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# **Implied Probability Distributions: Estimation, Testing and Applications**

A thesis presented

by

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to

The Faculty of Finance

in partial fulfilment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Finance

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*To my parents, Eleni and George*

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# List of Abbreviations

API	American Petroleum Institute
BAW	Barone-Adesi and Whaley
BSM	Black-Scholes-Merton
CBOE	Chicago Board of Options Exchange
CME	Chicago Mercantile Exchange
EMU	European Monetary Union
<i>ESE</i>	Edgeworth Series Expansion
Fed	Federal Reserve Bank
FOMC	Federal Market Open Committee
FT	Financial Times
FX	Foreign Exchange
GBM	Geometric Brownian Motion
IEA	International Energy Agency
IMF	International Monetary Fund
<i>LGN</i>	Log-normal distribution
MSE	Mean Squared Error
<i>MLN</i>	Mixture of two Log-normal distributions
NYMEX	New York Mercantile Exchange
OPEC	Organisation of Petroleum Exporting Countries
PDF	Probability Distribution Function
RND	Risk Neutral Density
SDE	Stochastic Differential Equation
SPD	State Price Density
UN	United Nations
WTI	West Texas Intermediate

# List of Symbols

$t$	Current time
$T$	Maturity time of the option
$\tau$	Time left until option expires ( $= T - t$ )
$S_t$	Current asset price
$S_T$	Asset price at maturity of the option
$r$	Risk free interest rate
$q$	Continuous time dividend yield
$K$	Option's strike price
$\sigma^2$	Variance
$f(\cdot)$	Probability Density Function
$F(\cdot)$	Cumulative Density
$\alpha_j(F)$	$j^{\text{th}}$ moment of $F$ about the origin
$\kappa_j(F)$	$j^{\text{th}}$ semi-invariant or cummulant of $F$
$\phi(F, \tau)$	Characteristic Function of $F$
$E_t[\cdot]$	Expectation operator based on information available at time $t$
$N(\cdot)$	Cumulative Normal distribution
$c(K)$	Theoretical value of a European call option with strike price $K$
$p(K)$	Theoretical value of a European put option with strike price $K$
$C(K)$	Theoretical value of an American call option with strike price $K$
$P(K)$	Theoretical value of an American put option with strike price $K$
$\equiv$	Equals by definition
$\sum_{i=0}^n x$	Summation operator over variable $x$ from 0 to $n$
$APD_{Z,t}$	Average Percentage Deviation of statistic $Z$ at time $t$
$RCV_{Z,t}$	Robust Coefficient of Variation of statistic $Z$ at time $t$
$R_{APR}$	Absolute Risk Preferences function
$R_{RPR}$	Relative Risk Preferences function



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# Declaration

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# Abstract

A relatively large number of authors have proposed alternative techniques for the estimation of implied risk-neutral densities. As a general rule, an assumption for a theoretical equilibrium option pricing model is made and with the use of cross-sections of observed options prices point estimates of the risk-neutral probability densities are obtained.

The present study is primarily concerned with the estimation of implied risk-neutral densities by means of a semi-parametric *Edgeworth Series Expansion* probability model as an alternative to the widely criticized log-normal parameterization of the Black, Scholes and Merton model. Despite the relatively early introduction of this type of models in academic literature in the early '80s, it was not until the mid '90s that people started showing interest in their applications. Moreover, no studies by means of the *Edgeworth Series Expansion* probability model have so far been conducted with American style options.

To this end, the present work initially develops the general theoretical framework and the numerical algorithm for the estimation of implied risk-neutral densities of the *Edgeworth Series Expansion* type from options prices. The technique is applicable to European options written on a generalized asset that pays dividends in continuous time or American futures options.

The empirical part of the study considers data for the Oil and the Interest rates markets.

The first task in the empirical investigation is to address general concerns with regard to the validity of an implied risk-neutral density estimation technique and its ability to stimulate meaningful discussion. To this end, the consistency of the *Edgeworth Series Expansion* type implied densities with the data is checked. This consistency is viewed in a broader sense: internal consistency - adequate fit to observed data - and economic rationale of the respective densities. An analysis is, therefore, performed to examine the properties of the implied densities in the presence of large changes in economic conditions. More specifically, the ability of the implied *Edgeworth Series Expansion* type implied densities to capture speculation over future eventualities and their capacity to immediately reflect changes in the market sentiment are examined. Motivated by existing concerns in the literature that the differences between the estimates from an alternative parameterization and the log-normal Black-Scholes-Merton parameterization may be apparent - better fit to observed data - but not significant

in a statistical sense, the hypothesis that the use of the *Edgeworth Series Expansion* model is able to offer a statistically significant better fit compared to the log-normal parameterization is also tested. The information conveyed by the implied probability distributions, recovered by the *Edgeworth Series Expansion* model, proves to be consistent with the market commentary of the study period, thus, the *Edgeworth Series Expansion* type implied densities can be seen as economically sensible. The implied distributions are also shown to be able to capture the general market sentiment as well as able to incorporate isolated events with a significant impact on the market. In addition the model is found to offer a statistically significant better fit to observed option prices and can, therefore, be considered a superior means - compared to the log-normal Black-Scholes-Merton parameterization - of extracting information implied in option prices.

The second task in the empirical investigation is to assess the performance of the *Edgeworth Series Expansion* type implied densities in a more quantitative basis. Typical studies assess alternative techniques for the estimation of implied densities on the basis of the goodness of fit achieved to observed options cross-sections. The goodness of fit, however, should not be the sole criterion - not even the prior one. Implied risk-neutral densities estimation techniques defined by a number of parameters larger than the respective number of parameters used by other competitive techniques are expected to result in more accurate fit of the data. A fair comparison seems to be one that assesses also the robustness of these techniques or, more precisely, the degree of confidence that can be placed on the summary statistics calculated off the implied distributions. The performance of the *Edgeworth Series Expansion* in that respect is examined on a relative basis. A mixture of two log-normals specification, being a very commonly used parametric model which has already been studied in terms of stability, is used as a comparative benchmark. The results are highly supportive of the superior performance of the *Edgeworth Series Expansion* model which is found to be more stable than the mixture of two log-normals specification; it is found more capable of estimating on average densities whose summary statistics converge to the original solution; and also capable of estimating statistics with relatively low dispersion.

Having examined the 'goodness' of an *Edgeworth Series Expansion* type probability model in a number of ways proposed in the current literature, it is natural to seek an application within the framework implied densities are acknowledged to be suitable for, that is in addressing a more general fundamental economic question. To this end the model is, finally, used to infer investor risk preferences functions from option prices. The estimated risk preferences functions are found consistent with existing empirical evidence and consistent with the market conditions of the study period.

This is the first time an *Edgeworth Series Expansion* type probability model is studied in the context of American options. The stability of the model is investigated for the first time. It is also the first time that the robustness of a model belonging to the semi-parametric family of densities parameterisations is examined. The study's contribution is further enhanced with the exploration of the *Edgeworth Series Expansion*

type probability model's ability to properly reflect/capture investor's risk preferences. It is also the first a semi-parametric model is used in such an exercise.





# Preface

Financial and economic researchers have recently become extremely interested in exploiting the forward looking nature of derivatives and become increasingly sophisticated in their attempts to analyse market expectations embedded in traded derivative securities. Efficient market hypothesis considerations imply that market asset prices contain all the available information - including costly information as well - and can be used to enhance the forecasts of both informed and uninformed participants. Motivated by this fundamental principle and by the forward looking nature of exchange traded derivatives researchers have attempted to assess the information content of Futures and Options prices. On the one hand they have investigated the lead-lag relationships between the Futures and the underlying spot markets. On the other hand a large body of work has focused on Option prices trying to recover either the risk-neutral stochastic process followed by the underlying asset or the risk-neutral density function from which the asset price at expiration will be drawn.

The present study provides an in-depth analysis in the area of implied risk-neutral densities.

Applications of implied risk-neutral densities have been well documented in academic literature. One of the uses suggests that implied distributions be used to price illiquid, exotic or over-the-counter options consistently with exchange traded, vanilla options which also helps to develop more trivial hedging strategies. On the other hand implied distributions prove useful when a more qualitative/fundamental

economic question is addressed as they help to reveal ‘market sentiment’. This is useful for the policy-stance of a central bank and for investors who may wish to take positions based upon the difference between their forecast of the distribution and the consensus of the market.

To be able to make any analysis based on implied distributions one should be equipped with an option pricing model that adequately describes observed option prices. To this end one should be very cautious and make sure that a model is constructed so that it adequately describes observed options data but at the same time is able to recover implied risk-neutral densities that are economically sensible. Thus, the purpose of the present study is threefold: firstly, to develop a model along the lines of modern option pricing theory which succeeds in describing observed prices with high accuracy; secondly, to explore its ability to reflect/capture the information content of option prices; and thirdly to suggest potential applications of implied risk-neutral densities.

The study is organised as follows: the Black-Scholes-Merton pricing framework, being the foundation of nearly every option pricing analysis as well as of the one presented herein, is discussed in Chapter 1. Chapter 2 offers a comprehensive review of the studies that followed the seminal work of Black, Scholes and Merton in the option pricing literature. Chapter 3 substantiates the need to pursue research in the

area of options implied risk-neutral densities and identifies the contribution of a new methodology. Chapter 4 presents the theoretical framework within which the new method is developed and describes the procedure followed for the estimation of the implied risk-neutral densities. Chapter 5 examines the internal consistency of the model - by means of a statistically significant improved fit compared to the Black-Scholes-Merton parameterisation - and also its ability to recover economically sensible risk-neutral densities in an attempt to - rather qualitatively - assess its performance. Chapter 6 investigates the sensitivity of implied risk-neutral densities in the presence of measurement errors in an attempt to offer a more quantitative assessment of the model. Chapter 7 illustrates a potential application of implied risk-neutral densities where the information content of option prices is explicitly quantified and illustrates its implication with the use of the proposed methodology. Finally, Chapter 8 concludes the study by addressing issues related to the use of implied risk-neutral densities in general and the application of the proposed methodology, and also suggests topics for future research.

# Chapter 1

## The Black-Scholes-Merton Asset Pricing Framework

The pricing of options contracts is an issue that has received economists' attention as early as the beginning of the century. The first attempt to tackle the problem mathematically, rather than by intuition, was made by Luis Bachelier in 1900. The proposed model made a key assumption, that the prices of financial assets change randomly, but it also relied on an interest rate of zero and allowed share prices to be negative. It was not until 1973 that a generally acceptable options pricing model was derived. Pre-1973 attempts to value options approached the problem by estimating the expected value of the option at maturity and then discounting its value back in time. This obviously requires the choice of a risk premium to use in the discounting which differs from person to person and from asset to asset.

The key breakthrough of Black, Scholes and Merton (BSM hereafter) option pricing model is that it requires no explicit use of a risk premium. The concept behind the model is simple: *it is possible to construct a portfolio which consists of shares and risk-free bonds with the same total risk as the options on those shares; since the risk of the two sets of assets is the same and the investment is the same, then the returns must be the same over a short period, otherwise there should be opportu-*



*nities for arbitrage.* The elegance and the simplicity of the resulted pricing formula yielded its dominance among existing, at that time, pricing models. Moreover all options models of the past nearly 30 years have been refinements of the basic BSM model.

## 1.1 The Black-Scholes-Merton model

In the BSM asset pricing framework the following assumptions are made:

- The short-term interest rate is known and is constant through time.
- The underlying asset pays no dividends or other distributions.
- The option is 'European'.
- The asset price follows a random walk in continuous time with a variance rate proportional to the square of the price. Thus the distribution of possible asset prices at the end of any finite interval is log-normal. The variance rate of the return on the asset is constant.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.
- There are no penalties for short selling.
- There are no transaction costs in buying or selling the asset or the option.

The value of the option will then depend only on the price of the underlying asset and time, and on variables that are taken to be known constants. It is therefore possible to create a hedged position consisting of a long position in the asset and a short position in the option, whose value will not depend on the price of the asset, but will depend only on time and the values of the known constants.

To formally express this denote by  $v(S, t)$  the value of the option as a function of the asset price  $S$  and time  $t$ . The number of the options that must be shorted against one unit of the asset long is:

$$\frac{1}{v_S(S, t)} \quad (1.1)$$

where  $v_S(S, t)$  is the partial derivative of the option price with respect to the asset price. The latter is referred to as the ‘delta’ of the option and expresses how much the value of the option is altered by a unit change in the current asset price.

If the asset price changes by a small amount  $\Delta S$  the option price will change by an amount  $v_S(S, t) \times \Delta S$ , and the number of options given by Equation (1.1) will change by an amount  $\Delta S$ . Thus the change in the value of a long position in the stock will be approximately offset by the change in value of a short position in  $1/v_S(S, t)$  options. This means that for reasonably small increases in the asset price, the profit the investor makes on the asset will be the same as the loss he incurs on the options, and vice versa for decreases in the asset price. As the portfolio thus constructed is risk-free, it must yield exactly the same return as a risk-free treasury

bond which matures on the same date as the option does. If it did not, arbitrage trading would begin to eliminate the possibility of making risk-free profits.

If  $P$  is the value of the portfolio consisting of one unit of the asset long and  $1/v_S(S, t)$  options and  $r$  is the treasury bond rate, following the above, the change  $\Delta P$  in its value in the time interval  $\Delta t$  is given by:

$$\Delta P = rP\Delta t$$

Following the BSM assumptions, listed on page 5, the asset price evolution over time can be described by a *Stochastic Differential Equation* (SDE hereafter) as follows:

$$dS = \mu S dt + \sigma S dX \quad \text{in continuous time} \quad (1.2)$$

or

$$\Delta S = \mu S \Delta t + \sigma S \Delta X \quad \text{in discrete time} \quad (1.3)$$

where  $\mu$  is the expected return of the asset,  $\sigma$  is the standard deviation of the asset price returns and  $dX$  is a Wiener process. The process is known as a *Geometric Brownian Motion* (GBM hereafter). From Ito's lemma, it follows that a process



followed by a function  $v$  of  $S$  and  $t$  is :

$$dv = \left( v_S \mu S + v_t + \frac{1}{2} v_{SS} \sigma^2 S^2 \right) dt + v_S \sigma S dX \quad (1.4)$$

or

$$\Delta v = \left( v_S \mu S + v_t + \frac{1}{2} v_{SS} \sigma^2 S^2 \right) \Delta t + v_S \sigma S \Delta X \quad (1.5)$$

which implies that both  $S$  and  $v$  are affected by the same underlying source of uncertainty.

If

$$P = S - \frac{1}{v_S(S, t)} v(S, t)$$

is the value of the portfolio that consists of one share long and  $1/v_S(S, t)$  options short, and

$$\Delta P = \Delta S - \frac{1}{v_S(S, t)} \Delta v(S, t)$$

is the change in its value, the process followed by the portfolio value, given Equations (1.2) to (1.5), is

$$\begin{aligned} \Delta P &= \mu S \Delta t + \sigma S \Delta X - \frac{1}{v_S} \left[ \left( v_S \mu S + v_t + \frac{1}{2} v_{SS} \sigma^2 S^2 \right) \Delta t + v_S \sigma S \Delta X \right] \\ &= \frac{1}{v_S} \left( -v_t - \frac{1}{2} v_{SS} \sigma^2 S^2 \right) \Delta t \end{aligned}$$

Since the portfolio is riskless it can only yield the risk free interest rate  $r$ .

Therefore:

$$\Delta P = r P \Delta t$$

By substituting and dropping out  $\Delta t$  we get:

$$\begin{aligned} \frac{1}{v_S} \left( -v_t - \frac{1}{2} v_{SS} \sigma^2 S^2 \right) &= rP \\ \frac{1}{v_S} \left( -v_t - \frac{1}{2} v_{SS} \sigma^2 S^2 \right) &= r \left[ S - \frac{1}{v_S} v(S, t) \right] \Rightarrow \end{aligned}$$

$$rv(S, t) = v_t + rSv_S + \frac{1}{2} v_{SS} \sigma^2 S^2 \quad (1.6)$$

which is the SDE followed by the value of an option at each point of time  $t$ .

The significance of Equation (1.6) lies on the fact that  $\mu^1$  is eliminated, thus creating a riskless position. As a result the option can be priced irrespective of investors' views towards risk. The resulting differential equation can be solved subject to the boundary conditions that the nature of the option each time implies. In fact it is the heat-transfer equation of physics whose solution is given by Churchill (1963). In the simplest case of a European call option the solution is the following:

$$v(S, t) = SN(d_1) - Ke^{r(T-t)}N(d_2) \quad (1.7)$$

with

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

---

<sup>1</sup>  $\mu$  depends on risk preferences. The higher the level of risk aversion by investors, the higher  $\mu$  be for any given asset.

where  $N(\cdot)$  is the cumulative normal density function,  $S$  is the current price of the underlying asset,  $K$  is the strike price of the option,  $r$  is the treasury bond rate and  $\sigma$  is the expected volatility of the underlying asset's returns upon maturity of the option in  $\tau(= T - t)$  units of time. Equation (1.7), implies that the value of the option is given by the difference between the expected asset price - the first term on the right-hand side - and the expected cost - the second term - if the option is exercised.

A few years later Black (1976), developed a formula for the pricing of commodity options contracts. The formula serves as an extension to the original BSM formula<sup>2</sup> and it is derived under the same theoretical framework. A key assumption to the derivation of Black's formula is that the expected return on a futures contract is zero, since no initial capital needs to be engaged to enter the position. A riskless hedge can be created, as in the case of the original model, by taking a long position in the option and a short position in the futures contract with the same transaction date. Since the value of a futures contract is always zero, the equity in this position is just the value of the option. The formula Black comes up with is very similar to the one presented in Equation (1.7) and reads:

$$v(F, t) = e^{-r(T-t)} [FN(d_1) - KN(d_2)]$$

---

<sup>2</sup> To pay equal tribute to all three authors the present study uses the acronym BSM to refer to the general framework rather than one of the two models. The reader may bear in mind that the relevant modifications are made when necessary.

with

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where  $F$  is the current price of the underlying futures contract and the remaining symbols correspond to the ones used in Equation (1.7).

## 1.2 Conclusions

The concept behind the BSM model is simple: *it is possible to construct a portfolio which consists of shares and risk-free bonds with the same total risk as the options on those shares; since the risk of the two sets of assets is the same and the investment is the same, then the returns must be the same over a short period, otherwise there should be opportunities for arbitrage.*

The Black, Scholes and Merton option pricing model is a closed form model which calculates the value of a European option - call or put - as a function of three observables - the current value of the underlying asset of the option, the strike price, and the time to expiry - and one unobservable - the expected volatility of the underlying asset. Following the same principles Black (1976) derived an option pricing model to value options written on futures contracts. The key breakthrough of BSM's and subsequently Black's option pricing models is that they require no explicit use of a risk premium.



## Chapter 2

# Literature Review

The elegance and the simplicity of the BSM pricing formula yielded its dominance among existing, at that time, pricing models. Moreover all options models of the past nearly 30 years have been refinements of the basic BSM model.

Over the years several authors have tested the validity of the model on empirical grounds. Black and Scholes (1972) and Galai (1977) tested whether it was possible to make excess returns - higher than the risk-free interest rate - by selling overvalued and buying undervalued options. A number of subsequent studies were carried out in the same spirit and include: Black (1975), Chiras and Manaster (1978) who used CBOE data, MacBeth and Merville (1979) with data on individual stocks, Rubinstein (1985) also with data on individual stocks from the CBOE, Shastri and Tandon (1986) and Bodurtha and Courtadon (1987) on foreign currency data, Shastri and Tandon (1986) on futures options data and Chance (1986) on index options data. With little variation in the results, the above studies conclude that *the BSM formula systematically misprices options across strike prices.*

To deal with this deficiency several of the original - and somewhat restrictive - assumptions behind the initial derivation of the BSM formula have been relaxed in the subsequent literature. This chapter reviews a large body of studies that have re-considered one or more of the following assumptions:

- the underlying asset pays no dividends or other distributions
- the variance rate of the return on the asset and short-term interest rate is known and is constant through time
- the asset price follows a random walk in continuous time with a variance rate proportional to the square of the price, or equivalently
- the distribution of possible asset prices at the end of any finite interval is log-normal.

## 2.1 The underlying asset pays dividends or other distributions

In real life financial markets it is rare to find any financial asset that pays no dividends or distributions<sup>3</sup>. The BSM option valuation formula can be extended to incorporate this feature and relatively simple modifications can be made to value options on stocks that pay a continuous dividend yield, options on stock indices and currencies. Garman and Kohlhagen (1983) and Biger and Hull (1983) derived an option valuation formula for options on currencies under the assumption that the stochastic process that exchange rates follow is:

$$dS = (r - r_f) S dt + \sigma S dX$$

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<sup>3</sup> The only exception being the Futures contracts.

where  $S$  is the value of an FX security,  $r$  is the risk free interest rate of the domestic currency,  $r_f$  is the risk free interest rate of the foreign currency and  $\sigma$  is the exchange rate's returns expected volatility.

If  $r_f$  is replaced by  $q$ , which represents a continuous dividend yield rate, a valuation formula can be derived for options on stocks and options on stock indices with a continuous dividend yield [Merton (1973)].

The arguments that underlie the above modifications have enjoyed broad acceptance among researchers and practitioners and is now standard practice to replace the expected return of currencies with  $(r - r_f)$  and general assets with expected dividends  $q$  with  $(r - q)$  when valuation models are derived under the risk neutrality argument.

## **2.2 Non-constant variance and non-constant interest rates**

One assumption in the BSM model that is clearly not realistic is that the variance rate of the return on the asset is constant. Black (1976) examined the relationship between asset prices and volatility and found a strong tie between them. Christie (1982), Schwert (1989a,b), Cheung and Ng (1992) have also identified similar patterns in volatility - stock return volatility rises after stock prices fall and vice versa (leverage effect)- while Giamouridis and Tamvakis (2001) found an inverse relationship to hold in the commodity markets.

The introduction of the Autoregressive Conditional Heteroscedasticity (ARCH) and the Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models by Engle (1982) and Bollerslev (1987) respectively, gave researchers a formal statistical tool to model the behaviour of stock market returns in a more realistic framework. The model imposes a structural form on the conditional variance, which is expressed as a function of past squared errors and past conditional variances.

A typical GARCH (1,1) specification, for example, relates the variance of asset returns at time  $i\Delta t$  to the variance of asset returns in time  $(i - 1) \Delta t$  and past errors in the following way:

$$\sigma_i^2 = \omega + \alpha\epsilon_{i-1}^2 + \beta\sigma_{i-1}^2$$

where  $\sigma_i$  is the volatility of the asset at time  $i\Delta t$  and  $\epsilon_i$  is the past error. Variations of the above models include Nelson's (1991) Exponential GARCH (EGARCH) which allows the conditional variance to respond asymmetrically to positive and negative innovations and Zakoian's (1990) Threshold GARCH (T-GARCH) model where the specification of the conditional variance also allows to test the significance of the asymmetric effect.

The above mentioned models are generally formulated in discrete time and postulate a log-price process for the stock which has a conditional variance (Conditional Heteroscedasticity) depending on exogenous and lagged endogenous variables, and past residuals.



A quite different approach, still based on the same principle, evolves with the use of stochastic calculus techniques. Models in this family include Hull and White (1987), Scott (1987) and Wiggins (1987), and Melino and Turnbull (1990, 1995).

The general framework uses the following specification for the risk-neutral behaviour of asset prices:

$$dS = \mu S dt + \sigma S dX$$

$$d\sigma = \gamma(\sigma) dt + \delta(\sigma) dW$$

where  $dX$  and  $dW$  are Wiener processes with covariance

$$dXdW = \rho dt$$

for some correlation  $-1 < \rho < 1$ .

These models have generally the disadvantage that no closed form solution exists and the use of extensive numerical techniques is required to solve two-dimensional partial differential equations. What makes these models even less appealing is the fact that volatility is not a traded asset, therefore, the associated risk can not be hedged in this setting. In addition, the introduction of two Brownian motions makes the model incomplete and no unique prices for stock options exist [Hobson and Rogers (1998)].

Stein and Stein (1991) relaxed the assumption of correlated Wiener processes and developed a closed form solution for the pricing of options contracts. They

assumed that volatility is driven by an Ornstein-Uhlenbeck [or AR(1)] process. The processes they proposed for the underlying asset and the volatility are the following:

$$dS = \mu S dt + \sigma S dX$$

$$d\sigma = -\delta(\sigma - \theta) dt + k dW$$

where  $S$  is the asset price,  $\sigma$  is the expected volatility of the asset return,  $k, \mu, \delta$  and  $\theta$  are constants, and  $dX$  and  $dW$  are two independent Wiener processes. Due to this specification, however, the model cannot capture important skewness effects that arise from the correlation of asset returns and volatilities. Heston (1993), also used the Ornstein-Uhlenbeck specification of Stein and Stein (1991) and provided a closed form solution by making an assumption for the ‘price of the volatility risk’ and by guessing a functional form for the option price. Hobson and Rogers (1998) proposed an alternative model, where the instantaneous volatility was specified in terms of exponentially weighted moments of the historic log-price. This introduces a feedback effect into the volatility process: present shocks in the asset price result in higher future uncertainty.

A number of research papers have also studied models with stochastic interest rates. Models that incorporate non-deterministic interest rates include Stein and Stein (1991), Miltersen and Schwartz (1998) and Hilliard and Reis (1998).

Both the GARCH (n,m) and the stochastic volatility specifications generally result in skewed distributions for the terminal asset prices. While the tails of the dis-

tribution can be fat, depending for example on the correlation between the volatility and the asset price in the latter specification, multimodal shapes can not be incorporated unless jump components are included in the diffusion (see Section 2.3).

In conclusion, for options that last less than a year, the pricing impact of a stochastic volatility is fairly small in absolute terms (although in percentage terms it can be quite large for deep-out-of-the-money options). It becomes progressively larger as the life of the option increases, [Hull (1999)]. The hedging performance on the other hand is a factor that has to be considered in evaluation of the performance of pricing models. Stochastic volatility models can be hedged only by using other volatility depended instruments, such as other options. A fair comparison between stochastic and deterministic volatility models should therefore include the comparison of delta- and vega-neutral position of deterministic models with delta-neutral stochastic models. Bakshi, Cao and Chen (1997) run such tests and conclude that BSM delta- plus vega-neutral strategy performs no worse than the other models' delta-neutral hedges. Studies, on the other hand, that use stochastic interest rate specifications seem to add complexity to the valuation problem without adding any significant degree of accuracy. Moreover, interest rates are not expected to change significantly over short periods, thus making such models inappropriate for short-term options contracts.

## 2.3 Alternative Stochastic Processes

### 2.3.1 Continuous-Time Models

In the BSM framework the asset price is assumed to follow a random walk in continuous time with a variance rate proportional to the square of the asset price. The equation that describes asset price dynamics is the following:

$$dS = \mu S dt + \sigma S dX$$

Cox and Ross (1976) proposed an alternative specification for the asset price dynamics. In their setting the asset prices change according to the following SDE:

$$dS = \mu S dt + \sigma S^{1-\alpha} dX$$

where  $0 \leq \alpha \leq 1$ . The model - known as the Constant Elasticity of Variance model (CEV) - allows the *leverage effect* to be incorporated since the volatility of the asset  $\sigma S^{-\alpha}$  decreases as the asset price increases. Its weakness, however, is that it allows asset prices to become negative.

Merton (1976), on the other hand, pointed out that the use of the standard BSM process for the asset price implies that in a short interval of time the stock price can only change by a small amount. It would therefore be more realistic to use a process which would allow for a positive probability of an asset price change of extraordinary



magnitude (e.g. some important information on the asset arrives). He suggested the dynamics of the asset price changes be composed of two types of changes:

- the ‘normal’ vibrations in price, which are modelled by a standard GBM, and
- the ‘abnormal’ vibrations in price, which are modelled as a ‘jump’ process

in the following equation:

$$dS = (\alpha - \lambda k) S dt + \sigma S dX + S dq$$

where  $\alpha$  is the instantaneous expected return on the asset,  $\lambda$  is the rate at which jumps occur,  $k$  is the average jump size measured as a proportional increase in the asset price,  $\sigma^2$  is the instantaneous variance of the return conditional on no arrivals of important new information,  $dX$  is a standard Wiener process, and  $dq$  is a Poisson process generating the jumps. The processes  $dq$  and  $dX$  are assumed to be independent.

The ‘Poisson’ element is included in the equation to incorporate the arrival of a new piece of information about the asset. The arrivals of news are independently and identically distributed and during a small time interval  $(t, t + h)$  :

- the probability that an event occurs is  $\lambda h + O(h)$
- the probability that an event does not occur is  $1 - \lambda h + O(h)$
- the probability that more than one events occur is  $O(h)$

where  $O(h)$  is the asymptotic order symbol defined by  $\psi(h) = O(h)$  if

$$\lim_{h \rightarrow 0} \left[ \frac{\psi(h)}{h} \right] = 0$$

The above mentioned model contains the BSM model as a special case, when  $\lambda = 0$ , which translates to the fact that no events occur. The main drawback stems from the fact that the risk of jumps in the underlying asset's return is not diversifiable.

Naik and Lee (1990) pointed out that since neither the no-arbitrage argument nor Merton's approach is appropriate to price options under a jump process framework, a model can be derived for European calls and puts by placing restrictions on investors' preferences. The model shares its own disadvantages, the main one being its principle of derivation, namely its dependence upon investors' preferences. Bates (1991) derived the American analogue of Merton's model and carried out empirical tests using data on the S&P 500 Index. One can easily argue that a jump in the market index or a large stock portfolio would also include a jump of the same, or even larger, magnitude in all of the stock prices meaning that such a risk cannot be diversified away resulting in a quite considerable disadvantage.

In the same line of research Bates (1996) extended the jump-diffusion model to incorporate stochastic volatility and Scott (1997) to incorporate stochastic volatility and stochastic interest rates as well. Bakshi, Cao and Chen (1997) used a stochastic volatility, stochastic interest rate and random jumps model which contained many known option pricing models as special cases. To test its validity and per-

formance they run a ‘horse race’ among known option pricing formulas including BSM, stochastic volatility (SV), stochastic volatility with jumps (SVJ) and stochastic volatility stochastic interest rates (SVSI). Overall they found models that admit stochastic volatility improve greatly the BSM formula. The inclusion of stochastic volatility, and random jumps improved the performance of short term option pricing whereas longer maturity options were priced more accurately with the stochastic interest rates consideration<sup>4</sup>. This was in a way bound to happen as asset prices seem to experience abrupt changes in short time intervals which tend to cancel out in the long run whereas interest rates are not expected to fluctuate significantly within a short time period. As far as the hedging performance is concerned, it was concluded that adding either random jumps or stochastic interest rates did not improve the SV model’s performance despite the additional complexity they required.

Motivated by the work of Mandelbrot (1997), finally, suggesting that pure jump models are able to capture both rare large moves - ‘abnormal’ vibrations in price - and frequent small moves - ‘normal’ vibrations in price - several authors have recently focused attention on pure jump models in the Lèvy class. The latter can capture frequent small moves through the use of a Lèvy density whose spatial integral is infinite. Three such processes are presented in Carr et al. (2001a): the normal inverse Gaussian model of Barndorff-Nielsen (1998), the symmetric variance gamma model studied in Madan and Seneta (1990), and, finally, the generalised variance

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<sup>4</sup> This result is probably due to the inclusion of random jumps as the stochastic volatility models do not perform extremely well in short-maturity options, see Section 2.2, p. 18.



gamma model developed by Carr et al. (2001b). Carr et al. (2001a) identify a potential shortcoming of the former methodologies in that their specification fails to accommodate the volatility persistence (volatility clustering) observed in financial time series and try extend the models by incorporating stochastic and mean reverting volatilities.

### 2.3.2 Discrete-Time Models

The models in this category follow from the seminal work of Cox, Ross and Rubinstein (1979). The model assumes asset price dynamics as illustrated in the following:

$$S \begin{cases} Su \text{ with probability } p \\ Sd \text{ with probability } 1 - p \end{cases}$$

The stochastic process of the asset price - which is assumed to pay no dividends - is modelled in discrete time intervals  $dt$ . Within this small interval of time the asset price has a probability  $p$  of moving from  $S$  to  $Su$ , for  $u > 1$ , and a probability of  $1 - p$  of moving from  $S$  to  $Sd$ ,  $d < 1$ . Over the life of the option the asset price is assumed to be composed by a large number of the binomial movements described above. The binomial tree describing the asset price dynamics is the discretization of the continuous-time BSM model's Brownian motion.

The parameters of the tree are determined through risk-neutral valuation arguments, that the expected return of the asset should equal the risk-free interest rate and also by requiring  $u = 1/d$ . The latter ensures that the tree is recombining, which



means that an up movement followed by a down movement leads to the same asset price as down movement followed by an up movement.

The model in its initial form received great acceptance among academics and practitioners as it provided a very useful tool for the pricing of products with more complicated pay-offs i.e. options with American style exercise features. The advantage of the methodology lies on the fact that the whole stochastic process of the underlying asset can be ‘visualised’.

With the quality of available data sets vastly improved in the recent years researchers have tried to take advantage of the latter and learn more about the stochastic process of the asset from the observed option prices. Advances in discrete-time modelling, therefore, mainly include extensions of the original Cox, Ross and Rubinstein (1979) model which are utilised to recover the whole risk-neutral stochastic process of the underlying security using option prices data<sup>5</sup>. Standard practice suggests the estimation of the risk-neutral distribution of the terminal asset price first; and then the estimation of a stochastic process, in binomial tree form, that results in the same terminal distribution that is implied in the option prices.

A first attempt to address this issue was made by Rubinstein (1994) and was generalised later by Jackwerth (1997). The proposed methodology assumes a prior distribution - the binomial - as an input. The terminal distribution is then recovered in discrete points through an optimization procedure which essentially estimates the

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<sup>5</sup> These models also fall within the class of ‘implied models’ discussed extensively in Section 2.4.

distribution that deviates the least from the prior so that the treasury bond, the underlying asset, and the option are priced consistently. The methodology is formally described in the following equations:

$$\min_{P_j} \sum_j (P_j - P_j^i)^2$$

subject to:

$$\sum_j P_j = 1 \text{ and } P_j \geq 0 \text{ for } j = 0, \dots, n$$

$$S^b \leq S \leq S^a \text{ where } S = \frac{\left( q^\tau \sum_j P_j S_j \right)}{r^\tau}$$

$$C_i^b \leq C_i \leq C_i^a \text{ where } C_i = \frac{\left( \sum_j P_j \max [0, S_j - K_j] \right)}{r^\tau} \text{ for } i = 1, \dots, m$$

with

$j$  the ending binomial nodes from lowest to highest

$P_j$  implied (posterior) ending risk-neutral probabilities

$P_j^i$  prespecified (prior) ending nodal log-normal risk-neutral probabilities

$S_j$  underlying (ex-payout) asset prices at the end of standard binomial tree

$S_j^b$  ( $S_j^a$ ) current (observed) bid (ask) underlying asset price

$C_j^b$  ( $C_j^a$ ) current (observed) bid (ask) underlying option price with striking price  $K_i$

$q$  observed annualised payout return

$r$  observed annualised riskless return

$\tau$  time to expiration

Jackwerth (1997) identified two drawbacks of the methodology in that options expiring earlier than the terminal date are not used in constructing the tree and also that the equal path probabilities assumption was rather arbitrary. He introduced a simple weight function which allowed a more flexible backward construction. As a consequence information from times other than the end of the tree and also from American and exotic options could be incorporated.

As an alternative to the above backward constructed trees, Derman and Kani (1994) introduced a forward constructed tree. Derman and Kani's (1994) tree can incorporate options with different maturities but requires extensive interpolations and extrapolations of the observed option prices. The method requires a quite substantial amount of data to perform the relevant interpolations and extrapolations. An additional drawback of the methodology is that it may give rise to negative probabilities which due to the fact that are remedied with add-hoc techniques result in an overall numerically unstable methodology.

Several improvements to Derman and Kani's (1994) methodology have subsequently been proposed. Derman, Kani and Chriss (1996) recommended the use of trinomial trees while Barle and Cakici (1998) suggested some technical improvements on the original model i.e. aligning the center nodes of the tree with the forward price. Dupire (1994) proposed alternative models in the spirit of Derman and Kani (1994). Brown and Toft (1999) developed a semi-recombining tree by estimating risk-neutral densities for every traded expiration class of options.

A further development extends the implied binomial tree models to multiple factors considerations. Derman and Kani (1998) fit a trinomial tree to observed option prices and allowed the transition probabilities from node to node to change according to a separate stochastic process. Ledoit and Santa-Clara (1998) in a similar approach modelled stochastic implied volatilities as opposed to the stochastic local volatilities in the former paper. Britten-Jones and Neuberger (2000), finally, proposed the use of an implied trinomial tree where the stochastic volatility component was restricted to be a Markov process.

Despite the nearly perfect fit of cross-sectional data the above models provide and the attractiveness of a market implied stochastic process for the asset, implied tree methodologies have been subject to extensive criticism both on theoretical and on empirical grounds.

Jackwerth (1999) identified a theoretical weakness of the implied tree models in that the methodology estimates a stochastic process which is consistent with the terminal distribution of the asset prices. No such unique stochastic process, however, exists [see also Melick and Thomas (1997)]. The asset price can evolve according to either a diffusion process, a jump process, or any number of alternative processes, which can involve jumps or additional stochastic factors such as stochastic interest rates or stochastic volatility.

Dumas, Fleming and Whaley (1998) performed empirical tests on the Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) models. They developed



deterministic volatility functions, which were polynomials in time and asset price, and used these functions as volatility surfaces of generalised diffusions. Dumas, Fleming and Whaley (1998) found that the parameter estimates were unstable and most importantly that despite the imposed complexity in the implied trees the hedging performance was less reliable than an ex-post ad-hoc procedure that smoothes BSM model implied volatilities, across exercise prices and times to expiration. Jackwerth and Rubinstein (1998) compare implied binomial trees, parametric models (to be reviewed in Section 2.4) and naive trader rules of thumb<sup>6</sup> and found that none of the models performed extraordinary well and that implied binomial trees performed as well as the parametric models and the naive trader rules.

## **2.4 Alternative distributional assumptions - The concept of implied risk-neutral Probability Density Functions**

The assumption regarding the stochastic process that the underlying asset's returns follow, implies that at the end of any finite interval the stock price is log-normally distributed. The symmetry that this assumption implies is rejected in unconditional returns distributions which seem to be skewed and highly leptokurtic. Westerfield (1977), Tucker and Pond (1988), and others all report severe departures from normality for either daily, weekly or monthly exchange rate returns. Harvey and Siddique (2000) report similar patterns for U.S. monthly stock returns. In addition, the

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<sup>6</sup> These naive rules assume that implied volatility is a function of strike price and does not change as time passes.

log-normal distribution has unanimously failed to describe observed option prices [see for example Ritchey (1990), Rubinstein (1994) and Bahra (1997)].

The models described in Sections 2.2 and 2.3 result in more realistic terminal distributions. The stochastic volatility models offer a flexible distributional structure in which the correlation between the two diffusion processes controls the level of skewness and the volatility variation coefficient controls the level of kurtosis. The jump diffusion models assert that it is occasional, discontinuous jumps and crashes that cause the negative implicit skewness and high implicit kurtosis, to exist in option prices, [Bakshi, Cao and Chen (1997)].

Alternatively the form of the terminal distribution can be explicitly defined to accommodate the features observed in realised asset returns distribution, using a rather more flexible parametric or a totally non-parametric form.

Ross (1976) and Cox and Ross (1976) have shown that in a market that offers no arbitrage possibilities, a set of state space prices or Arrow-Debreu prices that support current values should exist. Formally expressed, let  $\vartheta_T$  be a basic state of nature at time  $T$  as perceived for the current time,  $t$ , and let  $Q_{T\vartheta}$  be the distribution of prices of contingent claims to wealth if state  $\vartheta$  occurs at time  $T$ . If an asset offers returns of  $\chi_{\vartheta}$ , then its current value  $V_{\chi}$ , will be given by

$$V_{\chi} = \int_t^T \int_{\vartheta} \chi_{T\vartheta} dQ_{t\vartheta} \quad (2.1)$$

or if the state and time spaces are discrete then

$$V_x = \sum_T \sum_{\vartheta} X_{T\vartheta} q_{T\vartheta} \quad (2.2)$$

where  $q_{t\vartheta}$  is the value of an ‘elementary’ contingent claim<sup>7</sup>. Ross (1976) suggests that no arbitrage possibilities exist if and only if (2.1) or (2.2) holds.

Following the above, the value of an option in a market that no arbitrage possibilities<sup>8</sup> exist can be expressed with the following:

$$V = e^{-r(T-t)} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T \quad (2.3)$$

or if the state and time spaces are discrete then

$$V = e^{-r(T-t)} \sum_{S_T} q_{S_T} g(S_T) \quad (2.4)$$

where  $S_T$  is the asset price at the maturity of the option (the possible states of  $S$  at time  $T$ ),  $g(S_T)$  is the payoff function (contingent on  $S_T$ ) and  $f(S_T)$  is the risk-neutral probability distribution function of the stock price at maturity (the state price density of states of  $S_T$ ). Given an explicit formula for the probability distribution function a closed form pricing formula can be derived. Hence the knowledge of the terminal distribution would be an adequate means for pricing options and vice

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<sup>7</sup> An ‘elementary’ contingent claim on any security or portfolio of securities is defined as a security that pays one unit of wealth at a given date in  $T$  periods, if the value of the portfolio is  $M$  at that time; if the value of the portfolio is not  $M$  in  $T$  periods, the elementary claim expires paying nothing.

<sup>8</sup> Cox and Ross (1976) suggest that a world free of arbitrage possibilities is not quite the same as saying that the world is in equilibrium but this will have no effect on the value of the option.

versa the knowledge of option prices would be sufficient to recover the implied PDF, within a risk-neutral framework.

Gastineau (1975) presented the Gastineau and Madansky model for the pricing of options. The model essentially uses Equation (2.3) where  $f(S_T)$  is estimated and numerical integration techniques are then used to obtain the option price. Jarrow and Rudd (1982) criticised this approach as somewhat complex due to the analytical intractability of the density function. Cont (1998) also criticises an approach based on the estimation of  $f(S_T)$  with historical data and calculates option prices using (2.3). Jarrow and Rudd (1982) proposed an alternative - more tractable - distributional form and derived option pricing formulas, and tested, in Jarrow and Rudd (1983), the methodology. The model was found to consistently outperform the BSM model.

An approach developed in Breeden and Litzenberger (1978) complemented the work of Cox and Ross (1976) and set the foundations for one of the most celebrated areas of modern finance - the study of the information content of asset prices. Breeden and Litzenberger (1978) considered an 'elementary' contingent claim and expressed its price in terms of the prices of European call options on the underlying



portfolio<sup>9</sup>. The pricing function for the elementary claim is proved to be:

$$\frac{P(M, T; \Delta M)}{\Delta M} = \frac{[c(M + \Delta M, T) - c(M, T)] - [c(M, T) - c(M - \Delta M, T)]}{(\Delta M)^2} \quad (2.5)$$

and in the limit as the step size tends to zero,

$$\lim_{\Delta M \rightarrow 0} \frac{P(M, T; \Delta M)}{\Delta M} = \frac{\partial^2 c(X, T)}{\partial X^2} \Big|_{X=M} \quad (2.6)$$

where  $P(M, T; \Delta M)$  is the price of the call portfolio that gives a payoff of one unit of wealth if  $M$  occurs in  $T$  periods,  $\Delta M$  is the step size - the distance between two consecutive values of the underlying portfolio - and  $c(X, T)$  denotes the price of a call option with strike  $X$  maturing in  $T$  periods. Thus (2.5) gives the pricing function of an elementary claim on  $M$  maturing in  $T$  periods in the discrete case, and (2.6) gives the pricing function for continuous  $M$ . Equation (2.6) is interpreted as follows:

**Proposition 2.1**     *Within a time-state preference framework, if European options prices, with the same time-to-expiration and for strike prices spanning from zero to infinity, existed for a single underlying asset, the entire risk-neutral probability density<sup>10</sup> for that expiration date could be inferred; the risk-neutral probability density*

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<sup>9</sup> The authors highlighted the fact that an essential condition for this analysis to be valid is that there exists a one-to-one mapping between aggregate consumption at a given time and the level of the market portfolio at that time. The latter ensures that the price on an elementary claim on aggregate consumption will equal the price of an elementary claim on the corresponding level of the market portfolio. Rubinstein (1994) and Jackwerth and Rubinstein (1996), however, extended the interpretations of Breeden and Litzenberger to the case where options are written on securities or security indices.

<sup>10</sup> The terms **risk-neutral probability distribution**, **probability distribution function** and **state-price density** are usually used in the option pricing literature to define the same entity. The same applies to the present study. Through out the study the acronyms **RND** (Risk Neutral Density), **PDF**

*would be obtained by calculating the second derivative of each option price with respect to its strike price.*

The immense contribution of the above proposition lies on the fact that given a set of options on an underlying asset it is possible to estimate the equilibrium or the aggregate risk-neutral distribution implied in the options prices for the expiration date of the option.

Starting in the early 1990s financial and economic researchers have become extremely interested in exploiting the forward looking nature of options and became increasingly sophisticated in their attempts to analyse market expectations embedded in traded options contracts. The ‘implied’ information can prove important when a more qualitative/fundamental economic question is addressed as it helps to reveal ‘market sentiment’. This is useful for the policy-stance of a central bank<sup>11</sup> (e.g. Federal Reserve Bank, Bank of England etc.) and also for investors who may wish to take positions based upon the difference between their forecast of the distribution and the consensus of the market. On the other hand implied distributions can also be used to price illiquid, exotic or over-the-counter options consistently with exchange traded, vanilla options.

The advantages and disadvantages of implied distributions over unconditional distributions estimated from time series data, for applications similar to those de-

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(Probability Distribution Function) and **SPD** (State Price Density) will be used interchangeably.

<sup>11</sup> See Bliss and Panigirtzoglou (2000) for a comprehensive list of Central Banks’ research papers on implied distributions.

scribed above, have been discussed extensively in various studies. Chang and Melick (1999), despite being sceptical on what the implied PDFs really represent and despite admitting that the risk of overinferencing the implied PDFs is always present, conclude that ‘... an analytical approach based on PDFs [...] has much to recommend over alternatives based on time series data’. A series of key advantages are identified; firstly the implied PDFs’ forward looking nature allows any uncertainty inherent in the financial markets to be captured; secondly, implied PDFs can be easily estimated with the use of substantially short time-series data sets; thirdly, they are capable of immediately reflecting a change in the market sentiment; and finally, for certain regions representing a large percentage of total probability, it has been shown that they are relatively free of mathematical priors imposed by a specific economic model or structure<sup>12</sup>.

In addition to the models reviewed in Section 2.3.2 (pp. 23 - 28) which fall in the class of ‘implied’ models several methods have been developed in this area over the years. Jackwerth (1999) classifies the models in two main categories: the *non-parametric* methods, which don’t make any specific assumption on the form of the RND and allow more general functions and the *parametric* methods, which aim to find a parametric distribution more flexible than the log-normal. The following paragraphs review the models in each category.

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<sup>12</sup> See Campa, Chang and Reider (1998) and Coutant, Jondeau and Rockinger (2001).



### 2.4.1 Non-Parametric models

Non-parametric methods in general enable us to avoid the strong assumption regarding the distribution of the underlying asset's returns, by using model-free statistical methods based on very few assumptions about the process that generates the data. Three types of non-parametric methods have been proposed in the context of the study of option prices: kernel regression, maximum entropy and curve fitting techniques

The kernel regression technique was introduced by Ait-Sahalia and Lo (1998). It is based on Equation (2.6), and uses kernel regression methods to calculate a smooth estimator of the relationship between the option price and the strike price. The second derivative of this function then gives the RND. The method makes use of a cross-sectional time-series of option prices and performs kernel regression across five dimensions: stock price, strike price, time to expiration, interest rate, and dividend yield. In the same paper they also used a reduced version with three dimensions: the forward price, the strike price and the time to expiration. Pritsker (1997) and Rookley (1997) have further modified and further applied the methodology. Ait-Sahalia and Lo (2000), finally, proposed a semi-parametric form of their initial model which assumes that the pricing function is given by the parametric BSM formula except that the implied volatility parameter of that option is a non-parametric function of 'moneyness' (strike price/current asset price). The maximum entropy method, presented in Buchen and Kelly (1996) and Stutzer (1996), is a method for estimating



the RND based on a statistical mechanics/information theoretic approach. Equations (2.3) and (2.4) are essentially utilised. The method searches for a RND  $P_i$  which, given a prior distribution  $O_i$ , maximises the cross-entropy:

$$-\sum P_i \ln \left( \frac{P_i}{O_i} \right)$$

subject to constraints such as positivity of the probabilities, summing of the probabilities to one, and correct pricing of options and the underlying asset. Buchen and Kelly (1996) used the log-normal as the prior whereas Stutzer (1996) used the empirical distribution of historical asset returns.

Curve fitting methods, finally, include methods in which either the function of implied volatilities across strike prices or the RND itself is approximated by some general function. Shimko (1993) utilised the Breeden and Litzenberger (1978) framework by introducing a quadratic polynomial fitting of the implied volatility smile. Malz (1997) applied a quadratic polynomial fitting to the function that relates implied volatilities with options' deltas rather than strike prices. Both models make use of the BSM formula to derive implied volatilities but do not require it to be correct. Brown and Toft (1999) extended Shimko's (1993) approach by using 7<sup>th</sup>-order splines to approximate the implied volatility smile. Campa, Chang and Reider (1998) and Aparicio and Hodges (1998) proposed the use of cubic and cubic-B splines respectively to fit the implied volatility smile instead. Rosenberg (1996) and Rosenberg and Engle (1997) used a polynomial fitted to the log-smile to prevent

negative implied volatilities while Jackwerth (2000) maximised the smoothness of the smile. Studies that directly approximate the RND include Mayhew (1995), who approximated the RND with cubic splines, and Hartvig, Jensen and Pedersen (1999) who built up the logarithm of the RND from piece-wise linear segments.

In a non-parametric fashion Neuhaus (1995) also develops a technique which can not be classified in any of the above categories. He chose to use the first derivative of  $P(M, T; \Delta M)$  in Equation (2.6), p. 32, to recover the Cumulative density. The derivatives are numerical and discrete in that only the available strikes are used. This technique, however, allows for probability calculations at and between strike prices and no information is recovered for the strikes in the regions below the lowest and beyond the highest traded strikes.

However flexible non-parametric models may be, numerous disadvantages emerge in their application. Kernel regression techniques, for example, are highly data intensive and difficult to use in real time applications. Maximum entropy methodologies, on the other hand, lack smoothness constraints and as a consequence they result in multi-modal spiky RND shapes. Curve-fitting techniques, finally, do not guarantee in general that the resulting probabilities are positive and the fact that this condition needs to be checked independently further increases the complexity of the calculations.

## 2.4.2 Parametric Models

The parametric models can be divided into two groups. First, the *fully parametric* models which make an explicit assumption for the RND and the *semi-parametric* models which assume a more flexible functional form and as a consequence allow more flexible shapes for the RNDs.

### Fully Parametric models

Sherrick, Garcia and Tirupattur (1996) proposed the use of the Burr type III distribution - as an alternative to the log-normal distribution - which allows a wide range of skewness and kurtosis values. The Burr type III distribution covers all the space regions in the skewness-kurtosis plane occupied by Pearson types IV, VI and bell-shaped curves of Pearson type I, gamma, Weibull, normal, log-normal, exponential and logistic distributions [see Rodriguez (1977)]. The respective closed form solution results in three parameters instead of two in the case of the log-normal distribution. The model has received very little attention over the years. In the same spirit Aparicio and Hodges (1998) used generalised beta functions of the second kind. The parametric family was introduced in the area of finance by Bookstaber and McDonald (1987) in an attempt to estimate a general distribution for security price returns. Several parametric forms are nested within the generalised beta functions family - including the log-normal, gamma and exponential distributions and several Burr type distributions - which is fully specified by four parameters. Alternatively Posner and



Milevsky (1998) explored the Johnson family. The variety of shapes given by the latter is referred to as ‘...quite as great as that of the Pearson system’, in Stuart and Ord (1987), and is also defined by four parameters.

A large body of research in this area, on the other hand, has focused on the mixture methods. Mixture methods achieve greater flexibility by drawing with different probabilities from several simple distributions, each with a distinct parameterisation. Models of this type were introduced in the finance literature by Ritchey (1990), who assumed that the RND can be assumed to belong to the family of the  $k$ -component mixture of log-normal distributions. Bahra (1997), Malz (1996, 1997) and Gemmill and Saflekos (2000), used a mixture of two log-normal distributions and Melick and Thomas (1997) used a mixture of three log-normal distributions. Parameterisation in the  $k$ -component mixture of log-normal distributions family appears very appealing since a desired degree of skewness as well as excess kurtosis can be obtained in the resulting distributions. The main drawback, however, is the estimation of large number of parameters, which in the simplest case of the mixture of two log-normal distribution are five. The number of parameters grows very fast and eight parameters need to be estimated in the case of a mixture of three log-normal distributions. In addition, Cont (1998) argues that by construction a mixture of log-normal distributions has thin tails unless one allows high values of variance. Finally, the estimation of a mixture of log-normals model has been found to be sensitive to large measurement errors (see Chapter 6).



### Semi-parametric models

This category includes the expansion methods. As a general rule the RND is approximated by an expansion around a known statistical density in the same fashion as a mathematical function is approximated by a Taylor series expansion around a point. The general form of the assumed RND is:

$$P(S_T - S_t \leq x) = P_0(x) + \sum_{k=1}^{\infty} u_k P_k(x) \quad (2.7)$$

The first term of the expansion  $P_0(\cdot)$  corresponds to the base distribution and the second term adds successive corrections to account for differences between  $P_0(\cdot)$  and the true RND. The series is then truncated to a finite number of corrections which gives a parametric, usually analytically tractable functional form for the RND and enables the derivation of closed form option pricing formulae.

Abadir and Rockinger (1998) used Kummer functions as a basis for the RND. The resulting analytic expression includes the Normal, Gamma, Inverse Gamma and mixtures as special cases. The option price function across strike prices was derived in closed form. The methodology, however, requires the estimation of seven parameters, which is considered relatively large for this type of models. Madan and Milne (1994) price contingent claims as elements of a separable Hilbert space whereas Abken et al. (1996a, 1996b) specialised the Hilbert space basis to the family of Hermite polynomials and approximate the RND with Hermite polynomials up to the 4<sup>th</sup>-order. The latter results in a four-parameters model. Jondeau and Rockinger

(2000) and Backus et al. (1997) restrict their attention to the class of Gram-Charlier expansions.

Jarrow and Rudd (1982) used a different class of expansions, the Edgeworth series expansions. The log-normal distribution was assumed to be the base of the expansion. Corrado and Su (1996, 1997), Longstaff (1995) and Rubinstein (1998) subsequently used the Edgeworth series expansions around a log-normal, a normal and a binomial distribution respectively. Jarrow and Rudd (1983) performed empirical tests to examine the validity of the methodology, when used to price individual stock options. Corrado and Su (1996, 1997) also tested the model using options data on the S&P 500 index. They showed that a model of this type outperforms the simple BSM pricing model, by providing a significantly better fit to observed option prices and by performing better when used to predict option prices one period ahead.

The expansion methods result generally in tractable analytic expressions for the RND which in turn allows the derivation of closed form option pricing formulas. Despite being similar in spirit with more complicated models i.e. the implied binomial trees methodology of Rubinstein (1984) [Ait-Sahalia and Lo (2000)] they are less data-intensive and require less complex estimation procedures. As a disadvantage, Jackwerth (2000) identifies the possibility of estimating negative risk-neutral probabilities especially in the tails. Cont (1998) suggests that this drawback should not be seen as prohibitive for the whole span of strike prices but for strike prices too far from the money. To overcome the possibility of negative probabilities, Jondeau

and Rockinger (1998) provide an algorithm which guarantees positive probabilities in Gram-Charlier expansions. The present study introduces additional constraints in the estimation of the RND which also guarantees positive probabilities in Edgeworth series expansions.

## 2.5 Conclusions

The elegance and the simplicity of the BSM pricing formula yielded its dominance among existing, at that time, pricing models. Moreover all options models of the past nearly 30 years have been refinements of the basic BSM model.

This chapter examined the literature that followed the seminal work of BSM which - with little variation in the results - concludes that the BSM formula systematically misprices options across strike prices. The studies were classified with regard to the assumption behind the original derivation of the BSM formula they relax. Four possible categories were identified in that respect: studies that assume the underlying asset pays some dividends, studies where the variance rate of the return on the asset and short-term interest rate is not constant through time, studies where the asset price follows alternative stochastic processes and finally studies which relax the assumption that the distribution of possible asset prices at the end of any finite interval is log-normal.



## Chapter 3

# Motivation for Research and Thesis Outline

The present work is motivated by the increasing interest in the study of the information content of option prices and the need for formally addressing issues that have recently emerged as research and applications in the area becomes more and more sophisticated.

The study of the information content embedded in option prices focuses on the recovery of either the stochastic process followed by the underlying asset price or the density from which the asset price at expiration is drawn. While some benefits in the first exercise are acknowledged<sup>13</sup>, the latter approach has mainly received the attention of researchers and practitioners as to being somewhat more advantageous for a number of reasons. The terminal density of the underlying asset - by construction - encompasses many stochastic processes, thus, allowing for a more general exercise. This is of particular interest in situations where interest is focused on possible asset price outcomes i.e. market crashes, interest rate cuts, currency devaluations etc. Moreover a reasonably flexible assumption for the functional form of the terminal density can accommodate a wide variety of shapes for the terminal distribution.

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<sup>13</sup> See Bates (1991 and 1996a and 1996b) and Malz (1996) and the models examined in Section 2.3.2, p.23.



The study of implied RNDs, on the other hand, has much to recommend over alternative approaches based on time series data. A series of key advantages are generally identified. Firstly the implied densities are estimated from market prices of options which are forward-looking and therefore allow any uncertainty inherent in the financial markets to be incorporated. Secondly, implied densities can be easily estimated with the use of substantially short time-series data sets; in many cases one cross-section of observed options prices is considered enough to recover the entire risk-neutral density of the underlying asset. Thirdly, they are capable of immediately reflecting a change in the market sentiment and also incorporate ‘multiple scenarios’ speculations. Fourthly, rather than calculating a forecast for the future price of an asset - econometric approach - the entire distribution is estimated, which allocates certain probabilities over all possible outcomes in a risk-neutral framework. This allows the pricing of literally any claim contingent on future outcomes of the asset price. Finally, for certain regions representing a large percentage of total probability, it has been shown that they are relatively free of mathematical priors imposed by a specific economic model or structure.

The aim of the study is threefold:

- to develop a technique for the estimation of implied RNDs which can incorporate the characteristics of modern financial markets

- to develop realistic and economically sensible tests for the validation of implied RNDs and implied RNDs estimation techniques - on qualitative as well as quantitative grounds - and conduct them with the developed technique
- to present an empirical use of implied RNDs and illustrate it with the use of that technique

In trying to develop a RND estimation method in the spirit of this work the Cox and Ross's (1976) framework is employed. The value of an option is expressed as:

$$V = e^{-r(T-t)} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T \quad (3.1)$$

Expression (3.1) is the fundamental option's risk-neutral valuation equation which expresses the value of an option as the discounted integral of the product of the payoff function  $g(S_T)$  and the density function  $f(S_T)$  from which the asset price at expiration is drawn. According to Melick and Thomas (1999) it has a number of plausible solutions and it is the *a priori* structure, that is to say the functional form of the estimated density, that allows us to choose one particular PDF. If  $f(S_T)$  is replaced, for example, with a log-normal parameterisation, Equation (3.1) degenerates to the BSM formula.

The log-normal parameterisation, however, has proved to be not adequate to describe observed prices and various alternatives have been proposed. Given the large number of alternative structures that can be assumed for the terminal PDF it

is natural to seek a parameterisation flexible enough to encompass a wide variety of shapes.

A large family of models that satisfy this criterion is the *semi-parametric* family, presented in Section 2. The probability models of this family offer a great flexibility with regard to the shape of the PDF and can be thought of as a smooth approximation of all the potential shapes which serve as solutions to Equation 3.1.

The present study assumes a generalised *Edgeworth Series Expansion* (*ESE* hereafter) probability model for the parameterisation of  $f(S_T)$  as in Jarrow and Rudd (1982). The *ESE* provides a method for finding a series expansion of a non-Gaussian probability distribution of which kind the empirical distribution of asset log-returns has been found to be. The most important features of the *ESE* probability specification that prove extremely useful in empirical applications and make it an appealing parameterisation are:

- the ability to select from a broad range of reference distributions, providing flexibility in finding one that closely approaches the distribution to be approximated
- the fact that by construction the coefficients in the expansion are simple functions of the moments of the given and the approximating distributions. As a result the parameters that define the PDF have a physical meaning as opposed to being abstract mathematical quantities as in the case of other

parametric families of distributions i.e. the parameters that define a mixture of  $k$  log-normal distribution or a Pearson type density

- the ability to derive closed form expressions for the theoretical option value

Despite this advantageous features existing literature has not explored the potential of the *ESE* parameterisation for the study of implied RNDs. While other parametric families have been examined as to how well they do in pricing options and also in analysing market conditions and market expectations, i.e. the mixture of two log-normals, the Burr III, the Hermite polynomials etc., existing studies have only looked into the use of *ESE* type parameterisations in option pricing. This work aims to complement the existing literature in that sense and also address issues related to the estimation and use of implied RNDs having the *ESE* probability parameterisation as a reference model.

The remaining of the study is organised as follows:

Chapter 4 is focused on the introduction of the *ESE* model. The chapter starts with a description of the data set used in the empirical applications. The functional form of the PDF is then demonstrated and its fundamental statistical properties are discussed. Following, generalised European option pricing formulae are derived for call and put options written on a general asset that pays dividends in continuous time. Pricing formulae for American options are then derived as weighted sums of upper and lower bounds. Given the latter, implied RNDs can be estimated following an



algorithm that minimises the sum of squared discrepancies between theoretical and observed option prices subject to a certain set of constraints. Finally, Chapter 4 discusses the data used for the empirical investigation of the study.

Chapters 5, 6 and 7 compose the empirical part of the study.

Chapter 5, aims to assess the validity of the *ESE* model in the context of the study of implied PDFs. To defend criticisms on the ability of the model to stimulate meaningful interpretations it is essential that the consistency with the data is checked. Chang and Melick (1999) view this consistency with the data in a broader sense: internal consistency - adequate fit to observed data - and economic rationale of the respective PDFs. In that respect Chapter 5 examines the properties of the implied RNDs in the presence of large changes in economic conditions. The ability of the implied RNDs recovered with the *ESE* model to capture speculation over future eventualities and their capacity to immediately reflect changes in the market sentiment are examined. Finally, following the argument of Melick and Thomas (1997) that the differences between the estimates from an alternative parameterisation and the log-normal BSM parameterisation may be apparent - better fit to observed data - but not significant in a statistical sense, this chapter also examines whether the use of the *ESE* model is able to offer a statistically significant better fit compared to the log-normal BSM parameterisation.

Chapter 6 assesses the performance of the *ESE* model on a more quantitative basis. Typical studies assess alternative techniques for the estimation of RNDs on the

basis of the goodness of fit achieved to observed options cross sections. The goodness of fit, however, should not be the sole criterion - not even the prior one. RNDs estimation techniques defined by a number of parameters larger than the respective number of parameters used by other competitive techniques are expected to result in more accurate fit of the data. A fair comparison seems to be one that assesses also the robustness of these techniques or, more precisely, the degree of confidence that can be placed on the summary statistics calculated off the implied distributions. Chapter 6 addresses this issue on a relative basis. A mixture of two log-normals specification, being a very commonly used parametric model which has already been studied in terms of stability, is used as a comparative benchmark.

Chapter 7 illustrates an application where the information content of option prices is explicitly quantified, rather than be qualitatively assessed. A fundamental principal of economic theory is employed: *in the absence of arbitrage, all asset prices can be expressed as the expected value of the product of the pricing kernel (a preference function) and the asset payoff*. It follows then that, the pricing kernel, coupled with a probability model for the future states, gives a complete description of asset prices, expected returns and risk preferences. Chapter 7 solves the inverse of the equilibrium asset pricing model to identify preference parameters - *given asset prices and a probability model for futures states what can be inferred about investors' risk preferences*. Using the *ESE* probability model this chapter derives risk aversion functions and compares them with the market conditions of the study period.

Chapter 8 discusses issues related to the limitations of the methodology presented in the thesis as well as the applications performed. It concludes the thesis by presenting the general conclusions and suggests a number of issues that need to further be investigated in subsequent studies in the area.

# Chapter 4

## Data and Central Methodology

The present work aims to provide an in-depth analysis of the estimation, testing and applications of implied probability distributions. The empirical analysis is performed by means of a semi-parametric *ESE* model.

This chapter presents the data used in the empirical exercise and develops the *ESE* model. It begins with a description of the data set in Section 4.1. Section 4.2 offers a detailed presentation of the methodology used to recover the implied RNDs. More specifically, Section 4.2.1 presents the functional form which is assumed to adequately approximate the true RND. Section 4.2.2 derives closed form generalised pricing formulae for European options while Section 4.2.3 demonstrates the modifications that need to be made in order to extend the methodology to incorporate the early exercise feature embedded in American options. Section 4.2.4 describes the algorithm for the estimation of the implied RNDs and Section 4.2.5 discusses the constraints of the optimization algorithm. Finally, Section 4.2.6 demonstrates the evolution of *ESE* implied PDFs over the sample period. Derivations and proofs are appended in 4.A.



## 4.1 Data

The present study aims to develop a methodology for the estimation of implied RNDs and to address a number of issues related to their use in financial practice. It is therefore essential that the empirical investigation is not limited to a single market. Space constraints, on the other hand, do not allow an analysis like the one performed, to be carried out over all possible markets.

It was, therefore, decided, that two markets-contracts be considered. Among possible candidates the Eurodollar contract - traded on the Chicago Mercantile Exchange (CME) - and the WTI (West Texas Intermediate) contract - traded on the New York Mercantile Exchange (NYMEX) - were thought to be suitable for the nature of the study. The former are American style options on the Eurodollar time deposit futures contract. The CME's Eurodollar futures contract reflects the London Interbank Offered Rate (LIBOR) for a three-month, \$1 million offshore deposit. Eurodollar options listings include March, June, September, December, expirations and also six months in the March quarterly cycle and two serial months not in the March cycle. The options on NYMEX's WTI futures are also American style. A WTI futures contract calls for delivery of 1000 barrels of light sweet (low-sulfur) crude oil. WTI options contracts are quoted for the first twelve consecutive months, plus three long-dated options at 18, 24, and 36 months out on a June/December cycle.

Following are the main reasons that led to this decision.

### *Liquidity*

Liquidity considerations may affect the results of the study and may consequently lead to various misinterpretations. The above choice allows investigation to be made in two very distinct markets in that sense. The Eurodollar futures, on the one hand, are the most liquid exchange-traded contracts in the world when measured in terms of open interest<sup>14</sup> which ensures that any results are rather representative of a *liquid* market. The WTI futures contract, on the other hand, is the world's largest-volume futures contract trading on a physical commodity<sup>15</sup>. It offers excellent liquidity and price transparency which makes it appropriate for a pricing benchmark. This allows the study of a rather *illiquid* market - a commodity market - in the least 'costly' way with the fear of 'corner solutions' due to this characteristic limited to a minimum.

### *Market conditions*

The *ESE* model - which includes the log-normal distribution as a special case - is expected to perform best when the implied distributions substantially deviate from a log-normal. This is likely to happen during 'unsettled' periods i.e. market crashes, general elections, wars etc. The study is carried out for data throughout the year 1998. This year was a very 'rich' year in terms of eventualities in both markets. During the fall of 1998 the Federal Reserve lowered the federal funds rate by a to-

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<sup>14</sup> "How to Get Started Trading CME Interest Rate Products", CME publication.

<sup>15</sup> Source NYMEX.

tal of 75 points on three occasions - following the regularly Federal Market Open Committee (FOMC) meetings on September 29, 1998 and November 17, 1998 and following a conference call meeting of the FOMC on October 15, 1998. The whole of 1998 was, on the other hand, an unsettled year for the oil market as well. The price of WTI fluctuated between a maximum of \$17.81 / barrel on January 29, 1998 and a minimum of \$10.70 / barrel on December 21, 1998 an overall drop of approximately 40% of the maximum price. The fact, finally, that the study is performed in 1998 for both markets ensures that the findings are affected to a minimum by a possible change in the general market conditions, a concern expressed by Jackwerth (1999).

### *Originality*

Last but not least in its own mean the data set should enhance the originality of the study. While some studies have been undertaken to examine the information content of Eurodollar futures options [Madan and Milne (1994), McManus (1999)], to the author's knowledge, only Melick and Thomas (1997) have carried out similar research for the WTI market. Originality in that sense, however, is ensured with the use of this data set in the applications of Chapters 6 and 7.

The WTI options data set was obtained from NYMEX. It contains daily observations of Bid and Ask Prices, Settlement Prices, Trading Volumes and Open Interest for the all traded expirations of the WTI options contract for the period January 1, 1998 through to December 31, 1998. The data set was refined so that it did not con-



tain meaningless information. Options with trading volume of less than five contracts were excluded due to illiquidity reasons. Also on a given day, contracts with different strike prices, recorded to trade at the same premium, were excluded and the data set was also checked for arbitrage restrictions involving monotonicity, slope, concavity, and put-call parity. Finally the last three trading days of the options contracts were not included in the data set, due to any undesirable noise information that they may have conveyed. The total number of options retained for the estimation and the analysis of the implied PDFs varies between 10 and 23 - the average being 16. The minimum recorded strike price for a call option was \$12.5 / barrel whereas the maximum \$25 / barrel. The respective figures for put options were \$ 11 and \$ 22 / barrel. Over the whole sample, the bid-ask spread was between minimum of \$ 0.01 / barrel for out-of-the-money options and a maximum of \$ 0.7 / barrel for in-the-money options.

Settlement prices were used to represent the value of the options. The settlement price of the WTI options is determined at the end of each day by a settlement committee made up of roughly 20 market participants. The committee frequently relies on the average of bid and ask prices, during the last minutes of trading, as starting points for the settlement prices. Heavily traded options are priced first, with put-call parity used to price low volume options at the same strike when the futures markets has settled [Melick and Thomas (1997)].

Data for the underlying WTI futures contract were obtained from Datastream.



The data set for the Eurodollar market<sup>16</sup> consists of daily observations of the Eurodollar futures option settlement price for the period September 1, 1998 through to November 30, 1998 for the December 1998 contract, a total of 61 trading days. The original data set excluded any option that had no open interest, exercises, or volume, on a given day. The remaining options were checked to ensure that they satisfied arbitrage restrictions involving monotonicity, slope, concavity, and put-call parity (within ranges that would result from the transactions costs involved in eliminating the arbitrage possibility). The data set was further filtered and options with different strike prices that were recorded to trade at the same premium were excluded. The total number of options - calls and puts - retained for analysis on each day ranges from 20 to 29, the average number of options used being 25. The minimum recorded strike price for a call option in the entire period was 4.5% whereas the maximum 6%. The respective figures for put options were 3.5% and 8%. Although it is not possible to provide exact figures for the bid-ask spread for the sample period, as only settlement prices were available, the bid-ask spreads for the contract under consideration are typically 0.005 and rarely exceed 0.01 under normal market conditions<sup>17</sup>.

For the estimation of the implied PDFs options settlement prices were used as the values of the options. Settlements for the Eurodollar options contracts are deter-

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<sup>16</sup> The data set was originally used at the BIS workshop 'Estimating and Interpreting Probability Density Functions', Switzerland, June 1999.

<sup>17</sup> Peter Barker from the CME's Interest Rate products marketing department is acknowledged for providing this information and also for his description of the settlement prices.

mined by the Pit Committee<sup>18</sup>, based on the levels of resting orders and spread/volatility relationships that existed in the market at the close of trading.

Data for the underlying Eurodollar futures contract were collected from Datastream and also represent settlement prices. The option strikes and the futures prices were subtracted from 100 with calls redefined as puts and puts redefined as calls, so that the RNDs are estimated with regard to the more intuitive underlying interest rate as opposed to the artificial index price<sup>19</sup>.

Treasury bonds data were also obtained from Datastream. Standard practice suggest that Treasury-bills with expiration as close as possible to the expiration of the option contracts are used to proxy for the risk-free interest rate.

Finally, market news data were collected from two individual sources. The Financial Times of London newspaper; and the FT Discovery (the FT Electronic Publishing), which covers a wide variety of news vendors. A large number of relevant articles were reviewed, and those of greater importance were used for the analysis.

Additional information on the data used for the analysis in the subsequent chapters is given when necessary.

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<sup>18</sup> See [www.cmerulebook.com](http://www.cmerulebook.com) for further details.

<sup>19</sup> The CME interest rate contracts are traded using a price index, which is derived by subtracting the futures' interest rate from 100.00. For instance, an interest rate of 5.00 percent translates to an index price of 95.00. Given this price index construction, if interest rates rise, the price of the contract falls and vice versa. Strike prices of the options contracts are also quoted in the same fashion.

### 4.1.1 Implied Volatility Patterns

Consideration of alternative PDFs parameterisation is motivated by apparent deficiencies of the BSM model. These deficiencies are most commonly expressed in options cross sections as the relation between the BSM implied volatility and the option exercise price, Dumas et al. (1998). This section illustrates this relation for the data set described above.

Following the suggestion of Broadie and Detemple (1996), a 200-step binomial tree - modified to account for the early exercise premium - is used to calculate the implied volatility. The modified binomial tree is the American analogue of the BSM model and assumes a log-normal distribution for the underlying asset.

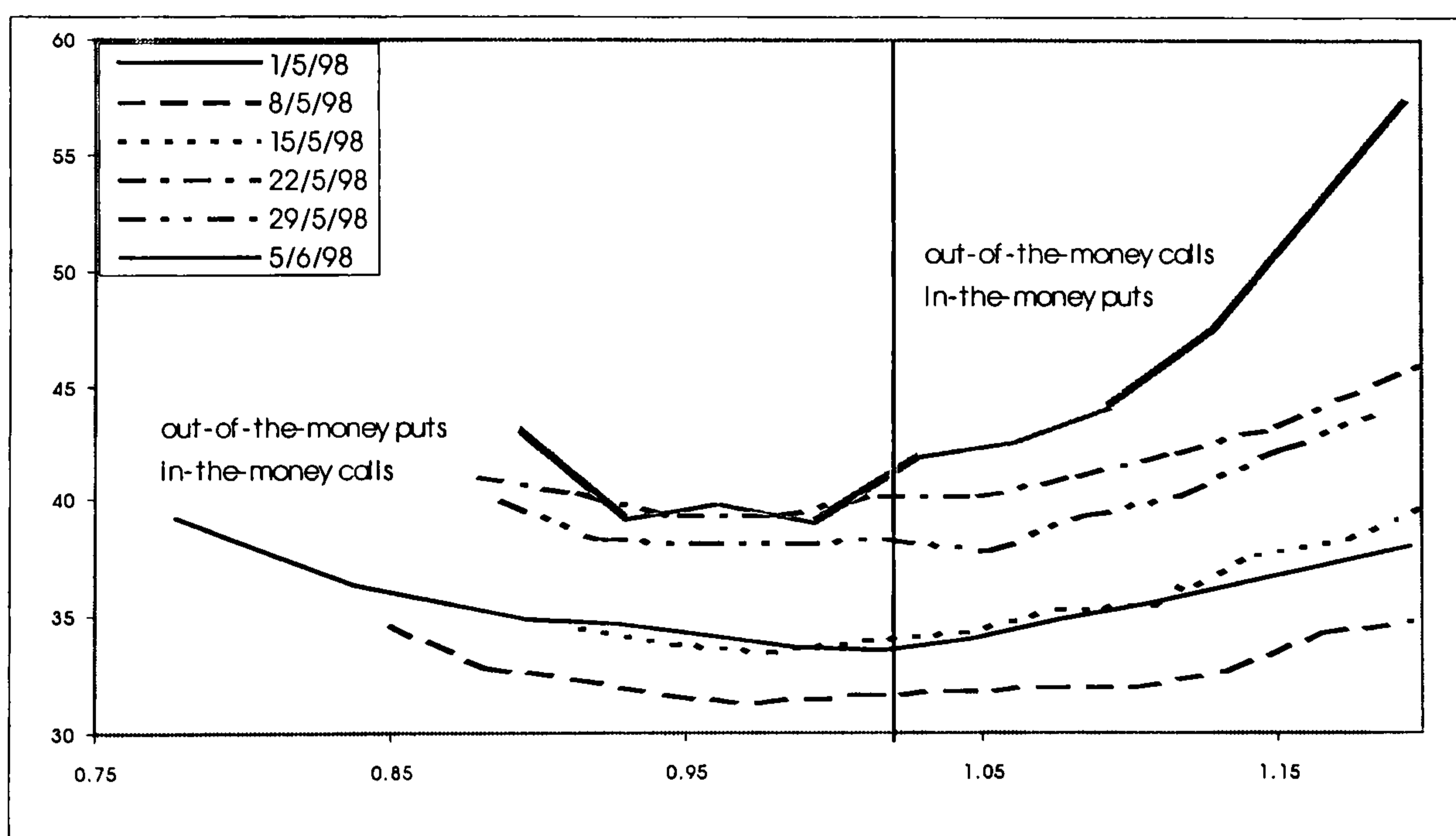


Figure 4.1: Implied Volatilities (%) across moneyness (strike price/futures price) calculated from July 98 WTI futures options from May 1, 1998 through to June 5, 1998



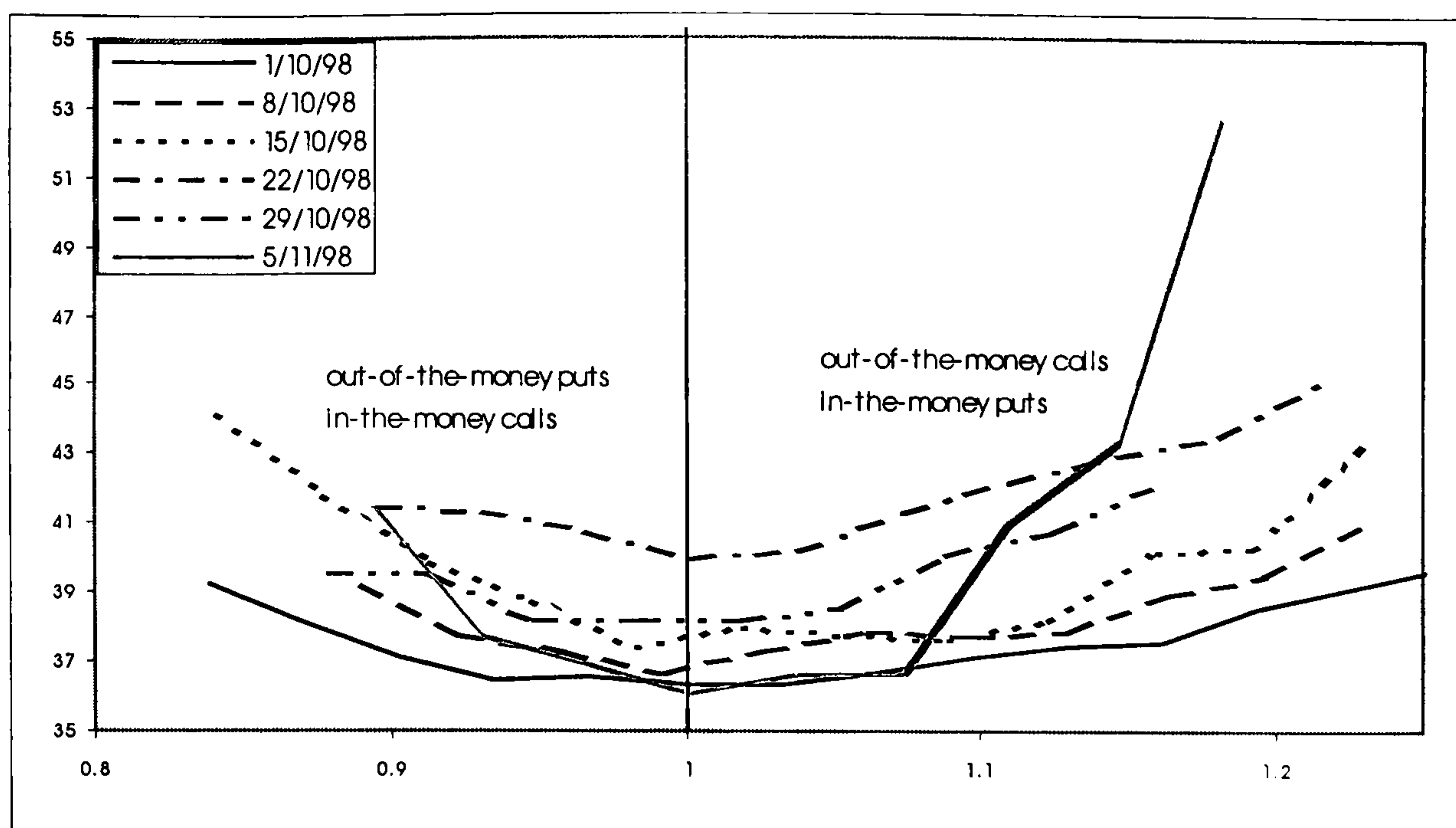


Figure 4.2: Implied Volatilities (%) across moneyness (strike price/futures price) calculated from December 98 WTI options from October 1, 1998 through to November 5, 1998

For the WTI futures options two contracts are considered - the July 98 and the December 98. Figure 4.1 plots volatilities implied by the July 98 contract across moneyness (strike price/futures price) for the period May 1, 1998 through to June 5, 1998. The volatilities do not all lie on a horizontal line as the underlying model suggests. This pattern is often called the volatility 'smile'. While a profound 'smile' is not clearly apparent in Figure 4.1, out-of-the-money calls and in-the-money puts do imply higher volatilities than out-of-the-money puts and in-the-money calls. A similar pattern is also shown in Figure 4.2 which plots volatilities implied by the December 98 WTI options from October 1, 1998 through to November 5, 1998. The results presented in Figures 4.1 and 4.2 suggest that the assumed probability model - the *LGN* model - is not capable of assigning the required probability mass



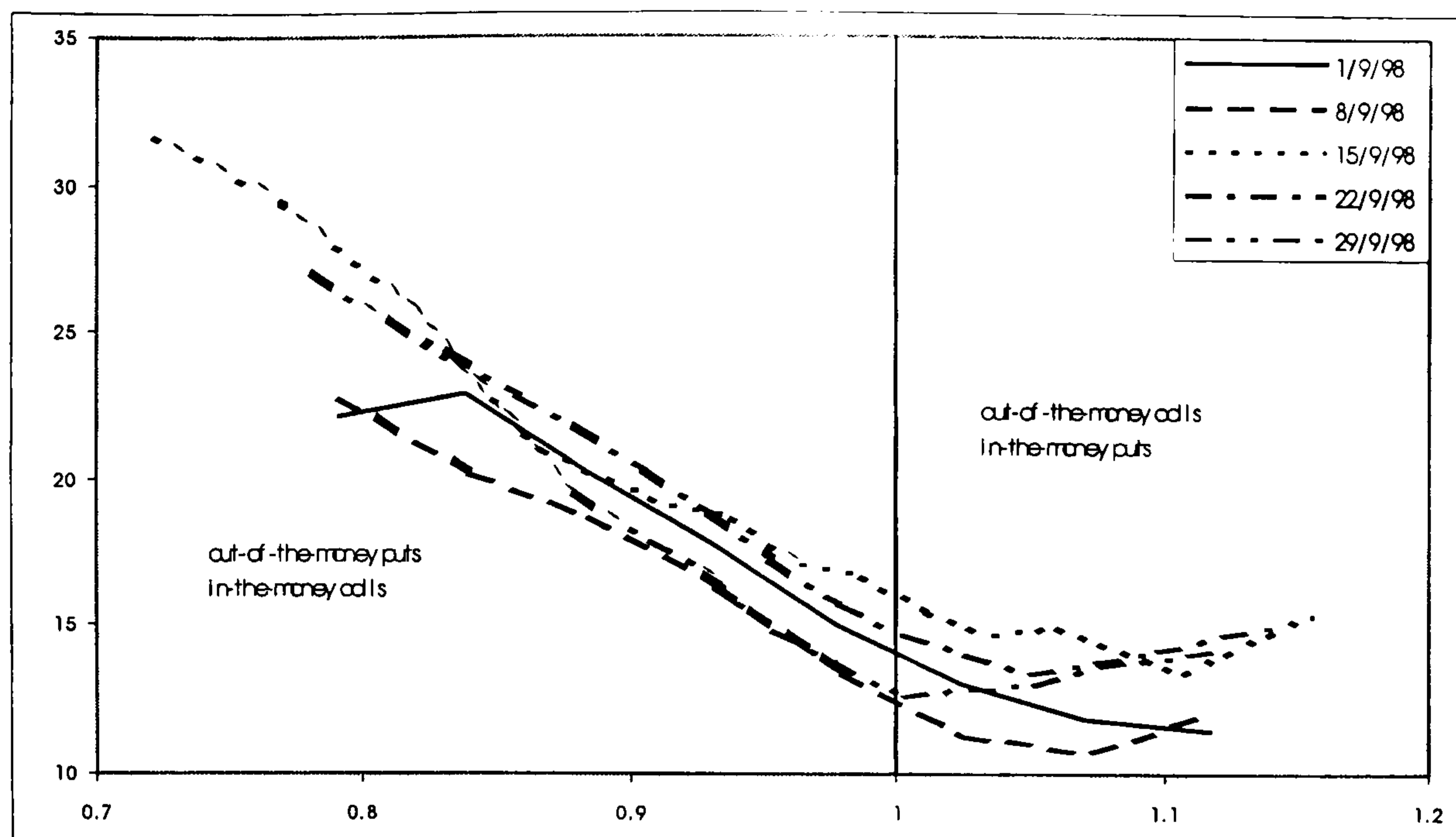


Figure 4.3: Implied Volatilities (%) across moneyness (strike price/futures price) calculated from December 98 Eurodollar options from September 1, 1998 through to September 29, 1998

in the far right tail area, an indication of small positive skewness and leptokurtosis in the data.

Figures 4.3, 4.4 and 4.5 show volatility 'smiles' calculated for December 98 Eurodollar futures options for the periods September 1 to September 29, October 6 to October 27, and November 3 to November 24, 1998 respectively. In contrast to the patterns shown in Figures 4.1 and 4.2 the Eurodollar volatility 'smiles' are much more intense. Out-of-the-money puts and in-the-money calls imply substantially higher volatilities than out-of-the-money calls and in-the-money puts. This suggests that the underlying probability model - the *LGN* model - is not able to incorporate the probability mass in the far left tail area, an indication of negative skewness and leptokurtosis in the data.

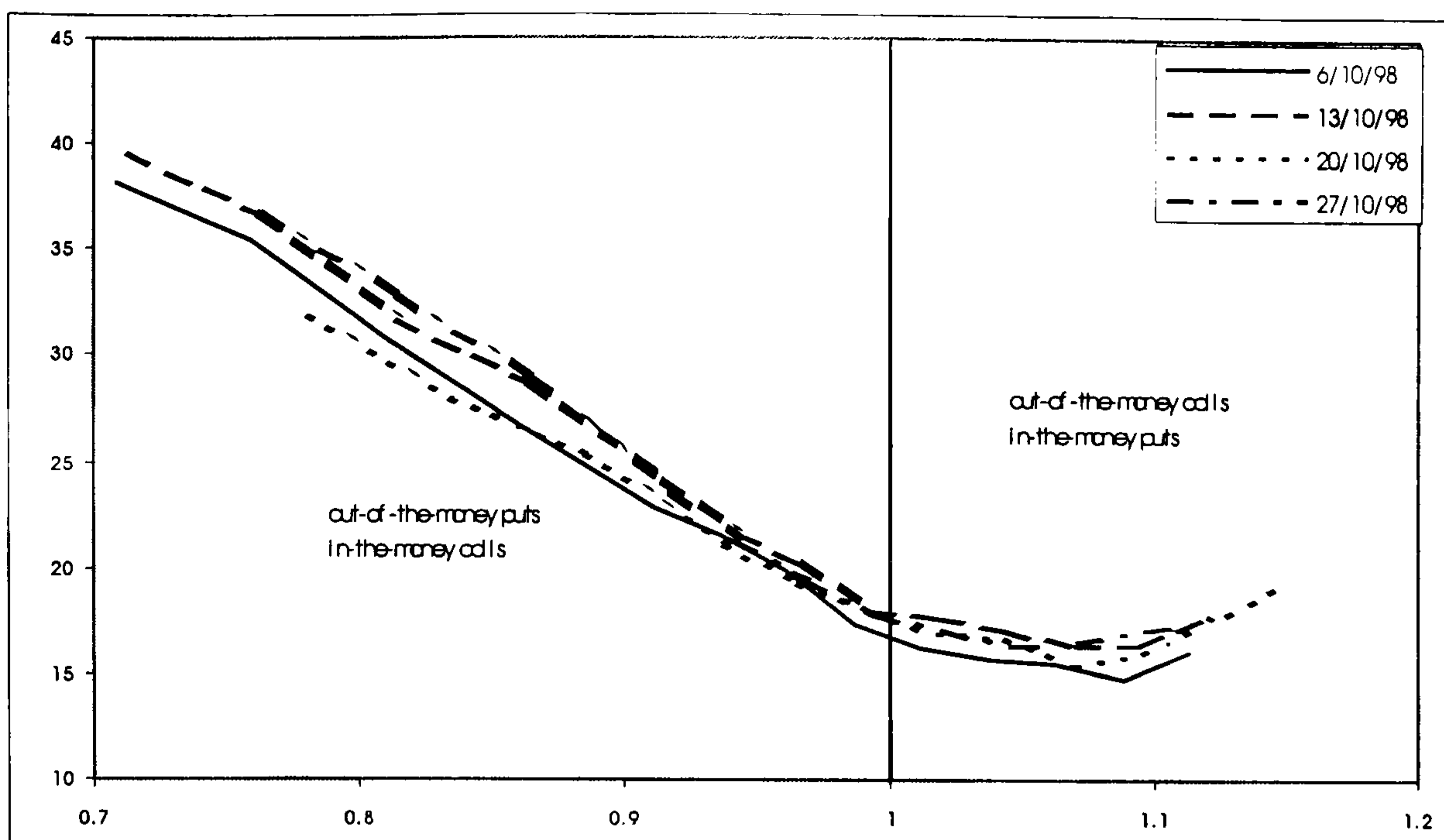


Figure 4.4: Implied Volatilities (%) across moneyness (strike price/futures price) calculated from December 98 Eurodollar options from October 6, 1998 through to October 27, 1998

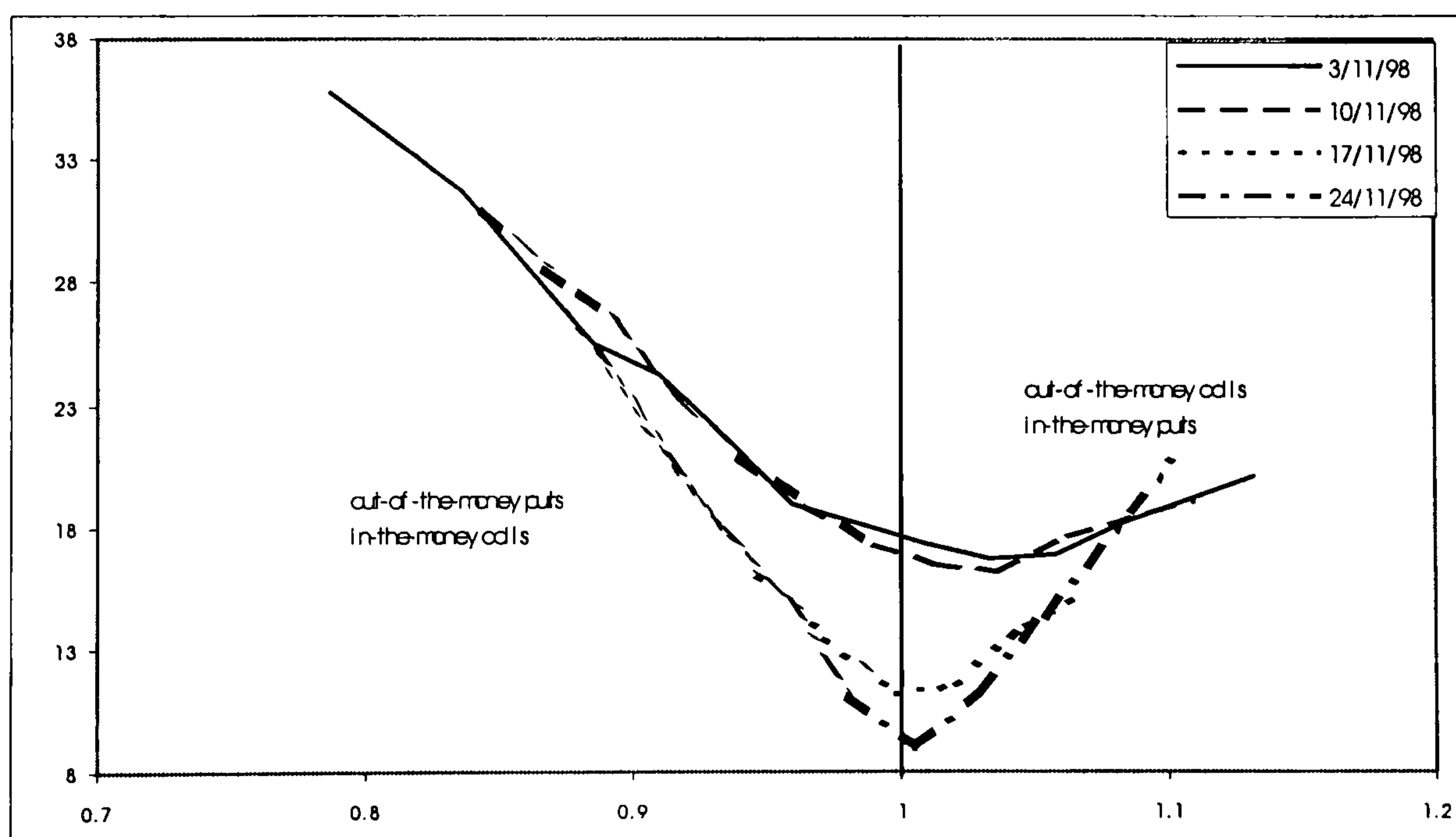


Figure 4.5: Implied Volatilities (%) across moneyness (strike price/futures price) calculated from December 98 Eurodollar options from November 3, 1998 through to November 24, 1998

## 4.2 Methodology

### 4.2.1 Functional form of the RND

It is well known in mathematical and physical works that functions can often be usefully expressed as a series of terms such as powers of the variable (Taylor's series) or trigonometrical functions (Fourier's series). In the same spirit density functions can be represented by series expansions.

The problem is formulated as follows<sup>20</sup>. It is desired to find a probability density  $f(S_T)$  with cumulative density  $F(S_T)$  of a stochastic variable  $S_T$  (the value of the asset price at time  $T$ ). Given that moments about the origin  $\alpha_j(F)$  exist the characteristic function of the probability density function  $f(S_T)$  is given by:

$$\phi(F, \tau) = \exp \left\{ \sum_{j=1}^{\infty} \kappa_j(F) \left[ \frac{(i\tau)^j}{j!} \right] \right\} \quad (4.1)$$

where  $\kappa_j(\cdot)$  are the semi-invariants or cumulants of a distribution. The cumulants can be expressed as a polynomial in  $\alpha_j(\cdot)$ . An important characteristic of the cumulants is that the cumulant of  $n$  independent, identically distributed variates is simply  $n$  times the cumulant of the basic distribution.

Now let  $\kappa_j(F)$  be given as a sum of the cumulant associated with a reference distribution  $\kappa_j(A)$  and an error cumulant  $\kappa_j(E)$

$$\kappa_j(F) = \kappa_j(A) + \kappa_j(E) \quad (4.2)$$

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<sup>20</sup> The approach was introduced in the finance literature by Jarrow and Rudd (1982).

The characteristic function of  $f(S_T)$  can then be expressed as:

$$\phi(F, \tau) = \exp \left\{ \sum_{j=1}^{\infty} \kappa_j(F) \left[ \frac{(i\tau)^j}{j!} \right] \right\} \exp \left\{ \sum_{j=1}^{\infty} \kappa_j(E) \left[ \frac{(i\tau)^j}{j!} \right] \right\} \quad (4.3)$$

Formally expanding (4.3) in power series results in

$$\begin{aligned} \phi(F, \tau) = \phi(A, \tau) & \left\{ 1 + \sum_{j=1}^{\infty} \kappa_j(E) \left[ \frac{(i\tau)^j}{j!} \right] + \frac{1}{2} \left\{ \sum_{j=1}^{\infty} \kappa_j(E) \left[ \frac{(i\tau)^j}{j!} \right] \right\}^2 \right. \\ & \left. + \frac{1}{6} \left\{ \sum_{j=1}^{\infty} \kappa_j(E) \left[ \frac{(i\tau)^j}{j!} \right] \right\}^3 + \dots \right\} \end{aligned} \quad (4.4)$$

The desired series expansion of the probability density function  $f(S_T)$  in terms of the reference probability distribution  $a(S_T)$  is obtained by taking the Fourier transform of (4.4),

$$f(S_T) = a(S_T) + \sum_{j=1}^{\infty} E_j \frac{(-1)^j}{j!} \frac{d^j a(S_T)}{dS_T^j} \quad (4.5)$$

where  $E_j$  are the error terms given in terms of the difference between the cumulants of the  $f(S_T)$  and the  $a(S_T)$  distributions. The first four coefficients are:

$$E_1 = (\kappa_1(F) - \kappa_1(A))$$

$$E_2 = (\kappa_2(F) - \kappa_2(A)) + E_1^2$$

$$E_3 = (\kappa_3(F) - \kappa_3(A)) + 3E_1(\kappa_2(F) - \kappa_2(A)) + E_1^3$$

$$\begin{aligned} E_4 = & (\kappa_4(F) - \kappa_4(A)) + 4(\kappa_3(F) - \kappa_3(A))E_1 + 3(\kappa_2(F) - \kappa_2(A))^2 \\ & + 6E_1^2(\kappa_2(F) - \kappa_2(A)) + E_1^4 \end{aligned}$$



A finite series expansion of order 4 for the  $f(S_T)$  in terms of  $a(S_T)$  is expressed as

$$\begin{aligned}
 f(S_T) = & a(S_T) - \frac{(\kappa_1(F) - \kappa_1(A))}{1!} \frac{da(S_T)}{dS_T} \\
 & + \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \frac{d^2a(S_T)}{dS_T^2} - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3a(S_T)}{dS_T^3} \\
 & + \frac{(\kappa_4(F) - \kappa_4(A)) + 3(\kappa_2(F) - \kappa_2(A))^2}{4!} \frac{d^4a(S_T)}{dS_T^4} + \varepsilon_N(S_T)
 \end{aligned} \tag{4.7}$$

Cumulants are similar to moments<sup>21</sup>. In fact, the first cumulant of a distribution is equal to its mean, the second is equal to its variance, the third and the fourth are measures of skewness and kurtosis respectively. The leading term in Equation (4.7) is the *reference* or *approximating* distribution which is adjusted to reflect any differences between its cumulants and the cumulants of the *true* distribution. The second term in the expansion adjusts  $a(S_T)$  to account for the differences between the mean of the *true* and the *approximating* distributions. Similarly the third and the fourth terms adjust  $a(S_T)$  to reflect any differences between the variance and the skewness of  $f(S_T)$  and  $a(S_T)$  respectively. The fifth term corrects for differences in the kurtosis and the variance of the two distributions. The residual error  $\varepsilon_N(S_T)$

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<sup>21</sup> The characteristic function of a probability distribution  $\phi(\tau) = E(e^{i\tau x})$  is calculated with the following Taylor series expansion

$$\phi(\tau) = \exp \left\{ \sum_{r=1}^{\infty} \frac{\kappa_r \tau^r}{r!} \right\} = \sum_{r=0}^{\infty} \frac{\mu_r \tau^r}{r!}$$

Whereas  $\mu_j$  are the coefficients of  $\frac{(it)^j}{j!}$ ,  $\kappa_j$  are the coefficients of  $\frac{(it)^j}{j!}$  in  $\log \phi(\tau)$ .

contains any remaining difference between the moments of the two distributions of order higher than 4.

Selection of a two parameter reference distribution allows any two cumulants of the true and the approximating distribution to be matched to corresponding cumulants of the distribution to be approximated. The two parameter log-normal distribution, whose pre-eminence in the option pricing literature is an undoubted fact, seems to be a very appealing candidate for the role of the *reference* distribution  $a(S_T)$ . This choice also ensures that higher order terms in the remainder  $\varepsilon_N(S_T)$  become negligible<sup>22</sup>. Setting the first cumulants of the *true* and the *reference* distributions equal, the desired functional form of the *true* density becomes:

$$f(S_T) = a(S_T) + \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \frac{d^2 a(S_T)}{dS_T^2} + \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3 a(S_T)}{dS_T^3} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^4 a(S_T)}{dS_T^4} \quad (4.8)$$

where

$$a(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$$

is a log-normal distribution with  $\mu = \log S_t + (r - q - \sigma^2/2)\tau$ ,  $S_t$  being the current price of the underlying asset,  $r$  the risk-free interest rate,  $q$  the continuous dividend yield and  $\sigma$  the expected volatility of the underlying asset's returns.

A natural question that arises when such expansions are truncated to a finite number of terms is whether the number of terms considered is adequate to ensure

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<sup>22</sup> See Jarrow and Rudd (1982) for an analysis on the relative size of the residual error  $\varepsilon_N(S_T)$ :

convergence of the series. Schleher (1977) suggests that the series converges for a small number of terms and then diverges. In trying to identify a rule of thumb for the number of terms that need to be considered in the expansion he quotes that ‘... the ability to pick a reference distribution that closely matches the distribution to be approximated invariably results in high accuracy being attained by including error terms one beyond the highest order matching cumulant.’ The choice to truncate (4.7) up to the 4<sup>th</sup>- order term is on the other hand also intuitively substantiated. The third and the fourth terms in (4.8) adjust  $a(S_T)$  to reflect any differences between the skewness and the kurtosis of  $f(S_T)$  and  $a(S_T)$ . These differences have well been documented in the literature of empirical distributions of asset price returns and most of the density parameterisations proposed aim to primarily account for these features. The assumption  $\kappa_1(F) = \kappa_1(A)$  is predetermined by the risk-neutrality argument.

Although a satisfactory approximation of the *true* distribution can be achieved with a finite number of terms as suggested above, there is one inherent disadvantage that has to be dealt with when the methodology is used. The sum of a finite number of terms of the series in Equation 4.8 may give negative frequencies, particularly near the tails. An investigation by Barton and Dennis (1952) offers a locus of possible combinations of skewness and kurtosis coefficients guaranteeing density positivity for a similar expansion, the Gram-Charlier expansion. It follows that finite series expansions perform well in cases of moderate skewness - within the range of  $-1.2$  to  $+1.2$ . While the methodology used in the present study ensures that the



finite series approximation of Equation 4.8 satisfies conditions in order to represent a density function (see Section 4.2.5), Stuart and Ord (1987) suggest that we can overcome inadequacies of this nature by using alternative series expansions. Potential candidates include the expansion presented in Gray et al. (1975) and the saddlepoint approximation of Barndorff-Nielsen and Cox (1979) which can possibly improve on Edgeworth expansion in the tails of the distribution.

#### 4.2.2 Generalised pricing formulae for European options

For ease of reference the framework developed in Section 2.4 is recalled. Ross (1976) and Cox and Ross (1976) have shown that in a market that offers no arbitrage possibilities, the value of an option can be expressed with the following:

$$V = e^{-r\tau} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T$$

where  $S_T$  is the asset price at the maturity of the option,  $g(S_T)$  is the payoff function and  $f(S_T)$  is the risk-neutral probability distribution function of the asset price at maturity. Let the terminal payoff of a European call option on the asset maturing at time  $T$  be  $\max(S_T - K, 0)$ , given a terminal asset price  $S_T$  and a strike price  $K$ . Assuming that the interest rate  $r$  is constant over the remaining life of the option  $\tau = (T - t)$ , then the price of the call option is the discounted pay-off (conditional upon finishing in the money), multiplied by the probability of finishing in the money:

$$c(K) = e^{-r\tau} \int_K^{\infty} f(S_T)(S_T - K)dS_T \quad (4.9)$$



where  $f(S_T)$  is the RND function of the underlying asset price at maturity.

Similarly, the current price of a put option with terminal pay-off function  $\max(K - S_T, 0)$  is:

$$p(K) = e^{-r\tau} \int_0^K f(S_T)(K - S_T)dS_T \quad (4.10)$$

Given a tractable analytic functional form for the probability distribution function  $f(S_T)$  closed form pricing formulae can be derived for the options' values.

The present study makes the assumption that the true RND is adequately approximated by an the *ESE* specification presented in the previous section. These assumptions allow the derivation of closed form expressions for the values of European style call and put options written on an asset that provides a continuous dividend yield (or any other distribution) at a rate  $q$ . It follows that the option valuation formulae for a call or a put option with strike price  $K$  are the following:

$$c_F(K) = c_A(K) + e^{-r\tau} \left[ \frac{(\kappa_2(F) - \kappa_2(A))}{2!} a(K) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} + \frac{(\kappa_4(F) - \kappa_4(A)) + 3((\kappa_2(F) - \kappa_2(A))^2)}{4!} \frac{d^2a(K)}{dK^2} \right] + \epsilon_{c(K)}, \quad (4.11)$$

$$p_F(K) = p_A(K) + e^{-r\tau} \left[ \frac{(\kappa_2(F) - \kappa_2(A))}{2!} a(K) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} + \frac{(\kappa_4(F) - \kappa_4(A)) + 3((\kappa_2(F) - \kappa_2(A))^2)}{4!} \frac{d^2a(K)}{dK^2} \right] + \epsilon_{p(K)}, \quad (4.12)$$

where

$$c_A(K) = S_t e^{-q\tau} N(d_1) - e^{-r\tau} K N(d_2)$$

$$p_A(K) = K e^{-r\tau} N(-d_2) - S_t e^{-q\tau} N(-d_1)$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

and

$$N(\cdot) = \text{cumulative standard normal}$$

Expressions (4.11) and (4.12) give the approximate value of the option based on a leading term, which corresponds to the BSM value of the option and adjustment terms. The first term corrects for possible differences between the true and the log-normal distributions while the second and the third adjustment terms correct for skewness and kurtosis differences respectively. The terms  $\epsilon_{c(K),N}$  and  $\epsilon_{p(K),N}$  contain any residual error due to the finite number of adjustment terms in (4.11) and (4.12).

### 4.2.3 Generalised pricing formulae for American options

The early exercise premium that is embedded in American options does not allow the application of techniques described in Section 4.2.2. The American option value depends on the entire stochastic process of the underlying asset, thus, making difficult the recovery of implied RNDs. To deal with this deficiency the value of an American

option can be expressed as a combination of an upper and a lower bound, which restrict it within a very tight range.

Chaudhury and Wei (1994) and Melick and Thomas (1997), present upper and lower bounds which restrict the value of the option within a very tight range. The equations giving the upper  $[\overline{C}, \overline{P}]$  and lower  $[\underline{C}, \underline{P}]$  bounds for call and put options, respectively, are the following:

$$\overline{C}(K) = E_t [\max \{0, S_T - K\}] \quad (4.14a)$$

$$\underline{C}(K) = \max \{E_t [S_T] - K, e^{-r\tau} E_t [\max \{0, S_T - K\}]\} \quad (4.14b)$$

$$\overline{P}(K) = E_t [\max \{0, K - S_T\}] \quad (4.14c)$$

$$\underline{P}(K) = \max \{K - E_t [S_T], e^{-r\tau} E_t [\max \{0, K - S_T\}]\} \quad (4.14d)$$

where  $S_T$  denotes the arbitrary asset price at the expiration of the option and  $K$  denotes the option's strike price. Time periods are indexed prior to the option's expiration so that  $\tau = T - t$  expresses the total time to expiration.  $E_t [\cdot]$  represents the expectation taken with respect to the RND that the asset returns are assumed to satisfy at current time  $t$ ;  $r$  is the risk free interest rate.

Equation (4.14a) is equal to the undiscounted otherwise equivalent<sup>23</sup> European call option price whereas Equation (4.14b) calculates the lower bound as the maximum between the call option's intrinsic value and the value of an otherwise European style call option. Similarly, Equations (4.14c) and (4.14d) represent the undiscounted

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<sup>23</sup> Equivalent in the sense that the option has exactly the same characteristics with the American option apart from the early exercise feature.



otherwise equivalent European style put option and the maximum of the option's intrinsic value and the value of an otherwise European style put option respectively. Using the bounds described by Equations (4.14a) - (4.14d) the prices of American options can be expressed in term of 'weighted' prices of otherwise equivalent European style options and the formulas derived in Section 4.2.2 can therefore be used for our analysis<sup>24</sup>.

Applying this technique to derive implied PDFs from American options has both costs and benefits. The major cost is that the methodology is limited to American style futures options - of which kind, however, most of the Exchange traded contracts are. Chaudhury and Wei (1994) suggest that while Equation (4.14a), is also an upper bound when the optioned asset is a stock with no dividends prior to maturity the case is not the same with Equation (4.14c). On the other hand the most important benefit of the technique arises from its flexibility, generality and directness. A large number of alternative probability models can be considered as the bounds of the option values are expressed in terms of the terminal distribution alone. The flexibility of the *ESE* parameterisation ensures additional flexibility<sup>25</sup>. The main

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<sup>24</sup> The expressions that give the value for the upper and lower bounds essentially differ only by the discount factor. An immediate implication is that the bounds will be extremely tight in situations when interest rates (as expressed by  $r$ ) are low and maturity is close. Even if interest rates are high and maturity time is long, the above equations adequately bound the American option price, as discussed in Melick and Thomas (1997).

<sup>25</sup> Longstaff (1995) suggests that this general option pricing model includes many other option pricing models as special cases. Examples of models that are nested within this general model include: the Black and Scholes (1973) model, the Merton (1973) stochastic interest rate model, the Merton (1976, eq. 17) jump diffusion model, the Merton (1976, eq. 18) jump diffusion model. In addition, since the risk-neutral density can match the first four moments of any continuous density, the four parameters of the model can be chosen in such a way that closely approximates most existing option



advantage of this set of bounds is that they are expressed in terms of the terminal distribution alone<sup>26</sup>.

The present study, however, deals primarily with the inverse problem: *given a set of observed option prices what are the parameters of an ESE specification for the RND that best fit the data.*

In connection to the latter, the methodology will prove useful when a more qualitative/fundamental economic question is addressed<sup>27</sup>. Examples of such studies include Leahy and Thomas (1996), who derive the RND of the Canadian dollar - US dollar exchange rate during the October 1995 referendum on Quebec; Melick and Thomas (1997), who estimate the PDFs implied by crude oil options during the Gulf War to recover the market sentiment during the crisis; Adao and Barros Luis

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pricing models.

<sup>26</sup> An alternative technique would be to calculate implied volatilities using an 'American' BSM model i.e. the BAW approximation or a binomial tree model and use them to calculate pseudo-European option prices with the BSM model. The implied PDFs could then be calculated using the pseudo-European prices without any adjustment.

While such a technique is set out to account for the early exercise premium some inherent deficiencies can be identified. The technique involves one extra step - that of calculating 'American' BSM implied volatilities - compared to the technique used in the present study, which makes it more complicated as a whole and gives rise to some concerns. A minor concern lies on the fact that the estimation of implied volatilities is done through numerical procedures. Any errors associated with the numerical algorithm used, therefore, may be carried through and may eventually distort the shape of the implied PDFs. A major concern - especially in the context of implied PDFs - arise from the fact that the technique assumes that the early exercise premium could be solely attributed to the implied volatility. This obviously follows from the distributional assumptions of the BAW and the binomial trees models suggesting that the distribution of the underlying asset is about log-normal and log-normal in the limit respectively. And while a more flexible parametrisation would help to eventually recover some part of this information from the pseudo-European prices we would not know what amount of the original information is missed.

<sup>27</sup> The use of the methodology may not seem ideal in the context of pricing options contracts, as a naive trader's model which assumes a constant implied BSM volatility smile can possibly work equally well, as Jackwerth and Rubinstein (1998) show for implied binomial trees, a number of parametric models and naive trader rules.

(1999), who examine the evolution of the expectations of interest rates convergence around the transition of Italy and Spain towards the EMU; and Gemmill and Saflekos (2000), who study RNDs implied by the Equity Index options around crashes, the British general elections, and extraordinary events.

#### **4.2.4 Estimation of the implied RND**

The estimation procedure essentially extends the methodology presented in Corrado and Su (1996). Edgeworth series expansions, being polynomial approximations, have the drawback, as already highlighted, of possibly yielding negative values for the RNDs for certain combinations of parameters. Moreover, there does not seem to be an easy and analytic characterization of those parameters for which the density will not take negative values. In addition there is no condition that ensures the series expansion expressed by (4.8) integrates to unity.

The approach proposed in this study, rather than deriving complicated mathematical expressions to guarantee positivity, imposes the restriction the density be always positive as a constraint implicit in the constraint optimization algorithm, used to recover the RNDs' parameters. In addition, the constraint that the recovered density's integral over its domain equals unity is being imposed. The above mentioned constraints are sufficient exclusively with regard to the RND as they allow RNDs with physical as well as mathematical meaning to be recovered<sup>28</sup>.

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<sup>28</sup> The imposed constraints further reduce the degrees of freedom of the system which as Melick (1999) pp.3-4 discusses is of complicated nature.

Jarrow and Rudd (1982) argue that imposing the condition  $\kappa_1(F) = \kappa_1(A)$  is predetermined by risk-neutrality arguments. This leaves the second cumulants of the true and the approximating RNDs unrestricted and Jarrow and Rudd (1982) suggest equating the second cumulants as well. This argument is also justified on numerical grounds by Corrado and Su (1996) who notice that, without this condition, there will exist a problem of multicollinearity between the second and the fourth moments in the estimation of the implied model.

Following the above, let the value of a call option<sup>29</sup> on a given date be  $c(K)$ .

By dropping the remainder term  $\epsilon_{c(K),N}$  Equation (4.11)

$$c_F(K) = c_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} - \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K)}{dK^2} \right]$$

can be written in a simplified way as

$$c_F(K) = c_A(K) + \lambda_1 Q_3 + \lambda_2 Q_4 \quad (4.15)$$

where the terms  $\lambda_i$  for  $i = 1, 2$  and  $Q_j$  for  $j = i + 2$  are defined as follows

$$\lambda_1 = \gamma_1(F) - \gamma_1(A) \quad (4.16a)$$

$$\lambda_2 = \gamma_2(F) - \gamma_2(A) \quad (4.16b)$$

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<sup>29</sup> Identical procedures are employed to express the value of a put option.



and

$$Q_3 = - [S_t e^{(r-q)\tau}]^3 \left( e^{\sigma^2 \tau} - 1 \right)^{\frac{3}{2}} \frac{e^{-r\tau}}{3!} \frac{da(K)}{dK}$$

$$Q_4 = [S_t e^{(r-q)\tau}]^4 \left( e^{\sigma^2 \tau} - 1 \right)^2 \frac{e^{-r\tau}}{4!} \frac{d^2 a(K)}{dK^2}$$

In Equation (4.16a)  $\gamma_1(F)$  and  $\gamma_1(A)$  are skewness coefficients. Similarly in Equation (4.16b)  $\gamma_2(F)$  and  $\gamma_2(A)$  are excess kurtosis coefficients. Skewness and excess kurtosis coefficients are defined in terms of the cumulants as follows

$$\gamma_1(A) = \frac{\kappa_3(A)}{\kappa_2^{3/2}(A)} \quad \text{and} \quad \gamma_2(A) = \frac{\kappa_4(A)}{\kappa_2^2(A)}$$

where the cumulants of the approximating (log-normal) distribution  $A$  are given by

$$\kappa_1(A) = S_t e^{(r-q)\tau}$$

$$\kappa_2(A) = [\kappa_1(A)\varrho]^2$$

$$\kappa_3(A) = [\kappa_1(A)\varrho]^3 (3\varrho + \varrho^3)$$

$$\kappa_4(A) = [\kappa_1(A)\varrho]^4 (16\varrho^2 + 15\varrho^4 + 6\varrho^6 + \varrho^8)$$

and  $\varrho \equiv \left( e^{\sigma^2 t} - 1 \right)^{\frac{1}{2}}$ ,  $\sigma$  being the (implied) volatility of the underlying asset.

Let now  $z(\sigma, \lambda_1, \lambda_2)$  denote a vector of the unknown parameters of  $f[\cdot]$  and let  $(w_1, w_2)$  denote the weights that describe where the actual option price falls between the bounds. Combining equations (4.14a) - (4.14d) with (4.11) and (4.12) and weighting the bounds<sup>30</sup> results in the following pricing equations for the American

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<sup>30</sup> Two weights are used; one for options in-the-money and one for options out-of-the-money. This is justified by the fact that pricing errors are expected to vary systematically with the degree to which



futures options, in terms of the 5 estimated parameters  $(\hat{z}, \hat{w}_1, \hat{w}_2)$ , four observables  $S_t, K, r$  and  $q$ , and an error term:

$$C(K) = \hat{w}_{it} \bar{C}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{C}(K; \hat{z}) + \tilde{\epsilon}'_{C(K)} \quad (4.19)$$

$$P(K) = \hat{w}_{it} \bar{P}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{P}(K; \hat{z}) + \tilde{\epsilon}'_{P(K)} \quad (4.20)$$

where

$$i = \begin{cases} 1 & \text{if } \begin{cases} \text{call and } K < S_t \\ \text{put and } K > S_t \end{cases} \\ 2 & \text{otherwise} \end{cases}$$

#### 4.2.5 Constraint Optimization

On any given day, the implicit parameters  $(\hat{z}, \hat{w}_1, \hat{w}_2)$  are estimated by minimizing the sum squared errors:

$$\min_{(\hat{z}, \hat{w}_1, \hat{w}_2)} \sum_{K_i} \{ [C_{OBS}(K_i) - C(K_i)]^2 + [P_{OBS}(K_i) - P(K_i)]^2 \} \quad (4.21)$$

with

$$E_t [S_T] \equiv S_t e^{(r-q)\tau} \quad \text{risk-neutrality argument} \quad (4.22)$$

where  $C(K_i)$  and  $P(K_i)$  are the theoretical call and put prices, respectively, as expressed in Equations (4.19) and (4.20). In the objective function, both the call and put options are included. This is not common in existing studies, e.g. Corrado and Su (1996, 1997) use only the call or only the put options to back out the implied parameters of their model, assuming that the Put/Call parity holds. This approach has

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an option is in- or out-of-the-money as pointed out by Melick and Thomas (1997).

the inherent disadvantage that parameters estimated separately for call or put options may possibly yield two different RNDs if the Put/Call parity is not satisfied. This implies that market participants have different perception of future outcomes when they trade different type of options or, on the other hand, that their view towards risk is depended upon the type of contract they trade. The latter is in contrast with the risk neutral valuation approach, a fundamental assumption of which is that risk aversion is common among agents.

The estimation procedure differs from Corrado and Su (1996) also in that it imposes additional constraints on the parameter values recovered from (4.21) so that Equation (4.8) is always a valid PDF. Two additional conditions should be satisfied for Equation (4.8) to represent a valid PDF. It should always be positive and have an integral over its domain equal to one. Formally:

$$f(S_T) \geq 0 \quad \forall S_T, S_T \in [0, \infty) \quad (4.23)$$

$$\int f(S_T) = 1 \quad (4.24)$$

These two constraints should always be imposed whether the parameters recovered with Equation (4.21) are used only for pricing purposes as in Corrado and Su (1996), or in order to recover the risk neutral probability density function.

The optimization is performed using MATLAB<sup>®</sup>'s constrained optimization routine which requires the function to be minimized, upper and lower bounds for

the parameters to be estimated, and any other constraints that should apply as inputs. Bounds for the parameters were set so that  $0.01 \leq \sigma \leq 0.7$ ,  $-2 \leq \lambda_1 \leq 2$ , and finally  $0 \leq \lambda_2 \leq 7$ . The additional constraints included Equations (4.23) and (4.24), the latter one requiring the associated integral to equal unity with a tolerance of  $\pm 0.001$ . The algorithm focuses on the relevant Kuhn-Tucker equations and attempts to compute directly the Lagrange multipliers, using a quasi-Newton updating procedure. Methods of this nature are commonly referred to as Sequential Quadratic Programming methods and represent the state-of-the-art in non-linear programming methods<sup>31</sup>.

To examine the impact of not imposing the constraints given by Equations (4.23) and (4.24), the optimization procedure was run with - and without - imposing the constraints. The results indicated that it is essential that the constraints be imposed. Figures 4.1 and 4.2 show how the shape and the information conveyed by the risk neutral density function would significantly differ, depending on the way the optimization was carried out. For economy of space results are plotted only for one of the contracts examined.

In Figure 4.1a the parameters were recovered without imposing the constraint that Equation (4.8) be always positive. This has led to negative values of the PDF for certain values of the underlying asset, which is mathematically not feasible. An apparent practical implication of recovering a PDF with the shape of Figure 4.1a

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<sup>31</sup> See Schittowski (1985).

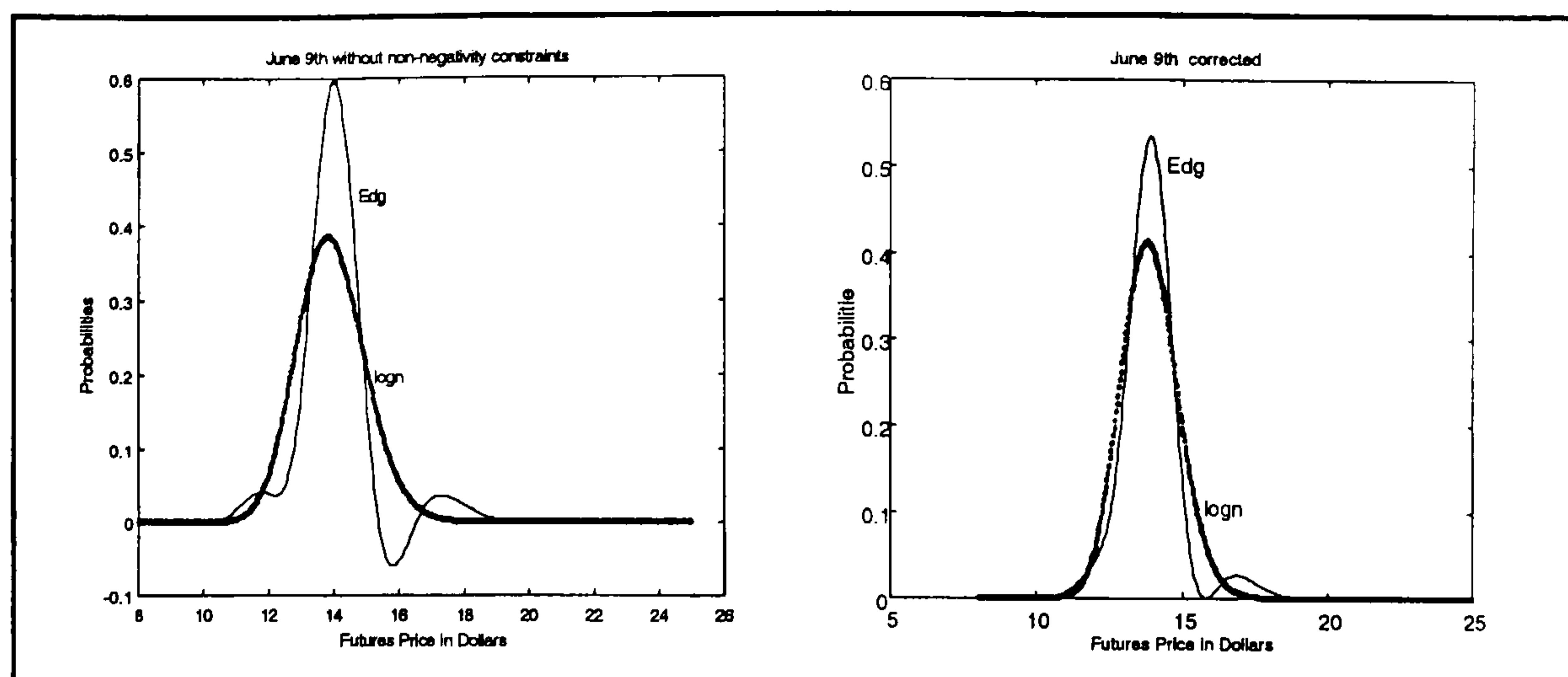


Figure 4.1: Probability distributions recovered for the WTI July 98 futures contracts on June 9th without and with positivity constraints

would be the overestimation of the probability that the underlying asset takes values below 15, resulting in overpricing of put options with strike prices within that range. As is obvious from Figure 4.1a, the distribution is more peaked in the former, having greater mass below the value of 15 to compensate for the fact that it becomes partly negative beyond that point, while forced to maintain an integral equal to 1 at all times.

A situation very similar to the one described in the previous paragraph arises if the restriction, that the integral of the recovered distribution function be equal to one, is not imposed.

In Figure 4.2a, the parameters have been recovered without imposing the constraint that the integral of the distribution recovered be equal to one. Imposing this restriction leads to a significantly different shape for the implied risk neutral PDF depicted in Figure 4.2b. The shape of the distribution in this figure approaches more that of the fitted log-normal. It is less leptokurtic and its right tail is considerably



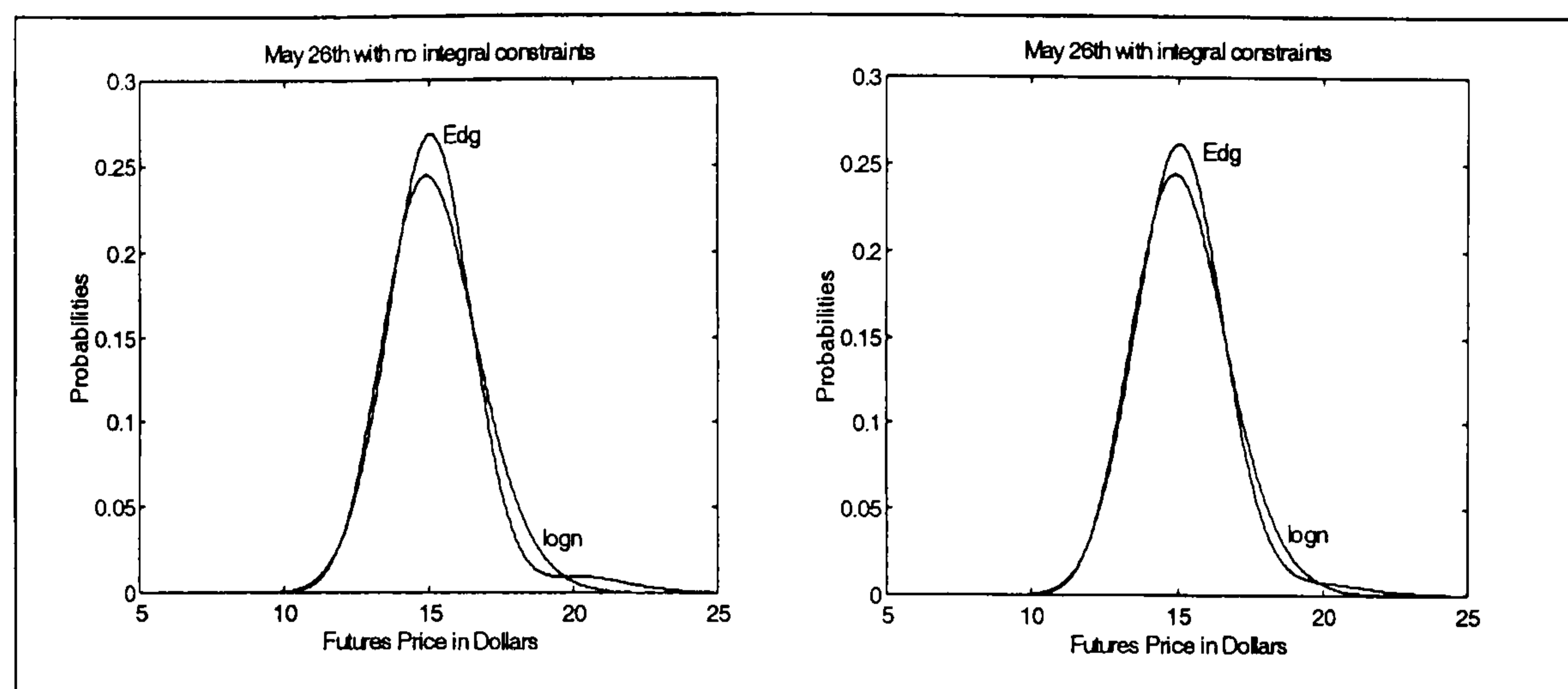


Figure 4.2: Probability distributions recovered for the WTI July 98 futures contracts on May 26th without and with integral constraints

smoother. For a policy maker, this implies that, relying on a distribution of Figure 4.2a to extract the market sentiment, he would find that there are two possible scenarios in the market, indicating a strong bimodality, while the situation would be much more ‘normal’ than would have been thought.

#### 4.2.6 Evolution of *ESE* implied PDFs

This section demonstrates the evolution of *ESE* implied PDFs over time. Figure 4.3 shows the evolution of the PDF implied by the July 98 WTI futures options contract from May 1, 1998 through to June 5, 1998 on a weekly basis. Figure 4.4 plots PDFs implicit in options on December 1998 futures contracts for the period October 1, 1998 through to November 5, 1998 weekly. While Figures 4.3 and 4.4 suggest that the *ESE* implied PDFs are not too far from being log-normal (*LGN*) a positive skewness and higher-than-the log-normal kurtosis patterns are apparent. Figures 4.5 and 4.6 show the evolution the PDF implied in December 98 Eurodollar

futures options for the period September 1, 1998 to November 17, 1998 weekly. The difference between the *ESE* and the BSM *LGN* implied PDFs are somewhat more profound than in Figures 4.3 and 4.4. The data shows large negative skewness and leptokurtosis which result in the allocation of higher probabilities in the states below approximately 4.5% compared to the probabilities assigned by the BSM model. The shape of the implied PDFs is also explained by the patterns of the volatility ‘smiles’ in Figures 4.1 to 4.5.

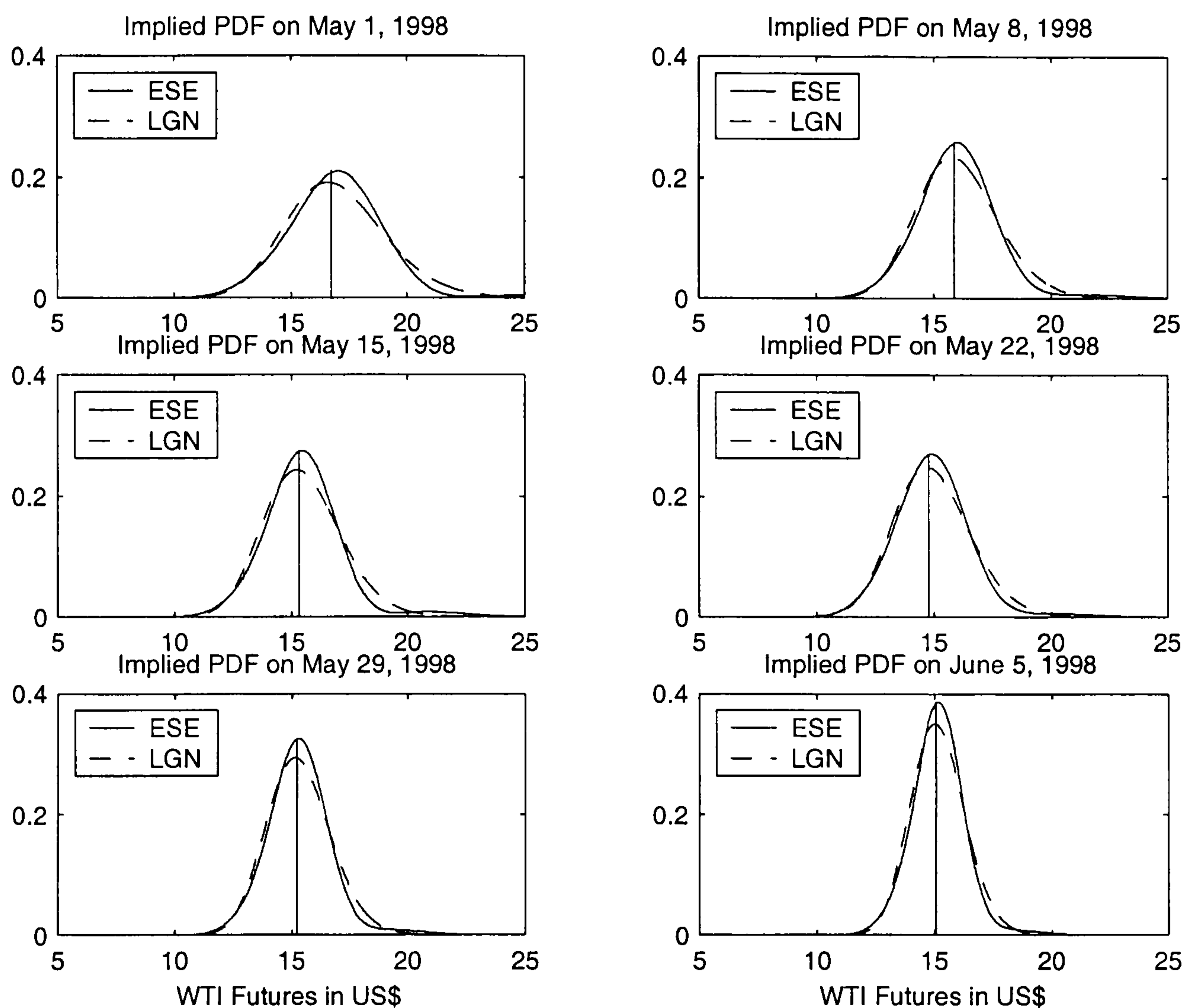


Figure 4.3: Implied PDFs calculated from July 98 WTI options from May 1, 1998 through to June 5, 1998 for the *ESE* and the *LGN* parametrisations

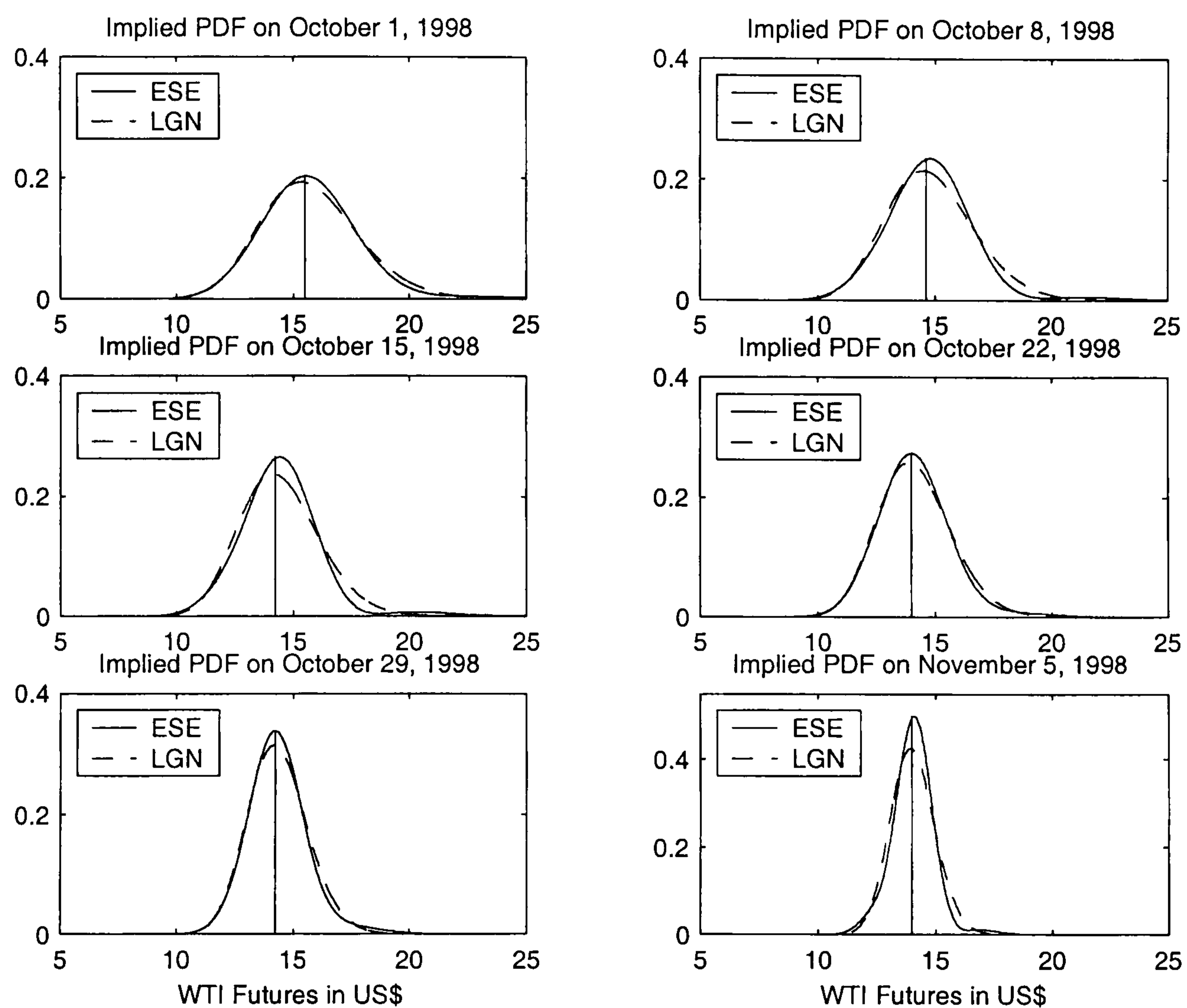


Figure 4.4: Implied PDFs calculated from December 98 WTI options from October 1, 1998 through to November 5, 1998 for the *ESE* and the *LGN* parametrisations



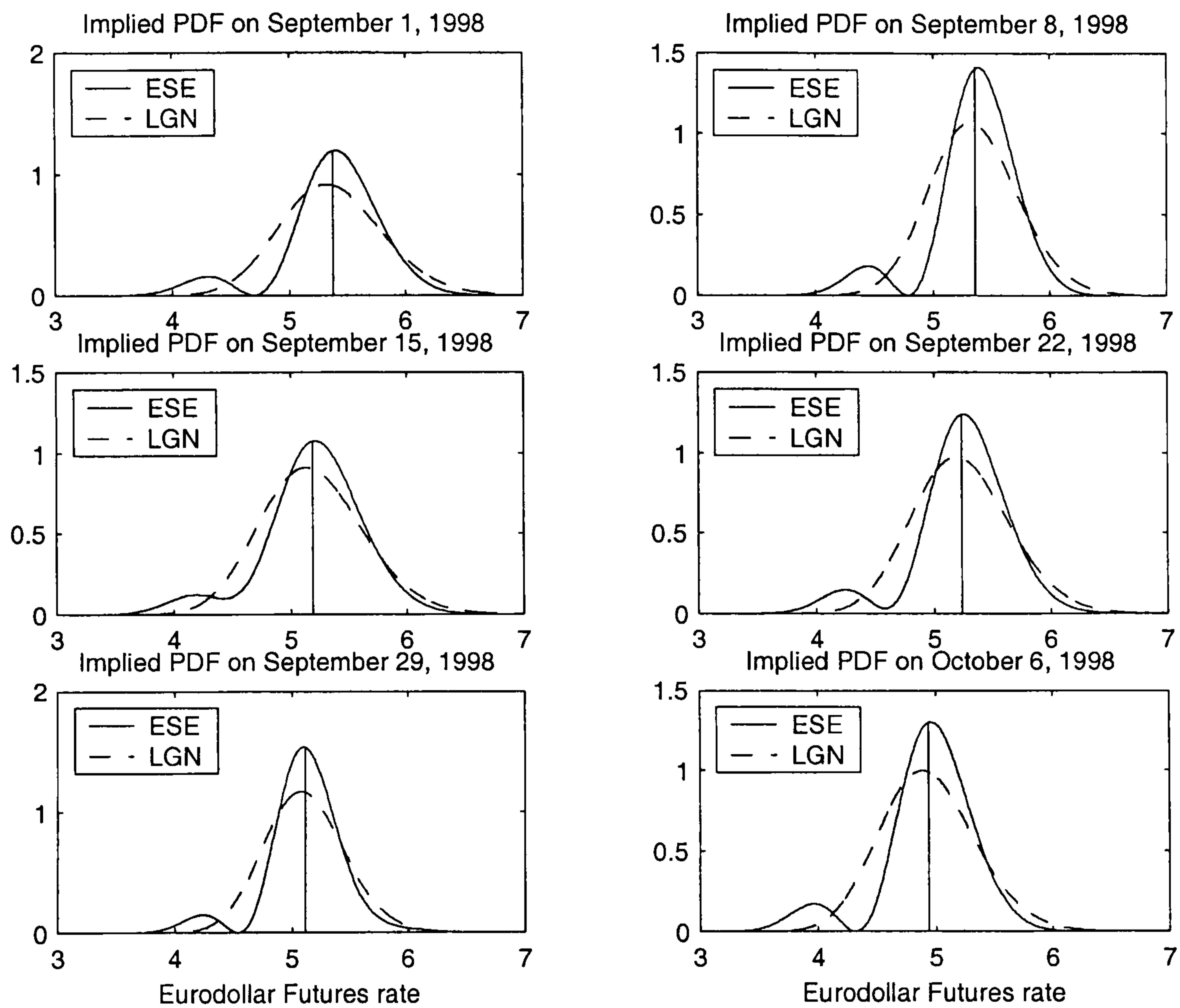


Figure 4.5: Implied PDFs calculated from December 98 Eurodollar options from September 1, 1998 through to October 6, 1998 for the *ESE* and the *LGN* parametrizations

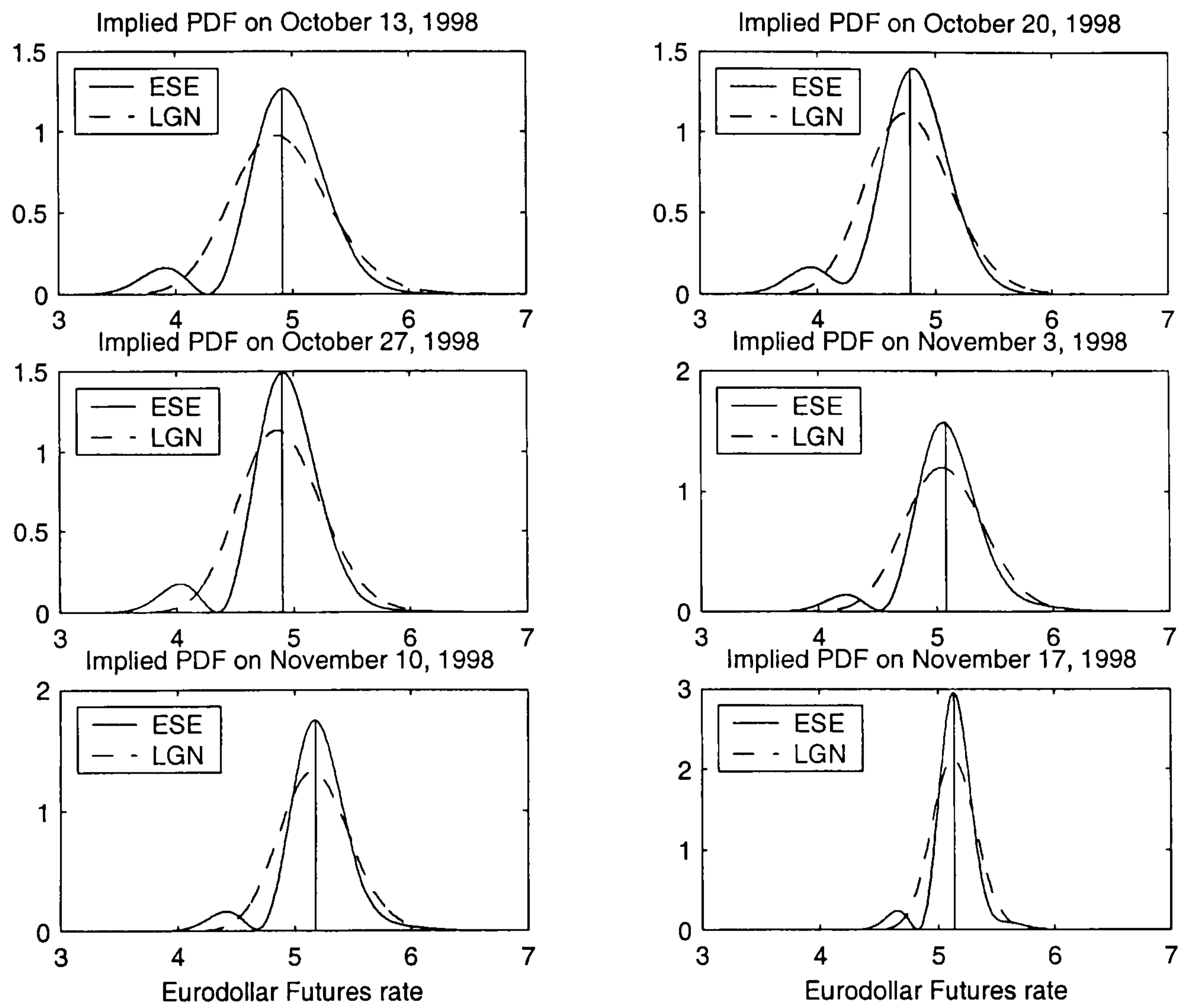


Figure 4.6: Implied PDFs calculated from December 98 Eurodollar options from October 13, 1998 through to November 17, 1998 for the *ESE* and the *LGN* parametrizations

## 4.A Derivations and proofs

### 4.A.1 Asymptotic Distribution Expansion

The Edgeworth group of asymptotic distribution expansion is based on the approximation of the characteristic function of a probability density. Once the characteristic function has been approximated the probability distribution function can be obtained with the application of the inverse Fourier transform.

Consider an arbitrary distribution  $F(S_T)$  called the *true* distribution and a given probability distribution  $A(S_T)$ , called the *approximating*<sup>32</sup>. The analysis that follows is carried out for the restricted class of distributions with continuous density functions, that is  $dA(S_T)/dS_T = a(S_T)$  and  $dF(S_T)/dS_T = f(S_T)$  exist.

The  $j^{\text{th}}$  moment  $\alpha_j(F)$ , the  $j^{\text{th}}$  central moment  $\mu_j(F)$ , and the characteristic function  $\phi(F, \tau)$  of the *approximated distribution*  $F(S_T)$ , are given by

$$\begin{aligned}\alpha_j(F) &= \int_{-\infty}^{\infty} S_T^j f(S_T) dS_T \\ \mu_j(F) &= \int_{-\infty}^{\infty} (S_T - \alpha_j(F))^j f(S_T) dS_T \\ \phi(F, \tau) &= \int_{-\infty}^{\infty} e^{i\tau S_T} f(S_T) dS_T\end{aligned}$$

where  $i^2 = -1$  and  $\alpha_j(F)$  exists for  $j \leq n$ .

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<sup>32</sup> The derivations for the call option formulas herein are presented in Jarrow and Rudd (1982).

Given  $\alpha_n(F)$  exists, the first  $n - 1$  cumulants  $\kappa_j(F)$  from  $j = 1, \dots, n - 1$  also exist. These are defined by

$$\log \phi(F, \tau) = \sum_{j=1}^{n-1} \kappa_j(F) \left\{ \frac{(i\tau)^j}{j!} \right\} + o(\tau^{n-1}) \quad (4.25)$$

where

$$o(\tau^{n-1}) \text{ satisfies } \lim_{\tau \rightarrow 0} o(\tau^{n-1})/\tau^{n-1} = 0$$

From (4.25), letting  $N = \inf(n, m)$ ,

$$\log \phi(F, \tau) = \sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A)) \left\{ \frac{(i\tau)^j}{j!} \right\} + \sum_{j=1}^{N-1} \kappa_j(A) \left\{ \frac{(i\tau)^j}{j!} \right\} + o(\tau^{N-1})$$

But for the approximating distribution

$$\sum_{j=1}^{N-1} \kappa_j(A) \left\{ \frac{(i\tau)^j}{j!} \right\} = \log \phi(A, \tau) + o(\tau^{N-1})$$

and by substitution

$$\log \phi(F, \tau) = \sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A)) \left\{ \frac{(i\tau)^j}{j!} \right\} + \log \phi(A, \tau) + o(\tau^{N-1}) \quad (4.26)$$

By taking exponentials and using  $e^{o(\tau^{N-1})} = 1 + o(\tau^{N-1})$ , equation (4.26) transforms into

$$\phi(F, \tau) = \exp \left\{ \sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A)) \left\{ \frac{(i\tau)^j}{j!} \right\} \right\} \phi(A, \tau) + o(\tau^{N-1}) \quad (4.27)$$

Stuart and Ord (1987), p.228, discuss the sufficient conditions that a density function with a continuous derivative should satisfy in order to be developed in a series. Jarrow and Rudd (1982) suggest that the fact that  $\exp \{ \cdot \}$  is an analytic function ensures



that  $\phi(F, \tau)$  can analytically be expanded as an infinite polynomial. Consequently there exists  $E_j, j = 0, 1, \dots, N - 1$ , such that

$$\exp \left\{ \sum_{j=1}^{N-1} (\kappa_j(F) - \kappa_j(A)) \left\{ \frac{(i\tau)^j}{j!} \right\} \right\} = \sum_{j=1}^{N-1} E_j \frac{(i\tau)^j}{j!} + o(\tau^{N-1}) \quad (4.28)$$

The first five coefficients are

$$E_1 = (\kappa_1(F) - \kappa_1(A)) \quad (29a)$$

$$E_2 = (\kappa_2(F) - \kappa_2(A)) + E_1^2 \quad (29b)$$

$$E_3 = (\kappa_3(F) - \kappa_3(A)) + 3E_1 (\kappa_2(F) - \kappa_2(A)) + E_1^3 \quad (29c)$$

$$E_4 = (\kappa_4(F) - \kappa_4(A)) + 4 (\kappa_3(F) - \kappa_3(A)) E_1 + 3 (\kappa_2(F) - \kappa_2(A))^2 + 6E_1^2 (\kappa_2(F) - \kappa_2(A)) + E_1^4 \quad (29d)$$

The substitution of (4.28) into (4.27) and the fact that  $\lim_{\tau \rightarrow 0} \phi(A, \tau) = 1$ , give

$$\phi(F, \tau) = \sum_{j=1}^{N-1} E_j \frac{(i\tau)^j}{j!} \phi(A, \tau) + o(\tau^{N-1}) \quad (4.30)$$

The inverse Fourier transform of (4.30) using also

$$\begin{aligned} f(S_T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau S_T} \phi(F, \tau) d\tau \\ a(S_T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau S_T} \phi(A, \tau) d\tau \\ (-1)^j \frac{d^j a(S_T)}{dS_T^j} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\tau S_T} (i\tau)^j \phi(A, t) d\tau \end{aligned}$$

gives

$$f(S_T) = a(S_T) + \sum_{j=1}^{N-1} E_j \frac{(-1)^j}{j!} \frac{d^j a(S_T)}{dS_T^j} + \varepsilon_N(S_T) \quad (4.31)$$

where

$$\varepsilon_N(S_T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau S_T} (i\tau)^j o(\tau^{N-1}) d\tau$$

The error term exists since  $\sum_{j=0}^{N-1} E_j \frac{(-1)^j}{j!} \frac{d^j a(S_T)}{dS_T^j}$ ,  $a(S_T)$  and  $f(S_T)$  are finite.

The approximated probability distribution is then given by substituting Equations (4.29a) - (4.29d) into (4.31)

$$\begin{aligned} f(S_T) = & a(S_T) - \frac{(\kappa_1(F) - \kappa_1(A))}{1!} \frac{da(S_T)}{dS_T} \\ & + \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \frac{d^2 a(S_T)}{dS_T^2} - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3 a(S_T)}{dS_T^3} \\ & + \frac{(\kappa_4(F) - \kappa_4(A)) + 3(\kappa_2(F) - \kappa_2(A))^2}{4!} \frac{d^4 a(S_T)}{dS_T^4} + \varepsilon_N(S_T) \end{aligned} \quad (4.32)$$

where  $a(S_T)$  is the log-normal distribution defined by

$$a(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$$

$\mu = \log S_t + (r - q - \sigma^2/2)\tau$ ,  $S_t$  being the current price of the underlying asset. The error  $\varepsilon_N(S_T)$  captures terms neglected in the expansion.

Terms contained in (4.32) are calculated by the following expressions

$$\frac{da(S_T)}{dS_T} = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \left( \frac{dg(S_T)}{dS_T} - g(S_T) \frac{1}{S_T} \right)$$

$$\frac{d^2a(S_T)}{dS_T^2} = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \left( \frac{d^2g(S_T)}{dS_T^2} - 2 \frac{dg(S_T)}{dS_T} \frac{1}{S_T} + 2g(S_T) \frac{1}{S_T^2} \right)$$

$$\frac{d^3a(S_T)}{dS_T^3} = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \left( \frac{d^3g(S_T)}{dS_T^3} - 3 \frac{d^2g(S_T)}{dS_T^2} \frac{1}{S_T} + 6 \frac{dg(S_T)}{dS_T} \frac{1}{S_T^2} - 6g(S_T) \frac{1}{S_T^3} \right)$$

$$\begin{aligned} \frac{d^4a(S_T)}{dS_T^4} = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} & \left( \frac{d^4g(S_T)}{dS_T^4} - 4 \frac{d^3g(S_T)}{dS_T^3} \frac{1}{S_T} + 12 \frac{d^2g(S_T)}{dS_T^2} \frac{1}{S_T^2} \right. \\ & \left. - 24 \frac{dg(S_T)}{dS_T} \frac{1}{S_T^3} + 24g(S_T) \frac{1}{S_T^4} \right) \end{aligned}$$

where  $g(S_T) = \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$ .

#### 4.A.2 European Options Formulae

The value of a European option is obtained by solving the generalized Cox and Ross (1976) option pricing equation

$$c(K) = e^{-r\tau} \int_K^{\infty} f(S_T)(S_T - K)dS_T$$

which gives the value of a *call options*, where  $f(S_T)$  is the RND function of the underlying asset price at maturity and

$$p(K) = e^{-r\tau} \int_0^K f(S_T)(K - S_T)dS_T$$

which gives the value of a *put option*.

Substituting Equation (4.32) in the above we obtain the pricing formula for a *call option*

$$\begin{aligned}
c_F(K) = & c_A(K) + e^{-r\tau} \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \int_{-\infty}^{\infty} \max[0, S_T - K] \frac{d^2 a(S_T)}{dS_T^2} dS_T \\
& - e^{-r\tau} \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \int_{-\infty}^{\infty} \max[0, S_T - K] \frac{d^3 a(S_T)}{dS_T^3} dS_T \\
& + e^{-r\tau} \frac{(\kappa_4(F) - \kappa_4(A)) + 3(\kappa_4(F) - \kappa_4(A))^2}{4!} \\
& \times \int_{-\infty}^{\infty} \max[0, S_T - K] \frac{d^4 a(S_T)}{dS_T^4} dS_T + \epsilon_{c(S_T),N} \tag{4.33}
\end{aligned}$$

and also for a *put option*

$$\begin{aligned}
p_F(K) = & p_A(K) + e^{-r\tau} \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \int_{-\infty}^{\infty} \max[K - S_T, 0] \frac{d^2 a(S_T)}{dS_T^2} dS_T \\
& - e^{-r\tau} \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \int_{-\infty}^{\infty} \max[K - S_T, 0] \frac{d^3 a(S_T)}{dS_T^3} dS_T \\
& + e^{-r\tau} \frac{(\kappa_4(F) - \kappa_4(A)) + 3(\kappa_4(F) - \kappa_4(A))^2}{4!} \\
& \times \int_{-\infty}^{\infty} \max[K - S_T, 0] \frac{d^4 a(S_T)}{dS_T^4} dS_T + \epsilon_{p(S_T),N} \tag{4.34}
\end{aligned}$$

Expressions (4.33) and (4.34) involve the calculation of integrals of the following type, for  $j \geq 2$ :

$$\int_K^{\infty} (S_T - K) \frac{d^j a(S_T)}{dS_T^j} dS_T \tag{4.35}$$

and of the type:



$$\int_0^K (K - S_T) \frac{d^j a(S_T)}{dS_T^j} dS_T \quad (4.36)$$

respectively.

These can be integrated by parts. For  $j \geq 2$ , equation (4.35) gives:

$$\begin{aligned} \int_K^\infty (S_T - K) \frac{d^j a(S_T)}{dS_T^j} dS_T &= \left[ (S_T - K) \frac{d^{j-1} a(S_T)}{dS_T^{j-1}} \right]_K^\infty - \int_K^\infty (S_T - K)' \frac{d^{j-1} a(S_T)}{dS_T^{j-1}} dS_T \\ &= \lim_{x \rightarrow \infty} x \frac{d^{j-1} a(x)}{dS_T^{j-1}} - K \lim_{x \rightarrow \infty} \frac{d^{j-1} a(x)}{dS_T^{j-1}} \\ &\quad - \lim_{x \rightarrow \infty} \frac{d^{j-2} a(x)}{dS_T^{j-2}} + \frac{d^{j-2} a(K)}{dS_T^{j-2}} \end{aligned}$$

and similarly equation (4.35) gives:

$$\begin{aligned} \int_0^K (K - S_T) \frac{d^j a(S_T)}{dS_T^j} dS_T &= \left[ (K - S_T) \frac{d^{j-1} a(S_T)}{dS_T^{j-1}} \right]_0^K - \int_0^K (K - S_T)' \frac{d^{j-1} a(S_T)}{dS_T^{j-1}} dS_T \\ &= -K \lim_{x \rightarrow 0} \frac{d^{j-1} a(x)}{dS_T^{j-1}} + \frac{d^{j-2} a(K)}{dS_T^{j-2}} - \lim_{x \rightarrow 0} \frac{d^{j-2} a(x)}{dS_T^{j-2}} \end{aligned}$$

For the log-normal distribution

$$a(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right] \quad (4.37)$$

it is known [see Kendall and Stuart (1977), p.180] that for  $u > 0$

$$\lim_{x \rightarrow \infty} x^u a(x) = 0$$

and also it can be proved that for  $j \geq 2$

$$\lim_{x \rightarrow 0} \frac{d^{j-1} a(x)}{dS_T^{j-1}} \lim_{x \rightarrow 0} \frac{d^{j-2} a(x)}{dS_T^{j-2}} = 0$$

Therefore

$$\int_0^K (K - S_T) \frac{d^j a(S_T)}{dS_T^j} dS_T = \int_K^\infty (S_T - K) \frac{d^j a(S_T)}{dS_T^j} dS_T = \frac{d^{j-2} a(K)}{dS_T^{j-2}}$$

The expressions for the approximate *call option* price is then given by

$$c_F(K) = c_A(K) + e^{-r\tau} \left[ \frac{(\kappa_2(F) - \kappa_2(A))}{2!} a(K) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} + \frac{(\kappa_4(F) - \kappa_4(A)) + 3((\kappa_2(F) - \kappa_2(A))^2)}{4!} \frac{d^2 a(K)}{dK^2} \right] + \epsilon_{c(K),N}$$

and for the *put option* price

$$p_F(K) = p_A(K) + e^{-r\tau} \left[ \frac{(\kappa_2(F) - \kappa_2(A))}{2!} a(K) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} + \frac{(\kappa_4(F) - \kappa_4(A)) + 3((\kappa_2(F) - \kappa_2(A))^2)}{4!} \frac{d^2 a(K)}{dK^2} \right] + \epsilon_{p(K),N}$$

where

$$c_A(K) = S_t e^{-q\tau} N(d_1) - e^{-r\tau} K N(d_2)$$

$$p_A(K) = K e^{-r\tau} N(-d_2) - S_t e^{-q\tau} N(-d_1)$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

and

$N(\cdot)$  = cumulative standard normal

## Chapter 5

# Economic Rationale and Significance of the Internal Consistency of the *Edgeworth Series Expansion* model

This chapter attempts to empirically assess the *ESE* model. Typical studies in this direction compare option pricing methodologies in terms of the Mean Squared Errors (MSE) - the Sum of Squared Errors given by Equation (4.21), p. 76, divided by the number of contracts used for its calculation. Coutant et al. (2001), for example, compared a number of methodologies including a single log-normal, a mixture of three log-normals, a Hermite expansion (similar to Edgeworth expansion) and a Maximum Entropy model. McManus (1999), also compared a number of models including a single log-normal, a mixture of two log-normals, a jump diffusion, a Hermite expansion and a Maximum Entropy model. Corrado and Su (1996) compared the *ESE* with a single log-normal model.

A number of other issues, however, may arise with regard to the use of alternative parameterisations for the RND. One can argue that a particular parameterisation fits observed prices very well due to purely mathematical reasons and at the same time offers no intuition on the 'physics' of the problem. Moreover, while better fit is achieved it is interesting to examine what can one say on the statistical significance of this better fit.



This chapter addresses the above issues in order to explore the validity of the *ESE* model. The information conveyed by the implied probability distributions, recovered by the *ESE* method is ‘compared’ with the market conditions at the time to ensure that the model is able to capture the general market sentiment and also able to incorporate isolated events causing a significant impact on the market. In addition, the model’s explanatory ability is tested against that of the corresponding nested single log-normal to check whether it offers a statistically significant better fit to observed option prices and can, therefore, be considered a superior means of extracting information implied in option prices.

The remaining of the chapter is organized as follows. Section 5.1 summarises the *ESE* methodology for the estimation of implied RNDs and incorporates the modifications so the methodology can be used with American futures options. Section 5.2 discusses the data set used for the analysis. Section 5.3 describes the application and discusses the results and, finally, Section 5.4 concludes.

## **5.1 Estimation of the implied RND**

The implied RNDs are derived by means of an *ESE* model. Chapter 4 comprehensively presents the methodology for the estimation of implied RNDs from options on a generalised asset that pays continuous dividends at a rate  $q$ . The contracts, however, considered in the present study are options on futures. Similar mathematical

formulas can be derived by substituting in Equations (4.11) - (4.22),

$$q \equiv r$$

The respective formulas are now derived for the *ESE* specification with the following analytic form:

$$f(S_T) = a(S_T) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3 a(S_T)}{dS_T^3} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^4 a(S_T)}{dS_T^4} \quad (5.1)$$

In Equation (5.1)  $a(S_T)$  is the log-normal density defined by

$$a(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$$

with mean

$$\mu = \log S_t - \sigma^2/2\tau$$

and variance

$$\sigma^2 = \sigma \sqrt{\tau}$$

where  $S_t$  is the current price of the underlying futures (time  $t$ ),  $S_T$  is the price of the underlying futures upon maturity of the option,  $\tau (= T - t)$  is the time between now and the expiry date of the option, and  $\sigma$  the expected volatility of the returns of underlying futures contract.

Following the argument of Ross (1976) and Cox and Ross (1976) the value of an option in a market that offers no arbitrage possibilities can be expressed with the

following:

$$V = e^{-r\tau} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T$$

where  $g(S_T)$  is the pay-off function of the option and  $f(S_T)$  the distribution of asset prices at the end of any interval between  $t$  and  $T$ .

The above framework results in the following option pricing formulae

$$c_F(K) = c_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} - \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K)}{dK^2} \right] \quad (5.2)$$

$$p_F(K) = p_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} - \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K)}{dK^2} \right] \quad (5.3)$$

In Equations (5.2) and (5.3)

$$c_A(K) = e^{-r\tau} [S_t N(d_1) - KN(d_2)]$$

$$p_A(K) = e^{-r\tau} [KN(-d_2) - S_t N(-d_1)]$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

and

$$N(\cdot) = \text{cumulative standard normal}$$

Recalling the analysis in Chapter 4 (pp.69-73) an American style option can be expressed as a weighted sum of the following bounds:

$$\overline{C}(K) = E_t [\max \{0, S_T - K\}]$$

$$\underline{C}(K) = \max \{E_t [S_T] - K, e^{-rT} E_t [\max \{0, S_T - K\}]\}$$

where  $\overline{C}(K)$  and  $\underline{C}(K)$  are the upper and lower bounds for a call option and

$$\overline{P}(K) = E_t [\max \{0, K - S_T\}]$$

$$\underline{P}(K) = \max \{K - E_t [S_T], e^{-rT} E_t [\max \{0, K - S_T\}]\}$$

and  $\overline{P}(K)$  and  $\underline{P}(K)$  are the upper and lower bounds for a put option. The expectations in the above are taken with respect to the distribution the terminal asset price is assumed to be drawn from.

Weighting the upper and lower bounds the values of call and put options are expressed as

$$C(K) = \hat{w}_{it} \overline{C}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{C}(K; \hat{z}) + \hat{\epsilon}'_{C(K)}$$

$$P(K) = \hat{w}_{it} \overline{P}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{P}(K; \hat{z}) + \hat{\epsilon}'_{P(K)}$$

where

$$i = \begin{cases} 1 & \text{if } \begin{cases} \text{call and } K < S_t \\ \text{put and } K > S_t \end{cases} \\ 2 & \text{otherwise} \end{cases}$$



Implied RNDs are then recovered by minimising the sum squared errors

$$\min_{(\hat{z}, \hat{w}_1, \hat{w}_2)} \sum_{K_i} \{ [C_{OBS}(K_i) - C(K_i)]^2 + [P_{OBS}(K_i) - P(K_i)]^2 \}$$

subject to

$$E_t [S_T] \equiv S_t$$

and

$$f(S_T) \geq 0 \quad \forall S_T, S_T \in [0, \infty)$$

$$\int f(S_T) = 1$$

where  $S_t$  is the current price of the underlying futures contract.

## 5.2 Data

This chapter examines whether the qualitative information, recovered from option prices, is consistent with the market commentary of the study period in order to examine whether the *ESE* model is capable of recovering economically sensible RNDs. The model is also compared with a single log-normal model to investigate whether the additional complexity introduced is also associated with significantly - in a statistical sense - superior explanatory ability.

Since the primary task is to investigate if the implied distributions are consistent with the market commentary, it is not necessary to examine the whole of the data set in the following analysis. It is, however essential that the most 'reliable' market information is used. Heavily traded contracts are said to be more 'reliable' in that

respect. While volume data are available for the WTI options and we can be confident that near-month contracts are the most heavily traded ones we hypothesise that closest-expiration Eurodollar options contracts are in line with the former prerequisite. An additional criterion suggests that periods with increased market activity - in terms of 'events' - should be considered, as it is more likely that implied distributions deviate from a log-normal during such periods. Finally, the selection should also account for the contracts listings.

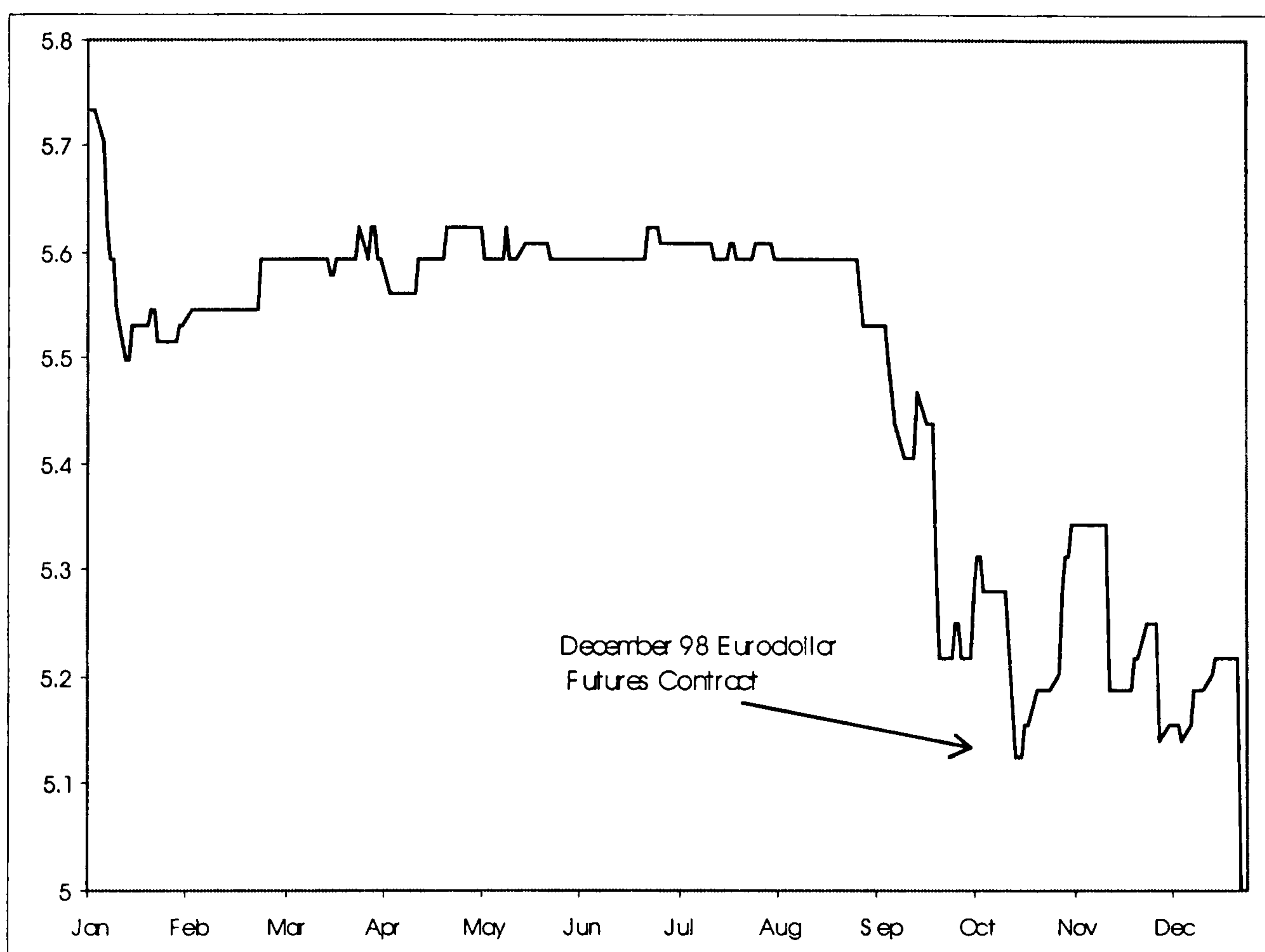


Figure 5.1: Eurodollar rate from January 2, 1998 to December 31, 1998

During 1998 the Eurodollar market was fairly stable until the end of August when a rather turbulent period is observed which extends until the end of the year (see

Figure 5.1 and discussion on p. 53). For the study of this period the December '98 Eurodollar futures options seems to satisfy the criteria set above fairly well. The WTI contracts were examined on a selective basis. Contracts due to expire in a month when oil prices followed an 'abnormal' pattern were selected. Figure 5.2 suggests that two appropriate candidates for this task were the July '98 and the December '98 futures contracts and, consequently, the respective options contracts.

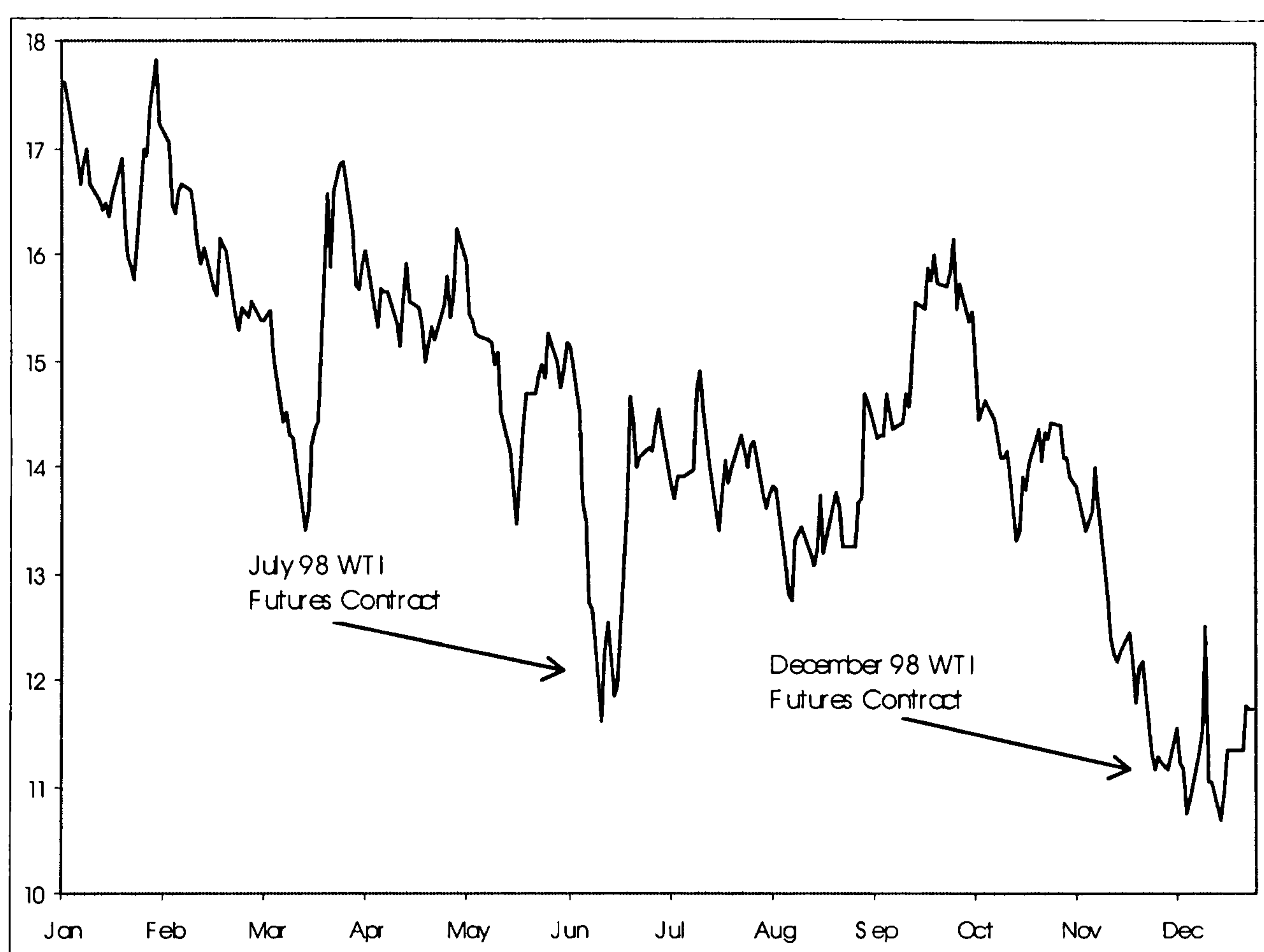


Figure 5.2: Near month WTI futures contract in \$/barrel from January 2, 1998 to December 31, 1998

For the model's statistical comparison, however, it is essential that the largest possible amount of available information is used and therefore all WTI options con-

tracts are also considered for the whole of the period starting on January 2, 1998 through to December 31, 1998. The latter include all appropriate trading days for every traded expiration month.

## **5.3 Application and results**

This section examines the properties of the implied RNDs in the presence of large changes in economic conditions. A number of events are considered and the impact on the implied RNDs is investigated. Section 5.3.1 explores the ability of the implied RNDs recovered with the *ESE* model to capture speculation over future eventualities whereas Section 5.3.2 examines their capacity to immediately reflect changes in the market sentiment. Finally Section 5.3.3 examines whether the use of the *ESE* model is able to offer a statistically significant better fit when compared with a single log-normal BSM model.

### **5.3.1 Multimodal scenarios**

The data for the WTI July futures options span a time period of around 40 days starting from May 1, through to June 10, 1998. During this period oil prices were sliding, due to increased supply in the oil market while, at the same time, demand was at relatively low levels. Demand in Asia had fallen due to the Asian crisis. The International Energy Agency (IEA), expected demand for oil for the whole year to increase only by 1.7%, while the percentage increase the previous year was 2.7%.



In a meeting which took place in Riyadh in March, OPEC and non-OPEC countries agreed to make a total cut in production of 1.5m barrels per day, in order to stabilize the oil market.

During May, there was growing concern as to whether it was feasible for oil prices to sustain the then current levels without any further action on behalf of the oil-producing countries. Oil prices were expected to drop even more, unless further cuts in production were made which would stabilize or even raise them. This 'bimodality' in traders' beliefs could not obviously be detected simply by observing the price of oil futures contracts. However, the use of an implied distribution is assumed to offer greater insight into investors' available information and as a consequence be capable of revealing the bimodal scenario that was dominant in investors' expectations at the time.

As can be seen in Figure 5.3, the distribution recovered using the *ESE* model does, indeed, capture the bimodal scenario. It assigns a greater mass than a fitted log-normal does in values around the mean (thus being more leptokurtic) while, at the same time, it also assigns larger probability mass at the right tail, consistent with the scenario of further anticipated future cuts. This bimodal pattern is a standard feature of most of the probability density functions recovered for that period.

Despite OPEC's decision on the 24th June for further cuts in oil production, the market consensus did not dramatically change in the subsequent months. Implied distributions recovered for the period October-November '98, exhibited very similar

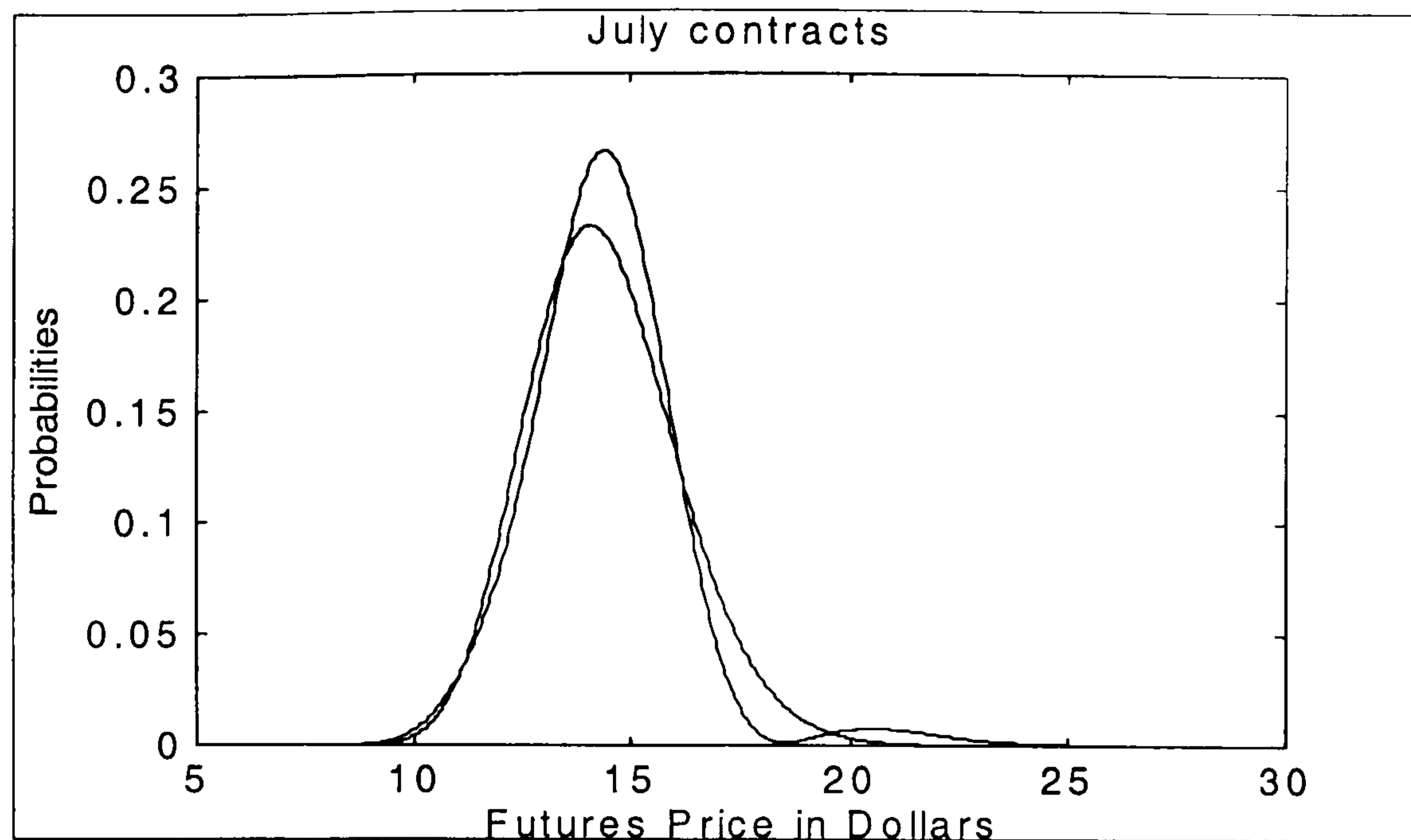


Figure 5.3: Probability distribution implied by July '98 WTI options contracts on May 12, 1998

characteristics to those implied by the July contracts. The Fall of '98 was a period of great uncertainty for oil prices. Prices declined, mainly due to increasing supplies, since non-OPEC members were not willing to comply with the organization pumping limits. In addition to that, Venezuela, one of the major oil exporting countries, announced that the previously decided exploration cuts would cease to apply. The International Monetary Fund (IMF) lowered its predictions for oil prices for the current - and the following - year and the American Petroleum Institute (API) reported a surge in US crude oil stocks. Iraq's confrontation with the UN arms inspectors, and the forecast for a cold northern winter on the other hand, gave rise to speculation for higher oil prices. A bimodal pattern that was present in most of the recovered distributions from December WTI options contracts, similar to that observed in the July contracts' distributions, further confirms the consistency of the *ESE* model.

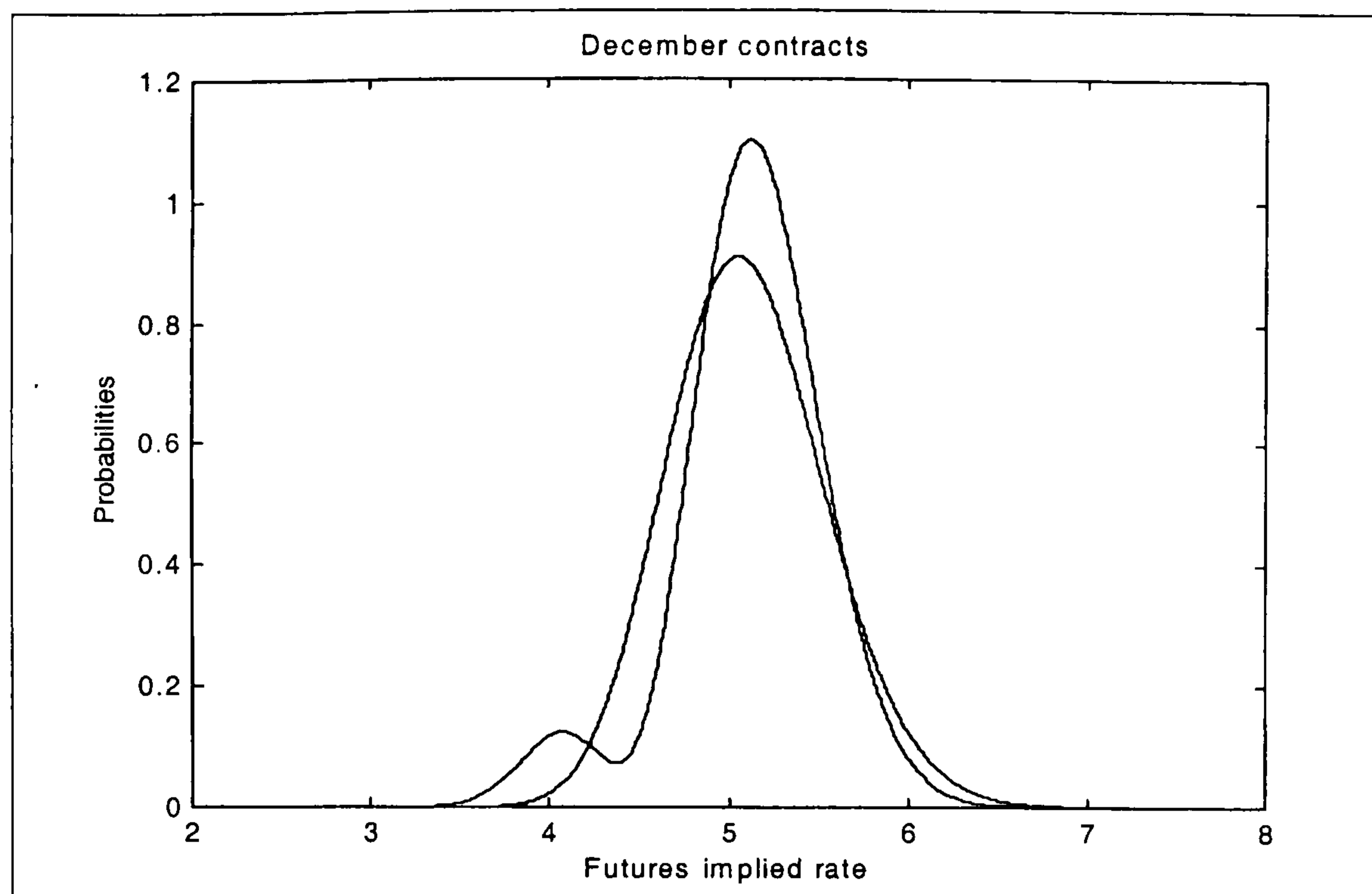


Figure 5.4: Probability distribution implied by December '98 Eurodollar options contracts on September 24, 1998

The data for the Eurodollar futures options span a time period of 61 days starting from September 1, through to November 30, 1998, a period which included 3 meetings of the FOMC. The FOMC makes decisions with regard to the Federal Reserve's interest rate policy. The study period coincides with an international market turmoil the effects of which were likely to affect the US economy. In the event, Alan Greenspan, the chairman of the Federal Reserve, warned that '... the US cannot remain an oasis of prosperity unaffected by a world that is experiencing increased stress', giving rise to speculation over interest rate cuts. The dollar sank sharply after Alan Greenspan said the risk of an economic slowdown in the US had increased. The news 'hit' the market on September 24, 1998 just five days before the scheduled meeting of the FOMC. In the event, expectations the Fed may cut interest rates in-

tensified. 'Deteriorating foreign economies and their spillover to domestic markets have increased the possibility that the slowdown in the growth of the American economy will be more than sufficient to hold inflation in check' Alan Greenspan told a Senate committee. Figure 5.4 plots the RND implied by December '98 Eurodollar options contracts on September 24, 1998. A bimodal pattern - hump on the left side of the distribution - reveals expectations consistent with the fear of interest rates cut, confirming the economic sensibility of the implied RNDs recovered with the *ESE* model. This scenario would not be captured with a single log-normal parameterisation. Similar shapes are recovered in the subsequent days.

### **5.3.2 Isolated events**

Market sentiment reflects investors' expectations about future values of the underlying asset, which are formed, based on available information. However, news arrives randomly in the market on an every day basis, forcing investors to constantly revise their expectations in the light of this newly arrived information. This section investigates whether implied distributions recovered with the *ESE* model are consistent with this feature, whether in other words implied RNDs estimated with the *ESE* model are able to reflect the extent to which important news arrival has an impact on market participants' reactions. To this end, a series of isolated events, and the information one could infer by recovering the implied distributions of option prices, is examined.



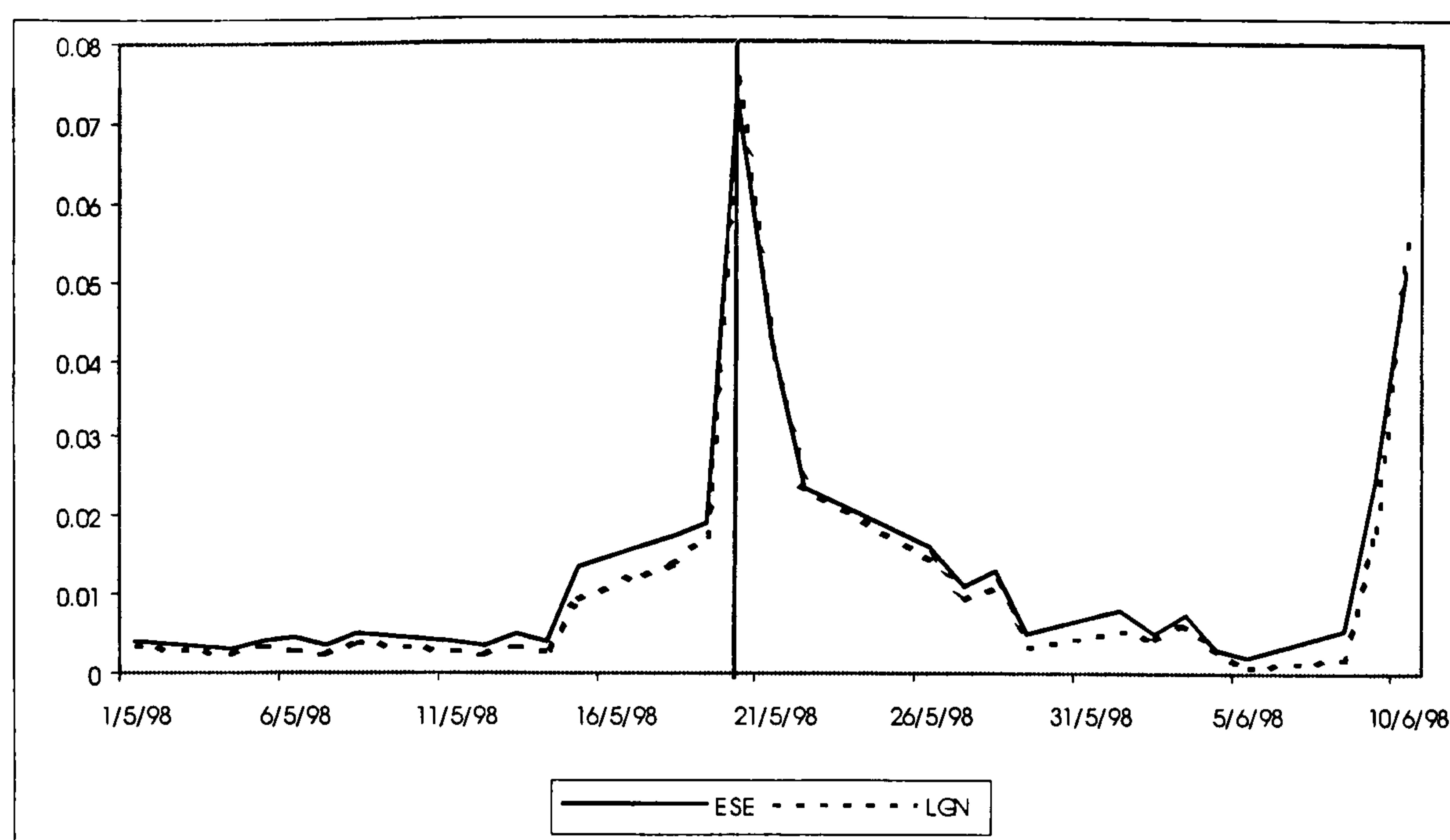


Figure 5.5: Implied probabilities for the July WTI futures contract trading below **\$12 / barrel** upon expiration of the July WTI option contract for the period May 1, 1998 through to June 12, 1998. The line indicates the date when the American Petroleum Institute published data showing an 8.79m barrel increase in US crude stocks

On May 20, 1998, the American Petroleum Institute published data showing an 8.79m barrel increase in US crude stocks, when the market expected a fall of roughly 2.5m barrels. Stocks went up to 353.13m barrels, the biggest stock of crude oil since August 1993. The news was reflected in the implied distribution of WTI futures prices. Figures 5.5 and 5.6 plot the probability of the WTI being less than \$12 or less than \$10 per barrel respectively as calculated by the *ESE* and the BSM *LGN* models. A peak was observed on that day with the probability of WTI being less than \$12 being three almost times higher than it had been the previous day for both models. The probability of the WTI being less than \$10 per barrel is found to be nearly twenty times larger than its corresponding value the previous day when calculated with an

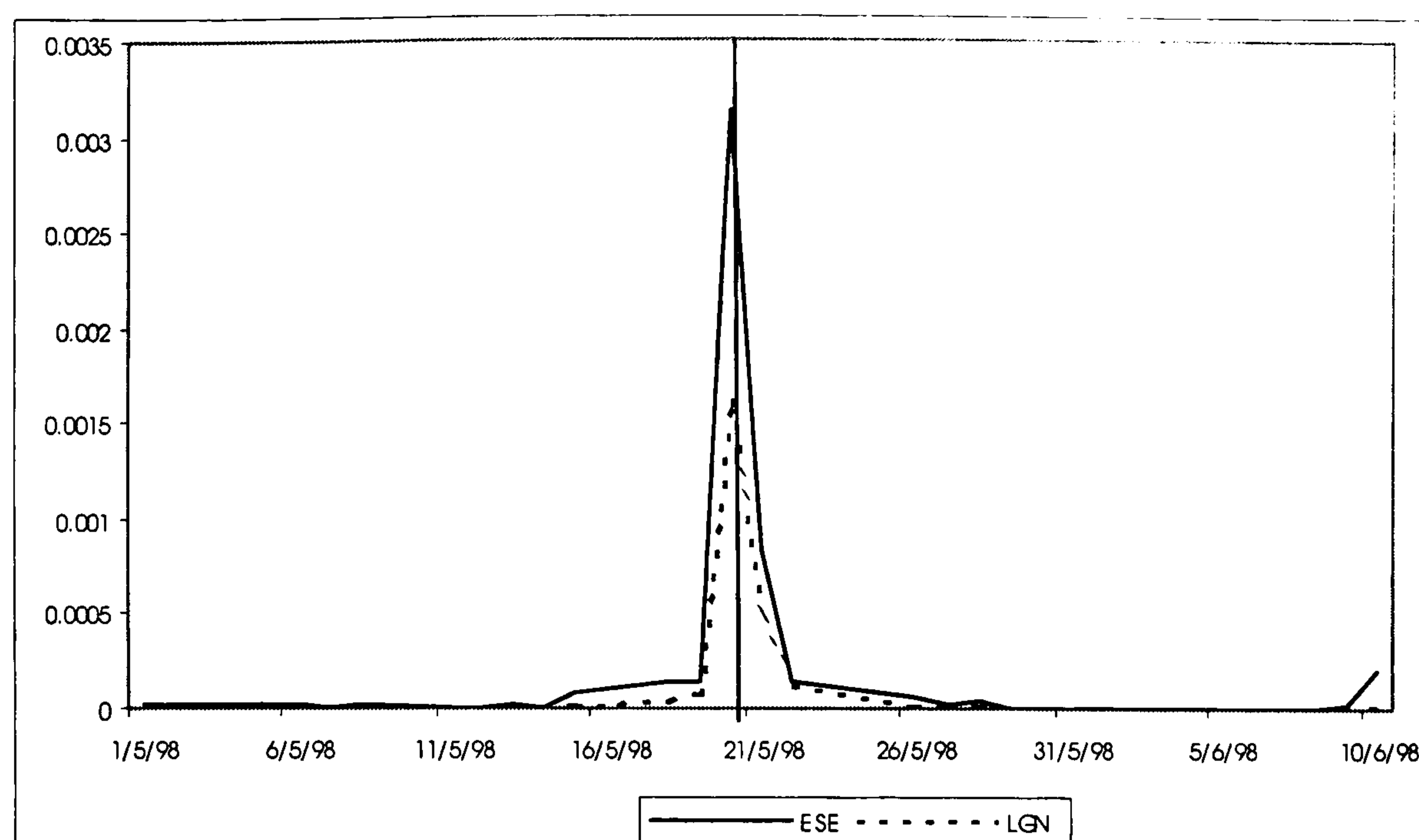


Figure 5.6: Implied probabilities for the July WTI futures contract trading **below \$10 / barrel** upon expiration of the July WTI option contract for the period May 1, 1998 through to June 12, 1998. The line indicates the date when the American Petroleum Institute published data showing an 8.79m barrel increase in US crude stocks

*ESE* model. While the BSM model also shows a peak on the that day, it fails to capture the magnitude, due to its incapability to assign high probabilities in the tails.

Examining the December WTI options contracts, an exactly similar situation arises on October 20, 1998, when Venezuela's energy minister announced that his country's oil output would return to normal levels in the second half of 1999, after the two production cuts in 1998. This was in contrast to traders' hopes for a sustained reduced oil production throughout the whole of 1999. This is highlighted in Figures 5.7 and 5.8, where probabilities exhibit a 'jump' around the October 20, 1998. The difference between the calculated probabilities by the *ESE* and the BSM *LGN* models is not as big as above - if at all different. This is somewhat expected as the implied

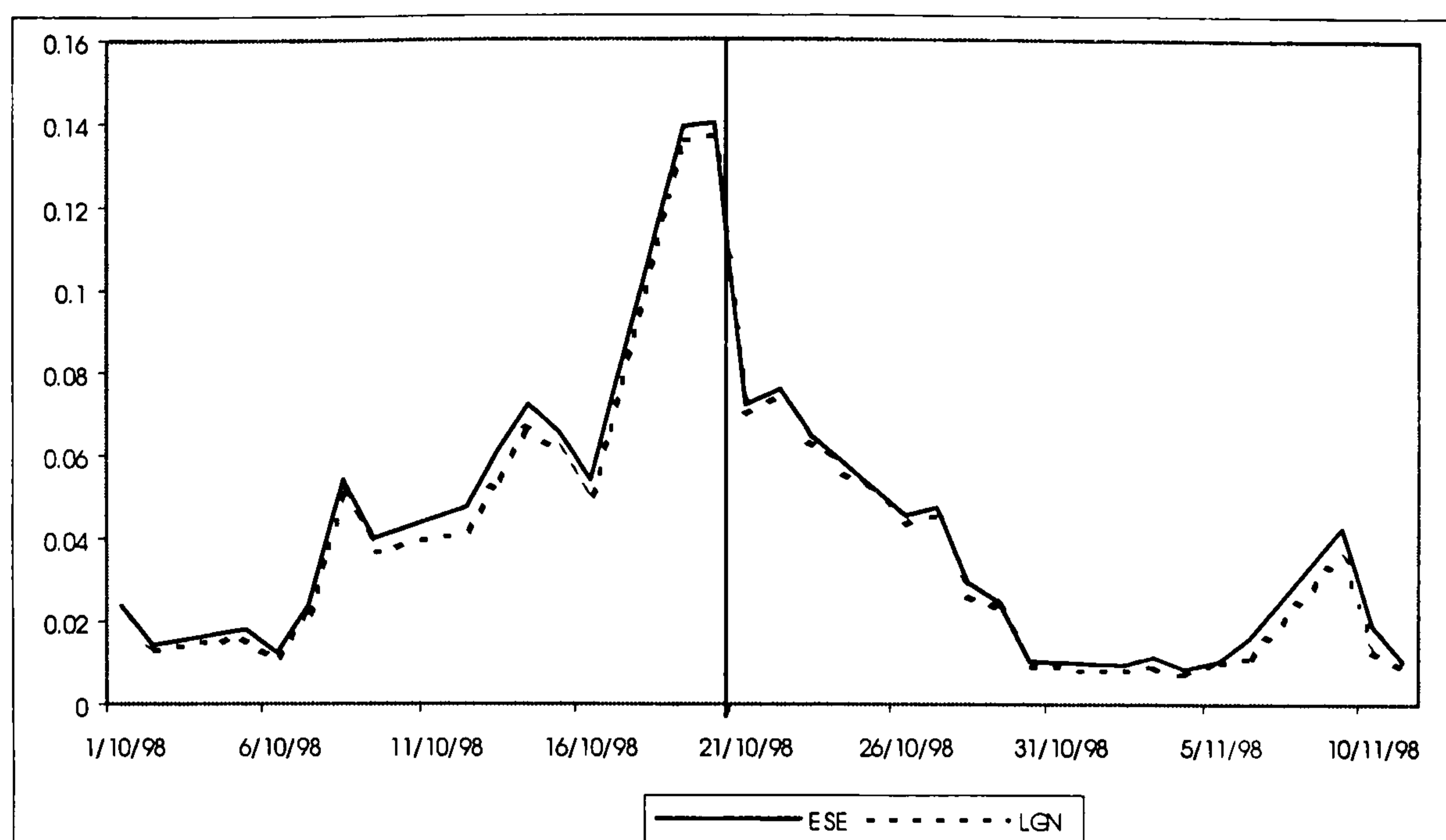


Figure 5.7: Implied probabilities of the December WTI futures contract trading **below \$12 / barrel** upon expiration of the December WTI option contract for the period October 1, 1998 through to November 11, 1998. The line indicates the date when Venezuela's energy minister announced that his country's oil output would return to normal levels in the second half of 1999, after the two production cuts in 1998

PDFs during the study period were right skewed and leptokurtic (see Figure 4.4) assigning higher probabilities, mainly, in the right tail.

On Thursday, June 4, an unexpected meeting took place in Amsterdam between Saudi Arabia, Venezuela and Mexico. The three oil-exporting countries announced that they would reduce production by a total of half a million barrels per day. The initial reaction of the markets was positive; prices rose on the Friday. During the weekend that followed, however, investors realized that the cuts were not of the extent that they expected and that the possibility of further cuts seemed more distant now than before<sup>33</sup>.

<sup>33</sup> Hopes for further cuts were placed at the June 24th meeting of the OPEC countries.

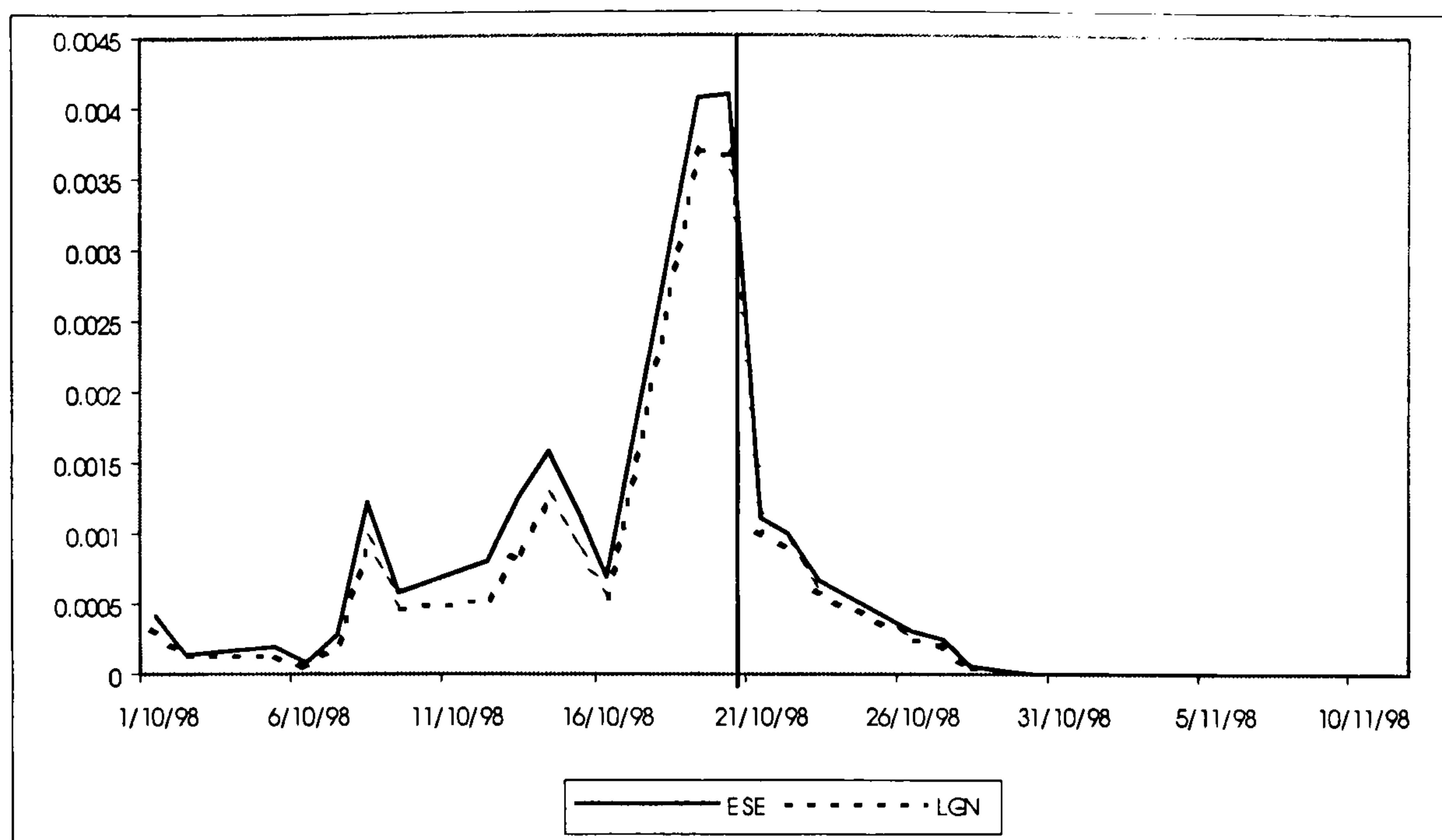


Figure 5.8: Implied probabilities of the December WTI futures contract trading **below \$10 / barrel** upon expiration of the December WTI option contract for the period October 1, 1998 through to November 11, 1998. The line indicates the date when Venezuela's energy minister announced that his country's oil output would return to normal levels in the second half of 1999, after the two production cuts in 1998

A similar pattern is observed for the Eurodollar contracts. Over the period September 1, 1998 through to November 30, 1998 the FOMC lowered the target federal funds rate by a total of 75 basis points on three occasions - following the regularly meetings on September 29, 1998 and November 17, 1998 and following a conference call meeting of on October 15, 1998. Figures 5.9 and 5.10 plot implied probabilities for the Eurodollar futures trading below 5% and 4.5% respectively upon expiration of the December Eurodollar option contract as calculated by the *ESE* and the BSM *LGN* models. The vertical lines indicate the three occasions when Federal Reserve lowered the federal funds rate. Melick (1999) argues that while the September 29 cut was somehow expected the rate cut on October 15 came as a great surprise to fi-



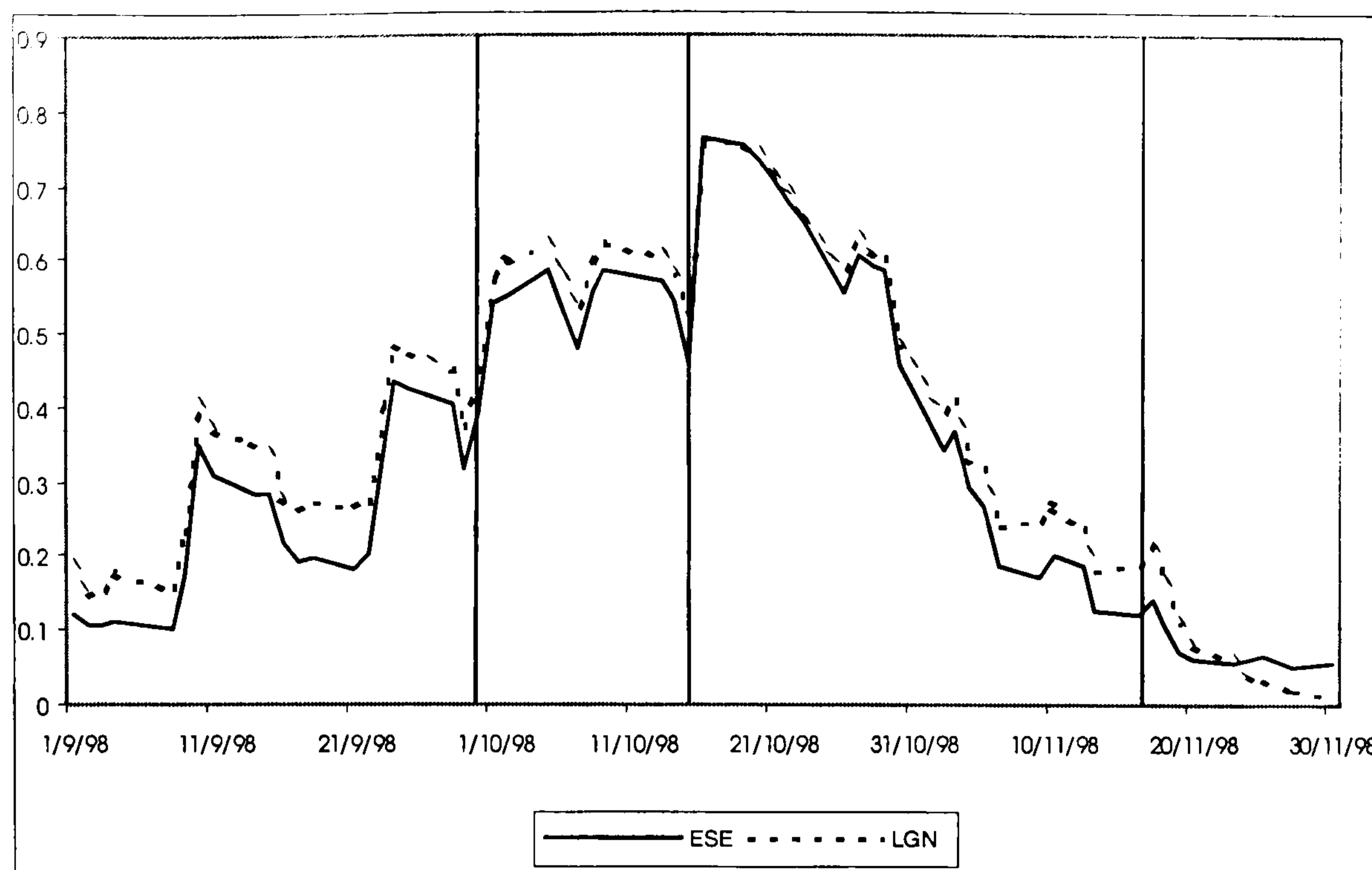


Figure 5.9: Implied probabilities for the Eurodollar futures trading **below 5%** upon expiration of the December Eurodollar option contract for the period September 1, 1998 through to November 30, 1998 for the December 1998 contract. The lines indicate the three occasions when Federal Reserve lowered the federal funds rate by 25 basis points each time

financial market participants. He justifies his results on the basis of an increase in the dispersion of the percentiles following each of the changes. In a similar fashion Figure 5.10 plot an abrupt jump in the implied risk neutral probability associated with declines of the rate below 4.5% whereas somewhat regular changes occur on the days following the September 29 and November 17 cuts. The difference in the probabilities calculated by the *ESE* and the *LGN* models is now more profound especially in Figure 5.10. In addition, the probabilities calculated with the BSM for the rate below 4.5% occurring, seem to fluctuate more, relatively to the respective ones

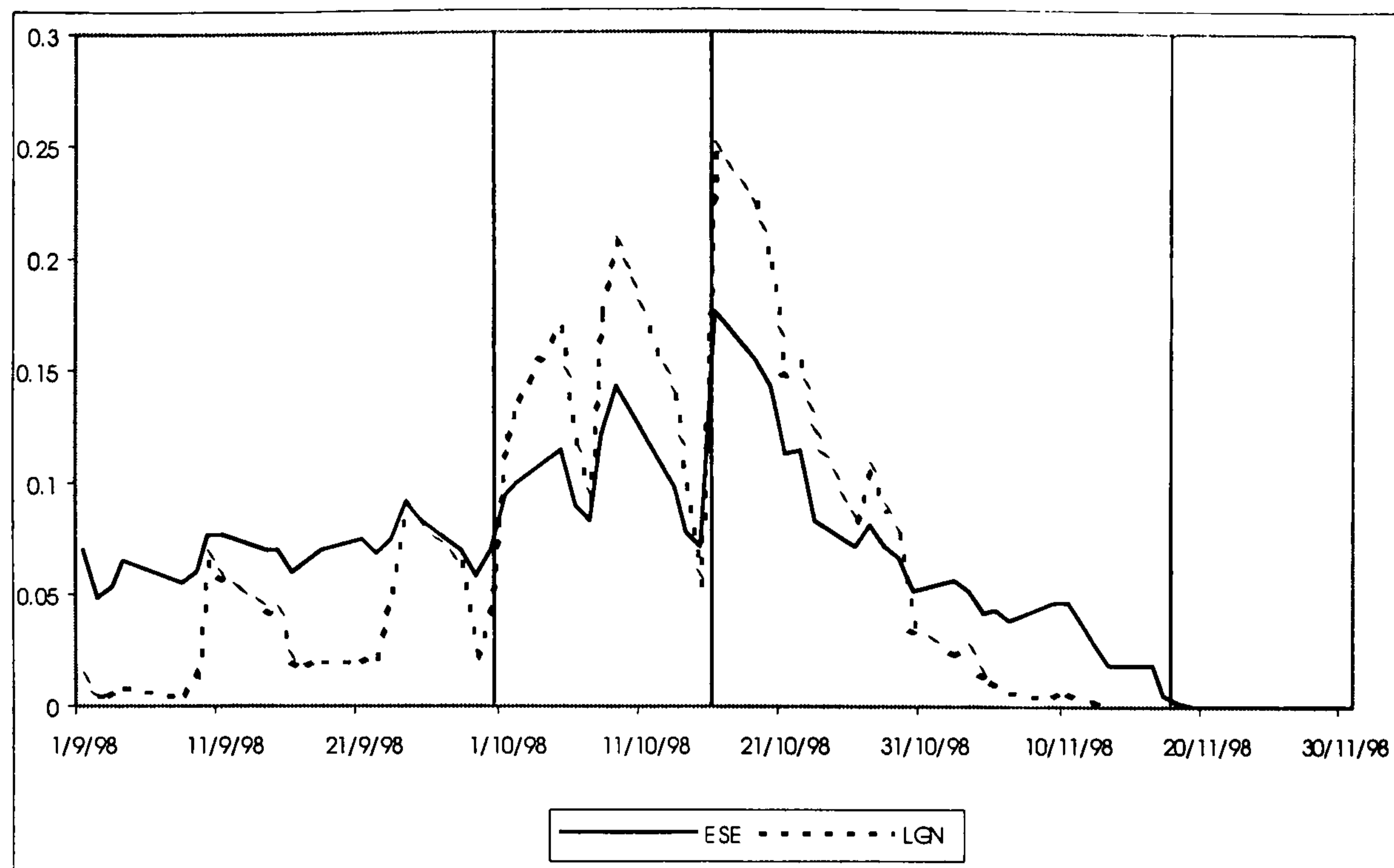


Figure 5.10: Implied probabilities for the Eurodollar futures trading **below 4.5%** upon expiration of the December Eurodollar option contract for the period September 1, 1998 through to November 30, 1998 for the December 1998 contract. The lines indicate the three occasions when Federal Reserve lowered the federal funds rate by 25 basis points each time

calculated with the *ESE* model, an indication of instability with regard to how much probability the former assigns in the left tail.

Another typical feature of implied RNDs is clearly evident in Figures 5.11 and 5.12 which plot the inter-quartile ranges<sup>34</sup> of the recovered distributions across time. As expiration of the options approach uncertainty over the terminal value of the underlying asset tends to fall and as a consequence the largest mass of the implied RNDs is concentrated around the mean, Melick and Thomas (1999). This is reflected

<sup>34</sup> The inter-quartile range of a distribution is defined as the difference between the 75 and 25 percent quartiles. It is the interval that contains the central fifty percent of the distribution.

in Figures 5.11 and 5.12 which show that the inter-quartile range is decreasing with time.

### 5.3.3 A Statistical Comparison

This section examines the goodness of fit of the *ESE* model versus that of the fitted single log-normal one. Even though there is enormous literature documenting the superiority of alternative parametric and non-parametric models over the BSM model<sup>35</sup>, to the author's knowledge, existing studies [the only exemption being Melick and Thomas (1997)], make comparisons in terms of the Mean Squared Errors (MSE) and offer no discussion on the statistical significance of the accuracy that is achieved by using a model with additional to the BSM parameters. Although a comparison with the BSM would favour literally any alternative parameterisation it could be the case that even an unfair comparison of this nature results in differences that are not significant in a statistical sense. My concern is to examine whether the use of the *ESE* model is able to offer a statistically significant better fit - not just a better fit.

To this end, a single log-normal model is estimated, using the same optimization procedure that was used for the *ESE* model. Option prices are expressed as a combination of the bounds described in Equations (4.14a)-(4.14d) but, this time, the expectation is taken over a single log-normal distribution. This implies that the opti-

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<sup>35</sup> Recent papers include Coutant et al. (2001), who compared a single lognormal, a mixture of three log-normals, a Hermite expansion and a Maximum Entropy model and McManus (1999), who compared a single lognormal, a mixture of two log-normals, a jump diffusion, a Hermite expansion and a Maximum Entropy model.

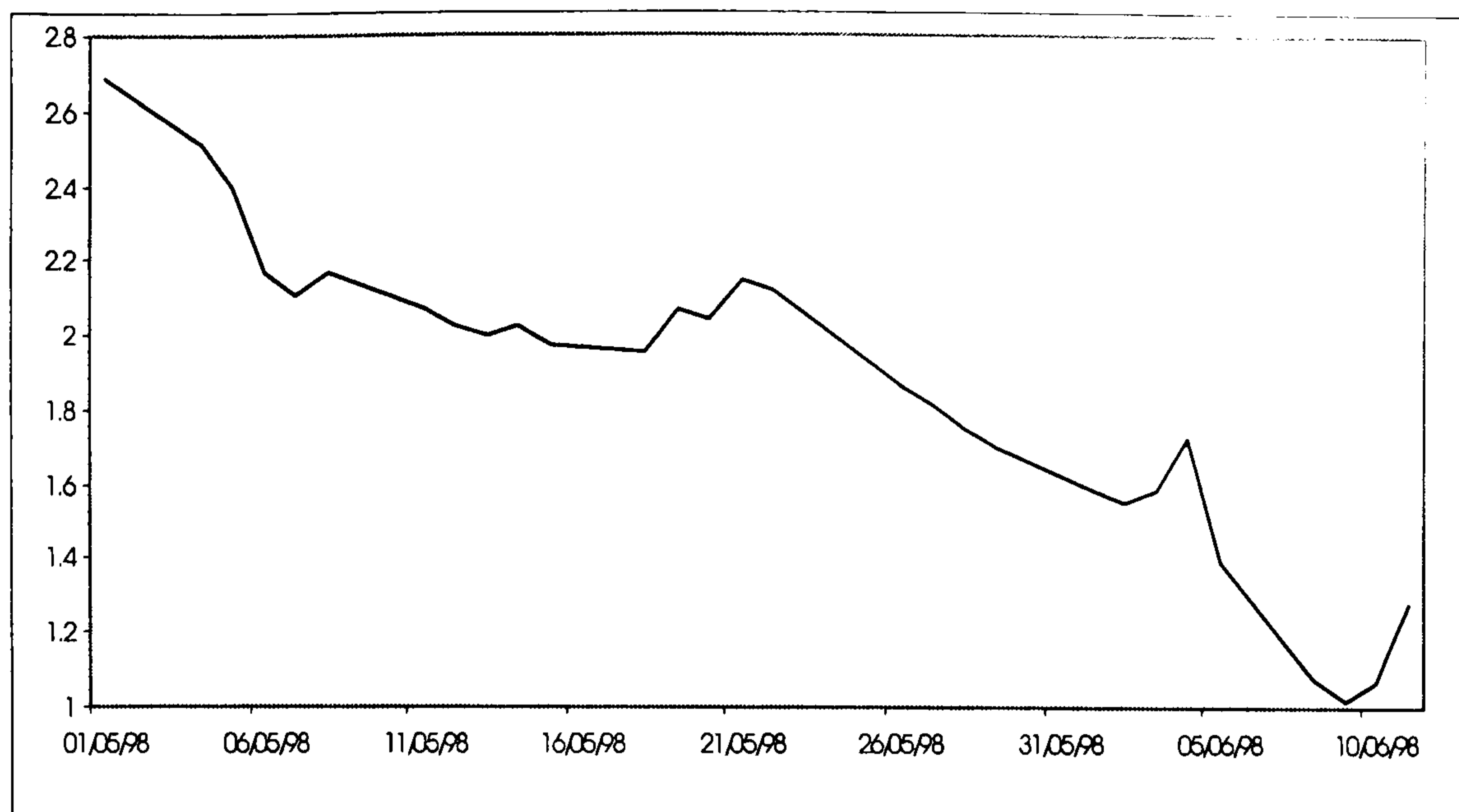


Figure 5.11: Interquartile range for the July '98 WTI futures options for the period May 1, 1998 through to June 12, 1998

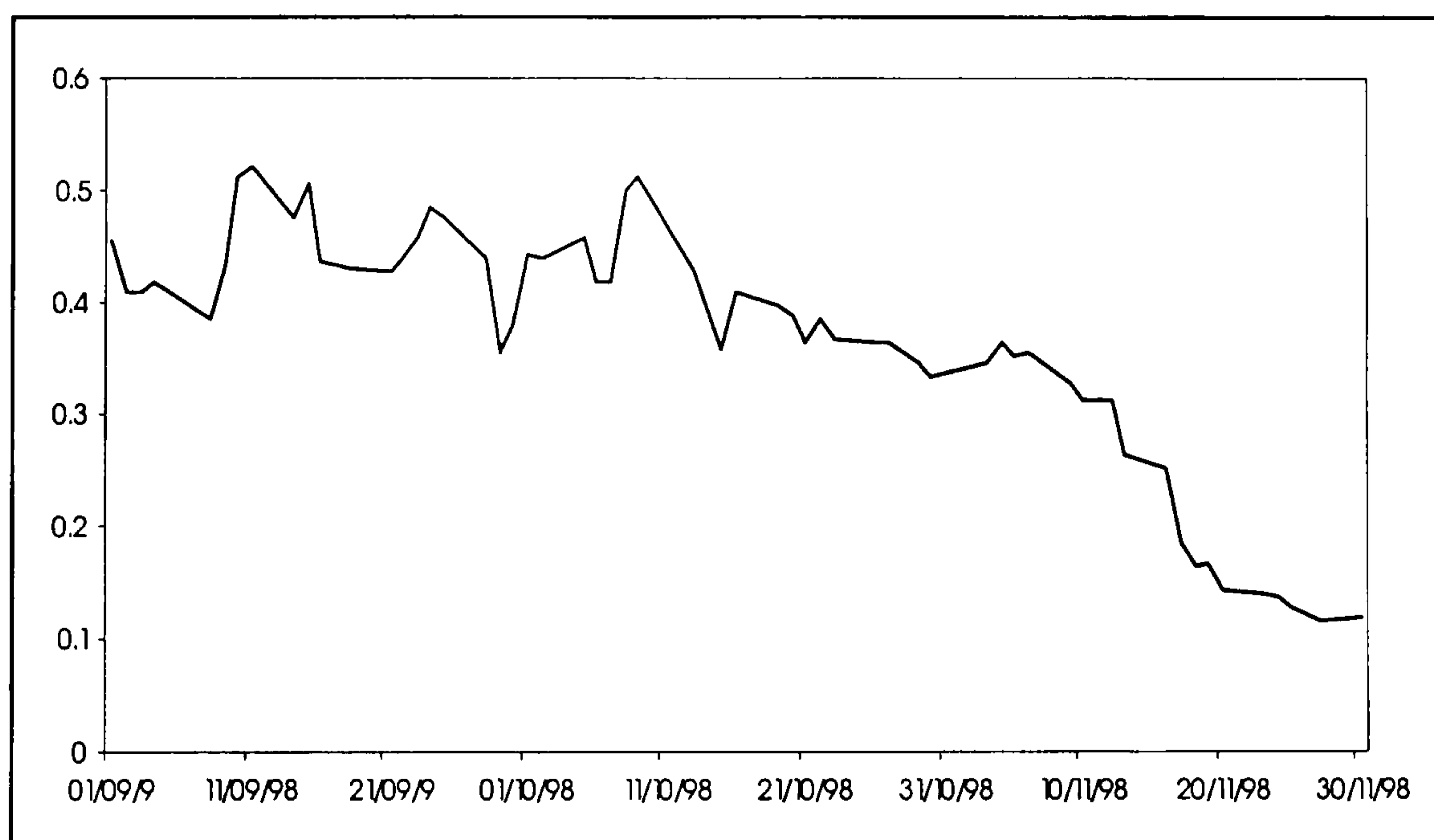


Figure 5.12: Interquartile range for the December '98 Eurodollar futures options for the period September 1, 1998 through to November 30, 1998



mization procedure involves the estimation of three parameters, namely, the implied standard deviation  $\sigma$  and the two weighting factors  $w_1$  and  $w_2$  (since the mean is set equal the current futures price to ensure risk-neutrality). As the log-normal distribution is a special case of the *ESE* density the BSM model can be seen as a nested one within the *ESE* model.

The *ESE* model is expected to provide a better fit to observed option prices than the nested single log-normal one, since it involves the estimation of additional parameters. We hypothesise that *the single log-normal distribution could adequately describe the distribution of the underlying asset returns. If the latter proves to be true the use of terms correcting for skewness and kurtosis in the proposed method would not significantly improve the fit to observed option prices.*

To examine the significance of these terms, a test that compares two competing non-linear models needs to be employed. Let  $V^{ESE}[\sigma, \lambda_1, \lambda_2, w_1, w_2 | \Omega]$  denote the option valuation formula for a call or a put option that results from the *ESE* method.  $\Omega$  represents an information matrix that contains strike prices, interest rates and times to maturity. If  $V^{BSM}[\sigma, w_1, w_2 | \Omega]$  is the corresponding formula consistent with the BSM assumptions, then it is obvious that the latter formula arises from the former, by imposing the restrictions that  $\lambda_1 = \lambda_2 = 0$ .

Since the two option pricing formulae do not represent different functional forms -  $V^{BSM}[\cdot]$  is nested within  $V^{ESE}[\cdot]$  - a Likelihood-Ratio (LR) test, testing the hypothesis that the two coefficients are jointly zero, can be applied. It will effectively

test whether the *ESE* method provides superfluous information for the description of the underlying asset's distribution or whether the additional complexity introduced by the *ESE* model is, indeed, useful in explaining distributional characteristics of option prices.

In the case of call options the problem is formulated as follows. The unrestricted *ESE* model

$$C_{obs} = C_{th}(\sigma, \lambda_1, \lambda_2, w_1, w_2 | \Omega) + \varepsilon_{C,UR} \quad (5.7)$$

is estimated and we wish to test the restrictions  $\lambda_1 = \lambda_2 = 0$  resulting in the BSM model

$$C_{obs} = C_{th}(\sigma, 0, 0, w_1, w_2 | \Omega) + \varepsilon_{C,R} \quad (5.8)$$

Under this setting the null hypothesis is that the restrictions  $\lambda_1 = \lambda_2 = 0$  are supported by the data. The error terms  $\varepsilon_{C,UR}$  and  $\varepsilon_{C,R}$  capturing any differences between the observed and the theoretical value of the option are assumed to be sufficiently small (see also discussion in Section 4.2.1, p. 62) and, given the large number of observations, be asymptotically normally distributed with 0 mean. The maximum value of the likelihood of the unrestricted model,  $L_{UR}$ , is obtained by maximising the likelihood function consistent with Equation 5.7, and maximum value of the likelihood of the restricted model,  $L_R$ , is given by maximising the likelihood function of the constrained model presented in Equation 5.8. Similar formulation applies for the put options case.

Under general conditions, the statistic

$$\xi_{LR} = -2(L_R - L_{UR})$$

can be shown to have a limiting  $\chi^2(m)$  under the null hypothesis, where  $m$  is the number of parameter restrictions [Engle (1984)]. The restricted model (the BSM model), imposes two restrictions: the first is that the skewness of the distribution is 0 and the other that the kurtosis of the distribution is a known function of its variance. The number of parameter restrictions are, therefore, two -  $m = 2$ , and the critical values for the  $\chi^2(2)$  at the 95% and 99% levels are 5.99 and 9.21, respectively. A statistic greater than 5.99 implies that the null hypothesis that the restrictions apply, can be rejected.

The test was carried out separately for call and put options since they have different valuation formulas.

Table 5.1 reports the results of the LR test, carried out for the data set of WTI futures options while Table 5.2 contains the respective results for the Eurodollar futures options data set.

The test indicated that, in both cases, the coefficients which correct the RND assumed within the BSM for non-zero skewness and excess kurtosis, are jointly statistically significant at the 5% and 1% level for each of the contracts examined. This implies that the proposed functional form for the distribution of the underlying asset's

Likelihood-Ratio Test				
Call Options				
<b>Restricted:</b> $C_{obs} = \hat{C}_{th}(\sigma, 0, 0, w_1, w_2   \Omega)$				
<b>Unrestricted:</b> $C_{obs} = \hat{C}_{th}(\sigma, \lambda_1, \lambda_2, w_1, w_2   \Omega)$				
Contracts	Obs.	Log-Likelihood Restricted	Log-Likelihood Unrestricted	$\xi_{LR}$
May	1051	967.74	1211.82	488.16
Jun	1014	919.54	1126.39	413.70
Jul	961	841.32	1120.88	559.12
Aug	837	1002.71	1107.35	209.28
Sep	837	540.01	782.49	484.96
Oct	694	735.16	946.36	422.40
Nov	590	655.64	869.23	427.18
Dec	1789	526.50	1196.79	1340.58
Put Options				
<b>Restricted:</b> $P_{obs} = \hat{P}_{th}(\sigma, 0, 0, w_1, w_2   \Omega)$				
<b>Unrestricted:</b> $P_{obs} = \hat{P}_{th}(\sigma, \lambda_1, \lambda_2, w_1, w_2   \Omega)$				
Contracts	Obs.	Log-Likelihood Restricted	Log-Likelihood Unrestricted	$\xi_{LR}$
May	916	700.06	788.04	175.97
Jun	643	536.20	631.55	190.70
Jul	638	452.68	587.96	270.56
Aug	588	394.17	517.49	246.64
Sep	597	322.43	552.21	459.56
Oct	456	507.24	572.46	130.46
Nov	438	483.78	653.78	339.99
Dec	935	422.35	534.76	224.82

Table 5.1: Likelihood-Ratio test for all traded expirations of WTI call and put options for the period January 1, 1998 to December 31, 1998



Likelihood-Ratio Test				
Call Options				
<b>Restricted:</b> $C_{obs} = \hat{C}_{th}(\sigma, 0, 0, w_1, w_2   \Omega)$				
<b>Unrestricted:</b> $C_{obs} = \hat{C}_{th}(\sigma, \lambda_1, \lambda_2, w_1, w_2   \Omega)$				
Contracts	Obs.	Log-Likelihood Restricted	Log-Likelihood Unrestricted	$\xi_{LR}$
Dec	473	1370.04	1412.82	85.56
Put Options				
<b>Restricted:</b> $P_{obs} = \hat{P}_{th}(\sigma, 0, 0, w_1, w_2   \Omega)$				
<b>Unrestricted:</b> $P_{obs} = \hat{P}_{th}(\sigma, \lambda_1, \lambda_2, w_1, w_2   \Omega)$				
Contracts	Obs.	Log-Likelihood Restricted	Log-Likelihood Unrestricted	$\xi_{LR}$
Dec	1022	2997.26	3193.60	392.68

Table 5.2: Likelihood-Ratio test for December Eurodollar call and put options for the period September 1, 1998 to November 30, 1998

returns serves as a more realistic representation of the ‘true’ RND than the standard single log-normal assumption.

Melick and Thomas (1997) performed similar tests under the assumption that the underlying asset returns distribution is a mixture of three log-normal distributions. In their tests, however, they compared their model with the Barone-Adesi and Whaley (1987) model, arguing that it is the most commonly used model among practitioners. Available evidence<sup>36</sup>, however, examining the relative performance of alternative American option pricing models highlights that, even though the BAW model is the fastest such model, it is also the least accurate one.

<sup>36</sup> See Broadie and Detemple (1996).

The specification of the single log-normal model, as suggested in this study, makes it consistently comparable with the model against which it is being tested. Both models use the same bounds for the American option price being different only in the assumed functional form of the distribution. Intuitively, this seems to be a relatively more valid way to assess the distributional differences through the examination of option prices.

## 5.4 Conclusions

This chapter examined the economic sensibility of the implied RNDs recovered with the *ESE* model and also the statistical significance of the improved fit of the *ESE* parameterisation when compared with the standard BSM single log-normal assumption.

The information conveyed by the implied probability distributions, recovered by the *ESE* method, proved to be consistent with the market commentary at the time. The implied distributions were also shown to be able to capture the general market sentiment as well as able to incorporate isolated events with a significant impact on the market.

In addition, the model's explanatory ability was tested against that of the corresponding nested single log-normal. It was found to offer a statistically significant better fit to observed option prices and can, therefore, be considered a superior means of extracting information implied in option prices.

## Chapter 6

# Testing the Effect of Measurement Errors on the Estimation of RNDs

Chapter 5 addresses concerns expressed in Chang and Melick (1999) that implied PDFs should be consistent with available data both internally - adequate fit - and 'externally' - economically sensible - in order to assess the validity of the introduced *ESE* methodology.

The above criteria should not, however, be used exclusively when one tries to identify a generally 'good' model for the estimation of implied RNDs. Observed option prices, which are used as inputs in the estimation, are subject to various errors; and, as a consequence, deviations from what one would expect to obtain under the models' assumptions, are present. It is natural then that the effect of these measurement errors is also examined and the robustness of RNDs estimation techniques or the degree of confidence that can be placed on the summary statistics calculated off the implied RNDs. Implied distributions would be a far more reliable tool for financial analysis if they were accompanied by some kind of confidence assessment, rather than just been taken for granted.

To further explore the validity of the proposed model this chapter examines the stability of the implied RNDs estimated with the *ESE* method. The mixture of two log-normals (*MLN* hereafter) specification, being a very commonly used parametric

model and also studied in terms of stability in other studies, is used as a comparative measure. This application also helps to see some more general features of implied RNDs.

The chapter is organised as follows. Section 6.1 sets the theoretical framework of the study: Subsection 6.1.1 offers a literature review of the subject area; Subsection 6.1.2 develops the models used for the empirical investigation. Section 6.2 includes the description of the market examined and the data set used. Section 6.3 discusses the hypothesis under question. Section 6.4 discusses the results of the study and Section 6.5 concludes.

## **6.1 Theoretical Framework**

### **6.1.1 Literature review**

Söderlind and Svensson (1997) were, to the author's knowledge, the first researchers to raise the issue of the stability of implied RNDs. To test the confidence that can be placed on implied RNDs, they calculated 95% point-by-point confidence intervals for the RNDs using the Delta method. They assumed that a mixture of  $n$  normal distributions was the 'correct' model for the underlying asset's log-prices and that actual option prices differ from theoretical prices by a random error term. The parameters of the model were then estimated in a non-linear least squares fashion and the possibility of heteroscedastic price errors was taken into account by using a



heteroscedastic-consistent estimator for the covariance matrix. The methodology resulted in quite narrow confidence intervals, across all strike prices. The authors also proposed the use of Monte Carlo simulation as an alternative method for constructing confidence intervals.

Melick and Thomas (1999) performed Monte Carlo simulations<sup>37</sup> in a study where a modified boot-strapping method was also utilized. A mixture of two log normal distributions was assumed to represent the underlying asset's PDF. For the Monte Carlo approach a point estimate of the parameter vector was obtained along with a covariance matrix from the Hessian of the constrained maximum likelihood. The error between the estimated and the true parameters was assumed to follow a multivariate normal distribution with zero mean and a covariance matrix equal to the covariance matrix obtained from the maximum likelihood estimation. Sampling from that distribution allowed them to construct confidence bands for the true parameters. In contrast to the former, the boot-strapping methodology required no particular structure for the generating process of the error terms. A pseudo-sample of observations was created by drawing with replacement from the original set and the model was estimated many times based on this pseudo-sample. The authors were eventually very reluctant to draw any definite conclusions on the findings.

In a more recent study, Söderlind (2000) also questioned the uncertainty about the 5th and 95th percentile of estimated risk-neutral distributions for Short sterling

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<sup>37</sup> Monte Carlo simulations are claimed to have wider applicability compared to the Delta method, since the latter requires derivatives of the function of interest, which are not always available.

contracts. His tests assumed that the underlying asset's log-price distribution was a mixture of two normal distributions. The parameters of the model were estimated by generating discrepancies between theoretical and observed option prices in two different ways: by sampling pseudo-random numbers from an i.i.d. normal distribution, with the same variance as the original price errors, and by boot-strapping the original errors. In either case the 90% confidence interval was found to be very narrow.

A slightly different approach was taken by Cooper (1999) and Bliss and Panigirtzoglou (2000). Rather than making any assumptions on the error generating process, they concentrated on the observed prices and the small errors in prices that may be present in real world, due to the existence of discrete tick size intervals. The methodology both studies used is described in Bliss and Panigirtzoglou (2000). Observed option prices are perturbed by a random number, which is uniformly distributed between  $-1/2$  and  $+1/2$  a 'tick size', and implied distributions are re-estimated from the perturbed options cross-sections. Cooper (1999), applied the methodology with simulated options prices assuming that 'true' option prices can be generated by Heston's (1993) stochastic volatility model. Implied distributions were estimated using a smoothed volatility smile method<sup>38</sup> and a model assuming a mixture of two log-normal distributions for the underlying asset. He found that the smoothed volatility smile model was superior to the mixture of two log-normal distributions model in terms of stability. Bliss and Panigirtzoglou (2000), arrived at the same conclu-

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<sup>38</sup> See Campa, Chang and Reider (1998).

sion using observed options cross-sections. The results presented in Cooper (1999) and Bliss and Panigirtzoglou (2000), however, are somehow questionable. Cooper (1999), made use of close intervals between strike prices which may have favoured the smile approach<sup>39</sup>, whereas Bliss and Panigirtzoglou (2000) used different weighting schemes<sup>40</sup> to formulate the sum of squared errors which may also have favoured the smoothed volatility smile method. On the other hand a fair comparison of two models should also take into account qualitative factors, as for example the degrees of freedom of the two models and, consequently, the computational capacity that needs to be engaged for the estimation of the model's parameters.

### 6.1.2 Development of the *ESE* and *MLN* models

Following are the theoretical models employed.

The derivation of implied RNDs with the *ESE* model is thoroughly described in Chapter 4 and the relevant adjustments are presented in Chapter 5 to modify the methodology so it can be used with the data set of the study. For ease of reference, however, the procedure followed to derive implied RNDs with the *ESE* model is briefly demonstrated below, along with the respective procedure needed for the estimation of RNDs under the *MLN* probability specification.

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<sup>39</sup> See Neuhaus (1999).

<sup>40</sup> The authors claim that vega weighting the implied volatility errors in the smoothed volatility smile method is equivalent to equally weighting the fitted price errors of the options in the mixture of two lognormal distributions method. This is rather arbitrary if not incorrect when only one type of contracts - calls or puts - is used in the estimation of the implied RNDs. Away from the money options have low vegas which in turn means that the approach assigns low weights to pricing errors of these options. This introduces a bias in favour of the smoothed volatility smile method.



The *ESE* parameterisation assumes the following analytic representation for the RND

$$f(S_T) = a(S_T; \sigma^{ESE}, \mu^{ESE}) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3 a(S_T; \sigma^{ESE}, \mu^{ESE})}{dS_T^3} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^4 a(S_T; \sigma^{ESE}, \mu^{ESE})}{dS_T^4} \quad (6.1)$$

Alternatively, following the work of Ritchey (1990), implied RNDs can be represented as a  $k$ -component mixture of log-normal distributions. A mixture of two log-normal distributions is employed in this chapter ( $k = 2$ ). The analytic form of this parameterisation is the following<sup>41</sup>:

$$f(S_T)_{MLN} = \theta a(S; \sigma_1^{MLN}, \mu_1^{MLN}) + (1 - \theta) a(S_T; \sigma_2^{MLN}, \mu_2^{MLN}) \quad (6.2)$$

where  $\theta$  is weighting factor ( $0 \leq \theta \leq 1$ ) and  $\sigma_i, \mu_i, i = 1, 2$  are the volatilities and the means of the two independent log-normal distributions.

In Equations (6.1) and (6.2)  $a(S_T; \sigma, \mu)$  is the log-normal density defined by

$$a(S_T; \sigma, \mu) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$$

with mean

$$\mu = \log S_t - \sigma^2/2\tau$$

where  $S_t$  is the current price of the underlying futures (time  $t$ ),  $S_T$  is the price of the underlying futures upon maturity of the option,  $\tau (= T - t)$  is the time between

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<sup>41</sup> For a detailed derivation see Bahra (1997) and for empirical applications also Bahra (1997), Malz (1996,1997), Gemmill and Saflekos (2000).



now and the expiry date of the option, and  $\sigma$  the volatility of the underlying futures contract.

Following the argument of Ross (1976) and Cox and Ross (1976) the value of an option in a market that no arbitrage possibilities can be expressed with the following:

$$V = e^{-r\tau} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T$$

where  $g(S_T)$  is the payoff function of the option and  $f(S_T)$  the distribution of asset prices at the end of any interval between  $t$  and  $T$ .

The above framework results in the following option pricing formulae

$$c_F^{ESE}(K) = c_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K; \sigma^{ESE}, \mu^{ESE})}{dK} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K; \sigma^{ESE}, \mu^{ESE})}{dK^2} \right] \quad (6.3)$$

$$p_F^{ESE}(K) = p_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K; \sigma^{ESE}, \mu^{ESE})}{dK} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K; \sigma^{ESE}, \mu^{ESE})}{dK^2} \right] \quad (6.4)$$

for the ESE model, and

$$c^{MLN}(K) = e^{-r\tau} \left\{ \theta \left[ e^{\mu_1 + \frac{\sigma_1^2}{2}} N(d_1^{MLN}) - KN(d_2^{MLN}) \right] + (1 - \theta) \left[ e^{\mu_2 + \frac{\sigma_2^2}{2}} N(d_3^{MLN}) - KN(d_4^{MLN}) \right] \right\} \quad (6.5)$$

$$p^{MLN}(K) = e^{-r\tau} \left\{ \theta \left[ -e^{\mu_1 + \frac{\sigma_1^2}{2}} N(-d_1^{MLN}) - KN(-d_2^{MLN}) \right] + (1 - \theta) \left[ -e^{\mu_2 + \frac{\sigma_2^2}{2}} N(-d_3^{MLN}) - KN(-d_4^{MLN}) \right] \right\} \quad (6.6)$$

for the *MLN* model.

In Equations (6.3) and (6.4)

$$c_A(K) = e^{-r\tau} [S_t N(d_1^{ESE}) - KN(d_2^{ESE})]$$

$$p_A(K) = e^{-r\tau} [KN(-d_2^{ESE}) - S_t N(-d_1^{ESE})]$$

$$d_1^{ESE} = \frac{\log\left(\frac{S_t}{K}\right) + \frac{(\sigma^{ESE})^2}{2}\tau}{\sigma^{ESE}\sqrt{\tau}}, \quad d_2^{ESE} = d_1^{ESE} - \sigma^{ESE}\sqrt{\tau}$$

whereas in Equations (6.5) and (6.6)

$$d_1^{MLN} = \frac{\ln\left(\frac{S_t}{K}\right) + \frac{(\sigma_1^{MLN})^2}{2}\tau}{\sigma_1^{MLN}\sqrt{\tau}}, \quad d_2^{MLN} = d_1^{MLN} - \sigma_1^{MLN}\sqrt{\tau}$$

$$d_3^{MLN} = \frac{\ln\left(\frac{S_t}{K}\right) + \frac{(\sigma_2^{MLN})^2}{2}\tau}{\sigma_2^{MLN}\sqrt{\tau}}, \quad d_4^{MLN} = d_3^{MLN} - \sigma_2^{MLN}\sqrt{\tau}$$

and

$$N(\cdot) = \text{cumulative standard normal}$$

Recalling the analysis in Chapter 4 (pp.69-73) an American style option can be approximated by a weighted sum of the following bounds:

$$\bar{C}(K) = E_t[\max\{0, S_T - K\}]$$

$$\underline{C}(K) = \max\{E_t[S_T] - K, e^{-r\tau} E_t[\max\{0, S_T - K\}]\}$$

where  $\overline{C}(K)$  and  $\underline{C}(K)$  are the upper and lower bounds for a call option and

$$\overline{P}(K) = E_t[\max\{0, K - S_T\}]$$

$$\underline{P}(K) = \max\{K - E_t[S_T], e^{-rt} E_t[\max\{0, K - S_T\}]\}$$

and  $\overline{P}(K)$  and  $\underline{P}(K)$  are the upper and lower bounds for a put option. The expectations in the above are taken with respect to the distribution the terminal asset price is assumed to be drawn from - the *ESE* and the *MLN*.

Weighting the upper and lower bounds the values of call and put options are expressed as

$$C(K) = \hat{w}_{it} \overline{C}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{C}(K; \hat{z}) + \tilde{\epsilon}'_{C(K)}$$

$$P(K) = \hat{w}_{it} \overline{P}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{P}(K; \hat{z}) + \tilde{\epsilon}'_{P(K)}$$

where

$$i = \begin{cases} 1 & \text{if } \begin{cases} \text{call and } K < S_t \\ \text{put and } K > S_t \end{cases} \\ 2 & \text{otherwise} \end{cases}$$

Implied RNDs are then recovered by minimising the sum squared errors

$$\min_{(\hat{z}, \hat{w}_1, \hat{w}_2)} \sum_{K_i} \{ [C_{OBS}(K_i) - C(K_i)]^2 + [P_{OBS}(K_i) - P(K_i)]^2 \} \quad (6.10)$$

## 6.2 Data

The analysis is carried out on a subset of the data set described in Chapter 4.

This consists of the full data set of Eurodollar options - a period starting on September 1, 1998 finishing on November 30, 1998 for the December 1998 contract - and data for the July 1998 and December 1998 WTI futures option contracts for the periods May 1 to June 10, 1998 and September 1 to November 11, 1998 respectively.

The data sets were chosen so that they corresponded to 'unsettled' periods when the behaviour of implied RNDs is likely to be more sensitive (see Section 5.3).

### **6.3 Hypothesis and application**

Observed option prices, which are used as inputs in the estimation, are subject to various errors; and, as a consequence, deviations from what one would expect to obtain under the models' assumptions, are present. This section discusses the nature and the possible sources of measurement errors that may affect the robustness of implied RNDs estimation techniques.

Bliss and Panigirtzoglou (2000) and Melick and Thomas (1998), argue that these errors mainly include:

1. Errors due to the use of asynchronous quotes for the option and the underlying asset.
2. Errors due to possible liquidity premia, arising from the potential impact of differential liquidity on prices.
3. Errors in recording and reporting the data.



4. Errors arising from quoting, trading and reporting prices in discrete increments, rounded to the nearest tick.

The use of end-of-day settlement prices in most of the existing studies, mitigates the first two sources of errors but, the existence of the others remains a problem. The fact also that implied distributions essentially represent the solution to a rather complicated numerical problem, further increases the possibility of obtaining PDFs which, on the one hand, correspond to a solution of the mathematical problem but, on the other, may be 'financially' irrelevant to the situation under examination.

The robustness of the *ESE* model, as well as any other methodology, can be tested by examining the sensitivity of the resulting RNDs in the presence of errors 3 and 4. The methodology employed consists of two main steps: first, the recorded options prices are perturbed by a random quantity, and then implied PDFs are repeatedly re-estimated from the 'artificial' options cross-sections.

The first step aims to simulate errors 3 and 4 by perturbing the observed option prices by a random quantity between  $-1/2$  and  $+1/2$  a tick, independently generated with a continuous uniform distribution. For every trading day a set of 100 'observationally equivalent' options cross-sections are produced. The magnitude of the perturbation was chosen so that it reflected the possible measurement error, especially error 4<sup>42</sup>. The size of the tick for the Eurodollar option under examination is

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<sup>42</sup> The fact that the same magnitude of perturbations is used across strike prices may raise concerns as the disturbances could potentially be large changes for the lowest-priced options and rather small changes for the largest-priced options. An alternative would be to use a moneyness based weighting

1/4 basis points or 0.25%, whereas the size of the tick for the WTI option contract is \$ 0.01.

The second step included the estimation of implied PDFs from the artificially produced options cross-sections, using the ESE and the *MLN* methods.

As it is literally impossible to examine the stability of the actual PDFs, the stability of their descriptive statistics could be assessed instead. An extended set of what is referred to as ‘a standard set of results’ by Melick (1999) is considered, which also included the skewness and the kurtosis coefficients. The full set included the computation of the following descriptive statistics for the implied PDFs:

$\hat{\sigma}$  : The standard deviation or the second central moment of the distribution.

$\widehat{Sk}$  : The skewness coefficient defined as the third central moment normalized by the cube of the standard deviation.

$$\widehat{Sk} = \frac{\hat{m}^3}{\hat{\sigma}^3}$$

$\widehat{Kurt}$  : The kurtosis coefficient, defined as the fourth central moment normalized by the square of the variance

$$\widehat{Kurt} = \frac{\hat{m}^4}{\hat{\sigma}^4}$$

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scheme for the perturbations. Although the study acknowledges that large disturbances in option prices would affect lowest-priced options more, this choice is driven by two facts. Firstly, an option, for example, on the WTI futures with recorded settlement price of \$1.81 is ‘observationally equivalent’ to any option that is worth any price within the range \$1.805 and \$1.8149. Secondly, both calls and puts are used in the estimation of the implied PDFs from Equation 6.10. In-the-money contracts are bound to contribute more in the Sum of Squares, thus, minimising any potential distortions arising from using disturbances of the same magnitude.

$\hat{X}_n$  : The percentiles  $\hat{X}_{005}, \hat{X}_{01}, \hat{X}_{05}, \hat{X}_{10}, \hat{X}_{25}, \hat{X}_{50}, \hat{X}_{75}, \hat{X}_{90}, \hat{X}_{95}, \hat{X}_{99}$  and  $\hat{X}_{995}$ .  $\hat{X}_n$  is the solution to the equation  $n = F(\hat{X})$ ,  $F$  been the cumulative density, using interpolation where necessary.

The estimation procedure, as well as the computation of the statistics listed above, involves the use of numerical techniques, thus, increasing the possibility of suspect outliers. To further proceed with the analysis all values outside the range defined by the 0.5% and 99.5% percentiles of each of the daily distributions of the 100 observations were excluded, thus placing 99% confidence in the results. In addition, if a recovered distribution, within a daily series, resulted in at least one outlier summary statistic, it was excluded from the series. This process led to removing 589 observations (9.7% of the sample) for the parameters recovered with the ESE method for the Eurodollar contracts, and 770 observations (12.6% of the sample) for the parameters recovered with the *MLN* method. For the WTI contracts the filtering process resulted in removing 294 observations (10.5% of the sample) for the parameters recovered with the ESE method for the July options, and 287 observations (10.3% of the sample) for the parameters recovered with the *MLN* method. The respective numbers of observations removed from the December contracts were 296 (9.9% of the sample) and 351 (11.7% of the sample). The fact that in general a larger number of outliers was removed from the estimates produced with the *MLN* method, indicates a relatively high dispersion in the sample of the estimates.



The robustness of the *ESE* and the *MLN* estimation techniques was then assessed in terms of the:

*Stability of the convergence to the original solution*

This refers to the ability of the model to estimate, on average PDFs whose summary statistics converge to the respective statistics, calculated from the original data. To quantify this, the Absolute Percentage Deviation (*APD* hereafter) measure was constructed, which is given by the following equation:

$$APD_{Z,t} = \left| \frac{Z_{unperturbed,t} - \bar{Z}_{perturbed,t}}{Z_{unperturbed,t}} \right| \quad (6.11)$$

Equation (6.11) expresses the discrepancy between the summary statistic  $Z$ , at date  $t$ , as computed from the implied PDF of the unperturbed data, and the trimmed mean of the respective statistic taken over the 100 observations resulting from the perturbed option prices also at date  $t$ , as a percentage of  $Z_{unperturbed,t}$ . This measure is less sensitive to the presence of outliers, as 0.5% on each side of the distribution of  $Z_{perturbed,t}$  is excluded.

Since the objective is to measure the magnitude of this deviation, the absolute operator was considered.

*Stability at the solution*

The objective is to examine the stability of the estimates themselves, by examining the dispersion of a particular statistic within a daily series. Typical measures



of dispersion (i.e. the standard deviation), are expressed in terms of units of the variate, thus, making it difficult to compare dispersions in different populations. To overcome this deficit, the Karl Pearson's coefficient of variation, defined by

$$CV = \frac{\tilde{\sigma}}{\mu'_1}$$

where  $\tilde{\sigma}$  is the standard deviation of the population and  $\mu'_1$  is the first centralized moment, was used. To account for potential biases due to outliers in the sample,  $\tilde{\sigma}$  was replaced with a less sensitive dispersion measure, namely the Pearson and Tukey (1965) robust measure of dispersion, defined by

$$\tilde{\sigma}_{PT} = \frac{X_{95} - X_{05}}{3.25}$$

and  $\mu'_1$  with the trimmed mean  $\bar{Z}_{perturbed,t}$ .

The *RCV* (Robust Coefficient of Variation) is consequently defined by:

$$RCV_{Z,t} = \left| \frac{\tilde{\sigma}_{PTZ,t}}{\bar{Z}_{perturbed,t}} \right| \quad (6.12)$$

The quantities  $\sigma_{PTZ,t}$  and  $\bar{Z}_{perturbed,t}$  now refer to the distribution of the summary statistics alone (100 observations per day). The *RCV* measures the dispersion of the estimated statistic and is free of any possible biases associated with the magnitude of the statistic under examination. By construction the lower the *RCV*, the less dispersed the calculated summary statistics are. The absolute is taken for reasons mentioned earlier in the text.

The measures defined by Equations (6.11) and (6.12) are expressed in percentage terms and are free of any measurement unit. This provides great flexibility to the present analysis for two reasons: firstly, it allows to assess the performance of the models, regardless of how close the summary statistics of the implied distributions are with the summary statistics of the 'market' distribution or, in other words, of how well the assumed RND approximates the true RND<sup>43</sup>; and secondly it makes it possible to compare the summary statistics amongst themselves and identify those upon which we can place more confidence.

It should be noted, however, that since there are no existing benchmarks for the above mentioned measures it is not possible to identify the methodology that performs well in absolute terms, or draw any conclusions on the significance of the impacts of possible model imperfections. The present analysis is mainly qualitative and aims to discover whether the *ESE* model performs better than the *MLN* model - which is a well acknowledged model in the RNDs literature - in terms of stability, and how confident, in general, we can be when using summary statistics calculated from these specific forms of implied PDFs.

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<sup>43</sup> The investigation also included the calculation of the Sum of Squares as given by Equation (6.10) which yielded very similar errors. As the goodness of fit is not an issue examined in this chapter for economy of space these results are not reported.

## 6.4 Results and Discussion

Tables 6.1, 6.2 and 6.3 report average *RCV*s and average *APD*s for the summary statistics  $\hat{\sigma}$ ,  $\widehat{Sk}$ ,  $\widehat{Kurt}$  and  $\widehat{X}_n$ s of the implied distributions recovered from the July WTI, December WTI and the Eurodollar contracts, respectively. The general conclusion that can be drawn is that across the summary statistics the *ESE* method appears more stable than the *MLN* method, consistently leading to lower average *APD*s and lower average *RCV*s. Not only does the *ESE* method appear to ‘recover’ distributions that are not greatly affected by the presence of measurement errors, leading to ‘average’ distributions close to the ones recovered with the original data but also distributions which, despite the presence of such errors, are very close to each other. The only exception is the  $\widehat{Sk}$  parameter for both WTI contracts which calculates similar - slightly worse - average *RCV*s and average *APD*s. The fact that two different data sets are used, suggests that the latter can be attributed to the properties of the model rather than to features of the data set used. On the other hand, respective statistics appear with average *APD*s and *RCV*s of the same magnitude which, further, supports the former argument.

To proceed with the analysis the results are divided in two subsets: one subset consists of the  $\hat{\sigma}$ ,  $\widehat{Sk}$ ,  $\widehat{Kurt}$  and the other of the  $\widehat{X}_n$ s. The reasoning behind this is that average *APD*s and *RCV*s for the first subset are, in general, of higher magnitude compared to those for the percentiles.

Descriptive Statistic	APD (%)		RCV (%)	
	<i>ESE</i>	<i>MLN</i>	<i>ESE</i>	<i>MLN</i>
$\hat{\sigma}$	0.300	0.463	0.33	0.71
$\widehat{Sk}$	5.933	4.353	8.98	9.49
$\widehat{Kurt}$	1.694	4.835	1.72	7.50
$\hat{X}_{005}$	0.740	0.647	0.45	1.06
$\hat{X}_{01}$	0.896	0.500	0.56	0.77
$\hat{X}_{05}$	0.122	0.136	0.12	0.24
$\hat{X}_{10}$	0.109	0.046	0.09	0.12
$\hat{X}_{25}$	0.054	0.090	0.05	0.16
$\hat{X}_{50}$	0.010	0.027	0.03	0.06
$\hat{X}_{75}$	0.035	0.080	0.04	0.15
$\hat{X}_{90}$	0.022	0.087	0.03	0.15
$\hat{X}_{95}$	0.034	0.062	0.06	0.18
$\hat{X}_{99}$	0.130	0.390	0.15	0.60
$\hat{X}_{995}$	0.156	0.574	0.17	0.87
No Obs.	2,506	2,513		

Table 6.1: Average *APDs* and *RCVs* of July WTI futures options implied PDFs summary statistics

The skewness coefficient  $\widehat{Sk}$  proves to be by far the worst descriptive statistic in terms of stability. The *ESE* computed average *APDs* are 5.933%, 12.019%, and 0.892% for the July WTI, the December WTI and the Eurodollar contracts. These figures are approximately 3.5, 90 and 3 times higher when compared to the respective figures calculated for the kurtosis coefficient. The *MLN* model computes the respective numbers as 4.353%, 9.529% and 14.505%, being approximately about the same magnitude as  $\widehat{Kurt}$  for the first two and about 6 times higher for the third. The dispersion of  $\widehat{Sk}$  is also disappointing as both models produce quite high *RCVs* for both data sets.



Descriptive Statistic	APD (%)		RCV (%)	
	<i>ESE</i>	<i>MLN</i>	<i>ESE</i>	<i>MLN</i>
$\hat{\sigma}$	0.041	0.652	0.27	1.13
$\widehat{Sk}$	12.019	9.529	30.98	19.99
$\widehat{Kurt}$	0.131	8.841	1.22	10.51
$\hat{X}_{005}$	0.042	1.244	0.30	1.66
$\hat{X}_{01}$	0.047	0.913	0.36	1.30
$\hat{X}_{05}$	0.013	0.168	0.11	0.35
$\hat{X}_{10}$	0.009	0.083	0.07	0.14
$\hat{X}_{25}$	0.006	0.137	0.04	0.24
$\hat{X}_{50}$	0.004	0.042	0.03	0.17
$\hat{X}_{75}$	0.004	10.209	0.03	6.91
$\hat{X}_{90}$	0.005	0.138	0.03	0.22
$\hat{X}_{95}$	0.008	0.089	0.06	0.18
$\hat{X}_{99}$	0.014	0.827	0.13	0.80
$\hat{X}_{995}$	0.016	0.909	0.14	1.15
No Obs.	2,704	2,649		

Table 6.2: Average *APDs* and *RCVs* of December WTI futures options implied PDFs summary statistics

The kurtosis coefficient  $\widehat{Kurt}$  is the second worst descriptive statistic, as it proves to be fairly sensitive to measurement errors. The *ESE* computed average *APDs* of 1.694%, 0.131% and 0.310% for the July WTI, December WTI and Eurodollar contracts respectively. These values are approximately 2, 3 and 2 times higher, when compared to the third worst statistics, the  $\hat{X}_{01}$  for the WTI contracts and the  $\hat{\sigma}$  for the Eurodollar contracts respectively. The *MLN* model computes the respective numbers as 4.835%, 8.841% and 2.601% being approximately 7, 7 and 4 times higher than the third worst statistic, the  $\hat{X}_{005}$ . The dispersion of  $\widehat{Kurt}$  also

Descriptive Statistic	APD (%)		RCV (%)	
	<i>ESE</i>	<i>MLN</i>	<i>ESE</i>	<i>MLN</i>
$\hat{\sigma}$	0.133	0.272	0.35	0.70
$\widehat{Sk}$	0.892	14.505	6.16	41.91
$\widehat{Kurt}$	0.310	2.601	0.84	4.29
$\hat{X}_{005}$	0.047	0.585	0.11	0.49
$\hat{X}_{01}$	0.055	0.338	0.10	0.48
$\hat{X}_{05}$	0.070	0.116	0.18	0.37
$\hat{X}_{10}$	0.020	0.076	0.06	0.15
$\hat{X}_{25}$	0.005	0.049	0.02	0.10
$\hat{X}_{50}$	0.003	0.024	0.01	0.06
$\hat{X}_{75}$	0.006	0.035	0.02	0.08
$\hat{X}_{90}$	0.009	0.043	0.02	0.07
$\hat{X}_{95}$	0.012	0.054	0.04	0.12
$\hat{X}_{99}$	0.037	0.136	0.12	0.28
$\hat{X}_{995}$	0.049	0.192	0.15	0.38
No Obs.	5,511	5,330		

Table 6.3: Average *APDs* and *RCVs* of Eurodollar futures options implied PDFs summary statistics

seems to be high for both models and data sets: the average *RCVs* are 1.72%, 1.22% and 0.84% for the *ESE*, and 7.50%, 10.51% and 4.29% for the *MLN* model, for the July WTI, December WTI and Eurodollar data sets respectively. In any case the *ESE* model produces implied PDFs which lead to stable  $\widehat{Kurt}$  compared to the *MLN* model, but little confidence can be placed in comparison to the statistics included in the complementary subset.

The standard deviation  $\hat{\sigma}$  of the implied PDFs provided somehow mixed results. While the *APD* and the *RCV* for the  $\hat{\sigma}$  calculated with the *ESE* model for the July WTI and the December WTI contracts are quite low and comparable to those

computed for what are considered as relatively stable descriptive statistics (0.300%, 0.041% and 0.33%, 0.27% respectively compared to 0.896%, 0.047% and 0.56%, 0.36% for  $\hat{X}_{01}$ ), the case is not the same when the Eurodollar contracts are examined. The respective *APD* and *RCV* are higher, 0.133% and 0.35%, and much higher than for any other statistic except for the  $\widehat{Sk}$  and the  $\widehat{Kurt}$ . The picture is not as mixed in the case of the *MLN* model which consistently computes *APDs* and *RCVs* that compare favourably to  $\hat{X}_{005}$  and  $\hat{X}_{01}$  and also the  $\widehat{Sk}$  and the  $\widehat{Kurt}$  coefficients (except for one case). The average *APDs* and *RCVs*, however, for the  $\hat{\sigma}$  of implied PDFs estimated with the *ESE* model are clearly lower than the respective ones of implied PDFs, estimated with the *MLN* model.

The second subset of results examined includes the percentiles  $\hat{X}_{005}$  to  $\hat{X}_{995}$  of the implied PDFs. Examination of the percentiles of the implied PDFs recovered with the *ESE* model suggests that they offer a reliable means for implied PDFs interpretation; the obtained *APDs* and *RCVs* for the two data sets are very similar for the range defined by  $\hat{X}_{005}$  to  $\hat{X}_{95}$  and quite similar for percentiles outside this range, thus, make convincing the argument that the results are not specific to the data set used. For the percentiles of the implied PDFs estimated with the *MLN* model, it also appears that the majority of them offer a greater degree of stability compared to the other statistics examined, the two exceptions being the tail percentiles  $\hat{X}_{005}$  to  $\hat{X}_{01}$  which seem to perform worse than the  $\hat{\sigma}$ ; the whole set of percentiles though offer far greater stability than the skewness and the kurtosis coefficients. The sta-



bility of the percentiles of the PDFs estimated with *ESE* model in general appears to be superior to that of the *MLN* model, the only exception being the *APD* calculated for  $\hat{X}_{10}$  for the July WTI options. As far as the *RCV*s are concerned, there is no exception to object the superiority of the *ESE* model.

Overall, the analysis confirms existing evidence, presented in Cooper (1999) and Bliss and Panigirtzoglou (2000), that the skewness and the kurtosis summary statistics of a two log-normal distribution are strongly influenced by small disturbances to option prices<sup>44</sup>. Even though the influence is not that strong in the case of the *ESE* method, one should be very reluctant to use these summary statistics to study the behaviour of the implied PDFs. On the other hand, the percentiles seem to be a much safer means for implied PDFs' analysis and interpretation. Not only is there a great degree of stability of the convergence to the original solution, as expressed by a low mean *APD* present, but also the dispersions around the means of the statistics' distributions are very low, irrespective of the model assumed. This complements existing evidence, reported in Campa, Chang and Reider (1998), Coutant, Jondeau and Rockinger (2001) and Melick (1999), that for certain regions representing a large percentage of total probability (for example between  $\hat{X}_{10}$  and  $\hat{X}_{90}$ ), implied PDFs are relatively free of mathematical priors imposed by a specific economic model or structure, and highlight the fact that the stability of these regions is

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<sup>44</sup> Cooper (1999), attributes the instability of the mixture-of-two log-normal distributions of options with less than three months to mature (as those examined in the current study), mainly to the existence of 'spikes' in the distribution. The 'spiked' distributions are recovered when the estimated variance of one of the two log-normal distributions falls to a very low level.



also relatively irrespective of the model used . The percentiles  $\hat{X}_{005}$  to  $\hat{X}_{05}$  and also  $\hat{X}_{95}$  to  $\hat{X}_{995}$  appear with higher average *APDs* and *RCVs*, still the *ESE* model providing the lower ones. This also supports the argument of Melick (1999) that the tail percentiles are sensitive to the choice of the estimation technique (see for example in Tables 6.1, 6.2 and 6.3 the rise in the *APDs* and the *RCVs* of both methodologies as we move further in the tails).

## 6.5 Conclusions

This chapter investigated the issue of the robustness of implied PDFs estimated with the *ESE* model developed in Chapter 4. An *MLN* probability model is used for comparative purposes following its dominance among parametric models and, also, due to the attention it has received in similar studies.

Errors that may distort the implied RNDs and consequently affect the robustness of the estimation techniques were replicated with small random perturbations in observed option prices. Implied RNDs were then re-estimated and their properties were summarised in a number of statistics.

The results are highly supportive of the superior performance of the *ESE* model. The *ESE* model is found to be more stable than *MLN*; it is found more capable to estimate on average PDFs whose summary statistics converge to the original solution; and also capable to estimate statistics with relatively low dispersion. It can, therefore, be said that the *MLN* method is more likely to be affected by errors in

recording and reporting the data, as well as by errors arising from quoting, trading and reporting prices in discrete increments, rounded to the nearest tick, than the *ESE* model.

This chapter sheds also some light with regard to implied RNDs in general. Firstly, the analysis suggests that the skewness and kurtosis summary statistics of the implied PDFs should be used very moderately, if at all, as they are subject to large measurement errors. Secondly, the results provide strong evidence for the stability of alternative summary statistics such as the percentiles. Both models were found to perform reasonably well between the 10% and the 90% percentiles, much better compared to statistics referring to the higher moments of the implied PDFs. The MLN failed to produce robust estimates outside this region, which implies that the model is more sensitive to the estimation of the tails of the PDFs than the *ESE* one.

As a conclusion, the results suggest that one could generally rely on implied PDFs, but has to be very cautious with the statistics used to interpret the information embedded in option prices.

## Chapter 7

# Inferring Investors' Risk Preferences by Means of Implied RNDs

The present study has thus far employed a number of tests to validate the use of the *ESE* model for the estimation of RNDs. The model has proved to be consistent with the market commentary and able to reflect the market conditions. It has also proved to be a superior model in terms of stability over the widely acknowledged *MLN* model. A natural step forward would, therefore, be to explore the uses of the *ESE* model.

Implied RNDs are used in financial practice to price illiquid, exotic or over-the-counter options consistently with exchange traded, vanilla options. On the other hand implied distributions are used extensively by traders and policy makers to qualitatively assess market beliefs on future movements of various securities. This chapter illustrates an application where the information content of option prices, rather than qualitatively be assessed, is explicitly quantified.

A fundamental principal of economic theory is employed: *in the absence of arbitrage, all asset prices can be expressed as the expected value of the product of the pricing kernel (a preference function) and the asset payoff*. It follows then that the pricing kernel, coupled with a probability model for the future states, gives a complete description of asset prices, expected returns and risk preferences.

This chapter solves the inverse of the equilibrium asset pricing problem to identify preference parameters - *given asset prices and a probability model for futures states what can be inferred about investors' risk preferences?*

Standard empirical approaches in this area are classified in two main categories: the studies that estimate consumption-based pricing kernels using a known parametric form data on aggregate consumption; and the studies that use a proxy for consumption e.g. a stock market index, and estimate mainly a non-parametric return-based pricing kernel. A significant problem with the former approaches stems from the poor fit that consumption-based pricing kernels provide to financial market data (the 'asset pricing puzzle') at economically plausible preference parameters, Rosenberg (2000). The latter approaches, on the contrary, avoid the use of aggregate consumption data or a parametric pricing kernel specification, thus allowing for more flexibility in the model's specification and consequently resulting in a rather better rationalisation of observed asset prices.

The present study falls within the second class. Pricing kernels are derived as the state-price-per-unit probabilities; and risk preferences parameters are estimated on days with certain market conditions.

The chapter is organised as follows: Section 7.1 demonstrates the theoretical framework and derives measures for investors' risk preferences. Section 7.2 reviews the existing literature. Section 7.3 is divided in Section 7.3.1 which describes the data set, and Sections 7.3.2 and 7.3.3 which present the methodologies used for the



estimation of the risk-neutral and the statistical densities respectively. Section 7.4 discusses the empirical results and Section 7.5 concludes.

## 7.1 Theoretical Framework - Risk Aversion and Investors' Risk Preferences

### 7.1.1 Risk Aversion

The fundamental investment-selection problem for an individual is to determine the optimal allocation of his wealth among the available investment opportunities. Under the expected utility hypothesis each individual's consumption and investment decision is characterised as if he determines the probabilities of possible asset pay-offs, assigns an index to each possible consumption outcome, and chooses the consumption and investment policy to maximise the expected value of the index.

To formally express this consider an economy with complete markets. This allows the existence of a 'composite' consumer who wishes to maximise a utility function of aggregate consumption [Constantinides (1982)]. The 'composite' consumer is endowed with wealth of 1 unit and has an investment horizon of  $T$ . The optimality of the competitive equilibrium leads to the formulation of the 'composite' consumer's utility maximisation problem which is expressed by the following, Leland (1980):

$$\underset{Y(\cdot)}{\text{Maximise}} \int_{-\infty}^{+\infty} U(W_T) p(W_T) dW_T \quad (7.1)$$

subject to the budget constraint

$$\frac{1}{(1+r)^\tau} \int_{-\infty}^{+\infty} W_T f(W_T) dW_T = 1 \quad (7.2)$$

where

$W_T$  end-of -period wealth

$U(\cdot)$  a von Neumann-Morgenstern utility function

$f(\cdot)$  pricing function at the initial time period for 1 unit of wealth delivered at the terminal time period, contingent on the value  $W_T$  (known also as the *State Price Density*)

$p(\cdot)$  investor's probability density function over terminal portfolio values

$r$  risk-free interest rate

$\tau$  time until the end of the investment horizon

The model defined by Equations (7.1) and (7.2) simply seeks the allocation schedule of the 'composite' consumer's wealth that maximises his expected utility subject to the budget constrained.

The solution to (7.1) is straightforward. For every level of end-of-period wealth  $W_T$ :

$$U'(W_T) = \frac{\lambda}{(1+r)^\tau} \frac{f(W_T)}{p(W_T)} \quad (7.3)$$

where  $\lambda$  is the Lagrange multiplier for the budget constraint (7.2)<sup>45</sup>.

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<sup>45</sup> The ratio  $\frac{f(W_T)}{p(W_T)}$  is equivalent to the *pricing kernel* variable  $M_{t+1}$  in Equation (8.1.3) Campbell et

Consider now an investor with the above mentioned characteristics who wishes to allocate his wealth among a risky asset (the market portfolio) and a riskless security. Merton (1990) suggests that the equilibrium expected return on the risky asset exceeds the return on the riskless security; the best investment strategy would therefore be to invest all of the investor's wealth in the risky asset. Thus, if  $S$  is the risky asset the following relation has to hold in equilibrium:

$$U'(S_T) = \frac{\lambda}{(1+r)^t} \frac{f(S_T)}{p(S_T)} \quad (7.4)$$

For the purpose of the present study a measure to quantify investors' risk aversion is sought. A possible candidate would be to calculate the rate of change of  $U'(S_T)$ , i.e.,  $U''(S_T)$ . Arrow (1970) discussed the disadvantages of this measure and proposed two measures based on  $U''(S_T)$  which he modified to remain invariant under positive linear transformations of the utility function. Arrow's measures<sup>46</sup> for risk aversion are given by the following equations:

$$R_A = -\frac{U''(S_T)}{U'(S_T)} \quad \text{absolute risk aversion} \quad (7.5)$$

and

$$R_R = -S_T \frac{U''(S_T)}{U'(S_T)} \quad \text{relative risk aversion} \quad (7.6)$$

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al. (1997), p. 294, also referred to as the *stochastic discount factor*.

<sup>46</sup> Also developed independently by Pratt (1964). The measures are usually referred to as Arrow-Pratt's risk aversion measures.

Differentiating (7.4),

$$U''(S_T) = \frac{\lambda}{(1+r)^\tau} \frac{f'(S_T)p(S_T) - p'(S_T)f(S_T)}{[p(S_T)]^2}$$

Equations (7.5) and (7.6) can be rewritten as:

$$R_A = \frac{p'(S_T)}{p(S_T)} - \frac{f'(S_T)}{f(S_T)} \quad (7.7)$$

and

$$R_R = S_T \left( \frac{p'(S_T)}{p(S_T)} - \frac{f'(S_T)}{f(S_T)} \right) \quad (7.8)$$

which express the risk aversion parameters as functions of the state-price or risk-neutral density  $f(\cdot)$  and the statistical density  $p(\cdot)$ .

## 7.1.2 Risk Preferences

The majority of the studies reviewed in Section 7.2 share a methodological assumption which is rather arbitrary, thus, exposing the interpretation of the empirical findings to potential criticism. Motivated by the work of Breeden and Litzenberger (1978) - who first showed that given a set of options on aggregate consumption dense in the set of possible strike prices, a state-price density could be calculated - existing studies assume that a security index adequately proxies for aggregate consumption and they use options on a stock market index to derive the state-price density and consequently risk aversion functions. The S&P 500, used in all but one of the studies, represents roughly the 50% of public US equity capital and may raise objections as to whether it serves as a suitable proxy. In a sense, deriving risk aversion parameters



in this fashion and interpreting them as representative for the ‘composite’ investor is somewhat similar to, for example, deriving implied distributions from S&P 500 options and using them to price options written on any other traded security.

None of the existing studies has discussed this issue. Developing however a theoretical framework within which the above matter is fully resolved is far beyond the scope of this study. We, therefore, decided to draw my attention on the interpretation of the risk aversion functions and tried to improve our insight on the parameters calculated by Equations (7.7) and (7.8).

The methodology used in this article derives a state-price density from asset returns and options written on that asset which obviously do not represent the space of all traded securities but a subset defined by the asset under examination - Eurodollar and WTI futures in the present study. As a consequence the state-price density essentially corresponds to an empirical projection of the general state-price density onto the space defined by the assets’ pay-offs. This simply means that the state-price density estimated from Eurodollar and WTI futures options - or options on stock market indices - is particular to that market and should not be used to price all securities but a subset of securities with pay-offs contingent on the value of the Eurodollar and WTI futures contracts respectively. Equivalently, the estimated risk aversion should not be viewed as the individual’s risk aversion related to the preferences over aggregate consumption.

This chapter, therefore, estimates an empirical projection of the general risk aversion onto the space defined by the Eurodollar and WTI futures pay-offs. While investors' preferences over aggregate general consumption are not estimated, measures or indicators of investors' preferences over the aggregate consumption of the good under discussion are. With these theoretical considerations, Equations (7.7) and (7.8) are redefined as:

$$R_{APR} = \frac{p'(S_T)}{p(S_T)} - \frac{f'(S_T)}{f(S_T)} \quad \text{absolute risk preferences} \quad (7.9)$$

and

$$R_{RPR} = S_T \left( \frac{p'(S_T)}{p(S_T)} - \frac{f'(S_T)}{f(S_T)} \right) \quad \text{relative risk preferences} \quad (7.10)$$

## 7.2 Literature Review

Several approaches have so far been proposed for the estimation of Equations (7.9) and (7.10). The standard setting assumes a parametric or a non-parametric form for  $f(\cdot)$  and estimates are produced using options cross-sections. Alternatively a stochastic process is assumed and its parameters are estimated using contemporaneous option prices. To derive  $p(\cdot)$  similar procedures are usually employed and the parameters of the distribution assumed or the parameters of the stochastic process considered are estimated using historical asset returns data.

To the authors' knowledge Jackwerth (2000) was the first author who derived risk aversion functions from options prices and realised asset returns. He used a

modified version of the method developed in Jackwerth and Rubinstein (1996) to derive the risk neutral probability distribution and a Gaussian kernel density to calculate the subjective probability distribution of the S&P 500. Using a methodology similar to the one presented in Section 7.1.1 he calculated the average absolute risk aversion functions across wealth pre- and post- '87 crash. The findings of the study post-crash were not consistent with the assumptions of the model used. Several reasons that could have caused this inconsistency were examined and mostly ruled out. The inconsistency was attributed to mispricing of options in the market and that hypothesis was also confirmed on empirical grounds.

Ait-Sahalia and Lo (2000) also studied empirical risk aversion functions within a similar framework. They estimated the implied density non-parametrically. The statistical density was also estimated non-parametrically and risk aversion functions were calculated for the study period, in a fashion similar to Jackwerth (2000). Even though the implied risk aversion was found positive across wealth levels, Jackwerth (2000) replicated the study and claimed that the results were very sensitive to the model's assumptions, i.e. the bandwidth in the kernel estimation and the use of overlapping vs. non-overlapping returns in the estimation of the statistical density.

Rosenberg and Engle (2000) rather than estimating average risk aversion functions allowed for time variance to account for the cases when risk aversion could deviate from its average. This seems a more rational approach and can be substantiated with many arguments the main one being the expected variation of investors'



risk attitude under different states of economy: an individual is not expected to demand the same amount of compensation for a certain type of risk under economy expansions and recessions. The study used a stochastic volatility model - an asymmetric GARCH specification - to estimate the parameters of a pricing kernel that rationalised a cross-section of contemporaneous option prices. Two pricing kernel specifications were considered: a power function pricing kernel and a kernel defined by the exponential of an orthogonal polynomial expansion. Jackwerth (2000) argued that the shapes of the implied distributions were very close to a log-normal specification and the methodology as a whole bore the risk of not matching options' cross-sections very well.

Finally, Coutant (2000), estimated time-varying risk aversion functions using a semi-parametric form for the implied density and a GARCH specification - which accounted for asymmetric effects - for the estimation of the statistical density. The model was applied with data on the CAC 40 and risk aversion functions were estimated. None of the patterns found in Jackwerth (2000) and Rosenberg and Engle (2000) were present in Coutant (2000). What also raises some questions in the study is the inconsistency between the specifications of the implied density - which was assumed to be of a semi-parametric form - and the statistical density which was assumed to be the 'result' of a stochastic volatility process (see discussion in Section 7.3.3).



## 7.3 Data and Estimation

### 7.3.1 Data

The analysis is carried out on a subset of the data set described in Chapter 4. Additional data were collected for the underlying Eurodollar futures and the WTI futures contracts.

The Eurodollar futures complementary data consist of daily observations of the Eurodollar futures settlement price for the period September 1, 1994 through to November 30, 1998 for the December 1998 contract. Additional data for the WTI futures were also collected for the period September 16, 1994 to November 16, 1998 daily for the near month and the 2nd near month forward contract. In both cases the data were obtained from Datastream.

Following the suggestion of Jackwerth (2000) the analysis was focused on the centre of the distributions. Therefore options with moneyness (strike price / index asset level) between 0.84 and 1.12 - where relative errors are expected to be lower - were considered. This is also justified by the findings of Chapter 6. For certain regions representing a large percentage of total probability (for example between the 10th and the 90th percentile), implied RNDs are relatively free of mathematical priors imposed by a specific economic model or structure, and highlight the fact that the stability of the implied densities in these regions is also relatively irrespective of the model used. Thus, any differences in the shape of the risk preferences mea-

ifferences among alternative model specifications should not be attributed to the different specifications of the models.

### 7.3.2 Estimation of the Risk-Neutral Density

The implied RNDs are derived with an *ESE* model. The derivation of implied RNDs with the *ESE* model is thoroughly described in Chapter 4. For ease of reference, however, a number of key points are demonstrated below.

The *ESE* specification assumes the following analytic representation for the RND:

$$f(S_T) = a(S_T) - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3 a(S_T)}{dS_T^3} + \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^4 a(S_T)}{dS_T^4} \quad (7.11)$$

In Equation (7.11)  $a(S_T)$  is the log-normal density defined by

$$a(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp \left[ -\frac{1}{2} \left( \frac{\log S_T - \mu}{\sigma \sqrt{\tau}} \right)^2 \right]$$

with mean

$$\mu = \log S_t - \sigma^2/2\tau$$

and variance

$$\sigma^2 = \sigma \sqrt{\tau}$$

where  $S_t$  is the current price of the underlying futures (time  $t$ ),  $S_T$  is the price of the underlying futures upon maturity of the option,  $\tau (= T - t)$  is the time between now and the expiry date of the option, and  $\sigma$  the expected volatility of the returns of underlying futures contract.

Following the argument of Ross (1976) and Cox and Ross (1976) the value of an option in a market that offers no arbitrage possibilities can be expressed with the following:

$$V = e^{-r\tau} \int_{-\infty}^{\infty} g(S_T) f(S_T) dS_T$$

where  $g(S_T)$  is the pay-off function of the option and  $f(S_T)$  the distribution of asset prices at the end of any interval between  $t$  and  $T$ .

The above framework results in the following option pricing formulae

$$c_F(K) = c_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} - \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K)}{dK^2} \right] \quad (7.12)$$

$$p_F(K) = p_A(K) - e^{-r\tau} \left[ \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{da(K)}{dK} - \frac{(\kappa_4(F) - \kappa_4(A))}{4!} \frac{d^2a(K)}{dK^2} \right] \quad (7.13)$$

In Equations (7.12) and (7.13)

$$c_A(K) = e^{-r\tau} [S_t N(d_1) - KN(d_2)]$$

$$p_A(K) = e^{-r\tau} [KN(-d_2) - S_t N(-d_1)]$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

and

$$N(\cdot) = \text{cumulative standard normal}$$

Recalling the analysis in Chapter 4 (pp.69-73) an American style option can be expressed as a weighted sum of the following bounds:

$$\overline{C}(K) = E_t [\max \{0, S_T - K\}]$$

$$\underline{C}(K) = \max \{E_t [S_T] - K, e^{-r\tau} E_t [\max \{0, S_T - K\}]\}$$

where  $\overline{C}(K)$  and  $\underline{C}(K)$  are the upper and lower bounds for a call option and

$$\overline{P}(K) = E_t [\max \{0, K - S_T\}]$$

$$\underline{P}(K) = \max \{K - E_t [S_T], e^{-r\tau} E_t [\max \{0, K - S_T\}]\}$$

and  $\overline{P}(K)$  and  $\underline{P}(K)$  are the upper and lower bounds for a put option. The expectations in the above are taken with respect to the distribution the terminal asset price is assumed to be drawn from.

Weighting the upper and lower bounds the values of call and put options are expressed as

$$C(K) = \hat{w}_{it} \overline{C}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{C}(K; \hat{z}) + \hat{\epsilon}_{C(K)}$$

$$P(K) = \hat{w}_{it} \overline{P}(K; \hat{z}) + (1 - \hat{w}_{it}) \underline{P}(K; \hat{z}) + \hat{\epsilon}_{P(K)}$$

where

$$i = \begin{cases} 1 & \text{if } \begin{cases} \text{call and } K < S_t \\ \text{put and } K > S_t \end{cases} \\ 2 & \text{otherwise} \end{cases}$$



Implied RNDs are then recovered by minimising the sum squared errors

$$\min_{(\hat{z}, \hat{w}_1, \hat{w}_2)} \sum_{K_i} \{ [C_{OBS}(K_i) - C(K_i)]^2 + [P_{OBS}(K_i) - P(K_i)]^2 \}$$

subject to

$$E_t [S_T] \equiv S_t$$

and

$$f(S_T) \geq 0 \quad \forall S_T, S_T \in [0, \infty)$$

$$\int f(S_T) = 1$$

where  $S_t$  is the current price of the underlying futures contract.

### 7.3.3 Estimation of the Statistical Density

In order to estimate the investor's probability density function over terminal portfolio values historical time series of the underlying securities are used. To maintain consistency with the considerations of existing studies a 4-year frame is selected as in Jackwerth (2000).

One approach to density estimation is *non-parametric*<sup>47</sup> which allows the estimated density to accommodate rather flexible shapes. Alternatively *parametric* estimation techniques can be used in which the terminal asset price is assumed to be drawn from one of a known parametric family of distributions e.g. the log-normal family. The present study employs a technique which falls somewhere in the mid-

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<sup>47</sup> See Silverman (1986) for a review of non-parametric density estimation techniques.

dle. The terminal statistical distribution  $p(\cdot)$  is assumed to be well approximated by an *ESE* specification similar to the one described by (7.11), thus, allowing for flexible distributional patterns. The functional form of  $p(\cdot)$  under the same considerations of Section 7.3.2 and Chapter 4, Section 4.2.1, is given by:

$$p(S_T) = g(S_T) - \frac{(\kappa_3(P) - \kappa_3(G))}{3!} \frac{d^3 g(S_T)}{dS_T^3} + \frac{(\kappa_4(P) - \kappa_4(G))}{4!} \frac{d^4 g(S_T)}{dS_T^4} \quad (7.17)$$

where  $g(\cdot)$  is a log-normal distribution with mean  $\mu$  and variance  $\sigma^2$ . The density  $p(\cdot)$  is then estimated by finding estimates of the parameters of Equation (7.17) from the underlying futures returns time series.

The methodology used to derive estimates for the parameters of Equation (7.17) is similar to the one presented in Jarrow and Rudd (1983). The methodology exploits two properties inherent in the model specification defined in (7.17):

- the cumulants of the approximating log-normal distribution  $g(\cdot)$  are simple functions of the instantaneous standard deviation of the underlying asset returns time series
- the cumulants of the true distribution  $p(\cdot)$  are simple functions of its empirical moments

To fully specify the parametric form of the statistical density we proceed in the following way.

Firstly, the instantaneous standard deviation,  $\sigma$ , of the underlying asset returns is estimated.  $\sigma$  is estimated as the sample standard deviation of the previous forty-five days of logarithmic returns. The standard deviation of these returns is the instantaneous standard deviation,  $\sigma$ , of the underlying security which is used in the BSM formula. This is essentially the standard deviation of the log-normal distribution around which the expansion is taken.

The cumulants of the log-normal distribution are then obtained as simple functions of the instantaneous standard deviation [see Jarrow and Rudd (1982)].

The cumulants of the underlying asset need then be calculated. These, as already mentioned being functions of the moments of order 3 and 4 of the actual returns distribution, are calculated from the skewness and the kurtosis coefficients of the actual distribution of the logarithmic returns.

Having specified the parameters of Equation 7.17 it is possible to calculate the statistical density. Jackwerth (2000), measures the actual distribution with a historical mean and assumes a risk premium of 8% to make it comparable to the risk neutral probability distribution. The present study does not directly assume a risk premium. To account for the fact, however, that the mean of the statistical density can be rather different from the mean of the future period associated with the RND, the current futures price is used as the mean of the statistical density. Although this choice may seem somewhat arbitrary, Jackwerth (2000) suggests that a risk premium

in the interval where most economists would expect it to lie over the long run (5-10% per year) is not expected to differentiate the results qualitatively.

The approach described in this section does not make use of any sophisticated econometric or statistical algorithm to derive the statistical density, thus being subject to potential criticisms for not being accurate enough. Two main reasons, however, justify the use of the former estimation procedure. Firstly, the assumption regarding the semi-parametric form of the density does not place any structure on the stochastic process of the underlying asset - it is well known that a stochastic process is consistent with one probability distribution whereas a probability distribution is consistent with more than one stochastic processes - thus allowing the dynamics of the economy to evolve without restrictions. Secondly, the methodology followed serves as the exact equivalent to deriving a density from its implied moments - the 'aggregate' investor described in Section 7.1.1 derives his equilibrium asset prices using an *ESE* distributional assumption, subject to a budget constraint which uses state-price densities of the same semi-parametric form.

## **7.4 Results and Discussion**

The present study considers an investor with investment horizons of one and two months. The dates considered for the Eurodollar market are: October 15, 1998 two months before the expiration of the December Eurodollar futures options, on the day of federal funds rate cut by 25 basis points; and November 16, 1998 one month



before expiration of the options contract, one day prior to the scheduled meeting of the FOMC, where the rate was also lowered by 25 basis points. The dates considered for the WTI market are: September 16, 1998 and October 16, 1998 two months and one month before the December 1998 options contracts' expiration. Autumn '98 was a period of great uncertainty for oil prices. Prices declined, mainly due to increasing supplies, since non-OPEC members were not willing to comply with the organization pumping limits. In addition, Venezuela, one of the major oil exporting countries, announced that the previously decided exploration cuts would cease to apply. The International Monetary Fund (IMF) lowered its predictions for oil prices for the current - and the following - year and the American Petroleum Institute (API) reported a surge in US crude oil stocks.

Figures 7.1 and 7.2 plot the absolute risk preferences functions given by Equation (7.9)<sup>48</sup> for an individual with one and two months investment horizons and relate them to the market conditions of the time. For reasons already mentioned a range of  $\pm 2.5$  standard deviations around the mean of the implied distributions is plotted.

Figures 7.1 and 7.2 plot risk preferences function across wealth. Within the framework of the present study the terminal wealth is defined as  $S_T/S_t$ . In this setting an expected wealth level  $\geq 1$  is associated with expectations of the value of the risky asset ending in equal or higher levels compared to the present level at the end of the investment horizon and vice versa for wealth level  $< 1$ .

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<sup>48</sup> These results are reported for economy of space. It is almost certain results from Equation 7.10 would have stimulated discussion along the same lines.

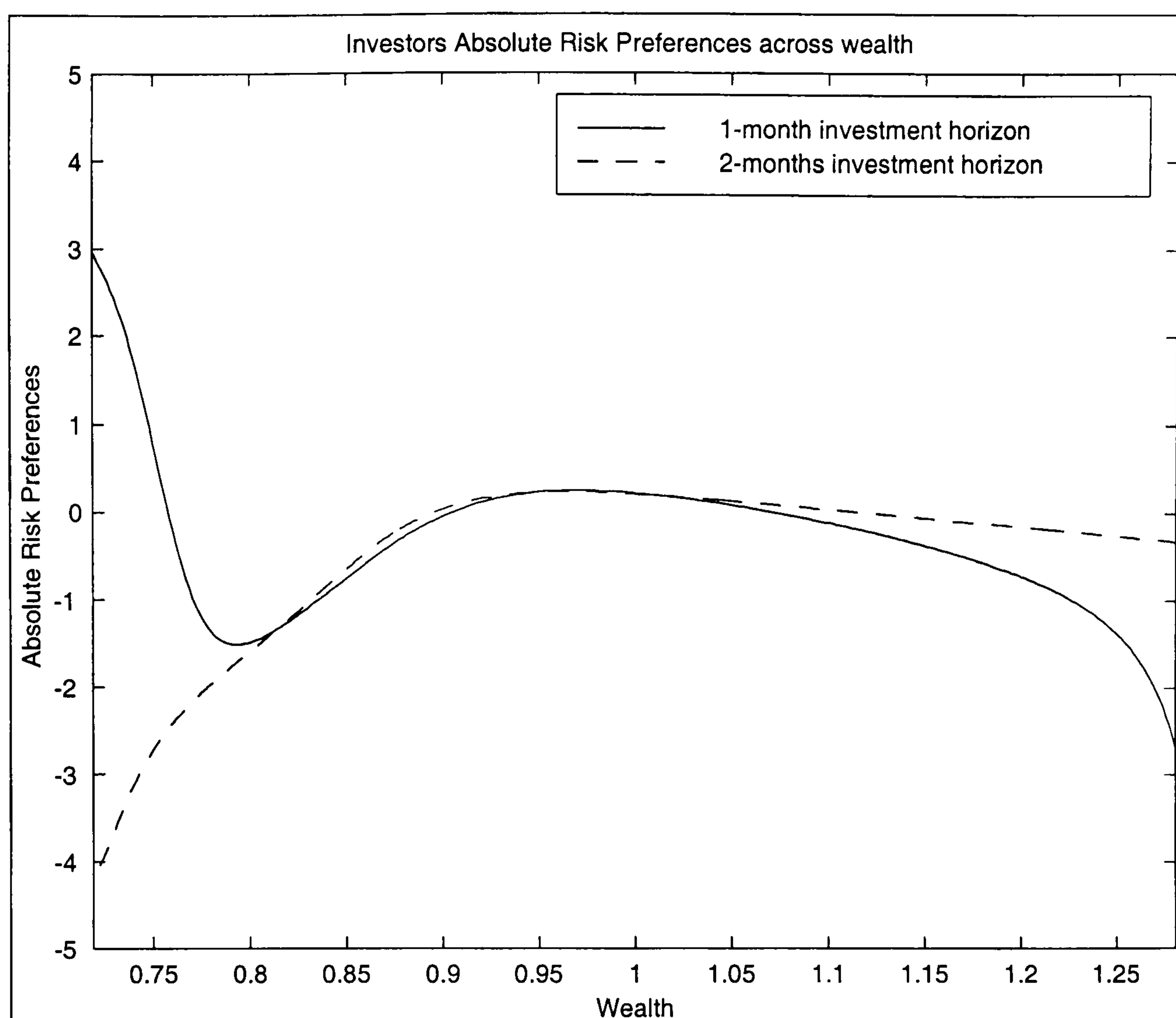


Figure 7.1: Absolute Risk Preferences functions across wealth for the WTI market calculated on September 16 and October 16, 1998

The patterns in Figures 7.1 and 7.2 are very similar to those presented in Jackwerth (2000) and also confirmed by Rosenberg and Engle (2000), for a data set on the S&P 500 index.

A region of negative risk aversion is found approximately over the range -10% to 3% for the two-months investment horizon which extends in the range between -6% to -1% and also from around 6% onwards for the one-month investment horizon in the case of the Eurodollar market. The pattern is fairly similar for the WTI market but shifted to lower levels of wealth (approximately between -25% and -15% and

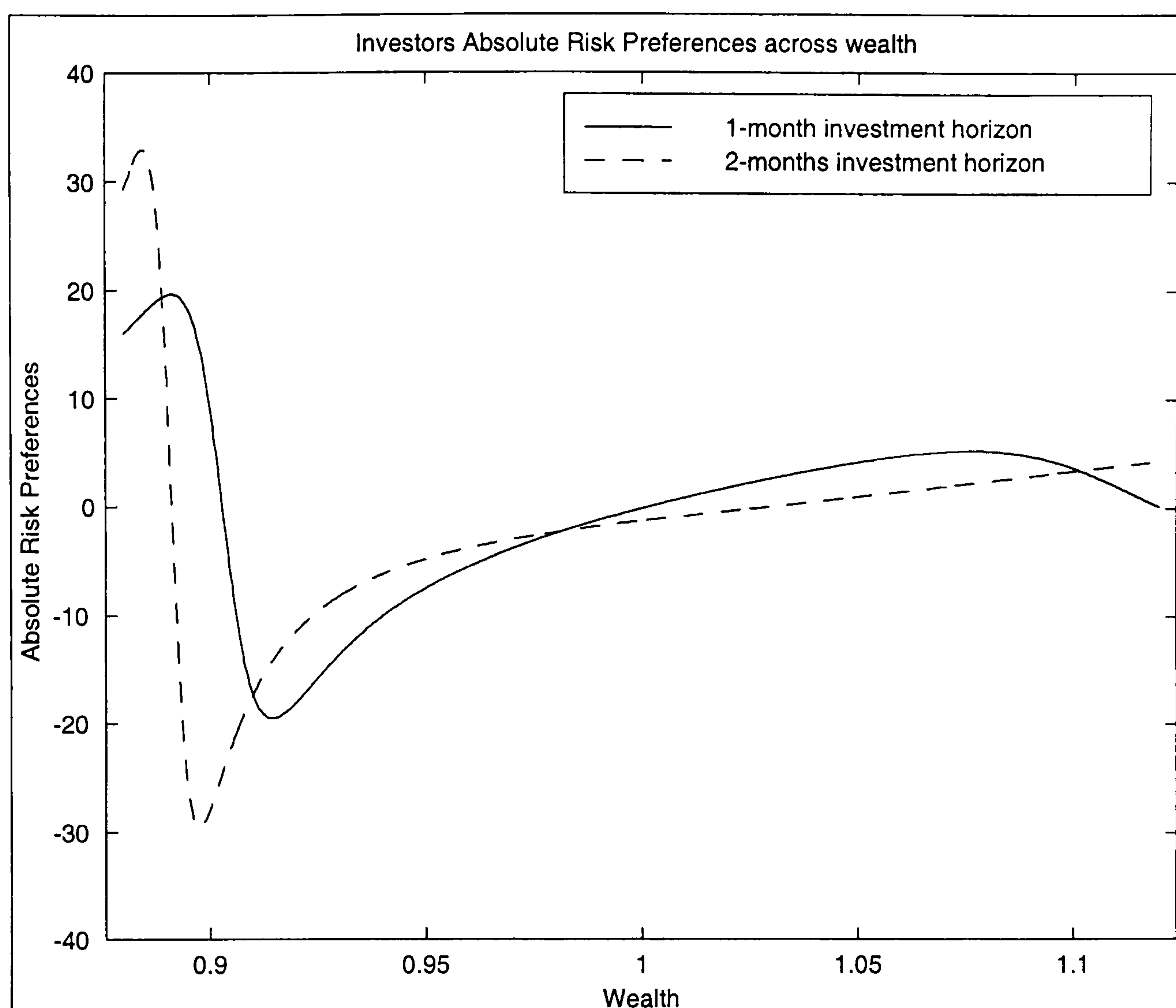


Figure 7.2: Absolute Risk Preferences functions across wealth for the Eurodollar market calculated on October 15 and November 16, 1998

from about 5% onwards) for the one-month investment horizon. For investment horizon of two months the graph was shifted to further lower levels of wealth (up to roughly -10%).

In all but one of the cases (the case of two-months investment horizon for the Eurodollar market) a globally decreasing relationship between risk preferences and wealth is found. This is consistent with economic theory, in contrast to the respective findings of Jackwerth (2000).

The results also relate to the market conditions.

For a large region of wealth levels - between approximately 0.98 and 1.1 - for the Eurodollar market, the absolute risk preferences for the one-month investment horizon are higher than those for the two-months investment horizon. This means that for the same level of wealth investors are willing to decrease the amount of risky asset held. On the other hand, risk preferences in the negative returns region up to roughly -9%, are at a lower level for the one-month investment horizon revealing that investors would require less compensation to bet against a market fall. These results confirm the findings of Melick (1999) who reported that '...The latter [interest rate cut of October 15] came as a great surprise to financial market participants' and also relate to the findings of Chapter 5. It seems however that the case was not the same for the cut that occurred on November 17 as the degree of risk aversion across wealth levels in the same region is shifted higher compared to the one corresponding to October 15.

For the WTI market the picture is somewhat the inverse. Iraq's confrontation with the UN arms inspectors in November 98, and the forecast for a cold northern winter on the other hand, gave rise to speculation for higher oil prices. This is reflected in the risk preference functions plotted in Figure 7.1. For wealth levels greater to 1, and for the same levels of wealth, the investor shows lower level of risk aversion in the case of 1-month investment horizon than in the case of 2-months. This is interpreted as a willingness to hold the risky asset, an indication that he expects the market to rise. The shape of the risk aversion preferences for negative returns



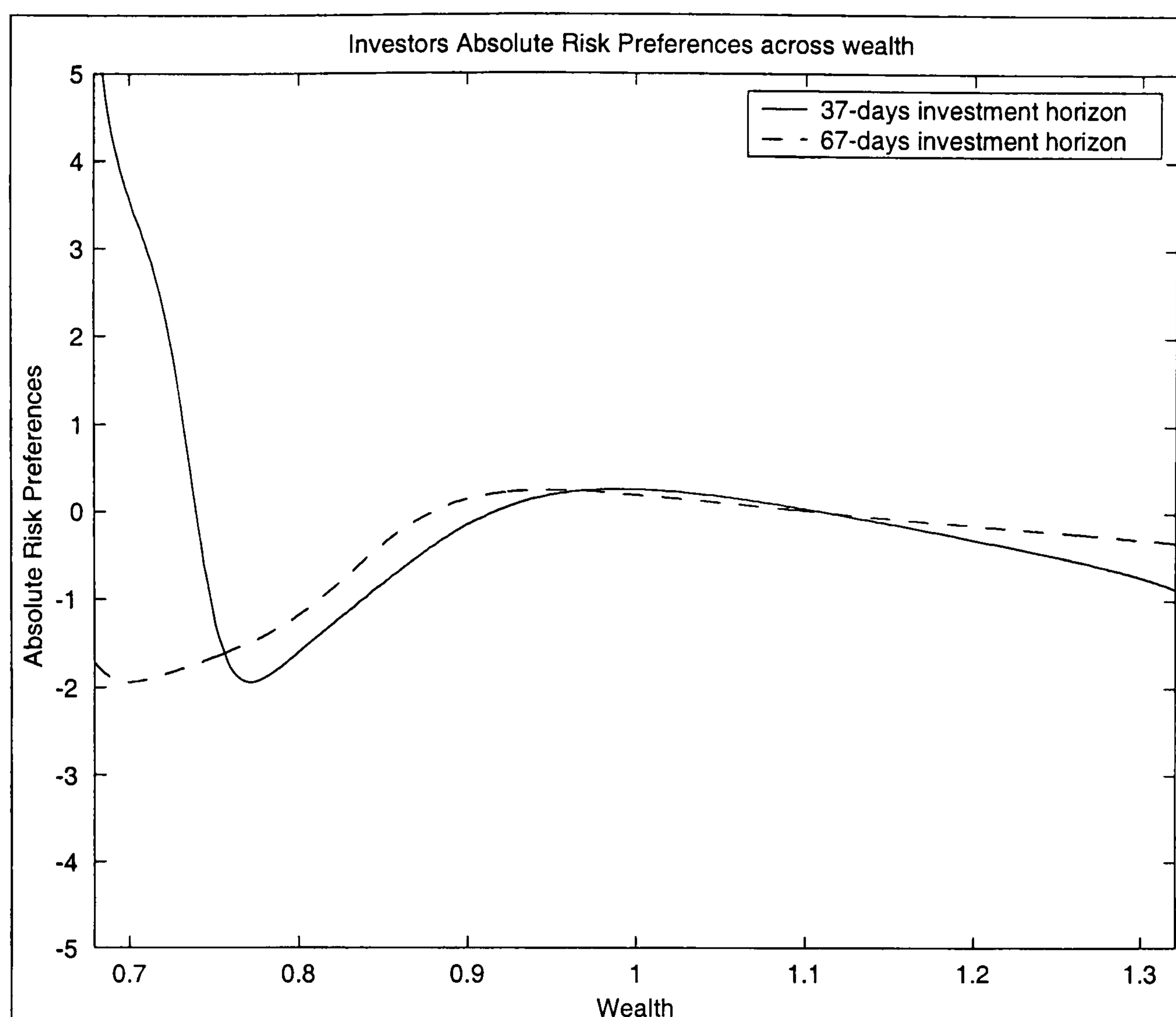


Figure 7.3: Absolute Risk Preferences functions across wealth for the WTI market calculated on September 9 and October 9, 1998

is almost identical. For the region, however, below  $-20\%$  returns it seems that an investor with a two-month horizon would require less compensation (as he expects a market decline) compared to the compensation that an investor with an one-month investor horizon would require, the latter expressed as higher risk aversion for the same levels of wealth. This probably reflects his expectation for a market rise.

To examine whether the imputed risk preferences are stable Equation (7.9) is calculated for one additional set of dates for each asset. The days considered correspond to days one week prior to the respective ones plotted in Figures 7.1 and

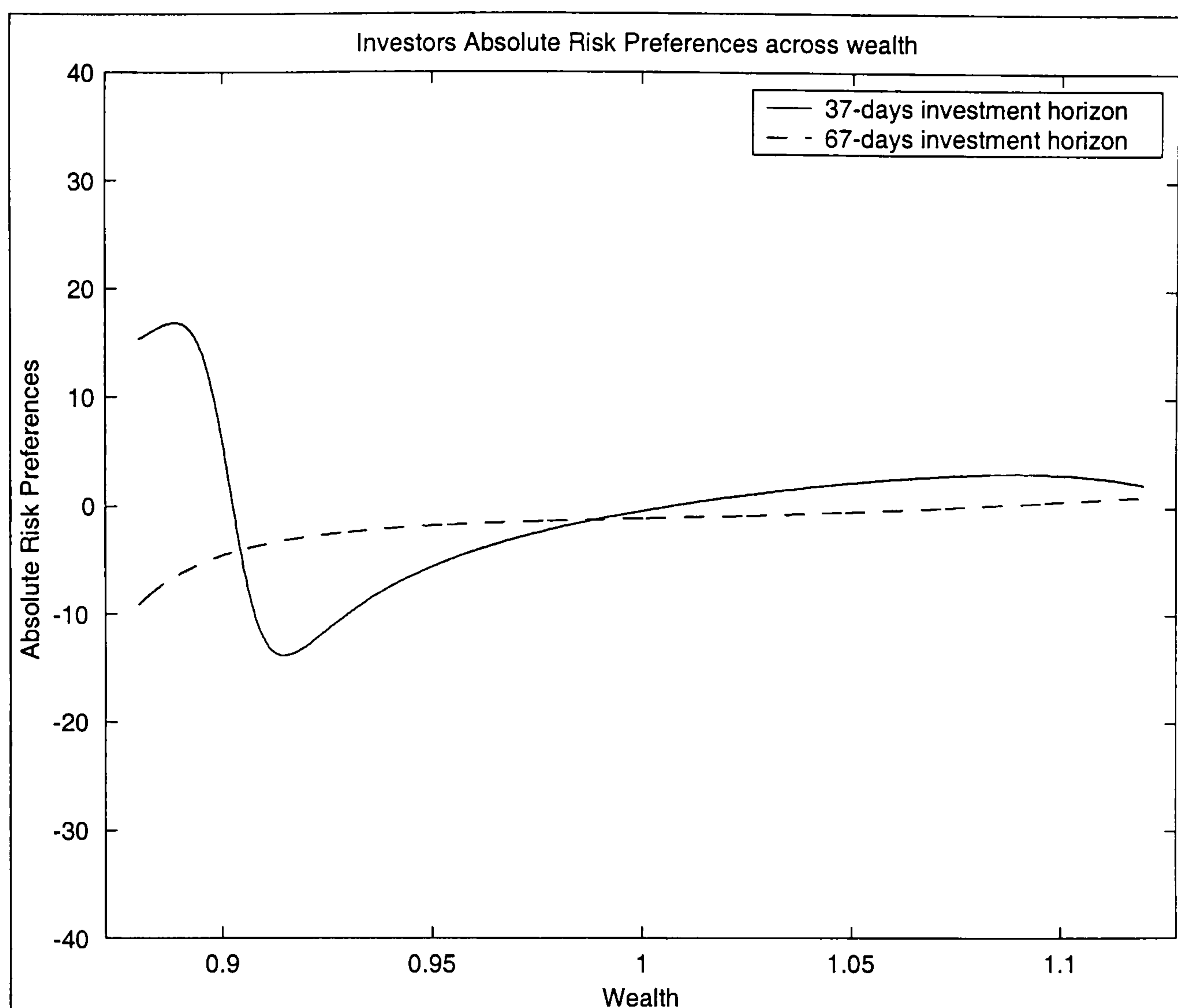


Figure 7.4: Absolute Risk Preferences functions across wealth for the Eurodollar market calculated on October 8 and November 9, 1998

7.2. Risk preferences for September 9 and October 9, 1998 for the WTI contract are shown in Figure 7.3 and for October 8 and November 9, 1998 for the Eurodollar contract are plotted in Figure 7.4. While the shapes of the risk preference functions are not identical, similar patterns persist. The shape of the functions are more intense in Figures 7.1 and 7.2, a rather expected feature as implied PDFs generally tend to be more leptokurtic when the time to maturity decreases (see also Figures 4.3 - 4.6 in Chapter 4 for the evolution of the respective PDFs).

Although the results presented above are somewhat encouraging and rather consistent with the market conditions, there are issues that require further attention. The risk preference function calculated by Equation 7.9 is the vertical distance of two downward sloping parametric functions -  $\frac{p'(S_T)}{p(S_T)}$  and  $\frac{f'(S_T)}{f(S_T)}$ . It follows then that risk preferences depend, not only on the parametric form assumed for the risk-neutral and the statistical densities, but also, on their relative position on the wealth axis. One immediate question that could, therefore, be posed is what confidence can be placed in attributing all of the deviation of the shape of the conditional distribution, from the unconditional distribution, to risk preferences. To the extent that both the risk neutral and the statistical densities are well approximated with the parametric forms presented in Sections 7.3.2 and 7.3.3 respectively the only obvious additional component of the risk preferences seem to be some part of the risk premium not correctly taken into account in the calculations in Section 7.3.3. It seems however that, as in Jackwerth (2000), a reasonably higher or lower risk premium than the one indirectly assumed in the study, would not dramatically change the shape of the risk preference function, although it would cause small upward or downward shifts respectively in Figures 7.1 - 7.4. The most serious concern though arises from the fact that the risk preferences plotted in the above figures are not consistent with economic theory in the entire wealth domain. If the assumptions with the representative investor presented in Section 7.1.1 hold true and more specifically if  $U(\cdot)$  is concave -  $U''(\cdot) < 0$  - and state-independent, given also that  $U(\cdot)$  is

increasing -  $U'(\cdot) < 0$  - since investors prefer more wealth to less,  $R_A$  in Equation 7.5 should not result in negative values under any circumstances. The findings of the present study suggest that one or even more of the underlying assumptions are violated in practice. Jackwerth (2000) suggests that the most likely explanation for similar patterns in his study is mispricing of options in the market but we would be very reluctant to accept that the mispricing persisted in the entire period that he studied - traders would have exploited any such opportunities. While empirically or theoretically examining the causes of the violations is beyond the scope of the present study we believe that future work should concentrate on examining whether  $U(\cdot)$  is indeed state-independent, a hypothesis also mentioned in Jackwerth (2000), and under what conditions its curvature can change sign.

## 7.5 Conclusions

This chapter illustrates an application where the information content of option prices, rather than qualitatively be assessed, is explicitly quantified.

The inverse of the equilibrium asset pricing problem is solved to identify preference parameters - *given asset prices and a probability model for futures states what can be inferred about investors' risk preferences?*

The estimated risk preferences functions were found consistent with existing empirical evidence. The risk preferences functions were also examined and found consistent with the market conditions of the study period.



Even though most of the results were encouraging we are left puzzled with two features of the risk preference functions: the inconsistency in the monotony; and the negative values in certain regions of wealth levels. While Jackwerth (2000) ruled out several reasons that could cause the latter and concluded that the most obvious reason was options mispricing, future research should examine this more deeply.

# Chapter 8

## Thesis Scope, Findings and Contribution, Extensions, and General Discussion

In concluding the thesis a number of general issues should be addressed. This final chapter summarizes the finding of the preceding analysis and presents some ideas for future research. It also addresses some limitations that may arise in the estimation and the interpretation of RNDs in the set-up of the present study and, finally, identifies the type of applications for which the use of RNDs is, if not the only, the best tool at hand.

### 8.1 Thesis Scope and Findings

The scope of the thesis is:

- to develop a technique for the estimation of implied RNDs which can incorporate the characteristics of modern financial markets
- to develop realistic and economically sensible tests for the validation of implied RNDs and implied RNDs estimation techniques - on qualitative as well as quantitative grounds - and perform them with that technique
- to present an empirical use of implied RNDs and illustrate it with the use of that technique

Along these lines Chapter 4 develops the general theoretical framework and the numerical algorithm for the estimation of implied risk-neutral densities of the *ESE* type from options prices. The *ESE* type parameterisation provides a method for finding a series expansion of a non-Gaussian probability distribution of which kind the empirical distribution of asset log-returns has been found to be. The most important features of the *ESE* probability specification that prove extremely useful in empirical applications and make it an appealing parameterisation are:

- the ability to select from a broad range of reference distributions provides flexibility in finding one that closely approaches the distribution to be approximated.
- by construction the coefficients in the expansion are simple functions of the moments of the given and the approximating distributions. As a result the parameters that define the PDF have a physical meaning as opposed to being abstract mathematical quantities as in the case of other parametric families of distributions i.e. the parameters that define a mixture of  $k$  log-normal distribution or a Pearson type density.

The technique is applicable to European options written on a generalized asset that pays dividends in continuous time or American futures options.

Chapters 5 and 6 develop realistic and economically sensible tests for the validation of implied RNDs and implied RNDs estimation techniques - on qualitative as

well as quantitative grounds - and apply them to assess the performance of the *ESE* model.

In Chapter 5 the information conveyed by the implied probability distributions, recovered by the *ESE* method is 'compared' with the market conditions at the time to ensure that the model is able to capture the general market sentiment and also able to incorporate isolated events causing a significant impact on the market. In addition, the model's explanatory ability is tested against that of the corresponding nested single log-normal to check whether it offers a statistically significant better fit to observed option prices. Implied RNDs of the *ESE* type are found to be able to capture the general market sentiment as well as able to incorporate isolated events with a significant impact on the market. In addition the *ESE* model is found to offer a statistically significant better fit to observed option prices and can, therefore, be considered a superior means - compared to the log-normal BSM parameterisation - of extracting information implied in option prices.

To examine the 'goodness' of the *ESE* methodology on a more quantitative basis, Chapter 6 addresses the issue of the robustness of implied RNDs in the presence of measurement errors. Observed option prices, which are used as inputs in the estimation, are subject to various errors; and, as a consequence, deviations from what one would expect to obtain under the models' assumptions, are present. It is natural then that the effect of these measurement errors is also examined and the robustness of RNDs estimation techniques or the degree of confidence that can be placed on



the summary statistics calculated off the implied RNDs is investigated. Errors that may distort the implied RNDs and consequently affect the robustness of the estimation techniques are replicated with small random perturbations in observed option prices. Implied RNDs are then re-estimated and their properties are summarized in a number of summary statistics. The results are highly supportive of the superior performance of the *ESE* model which is found to be more stable than the mixture of two log-normals specification; it is found more capable of estimating on average densities whose summary statistics converge to the original solution; and also capable of estimating statistics with relatively low dispersion.

Finally, Chapter 7 illustrates an application where the information content of option prices, rather than qualitatively be assessed - by 'reading' implied RNDs' graphs, is explicitly quantified. A fundamental principal of economic theory is employed: *in the absence of arbitrage, all asset prices can be expressed as the expected value of the product of the pricing kernel (a preference function) and the asset payoff*. It follows then that, the pricing kernel, coupled with a probability model for the future states, gives a complete description of asset prices, expected returns and risk preferences. This chapter solves the inverse of the equilibrium asset pricing model to identify the preference parameters - *given asset prices and a probability model for futures states what can be inferred about investors' risk preferences?* Using the *ESE* probability model this chapter derives risk aversion functions and compares them with the market conditions of the study period. The estimated risk aversion func-

tions are found consistent with existing empirical evidence and consistent with the market conditions of the study period.

## 8.2 Extensions

Throughout the development of the model and the subsequent empirical analysis we have identified a few points that may raise some criticisms. We have also identified a number of potential extensions which can - to some degree - complement the study and consequently shed some light on issues not covered herein

### Chapter 4

- In the development of the functional form of the RND the generalized *ESE* density given by Equation (4.7) p. 64 is truncated to the term of order 4 following the suggestions of existing studies. It may be the case that convergence of the series is attained - as suggested in Schleher (1977) - by including error terms one beyond the highest order matching cumulant which in my set-up is the second. On the other hand more terms may be needed in certain applications to allow the series to converge.
- While the consideration of a log-normal reference distribution is standard practice the use of other parameterisation i.e. mixture of two log-normal distributions, gamma distribution etc. may prove more effective.

- Imposing the constraints of the approximating density being positive and integrating to one in the optimization can make the interpretation of changes in RNDs more difficult, depending on whether these constraints become binding. This can be seen as a disadvantage. The alternative, though, to imposing the constraints would imply that no reliable statistics could be estimated off that distribution - when the density becomes negative and/or does not integrate to unity. The fact that an implied distribution estimated with the *ESE* methodology can take on negative values or not integrate to unity could be due in some circumstances solely to mathematical reasons. While imposing the constraints can be an advantageous practice, further investigation of this issue open to future research.
- While visual evidence is presented in Section 4.2.5 to demonstrate the importance of imposing the constraints, a statistical test of the importance of these constraints is not performed. To our knowledge such a test does not exist. Perhaps a formal econometric test can be constructed by comparing for example option pricing errors when the constraints are imposed and when they are not.

## Chapter 5

- Even though the model is found to be able to incorporate changes of the market conditions the study does not examine how useful it is to ‘predict’

future outcomes. A study in the spirit of Gemmill and Saflekos (2000) would shed some light in this direction.

- Although the shapes of the implied densities are found to change in shape on certain circumstances available statistical tools do not allow to test for the significance of this change. A potential extension would include the development of such formal tests

## Chapter 6

- The present study uses bounds to account for the early exercise premium of American options and assumes/hypothesises that, since the same methodology is used for the *ESE* and the *MLN* parameterization, the remaining instability can be solely attributed to the parameterization of the PDFs alone. An interesting issue would be to formally test this hypothesis.
- Another interesting issue would probably be to compare the different methodologies used to test the stability of implied PDFs reported in Section 6.1.1 by constructing standardized metrics similar to the ones presented in Section 6.3 and to see whether any of the methods under- or over-states the instability.
- The analysis of the results is done after removing 1% of the outliers which may raise some questions since the test aim to investigate what happens in



these extreme situations. To address this properly we recall the steps involved in the estimation of the statistics. Firstly, an optimization procedure is carried out to recover the parameters of the two models and secondly, further numerical techniques are employed to calculate the summary statistics listed in Section 6.3. We would like to stress the point that the calculation of the 11 percentiles ( $\hat{X}_{005}$  to  $\hat{X}_{995}$ ) in particular, rely on the numerical solution of an integral equation. The filtering is done only after both steps have been terminated. The numerical procedures mentioned above - optimization and calculation of summary statistics - require the input of initial values to arrive to a solution. The initial values I used, however, were not checked at each of the few thousand runs performed to ensure they constituted a proper set of initial values. We considered the algorithm 'efficient' enough to handle such deficiencies. It could, however, be the case that a non-suitable set of initial values results in failure of convergence of the numerical procedure. This type of failure can not be attributed to measurement errors i.e. data imperfections which is what the test is intended to investigate. On the other hand, the test is set out to assess the relative stability of the two models. The presence of outliers in the summary statistics can significantly distort the values of the metrics reported in Tables 6.1, 6.2 and 6.3 which can, as a consequence, bias any comparison. As the study aims to assess the stability of implied distributions which are in a sense economically sensible, we felt that filtering

the results - so that 1% is removed - would ensure the latter and, at the same time, would account for the above mentioned potential imperfections. Future research may investigate to what extent the inclusion of these 'extreme' observations affect the findings.

## Chapter 7

- Despite being mathematically valid the method used for the estimation of the statistical density in Section 7.3.3 may raise some questions as to how well it approximates the empirical density of asset returns. An alternative methodology would include the use of Maximum-Likelihood Estimation techniques to calculate the parameters of the density. This would also be an interesting exercise.
- Even though the method for deriving risk preference functions is comprehensively demonstrated and compared to the market conditions of the study period other empirical implications such as pricing or hedging performance are not examined. Modelling of these functions could probably also help to better understand the pattern of risk premia especially during unsettled periods (crashes, general elections, etc.)
- Examining the linkage of the risk preference functions to general business conditions would offer an alternative tool to examining the Fama and French

(1989) hypothesis that risk-premia are highest at business cycle troughs and lowest at business cycles peaks.

The above topics are open for future research.

### 8.3 Contribution

The contribution of the present work is mainly synthesised in the following:

- It studies, for the first time, the *ESE* specification in the context of RNDs implied by American futures options.
- It investigates, also for the first time, the stability of the *ESE* model. It is also the first time that the robustness of a model belonging to the semi-parametric family class of models is studied.
- The ability of the *ESE* parameterisation properly reflect/capture investor's risk preferences is explored for the first time. It is also the first a semi-parametric model is used in such an exercise.

## 8.4 General Discussion

The present work concludes by trying to identify the individuals or the organizations for whom or which the model presented and the findings illustrated are of great importance.

- One straightforward application includes the pricing of any claim - or pay-off - contingent on the terminal value of the asset for which the implied RND is estimated and which has the same time to expiration as the set of options used in the estimation. These claims can be other options with exotic pay-offs, illiquid or OTC options as well as a wide variety of other instruments with embedded options such as caps, floors, collars, collateralized mortgage obligations etc. which have Banks, Investment Banks, Hedge Funds, individual corporations and others on the long or the short positions of the contracts.
- The method helps to reveal market participants' expectations of future changes in the underlying asset. This could be useful for the policy stance of monetary authorities i.e. the Federal Reserve or regulatory and other bodies i.e. the OPEC, that wish to take immediate action in unsettled periods.
- Implied RNDs can also be used as a check on forecasts developed by other measures and allow contrarian investors who disagree with the consensus



shape of the distribution to take the consistent with their expectations positions in the market.

For this the type of applications the use of implied RNDs is, if not the only, the best tool at hand.

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