only be $\{(4256),(4356)\}+\{(425),(435)\}$. However, no matter which 24 we choose, we can not satisfy the toric condition: choices (1) and (6) do not give ϕ_{12} appearing two times; choice (2) can not give the consistent sign for fields $\phi_{12},\phi_{26},\phi_{61}$; choice (3) has the field ϕ_{64} appearing four times and so does choice (5), for the field ϕ_{54}; and finally choice (4) can not give consistent signs for fields $\phi_{12},\phi_{25},\phi_{51}$.

Having ruled out the case of breaking only one group, we consider the case that two symmetry generators are broken:

- Only the $\mathbb{Z}_3$ cyclic symmetry remains: We have orbits with field numbers $9,12,18$. From these three numbers we can not get 38.

- Only $\mathbb{Z}_2^{(1)}$ remains: We have orbits with fields $6,8,12$ which can give 38 by $12+12+8+6$, $12+8+6+6+6$, $8+8+8+8+6$ and $8+6+6+6+6+6$. It can be shown that there is solution which satisfies the toric condition, for example,

$$\begin{aligned}&(4256)-(4356)-(4\widetilde{25})+(4\widetilde{35})-(125)+(135)\\&+(726)-(736)-(7261\widetilde{35}),(7361\widetilde{25})\end{aligned}$$

  However, for all solutions, we must have at least one of the orbits with 8 fields, for example $\{(4256),(4356)\}$ where field ϕ_{56} couples to field ϕ_{64} two times. This hints that is not the correct choice for this kind of symmetry breaking because once again fields try to couple to different fields.

- Only $\mathbb{Z}_2^{(2)}$ remains: this is similar to the above, i.e., from orbits with fields 8 (or two orbits with fields 4), we will find two fields coupling to each other two times. So it hints again that it may not be the correct symmetry.

- The diagonal $\mathbb{Z}_2^{(3)}:(2,1)\leftrightarrow(3,7)$: this will turn out to be the correct symmetry preserved by the superpotential.

Now we have the correct symmetry $\mathbb{Z}_2^{(3)}$ to break, let us try to fix the superpotential. First we write down the orbits as

- (IIa). (1) $\{(125),(735)\}$; (2) $\{(725),(135)\}$; (3) $\{(126),(736)\}$;
  (4) $\{(726),(136)\}$; (5) $\{(425),(435)\}$; (6) $\{(426),(436)\}$;

- (IIb). (1) $\{(1256),(7356)\}$; (2) $\{(7256),(1356)\}$; (3) $\{(4256),(4356)\}$;

- (IIc). (1) $\{(125436),(735426)\}$; (2) $\{(725436),(135426)\}$; (3) $\{(125736),(735126)\}$; (4) $\{(725136),(135726)\}$; (5) $\{(425136),(435726)\}$; (6) $\{(425736),(435126)\}$;

- (IId). (1) $\{(126735),(736125)\}$; (2) $\{(726135),(136725)\}$; (3) $\{(126435),(736425)\}$; (4) $\{(726435),(136425)\}$; (5) $\{(426735),(436125)\}$; (6) $\{(426135),(436725)\}$.

There are four ways to get the number 38: $12+12+8+6$, $12+8+6+6+6$, $8+8+8+8+6$ and $8+6+6+6+6+6$. Let us consider them case by case:

- The case $8+6+6+6+6+6$: number 8 can come only from (IIb), where (IIb3) should be excluded because fields ϕ_{56},ϕ_{64} couple to each other two times. By relabelling, we can fix the number 8 to be the orbit $\{(7256),(1356)\}$. Since $(25,35)$ are doubly degenerate, we need them to show up four times, so (IIa1), (IIa2) and (IIa5) must be included. Then to complete fields $(42,12)$ we need to include (IIa6) and (IIa3). Putting the sign correctly we get the superpotential

$$W_{II} = [(7256)-(1356)]-[(736)-(126)]-[(426)-(436)]+[(4\widetilde{25})-(4\widetilde{35})] \\ -[(7\widetilde{25})-(1\widetilde{35})]-[(125)-(735)]$$

  where we use $(25,\widetilde{25})$ to distinguish the fields $(\phi_{25},\widetilde{\phi_{25}})$. This is in perfect agreement with our earlier results (q.v. (4.3)).

- Other cases: Notice that there are three fields $\phi_{72},\phi_{42},\phi_{12}$ which can not coexist in any orbit. To let every field appear twice, we need six terms in the superpotential. Since all other cases do not have six terms, the above $8+6+6+6+6+6$ case is the only allowed choice.

9. Conclusions and Prospects

The purpose of our writing is to implement the "unhiggsing mechanism" of finding the gauge theories living on the worldvolume of D-branes probing more general classes of singularities. In particular, we have addressed blow ups Y of singularities X (by a $\mathbb{P}^1$) whose corresponding probe theories are already known.

In order to do so, we have developed a field theoretic method to obtain the superpotentials once the matter content for the blowup geometry is at hand. The approach is based on identifying in each case the unhiggsing associated to the blown up 2-cycles[7] well-known in the literature (cf. e.g. [1, 3, 24]). Therefore from this standard result that acquisition of VEV's of spacetime fields is reflected

[7] In the case of toric singularities, the (p,q) web techniques discussed in [19] and the Inverse Algorithm of [6] are very computationally convenient for this purpose.

as blow downs in the geometry, while conversely unhiggsing corresponds to blowing up, we have devised a straight-forward algorithm of unhiggsing. The inputs to the procedure are the matter content and superpotential of X and the matter content of Y; the output is the superpotential (and hence the full theory) for Y.

As applications to our method, we venture into the unchartered waters of the non-toric higher del Pezzo's. Since we know that each dPk is $dP(k-1)$ blown up at a point and from the techniques of (p,q)-webs, we also know the matter content of the the higher $dP(k>3)$ [16], it seems that our unhiggsing procedure is perfectly adapted to this abovementioned problem of finding the full theory for $dP(k>3)$. Subsequently blow ups of $dP3$ were constructed along these lines, and the set of all the toric phases (with equal rank in all their gauge groups) closed under Seiberg dualities were found. As a confirmation, the inverse procedure of higgsing in the newly obtained gauge theory (as blow down of the singularity) was then thoroughly studied and indeed all the toric phases of $dP3$ were retrieved.

The geometry of the unhiggsed theory was then analyzed in detail. We found there that direct frontal-attack computation of the moduli space for the theory gives us a variety which we call **Pseudo** $dP4$ or $PdP4$. This is a toric variety, which is intimately related to $dP4$ in the sense that it is also $dP3$ blown up at a point, but one which is non-generic by having non-isolated singularities. In conclusion, the unhiggsing has provided a new set of theories supporting the Toric Duality/Seiberg Duality correspondence.

This program was repeated once more for a further blow up. The geometry of the moduli space is in this case a toric, non-generic pseudo-$dP5$. It is in fact the generalised conifold $C(2,2)$, which is a cone over $\mathbb{P}^2$ blown up at 5 non-generic points. Again Toric and Seiberg dualities coincide.

Finally we have systematically addressed the symmetries of these two new classes of gauge theories along the path of [9]. Indeed from considerations of the global symmetries alone we can obtain the superpotential by direct observation and the results are in perfect agreement with the superpotentials obtained from the unhiggsing method.

We have thus obtained the full theories for some pseudo dP's; of immediate concern is of course the question of finding the actual, generic dP's [38]. In principle, there are several reasons that explain why the direct unhiggsing method does not produce the true dP's. First, it is possible that the quivers found for the higher dP's are incomplete as the symmetric parts are missing from the (p,q)-web method. This is a general problem when the matter content is calculated just from the intersection numbers. Another possibility is that the four toric phases of $dP3$ are not directly related to the phase of $dP4$ given by the quiver in Figure 3. In other words, starting from this phase of $dP4$ we cannot higgs down to the four phases of $dP3$. In our example, it seems that it is the second reason accounts for our failure. In fact, it can be seen that the superpotential [8]

$$
\begin{aligned}
W = & [(125) + (735)] + [(1246) + (7346)] + [(1346)] + [(2467)] \\
& +[(2457) + (2567)] + [(1245) - (5673)] \ ,
\end{aligned}
$$

where we have grouped terms according to the Z_2 symmetry: $1 \leftrightarrow 3, 7 \leftrightarrow 2, 4 \leftrightarrow 6$ plus charge conjugation, does give the cone over $dP4$ as the moduli space. The question now becomes how do we know it is a brane probe theory if we cannot establish the relationship with known results[9] of $dP3$. Work on this issue is in progress [38].

Our unhiggsing technique thus stands yet another rung on the ladder toward the solution to general D-brane probe theories upon which we daily climb. Of course, the virtues of the unhiggsing method is appreciable; we are provided with a technique to address much more general situations than del Pezzo surfaces, such as arbitrary toric singularities, or even for singular manifolds with G_2 holonomy.

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