Risk Management and Decision Making in Defined Benefit Pension Schemes

by

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Declaration

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Abstract

A stochastic approach to decision-making in defined benefit pension schemes is presented. Existing decision-making tools in the form of actuarial valuations and asset and liability modelling are discussed. These tools are shown to be inadequate to fully address the objectives of the various stakeholders.

Pension fund control using a quadratic criteria with linear factors is studied in the case where the fund is invested in a risk-free asset and a risky asset. Optimal asset allocation strategies are shown to be counter-intuitive. The optimal strategy is shown to involve increasing the allocation in the risky asset as the fund deficit increases and increasing the allocation in the risk-free asset as the fund deficit decreases. It is further shown that increasing the weight on the linear factors leads to an increase in the optimal allocation in the risky asset.

A risk management approach to decision-making is presented. This is shown to be a more satisfactory decision-making tool in terms of setting the funding and investment strategies. The objectives of the stakeholders are addressed through downside risk measures and a performance measure for the cost. Methods of solving the problem are discussed: an indifference curve approach and a stochastic multi-objective approach leading to Pareto optimal solutions.

It is shown that, in the indifference curve approach, an "efficient region" exists. This efficient region is such that all funding and investment strategies outside this region are inefficient; that is, such strategies can be improved by choosing strategies in the region. On the other hand, in the multi-objective approach, pareto optimal investment strategies are located along an "efficient frontier".

An extension to the stochastic approach is presented. Optimal funding and asset allocation strategies, over a range of projection horizons, are determined by taking into account the probability of default by the sponsoring employer. It is shown that, over a short-term horizon, bond-only asset allocation strategies are optimal, whilst over a longer horizon equity-backed asset allocation strategies are optimal.

Table of Symbols

Pension Fund

- $f(t)$: the fund at time t
- $c(t)$: the contribution income for the year $(t, t+1)$ assumed to be payable at time t
- $NC(t)$: the Normal Cost at time t
- $AL(t)$: the Actuarial Liability at time t
- $B(t)$: the benefit outgo at time t
- $Adj(t)$: the adjustment for gains and losses at time t
- NC: the fixed part of the total contribution rate
- $\ddot{a}_{\overline{m|}}$: the present value of an annuity-due
- k: equals $1/\ddot{a}_{\overline{m|}}$ where m is the spread period
- f: the accrual rate
- $PS(y,t)$: the past service for a scheme member aged y at time t
- $S(y,t)$: the salary for a scheme member aged y at time t
- $N(y,t)$: the total number of scheme members aged y at time t
- ${}_tp_y$: the probability that (y) will survive for t years

Statistical

- $AR(p)$: autoregressive process of order p
- $ARCH(p)$: autoregressive process with conditional heteroscedasticity of order p
- $\mathcal{F}_t$: information history up to time t
- $E\big[X\big|\mathcal{F}_t\big]$: the expected value of random variable X given all the information up to time t
- $E_0\big[\,\cdot\,\big]$: expectation given information at time 0
- $E\big[X\big|Y\big]$: the expected value of X given Y
- $N(\mu,\, \sigma^2)$: normal distribution with mean μ and variance σ^2

Investment Modelling

- CAPM: Capital Asset Pricing Model.
- DS1: Dynamic asset allocation strategy 1.
- DS2: Dynamic asset allocation strategy 2.
- XMU: long-term mean parameter for economic series X.
- XA: autoregressive parameter for economic series X.
- XSD: standard deviation parameter for economic series X.
- YR(t): total return index at time t for asset class Y.

Chapter 1

Introduction

1.1 Problem Statement

The management of defined benefit pension schemes is a challenging exercise due to the complexities involved. Such pension schemes involve several crucial elements. Firstly, there is a promise to pay a wage-related benefit at retirement. Thus such a promise will be affected by various demographic and economic uncertainties. Secondly, there are various stakeholders to such schemes. Stakeholders will have their own objectives. Taken as a whole, such objectives can often be conflicting.

Furthermore, there is the fact that wage-related liabilities are non-marketable. This implies that such liabilities cannot be traded in existing financial markets. This is a problem since, at the very least, it means that matching assets do not exist for such liabilities.

The actuary's duty is to manage such schemes whilst taking into account the various uncertainties and also the objectives of the stakeholders. Actuaries have traditionally used actuarial valuations and, in recent times, asset and liability modelling techniques as decision-making tools.

However, such tools are inadequate in several respects. Firstly, actuarial valuations are a deterministic framework and thus do not adequately take into account the stochastic nature of the environment in which defined benefit pension schemes operate. Such frameworks also have, traditionally, placed more emphasis on funding strategies than on investment strategies.

Secondly, some common formulations of the objectives under the asset and liability modelling techniques are inadequate as they mostly only involve the probability but not the severity. Such techniques can be made more robust by, for instance, properly linking the investment decision with the funding decision; and also by reformulating some of the common objective functions.

Thirdly, actuarial decision-making tools have to be seen in the light of the recent Myners' review of institutional investment. Myners (2001) set forth principles codifying "...best practice for pension fund decision-making..."

Among others, he proposed that the pension fund's investment objective should take into account the trustees' attitude to risk and "...should not be expressed in terms which have no relationship to the fund's liabilities..." (p. 148).

He further proposed that "...[s]trategic asset allocation decisions should receive a level of attention...that fully reflect the contribution they can make towards achieving the fund's investment objective...[and also that]...[a]sset allocation should reflect the fund's own characteristics..." (p. 148).

The purpose of this thesis is to consider the problem of decision-making and risk management in a scheme-centred approach. We present a decision-making approach that addresses the shortcomings of the traditional tools. At the same time this approach is robust enough to stand-up to the 'Myners test'. By the Myners test we refer to the kind of questions that ought to be asked of any pension fund decision-making approach following the best practice principles set out by Myners (2001).

The objective functions that we consider are downside functions. We also make the modelling as realistic as possible in order to be directly applicable to practical situations. Hence, analytical solutions are not possible. As a result we employ Monte Carlo simulations. As an extension of the decision-making approach we present an approach which explicitly includes the probability that the sponsor might default on his/her pension obligations.

1.2 Overview of the thesis

This thesis is organized as follows: in Chapter 2 we review the traditional decision making process in defined benefit pension schemes. We consider how actuaries have traditionally addressed the main concerns of the stakeholders. We discuss the adequacy of the actuarial valuation as a decision-making tool. And we suggest possible ways of extending the approach.

In Chapter 3 we review the Asset and Liability Modelling approach to decision making in defined benefit pension schemes. We discuss various problems that could arise in using this approach and suggest possible solutions.

In Chapter 4 we present a dynamic programming approach to the problem of pension funding and investment. We consider a criterion with quadratic and linear factors and show that such a criterion leads to higher funding levels and lower contribution levels than a purely-quadratic criterion. We let the asset allocation be the control variable and show that the optimal strategy is counter-intuitive. We discuss possible extensions of this approach and problems that could arise when considering realistic scenarios.

In Chapter 5 we present the stochastic approach to decision making. Under this approach we show that the funding and investment strategies can be considered simultaneously. We present downside risk measures for solvency risk and contribution rate risk and a performance measure for the cost. We also show how choices can be introduced in the setting of the 'Normal contribution rate' and, thus, enabling us to consider a large spectrum of possible funding strategies. Each of these choices is combined with a choice for the asset allocation strategy.

An analysis of these combinations is undertaken by using the risk and performance measures over pre-set projection horizons. We also show how the choice of different projection horizons offers flexibility in the decision making process. Two methods of finding the solutions are considered: an indifference curve approach and a stochastic multi-objective approach.

In Chapter 6 we present a case study illustrating the stochastic approach to decision making. We show how to obtain strategies that are efficient under both the solvency risk and the contribute rate risk. We also show that such strategies are also efficient under the cost performance measure. We further undertake investigations into the effect of different amortization periods and initial funding levels. Also, we compare fixed and dynamic investment strategies and show that a counter-intuitive strategy is optimal.

In Chapter 7 we consider asset model risk in the decision making process. We present an asset model built on modern asset pricing theory using the idea of pricing kernels and compare the results to those obtained under the Wilkie model. We show that the general principles underlying the stochastic approach hold regardless of the asset model but that the location of the efficient strategies depend on the asset model.

In Chapter 8 we extend the stochastic approach to decision making by considering the probability that the sponsoring employer might default on the pension obligations. We design the probability of default by considering the return on a tangency portfolio. A default risk measure, which takes into account the probability of default event and the amount of defaulted deficit should a default occur, is introduced.

We show that for short-term projections investment strategies with full-weighting in bonds are optimal under the default risk. Whilst for medium-term and long-term projections we show that the optimal investment strategies under the default risk are not significantly different from the optimal strategies under the solvency risk.

In Chapter 9 we present some concluding remarks and possible ways of extending this thesis.

Chapter 2

A Review of Traditional Actuarial Valuation Techniques

2.1 Introduction

In this chapter we discuss actuarial valuation techniques. These techniques have traditionally been used as a decision-making tool in the management of defined benefit pension schemes. We present an assessment of the adequacy of these techniques in addressing the various aspects of such pension schemes.

The responsibility of the actuary in a defined benefit pension scheme is to manage the scheme with the goal of meeting stakeholders' objectives. The main stakeholders include the sponsoring employer, the members and trustees of the scheme, and the government. The sponsoring employer (or sponsor) might have the objective that the contributions be stable. Meanwhile, the objective of the members and trustees might be to ensure that the promised benefits are secure. On the other hand, the government might introduce legislation governing the running of such pension schemes.

These objectives indicate the type of risks faced by defined benefit pension schemes. Firstly, the contribution could be unstable. This implies that there is a risk that, from one valuation to the next, the sponsor could be requested to make highly unpredictable contributions. Secondly, the benefits could be insecure. This could be the case, for example in a wind-up, if the accumulated assets are inadequate to meet with the liabilities. These risks are generally referred to in the actuarial literature as 'contribution risk' and 'solvency risk'.

We assess the adequacy of actuarial valuations in the management of such risks and other aspects in a defined benefit pension scheme. We begin in Section 2.2 with some principles that have historically governed actuarial valuations. In Section 2.3 we consider factors that influence the way actuaries approach a valuation.

In Section 2.4 we consider the funding methods used by actuaries in deciding the sponsoring employer's payment plan. And in Section 2.5 we consider their categorization. In Section 2.6 we consider the sources and treatment of gains and losses in the pension fund. Lastly, in Section 2.7 we consider ways by which the traditional actuarial valuation approach can be extended.

2.2 Historical Aspects

Actuaries have traditionally approached the problem of management of defined benefit pension schemes through actuarial valuations of the pension fund. Such valuations have mainly been carried out periodically, for instance triennially. However, special valuations could also be undertaken, for instance, due to a significant change in the law governing pension schemes.

One of the earliest papers into the subject of actuarial valuations is King (1905). He sets out the principles for dealing with pension valuation problems. His principles have, over the years, essentially formed the basis of modern techniques in pension fund valuations.

King (1905) suggested that the first step is one of data collection for investigating the experience of the defined benefit scheme and for undertaking the valuation. He introduced the idea of supplying the required information on "cards". In modern times this could be thought of as computer spreadsheets.

He suggested that for each member of the pension scheme the required information would be their date of birth, date of entry into the scheme and age at entry, date of exit from the scheme and age at exit, their mode of exit (for example withdrawal, death or retirement), and any past contributions into the scheme. Such information would be gathered for all the members and, upon adoption of a scale of salaries, a service table would be drawn up. The service table would show, for each age group, the number of active members, the number of members withdrawing from the scheme (that is, withdrawals), the number of deaths, the number of retirees, and the salary scale.

Once the service table is done, King (1905) suggested that the particulars for the valuation could be prepared. This would include, for each age group, the age attained, the number of members, and the total contributions.

He further developed formulas, in the form of commutation functions, to be used in valuing future and past contributions, withdrawal benefits, and normal retirement benefits.

What is clearly missing from these principles is a specification of how to choose the valuation rate of interest, how to deal with the valuation of assets, and how to introduce the effects of price and wage inflation. This is understandable, nevertheless, since before the First World War interest rates were stable in Britain and inflation was not a major problem. Thus assets were only being taken at either the book value or market value (or the lower of the two).

Puckridge (1947) introduced a new principle in the valuation of pension schemes. He argued that assets and liabilities should be valued "...at the rate of interest which it is anticipated can be earned on future investments..." (p. 2). This principle was also supported by Gilley and Funnell (1960) who further observed that due to the effects of inflation "...the yield on the accumulated fund [bore] little relation to the yield obtainable on new moneys and the book value of the assets [bore] little relation to their market value..." (p. 44).

With the introduction of Puckridge's principle it was no surprise that the anticipated rate of return on equities was used as the valuation rate for liabilities. In the pre-war years pension funds were mainly invested in fixed-interest assets. However, in the post-war years, Gilley and Funnell (1960) observe that there was a shift in pension fund investment from fixed-interest assets into equities "...as a hedge against inflation..." (p. 53).

Seldon (1960) concurs by noting that the "...tendency to favour ordinary shares...rather than gilt-edged securities reflected the desire to offset inflation, avoid losses in the capital values of the...Government stocks, and increase yields..." (p. 17).

2.3 Modern approach to the Actuarial Valuation

The modern approach to the actuarial valuation of pension funds has essentially maintained the principles set by the early researchers like King (1905). The approach taken by actuaries has mainly been determined by two main factors: the purpose of the valuation and the setting of actuarial assumptions (or actuarial basis). In this section we consider each of these factors and we also consider the traditional approach to the choice of investment strategies.

2.3.1 Purposes of a Valuation

The purpose of a valuation significantly affects the way that the valuation itself is carried out and the way that assumptions are set and assets and liabilities are treated. In the actuarial literature the purposes of actuarial valuations have been described in different, but related, ways.

Gilley and Funnell (1960) observe that the "...purpose of a pension fund valuation is to compare the [fund's] expected future income and outgo with a view to determining the size of any adjustments which should be made to the future contributions or benefits of the fund, so that the income over its lifetime may as nearly as possible meet the expected benefit outgo or at least be unlikely to fall short of it...". In this case we could say that the valuation is being undertaken on an ongoing basis.

Heywood and Lander (1961) observe that the approaches in early actuarial valuations of pension funds were influenced by actuarial valuations of life funds. They observe that the actuarial valuation of a pension fund was naturally regarded as "...a process in which the first stage was to arrive at the difference between the present value of the liabilities on the one hand and the present value of the future premiums or contributions on the other, thus producing the net liability..." (p. 316).

They further consider the idea that the actuarial valuation of a pension fund is "...an instrument [for determining or checking] the pace of funding the benefits of the [pension] scheme ..." (p. 317). This idea is in a way a summary of that proposed by Gilley and Funnell (1960). This will be further discussed under 'Funding Methods'.

Daykin (1976) observes that the actuarial valuation of a pension fund "...is a means of assessing what progress is being made towards satisfying..." pension funding aims (p. 288). The main aim could be to accumulate enough assets to meet with the promised benefits. Thus through a valuation an assessment is made as to what progress has been made since the last valuation and what steps should be taken until the next valuation.

Worthington (1985) observes that the purpose of a valuation is "...to provide the actuary with all the information he requires to give sound actuarial advice to his client, normally the trustees or the employing company, to enable him to satisfy any legal requirements in relation to that scheme, ... and to provide any actuarial statements about the financial position of the scheme that may be required..." (p. 3). This 'sound actuarial advice' could be advice on required adjustments as noted by Gilley and Funnell; or advice on the net liability and pace of funding as noted by Heywood and Lander; or, indeed, advice on the fund's progress as concluded by Daykin.

Booth et al. (1999) note that the purpose of a valuation is "...to look at the long-term position..." of a pension fund and to meet with the needs of various stakeholders. We have already listed the main stakeholders; however, others include the tax authorities, accountants and financial regulators.

Thus in general an actuarial valuation has a three-fold purpose. Firstly, the valuation gives the actuary more insight into the position of pension scheme regarding various pre-set goals. Secondly, armed with the new insight the actuary is better able to recommend to the sponsoring employer and/or trustees the necessary way-forward for the pension scheme. Using an appropriate funding method the actuary will recommend a contribution rate applicable until the next valuation. Thirdly, following the valuation the actuary will be able to comply with the requirements of the various authorities.

Currently in the UK preparation of valuation reports must be made in compliance with a Guidance Note, called GN9, prepared by the Institute and Faculty of Actuaries. GN9 sets out the type of information that the actuary must include in the valuation report.

2.3.2 The Valuation Basis

The valuation basis is the set of assumptions which the actuary sets, using his/her judgement, in order to conduct the valuation. The choice of valuation assumptions is one of the most important areas of an actuarial valuation. This is because most of the deviations of predicted results from actual results stems from the deviations of the scheme's actual experience from that anticipated in the valuation assumptions.

The valuation assumptions normally fall into one of two groups: demographic assumptions and economic assumptions.

Demographic assumptions

Demographic assumptions concern mortality rates and rates of withdrawal. Rates of mortality are further subdivided into pre-retirement mortality (that is, concerning current members) and post-retirement mortality (that is, concerning pensioners). Whilst withdrawal rates can be subdivided into various categories such as withdrawal from the scheme due to ill-health and withdrawal due to termination of employment.

These assumptions can either depend on a specified mortality table showing the various rates of decrement or could be constructed by considering the scheme's past experience.

The effect of demographic assumptions on the valuation results could depend on the size of the scheme. The characteristics of small schemes differ significantly from those of a large scheme. Two such characteristics are 'nonhomogeneity' and 'limited exposure'. Small schemes tend to have a nonhomogeneous membership profile due to the small total number of members and also the number of members at each age group.

Shapiro (1983, p. 19) observes that "...[i]n many small plans the bulk of the total annual contribution is attributable to the funding of anticipated benefits of a few owner-employees. The financial demands on plan assets generally are dominated by the specific experience of these few. Such nonhomogeneity can disturb the cash flow...".

Thus, for example, the sudden exit of a member with considerable past service could easily lead to deficits in the scheme. Such an exit could also threaten the existence of the scheme since, for example, the sponsor might not be financially able to cover the deficit or, indeed, the withdrawing member might also be the owner of the business.

Shapiro further observes that "...[i]n small plans, actual experience is likely to deviate substantially from expected experience...[since, for example,] the historical data generally are insufficient to develop such things as decrement factors...[and also] the limited exposure would undoubtedly lead to considerable variation from anticipated experience..." (p. 19).

These observations imply that for pension schemes of a small size the deviation of the demographic experience from that anticipated on the valuation date could have a significant effect on the financial position of the scheme.

Economic assumptions

The economic assumptions concern three main areas. The first area is the return from investments. The actuary will have to make an assumption about the expected return from the fund's assets. Such an assumption would be required in the chosen actuarial funding method to value the pension liabilities (that is, as a valuation rate of interest).

Booth et al. (1999) note that "...because the fund is accumulated to meet the liabilities, [the return from investments] also represents the rate of return at which the value of liabilities is determined. That is, it is used to represent both the gross yield earned on the fund as well as the rate of return used to discount the liabilities..." (p.504).

The second economic area concerns future inflation outlook. This can be subdivided into price inflation and wage inflation. Price inflation can be taken as one of the main drivers of wage inflation. Also pensions-in-payment are directly affected by price inflation since they are supposed to increase in line with the Limited Price Indexation (LPI).

Wage inflation would concern the scheme members' future salary improvements. Traditionally, actuaries have either used salary rates specified in a service table or have to construct a table of salary rates specific to a given scheme. Assumptions about future salary improvements are mainly required by those actuarial funding methods where the benefits are valued with reference to future salaries. An example of such a method is the Projected Unit Credit method (see Section 2.5).

Most uncertainty in the valuation results arises from economic assumptions. Clark (1992) notes that "...in practice it is likely that the uncertainties in the economic elements of the [valuation] basis will far outweigh those connected with the decrements...". As noted above, the effect of demographic assumptions could depend on the size of the scheme. On the other hand, the effect of economic assumptions might not depend on the size of the scheme but on the differences between the return on assets and inflation.

Wright (1998) notes that "...the actual values chosen for each element of the valuation basis is not particularly important, but the relative differences between each are critical, particularly between [the rate of return on investments] and [the rate of earnings inflation]..." (p.867).

Thus, for example, if the return on investments is lower than anticipated and the wage inflation is higher than anticipated then we could get losses in the scheme. Such losses could arise from low accumulated assets (due to the low investment returns) and/or higher prospective benefits due to the higher salary levels.

The criteria for setting valuation assumptions have been widely discussed in the actuarial literature. Shapiro (1985) suggests that the valuation assumptions ought to be conservative, consistent, best estimate, prudent, precise and flexible (also see Thornton and Wilson (1992).

According to Shapiro, a valuation basis is said to be conservative if the basis tends to lead to actuarial gains. In this case the actuary will be in a position to know the net effect of his assumptions but may not know the extent of the actuarial gain.

On the other hand, a valuation basis is said to be a best estimate basis if the assumptions are not too different from some best estimates. For example, the inflation assumption could be set by considering the mean or median of the underlying statistical distribution for inflation.

Meanwhile, Shapiro observes that "...[a]ssumptions are prudent in the aggregate if the contribution they generate is appropriate, in the sense that it would be developed by a prudent actuary in similar circumstances..." (p. 490). The understanding here is that a prudent actuary would seek to adopt assumptions that are within the range commonly used by other actuaries.

2.3.3 Investment Strategy

In the traditional actuarial valuation approach the choice of investment strategy is treated, if at all, as a separate exercise from the valuation process itself. Indeed, in the early actuarial literature fairly little is said about how to approach the problem of choosing an investment strategy. This is even more noticeable if one compares it, for example, with the work on funding methods.

Clark (1992) observes that "...[traditionally] actuaries have provided limited advice on long-term or strategic asset allocation, either formally or informally. Historically this was based on the actuary's general knowledge of the particular plan's liabilities, the policy being pursued by other similar plans and the actuary's general knowledge of investments..." (p.28).

This observation contains two of the principles that have generally been used in assessing the suitability of a pension scheme's investment strategy. These are the scheme's liability profile and the assets available in the markets.

Actuaries have traditionally used the principle that real asset classes should be used to hedge real liabilities; and that as the membership profile of a scheme changes, the investment strategy should also be changed accordingly. Thus, for example, although a mainly equity-backed strategy might be used for a young scheme, a more bond- or cash-backed strategy might be necessary for a mature scheme with a high proportion of retirees. One of the reasons being that liquidity is necessary for a mature scheme with pensioners whilst for young schemes the main problem is one of matching the salary-related liabilities.

2.4 Spectrum of Funding Methods

In this section we consider the various funding methods used by actuaries in the management of defined benefit pension schemes. The subject of funding methods is a very wide area of research. Some of the early work on this subject include Trowbridge (1952, 1963) and Bowers et al. (1976, 1979).

Despite the immensity of this subject, the definition of a funding method is simple. For instance, Bronson (1949) defines a funding method as "...any method of financing the benefits of a plan..." (p. 250). Whilst Trowbridge (1952) defines a funding method as "...the budgeting scheme or the payment plan under which the benefits are to be financed..." (p. 17). Thus in seeking a funding method we are essentially seeking a financing method which best meets the objectives of the stakeholders.

There is a very wide spectrum of funding methods. The extreme funding methods are the so-called Pay-As-You-Go method and the Complete method:

2.4.1 Pay-As-You-Go (PAYG)

Under this method the sponsor only contributes to meet the benefit outgo. Thus no fund is accumulated up and the cost of pension provision is not met during the member's working life.

The cost for a new plan would start at zero but would steadily increase as members start retiring. A long period may elapse before the cost stabilizes. Furthermore, if the cost stabilizes it may do so at too high a level for the sponsor to sustain. This might also coincide with fluctuations in the employment levels and the sponsor's financial strength (Anderson (1992, p.6, 7)).

Thus such schemes are not allowed for private pension plans but are still used by some public pension providers.

2.4.2 Complete Method

Under this method the sponsor does not pay regular contributions to the scheme. However, at the inception of the pension plan the sponsor sets aside an amount large enough to meet all future benefits.

There are two clear drawbacks with such a method. Firstly, it is almost impossible to exactly calculate the necessary amount. And secondly, even if such an amount was calculable, it might be too high for most private sponsors.

2.4.3 Other funding methods

Due to the inherent problems of the PAYG and the Complete methods, different funding methods have been suggested in the literature. In the next section we will deal with the categorization of these funding methods; whilst in this section we consider the basic rationale and fundamental criteria of these funding methods.

The basic rationale behind all these funding methods is that, firstly, the cost of pension provision should be met during the member's working life. And secondly, the funding method should be such that a fund is accumulated at such a pace that a member's pension is exactly met at the time of retirement (Anderson (1992, p.6, 7)).

Booth et al. (1999, p.526-530) describe in detail four fundamental criteria for funding methods. These are security, stability, durability, and liquidity. Firstly, the security criterion concerns the security of the past service benefits, taking into account future expected salary improvements, in the event of, for instance, discontinuance of the scheme. That is, the chosen funding method should be such that at any given time the value of the assets should be large enough to cover the discontinuance liabilities.

Secondly, the stability criterion concerns the financial needs of the sponsor. For management purposes the sponsor would prefer a funding method which leads to a stable contribution rate. Stability might be measured, for example, in relation to the current salary roll. An unstable contribution rate could mean that the employer would face an uncertain future financial outflow.

Thirdly, a funding method would be said to be durable if significant sudden changes in the pension scheme's circumstances would not lead to significant changes in the contribution rate. A change in the circumstances could, for instance, be the closing of the scheme to new entrants as the case has been in the UK in recent years.

And lastly, the liquidity criterion refers to the ability of a funding method to ensure that enough cash is available to meet the benefit outgo. This should not be a problem if the contribution income is higher than the benefit outgo.

Furthermore, even though such may not be the case, the investment return might help to ensure that there is enough liquidity in the scheme. This would be especially the case if the contribution and investment incomes and the benefit outgo took place around the same time - at the end of the year, for example.

2.5 Categorization of Funding Methods

In the actuarial literature funding methods are categorized in several ways. The first way is to categorize funding methods as either Individual or Aggregate Funding Methods. Another way is to categorize funding methods as either Accrued Benefit or Prospective Benefit Methods. Furthermore, funding methods can also be broadly categorized by the way gains and losses are treated under each method. We consider the treatment of gains and losses in Section 2.6.

2.5.1 Individual and Aggregate Funding Methods

Funding methods are categorized as either Individual or Aggregate depending on whether the calculations of the Normal Cost and Accrued Liability are done individually for each member of the scheme or in aggregate.

Under the Individual funding methods, the Normal Cost is determined as the sum of individual Normal Costs for each member of the scheme. The Normal Cost would either be defined directly or else be derived from the definition of the Accrued Liability.

Individual funding methods include the Unit Credit Family of methods, the Entry Age Normal method and the Attained Age Normal method. Turner (1984) refer to these methods, together with the Aggregate method, as the basic methods.

Unit Credit family of methods

The most common methods under this family are the Current Unit Credit method and the Projected Unit Credit method. Under this family of methods the Normal Cost is derived from the definition of the Accrued Liability.

For each member of the scheme the Accrued Liability is defined as the present value of the member's accrued benefit (deferred until the normal retirement age). The difference between the Current Unit Credit and the Projected Unit Credit methods is that for the Projected Unit Credit method the accrued benefit is calculated by referring to the member's projected final salary; whilst under the Current Unit Credit method the accrued benefit is calculated by referring to the member's current salary; that is, the salary at the date of the valuation.

For the Unit Credit family of methods the Normal Cost for each member is calculated by considering the increase in the member's accrued benefit in the year following a valuation. This can be derived from the Accrued Liability by comparing the Accrued Liability at the valuation date and the Accrued Liability a year later (or at the next valuation).

Entry Age Normal method

Under the Entry Age Normal method the Normal Cost is defined directly and the Accrued Liability follows as a corollary. For each member, the Normal Cost is defined as the level contribution, calculated at the new entrant age, such that the present value of future Normal Costs would equal the present value of the member's future benefits, taking into account future salary improvements.

The Accrued Liability is then calculated, for each member, as the difference between the present value of future benefits and the present value of future Normal Costs, taking into account future salary improvements. This also works out as the present value of prior Normal Costs.

Attained Age Normal method

Similarly to the Entry Age Normal method the Normal Cost is defined directly. For each member, the Normal Cost is calculated as the level contribution at the attained age (that is, the member's age at the valuation date) such that the present value of future Normal Costs would equal the present value of future benefits (that is, benefits accruing after the valuation date), taking into account future salary improvements.

The Accrued Liability for a given member under this funding method is calculated in a similar way as under the Projected Unit Credit Method. That is, the Accrued Liability is calculated as the present value of the member's accrued benefit, taking into account the member's projected final salary.

Aggregate method

Colbran (1982, p. 361) notes that the idea underlying the Aggregate method is to "...produce a cost [...or contribution rate...] as a percentage of salary which if continued until the last member of the present [group] retires will provide a sum exactly equal to the liability to set up each member's retirement benefits and exhaust the fund for active employees...". If applied in isolation to a given member such a contribution rate might not work since it might be too low for some members or too high for others.

Thus under the aggregate method the current position of the scheme is considered by looking at the difference between the present value of benefits (due to both past and future service). We then decide a contribution rate required to finance this difference over the membership period by considering the total projected salaries.

That is, in order to produce the contribution rate, firstly a present value of all past and future benefits for all members is calculated. From this is subtracted the value of the pension scheme's assets. The difference is then divided by the present value of the total projected future salaries for all members. This gives the contribution rate.

2.5.2 Accrued and Prospective Benefit Methods

The second way is to categorize funding methods as either accrued benefit or prospective benefit methods.

Accrued Benefit methods tend to focus on the member's accrued benefits (since joining the pension scheme) with or without reference to future salary improvements. Clearly, the Unit Credit method, which focuses on the present value of accrued benefits without reference to future salaries, and the Projected Unit Credit method, which also focuses on the present value of accrued benefits but with future salary projections, both fall into this category.

Cooper and Hickman (1967) consider a family of accrued benefit methods by proposing a "pension purchase density function". Such a function works like a rule of allocation. It turns out that a uniform rule, without future salary improvements, leads to the Unit Credit method. Whilst a uniform rule, with future salary improvements, leads to the Projected Unit Credit method. By varying the rule of allocation we get a family of accrued benefit methods.

Pugh (2003) refers to such methods as being "security driven" in the sense that accrued benefit methods "...attempt to establish and maintain a sound relationship between the fund assets and the accruing liabilities...".

On the other hand, prospective benefit methods focus on the member's prospective benefits taking into consideration future salary improvements. Following our discussion above, the Entry Age Normal and the Attained Age Normal methods clearly fall into this category since they both focus on the present value of the member's future benefits.

Thus the aim of such methods is to find and maintain a level contribution rate. The calculation can either be individually for each member of the scheme or in aggregate and also either the entry or attained ages can be used.

Pugh (2003) observes that these methods "...are *contribution driven*, and the primary objective is stability of such contributions...". Thus the aggregate method would also fall into this category since, even though present values of both accrued and prospective benefits are calculated, the main focus is on the contribution rate.

2.6 Gains and Losses in a Pension Fund

2.6.1 Sources of gains/losses

One of the early papers on the analysis of gains and losses in a pension fund is Dreher (1959). He observes that "...if each valuation assumption exactly anticipated the experience under a retirement plan from year to year there would never be an actuarial gain or loss..." (p. 589). Thus the main source of actuarial gain or loss is the deviation of actual experience of the pension fund from the anticipated experience.

For a given intervaluation period, Dreher (1959) groups the main elements of an actuarial gain or loss under three categories. The first category of sources of gains and losses contains all those elements which influence the pension fund. These include the benefit and expense outgoes, and investment income.

The second category under Dreher's categorization of sources of gains and losses contains those elements affecting the future liabilities. These include new entrants, exits from the scheme, and salary changes. Depending on the rules of the scheme, exits might comprise deaths, withdrawals, early retirements (due, for instance, to ill-health), and normal retirements.

Finally, Dreher's third category consists of sources which affect the estimated actuarial liabilities but may not necessary depend on the experience during the intervaluation period. These sources of gain and loss include changes in benefit amounts and types, changes in the actuarial valuation method, changes in the asset valuation method, and changes in the actuarial assumptions underlying the pension fund valuation.

Dreher's approach is to identify and quantify the actuarial gain (with actuarial loss as a negative gain) from each of the factors outlined by considering the differences between the actual and the expected results. And then the total actuarial gain can be calculated as the sum of individual gains and losses.

Anderson (1971) approaches the problem from the point of view that the actuarial gain and loss is implicitly in the pension fund valuation method. Thus he derives the formulae for the actuarial gain under various valuation methods and from such formulae the sources of gain (and loss) are identified. (Also see Tino (1975) in the case of a projected benefit cost method).

2.6.2 Dealing with gains/losses

In this section we will talk about ways of dealing with gains and losses in a pension scheme. McGill et al. (1996, p. 525) observe that funding methods can also be categorized by the way gains and losses are treated under each method. Thus the choice of treatment of gains and losses is, implicitly, a foregone conclusion once the funding method has been chosen.

However, as we note below, care has be exercised in categorizing funding methods in this way. The treatment of gains and losses under the individual cost methods in the US practice differs from the UK practice. (see, for example, Dufresne (1986, Chapter 2) and Haberman (1994, pp. 5,6)).

The approach of finding the total actuarial gain by summing the components from various factors has long been recognized as unpractical by actuaries (see, for example, Trowbridge (1952, p. 37)). A more straight-forward method is to consider the unfunded accrued liability. This is defined as the difference between the accrued liability and the total accumulated assets on the valuation date.

Amortization method

This is also referred as a 'direct' approach. Under this approach, firstly, at the valuation date a gain or loss is calculated using Anderson's (1971) approach.

The idea behind the gain is that the sum of the unfunded accrued liability and Normal cost at the previous valuation, accumulated at the valuation rate of interest, should exactly match the sponsor's contribution, also accumulated at the valuation rate of interest, and the unfunded accrued liability at the current valuation, if assumptions are borne out in experience.

However, if assumptions differ from experience then a gain, positive or negative, will arise. Under the direct method, this gain is amortized over a given period by making an explicit adjustment to the Normal cost.

Booth et al. (1999, p. 658) note that under this method the adjustment "...is the total of the intervaluation losses arising during the last M years divided by the present value of an annuity for a term of M years...".

Spread method

The term 'spread method' is used differently in the US and in the UK. In the US this term refers to the treatment of gains and losses in the Aggregate method.

Anderson (1971) observes that "...[f]or the aggregate funding methods the term "gain" is undefined, and therefore no gain can be analyzed...". Rather, he suggests that in the case of the aggregate method we should be analyzing "...the net change in unit normal costs (normal cost per dollar of covered payroll or per employee) from one [valuation] to the next..." (p. 38).

We have already noted that the contribution rate under the aggregate method is calculated as the difference between present value of total benefits and assets (expressed as a percentage of total projected salaries). Any 'gains' and 'losses' arising due to assumptions differing from experience would thus be implicitly incorporated in the calculation of the present value of total benefits and the projected salaries.

Thus McGill et al. (1996, p. 525) conclude that "...[i]n effect, actuarial gains and losses are automatically, and without separate identification, spread over the future working lifetimes of all active participants as a component of the normal cost...". Hence by choosing the aggregate funding method we are implicitly choosing to treat gains and losses by the spread method.

On the other hand, in the UK practice the spread method also refers to the way gains and losses are dealt with under the individual funding methods. Under the spread method the adjustment is determined only from the unfunded accrued liability:

$$Adj(t) = \frac{1}{\ddot{a}_{\overline{m}|}}\Big(AL(t) - f(t)\Big). \tag{2.1}$$

Dufresne (1988) observes that "...this may be interpreted as 'spreading' the unfunded liability over a period of m years..." (p. 535).

2.7 Extending the Traditional Approach

The traditional approach to actuarial valuations is a good set of techniques which is backed-up by extensive actuarial research. It has thus served the actuarial community reasonably well as a vital tool for decision-making and can be used a benchmark.

The foremost advantage of the traditional approach is the ease with which results can be presented to the sponsor, trustees or other authorities. The traditional actuarial valuation is a deterministic framework. The advantage of such a framework is that since the valuation reports are prepared for sponsors, trustees or regulatory authorities it is easy for the recipients to understand the report. In the report the results of the valuation can easily be summed up in single numbers showing, for instance, if there is a surplus or deficit in the scheme and the required contribution.

Furthermore, in non-turbulent times, the traditional funding methods are capable of ensuring the security of benefits or stability of contributions that the stakeholders might require. Whilst in turbulent times experience might differ greatly from actuarial assumptions thus leading to gains and losses which may then need to be dealt with by amortization.

Thus, actuarial valuations will probably continue being employed by actuaries into the foreseeable, as long as defined-benefit pension schemes still exist. At the very least, actuarial valuations could be used as a benchmark for other techniques.

Also, although we have noted that actuaries have traditionally not invested as much effort in the techniques for dealing with the problem of investment strategy, traditional arguments on the choosing of the strategy have been good. For example, we have noted that one of the arguments has been that the investment strategy should be based on the maturity of the scheme.

However, problems do exist with the traditional approach in certain areas. Firstly, the use of present values of the accrued liability in the funding methods has been criticized. Ramsay (1993) observes that by employing the traditional funding methods it means that "...[i]n practice, the degree of assurance is never explicitly mentioned in terms of probabilities. [Thus]...actuaries cannot readily calculate such quantities as...the probability that the accumulation of contributions will ultimately provide sufficient funds to pay benefits or...the size of the fund needed to ensure that benefits are paid with a specified probability..." (p. 352).

He thus argues for the use of 'α-percentile' funding methods which "...*explicitly* include a degree of confidence (assurance) in payment of future benefits..." and, in so-doing, "...shift the valuation process away from expected values to percentiles..." (p. 355).

The notion that traditional funding methods do not give a 'degree of confidence' also suggests that the objectives of the stakeholders are not properly dealt with. In this thesis we argue that a proper way to address the objectives is through a target-based approach with well-constructed risk measures.

Secondly, the traditional approach does not explicitly take into account the stochastic nature of the environment in which pension schemes operate. Booth et al. (1999) observe that since the results of an actuarial valuation "...are presented without adequate reference to the random nature of the events being predicted, they can give a misleading impression of the accuracy..." (p. 502). In this thesis we thus advocate that stochastic methods be used as much as possible so that a full picture of the pension scheme be established and the interaction between various crucial factors be studied.

Gardener (1987) challenges the pension actuary "...to provide a coherent long-term investment strategy..." (p. 179). The lack of a rigorous approach to the problem of asset allocation strategy has led to the development of stochastic asset and liability modelling. In the next chapter will explore this technique.

In this thesis we argue that although the technique of asset and liability modelling has been successfully employed by actuaries for over a decade in the assessment of investment strategies, improvements can be made. We argue that there is still a lack of proper interaction between the decision concerning the funding strategy and that concerning the investment strategy. This lack of interaction implies that two separate decisions are made which result in poor performance compared to one simultaneous decision.

Chapter 3

Asset and Liability Modelling Approach

3.1 Introduction

In this chapter we assess Asset and Liability Modelling (ALM) as a decision-making tool for defined-benefit pension schemes. We give a general description of ALM and describe ways in which ALM can be extended.

ALM is a widely used technique in the financial sector. The underlying idea is to model the cash-flows representing both the assets and liability taking into account the stochastic nature of the cash-flows.

Lockyer (1990) notes that "...the aim of asset and liability modelling is to determine the spread of possible values in the medium and long-run and thereby adopt a strategy which has the highest probability of achieving a particular objective..." (p. 2).

Clark (1992) observes that "...the development of asset [and] liability modelling can, in part, be attributed to the desire by a number of actuaries to advise pension plan sponsors and/or trustees on strategic asset allocation based on a more scientific approach to the problem..." (p. 28).

Kemp (1996) provides a comprehensive review of stochastic asset and liability modelling for pension funds. He shows how ALM techniques can be extended to the cases where the investment strategy is dynamic and to cases where the investment strategy includes derivatives.

In defined benefit pension schemes, the liabilities arise from the promised retirement benefits whilst the value of the assets depend on the pension scheme's investment strategy and market returns. Depending on the scheme's benefit design, the promised benefits could depend on several stochastic factors. As noted in Chapter 2, actuaries have traditionally taken a deterministic view of the demographic and economic assumptions.

The demographic assumptions concern mortality rates and rates of withdrawal. Each of these factors can be modelled stochastically by making assumptions about the underlying stochastic process. However, depending on the objective of the modelling exercise, it is common for researchers to take the view that demographic assumptions would not be the main source of uncertainty and hence they are easier dealt with deterministically.

On the other hand, the economic assumptions concern the future price and wage inflation rates and the return from various asset classes.

This chapter is organized as follows: in Section 3.2 we review the various objectives of ALM for defined benefit pension schemes. In Section 3.3 we describe the general framework of pension fund ALM. Whilst in Section 3.4 we show various approaches to the stochastic modelling of demographic rates for ALM. And in Section 3.5 we describe how actuaries have generally approached the problem of modelling of economic factors affecting pension schemes. Lastly, in Section 3.6 we describe crucial extensions required in order to improve ALM as a decision-making tool.

3.2 ALM Objectives

The underlying idea of stochastic ALM is to address the needs of the sponsor and trustees/members whilst recognizing the stochastic nature of the pension liabilities and of the assets in which the pension funds are invested.

The common approach is to formulate an objective which meets the needs of the sponsor and/or trustees and members. Stochastic ALM is then used to find the investment strategy which best meets (that is, maximizes or minimizes) the objective.

The objectives in Lockyer's (1989) and (1990) papers are set as minimizing the contribution rate and maximizing the funding level (that is, assets divided by liabilities), respectively, for a given confidence level and for a given time horizon. Lockyer (1990) also suggests that a different objective for stochastic ALM would be to find an investment strategy which maximizes the pension increases in the case of a mature pension scheme.

Clark (1992) considers a stochastic ALM study in which the twin objectives are set as "...to maximise the surplus that can be expected to emerge in average conditions in the [pension] plan on an on-going basis, thereby maximising the funds available either for a reduction in the level of contributions required, pension increases or for benefit improvements..." and "...to minimise the variation in the sponsor's likely contribution in such circumstances..." (p. 40).

Yakoubov et al. (1999) illustrate an application of their investment model to an ALM problem where the goal is to find an optimal investment strategy in equities and conventional bonds. In their study they set the objectives as "...to maintain an adequate funding level in all but the worst circumstances, and...to have as low an employer contribution as possible..." (p. 262).

Booth et al. (1999, p. 570) observe that an objective for stochastic ALM could be formulated as to find an investment strategy to maximize probability of pension scheme surplus at the next valuation for a given contribution level.

Actuarial researchers have also applied stochastic ALM to broader aspects of defined benefit pension schemes. For instance, Wright (1998) considered the effect of margins in the valuation basis on the long-term security of the accrued benefits in a pension fund taking into account the legislative framework applicable to UK pension funds at the time. Using stochastic ALM he showed that the long-term security of accrued benefits is jeopardized if the actuary assumes a realistic basis (that is, using best estimate parameters without including explicit margins). This was due to the fact that, assuming a realistic basis, the probability of insolvency was not materially affected by the investment strategy. That is, under a realistic basis there was no significant trade-off between risk and reward.

However, under a prudent valuation (that is, best estimates plus explicit margins), Wright (1998) showed that the probability of insolvency tended to reduce as the proportion invested in less risky asset was increased. Therefore in this case an optimal investment strategy can be established. Thus he was able to show that stochastic ALM can be broadened to cover more aspects of defined benefit schemes.

3.3 ALM Framework

In conducting an ALM study actuaries carry out cash flow projections over a given horizon. For example, the length of the horizon may depend on the inter-valuation period. This is normally three years in the UK but may be shorter depending on the purpose of the valuation.

The common approach to ALM studies is to assume that a valuation along the lines described in Chapter 2 will be conducted at each valuation date. Meanwhile, during the inter-valuation period the return from the assets and the growth of the liabilities are assumed to stochastic.

Thus the actuary would need to make assumptions not just about the situation on the valuation date but also about the progress of the pension fund during the projection horizon. Kemp (1996) refers to the assumptions governing the progress during the projection horizon as "projection assumptions". He observes that "...projection assumptions are what we assume will have occurred between the start of the projection and the relevant valuation date... [and these assumptions]... describe some probability distribution about how we expect the future to evolve..." (p. 8).

Once these projection assumptions have been set, cash flow projections can then be conducted to the end of the projection horizon. Since these assumptions are mostly descriptions of probability distributions, the typical approach is to carry out several thousand simulations of the pension fund's evolution over the projection horizon.

At the end of the projection horizon, which might coincide with the next valuation date, a set of valuation assumptions would be required. Kemp (1996) observes that "...the valuation basis is the set of assumptions that we assume will be used by the actuary to carry out an actuarial valuation at the future point in time being considered...". He further observes that "...[i]n pension fund asset/liability modelling the valuation basis is often assumed to be largely constant, at least for long-term ongoing actuarial valuations...[t]here may also be an element of prudence incorporated within these assumptions, which may not be present in the projection assumptions...". This also is the approach taken by, for example, Wright (1998) although he further considers the effect of margins in the valuation basis.

In Chapter 2 we observed that the valuation assumptions encompass the demographic aspects and the economic aspects. Projection assumptions also encompass both of these aspects although the emphasis will depend on the objectives of the ALM study. In the following sections we describe how both aspects can be modelled stochastically.

3.4 Stochastic Modelling of Demographic Rates

Traditionally, stochastic modelling of demographic rates has mainly been applied to the modelling of life insurance contracts rather than pension schemes. This could be due to the fact that, compared to individual life insurance contracts, a large pension scheme could have several thousand members at each age. Thus reference might be made to the law of large numbers.

However, mortality is a stochastic process. We can not say with certainty the time of death or the chance of survival of a given life. In a pension fund we would extend this argument to other rates of decrement. The merits of including stochastic assumptions for mortality and other decrements in the ALM study depend on the study's objectives.

Clark (1992) notes that "...in practice it is likely that the uncertainties in the economic [assumptions] will far outweigh those connected with the decrements..." He, however, observes that "...the effect of a large number of withdrawals from pension plans during the 1980s was a significant contributory factor to the surpluses which arose..." He further notes that "...it might be thought desirable to allow for different new entrant and/or withdrawal assumptions depending on the economic climate prevailing during the course of a particular economic scenario..." (p. 17).

However, Booth et al. (1999) note that stochastic modelling of decrement rates might be fraught with danger due to a lack of data. They observe that "...[m]ost of the uncertainty inherent in the results of the pension scheme valuation arises due to the volatility of the investment yield available from the fund. The effect of the random nature of the interest rates is likely to be significantly greater than that of...the retirement or mortality experience. In addition, the data available to derive those decrements that do have a high degree of uncertainty, such as withdrawal rates, is quite poor and so any underlying assumed probability distribution will be unreliable..." (p. 570).

Nevertheless, in the following sections we describe various approaches to stochastic modelling of mortality rates. These approaches can easily be adapted to other types of decrement.

3.4.1 Monte Carlo approach

One of the early papers on stochastic mortality modelling is Boermeester (1956). He described a monte carlo approach to the estimation of the distribution of annuity costs. Under this approach death events are treated as Bernoulli trials as follows.

Let q_x^d be the probability that (x) will die within one year as depicted in a given mortality table. To conduct the Bernoulli trial we would need to generate a random number, say ${}^dZ$, between 0 and 1. Then we would define an indicator variable, I_x^d, such that:

$$I_x^d = \begin{cases} 1 & \text{if } {}^dZ < q_x^d; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Then $I_x^d = 1$ would indicate that (x) dies within the next year whilst $I_x^d = 0$ would indicate that (x) survives to age $x+1$.

3.4.2 Random walk-type models

Lee (2000) describes a stochastic mortality model where the probabilities of death are simulated as products of the probability of death as depicted in a given mortality table and a random factor. Let q_x^d be the probability that (x) will die within one year as depicted in a given mortality table. Then the probability that a life aged x at time t will die within one year is simulated as

$$^{sim}q_x^d = q_x^d \times \exp\left(Z_1(t)\right) \tag{3.2}$$

where the stochastic shocks $Z_1(t)$ are given by

$$\begin{aligned} Z_1(t) &= Z_2(t) + N(0, \sigma_1^2) \\ Z_2(t) &= Z_2(t-1) + N(0, \sigma_2^2). \end{aligned}$$

Czernicki et al. (2003) propose an additive model for stochastic mortality with both multiplicative and additive shocks to mortality:

$$^{sim}q_x^d = \alpha_1 \, q_x^d + \alpha_2 \tag{3.3}$$

where α_1 is a random variable with mean 1 and variance σ_1^2 and α_2 is a random variable with mean 0 and variance σ_2^2.

Thus these stochastic mortality models introduce shocks to the mortality assumptions. In the deterministic framework such shocks are not taken into account.

Other decrements in a pension scheme could be treated in a similar way to mortality. For instance, random shocks could be introduced in withdrawal rates and ill-health retirement rates.

3.5 Stochastic Modelling of Economic Variables

The subject of stochastic modelling of economic variables has been widely considered in the actuarial literature. As noted in Chapter 2, the main economic assumptions in a traditional pension scheme valuation are the rate of price inflation, rate of wage inflation and the return from investments.

These economic variables are treated in different ways in the actuarial literature. The common approach is to employ a single model with the economic variables being described by sub-models. One of the most widely known such model is the Wilkie (1995) model.

However, for the sake of comparison with the traditional, deterministic approach, in this section we consider how actuaries treat the stochastic nature of each of the economic variables mentioned above.

3.5.1 Price Inflation

The level of price of inflation affects the liabilities of a defined benefit pension scheme in several ways. Firstly, occupational pension schemes in the UK are required by legislation to increase pensions in payment in line with a Limited Price Index (LPI). The LPI varies in accordance with the Retail Price Index but is subject to a lower bound of 1 and a pre-set maximum value.

Secondly, it is widely recognized that price inflation significantly affects the levels of other economic variables. For example, the Wilkie (1995) model has a cascade structure where price inflation affects all the other variables.

Thus although the valuation assumption for the rate of price inflation may be deterministic, a stochastic approach is employed in setting the projection assumption.

Lockyer (1990) notes that "...by including a distribution for inflation..., and using these figures in the projections, the modeller can allow for changes in the liabilities caused by different inflation rates..." (p. 2). He further adds that modelling inflation stochastically may be more necessary in an economy with high inflation and discretionary pension increases.

Actuaries have modelled the force of price inflation in several ways. Price inflation has been modelled as either an autoregressive process, an autoregressive process with conditional heteroskedaticity, or an autoregressive process with regime switching. Another common approach is to model using vector autoregressive processes.

Autoregressive model

It is generally recognized amongst actuaries that the force of price inflation ought to be modelled as an autoregressive process. The general idea is that the price inflation in a given year will depend on the previous year's inflation and some random disturbance.

The common autoregressive model for the force of price inflation is the one suggested by Wilkie (1986, 1995). He suggested that price inflation be modelled as a first order autoregressive process, AR(1), as follows.

Let $I(t)$ be the force of inflation in the year $(t-1,t)$. Then, using Wilkie's original notation, we have:

$$I(t) = QMU + QA\left(I(t-1) - QMU\right) + QSD.QZ(t) \tag{3.4}$$

where $\{QZ(t)\}$ is a sequence of $i.i.d$ random variables from the standard normal distribution (that is, independent and identically distributed normal random variables with mean 0 and standard deviation 1).

The autoregressive model enhances the actuary's view of price inflation and is simple to justify. Equation 3.4 says that at any given time t the force of price inflation is equal to its long-term mean, QMU, plus an autoregressive factor plus some random shock.

As noted above, to conduct the traditional deterministic valuations the actuary assumes a constant force of price inflation. The actuary may choose to set the valuation assumption as equal to QMU because in the autoregressive model this represents the long-term mean force of inflation. The values for the parameters QMU, QA and QSD will depend upon the data used for estimation.

The Wilkie autoregressive model has been criticised by a number of researchers including Kitts (1990), Clarkson (1991), Geoghegan et al. (1992) and Huber (1997).

Geoghegan et al. (1992) observed that the autoregressive model failed to incorporate some crucial features in the inflation data. They noted that every price inflation model ought to incorporate ARCH effects in the UK inflation data ie heteroscedastic tendencies (we will come back to this point in the following section as we talk about the ARCH model). They also argued that the model did not incorporate "...the existence of large, irregular shocks, such as those of the mid-1970's..." and "...the possible non-normality of residuals, through asymmetry...".

They also observed that "...the (autoregressive) model provides for negative and positive movements in inflation with equal probability and it provides for a significant probability of negative inflation...in practice this might be unlikely...".

ARCH model

It is a well-known fact that high inflation levels are usually associated with high instability. Thus the idea behind ARCH models is to let the variance of inflation depend on past inflation levels. The presence of heteroscedastic tendencies in UK inflation data was shown by Engle (1982).

An autoregressive model with conditional heteroscedasticity was proposed by Geoghegan et al. (1992) in order to incorporate ARCH effects into the autoregressive model. Further investigation on the ARCH model was carried out in Wilkie (1995).

He states the ARCH model as:

$$I(t) = QMU + QA\left(I(t-1) - QMU\right) + QSD(t).QZ(t) \tag{3.5}$$

with

$$QSD(t)^2 = QSA + QSB.\big(I(t-1) - QSC\big)^2 \tag{3.6}$$

where $\{QZ(t)\}$ is a sequence of $i.i.d$ random variables from the standard normal distribution.

Wilkie (1995) observes that the ARCH model "...reflects the notion that, if the rate of inflation over any year has been unusually high, then the uncertainty about the rate of inflation in the following year is increased. It might be again high or it might be much lower. The same applies if the rate of inflation is unusually low, and this is made effective through the squared terms..." (p. 799).

The first problem with the ARCH model is whether or not it performs better than the autoregressive model as we might expect. Huber (1997) observes that the "...ARCH model is able to deal effectively with the problem of non-normality and heteroscedasticity..." and "...the ARCH model appears to describe the data better than the [autoregressive] model. Thus, it should generally be used in applications of the model, unless the ARCH effect is not significant for those particular applications".

Threshold autoregressive model

The main motivation behind a threshold autoregressive model for price inflation seems to be the fact that the inflation levels in a recession and out of a recession are always different: Clarkson (1991) refers to such periods as 'excited' and 'quiescent'; whilst Whitten and Thomas (1999) call such periods as 'high' and 'normal'. By classifying inflation levels into 'high' and 'normal' the obvious choice of a Threshold model is one with 2 regimes[1].

Whitten and Thomas (1999) set the threshold at 10% and found the following SETAR(2;1,0) model to be the most suitable for price inflation:

$$I(t) = \begin{cases} QMU1 + QA1\big(I(t-1) - QMU1\big) + QSD1.QZ(t), & \text{if } I(t-1) \leq QR \\ QMU2 + QSD2.QZ(t), & \text{if } I(t-1) > QR \end{cases} \tag{3.7}$$

where $\{QZ(t)\}$ is a sequence of $i.i.d$ random variables with from the standard normal distribution.

Thus the SETAR model says that in the normal periods the force of inflation follows an autoregressive process whilst in the high period the force of inflation follows a

[1] A simple class of threshold autoregressive models can be defined by:

$$X_t = \begin{cases} \mu_1 + \alpha(X_{t-1} - \mu_1) + \sigma_1\epsilon_t, & \text{if } X_{t-1} \leq d \\ \mu_2 + \beta(X_{t-1} - \mu_2) + \sigma_2\epsilon_t, & \text{if } X_{t-1} > d \end{cases}$$

where $\{\epsilon_t\}_{t=0}^{\infty}$ is a sequence of $i.i.d$ random variables with zero mean and variance 1; and d is the threshold.

random process with mean QMU2 and standard deviation QSD2.

The lack of data is a major hindrance to threshold autoregressive modelling. If more inflation data were available more 'precise' thresholds and regimes could be set.

For instance, Whitten and Thomas (1999) point out that apart from 10%, 0% would be a plausible choice for a second threshold. Setting thresholds at 0% and 10% would lead to a threshold autoregressive model with three regimes: negative inflation (ie inflation levels below 0%), normal inflation (ie inflation levels between 0% and 10%), and high inflation (ie inflation levels higher than 10%). This would probably be a SETAR(3;1,1,0) model.

3.5.2 Wage Inflation

Defined benefit liabilities are always designed to depend on the evolution of the scheme members' salaries over the course of their membership. In career-average salary schemes the benefits depend on some fraction of the member's average salary over their scheme membership. Whilst in final-salary schemes the benefits depend on some function of the member's average salary during some period of their membership, usually the final n months of the membership, where n is pre-specified in the scheme rules. Thus stochastic modelling for the wage inflation is crucial in ALM cash flow projections.

Various models have been suggested for wage inflation in the actuarial literature. In this section we review two of these models.

Wilkie's Wage Inflation Model

Wilkie (1995) approaches this problem by using a transfer function model. Under his model the force of wage inflation at time t depends on the force of price inflation at time t, the force of price inflation at time $t-1$, and an extra term defined as an autoregressive process.

Let $J(t)$ be the force of wage inflation at time t. Then

$$J(t) = WW1.I(t) + \left(1 - WW1\right) I(t-1) + WN(t) \tag{3.8}$$

where the extra autoregressive term is defined as

$$WN(t) = WMU + WA\left(WN(t-1) - WMU\right) + WSD.WZ(t). \tag{3.9}$$

One of the focal points of Wilkie's approach is the dependence of $J(t)$ on $I(t-1)$ instead of $J(t-1)$. This implies that the current force of wage inflation depends on the current and previous force of price inflation.

The coefficient $1 - WW1$ is introduced in order to get "unit gains" between the two inflation processes. That is, "...an unexpected change in prices will...produce a corresponding change in wages in the long run..." (Wilkie (1995, p. 810)).

Wilkie also suggests a vector autoregressive model in which the current force of wage inflation depends on its own previous values and those of the force of price inflation. Similarly, the force of price inflation depends on it own previous values and those of the force of wage inflation.

TY wage inflation model

Yakoubov et al. (1999) model the force of wage inflation in a similar way to Wilkie (1995) but with some crucial differences in the formulation. Using Wilkie's notation, the force of wage inflation is defined as

$$J(t) = WMU + \alpha_1^{sal}\left(J(t-1) - WMU\right) + \alpha_2^{sal}\left(I(t) - QMU\right) + WSD.WZ(t). \tag{3.10}$$

Under this model, the force of wage inflation at time t depends on the force of wage inflation at time $t-1$, the force of price inflation at time t, and a random term. This is a crucial difference with Wilkie's approach where, as observed above, instead of depending on $J(t-1)$, the force of wage inflation at time t depends on $I(t-1)$.

Essentially, the structure of this model implies that we no longer get "unit gains" as in Wilkie's approach. Hence, the current force of wage inflation could, depending on the values of the parameters α_1^{sal} and α_2^{sal}, be considerably influenced by the level of the previous values of the force of wage inflation and the current force of price inflation.

3.5.3 Return from Investments

A wide range of investment models have been suggested by actuarial researchers. The treatment of each asset class varies from model to model. Lee and Wilkie (2000) provide a comprehensive summary of how a number of UK investment models generate investment returns. Thus we will not go into the details for each model. We will, however, concentrate on how actuaries have moved on from the deterministic framework to the stochastic framework in the treatment of returns on pension fund investments.

The common denominator in all stochastic investment models is a generation of total return indices for the asset classes. In some models these total return indices are generated directly whilst in other models they are derived from other underlying series.

In stochastic ALM investment strategies are varied by changing the weights placed on the returns from different asset classes. For example, consider a case where the fund is to be invested in n assets, each with return in the year $(t-1,t)$ given by $1+r_i(t)$, $i=1,2,\ldots,n$. Then the total return in year $(t-1,t)$ would be given as a weighted sum:

$$MR(t)/MR(t-1)=\sum_{i=1}^{n}\alpha_i\Big(1+r_i(t)\Big) \tag{3.11}$$

where $MR(t)$ is the total return index at time t and the weights α_i signify the allocation in the ith asset and they sum to 1.

A Direct Approach

Different approaches are employed to model $1+r_i(t)$. The simplest approach is the random walk model where the rate of return follows a Log-Normal distribution

$$1+r_i(t)=\exp\Big\{\mu_i+\sigma_i N(0,1)\Big\}. \tag{3.12}$$

This is the approach taken by Smith (1996). He suggests that the rate of return on the ith asset class be modelled as

$$XR_i(t)/XR_i(t-1)=\exp\Big\{XMU_i+XSD_i XZ(t)+I(t)\Big\} \tag{3.13}$$

where $XR_i(t)$ is the total return index at time t for the ith asset class, the parameters XMU_i and XSD_i depend on the asset class, $XZ(t)$ is a sequence of $i.i.d$ random variables and $I(t)$ is the rate of inflation at time t.

The main weakness of this approach is that there is no separation of capital and income returns. This can be crucial where these are taxed differently and also where the investment is in a real asset class like Equities. In the discussion of Smith (1996), Professor Wilkie observes that "...[t]otal return models do not give you dividend yields and interest rates. If you use a completely total return model for your simulation, you do not actually know what the basis is for calculating valuation returns or minimum funding requirement returns..." (p. 1185).

Thus the 'Indirect approach' described below tends to be more preferable in stochastic ALM of pension funds.

An Indirect Approach

This is the approach used by, among others, Wilkie (1995) and Yakoubov et al. (1999). In this case the total return index is not modelled directly but split into a capital component and an income component.

Let us consider Wilkie's approach. In the case of equities he models the equity dividend index at time t, $D(t)$, and the equity dividend yield at time t, $Y(t)$.

Thus the equity price index at time t, $P(t)$, works out as

$$P(t) = D(t)/Y(t) \tag{3.14}$$

and the equity return index at time t, $PR(t)$, is calculated as

$$PR(t) = PR(t-1)\left\{P(t) + D(t)\,(1 - tax)\right\}/P(t-1). \tag{3.15}$$

Wilkie uses a similar approach to derive the return index for conventional long-term bonds. He models the yield on conventional bonds at time t, $C(t)$, and derives the return index at time t, $CR(t)$, as

$$CR(t) = CR(t-1)\left\{1/C(t) + (1 - tax)\right\}C(t-1). \tag{3.16}$$

With this approach the actuary has to specify models for the equity dividend index, equity dividend yield and the yield on fixed-interest bonds.

3.6 Extensions of the ALM approach

In recent times it has been observed that room for improvement does exist in most stochastic ALM exercises. Two of the areas where it has been argued that improvements can be made are in the construction of the objectives of the stochastic ALM and in the presentation of results. In this section we consider these and other areas where possibilities exist for improvement.

Most ALM studies have tended to focus on finding the investment strategies which would minimize the probability of failing to meet a given target. For example, the objective could be to minimize the probability of a shortfall at a given time horizon. This might then involve looking at the distribution of shortfall for a given investment strategy.

Chapman et al. (2001, p. 613) observe that the objectives of ALM can be improved by supplying "...a method of quantifying the value of the different [investment] strategies...". They further add that such a method would "...assess the frequency and the magnitude of the shortfalls...". In Chapter 5 we present a Risk Management approach with objectives constructed using risk measures that fully addresses these concerns.

Another area where more input is necessary is in the presentation of results of a stochastic ALM study. The problems in this area arise from several factors.

Kemp (1996) observes that stochastic ALM "...provides much more information than is available from an actuarial valuation. A valuation provides a single 'answer' at a set point in time (the valuation date). In contrast, asset/liability study provides three or more extra dimensions by...providing projections into the future (introducing a time dimension);...providing some estimate of the range of likely outcomes (a probabilistic dimension); and...indicating the effect of changing investment strategy (an asset mix dimension)..." (p. 7).

Another factor in the presentational problems might arise from the recipients of the stochastic ALM results. The kind of recipients would, of course, depend on the objectives of study.

Let us, for the sake of illustration, assume that the trustees of the pension scheme would like the actuary to undertake an ALM study of the scheme. The question arises as to how the results obtained would be presented, with clarity, to the trustees.

In the traditional deterministic framework the results are just single numbers. However, in the stochastic framework the results are statistical distributions. As standard statistics textbooks show, to present information about a statistical distribution one has to look at the measures of dispersion. These include the mean, standard deviation, median, and the percentiles.

Actuaries have thus tended to present the results of the stochastic ALMs using percentiles. Chapman et al. (2001, p. 612) observe that further work can be done on the presentation of the results since the use of percentiles can lead to "worst scenarios" being discarded. Thus, for example, the worst scenarios might lie in the tails of the distribution and this might require one to consider the 1st percentile or the 5th percentile. However, in most studies actuaries tend to have the 10th percentile as a lower cut-off point.

Chapman et al. (2001, p. 612) also observe that the presentation of results can be further improved by ensuring that the outcomes are always labelled with reference to the percentiles from which they originated. In Chapter 5 we construct a presentational framework which clarifies all these problems.

Another area that could be improved is the way funding and investment strategies are optimised. The funding strategy is usually established using the traditional actuarial valuations framework. Meanwhile, the investment strategy is established by using stochastic ALM. However, these decisions need not be treated separately. In

Chapter 5 we present a stochastic decision making approach under which these two key decisions are dealt with simultaneously.

It has also been suggested that the stochastic ALM approach ignores the risk of bankruptcy of the sponsoring employer. Thus there is a tendency, it has been argued, by this approach to favour higher-risk equity investment strategies.

In Chapter 8 we present a framework that addresses this problem by taking into account the probability of default by the sponsoring employer. It will be shown that for short projection periods the probability of default has a major effect on the pension fund's investment strategy.

Chapter 4

Stochastic Pension Fund Control

4.1 Introduction

In recent years the problem of pension funding and investment has been viewed as an optimal control problem. Four different types of optimal control problems have been considered in the actuarial literature depending on whether the horizon is finite or infinite and the time is discrete or continuous.

The quadratic-type performance criterion has been the most commonly used criterion. Such a criterion leads to analytically-tractable problems and explicit solutions for the optimal control functions. Problems involving quadratic criteria are formulated as 'tracking' problems. This means all deviations from the target are equally penalized.

Thus, for pension fund control, surpluses and deficits are treated in a similar way. Although this might be a convenient procedure, it does not reflect the way pension fund stakeholders might view surpluses and deficits in the scheme. Stakeholders might view surpluses as more acceptable than deficits. Hence in pension fund control we may require a performance criterion which does not penalize surpluses and deficits symmetrically.

For the sake of analytical tractability, in this paper we consider a quadratic criterion with linear factors. The linear factors are intended to penalize fund deficits and contribution excesses. We assume that the fund is invested in a risk-free asset and a risky asset and we show that the optimal asset allocation strategies are counter-intuitive: the optimal strategy involves increasing the allocation in the risky asset as the fund deficit increases and increasing the allocation in the risk-free asset as the fund deficit decreases. We further show that increasing the weight on the linear factors leads to an increase in the optimal allocation in the risky asset.

This chapter is organized as follows: in Section 4.2 we survey the actuarial literature on pension fund control. In Section 4.3 we consider a quadratic performance criterion with linear factors. Then in Section 4.4 we introduce the asset allocation choice as a control parameter. In Section 4.5 we show the effect of wage inflation on the growth of the actuarial liability, Normal Cost and the Benefit outgo. While in Section 4.6 we solve the pension fund control problem using Dynamic Programming and show that the optimal policy is counter-intuitive. In Section 4.7 we investigate the effect of various factors on the optimal policy. In Section 4.8 we solve the infinite-horizon problem. And in Section 4.9 we present a numerical implementation of the optimal policy for the finite-horizon problem. In Section 4.10 we present some concluding remarks. Lastly, in Appendix A we show the derivation of the solution to the finite-horizon problem.

4.2 Control with Quadratic Criteria

Benjamin (1984, 1989) explore the applicability of control theory to pension funds. Benjamin (1984) views the pension funding process as a problem in control theory where the real rate of return on investments is seen as the input signal whilst the recommended contribution rate is seen as the output signal. He assumes that the pension funding method is the aggregate method and that the valuation rate of interest is the average of past real investment rates of return. He employs control theory methods to study the effects of changes in the input signal (i.e. investment return) on the output signal (i.e. recommended contribution rate).

Benjamin (1989) further considers two applications of control theory to pension funds. The first application is a continuation of Benjamin (1984) with the aggregate funding method replaced by the projected unit method. The second application involves a minimum energy control problem where the aim is to obtain a "maximum smoothness path" for the recommended contribution rates i.e. a path which minimizes the year to year deviations in the contribution rates. Thus he suggested a criterion of the form

$$\min_{c(1),\ldots,c(T-1)} \sum_{t=1}^{T} \Big(c(t) - c(t-1)\Big)^2 \tag{4.1}$$

where $c(t)$ is the contribution rate at time t.

O'Brien (1987) models the pension fund process as a controlled diffusion process and the contribution rate is set as the control parameter. He applies dynamic programming to find an optimal control function which minimizes both the deviations of the funding level from a given target level and the cost as measured by the contribution.

He thus sets up a performance criterion of the form

$$\min_{c(0),\ldots,c(T-1)} E\left[\int_0^T e^{-\varphi t}\left(c(t)^2 + \beta\left[\eta A(t) - f(t)\right]^2\right)\Big|\mathcal{F}_0\right] \tag{4.2}$$

where $A(t)$ is the present value of future benefits for active members at time t and $f(t)$ is the fund at time t.

O'Brien (1987) shows that for this problem the optimal control function is an affine function of the present value of future benefits and the fund at time t.

Vandebroek (1990) aims to build on O'Brien (1987) by considering the problem of finding an optimal contribution function which minimizes the deviations between the present value of future benefits and the fund at time t and between the total salary at t and the annual contribution at t. However, unlike O'Brien (1987), Vandebroek (1990) considers a deterministic framework for the optimal control of the pension fund.

Hence Vandebroek (1990) considers a performance criterion of the form

$$\min_{c(0),\ldots,c(T-1)} \int_0^T e^{-\varphi t}\left\{[c(t) - \alpha W(t)]^2 + \beta[\eta A(t) - F(t)]^2\right\}dt \tag{4.3}$$

where η is the target fund ratio and α is the target contribution level.

Haberman and Sung (1994) generalize the approach O'Brien (1987) and Vandebroek (1990). They define the quadratic performance criterion in terms of 'contribution rate risk' and 'solvency risk'. Contribution rate risk is defined in terms of deviations of the contribution from a set contribution target. This contribution target need not be a fixed proportion of the total salary as in Vandebroek (1990). They point out that the contribution target could be set, for example, as the Normal Cost or as the expected value of the contribution.

They define the solvency risk in terms of deviations of the fund level from a set funding target. This target need not be a fixed proportion of the present value of future benefits for active members as in O'Brien (1987) and Vandebroek (1990). They point out that the target could be set, for example, as the Actuarial liability or as the expected value of the fund.

Haberman and Sung (1994) consider the problem of finding an optimal contribution function and they set up the stochastic pension fund control problem as

$$\min_{c(0),\ldots,c(T-1)} E\left\{ \sum_{t=0}^{T-1} v^t \left[\left(c(t) - CT_t\right)^2 + v\beta \left(f(t+1) - FT_{t+1}\right)^2 \right] \middle| \mathcal{F}_0 \right\}. \tag{4.4}$$

They show that the optimal contribution at time t is a linear function of the fund at time t and the benefit outgo at time t. Their result is thus similar to that of O'Brien (1987).

Extensive work has also been undertaken into the problem of finding optimal asset allocation policies when the criterion is of the quadratic type. For further details see Boulier et al. (1995, 1996), Cairns (1997, 2000), Siegmann and Lucas (1999), Josa-Fombellida and Rincón-Zapatero (2001).

One of the main results emerging from these studies is that in the case where the criterion is quadratic the optimal asset allocation policy is counter-intuitive. That is, the optimal strategy involves increasing the allocation in a risky asset as the fund decreases and vice versa. Owadally and Haberman (2004) also find a similar result.

They observe that "...[t]he contrarian strategy is evidently a consequence of the quadratic utility function implied in [the quadratic performance] criterion..., which is simplistic as it is symmetric and continuous, and does not admit solvency and full-funding constraints. The risk that the plan sponsor winds up the plan or defaults on pension obligations was also disregarded..." (p. 32).

These arguments thus imply that in considering the optimal stochastic control of pension funds we need an objective function which takes into account the various pension fund risks. The objective function should further take into account how the investor or decision maker views financial risk.

The obvious problem here is that the stochastic control problem could become mathematically intractable. Indeed this is one of the reasons why stochastic control problems with quadratic criteria have been widely considered in the literature. Bertsekas (1976) observes that the quadratic criterion is considered because "...it leads to an elegant analytical solution that can often be implemented with relative ease in...applications..." (p. 71).

4.3 Criterion with Quadratic and Linear Factors

Chang et al. (2003) suggest a performance criterion which takes into account both quadratic and linear deviations. They consider a finite-horizon, discrete-time performance criterion of the form

$$
\begin{aligned}
V\big(f(0)\big) &= E\Big[\sum_{t=0}^{T-1} \nu^t \Big\{\alpha_1 \Big(\frac{c(t)}{NC(t)} - 1\Big)^2 + \alpha_2 \nu \Big(1 - \frac{f(t+1)}{AL(t+1)}\Big)^2 \\
&\qquad + \alpha_3 \Big(\frac{c(t)}{NC(t)} - 1\Big) + \alpha_4 \nu \Big(1 - \frac{f(t+1)}{AL(t+1)}\Big)\Big\} \Big| f(0)\Big].
\end{aligned} \tag{4.5}
$$

A downside risk measure would be favoured by a decision maker because such a measure treats pension deficits and surpluses differently. Surpluses are treated as zero deficits and thus in the event of a pension surplus no cost is incurred. This applies to both the fund and contribution levels. For contribution we consider the excesses - that is, contribution above the target. A cost is incurred in the event of a positive excess whilst in the event of a negative excess no cost is incurred.

However, quadratic risk measures treat surpluses and deficits similarly. That is, fund deviations above and below the funding target are treated in the same way. The linear factors in criterion 4.5 are introduced to give more weight to pension fund deficits (that is, 'under-funding' risk) and contribution excesses (that is, 'over-contribution' risk). A similar criterion is used by Haberman and Vigna (2002) in the analysis of a defined contribution pension scheme.

Chang et al. (2003) use criterion 4.5 to obtain the optimal contribution policy in the case where the investment return follows an autoregressive process. They show that as the weight on the linear contribution factor increases, the optimal contribution decreases thus reducing the over-contribution risk but also leading to lower funding levels.

However, they further show that as the weight on the linear funding factor increases, the optimal contribution increases and also the funding level increases. They thus conclude that a positive weight on the linear factor can lead to higher funding levels and lower probabilities of insolvency (p. 225).

These results are intuitive. In order to reduce chances of under-funding risk, assuming all other things equal, we have to increase the sponsor's contribution. On the other hand, to reduce over-contribution risk, all other things equal, we have to reduce the sponsor's contribution and incur the penalty of pension deficits.

In this chapter we extend the framework of Chang et al. (2003) by considering the problem of seeking the optimal asset allocation policy for the finite-horizon and infinite-horizon cases. Our goal is to obtain optimal investment policy assuming that the contribution rates are set using an individual actuarial cost method. Our approach will closely follow that of Owadally and Haberman (2004).

4.4 Asset Allocation Problem

The fund dynamics can be described as

$$f(t+1) = \Big(1 + r_{t+1}\Big)\Big(f(t) + c(t) - B(t)\Big) \tag{4.6}$$

where r_{t+1} is the investment return in the year $(t, t+1)$.

We assume the pension fund can be invested in two assets:

- Asset 1 with return $r_1(t)$ distributed as $N\Big(\mu_1, \sigma_1^2\Big)$;
- Asset 2 with risk-free return μ_2, where $\mu_1 > \mu_2$.

We further assume that a proportion $\pi(t)$ of the fund is invested in the risky asset whilst $1 - \pi(t)$ is invested in the risk-free asset. Thus the total investment return in the year $(t, t+1)$ has Normal distribution with mean $1 + \pi(t)(\mu_1 - \mu_2) + \mu_2$ and variance $\pi(t)^2\sigma_1^2$.

The fund dynamics can be re-written as

$$f(t+1) = \Big(1 + \mu_2 + \pi(t)\left(r_1(t) - \mu_2\right)\Big)\Big(f(t) + c(t) - B(t)\Big). \tag{4.7}$$

Since the total returns $\{r_t\}$ are independent, $f(t)$ has the markov property. Hence

$$E\Big[f(t+1)\,\Big|\,\mathcal{F}_t\Big] = E\Big[f(t+1)\,\Big|\,f(t)\Big] \tag{4.8}$$

where $\mathcal{F}_t$ is the information history up to time t.

We seek a policy

$$\pi(t), \quad t = 1, 2, \ldots, T \tag{4.9}$$

which minimizes the downside risk criterion 4.5.

4.5 Liability growth

4.5.1 Salary growth

We assume a constant force of wage inflation, ω. Hence the wage index at time t is

$$W(t) = W(t-1)\,\exp(\omega) \tag{4.10}$$

where $W(0) = 1$.

We assume that the pension scheme is stable. Thus we get the Actuarial Liability and Normal Cost at time t as shown below.

4.5.2 Actuarial Liability

This is just the discontinuance liability. Let x be the entry age and NRA be the normal retirement age. Then

$$\begin{aligned}
AL(t) &= \frac{1}{\mathrm{f}} \sum_{y=x}^{NRA-1} N(y,t)\, PS(y,t)\, S(y,t)\, {}_{NRA-y|}\ddot{a}_y \\
&= \frac{1}{\mathrm{f}} \sum_{y=x}^{NRA-1} N(y,t-1)\, PS(y,t-1)\, S(y,t-1)\, e^{\omega}\, {}_{NRA-y|}\ddot{a}_y \\
&= e^{\omega}\, AL(t-1) \\
&= e^{\omega t}\, AL(0) \qquad (4.11)
\end{aligned}$$

where the deferred annuity is

$$\begin{aligned}
{}_{NRA-y|}\ddot{a}_y &= {}_{NRA-y}p_y\, (1+r)^{-(NRA-y)} \times \\
&\qquad \sum_{z=NRA}^{\infty} {}_{z-NRA}p_{NRA}\, (1+r)^{-(z-NRA)} \qquad (4.12)
\end{aligned}$$

and r is the valuation rate of interest.

4.5.3 Normal Cost

This is the present value of liabilities accruing over the year following the pension scheme valuation. We assume that only withdrawal and normal retirement benefits are paid. Hence

$$NC(t) = {}^{w}NC(t) + {}^{r}NC(t). \qquad (4.13)$$

For ${}^{r}NC(t)$ we get

$$\begin{aligned}
{}^{r}NC(t) &= \frac{1}{\mathrm{f}} \sum_{y=x}^{NRA-1} N(y,t)\, S(y,t)\, {}^{s}M_y^{rb}\, \frac{1}{{}^{s}D_y} \qquad (4.14) \\
&= \frac{1}{\mathrm{f}} \sum_{y=x}^{NRA-1} N(y,t-1)\, S(y,t-1)\, e^{\omega}\, {}^{s}M_y^{rb}\, \frac{1}{{}^{s}D_y} \qquad (4.15) \\
&= e^{\omega}\, {}^{r}NC(t-1) \qquad (4.16)
\end{aligned}$$

where ${}^{s}M_y^{rb}$ and ${}^{s}D_y$ are commutation functions.

Similarly for ${}^{w}NC(t)$. Hence

$$\begin{aligned}
NC(t) &= {}^{w}NC(t) + {}^{r}NC(t) \\
&= e^{\omega} \left[{}^{w}NC(t-1) + {}^{r}NC(t-1)\right] \\
&= e^{\omega}\, NC(t-1) \\
&= e^{\omega t}\, NC(0) \qquad (4.17)
\end{aligned}$$

We can write $NC(0)$ as

$$NC(0) = k_1\, AL(0) \qquad (4.18)$$

where k_1 takes a value between 0 and 1 and depends on the pension scheme's benefit design, population structure and the valuation rate of interest.

Thus

$$NC(t) = k_1\, e^{\omega t}\, AL(0). \tag{4.19}$$

4.5.4 Benefit Outgo

As for the Actuarial Liability and Normal Cost, the Benefit Outgo at time t can be rewritten as:

$$\begin{aligned} B(t) &= e^{\omega}\, B(t-1) \\ &= e^{\omega t}\, B(0). \end{aligned} \tag{4.20}$$

Similarly as for $NC(0)$ we can write $B(0)$ as

$$B(0) = k_2\, AL(0) \tag{4.21}$$

where k_2 takes a value between 0 and 1 and depends on the pension scheme's benefit design, population structure and the valuation rate of interest.

Thus

$$B(t) = k_2\, e^{\omega t}\, AL(0). \tag{4.22}$$

4.6 Solution by Dynamic Programming

4.6.1 Algorithm

We solve the problem using Bellman's principle of optimality (see Bertsekas (1976)). Essentially, the principle of optimality allows us to solve the dynamic problem as a number of sub-problems.

We define

$$\begin{aligned} J(f(t);t) &= \min_{\pi(t),\dots,\pi(T)} E\Bigg[\sum_{s=t}^{T-1} \nu^{s-t}\Bigg\{\alpha_1\left(\frac{c(s)}{NC(s)}-1\right)^2 + \alpha_2\,\nu\left(1-\frac{f(s+1)}{AL(s+1)}\right)^2 \\ &\qquad + \alpha_3\left(\frac{c(s)}{NC(s)}-1\right) + \alpha_4\,\nu\left(1-\frac{f(s+1)}{AL(s+1)}\right)\Bigg\}\,\Bigg|\, f(t)\Bigg]. \end{aligned} \tag{4.23}$$

We can write the Bellman optimality equation as:

$$J(f(t);t) = \min_{\pi(t)} E\left\{\alpha_1\left(\frac{c(t)}{NC(t)}-1\right)^2 + \alpha_2\,\nu\left(1-\frac{f(t+1)}{AL(t+1)}\right)^2\right.$$

$$+\alpha_3\Big(\frac{c(t)}{NC(t)}-1\Big)+\alpha_4\,\nu\Big(1-\frac{f(t+1)}{AL(t+1)}\Big)$$
$$+\nu\,J(f(t+1);t+1)\Big|f(t)\Big\}. \tag{4.24}$$

4.6.2 Trial solution

We consider the following trial solution and proceed inductively:

$$J(f(t);t) \;=\; P_t\,f(t)^2+Q_t\,f(t)+R_t. \tag{4.25}$$

This is satisfied as time T with $P_T=Q_T=R_T=0$ since there is no closing cost.

Now assume that the trial solution is satisfied at time $t+1$. Then at time t:

$$\begin{aligned} J(f(t);t) \;=\; \min_{\pi(t)} E\Big\{ & \alpha_1\Big(\frac{c(t)}{NC(t)}-1\Big)^2+\alpha_2\,\nu\Big(1-\frac{f(t+1)}{AL(t+1)}\Big)^2 \\ & +\alpha_3\Big(\frac{c(t)}{NC(t)}-1\Big)+\alpha_4\,\nu\Big(1-\frac{f(t+1)}{AL(t+1)}\Big) \\ & +\nu\,P_{t+1}\,f(t+1)^2+\nu\,Q_{t+1}\,f(t+1)+\nu\,R_{t+1}\Big|f(t)\Big\}. \end{aligned} \tag{4.26}$$

Thus

$$\begin{aligned} J(f(t);t) \;=\; \min_{\pi(t)} E\Big\{ & \alpha_1\Big(\frac{c(t)}{NC(t)}-1\Big)^2+\alpha_3\Big(\frac{c(t)}{NC(t)}-1\Big)+\nu\,R_{t+1}+(\alpha_2+\alpha_4)\nu \\ & +\nu\Big[Q_{t+1}-(2\alpha_2+\alpha_4)\frac{1}{AL(t+1)}\Big]f(t+1) \\ & +\nu\Big[P_{t+1}+\alpha_2\frac{1}{AL(t+1)^2}\Big]f(t+1)^2\;\Big|f(t)\Big\}. \end{aligned} \tag{4.27}$$

As a consequence of our assumption concerning the return on the risky asset we get

$$E\Big[f(t+1)\Big|f(t)\Big]=\Big(1+\mu_2+\pi(t)(\mu_1-\mu_2)\Big)\Big(f(t)+c(t)-B(t)\Big) \tag{4.28}$$

and

$$E\Big[f(t+1)^2\Big|f(t)\Big] \;=\; \Big(\pi(t)^2\sigma_1^2+(1+\pi(t)\mu_1+(1-\pi(t))\mu_2)^2\Big)\Big(f(t)+c(t)-B(t)\Big)^2. \tag{4.29}$$

Now we set J as $J(f(t);t)=\min_{\pi(t)} J$. Thus

$$
\begin{aligned}
J &= \alpha_1 \Big(\frac{c(t)}{NC(t)} - 1\Big)^2 + \alpha_3 \Big(\frac{c(t)}{NC(t)} - 1\Big) + \nu R_{t+1} + (\alpha_2 + \alpha_4)\,\nu \\
&\quad + \nu \Big[Q_{t+1} - (2\alpha_2 + \alpha_4)\frac{1}{AL(t+1)}\Big] \Big(1 + \mu_2 + (\mu_1 - \mu_2)\pi(t)\Big)\Big(f(t) + c(t) - B(t)\Big) \\
&\quad + \nu \Big[P_{t+1} + \alpha_2 \frac{1}{AL(t+1)^2}\Big] \Big(f(t) + c(t) - B(t)\Big)^2 \times \\
&\qquad \Big\{\pi(t)^2\,\sigma_1^2 + \Big(1 + \mu_2 + (\mu_1 - \mu_2)\pi(t)\Big)^2\Big\}.
\end{aligned} \tag{4.30}
$$

Hence

$$
\begin{aligned}
\frac{\partial J}{\partial \pi(t)} &= \nu \Big[Q_{t+1} - (2\alpha_2 + \alpha_4)\frac{1}{AL(t+1)}\Big] (\mu_1 - \mu_2) \Big(f(t) + c(t) - B(t)\Big) \\
&\quad + 2\nu \Big[P_{t+1} + \alpha_2 \frac{1}{AL(t+1)^2}\Big] \Big(f(t) + c(t) - B(t)\Big)^2 \times \\
&\qquad \Big\{\pi(t)\,\sigma_1^2 + (\mu_1 - \mu_2)\Big(1 + \mu_2 + (\mu_1 - \mu_2)\pi(t)\Big)\Big\}.
\end{aligned} \tag{4.31}
$$

And

$$
\frac{\partial^2 J}{\partial \pi(t)^2} = 2\frac{\nu}{AL(t+1)^2}\Big(\alpha_2 + P_{t+1}AL(t+1)^2\Big)\Big\{\sigma_1^2 + (\mu_1 - \mu_2)^2\Big\}\Big(f(t) + c(t) - B(t)\Big)^2. \tag{4.32}
$$

This is always greater than zero as long as

$$
P_{t+1} > -\alpha_2\, AL(t+1)^{-2}. \tag{4.33}
$$

We now set the first partial derivative equal to zero: $\frac{\partial J}{\partial \pi(t)} = 0$. Then

$$
\begin{aligned}
\pi(t) &= -\frac{(\mu_1 - \mu_2)\Big[Q_{t+1}AL(t+1) - (2\alpha_2 + \alpha_4)\Big]AL(t+1)}{2\Big(\alpha_2 + P_{t+1}AL(t+1)^2\Big)\Big(\sigma_1^2 + (\mu_1 - \mu_2)^2\Big)} \frac{1}{\Big(f(t) + c(t) - B(t)\Big)} \\
&\quad - \frac{(\mu_1 - \mu_2)(1 + \mu_2)}{\sigma_1^2 + (\mu_1 - \mu_2)^2}.
\end{aligned} \tag{4.34}
$$

We would expect

$$
Q_{t+1}AL(t+1) - (2\alpha_2 + \alpha_4) \;<\; 0 \tag{4.35}
$$

otherwise $\pi(t)$ would always be negative.

We rewrite the contribution at time t, $c(t)$, as

$$
c(t) = NC(t) + k\Big(AL(t) - f(t)\Big). \tag{4.36}
$$

This says that the contribution at time t is composed of a normal cost and an adjustment for gains/losses arising at the valuation date.

We also simplify the equations by writing

$$X_t = Q_t\, AL(t) - (2\,\alpha_2 + \alpha_4) \tag{4.37}$$
$$Y_t = \alpha_2 + P_t\, AL(t)^2 \tag{4.38}$$

$$\Omega_0 = (1+\mu_2)\sigma_1^2 \tag{4.39}$$
$$\Omega_1 = \sigma_1^2 + (\mu_1 - \mu_2)^2 \tag{4.40}$$
$$\Omega_2 = \sigma_1^2\,(\mu_1^2 - \mu_2^2) + (\mu_1 - \mu_2)^4 \tag{4.41}$$
$$\Omega_3 = -(\mu_1 - \mu_2)\,(1+\mu_2) \tag{4.42}$$
$$\Omega_4 = \sigma_1^2\,\Omega_3^2\,\Omega_1^{-2} + \Omega_0^2\,\Omega_1^{-2}. \tag{4.43}$$

We note that $\Omega_0, \Omega_1, \Omega_2, \Omega_4$ are positive; whilst Ω_3 is negative.

Replacing $\pi(t)$ in J then, as shown in the Appendix, J can be written as a quadratic in $f(t)$.

Thus the trial solution is satisfied at time t and the optimal cost is:

$$J(f(t);t) = P_t\, f(t)^2 + Q_t\, f(t) + R_t \tag{4.44}$$

where

$$P_t = \alpha_1\, k^2\, NC(t)^{-2} + \nu\,(1-k)^2\,\Omega_4\, AL(t+1)^{-2}\, Y_{t+1} \tag{4.45}$$

$$\begin{aligned} Q_t = & -k\, NC(t)^{-2}\left(2\alpha_1\, k\, AL(t) + \alpha_3\, NC(t)\right) \\ & + \nu\,(1-k)\,\Omega_0\,\Omega_1^{-1}\, AL(t+1)^{-1}\, X_{t+1} \\ & + 2\,\nu\,(1-k)\,\Omega_4\, AL(t+1)^{-2}\, Y_{t+1}\left(NC(t) + k\, AL(t) - B(t)\right) \end{aligned} \tag{4.46}$$

$$\begin{aligned} R_t = & \;\nu\, R_{t+1} + \nu\,(\alpha_2 + \alpha_4) + \left(\alpha_1\, k\, AL(t) + \alpha_3\, NC(t)\right) k\, AL(t)\, NC(t)^{-2} \\ & + \nu\,\Omega_0\,\Omega_1^{-1}\, AL(t+1)^{-1}\, X_{t+1}\left(NC(t) + k\, AL(t) - B(t)\right) \\ & + \nu\,\Omega_4\, AL(t+1)^{-2}\, Y_{t+1}\left(NC(t) + k\, AL(t) - B(t)\right)^2 \\ & + \frac{1}{4}\,\nu\left(\sigma_1^2 - \Omega_1\right)\Omega_1^{-1}\, X_{t+1}^2\, Y_{t+1}^{-1}. \end{aligned} \tag{4.47}$$

4.6.3 Optimal asset allocation policy

We can write the optimal asset allocation policy as

$$\begin{aligned}
\pi^*(t) &= \Omega_1^{-1}\Omega_3 - \frac{1}{2}(\mu_1-\mu_2)\,\Omega_1^{-1}\,AL(t+1)\,X_{t+1}\,Y_{t+1}^{-1}\times \\
&\qquad \Big\{(1-k)\,f(t)+NC(t)+k\,AL(t)-B(t)\Big\}^{-1} \\
&= \Omega_1^{-1}\Omega_3 - \frac{1}{2}(\mu_1-\mu_2)\,\Omega_1^{-1}\,AL(t+1)\,X_{t+1}\,Y_{t+1}^{-1}\,\chi_t^{-1} \qquad (4.48)
\end{aligned}$$

where

$$\begin{aligned}
\chi_t &= (1-k)\,f(t)+NC(t)+k\,AL(t)-B(t) \\
&= -(1-k)\,\big(AL(t)-f(t)\big)+AL(t)+NC(t)-B(t) \\
&= -(1-k)\,\big(AL(t)-f(t)\big)+e^{\omega t}\,\big(AL(0)+NC(0)-B(0)\big) \\
&= -(1-k)\,\big(AL(t)-f(t)\big)+(1+k_1-k_2)\,e^{\omega t}\,AL(0) \\
&= -(1-k)\,\big(AL(t)-f(t)\big)+\gamma_1\,e^{\omega t}\,AL(0) \qquad (4.49)
\end{aligned}$$

where $\gamma_1 = 1+k_1-k_2$.

Effect of Fund Deficit

As the deficit $AL(t)-f(t)$ increases, χ_t decreases since $1-k>0$. Hence the absolute value of $(\mu_1-\mu_2)\,\Omega_1^{-1}\,AL(t+1)\,X_{t+1}\,Y_{t+1}^{-1}\,\chi_t^{-1}$ increases. From Equations 4.35 and 4.37, X_{t+1} is negative. Thus $-\frac{1}{2}(\mu_1-\mu_2)\,\Omega_1^{-1}\,AL(t+1)\,X_{t+1}\,Y_{t+1}^{-1}\,\chi_t^{-1}$ has a positive sign.

So as the deficit increases, $\pi^*(t)$ the optimal allocation in the risky asset increases. Conversely, as the deficit decreases the allocation in the risky asset decreases.

Thus we get a 'contrarian' investment strategy as has been the case in the actuarial literature involving quadratic objective functions - see Section 4.2. This strategy entails shifting the fund into the risky asset as the funding position worsens (that is, as the deficit increases). On the other hand, we would shift the fund into the risk-free asset as the funding position improves (that is, as the deficit decreases).

4.7 Dissecting the optimal policy

To analyze the optimal asset allocation policy we write $\pi^*(t)$, P_{t+1}, and Q_{t+1} as:

$$\pi^*(t) = -\mathtt{Const}_1 - \mathtt{Const}_2\,\frac{\big[Q_{t+1}AL(t+1)-(2\alpha_2+\alpha_4)\big]}{\big[P_{t+1}AL(t+1)^2+\alpha_2\big]}\,\frac{AL(t+1)}{\big(f(t)+c(t)-B(t)\big)} \qquad (4.50)$$

$$P_{t+1} = \alpha_1\,\frac{k^2}{NC(t+1)^2} + \mathtt{Const}_3\,\big[P_{t+2}AL(t+2)^2+\alpha_2\big]\,AL(t+2)^{-2} \qquad (4.51)$$

$$
\begin{aligned}
Q_{t+1} \;=\; & -2\alpha_1 k^2 \frac{AL(t+1)}{NC(t+1)^2} - \alpha_3 \frac{k}{NC(t+1)} \\
& + \mathtt{Const}_4 \left[Q_{t+2}\, AL(t+2) - (2\alpha_2 + \alpha_4)\right] AL(t+2)^{-1} \\
& + \mathtt{Const}_5 \left[P_{t+2} AL(t+2)^2 + \alpha_2\right] AL(t+2)^{-2} \times \\
& \qquad\qquad \left[k\, AL(t+1) + NC(t+1) - B(t+1)\right]
\end{aligned} \tag{4.52}
$$

where the constants $\mathtt{Const}_1$, $\mathtt{Const}_2$, $\mathtt{Const}_3$, $\mathtt{Const}_4$, and $\mathtt{Const}_5$ are all positive.

Thus P_{t+1} depends on α_1 and α_2; whilst Q_{t+1} depends on α_1, α_2, α_3, and α_4.

4.7.1 Contribution linear factor

The weight α_3 affects Q_{t+1} through the factor $\alpha_3 \frac{k}{NC(t+1)}$. Since $\alpha_3 \frac{k}{NC(t+1)}$ is non-negative and we expect Q_{t+1} to be negative, then increasing α_3, decreases Q_{t+1} (i.e. makes Q_{t+1} more negative), all other things equal.

The effect of this is to make $Q_{t+1}\, AL(t+1) - (2\alpha_2 + \alpha_4)$ larger and negative. Hence, since $\mathtt{Const}_2$ is positive, the total effect is to increase the optimal allocation, $\pi^*(t)$, in the risky asset.

This agrees with intuition: the linear contribution cost factor penalizes overcontribution but rewards undercontribution. The only way the scheme can remain viable whilst the sponsor is undercontributing is when the investment returns are good. Thus, all other things equal, we would need to increase the allocation in the higher return asset.

4.7.2 Funding linear factor

The weight on the funding linear factor, α_4, affects $\pi^*(t)$ in two ways. Firstly, as can be seen explicitly in the equation for $\pi^*(t)$, increasing α_4 should increase $2\alpha_2 + \alpha_4$ and hence reduce $\left[Q_{t+1}AL(t+1) - (2\alpha_2 + \alpha_4)\right]$ i.e. makes this larger and negative since Q_{t+1} is expected to be negative. This should have the effect of increasing $\pi^*(t)$.

Secondly, from the formula for Q_{t+1}, increasing α_4 reduces Q_{t+1} making it larger and negative. Thus in the formula for $\pi^*(t)$ the expression $\left[Q_{t+1}AL(t+1) - (2\alpha_2 + \alpha_4)\right]$ would become larger and negative. The overall effect would thus be to increase $\pi^*(t)$.

This is intuitively obvious: increasing the weight on the linear funding cost factor would penalize underfunding whilst rewarding overfunding. All other things equal, this can only be achieved by increasing the allocation in the risky asset.

4.8 Infinite Horizon Problem

For the infinite horizon problem we follow closely the approach of Owadally and Haberman (2004). We assume that the population is stable and that there is no price and wage inflation. So $AL(t)$, $NC(t)$, $B(t)$ are constant and we denote them as AL, NC and B, respectively.

We write the performance criteria as:

$$\begin{aligned} V\big(f(0)\big) &= \lim_{T\to\infty} E\Big[\sum_{t=0}^{T-1} \nu^t \Big\{\alpha_1 \Big(\frac{c(t)}{NC}-1\Big)^2 + \alpha_2\,\nu\Big(1-\frac{f(t+1)}{AL}\Big)^2 \\ &\qquad + \alpha_3\Big(\frac{c(t)}{NC}-1\Big) + \alpha_4\,\nu\Big(1-\frac{f(t+1)}{AL}\Big)\Big\}\Big|\, f(0)\Big] \end{aligned} \tag{4.53}$$

The value function $J(f(t);t)$ does not depend on time, hence we write this as $J(f(t))$. For the Bellman optimality equation backwards induction is not necessary.

It can be shown that the solution in the infinite horizon case is the steady-state or equilibrium solution of the finite horizon case. As noted by Owadally and Haberman (2004), convergence as $T \to \infty$ is assured since we get a contraction mapping and the instantaneous costs are discounted.

To ensure that the costs are non-negative we could modify the performance criteria by adding constants: $\frac{\alpha_3^2}{4\alpha_1}$ and $\frac{\alpha_4^2}{4\alpha_2}\nu$ - the adequacy of these constants can be established by completing the square. This modification would only add a constant to the term independent of $f(t)$ in the optimal cost but would have no effect on the optimal asset allocation policy. Thus we disregard these constants.

Hence a trial solution is $J(f(t)) = P\,f(t)^2 + Q\,f(t) + R$ where $P_t \to P$, $Q_t \to Q$ and $R_t \to R$ as $t \to \infty$.

Thus J may be written as in Equation 4.30 with P_t, Q_t, R_t replaced by the constants P, Q, R, respectively. Thus

$$\begin{aligned} J &= \alpha_1\Big(\frac{c(t)}{NC}-1\Big)^2 + \alpha_3\Big(\frac{c(t)}{NC}-1\Big) + \nu R + (\alpha_2+\alpha_4)\,\nu \\ &\quad + \nu\Big[Q - (2\alpha_2+\alpha_4)\frac{1}{AL}\Big]\Big(1+\mu_2+(\mu_1-\mu_2)\pi_\infty(t)\Big)\Big(f(t)+c(t)-B\Big) \\ &\quad + \nu\Big[P + \alpha_2\frac{1}{AL^2}\Big]\Big(f(t)+c(t)-B\Big)^2 \times \\ &\qquad \Big\{\pi_\infty(t)^2\,\sigma_1^2 + \Big(1+\mu_2+(\mu_1-\mu_2)\pi_\infty(t)\Big)^2\Big\}. \end{aligned} \tag{4.54}$$

Hence

$$\frac{\partial J}{\partial \pi_\infty(t)} = \nu\Big[Q - (2\alpha_2+\alpha_4)\frac{1}{AL}\Big](\mu_1-\mu_2)\Big(f(t)+c(t)-B\Big)$$

$$+2\nu\left[P+\alpha_2\frac{1}{AL^2}\right]\left(f(t)+c(t)-B\right)^2\times$$
$$\left\{\pi_\infty(t)\,\sigma_1^2+(\mu_1-\mu_2)\left(1+\mu_2+(\mu_1-\mu_2)\pi_\infty(t)\right)\right\}. \tag{4.55}$$

And

$$\frac{\partial^2 J}{\partial\pi_\infty(t)^2} = 2\frac{\nu}{AL^2}\left(\alpha_2+P\,AL^2\right)\left\{\sigma_1^2+(\mu_1-\mu_2)^2\right\}\left(f(t)+c(t)-B\right)^2. \tag{4.56}$$

This is always greater than zero as long as

$$P\,AL^2+\alpha_2>0. \tag{4.57}$$

Set $\frac{\partial J}{\partial\pi_\infty(t)}=0$. Then the optimal asset allocation policy is

$$\begin{aligned}\pi_\infty^*(t) &= -\frac{(\mu_1-\mu_2)\left[Q\,AL-(2\alpha_2+\alpha_4)\right]AL}{2\left(\alpha_2+P\,AL^2\right)\left(\sigma_1^2+(\mu_1-\mu_2)^2\right)}\frac{1}{\left(f(t)+c(t)-B\right)}\\ &\quad-\frac{(\mu_1-\mu_2)(1+\mu_2)}{\sigma_1^2+(\mu_1-\mu_2)^2}.\end{aligned} \tag{4.58}$$

Now replace $\pi_\infty^*(t)$ in J. Then

$$\begin{aligned}J(f(t)) &= \nu\frac{\sigma_1^2(1+\mu_2)^2}{\sigma_1^2+(\mu_1-\mu_2)^2}\left[P+\alpha_2\frac{1}{AL^2}\right]\left(f(t)+c(t)-B\right)^2\\ &\quad+\nu\frac{\sigma_1^2(1+\mu_2)}{\sigma_1^2+(\mu_1-\mu_2)^2}\left[Q-(2\alpha_2+\alpha_4)\frac{1}{AL}\right]\left(f(t)+c(t)-B\right)\\ &\quad-\frac{1}{4}\nu\frac{(\mu_1-\mu_2)^2}{\sigma_1^2+(\mu_1-\mu_2)^2}\frac{\left[Q\,AL-(2\alpha_2+\alpha_4)\right]^2}{\left[\alpha_2+P\,AL^2\right]}\\ &\quad+\alpha_1\left(\frac{c(t)}{NC}-1\right)^2+\alpha_3\left(\frac{c(t)}{NC}-1\right)+\nu R+(\alpha_2+\alpha_4)\,\nu.\end{aligned} \tag{4.59}$$

We can rewrite the contribution at time t, $c(t)$, as

$$c(t) = NC+k\left(AL-f(t)\right). \tag{4.60}$$

Thus the coefficient of $f(t)^2$ in $J(f(t))$ is:

$$\alpha_1\frac{k^2}{NC^2}+\nu\frac{\sigma_1^2(1+\mu_2)^2}{\sigma_1^2+(\mu_1-\mu_2)^2}\left[P+\alpha_2\frac{1}{AL^2}\right](1-k)^2. \tag{4.61}$$

And the coefficient of $f(t)$ in $J(f(t))$ is:

$$
\begin{aligned}
&2\,\nu\,\frac{\sigma_1^2(1+\mu_2)^2}{\sigma_1^2+(\mu_1-\mu_2)^2}\left[P+\alpha_2\frac{1}{AL^2}\right](1-k)\left(k\,AL+NC-B\right)\\
&+\nu\,\frac{\sigma_1^2(1+\mu_2)}{\sigma_1^2+(\mu_1-\mu_2)^2}\left[Q-(2\alpha_2+\alpha_4)\frac{1}{AL}\right](1-k)\\
&-2\alpha_1\,\frac{k^2}{NC^2}\,AL-\alpha_3\,\frac{k}{NC}.
\end{aligned}
\tag{4.62}
$$

Thus the trial solution is satisfied at time t and the optimal cost is:

$$
J(f(t)) \;=\; P\,f(t)^2+Q\,f(t)+R \tag{4.63}
$$

where

$$
P \;=\; \alpha_1\,k^2\,NC^{-2}+\nu\,(1-k)^2\,\Omega_4\,AL^{-2}\left(\alpha_2\,+\,P\,AL^2\right) \tag{4.64}
$$

$$
\begin{aligned}
Q \;=\;& -k\,NC^{-2}\left(2\alpha_1\,k\,AL+\alpha_3\,NC\right)\\
&+\nu\,(1-k)\,\Omega_0\,\Omega_1^{-1}\,AL^{-1}\left(Q\,AL-\,(2\,\alpha_2\,+\,\alpha_4)\right)\\
&+2\,\nu\,(1-k)\,\Omega_4\,AL^{-2}\left(\alpha_2\,+\,P\,AL^2\right)\left(NC+k\,AL-B\right)
\end{aligned}
\tag{4.65}
$$

$$
\begin{aligned}
R \;=\;& \nu\,R\,+\,\nu\,(\alpha_2\,+\,\alpha_4)\,+\,\left(\alpha_1\,k\,AL+\alpha_3\,NC\right)k\,AL\,NC^{-2}\\
&+\nu\,\Omega_0\,\Omega_1^{-1}\,AL^{-1}\left(Q\,AL\,-\,(2\,\alpha_2\,+\,\alpha_4)\left(NC+k\,AL-B\right)\right.\\
&+\nu\,\Omega_4\,AL^{-2}\left(\alpha_2\,+\,P\,AL^2\right)\left(NC+k\,AL-B\right)^2\\
&+\frac{1}{4}\,\nu\left(\sigma_1^2-\Omega_1\right)\Omega_1^{-1}\left(Q\,AL\,-\,(2\,\alpha_2\,+\,\alpha_4)\right)^2\left(\alpha_2\,+\,P\,AL^2\right)^{-1}
\end{aligned}
\tag{4.66}
$$

where Ω_i, for $i=0,1,\cdots,4$ are as defined in the finite horizon problem.

Thus P and Q can be re-written as

$$
P \;=\; \left(\alpha_1\,k^2\,NC^{-2}+\alpha_2\,\nu\,(1-k)^2\,\Omega_4\,AL^{-2}\right)\left(1-\nu\,(1-k)^2\Omega_4\right)^{-1} \tag{4.67}
$$

$$
\begin{aligned}
Q \;=\;& \Big\{-\,k\,NC^{-2}\left(2\alpha_1\,k\,AL+\alpha_3\,NC\right)-\,\nu\,(1-k)\,\Omega_0\,\Omega_1^{-1}\,AL^{-1}\,(2\,\alpha_2\,+\,\alpha_4)\\
&+2\,\nu\,(1-k)\,\Omega_4\,AL^{-2}\left(\alpha_2\,+\,P\,AL^2\right)\left(NC+k\,AL-B\right)\Big\}\times\\
&\left(1-\nu\,(1-k)\,\Omega_0\,\Omega_1^{-1}\right)^{-1}.
\end{aligned}
\tag{4.68}
$$

And the optimal asset allocation policy can be written as:

$$\pi^*_\infty(t) = \frac{\Omega_3}{\Omega_1} - \frac{(\mu_1 - \mu_2)\,AL\left[Q\,AL - (2\alpha_2 + \alpha_4)\right]}{2\Omega_1\left[\alpha_2 + P\,AL^2\right]\left(\gamma_1\,AL - (1-k)\left[AL - f(t)\right]\right)}. \tag{4.69}$$

4.9 Numerical Implementation: Finite Horizon Problem

4.9.1 Model Scheme

We consider a defined-benefit model scheme with the following structure:

- Stable scheme with respect to age, pensionable salary in real terms and past pensionable service;
- Funding method: Projected Unit Method;
- Normal Retirement Age (NRA): 65;
- On retirement NRA, a pension of $\frac{1}{60}th$ of pensionable salary at retirement for each year pensionable service;
- On withdrawal before NRA, a deferred pension;
- On death in service no benefit;
- Early retirement no permitted.

Regular valuations: annual valuations with gains or losses spread over five years. Pension Scheme assumed initially fully-funded.

Discontinuance Liabilities: we assume that the valuation rate of interest (net of wage inflation) is 4%.

4.9.2 Investment Model

For simplicity we assume that the force of wage inflation, ω, equals zero. We further assume that the asset returns are as follows:

Asset 1 Return on Asset 1 assumed to be normally distributed with mean 4.5% and standard deviation 5%;

Asset 2 Risk-free return assumed to be 2%.

We will run 10,000 computer simulations each year for a projection period of 25 years for the finite-horizon problem.

4.9.3 Analysis of results

Distribution of Funding and Contribution Levels

In this section we consider the discontinuance funding level, $\frac{f(t)}{AL(t)}$, and the contribution level, $\frac{c(t)}{NC(t)}$. We show how the distribution of the discontinuance funding level and the distribution of the contribution level change with the inclusion of the linear factors. We analyse the results by considering the 10th, 25th, 50th, 75th and 90th percentiles.

Figures 4.1 and 4.2 show several percentiles of the funding level and contribution level, respectively. In each figure we consider four cases: first case where both linear factors are excluded (that is, $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$); the second case where the linear contribution factor is included but the linear funding factor is excluded (that is, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0.5$, and $\alpha_4 = 0$); third case where the linear contribution factor is excluded but the linear factor is included (that is, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0$, and $\alpha_4 = 0.5$); and the fourth case where both linear factors are included (that is, $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = \alpha_4 = 0.5$).

We have chosen these particular weights for illustrative purposes only. Our main results hold for different choices of weights.

These figures show that the inclusion of linear factors in the performance criteria significantly reduce chances of underfunding and overcontribution whilst increasing the chances of overfunding and undercontribution. In the cases where linear factors are included we get a significant shift of the distribution of the funding level towards higher levels (Figures 4.1) whilst the distribution of the contribution level shifts significantly towards lower levels (Figure 4.2).

For instance, consider the interquartile range (IQR) in each of the four cases: the IQR is from approximately 99% to 100% in the first case and in the fourth case the IQR is from 103% to 108%. Thus the IQR shifts towards higher funding levels as the linear factors are included.

A similar observation can be made for the distribution of contribution levels: the IQR is from 100% to 102% for the first case and from 70% to 86% in the fourth case. Thus the IQR shifts towards lower contribution levels following the inclusion of linear factors.

In the next section we show the average optimal asset allocation paths for each of these four cases to illustrate how the investment strategy changes with the inclusion of the linear factors.

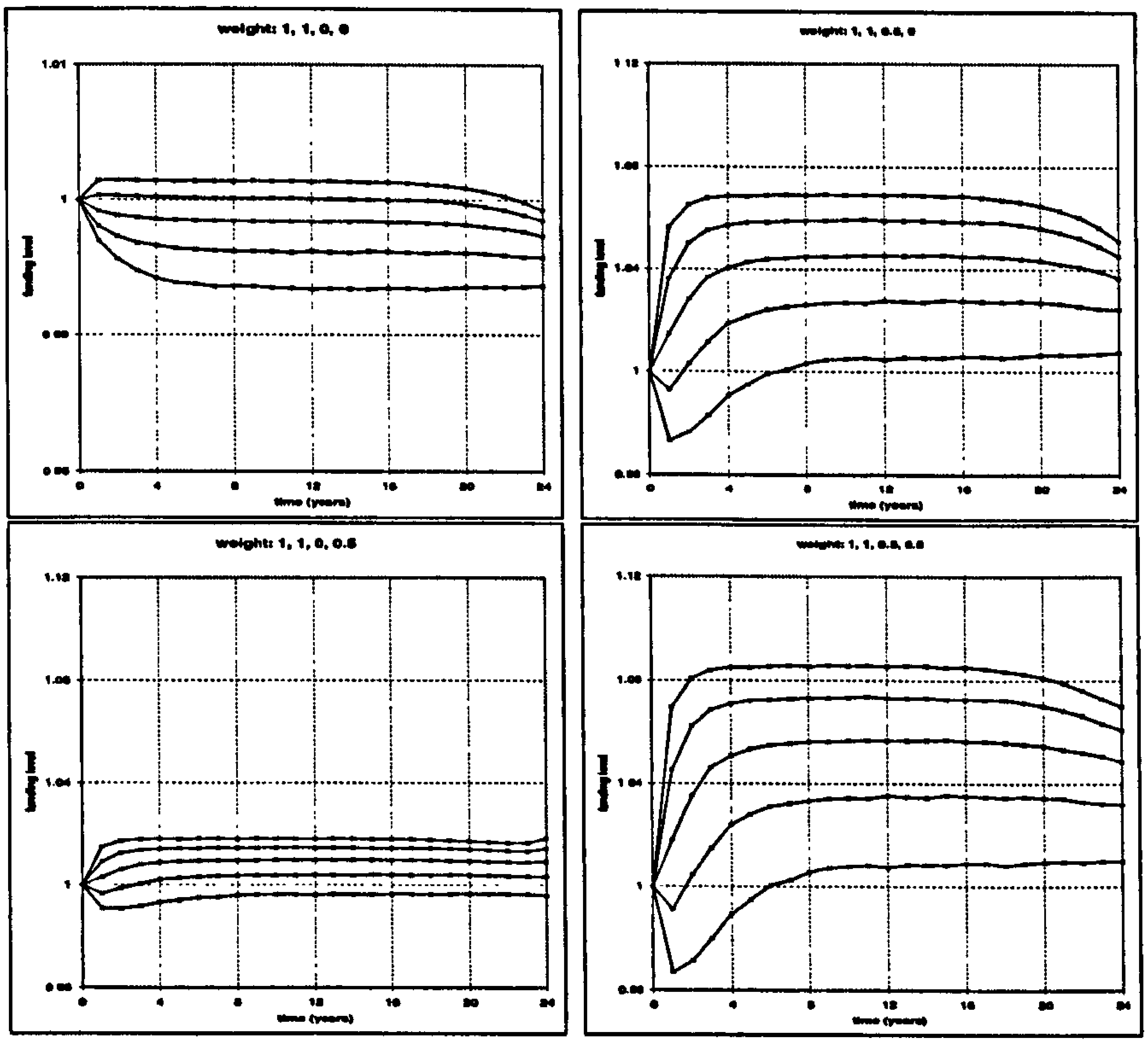

Figure 4.1: Percentiles for the discontinuance funding level: 10th, 25th, 50th, 75th and 90th percentiles (note change of scale for top left diagram).

Average Optimal Asset Paths

In this section we, firstly, illustrate how the investment strategy changes with the inclusion of the linear factors. To do so we will use the average optimal asset allocation paths for each of the four cases considered in the previous section. These paths are meant to show how 'on average' the investment strategy changes from one year to the next for a given set of weights.

Secondly, we investigate the effect of the linear funding factor on the average optimal asset allocation paths. For simplicity we will set this up as a minimum funding problem with the contribution factors excluded.

Figure 4.3 shows four average asset paths for each of the four cases considered in the previous section. For example, path A shows the average optimal allocation in the risky asset in the case where $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \alpha_4 = 0$. In this case we obtain an average optimal allocation of approximately 5% in the risky asset. This result is due to the fact that we have set the valuation rate of interest to be higher than the risk-free rate. On the other hand, setting the valuation rate equal to the risk-free rate would lead to the trivial result that the optimal strategy would involve full allocation in the risk-free asset.

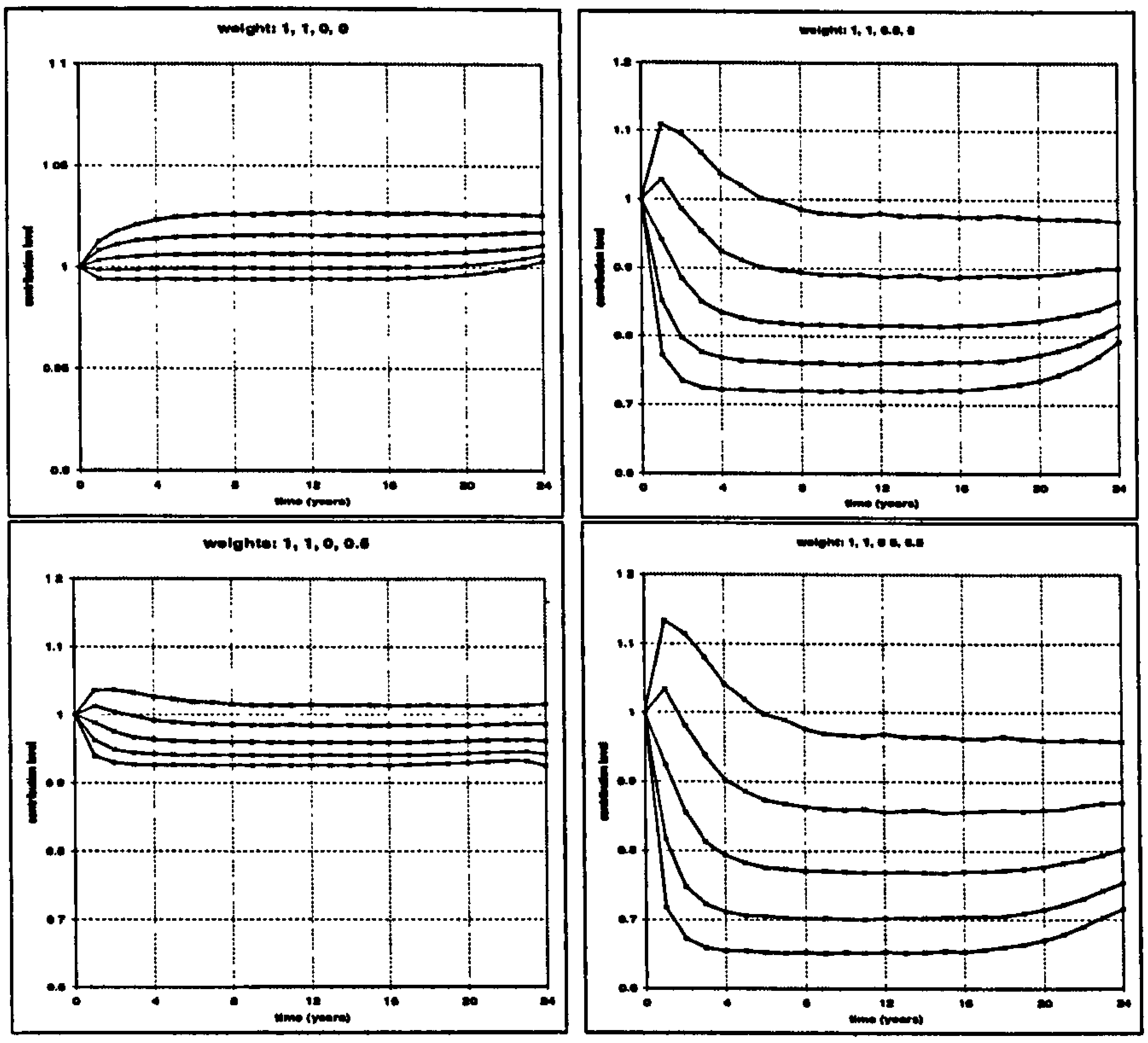

Figure 4.2: Percentiles for the contribution level: 10th, 25th, 50th, 75th and 90th percentiles (note change of scale for top left diagram).

Paths A, B, C and D in Figure 4.3 show that inclusion of the linear funding and contribution factors leads to investment strategies which 'on average' have a higher allocation in the risky asset. This confirms our observations in Section 4.7.

We further observe that the paths for the case where the linear contribution factor is included (that is, C and D) take longer to stabilize than the paths where this factor is excluded. Paths C and D also tend to be more affected by the zero closing cost condition as evidenced by looking at the progress of the paths for $t > 20$. This implies that care ought to be exercised in setting the linear contribution factor's weight, α_3, in relation to the other weights.

Figure 4.4 shows three average asset paths for cases where only the quadratic and linear funding factors are included. The weights in these cases are chosen only to illustrate the effect of the linear funding factor. We fix α_2 as 0.2 but consider three values for α_4: 0.01 (path A), 0.03 (path B), and 0.05 (path C).

These paths show that as we increase the weight on the linear funding factor, the optimal allocation in the risky asset increases 'on average'. Furthermore, in path C we initially short-sell the risk-free asset and the average path takes longer to stabilize.

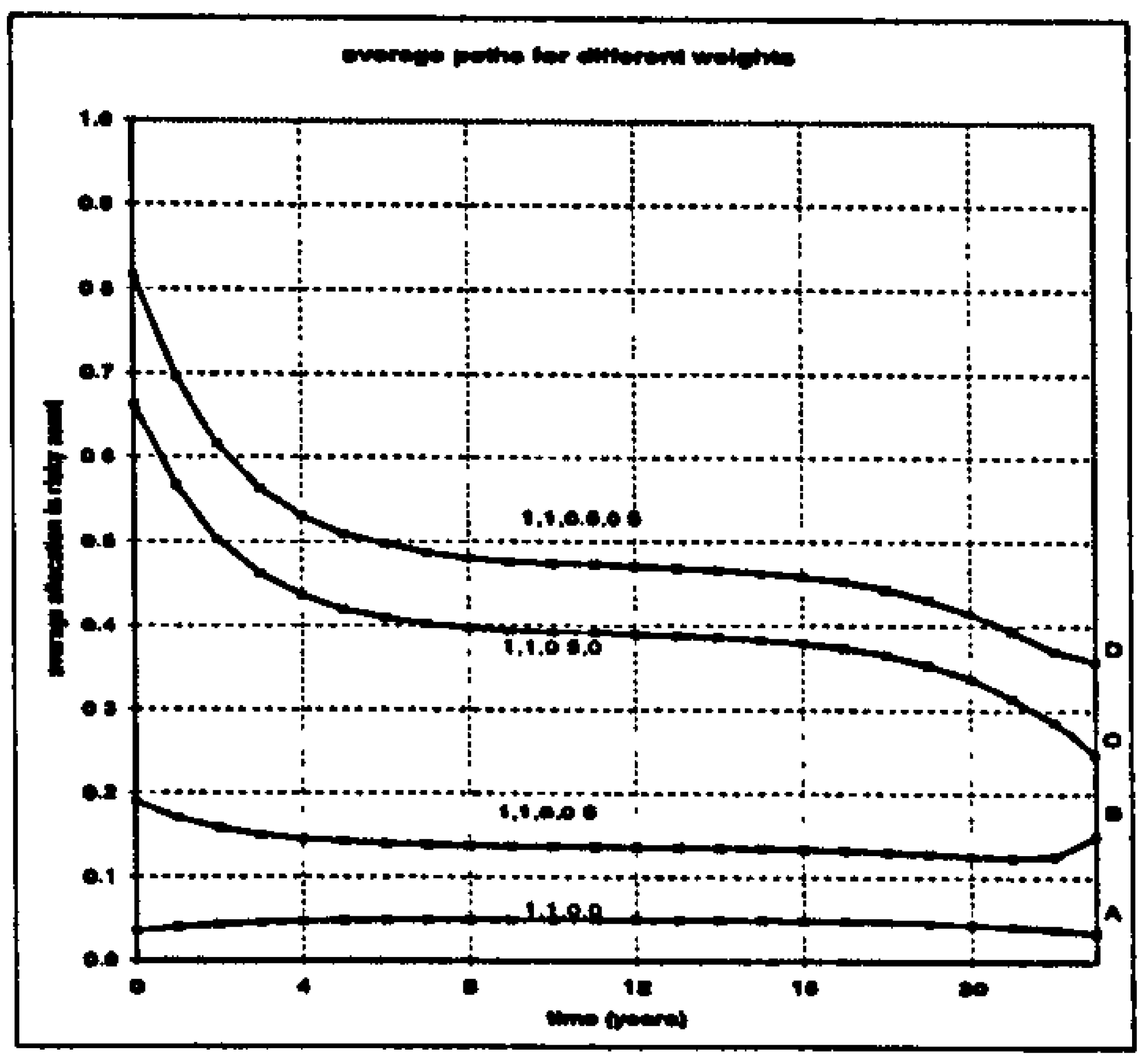

Figure 4.3: Average paths for the optimal allocation in the risky asset.

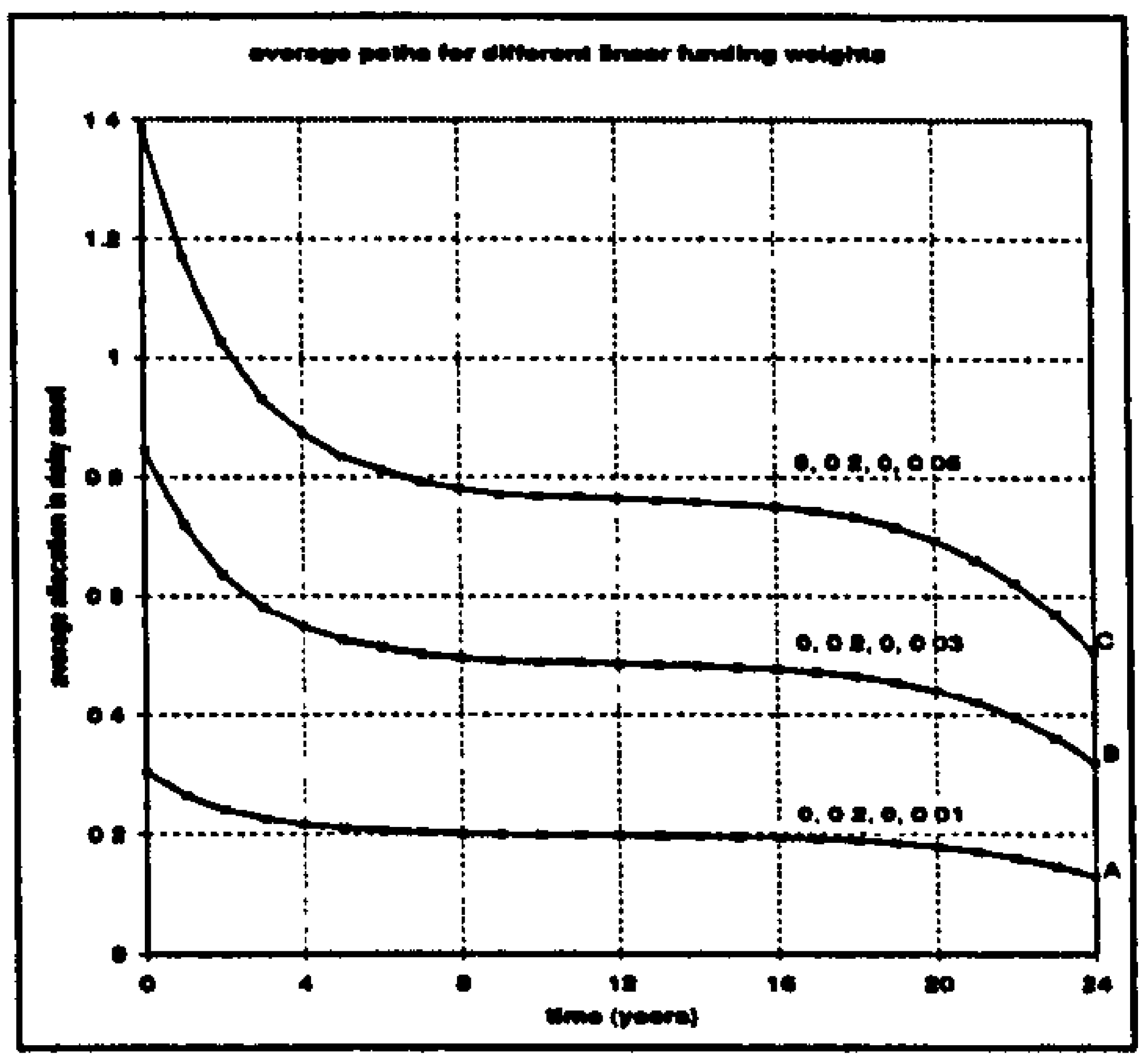

Figure 4.4: Average paths for the optimal allocation in the risky asset for different linear funding weights.

Effect of Initial Funding Level

In this section we study the effect of the initial funding level on the optimal asset allocation strategy. We consider three different initial funding levels: 90%, 100% and 110%. We amortize the initial deficit or surplus over 5 years in the 90% and 110% cases. In each of the three cases we disregard the contribution factors and set the weight on the quadratic funding factor as 0.2 whilst the weight on the linear funding factor is set as 0.01.

Figure 4.5 shows the average asset allocation paths for the three initial funding level cases. We observe that if the scheme is initially in deficit we would, on average, invest more in the risky asset - in this case we initially go long in the risky asset (that is, we short-sell the risk-free asset). On the other hand, if the scheme is initially in surplus we would invest more in the risk-free asset and, in our case, we short-sell the risky asset in the first few years.

We further observe that the three average paths converge after approximately 10 years. The time taken before convergence will depend on the size of initial deficit or surplus and on the period over which the deficit or surplus is amortized.

These results confirm the observations made in Section 4.6.3 regarding the effect of deficits on the optimal asset allocation strategy.

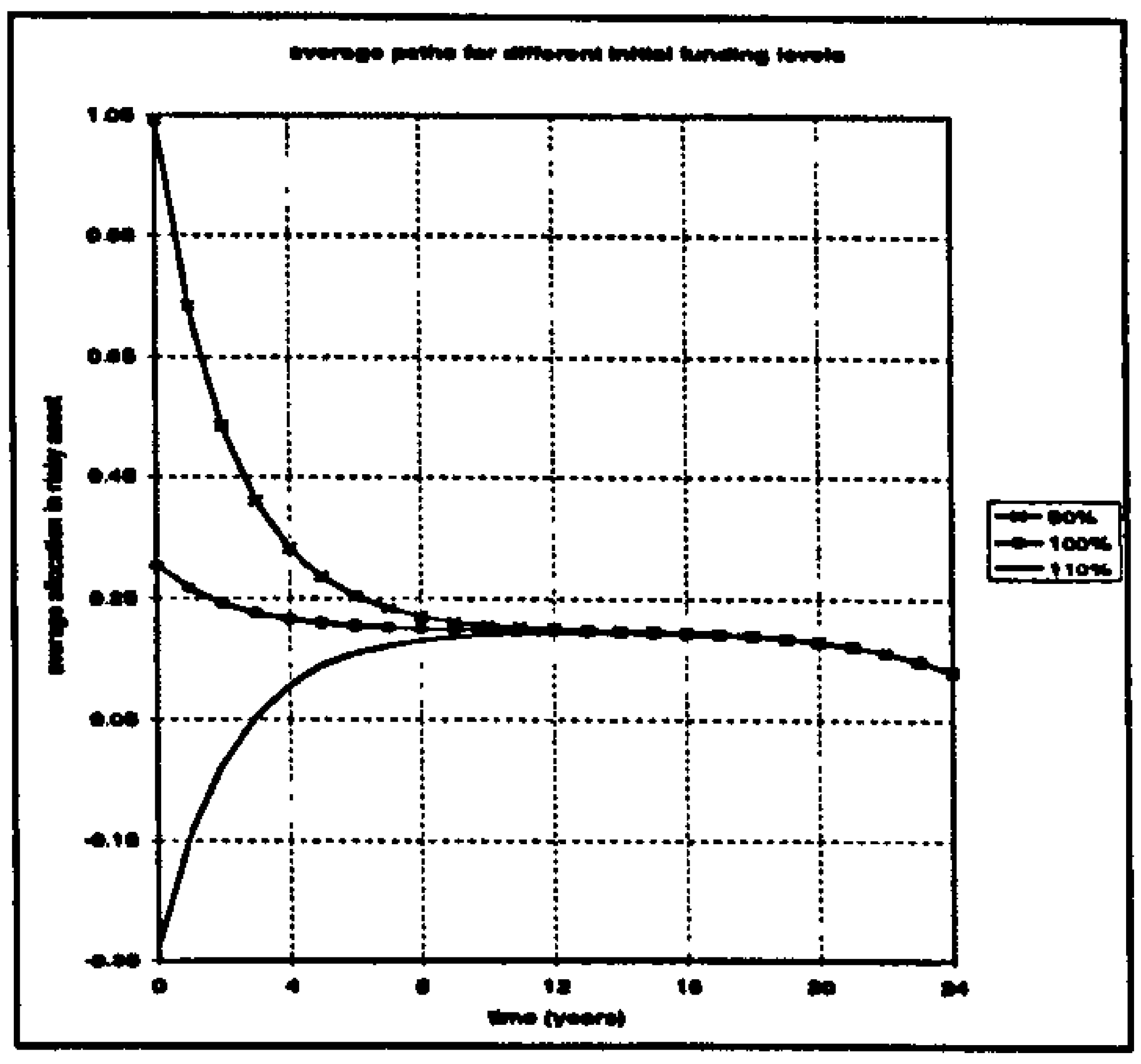

Figure 4.5: Average paths for the optimal allocation in the risky asset for different initial funding levels.

4.10 Concluding Remarks

In this chapter we have considered the problem of optimal asset allocation using a quadratic criterion with linear factors. We have shown that the optimal policy is counter-intuitive: we invest more in the risky asset as the deficit increases and vice versa.

We have further shown that higher funding levels and lower contribution levels can be achieved by including linear factors in the performance criteria. And that the inclusion of these linear factors leads, on average, to higher optimal allocations in the risky asset.

This work can be extended by considering more assets and different distributions of the return on risky assets. In addition, the optimal funding and asset allocation policies can be considered simultaneously.

The clear advantage of using the approach outlined in this chapter is that, in cases where an explicit analytical solution exists, it allows the decision-maker to clearly analyze the effect of crucial variables on the optimal policy. Thus, for instance, we have shown the effect of deficits and also the effect of the weights placed on the quadratic and linear factors in the performance criterion.

However, in some cases analytical-tractability is achieved at the expense of unrealistic simplifications. For instance, to simplify the mathematics, a zero or constant inflation assumption might be made for both prices and wages. This, coupled with an assumption that the population is stable, might lead to constant Benefit outgo, Actuarial Liability and Normal Cost.

Furthermore, the number of analytically-amenable criteria is fairly limited. Thus over the past decade actuarial researchers have mainly used the quadratic criterion. However, the decision-maker might favour other types of criteria for which analytical solutions for the optimal policy might be difficult to find.

Chapter 5

Risk Management Approach to Decision Making

5.1 Introduction

In this chapter we set out a framework for risk management and measurement for defined benefit pension schemes. This framework takes into consideration all the main factors affecting the decision making process in such pension schemes.

Firstly, unlike the traditional valuations, this framework fully recognizes the stochastic factors affecting the promised benefits. Secondly, the main stakeholders' objectives, for example security, stability and cost, are fully controlled through risk and performance measures.

In this framework both the investment and funding strategies are taken as crucial control variables. And thus the risk and performance measures are simultaneously minimized over these variables.

The stochastic approach also provides a way of considering the effect of other variables, for example the amortization period, on the decision making process. Also, projection periods are set and their effect on the investment and funding strategies are explicitly considered.

The idea of the 'decision maker' is very crucial to this stochastic framework. The decision maker could be an individual like the scheme actuary or a collective term for the trustees and/or the sponsoring employer. Myners (2001) identifies the trustees as the "...ultimate decision-makers for pension funds..." (p. 39).

This chapter is organized as follows: in Section 5.2 we present the aims of the stochastic approach. Whilst in Section 5.3 we consider the elements of this stochastic approach. We consider ways of presenting the results in Section 5.4. In Section 5.5

we formulate the stochastic pension fund problem as a multiobjective problem. And in Section 5.6 we present some concluding remarks.

5.2 Aims of the Stochastic Approach

The main objective of a stochastic approach is to recast the problem of decision making in a defined benefit pension scheme in a way which fully accounts for the risks and uncertainties in such schemes and also the various (often conflicting) objectives of the stakeholders. Under the stochastic approach we simultaneously evaluate all of the choices at the disposal of the main decision makers for a defined benefit pension scheme.

In so doing, we are able to meet with the needs of the various stakeholders in a more satisfactory way. These needs are also met simultaneously in the sense that any conflicting aims can be dealt with at the same time.

The choices available encompass the strategy for funding the promised benefits and the investment strategy for the fund. For the funding strategy a number of funding methods exist. The common goal amongst all these funding methods will involve the determination of the sponsor's contribution rate and how it changes over time.

Meanwhile, the choices for the investment strategy might involve the choice of asset classes, for example, equities, bonds (fixed-interest and index-linked), cash and property. Furthermore, the decision makers will have to make choices about proportions to be invested in each of the chosen asset classes and how these proportions will change over time.

In the traditional deterministic approach set out in Chapter 2, the funding and investment strategies are taken as separate decisions during the periodic valuations. Traditionally, actuaries have tended to concentrate more on the areas affecting the funding strategy. These areas include the choice of funding method, the pace of funding, and the strength of the actuarial valuation basis.

In the Asset and Liability Modelling approach, set out in Chapter 3, most of the emphasis has been on investment strategies with the funding strategy being treated as in the traditional approach. Thus the main application of ALM has been in the choice of optimal investment strategies and, in most cases, the objective has been to minimize the probability of insolvency or shortfall at some preset horizon, for example, 25 years.

A stochastic approach to decision-making involves the calculation of explicit and quantifiable risk and performance measures for all possible combinations of funding and investment strategies. These measures are constructed in such a way that the objectives of the stakeholders and the risks facing the scheme are taken into account.

5.3 Elements of the Stochastic Approach

In this section we introduce the main elements of the framework for the stochastic approach. We discuss the choice of the funding strategy, the investment strategy, the risk and performance measures, and the decision points and projection horizons.

5.3.1 Funding Strategy

In Chapter 2 we discussed how the funding strategy is chosen in the tradition actuarial valuation framework. We observed that although a spectrum of actuarial cost methods exists, one of two main class classifications is used to describe each method: individual versus aggregate cost methods, and accrued versus prospective benefit cost methods. Actuaries have traditionally chosen a funding method by considering the funding method's characteristics, the scheme design, and the legislative environment.

The common goal of these actuarial cost methods is to establish a recommended contribution rate. Under the individual funding methods the recommended contribution rate will essentially be made up of two factors: the normal cost and an adjustment for gains and losses, that is,

$$c(t) = NC(t) + Adj(t). \tag{5.1}$$

where $c(t)$ is the recommended contribution rate in year $(t, t+1)$, $NC(t)$ is the Normal cost and $Adj(t)$ is the adjustment for gains and losses.

Both of these factors, $NC(t)$ and $Adj(t)$, could vary from one valuation date to the next depending on the experience of the scheme during the intervaluation period and on the changes in the valuation basis.

Firstly, the changes in the adjustments depend on the changes in the intervaluation gains and losses, and hence, on the differences between the valuation basis and the actual experience of the scheme. Secondly, the development of the Normal cost over time will depend on the characteristics of the actuarial cost method and on the changes in the valuation basis.

In the stochastic approach, however, we deal with the recommended contribution rate in a simple way. We view the normal contribution rate and the adjustment as simply a method for dividing the total contribution rate into a fixed part and a variable part. We thus assume that the normal contribution rate remains fixed over time by definition.

Definition 5.1 *Normal contribution rate* We define the Normal contribution rate, NC, as the fixed part of the total contribution rate and is the rate payable if the scheme is fully-funded and there is no surplus or deficit.

Thus the total contribution rate can now be written as

$$c(t) = NC + Adj(t) \tag{5.2}$$

and changes in the recommended contribution rate are purely due to the intervaluation gains and losses.

This development offers the decision maker extensive choice in setting the funding strategy. The decision maker could, for example, set NC depending on the financial circumstances of the sponsoring employer. And, thus, offering great flexibility in the funding strategy. Furthermore, NC could be set as a factor of $NC(t)$. This not only offers a whole spectrum of choices but also offers a good reference point or benchmark.

However, the essence of the stochastic approach is that the choice of NC should not be made in isolation as is the case for the traditional actuarial valuations. In the stochastic approach NC is just one of the control variables.

Some actuarial researchers have argued that the choice of the contribution rate does not matter. For instance, Exley et al. (1997) argue that "...[t]he consequence of [setting a lower rate] is to shift contributions from one year to the next, which may have a small economic effect...overall..." (p. 856). Thus, assuming that the contribution rate is adjusted annually, a lower (higher) contribution in one year only leads to a higher (lower) contribution in the next year.

This argument can not be dismissed in some economic circumstances. For example, if a low contribution rate was to coincide with poor investment returns we could get a scenario where a higher contribution might be required. However, in our stochastic framework we consider the effect on the risk and performance measures.

In the stochastic framework choosing a high Normal contribution rate will mean that, all things equal, there will be less risk of scheme insolvency, less risk of increase in the current recommended contribution rate and a higher average contribution rate over a finite time horizon. Similarly, a lower Normal contribution rate will increase both risks of scheme insolvency and contribution rate excess but will reduce the average contribution rate. This gives the necessity for a stochastic approach to establish an optimal funding strategy.

We do not, however, completely ignore funding methods. The adjustment at time t, $Adj(t)$, must be made by considering the difference between the value of the fund at time t, $f(t)$, and a measure of the accrued liability, $AL(t)$, at time t:

$$Adj(t) = k\left(AL(t) - f(t)\right) \tag{5.3}$$

where k equals $1/\ddot{a}_{\overline{m}|}$ and m is the spread period.

To calculate the accrued liability a funding method is required. For this we propose that the Projected Unit Credit method be employed since it is a 'security-driven' model which aims at a certain level of funding and the accrued liability is calculated with reference to projected future salaries.

5.3.2 Investment Strategy

The investment strategy adopted by the pension scheme is one of the most important factors in the future progress of the fund. Traditionally, actuaries have argued that the investment strategy should depend on the liability profile or maturity of the scheme. That is, a young scheme might adopt a high expected return but volatile strategy backed by, for instance, equities. Whilst a mature scheme might adopt a low expected return but less volatile strategy backed, for instance, by bonds.

Furthermore, as shown in Chapter 4, analytical models show that counter-intuitive strategies are optimal. That is, strategies which entail shifting the pension fund into risky assets as the scheme deficit increases; and shifting into risk-free or less risky assets as the scheme surplus increases.

Seldon (1960) observes that "...the investment of pension funds demands a sensitive response to changing market conditions..." (p. 17). Thus, deciding how to allocate the fund is just one half of the coin. The other half concerns how these allocations will have to be changed as the fund progresses.

The investment strategy is also crucial to the cost of the pension plan. McGill (1962) observes that "...the investment returns on the pension fund assets are a substantial factor in reducing [the employer's]...contributions toward the cost of the plan. The higher the yield on the fund investments the lower the cost to the employer..." (p. 286).

McGill et al. (1996) add that the "...productive deployment of [the scheme's] assets reduces the direct cost of a defined benefit plan...". They further note that "...if the assets of a fully-funded defined benefit plan can be invested in a manner that earns on average a total return of 6 percent in a stable economic environment, about 70 percent of the plan's benefits will be paid from investment earnings, leaving only 30 percent to be met from contributions to the plan..." (p. 645).

We concur with these observations. But we further note that we also need to link the investment decision with the funding decision. In this case these two crucial decisions will be made, not separately, but simultaneously.

Following McGill et al.'s comments it is easy to see the advantages of this approach. On the one hand, good returns from assets reduce the sponsor's contributions. Whilst a high contribution could significantly increase the fund if returns are good thus leading to a reduction in the future excess contributions.

5.3.3 Risk and Performance Measures

We noted above that to evaluate the consequences of alternative strategies we need to use performance measures. In this section we present risk and performance measures which are necessary for a stochastic approach to decision-making in defined benefit pension schemes.

Downside Risk Measures

Downside risk measures have been considered by actuaries in various investment problems. Clarkson (1989) suggests several axioms for the measurement of investment risk. Firstly, he argues that "...[i]nvestment risk is a function both of the probability of the return being below a certain threshold and also of the severity of the financial consequences arising from these values of return..." (p. 145).

He thus suggests that the investment risk measure be calculated as

$$Risk = \int_{-\infty}^{L} W\left(\frac{L-r}{k}\right) g(r)\, dr \tag{5.4}$$

where $W(s)$ is a loss function, L is a pre-set threshold and $g(r)$ is the probability density function for the investment return.

Albrecht (1993) considers shortfall risk in the case where the returns follow normal and lognormal distributions. He observes that "...[i]n contrast to...the variance of returns, shortfall-risk is more consistent with the investors' intuitive perception of risk in that it focuses more on the real economical risk of an investor, whereas the variance...measure[s]...the volatility of financial assets..." (p. 417).

He defines shortfall risk in terms of the probability of the event that the return will be below some threshold. However, he does not include the severity of the event.

Thus this formulation of shortfall risk does not satisfy Clarkson's axioms.

Albrecht (1994) considers a more general approach to downside risk. He considers an approach similar to that of Clarkson (1989) by suggesting a power loss function. That is, setting $W(s) = s^n$ in Equation 5.4.

Setting $n = 0$ leads to the shortfall probability as in Albrecht (1993). Meanwhile, a linear loss function ($n = 1$) leads to the shortfall expectation; and a quadratic loss function ($n = 2$) leads to the shortfall semivariance. This approach to shortfall risk is similar to the lower partial moment approach considered, for instance, by Fishburn (1977). (Also see Albrecht, Maurer and Stephan (1995) and Albrecht, Maurer and Timpel (1995) for applications to options).

Another downside risk measure which has been widely covered in financial risk management is the Value at Risk (VaR). Dowd (1998) defines the VaR as "*...the maximum expected loss over a given horizon period at a given level of confidence...*" (p. 39). This expressed mathematically as

$$VaR_c(X) \quad = \quad inf\Big\{x \,\Big|\, Prob\Big(X \leq x\Big) \geq c\Big\} \tag{5.5}$$

where X is the random variable representing the loss and $1-c$ is the confidence level.

Artzner et al. (1997, 1999) criticize the VaR because it fails to satisfy some 'coherence axioms'. Let ρ, a mapping from the set.of loss random variables to real numbers, be a risk measure. Then ρ is said to be coherent if it satisfies the following axioms:

Axiom S: Subadditivity This is the property that for any loss random variables X and Y the risk of the sum of losses should be less than the sum of the individual risks. That is,

$$\rho\Big(X+Y\Big) \leq \rho\Big(X\Big) + \rho\Big(Y\Big). \tag{5.6}$$

Axiom H: Homogeneity This is a reverse property imposed following Axiom S:

$$\rho\Big(\lambda X\Big) = \lambda\,\rho\Big(X\Big) \tag{5.7}$$

for all positive real numbers ρ.

Axiom M: Monotonicity This ensures that ρ is a monotonic function (nonincreasing, in this case)

$$X \leq Y \Rightarrow \rho\Big(X\Big) \geq \rho\Big(Y\Big). \tag{5.8}$$

Axiom T: Translation Invariance This states that

$$\rho\Big(X+\alpha\, r\Big) = \rho\Big(X\Big) - \alpha \tag{5.9}$$

and in particular $\rho\Big(X+\rho(X)\,r\Big) = 0$.

Artzner et al. (1999) show that the VaR risk measure does not satisfy the subadditivity axiom (Axiom S). They conclude that "...with respect to the addition of risks..." the VaR creates "...severe aggregation problems...". They further note that the VaR "...prohibits diversification because...[it] does not take into account the *economic consequences* of the events, the probabilities of which it controls..." (p. 218).

They suggest that a coherent alternative to the VaR is the "TailVaR" or conditional tail expectation (CTE). Wirch and Hardy (1999) define this as "...the expected value of the loss given that the loss falls in the upper $(1-\alpha)$ tail of the distribution..." (p. 339).

In other words

$$CTE_\alpha(X) = E\left[X \,\middle|\, X > VaR_\alpha\left(X\right)\right] \tag{5.10}$$

where $VaR_\alpha\left(X\right)$ is as defined in Equation 5.5.

This risk measure is shown to be coherent. As an alternative to this measure, Wirch (1999) considers the Mean Excess Loss. The Mean Excess Loss is defined, for a benchmark x, as

$$MEL(x) = E\left[X - x \,\middle|\, X > x\right]. \tag{5.11}$$

The connection between the CTE and the MEL can easily be established by setting the benchmark x equal to the VaR. (see Albrecht et al. (2001, p. 4)).

Albrecht et al. (2001) observe that the shortfall expectation, defined above using a linear loss function, works out as the product of the probability of shortfall (loss) and the mean excess loss. (also see Maurer and Schlag (2002, pp.3,4)).

We apply these ideas in the formulation of the Solvency and Contribution risks below.

Solvency Risk Measure

The Solvency risk concerns the security of the benefits. Lee (1986) observes that "...the mere existence of a trust fund separate from the assets of the employer obviously does not guarantee security of pension rights. The size of the fund in relation to its liabilities is crucial..." (p. 158). Thus, the solvency risk is the risk that at time t the accumulated fund, $f(t)$, will be inadequate to meet with the accrued liability, $AL(t)$.

A common approach is to consider deficits: $AL(t) - f(t)$. Then the solvency risk is measured in terms of the probability of deficit: $P\left(AL(t) - f(t) > 0\right)$. As shown in Chapter 4 solvency risk can also be measured in terms of square deficits:

$\left(AL(t) - f(t)\right)^2$.

However, these approaches do not satisfy Clarkson's axioms. Firstly, the probability of deficit does not include the amount of deficit. Secondly, in the case of square deficits, the surpluses are negative deficits and hence, upon squaring, deficits and surpluses are treated similarly.

A more satisfactory approach would be to follow Axiom 1 of Clarkson and consider the probability of deficit and the severity of the deficit should it occur. An immediate consequence of this is to treat all surpluses as zero deficits (instead of negative deficits) and calculate the solvency risk as the expected (mean) shortfall, $MS_t(\cdot)$.

In the stochastic approach the solvency risk, given some initial information, will depend on the investment strategy, π, and the funding strategy, NC. Hence, we can write the Mean Shortfall risk at time t as

$$MS_t(\pi, NC) = E_0\left[max\left(AL(t) - f(t), 0\right)\right] \tag{5.12}$$

where $f(t)$ is the market value of the assets at time t, $AL(t)$ is the discontinuance liability at time t and $c(t)$ is the contribution rate at time t.

This risk measure is of the form

$$Prob\left(AL(t) - f(t) > 0\right) \times E\left[AL(t) - f(t) \,\middle|\, AL(t) - f(t) > 0\right]. \tag{5.13}$$

In a simulation setting this works out as follows. Consider N simulations where simulations $1, \ldots, N_D$ lead to (positive) deficits. Then

$$\begin{aligned} MS_t(\pi, NC) &= \frac{1}{N}\sum_1^N max\left(AL(t) - f(t), 0\right) \\ &= \frac{1}{N}\sum_1^{N_D}\left(AL(t) - f(t)\right) \\ &= \frac{N_D}{N}\frac{1}{N_D}\sum_1^{N_D}\left(AL(t) - f(t)\right) \\ &= Prob\left(AL(t) - f(t) > 0\right) \cdot E\left[AL(t) - f(t) \,\middle|\, AL(t) - f(t) > 0\right]. \end{aligned} \tag{5.14}$$

Thus this measure has the required form: it satisfies Clarkson's axioms and is a coherent risk measure in accordance with the coherence properties of Artzner et al. (1997, 1999).

Contribution Rate Risk Measure

The Contribution rate risk concerns the stability of the contribution plan. Lee (1986) observes that the "...employer will look for a contribution plan which will not be unduly disturbed by random fluctuations in the membership of the scheme..." (p. 159).

As in the case of the Solvency risk, the Contribution rate risk can also be formulated as a downside risk measure. In this case the decision maker would need a target for the contribution. An intuitive choice for this target would be, as in the theoretical analysis of Chapter 4, the normal contribution rate.

To formulate this risk measure we would need to consider the deviations: $c(t) - NC$, where $c(t)$ is the contribution rate in year $(t, t+1)$. The sponsoring employer would consider positive deviations as 'unacceptable'. Meanwhile, negative deviations would not be considered 'acceptable' since such deviations imply that $c(t)$ is less than the preset target of NC.

Then a risk measure could be constructed in a similar way to the Mean Shortfall using $E_0\Big[max\Big(c(t) - NC, 0\Big)\Big]$. However, instead of an end-of-horizon measure, we need to consider all deviations over a given horizon.

Thus we define the Excess Contribution rate risk as

$$ECR_T(\pi, NC) = \sum_{t=0}^{T-1} v^t \, E_0 \Big[max\Big(c(t) - NC, 0\Big)\Big]. \tag{5.15}$$

Hence, the Excess Contribution rate risk is just the sum of all the (discounted) expected excess contribution rates over a given projection horizon.

Average Contribution Rate

The sponsoring employer would also be concerned about the cost of benefit provision over a given horizon. Thus the decision maker would need to include a 'performance measure' of the various investment and funding strategies.

For this purpose the expected contribution rate, $E_0\Big[c(t)\Big]$, over a given year would be the natural choice. The reason being that we measure the cost in terms of the sponsor's "out-of-pocket contributions" (McGill (1962, p. 286)). Over a period longer than one year the expected annual contribution rates would need to be summed (or averaged).

Hence, we can calculate the average contribution rate performance measure over t years as

$$AC_t(\pi, NC) = \frac{1}{t} \sum_{s=0}^{t-1} E_0\Big[c(s)\Big]. \tag{5.16}$$

5.3.4 Decision Points and Projection Horizons

The choice of appropriate decision points and projection horizons is very critical in the stochastic framework. The optimal strategies for funding and investment will depend, to some extent, on the chosen projection horizons (see, example, Lockyer (1990, p.7) and Clark (1992, p.40) and the results contained therein).

There is an unresolved debate in the financial literature about long-term investments and 'time diversification'. Our goal in this thesis is not to endorse either side of the debate but to give a stochastic methodology for establishing optimal asset allocation strategies for a given set of asset classes and a given projection horizon.

In our stochastic approach the choice of the projection horizons and the factors that need to be taken into account is a crucial decision. We propose that projection horizons be classified as short-term, medium-term and long-term. Then, a financially weak sponsoring employer might choose the short-term horizon. Meanwhile, a financially strong sponsor might choose the long-term horizon.

Furthermore, for a given projection horizon, decision points have to be set. These decision points would be necessary for periodic reviews of the investment strategy and the funding strategy.

5.4 Presentation of Results

Due to the number of factors involved in the formulation of the stochastic problem, the presentation of results is a challenging exercise. In this section we consider presentational methods that are based on our construction of the risk and performance measures.

5.4.1 Multi-Dimensional Problem

Each of the risk and performance measures can be written as $Z = H(X, Y)$. Thus the results can be presented as three-dimensional surfaces. To illustrate the general results consider the surfaces in Figures 5.1 and 5.2. We depict here a case where the fund is allocated in equities and other assets.

Figure 5.1 shows a possible surface for the risk, either Mean Shortfall risk or Excess Contribution rate risk, at the end of some projection horizon. Each point on the surface shows the risk arising from a choice of the asset allocation and Normal contribution rate.This figure serves to illustrate several general principles.

Firstly, as the Normal contribution rate increases, the risk decreases and vice versa. Secondly, as the asset allocation is varied, the risk varies in such a way that we get an allocation leading to the lowest risk. Ultimately, this allocation will depend on such factors as the type of asset classes available, the maturity of the scheme and its funding level, and the projection period.

Figure 5.2 shows a possible surface for the average contribution rate over a given projection period. Each point on the surface shows the average contribution rate arising from a choice of the asset allocation and Normal contribution rate. Several general principles can also be illustrated from this graph.

Firstly, as the Normal contribution rate increases, the average contribution rate increases and vice versa. Secondly, as the asset allocation is varied, the average contribution rate varies in such a way that we get an allocation leading to the lowest average contribution rate.

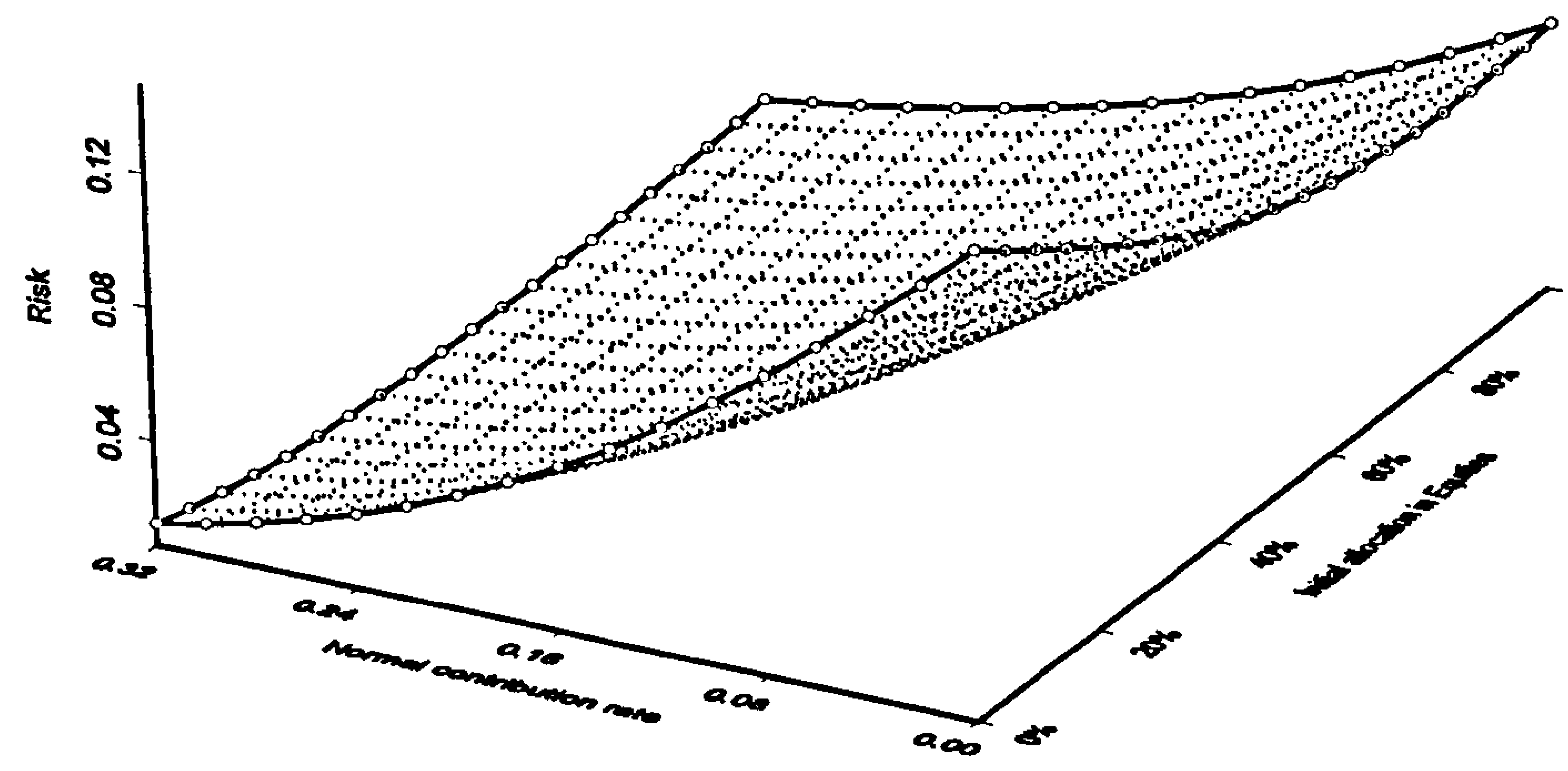

Figure 5.1: Risk at the end of T years

5.4.2 Indifference Curve Analysis

Presentation of results using 3-D surfaces might be mathematically correct but it offers little help to the decision-making process. What is necessary is to consider the lines of equal risk or cost: contours. This is illustrated in Figure 5.3.

We will refer to these contours as Indifference curves. Sloman (1999, p.115) defines an Indifference curve as follows:

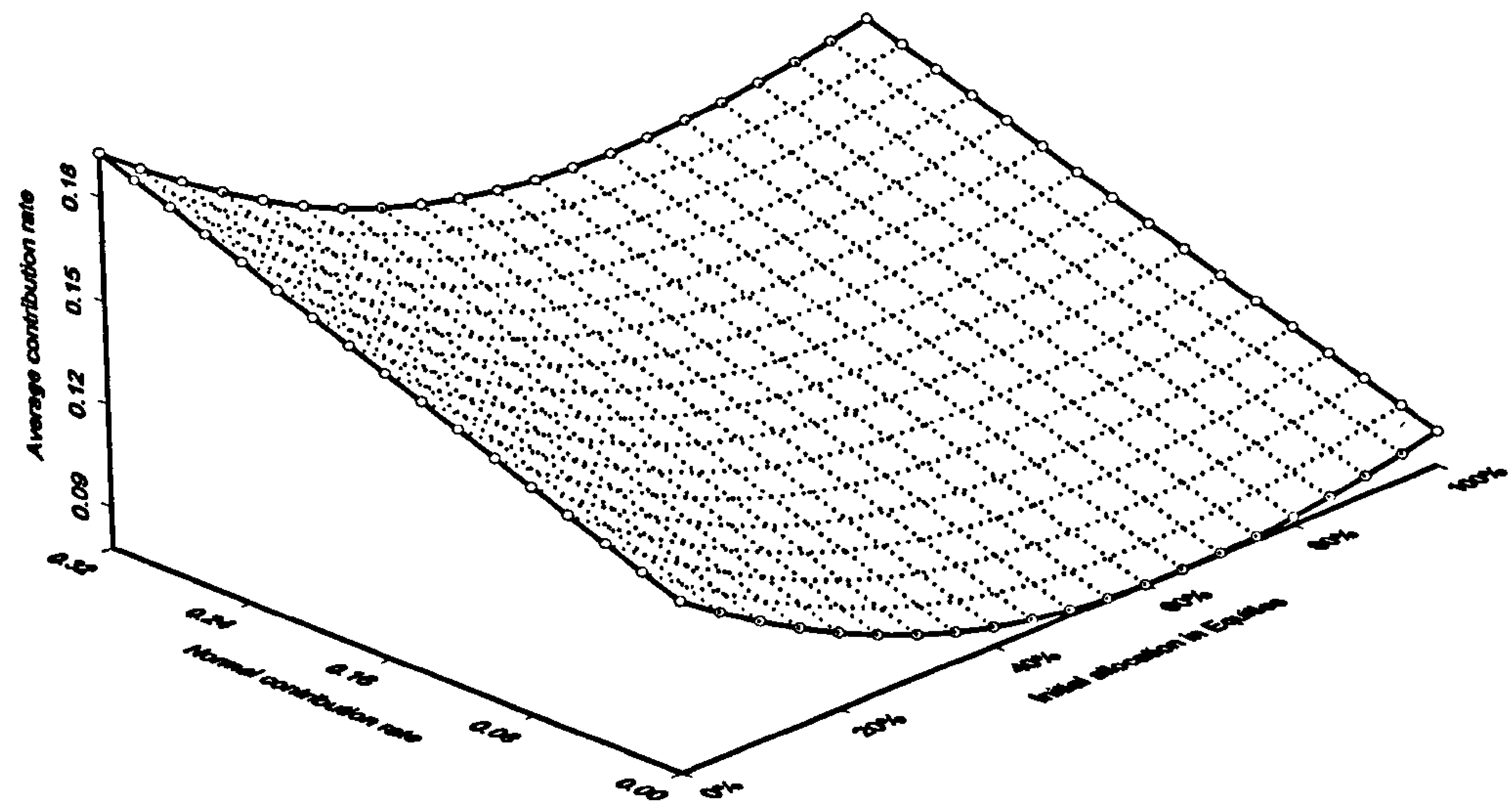

Figure 5.2: Average Contribution Rate at the end of T years

Definition 5.2 *Indifference Curve* An indifference curve is a line showing all those combinations of two goods which give the same level of utility.

The contours show combinations of asset allocation (π) and Normal contribution rate (NC) which lead to the same level of risk or cost. Hence, all things equal, the decision maker would be indifferent between combinations on the same contour.

Then the optimal investment strategies can be worked out as follows:

Definition 5.3 *Efficiency* An asset allocation π^* will be said to *efficient* for a given level h of the risk or performance measure $H(\pi, NC)$ if π^* lies on the intersection between the indifference curve (or contour) for h and the vertical $NC - H$ plane passing through some value of the Normal contribution rate.

We note that if we project onto the $\pi - NC$ plane, then instead of the $NC - H$ plane we get a horizontal line passing through some value of the Normal contribution rate. This idea is pursued further in Chapter 6.

5.5 Vector Stochastic Optimization

In this section we formulate the stochastic pension fund problem as a multiobjective problem.

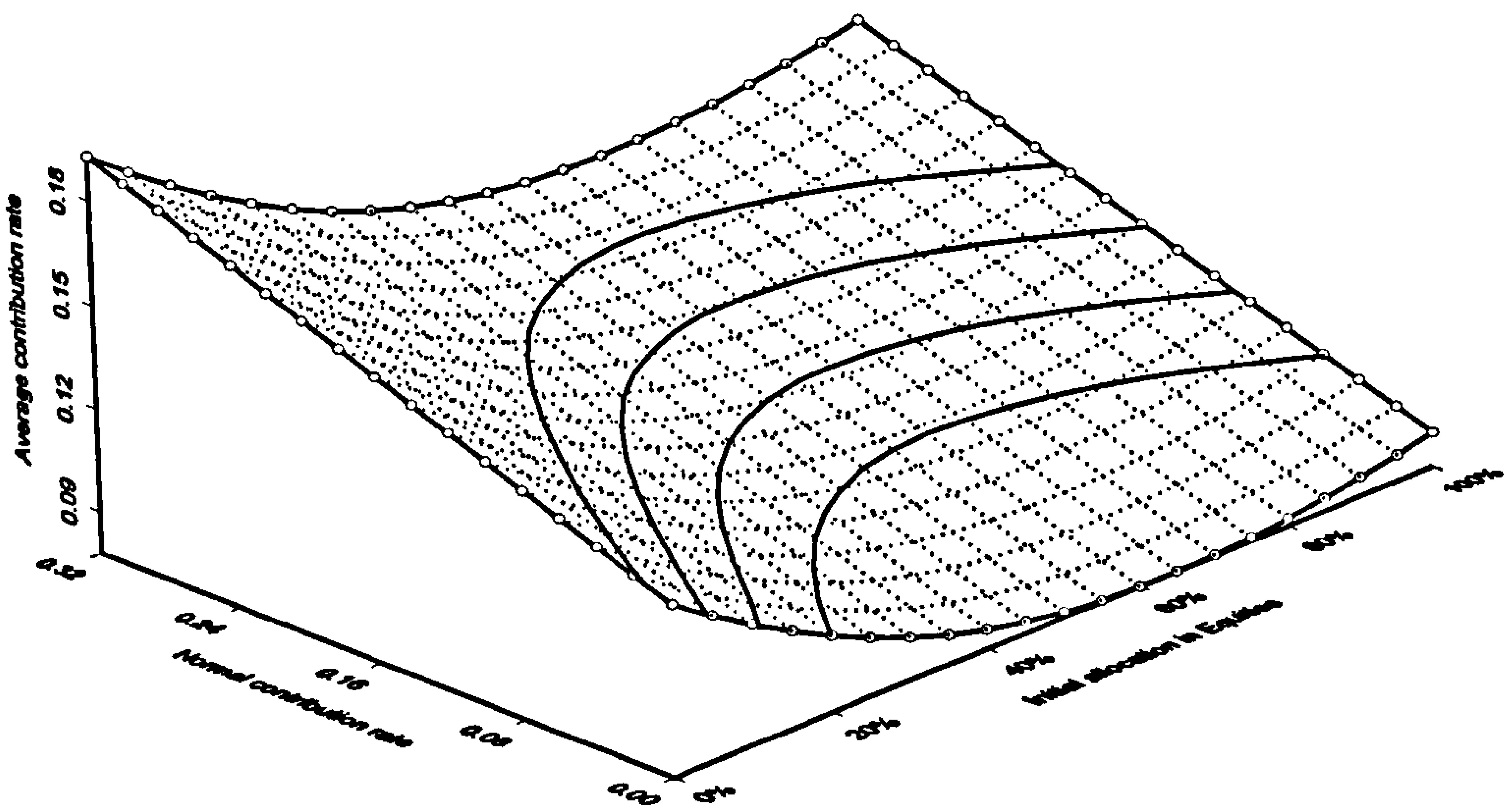

Figure 5.3: Average Contribution Rate at the end of T years with contours

5.5.1 Introduction

The underlying idea of vector stochastic optimization is to set up a multiobjective problem in order to obtain "Pareto optimal" solutions. Such solutions owe their name to the economist Vilfredo Pareto.

A multiobjective optimization problem can be set up as follows:

$$\min_{\pi} \left[H_1(\pi), H_2(\pi), \ldots, H_N(\pi)\right] \tag{5.17}$$

where the objective functions $H_i(\pi)$, $i = 1, \cdots, N$, are linear or nonlinear functions of the decision variables $\pi = (\pi_1, \ldots, \pi_n)$.

The objective functions are usually set up as conflicting goals of the decision maker and may also involve an element of randomness due to the decision maker's financial environment.

Eschenauer et al. (1990, p.10) define Pareto optimality as follows:

Definition 5.4 *Pareto-optimality* A decision variable π^* is said to be Pareto optimal for the multiobjective problem 5.17 if and only if there is no decision variable π with the characteristics

$$H_j(\pi) \leq H_j(\pi^*) \qquad \text{for all } j \in \{1, \ldots, N\}$$

and

$$H_j(\pi) \ < \ H_j(\pi^*) \qquad \text{for at least one } j \in \{1, \ldots, N\} \tag{5.18}$$

Thus a decision variable π^* is Pareto optimal if and only if any other decision variable π leads to worse values than those at π^* for at least one of the objective functions (Chankong and Haimes (1983, p. 115-116)).

Several methods of solving the multicriteria problem are described in the literature on optimization. These methods include

Method 1: Weighting

Under this method weights are introduced in order to reduce the multicriteria problem to a single criterion problem of the form

$$\min_{\pi} \sum_{i=1}^{N} w_i H_i(\pi) \tag{5.19}$$

where the weights could be set so that $\sum_{i=1}^{N} w_i = 1$.

Method 2: ϵ-constraint

Under this method the multiobjective problem is reduced to a single objective problem with $N-1$ constraints:

$$\begin{aligned} &\min_{\pi} H_i(\pi) \\ &\text{subject to } H_j(\pi) \leq \epsilon_j \quad \text{for } j = 1, \ldots, N, \quad j \neq i \end{aligned} \tag{5.20}$$

5.5.2 Application to Pension Fund problems

Formulation of the Problem

It is possible to formulate the stochastic pension fund problem as a multiobjective problem. The goal is to find pareto optimal solutions to this problem. For simplicity we only consider one decision variable : $\boldsymbol{\pi} = \pi$. The formulation can be done in several ways.

Firstly, consider the solvency and contribution rate risks. The multiobjective problem can be written as:

$$\min_{\pi} \Big[MS_T(\pi, NC), ECR_T(\pi, NC) \Big]. \tag{5.21}$$

Following Definition 5.4, the set of pareto optimal solutions will be the set of decision variables such that any other decision variable would either decrease the Mean Shortfall risk and increase the Excess Contribution Rate risk or increase the Mean Shortfall risk and decrease the Excess Contribution Rate risk.

Secondly, consider the solvency risk and the performance measure. The multiobjective problem can be written as:

$$\min_{\pi} \left[MS_T(\pi, NC), AC_T(\pi, NC)\right]. \tag{5.22}$$

The set of pareto optimal solutions will be the set of decision variables such that any other decision variable would either decrease the Mean Shortfall risk and increase the Average Contribution rate or increase the Mean Shortfall risk and decrease the Average Contribution rate.

Finding Pareto Solutions

Seeking analytic solutions to such problems would certainly be a very formidable task (except in the cases of very simple pension models). However, using computer simulations simplifies the analysis regardless of the complexity of the pension model. The use of computers to solve multiobjective problems is well-established in the literature - see, for example, Novak and Ragsdale (2003) for an application to a stochastic linear programming problem.

In our case, pareto optimal solutions can be obtained using the following steps:

Step 1: Formulate the pension model and set the evolution of the various crucial economic factors.

Step 2: Set the projection period and decision points.

Step 3: Fix the Normal contribution rate, NC.

Step 4: Project the fund for different values of π.

Step 5: Calculate the risk and performance measures.

Step 6: Find the Pareto solutions by considering the trade-offs of the two criteria for various values of π. These will be those values of π such that one of the measures can not be improved without worsening the second measure.

Step 7: Repeat Steps 4 to 6 for various choices of NC.

5.6 Concluding Remarks

In this chapter we have presented a stochastic approach to decision making in defined benefit pension schemes. We have considered two downside risk measures and a cost performance measure.

It is possible to extend this framework by considering other downside risk formulations for solvency and contribution rate risks. For example, depending on the population structure of the scheme, it might be more appropriate to consider the contribution amounts instead of the contribution rate.

In Chapter 6 we consider a case study to illustrate the ideas presented in this chapter.

Chapter 6

A Case Study of the Risk Management Approach

6.1 Introduction

In this chapter we present a case study in order to illustrate the stochastic approach set out in Chapter 5. We have considered the modelling framework in Appendix B. In this case study we carry out stochastic projections of the pension scheme over five 3-year periods. Thus the longest projection horizon is 15 years.

It is possible to consider longer horizons depending on the circumstances of the employer: a financially weak employer might prefer a short horizon whilst a financially strong employer might prefer a long horizon. In our case we chose to project only up to 15 years as this provided the best compromise between seeking a long-term horizon and the available computer power.

As will be discussed below, the 3-year periods were chosen as the intervaluation periods. This is a common approach to pension scheme valuations in the UK. At the end of every intervaluation period we measure, for each initial asset allocation (in equities and bonds) and Normal contribution rate, the Mean Shortfall risk, the Excess Contribution rate risk and the Average contribution rate. These performance measures have already been defined in Chapter 5. However, for the particular calculations we introduce some scaling factors: which necessitates some changes to the definitions.

Mean Shortfall risk

This is calculated at the end of every intervaluation period as follows. Let π be the initial allocation in equities (and hence, the initial allocation in bonds is $1-\pi$) and let NC be the Normal contribution rate. Then the Mean Shortfall risk, $MS_T(\pi, NC)$, at the end of T years is:

$$
\begin{aligned}
MS_T(\pi, NC) &= \frac{1}{f(0)} E_0\Big[\max\Big(L(T) - f(T), 0\Big)\Big] \\
&= \frac{1}{f(0)} Pr\Big(L(T) - f(T) > 0\Big)\, E\Big[L(T) - f(T) | L(T) - f(T) > 0\Big]
\end{aligned}
\tag{6.1}
$$

where $L(T)$ and $f(T)$ are the discontinuance liability and the market value of assets, respectively, at the end of T years; and $f(0)$ is the market value of assets at the start of the projections, and is used as a scaling factor.

Excess Contribution rate risk

The Excess Contribution rate risk, $ECR_T(\pi, NC)$, over T years is calculated as follows:

$$
ECR_T(\pi, NC) = \frac{1}{a(T)} \sum_{t=0}^{T} v^t\, E_0\Big[\max\Big(c(t) - NC, 0\Big)\Big]
\tag{6.2}
$$

where T is the time at the end of the period, and $a(T)$ represents the present value of a T year annuity-due calculated at a real rate of interest. We take this real rate of interest to be the difference between the long-term mean for the yield on undated fixed-interest gilts and the long-term mean for the force of earnings inflation.

Treatment of Gains and Losses

As noted above, we set the intervaluation period to be 3 years. We further set the spread period to be 3 years. The effect of other choices of these two parameters will be investigated later in the chapter.

Thus we amortize all surpluses and deficits over 3 years at the end of each 3-year period. That is, having set the Normal contribution rate at the start of the stochastic projections we adjust it every three years depending on whether surpluses or deficits have arisen during the intervaluation period. So a normal contribution rate of 0 does not imply a recommended contribution rate of 0. However, over the first period, for an initially fully-funded scheme the recommended contribution rate is just the normal contribution rate. Hence, even though the Mean Shortfall Risks can be calculated at the end of every period, we can only calculate the Excess Contribution rate risks at the end of the second, third, fourth and fifth periods.

The results in this chapter depend on the assumptions of our modelling framework. As a consequence, extensive sensitivity analysis is undertaken in the appendices.

This chapter is organized as follows: in Section 6.2 we consider the concept of indifference curves for the Mean Shortfall risk and the Excess Contribution rate risk. In Section 6.3 we consider the concept of an efficient region as a way of reconciling the optimal asset allocations under the two risk measures. Then in Section 6.4 we investigate the Average Contribution rate as a measure of the cost for the sponsoring employer. We carry out an analysis of different amortization periods on the optimal policies in Section 6.5. Whilst in Section 6.6 we compare the results for static and dynamic investment strategies. The effect of the initial funding level is investigated in Section 6.7. And in Section 6.8 we reconsider the problem of seeking optimal asset allocation strategies as a problem of seeking Pareto optimal policies in a multi-objective setting. We end in Section 6.9 with some concluding remarks.

6.2 The concept of indifference curves

In this section we consider the concept of indifference curves. We analyze the Mean Shortfall and Excess Contribution rate risks separately and construct indifference curves showing all choices of asset allocation and Normal contribution rate that lead to similar levels of risk. For illustrative purposes we will only consider the situation at the end of the second and fifth intervaluation periods.

6.2.1 Mean Shortfall Risk

Figure 6.1 shows the Mean Shortfall risk levels at the end of the second period (i.e. at the end of 6 years). We have shown the Mean Shortfall risk levels at the end of 6 years instead of at the end of 3 years in order to match the Excess contribution rate risk levels which can only be calculated from the end of 6 years (i.e. end of the second period). In this case the scheme is initially fully-funded and we employ a static asset allocation strategy (with annual rebalancing).

We illustrate our results by considering 4 curves although it is possible to draw a curve passing through any given point on the graph. Each curve shows all the combinations of initial allocation in equities and normal contribution rate that lead to a given Mean Shortfall Risk. For instance, all combinations along the top curve lead to a Mean Shortfall Risk of 0.05[1].

[1]It is not possible to consider all combinations of initial allocation in equities and normal contribution rate in the computer simulations. Hence for a given Mean Shortfall risk interpolations are carried out, where necessary, to calculate the combinations along the curves. To reduce the interpolation error simulations are carried out for 21 initial allocations in equities ie 0%, 5%, 10%...100%; and 17 normal contribution rates ie 0, 0.02, 0.04...0.32. So for a given normal contribution rate, 21 initial allocations in equities are considered.

As a further illustration, we consider six points on the graph: A, B, C, D, E, and F. The points A, B and C stand for asset allocation strategies with a high bond proportion whilst points D, E and F stand for asset allocation strategies with a high equity proportion.

Furthermore, points A and B have a similar Normal contribution rate; points B and C have a similar asset allocation strategy; points C, D and E have a similar Normal contribution rate; whilst points D and F have a similar asset allocation strategy.

Additionally, although points B and D have different asset allocations and Normal contribution rates, they lie on the same curve and thus lead to the same level of risk of 0.05. Similarly, points C and F both lie on the bottom curve and lead to the level of risk of 0.065.

Each point on the graph can thus be identified by 3 coordinates (x,y,z), where x is the initial allocation in equities, y is the normal contribution rate, and z is the Mean Shortfall Risk.

For example, point A has coordinates (7.6%, 0.196,0.06) and point D has coordinates (81%,0.146,0.05). This means that if our normal contribution rate is 0.196 and we initially invest 7.6% in Equities then we would expect a Mean Shortfall risk of 0.06 at the end of the second period. On the other hand if our normal contribution rate is 0.146 and we initially invest 75% in Equities then we would expect a Mean Shortfall risk of 0.05 at the end of the second period.

This analysis leads to the concept of indifference curves as explained in Chapter 5. The decision maker would be expected to be indifferent between a combination of 17% initial allocation in equities and normal contribution rate of 0.196 (i.e. point B) and a combination of 75% initial allocation in equities and normal contribution rate of 0.146 (i.e. point D). This is due to the fact that both combinations lead to the same level of Mean Shortfall risk.

We observe from Figure 6.1 that the two most important concepts of indifference curves hold in our case. Firstly, combinations along the lower curves lead to higher Mean Shortfall Risks, and vice versa. For example, point B and point D along the top curve lead to a lower Mean Shortfall Risk of 0.05, while point C and point F along the lowest curve lead to a higher Mean Shortfall Risk of 0.065. Thus a rational decision maker will, all things being equal, try to reach the highest possible indifference curve (Sloman (1999), page 117).

Secondly, it is impossible for two indifference curves to intersect, all things being equal (Sloman (1999), page 117). In our case it is impossible for a given combination

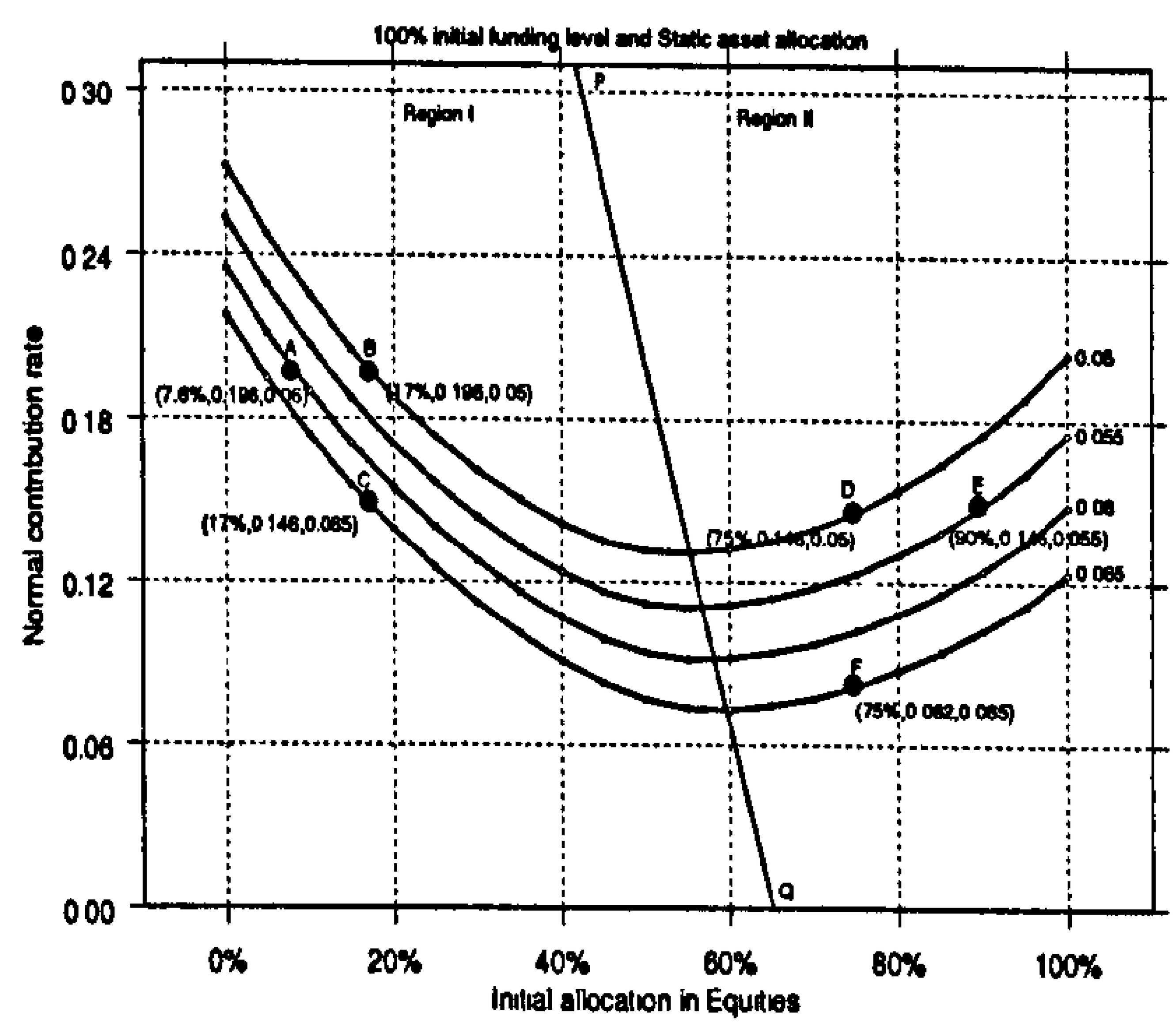

Figure 6.1: Mean Shortfall risk levels at the end of 6 years

of initial allocation in equities and normal contribution rate to lead to two different Mean Shortfall Risks.

As shown in Chapter 5, to find the optimal asset allocations we need the minimum points of the indifference curves. Hence, in Figure 6.1 we draw the line PQ through all the minimum points of the indifference curves. It is impossible to draw a line which passes 'exactly' through the minimum points unless if one solves the impossible problem of considering every contribution rate and every asset allocation. Hence, PQ is a best-fit line which passes either through or very close to the minimum points of the indifference curves. Thus PQ is a good approximation to the 'curve' which passes through all the minimum points. Hence we obtain two regions, region I and region II. Such a division is very important for decision making, as we will show below.

Decision-making under Mean Shortfall risk

Points A and B in Figure 6.1 have the same normal contribution rate of 0.196 but different initial allocation in equities of 7.6% and 17%, respectively. Furthermore, point A leads to a higher Mean Shortfall risk than point B i.e. 0.06 and 0.05, respectively.

This implies that a decision maker can improve their position by choosing point B (i.e. increasing initial allocation in equities) rather than point A. This also holds in general: that is, for positions in region I, for a given normal contribution rate a decision maker can improve their position by increasing their initial allocation in equities (i.e. by shifting horizontally from left to right). This means that for a given normal contribution rate the decision maker should choose points that are nearer to (or on) the line PQ than, for example, point A.

We also note that points D and E have the same normal contribution rate of 0.146 but different initial allocations in equities of 75% and 90%, respectively. A decision maker can improve their position by choosing point D (decrease initial allocation in equities) rather than point E. This is due to the fact that D leads to a lower Mean Shortfall risk than E.

Hence, for positions in region II, for a given normal contribution rate a decision maker can improve their position by decreasing their initial allocation in equities (i.e. by shifting horizontally from right to left). This means that for a given normal contribution rate the decision maker should choose points that are nearer to (or on) the line PQ than, for example, point E.

Other possible movements, like C to B and F to D, lead to lower Mean Shortfall Risk positions (i.e. Mean Shortfall Risk decreases from 0.065 to 0.05) but the normal contribution rate increases from 0.146 to 0.196 and 0.082 to 0.146, respectively. Whilst movements like A to C and E to F lead to higher Mean Shortfall risk positions (i.e. 0.06 to 0.065 and 0.055 to 0.065, respectively) but the normal contribution rate decreases from 0.196 to 0.146 and 0.146 to 0.082, respectively. Such movements can only be analysed by considering their effect on the Excess Contribution Rate Risk. This will be covered in a later section.

6.2.2 Excess Contribution Rate Risk

Having considered the Mean Shortfall risk we now turn to the Excess Contribution rate risk. We will show that the general ideas of indifference curves discussed under the Mean Shortfall risk also hold under the Excess Contribution rate risk.

Figure 6.2 below shows the Excess Contribution Rate Risk levels at the end of 6 years. Each curve shows all the <u>combinations</u> of initial allocation in equities and normal contribution rate that lead to a given Excess Contribution Rate Risk. For instance, all combinations along the top curve lead to an Excess Contribution Rate Risk of 0.035.

Each point on the graph can be identified by 3 coordinates (x,y,z), where x is the initial allocation in equities, y is the normal contribution rate, and z is the Excess Contribution Rate Risk. For example, point A has coordinates (8%, 0.157,0.04) and point F has coordinates (80%, 0.15,0.045). This means that if our normal contribution rate is 0.157 and we initially invest 8% in Equities then we would expect an Excess Contribution Rate risk of 0.04 at the end of the second period. On the other hand if our normal contribution rate is 0.15 and we initially invest 80%

in Equities then we would expect an Excess Contribution Rate Risk of 0.045 at the end of the second period.

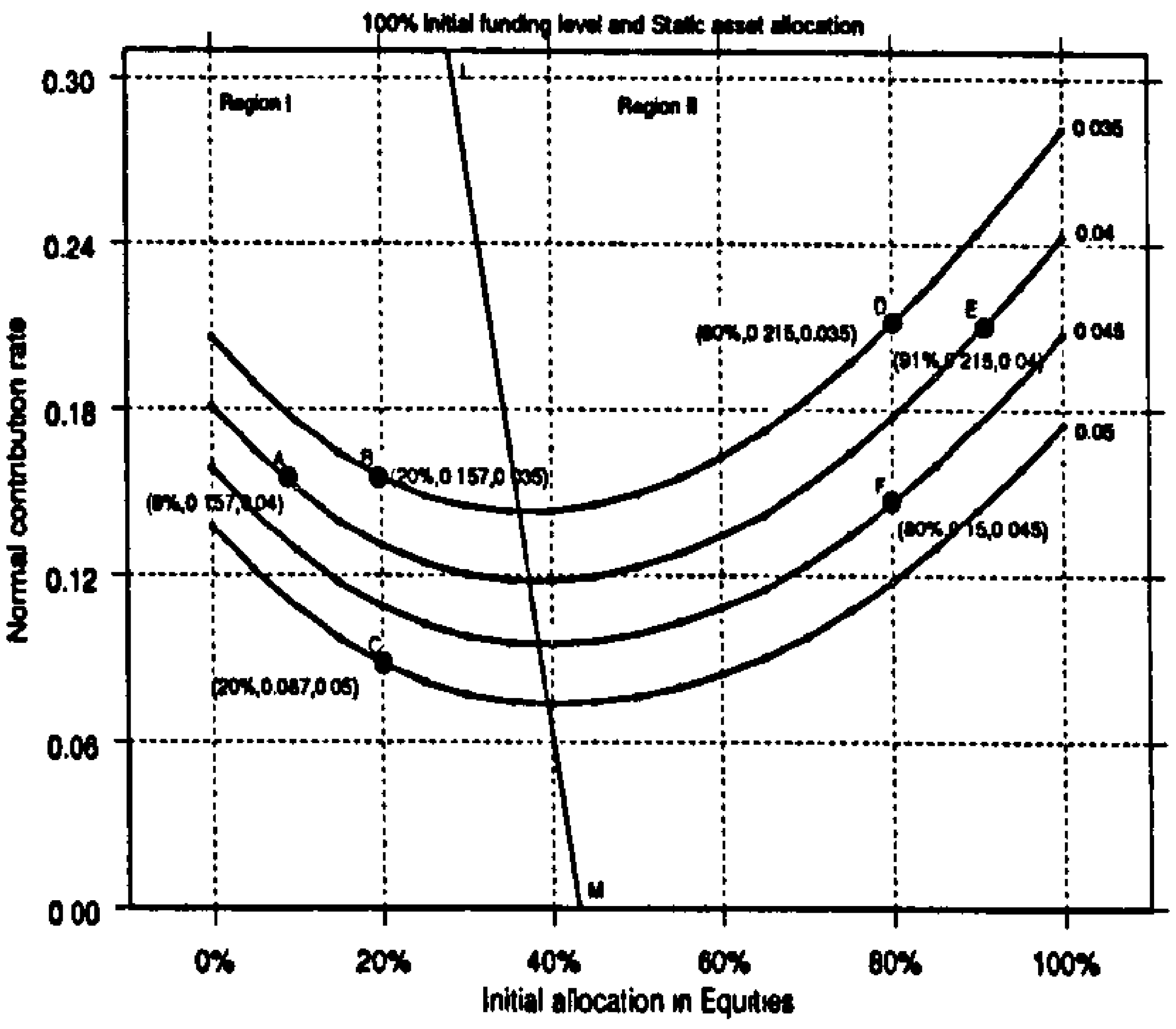

Figure 6.2: Excess contribution rate risk levels at the end of 6 years

The indifference curve concepts, discussed in Section 6.2.1, also hold in this case. For example, point B and point D along the top curve lead to a lower Excess Contribution Rate Risk of 0.035 whilst point C along the lowest curve leads to a higher Excess Contribution Rate Risk of 0.05. Furthermore, it is impossible for two indifference curves to intersect since a given combination of initial allocation in equities and normal contribution rate can only lead to one value of Excess contribution rate risk.

Decision-making under the Excess Contribution rate risk

As in the Mean Shortfall case, let LM be the best-fit line through all the minimum points of the indifference curves in Figure 6.2. Then, as we observed in the Mean Shortfall case, for positions in region I, for a given normal contribution rate the decision maker can improve their position by increasing the allocation in equities i.e. by moving from left to right towards (or onto) the line LM. While for positions in region II, for a given normal contribution rate the decision maker can improve their position by decreasing their allocation in equities i.e. by moving from right to left towards (or onto) the line LM.

Other possible movements, like C to B and F to D, lead to lower Excess Contribution Rate Risk positions (i.e. 0.05 to 0.035 and 0.045 to 0.035, respectively) but the normal contribution rate increases from 0.087 to 0.157 and 0.15 to 0.215, respectively. Whilst movements like A to C and E to F lead to higher Excess Contribution Rate Risk positions (i.e. 0.04 to 0.05 and 0.04 to 0.045, respectively) but the normal contribution rate decreases from 0.157 to 0.087 and 0.215 to 0.15, respectively. Such movements can only be analysed by considering their effect on the Mean Shortfall Risk. This will be covered in a later section.

6.2.3 Comparisons at the end of the Second Period

In this section we compare the asset allocations under the two risk measures by considering the situation at the end of the second intervaluation period. Figure 6.3 shows a comparison of Mean Shortfall and Excess Contribution Rate Risk levels at the end of 6 years. The minimum points for the curves for Mean Shortfall risk are in the region of 60% initial allocation in equities. On the other hand, the minimum points for the curves for Excess Contribution Rate Risk are centred around 40% initial allocation in equities.

This means that for a given normal contribution rate, we would initially invest more in equities if our decision were based only on the Mean Shortfall risk; and we would initially invest less in equities if our decision were based only on the Excess Contribution Rate Risk. This conflict has to be reconciled and in a subsequent section we will show how this can be done.

6.2.4 Comparisons at the end of the Fifth Period

In this section we conduct similar comparisons as in the previous section of the Mean Shortfall risk and the Excess Contribution rate risk results but for projections over 15 years.

Figure 6.3 also shows a comparison of Mean Shortfall and Excess Contribution Rate Risk levels at the end of 15 years. The minimum points for the curves for Mean Shortfall risk are in the region of 80% initial allocation in equities. On the other hand, the minimum points for the curves for Excess Contribution Rate Risk are centred around 55% initial allocation in equities.

This implies that, as we observed in Section 6.2.3, there is a conflict between decisions based only on Mean Shortfall Risk and decisions based only on Excess Contribution Rate Risk. Additionally, we observe that the minimum points are at a higher initial allocation in equities for the 15-year projections than for the 6-year projections (i.e. 80% versus 60% and 55% versus 40%). This means that if we were seeking to minimize risk we would initially invest more in equities if our projection period were 15 years than if our projection period were 6 years.

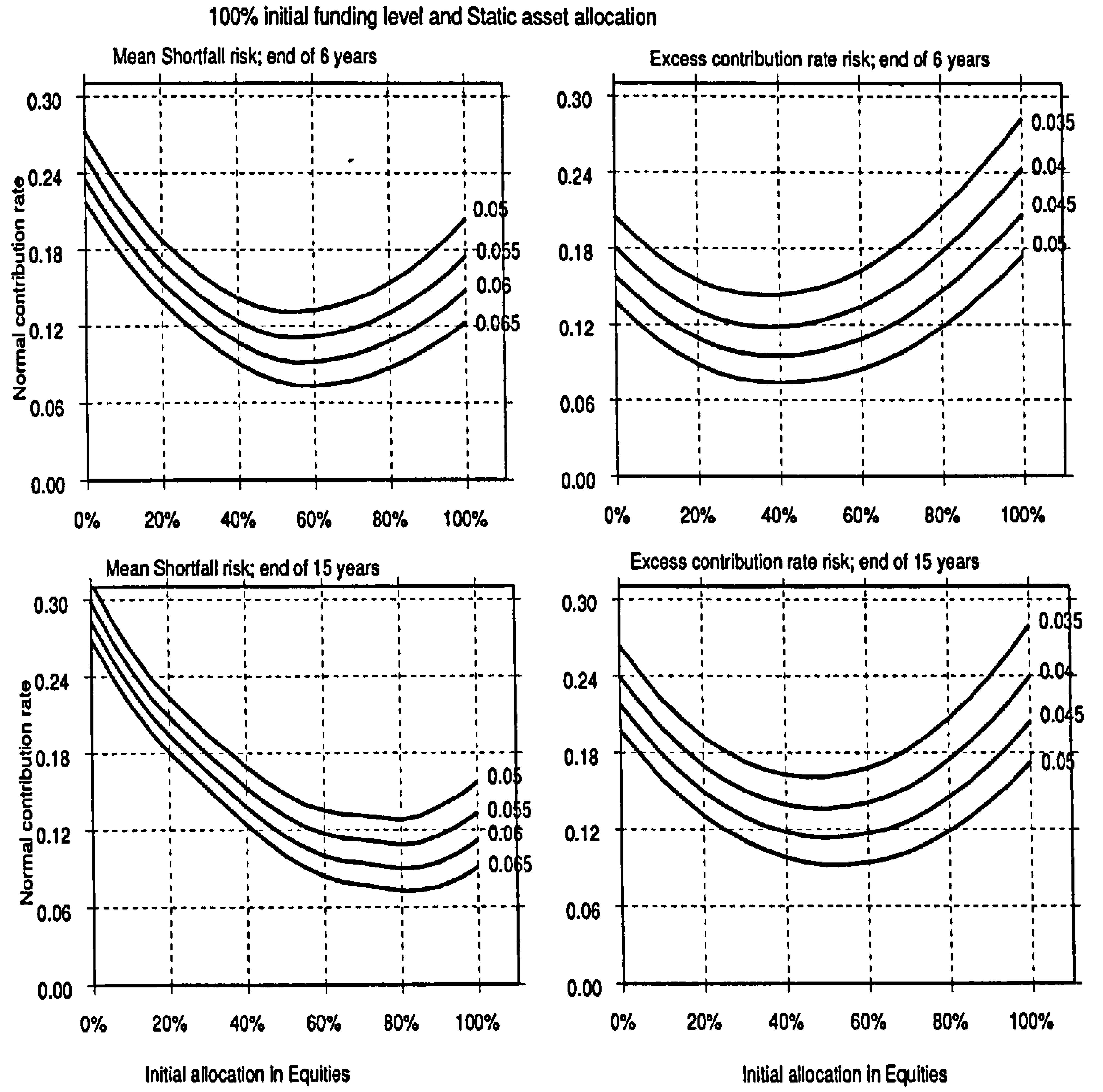

Figure 6.3: Comparison of risk levels at the end of 6 and 15 years

6.3 Decision making

In this section we introduce the idea of an Efficient Region as a way to reconcile the conflicts in the asset allocations under the Mean Shortfall risk and the Excess Contribution rate risk.

6.3.1 Reconciling the risks

In Section 6.2.1 we showed how the decision maker could choose a combination of initial asset allocation and normal contribution rate by considering only the Mean Shortfall Risk. And in Section 6.2.2 we showed how the decision maker could choose a combination of initial asset allocation and normal contribution rate by considering only the Excess Contribution Rate Risk. However, in Sections 6.2.3 and 6.2.4, we observed that there is a conflict between decisions based only on Mean Shortfall Risk and decisions based only on Excess Contribution Rate Risk.

We will now show how the two risks can be reconciled in the decision making process so that the decision maker can achieve an efficient combination of initial asset allocation and normal contribution rate.

In Figures 6.4 and 6.5 below we plot the Mean Shortfall and Excess Contribution Rate Risk levels at the end of 15 years on the same graph. Corresponding figures would emerge for comparisons at other time horizons. Thus we use the 15-year horizon for illustrative purposes only.
The lines LM and PQ are the best-fit lines through the minimum points for the Excess Contribution Rate and the Mean Shortfall Risk levels, respectively. LM and PQ are not necessarily parallel. These lines split each of the graphs into three regions. Region I contains all points which are to the left of the minimum points of both the Mean Shortfall and Excess Contribution Rate Risk levels. Points in Region II are to the right of Excess contribution rate risk minimum points and to the left of Mean Shortfall risk minimum points. Meanwhile points in Region III are to the right of both sets of minimum points.

In Figure 6.4, we identify points as (x,y) where x is the initial allocation in equities and y is the normal contribution rate. Thus for a given point the values x and y can be calculated from the x- and y-axes. In Figure 6.5, we identify points as (z,w) where z is the Mean Shortfall Risk and w is the Excess Contribution Rate Risk. Figures 6.4 and 6.5 must be used concurrently. Obviously, we could have shown all the information on one graph instead of two. However, we use two graphs in order to show, separately, the choices that are available and the consequences of such choices. Figure 6.5 shows the consequences of the choices made in Figure 6.4.

We consider some examples,

Example 1: for a normal contribution rate of 0.15 and initial allocation in equities of 7% we would expect a Mean Shortfall risk of 0.10 and Excess contribution risk of 0.055 after 15 years (point A in Figures 6.4 and 6.5);

Example 2: for a normal contribution rate of 0.215 and initial allocation in equities of 68% we would expect a Mean Shortfall risk of 0.029 and Excess contribution risk of 0.0315 after 15 years (G in Figures 6.4 and 6.5);

Example 3: for a normal contribution rate of 0.021 and initial allocation in equities of 60.5% we would expect a Mean Shortfall risk of 0.088 and Excess contribution risk of 0.069 after 15 years (E in Figures 6.4 and 6.5); and

Example 4: for a normal contribution rate of 0.15 and initial allocation in equities of 98% we would expect a Mean Shortfall risk of 0.051 and Excess contribution risk of 0.053 after 15 years (K in Figures 6.4 and 6.5).

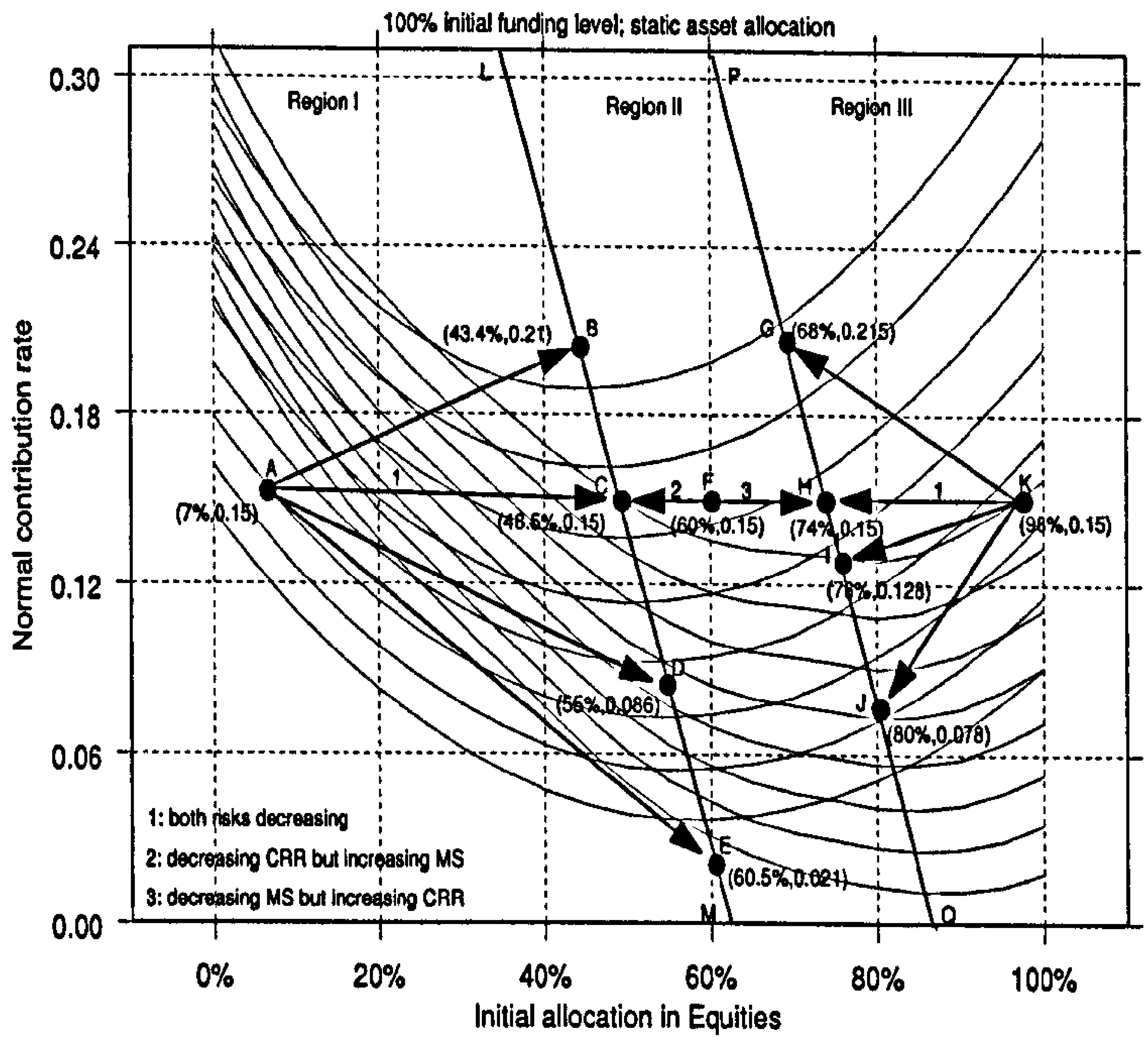

Figure 6.4: Mean Shortfall (MS) and Excess contribution rate (CRR) risk levels at the end of 15 years

6.3.2 Choosing an Efficient Combination

We now demonstrate how an efficient combination of initial asset allocation and normal contribution rate can be chosen. We do this by considering the three regions presented in Figure 6.5.

Region I

Assume, without loss of generality, that a decision maker chooses point A. Such a decision maker can change their position in several ways.

Firstly, by moving to higher points e.g. point B. Since we are dealing with indifference curves, this move would lead to a point which has lower Mean Shortfall risk and lower Excess contribution rate risk. Our example corresponds to arrow AB in Figure 6.5. In this case Mean Shortfall risk decreases from 0.10 to 0.035 and Excess contribution rate risk decreases from 0.055 to 0.0275. Moving from A to B can be effected by increasing the normal contribution rate from 0.15 to 0.21 and increasing the initial allocation in equities from 7% to 43.4%.

Secondly, by moving to points which lie horizontally to the right e.g. point C. Since we are moving horizontally towards the minimum points of both sets of indifference curves, both risks, Mean Shortfall Risk and Excess Contribution Rate Risk,

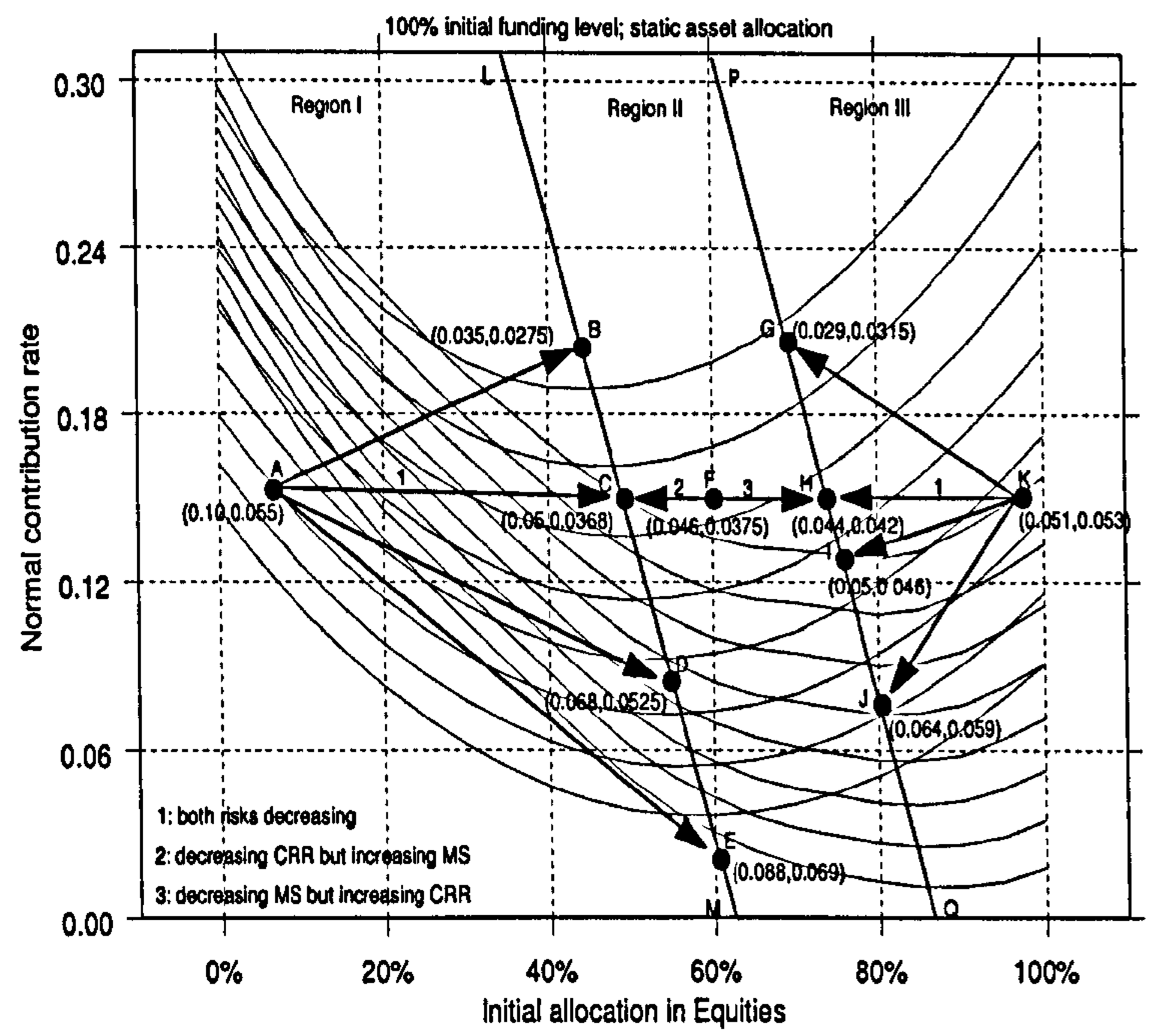

Figure 6.5: Mean Shortfall (MS) and Excess contribution rate (CRR) risk levels at the end of 15 years

will decrease. Our example corresponds to arrow AC in Figure 6.5. Mean Shortfall Risk decreases from 0.10 to 0.05 and Excess Contribution Rate Risk decreases from 0.055 to 0.0368. We can accomplish this by leaving the normal contribution rate unchanged and increasing the initial allocation in equities from 7% to 48.5%.

Thirdly, by moving to lower points. For example, moving from A to D or moving from A to E. We analyse these cases separately:

- moving from A to D: Mean Shortfall Risk decreases from 0.10 to 0.068 and Excess Contribution Rate Risk decreases from 0.055 to 0.0525. We can accomplish this by reducing the normal contribution rate from 0.15 to 0.086 and increasing the initial allocation in equities from 7% to 55%.

- moving from A to E: Mean Shortfall Risk decreases from 0.10 to 0.088 *but* Excess Contribution Rate Risk increases from 0.055 to 0.069. We can accomplish this by reducing the normal contribution rate from 0.15 to 0.021 and increasing the initial allocation in equities from 7% to 60.5%.

The examples above show that positions in region I can be improved by moving *towards* region II. However, as highlighted in the last example, we cannot just move arbitrarily to any point lower than A, for instance, without increasing at least one of the risks.

Region III

The scenario in region III is similar to that in region I. The main difference is that in region III all the movements involve reducing the initial allocation in equities:

moving from K to G is similar to moving from A to B;

moving from K to H is similar to moving from A to C; and

moving from K to I is similar to moving from A to D.

Moving from K to J is also similar to moving from A to E, but, in this case, *both* risks increase. The Mean Shortfall Risk increases from 0.051 to 0.064 and the Excess Contribution Rate Risk increases from 0.053 to 0.059.

Region II

We have observed that positions in regions I and III can be improved by moving *towards* region II. Now we will answer the crucial question: what happens if a decision maker chooses a position in region II?

Assume, without loss of generality, that the decision maker chooses point F i.e. a normal contribution rate of 0.15 and 60% initial allocation in equities. Then we would expect, after 15 years, a Mean Shortfall risk of 0.046 and Excess contribution rate risk of 0.0375. Can this position be improved?

Firstly, we can not move to lower points (e.g. D, E, I and J) without increasing *both* risks; secondly, moving to higher points (e.g. B and G) reduces both risks but involves increasing the normal contribution rate; and thirdly, moving horizontally, for example to C or H, maintains the normal contribution rate at 0.15. However, one of the risks decreases whilst the other risk increases:

- by moving from F to C the Mean Shortfall Risk increases from 0.046 to 0.05 and the Excess Contribution Rate Risk decreases from 0.0375 to 0.0368; and
- by moving from F to H the Mean Shortfall Risk decreases from 0.046 to 0.044 and the Excess Contribution Rate Risk increases from 0.0375 to 0.042.

It is important to notice that we can not only move from regions I and III *towards* region II but we can actually move *into* region II. For example,

Example 1: moving from A to F. In this case *both* risks decrease: the Mean Shortfall Risk decreases from 0.10 to 0.046 and the Excess Contribution Rate Risk decreases from 0.055 to 0.0375. (this move corresponds to increasing initial allocation in equities from 7% to 60%);

Example 2: moving from K to F. Again *both* risks decrease: the Mean Shortfall Risk decreases from 0.051 to 0.046 and the Excess Contribution Rate Risk decreases from 0.053 to 0.0375. (this move corresponds to decreasing initial allocation in equities from 98% to 60%).

6.3.3 Conclusions

This analysis leads us to the following conclusions. Firstly, positions in region I and region III are inefficient. This is because, for a given normal contribution rate, *both* the Mean Shortfall Risk and the Excess Contribution Rate Risk can be *reduced* by moving to a position in region II. And secondly, positions in region II cannot be improved without either increasing one of the risks or increasing the normal contribution rate.

6.4 Cost measure: Average Contribution Rate

In this section we show how the average contribution rate can be used as a performance measure in the context of indifference curves. As in the case of the Mean Shortfall risk and the Excess Contribution rate risk, the goal will be to find the asset allocation strategies which minimize the average contribution rate over a given projection horizon and for a given Normal contribution rate.

6.4.1 Average Cost

The sponsor will be interested in the average cost over the funding period. The average cost for a given asset allocation strategy and Normal contribution rate can be measured by considering the average contribution rate over a given projection horizon. The underlying idea is that strategies which lead to lower average contribution rates are less costly and hence optimal.

We calculate the average contribution rate over a given horizon as:

$$AC_T(\pi, NC) = \frac{1}{T} \sum_{t=0}^{T-1} E_0\Big[c(t)\Big] \tag{6.3}$$

where $c(t)$ is the contribution rate in the year $(t, t+1)$ payable at time t.

We only consider the results of projections over 6-year and 15-year horizons in order to compare with the results obtained for the Mean Shortfall and Excess Contribution rate risks. Figure 6.6 shows the average contribution rate levels at the end of 6 years and 15 years.

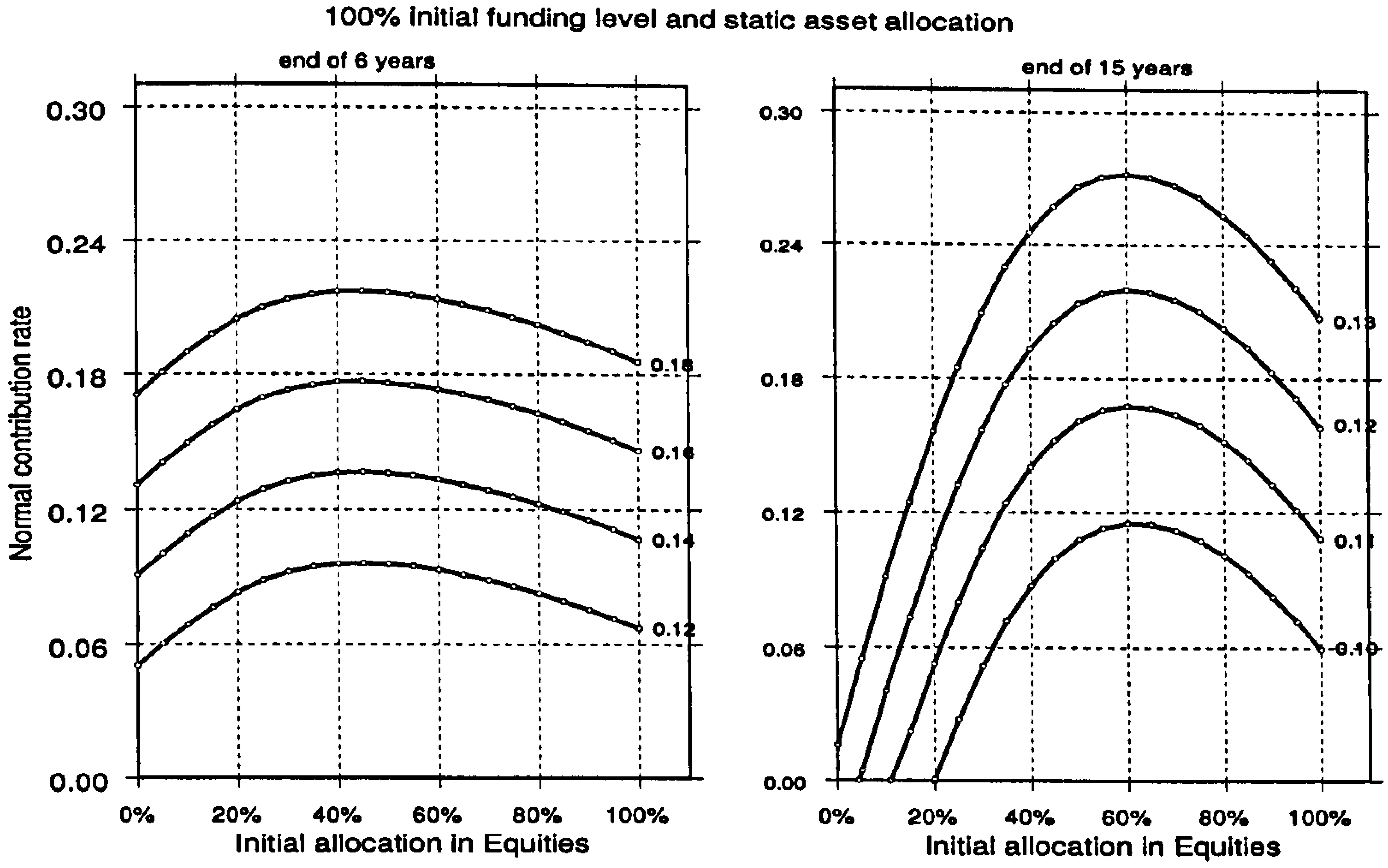

Figure 6.6: Average contribution rate levels at the end of 6 and 15 years

Each curve in Figure 6.6 shows all the combinations of initial allocation in equities and normal contribution rate that lead to a given average contribution rate.

We observe that combinations along higher curves lead to higher average contribution rates. For example, at the end of 6 years, combinations along the top curve lead to an average contribution rate of 0.18 whilst combinations along the lowest curve lead to a lower average contribution rate of 0.12. And at the end of 15 years, combinations along the top curve lead to an average contribution rate of 0.13 whilst combinations along the lowest curve lead to a lower average contribution rate of 0.10.

The curves in Figure 6.6 are indifference curves since the decision maker would be indifferent between all combinations along a given curve. These curves can be used as a measure of the cost. In our analysis of Mean Shortfall and Excess Contribution Rate Risks, we have concluded that a decision maker could improve their position by choosing combinations in the efficient region. For every choice of normal contribution rate and initial asset allocation, the decision maker could use Figure 6.6 to determine the average contribution rate over the projection period.

It is interesting to observe in Figure 6.6 that for a given normal contribution rate, if we move horizontally towards the *maximum* point we *reduce* the average contribution rate. The shape of the curves in Figure 6.6 is due to the zero lower bound on the recommended contribution rate. We would expect the shape of these curves to change if this lower bound was omitted.

6.4.2 Average contribution rate curves and the efficient region

In the reconciliation of the optimal asset allocations under the Mean Shortfall risk and the Excess Contribution rate risk in Section 6.3 we showed that asset allocations that are optimal under both risks are located *within* an efficient region. The reason for this was that a decision maker could improve his/her position by moving towards or *into* the efficient region.

We have further shown in Section 6.4.1 that the maximum points for the average contribution rate curves lead to the lowest average contribution rate for a given Normal contribution rate. That is, a decision maker could reduce the average contribution rate, for a given Normal contribution rate, by moving horizontally towards the maximum point.

In this section we investigate whether, for a given Normal contribution rate, the asset allocations that lead to the lowest average contribution rates coincide with the asset allocations in the efficient region. To this end, we depict the efficient region and the average contribution rate curves on the same axes. Figure 6.7 shows the average contribution rate levels and the efficient region at the end of 15 years.

We observe that the maximum points for the average contribution rate curves lie in the efficient region LMPQ. Therefore, Figure 6.7 implies that, by moving towards the efficient region, the decision maker will be *reducing* the Mean Shortfall Risk, the Excess Contribution Rate Risk and the average contribution rate.

Thus the asset allocations located in the efficient region are optimal not just for the two risk measures but also for the cost measure.

6.5 The Effect of Amortization Periods on the Indifference Curves

In this section we investigate the effect of amortization periods on the Mean Shortfall and Excess contribution rate risk levels. We consider the spread method and investigate two examples: spreading of surpluses and deficits over 3 years and over 12 years. In the previous sections we have assumed that surpluses and deficits are spread over 3 years.

We only examine the indifference curves at the end of the second and fifth projection

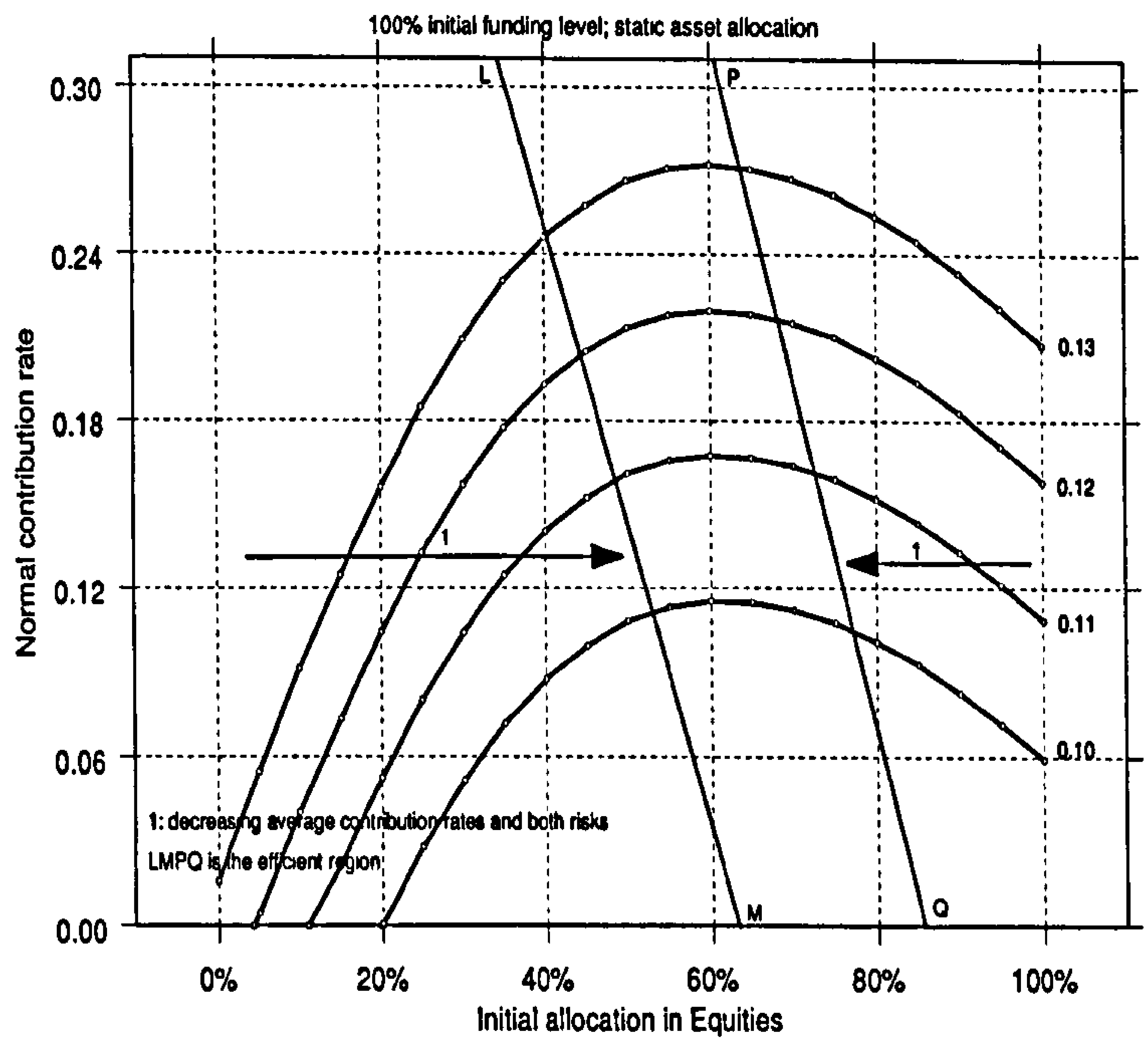

Figure 6.7: Average contribution rate levels and the efficient region at the end of 15 years

periods. The curves at the end of the first projection period are equivalent for the two cases of a 3 year and 12 year spread period since the pension scheme is initially fully-funded.

6.5.1 Mean Shortfall Risk

Figure 6.8 shows the Mean Shortfall Risk indifference curves at the end of 6 and 15 years.

End of 6 years

We observe that, firstly, for a given combination of normal contribution rate and initial allocation in equities, the Mean Shortfall Risk is lower for amortization over 3 years than for amortization over 12 years (except, perhaps, for very low values of initial allocation in equities). And secondly, the minimum points of the indifference curves are in the region of 60% initial allocation in equities in both cases.

End of 15 years

We observe that, firstly, for a given combination of normal contribution rate and initial allocation in equities, the Mean Shortfall risk is lower for amortization over 3 years than for amortization over 12 years, especially for initial allocation in equities higher than 20%. And secondly, the minimum points are in the region of 80% initial allocation in equities for amortization over 3 years and 60% for amortization 12 years.

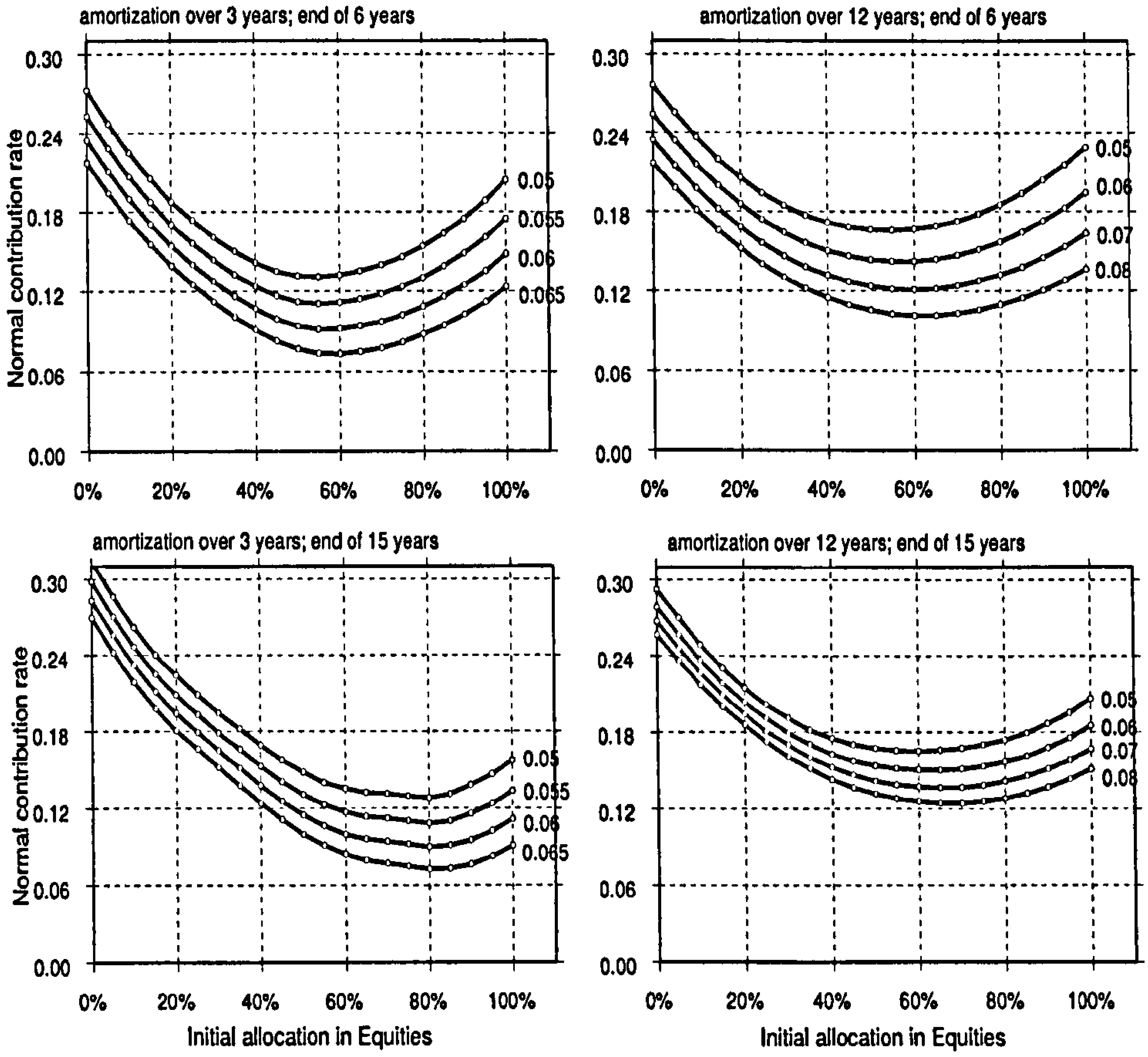

Figure 6.8: Comparison of Mean Shortfall risk levels at the end of 6 and 15 years (100% initial funding level; static asset allocation)

6.5.2 Excess Contribution Rate Risk

Figure 6.9 shows the Excess contribution rate risk indifference curves at the end of 6 and 15 years.

End of 6 years

We observe that, firstly, for a given combination of normal contribution rate and initial allocation in equities, amortization over 3 years leads to higher Excess contribution rate risk than over 12 years. And, secondly, the minimum points are in the region of 40% initial allocation in equities in both cases.

End of 15 years

We observe that, firstly, for a given combination of normal contribution rate and initial allocation in equities, amortization over 3 years leads to higher Excess contribution rate risk than over 12 years. And, secondly, the minimum points are in the region of 50% to 60% initial allocation in equities in both cases.

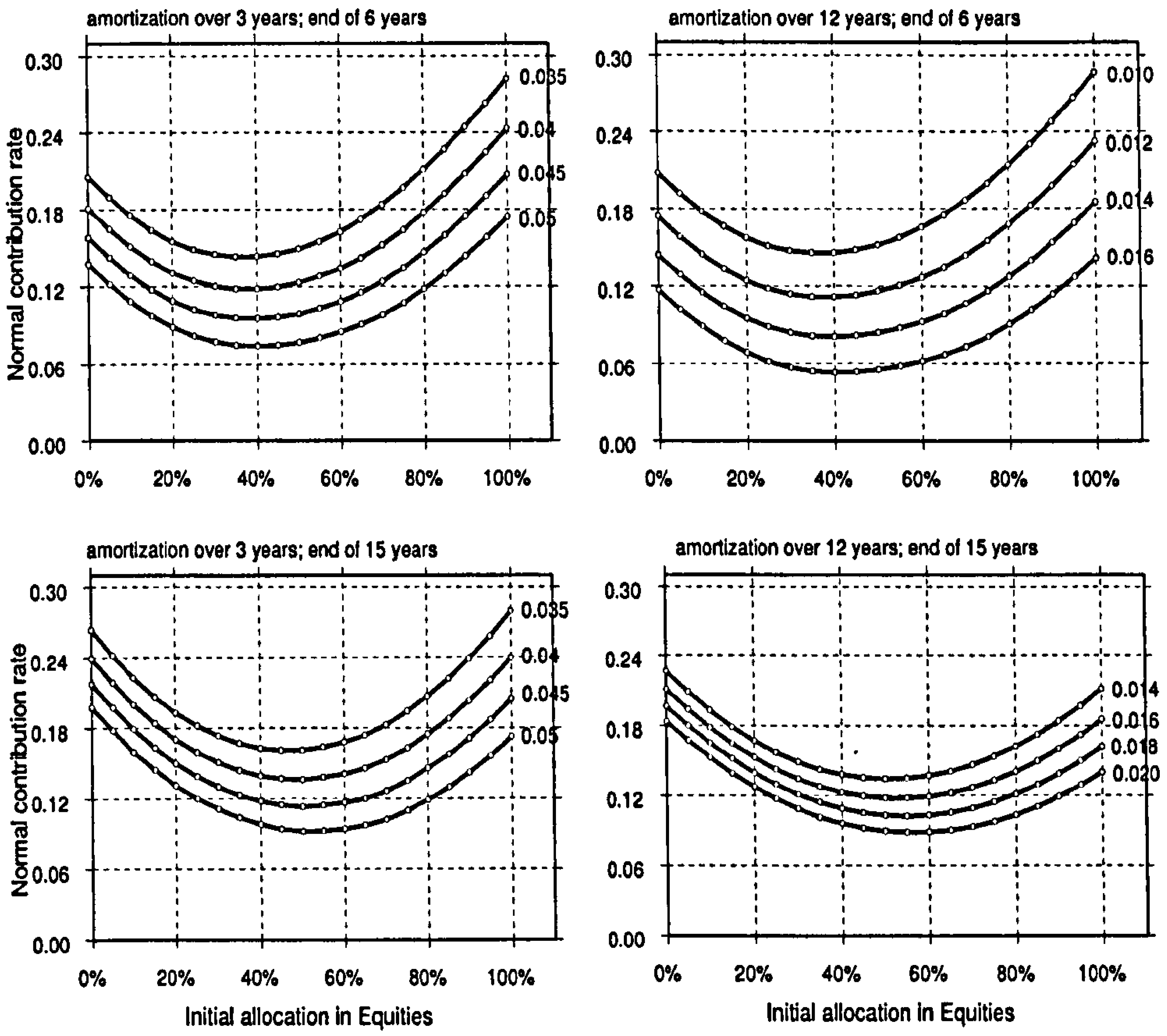

Figure 6.9: Comparison of Excess contribution rate risk levels at the end of 6 and 15 years (100% initial funding level; static asset allocation)

6.5.3 How do the Amortization Periods Compare?

These results are intuitive: if deficits and surpluses are amortized over 3 years, the scheme would be expected to attain full-funding more quickly than if the amortization was over 12 years. This explains the better Mean Shortfall risk position for amortization over 3 years than for the 12 years case.

However, the 'penalty' for this is the poor Excess Contribution Rate Risk positions for amortization over 3 years: we would expect higher recommended contribution rates for amortization over 3 years than over 12 years, all things being equal.

6.5.4 Efficient Regions

Figure 6.10 below shows the efficient regions for projections over 15 years for amortization over 3 and 12 years. ABCD is the efficient region for amortization over 12 years whilst LMPQ is the efficient region for amortization over 3 years.

The efficient region is much more tightly defined when amortization takes place over 12 years rather than over 3 years. Furthermore, the righthand boundary shifts from PQ to CD implying an increase in the bond proportion in the optimal asset allocations as the spread period increases.

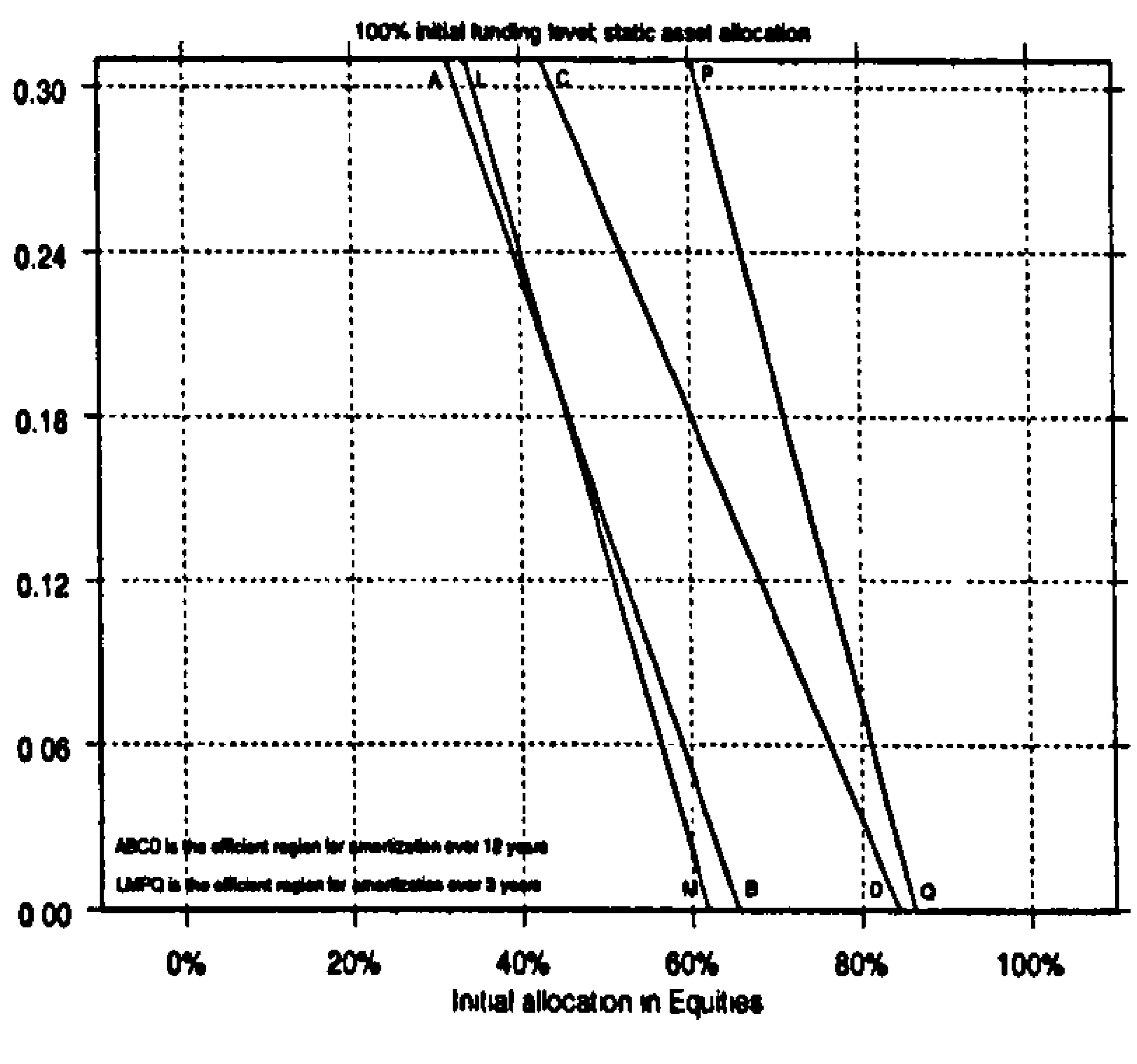

Figure 6.10: The efficient regions for 15-year projections for amortization over 3 and 12 years

6.6 Effect of Investment Strategies on the Indifference Curves

In this section we investigate the effect of investment strategies on the indifference curves. We consider a static strategy and two dynamic strategies:

Static asset allocation: this is the case we have been considering in the previous sections. The asset allocations are fixed and annually rebalanced;

Dynamic asset allocation (DS1): in this case the asset allocations are changed at the end of every 3 years as follows: for each 10% increase in the funding level (relative to the initial value), the allocation in equities is decreased by 5%, and vice versa; and

Dynamic asset allocation (DS2): in this case, for each 10% increase in the funding level (relative to the initial value), the allocation in equities is increased by 5%, and vice versa.

We note that DS1 is a counter-intuitive strategy where funds are switched into equities as the funding level becomes more adverse and into bonds as the funding level becomes more favourable. Although the choice of values for the percentage changes (that is, 10% and 5%) is arbitrary, the set up of these dynamic strategies will allow us to compare with the results for the dynamic programming problem in Chapter 4 and elsewhere in the actuarial literature.

6.6.1 Mean Shortfall Risk

Figure 6.11 shows the Mean Shortfall risk indifference curves at the end of 6 and 15 years[2]. We observe that for a given combination of normal contribution rate and initial allocation in equities, the Mean Shortfall risk is lowest under DS1 and highest under DS2 (except for very low values of initial allocation in equities).

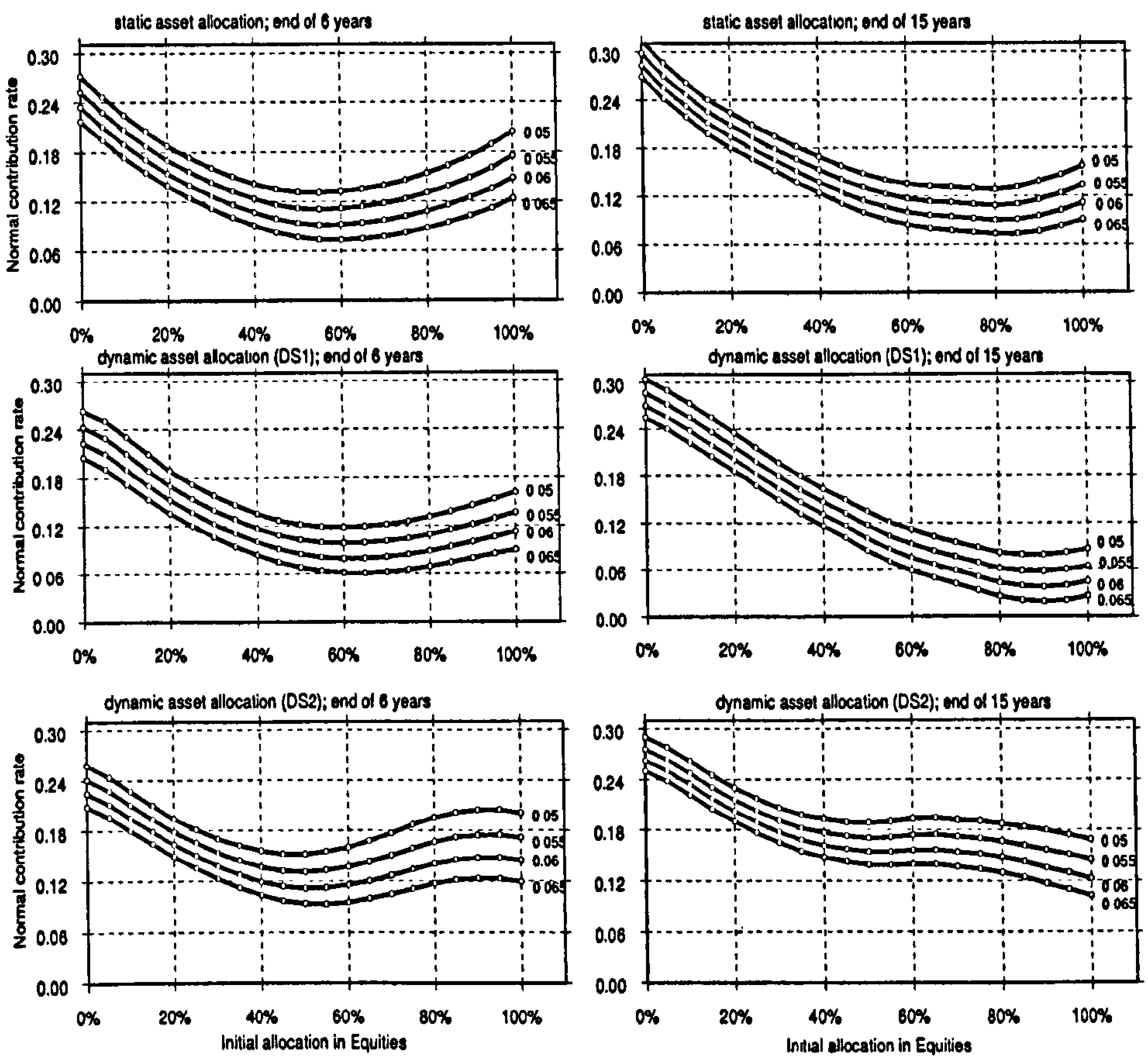

Figure 6.11: Comparison of Mean Shortfall risk levels at the end of 6 and 15 years (100% initial funding level)

[2]At the end of 3 years the indifference curves are similar since all asset allocations in the first period are as in the static case

6.6.2 Excess Contribution Rate Risk

In this case the results at the end of 6 years (i.e. end of the second period) the indifference curves are similar for all 3 asset allocations. However, as shown in Figure 6.12, for longer projection periods for a given combination of normal contribution rate and initial allocation in Equities, the Excess Contribution Rate Risk is lowest under DS1 and highest under DS2.

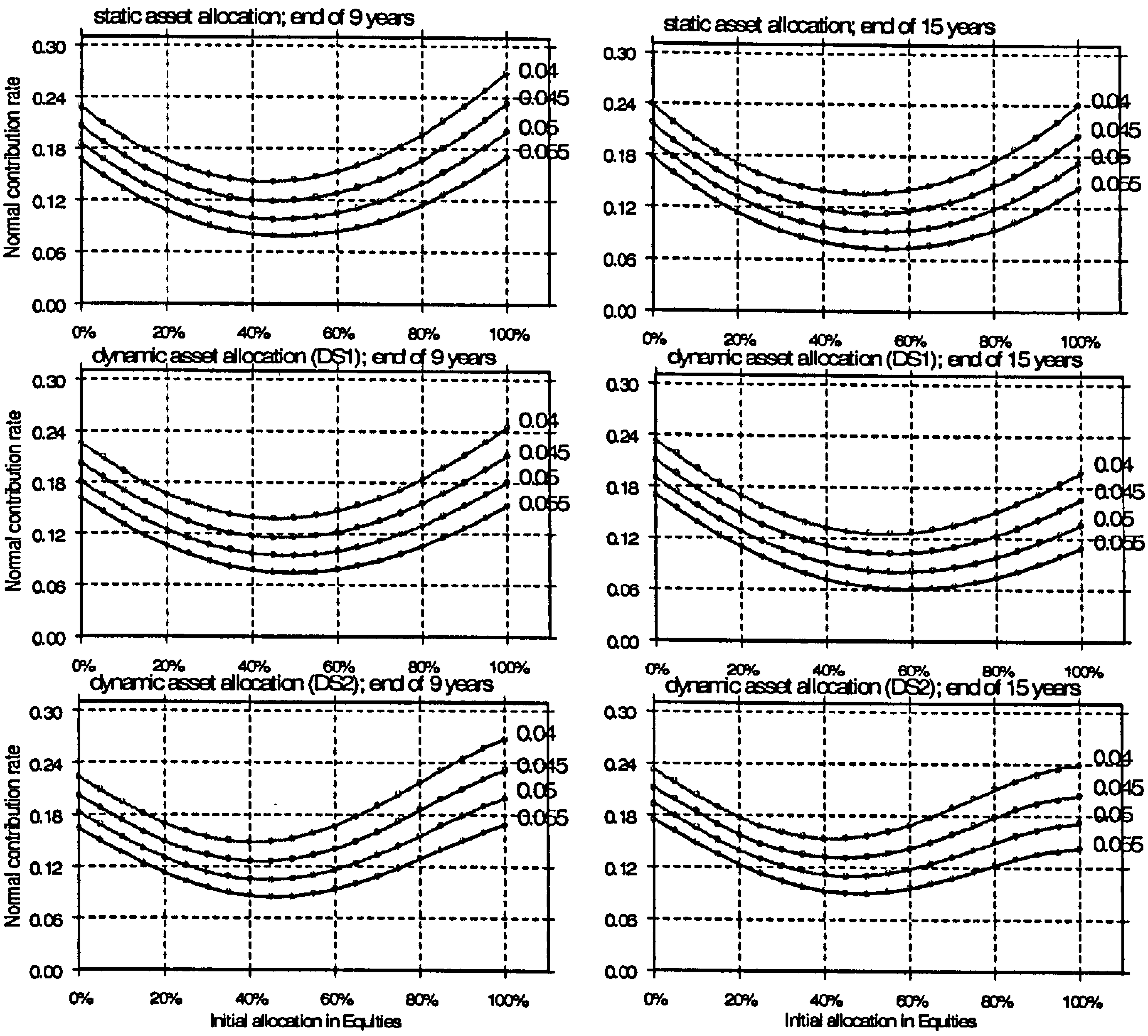

Figure 6.12: Comparison of Excess contribution rate risk levels at the end of 9 and 15 years (100% initial funding level)

6.6.3 How do the Investment Strategies Compare?

We have observed that the Mean Shortfall and Excess Contribution Rate Risks are lowest under DS1 and highest under DS2, for a given combination of normal contribution rate and initial asset allocation. This holds for all projection periods except for three-year projections for the Mean Shortfall risk and six-year projections for the Excess Contribution Rate risk where the results are similar.

Hence, we conclude that dynamic strategy DS1 performs consistently better than the other two strategies; whilst the static strategy performs better than dynamic strategy DS2. These results are consistent with the results emerging from theoretical models. In Chapter 4 we presented a solution for a dynamic programming problem involving a criterion with quadratic and linear factors. We showed that the optimal investment strategy is counter-intuitive in that it involves increasing the allocation in the risky asset as the pension fund deficit increases and vice versa. The analysis in this section further reinforce that finding.

6.7 Effect of the Initial Funding Level on the Indifference Curves

In Chapter 4 we showed that as the projection period increases we get a convergence in the optimal asset allocations for the cases where there are different levels of initial funding. In this section we study the effect of the initial funding level on the indifference curves.

6.7.1 Initially under-funded and over-funded schemes

We have so far analysed the results for an initially fully-funded scheme. We now consider the effect of other initial funding levels: a scheme initially in deficit and a scheme initially in surplus. We compare the results for the 100% initial funding level with the results for 80% and 120% initial funding levels.

We spread any surpluses and deficits, including the initial surplus or deficit, over 3 years. In our stochastic pension fund simulations, we use a recommended contribution rate, which is just the normal contribution rate plus an adjustment generated by the spread formula (as discussed in an earlier section).

A problem arises, however, if the adjustment is negative and larger, in absolute value, than the normal contribution rate. In practice this would mean a refund of contribution to the sponsor. In stochastic simulations, however, a refund of contributions would introduce unnecessary complications and, hence, we impose a lower bound of zero on the recommended contribution rate.

This feature has a significant impact on the results of a scheme which initially is in surplus. When the initial surplus, 20% in our case, is amortized over 3 years, the arising adjustment is so large and negative that in the first 3-year period we recommend a contribution rate of zero for most of the normal contribution rates. This has three significant implications:

Mean Shortfall Risk indifference curves would be impossible to calculate at the end of 3 years. The Mean Shortfall Risk would be the same, for a given asset allocation, for all normal contribution rates where the recommended contribution rate is zero. This complication only arises in the first 3-year period due to the initial surplus. However this does affect the Excess contribution rate risk calculations at the end of the second 3-year period, as we shall show below;

Excess Contribution Rate Risk indifference curves would be impossible to calculate at the end of 3 years[3]. This is due to the fact that there is no excess contribution in the first 3-year period since the scheme has an initial surplus. Thus whatever the initial funding position of the scheme, the Excess Contribution Rate Risk indifference curves cannot be calculated at the end of 3 years; and

Excess Contribution Rate Risk indifference curves would be impossible to calculate at the end of 6 years for a scheme, which has a (significant) initial surplus. This demands careful explanation. We have noted above that, at the end of 3 years, the Mean Shortfall Risk is the same, for a given asset allocation, for all normal contribution rates where the recommended contribution rate is zero in the first 3 years. This implies that when we spread surpluses and deficits at the end of 3 years the adjustments will be the same wherever we had a zero recommended contribution rate. In such cases, differences will only arise due to the initial asset allocation. Thus, at the end of 6 years, we would calculate the Excess Contribution Rate Risks but we would not be able to derive the indifference curves[4].

Figure 6.13 shows the Mean Shortfall Risk and the Excess Contribution Rate Risk indifference curves at the end of the 15 years for the 3 initial funding levels. We observe the following features:

Mean Shortfall Risk

Firstly, for a given combination of normal contribution rate and initial allocation in equities the Mean Shortfall risk is highest for the 80% case and lowest for the 120% case; and secondly, the minimum points are in the region of 70% to 85% initial allocation in equities.

Excess Contribution Rate Risk

We observe that, firstly, for a given combination of normal contribution rate and initial allocation in equities the Excess contribution rate risk is highest for the 80%

[3]For a scheme which is initially in deficit, we can calculate the Excess Contribution Rate Risk at the end of 3 years. But we cannot derive the indifference curves because this risk level is the same for all choices of the normal contribution rate, because there is only one value for the adjustment factor.

[4]However it is important to note that some of these problems arise due to the amortization period for the initial surplus or deficit; and/or the amount of the initial surplus or deficit.

initial funding case and lowest for the 120% initial funding case; and secondly, the minimum points are in the region of 50% to 60% initial allocation in equities.

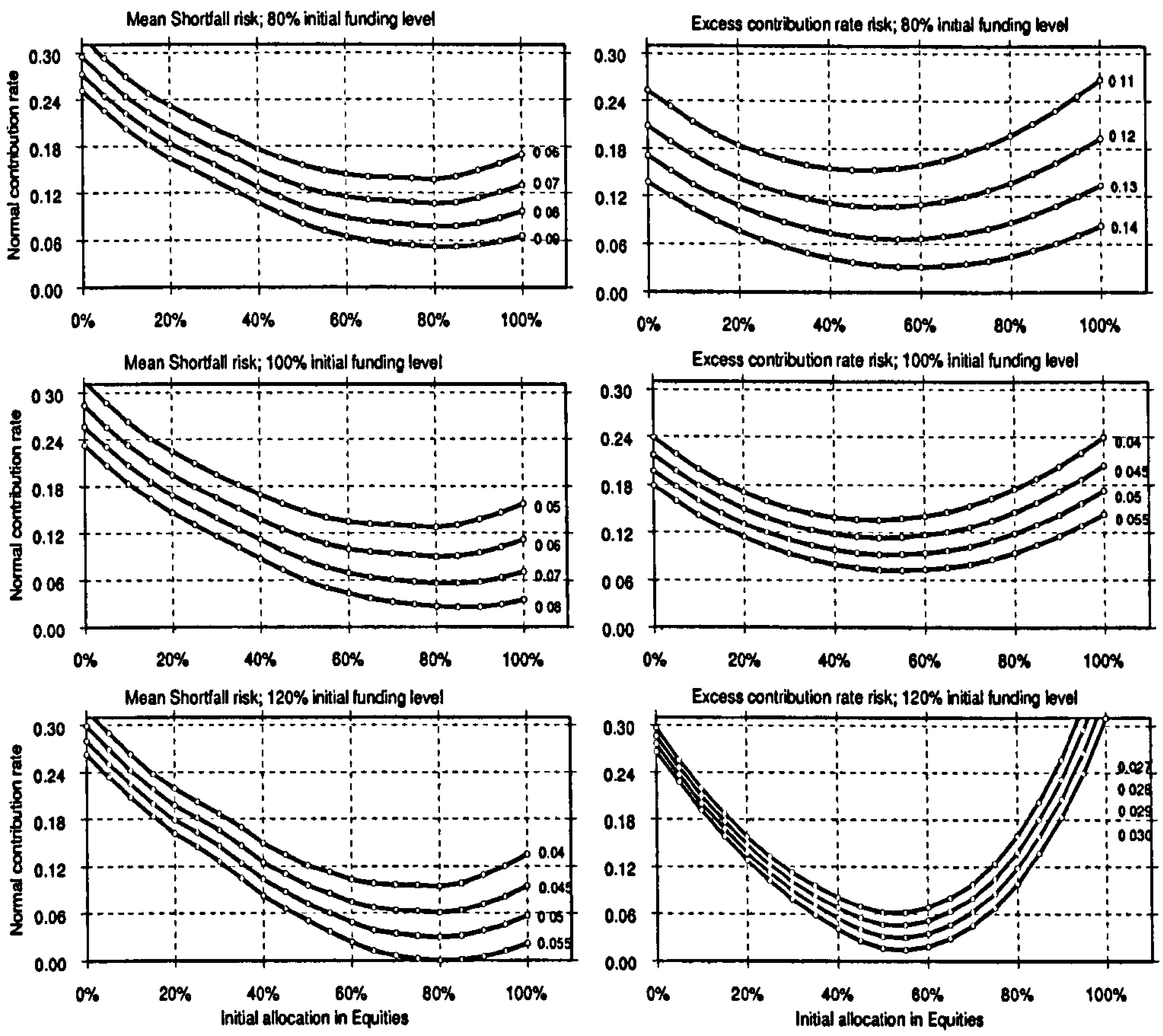

Figure 6.13: Mean Shortfall and Excess contribution rate risk levels at the end of 15 years (static asset allocation)

6.7.2 The Efficient Regions

As observed above, the minimum points in Figure 6.13 lie in similar regions for the three initial funding cases. Thus the efficient regions for the three cases are not significantly different.

This result is not surprising. we obtained a similar result for a theoretical model in Chapter 4. Hence, the findings in this section are due to the projection period: as the projection period increases we get a convergence in the optimal asset allocations.

6.8 Efficient Frontier Analysis

In this section we reconsider the problem of seeking optimal asset allocation strategies. However, instead of using indifference curves, we re-cast the problem as one of seeking Pareto optimal policies in a multi-objective setting.

As defined in Chapter 5, a strategy is said to be Pareto optimal if any other strategy leads to worse values for at least one of the objective functions. Although we have three objective functions we simplify the analysis by considering two objective functions at a time.

We firstly consider decision-making in the case where the Normal contribution rate is not a free variable. This analysis is extended to the case where the Normal contribution rate is a free variable. Furthermore, we consider different choices of asset classes: equities and fixed-interest bonds, and equities and index-linked bonds.

6.8.1 The Normal contribution rate as a fixed variable

We firstly consider the case where the decision-maker is not allowed to vary the Normal contribution rate and we assume, without loss of generality, that the Normal contribution rate is fixed at 10%. As before we carry out stochastic projections over a given period, say 15 years. We assume a fixed asset allocation (with annual re-balancing) in Equities and Fixed-Interest Bonds. A range of different initial asset allocations is considered and for each allocation we calculate the Mean Shortfall and Excess Contribution Rate risks.

Figure 6.14 shows a curve depicting the Mean Shortfall risk and the Excess Contribution Rate risk for 21 initial asset allocations. We plot the Mean Shortfall risk on the x-axis and the Excess Contribution Rate risk on the y-axis.

We can illustrate this graph by considering several examples:

Example 1: Point A represents the case where we invest 100% in equities. The Mean Shortfall risk is 0.111 whilst the Excess Contribution Rate risk is 0.089;

Example 2: Point B represents the case where we invest 60% in equities and 40% in bonds. The Mean Shortfall risk is 0.091 whilst the Excess Contribution Rate risk is 0.065;

Example 3: Point C represents the case where we invest 40% in equities and 60% in bonds. The Mean Shortfall risk is 0.0989 whilst the Excess Contribution Rate risk is 0.061; and

Example 4: Point D represents the case where we invest 100% in bonds. The Mean Shortfall risk is 0.157 whilst the Excess Contribution Rate risk is 0.080.

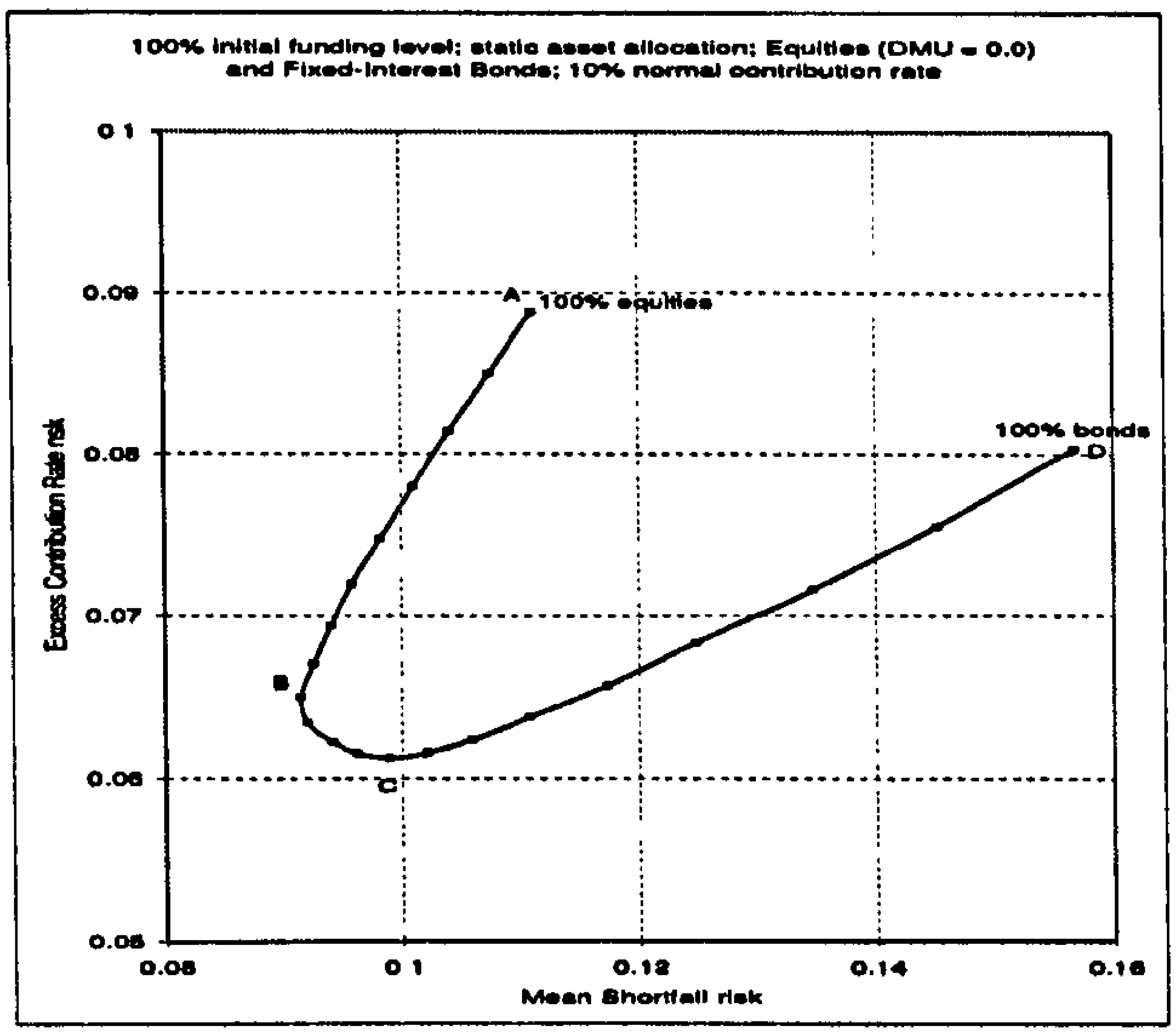

Figure 6.14: Mean Shortfall risk and Excess Contribution Rate risk trade-off at the end of 15 years for 10% Normal contribution rate.

Decision Making

Consider the arcs AB, BC and CD. We can draw similar conclusions as in Figure 3 with the three arcs representing the regions I, II and III, respectively (see Figures 6.4 and 6.5). Thus, we can conclude that asset allocations along AB and CD are inefficient because we can reduce both the Mean Shortfall and Excess Contribution Rate risks by shifting towards asset allocations along arc BC. Meanwhile, along arc BC, a change of asset allocation leads to a simultaneous reduction in one risk and increase in the other risk.

Alternatively, we could argue as follows. Firstly, consider asset allocations along arc AB. For a fixed Mean Shortfall risk we can find an asset allocation either on BC or on CD that has a lower Excess Contribution Rate risk. Hence asset allocations along AB are inefficient.

Secondly, consider asset allocations along arc CD. For a fixed Excess Contribution Rate risk we can find an asset allocation either on AB or BC that has a lower Mean Shortfall risk. Hence asset allocations along CD are inefficient.

And thirdly, consider asset allocations along arc BC. For a fixed Mean Shortfall risk the asset allocation with the lowest Excess Contribution Rate risk is along BC. Similarly, for a fixed Excess Contribution Rate risk the asset allocation with the lowest Mean Shortfall risk is along BC.

Hence, only the asset allocations along BC are efficient. Thus, BC is the efficient frontier when considering Mean Shortfall and Excess Contribution Rate risks for a given Normal contribution rate.

Performance Measure: Average Contribution Rate

In Section 6.4 we have shown how the Average Contribution Rate over the projection period could be used as a performance measure in Indifference Curve Analysis. In this section, we show how the Average Contribution Rate can be used in an Efficient Frontier Analysis.

In Figure 6.15 below, we plot the Mean Shortfall risk at the end of the projection period (15 years in this case) on the x-axis and the Average Contribution Rate over the projection period on the y-axis. The curve PQRS shows the Mean Shortfall risk and the Average Contribution Rate for each of the 21 asset allocations.

For example, point P shows that if we allocate 100% in equities (and 0% in fixed-interest bonds) the Mean Shortfall risk would be approximately 0.11 whilst the Average Contribution Rate would be approximately 14.2%. Point S shows that if we allocate 0% in equities (and 100% in fixed-interest bonds) the Mean Shortfall risk would be approximately 0.155 whilst the Average Contribution Rate would be approximately 14.6%.

Now, consider the arcs PQ, QR and RS. Firstly, consider asset allocations along arc PQ. For a fixed Mean Shortfall risk we can find an asset allocation either on QR or on RS that has a lower Average Contribution Rate. Hence asset allocations along PQ are inefficient.

Secondly, consider asset allocations along arc RS. For a fixed Average Contribution Rate we can find an asset allocation either on PQ or QR that has a lower Mean Shortfall risk. Hence asset allocations along RS are inefficient.

And thirdly, consider asset allocations along arc QR. For a fixed Mean Shortfall risk the asset allocation with the lowest Average Contribution Rate is along QR. Similarly, for a fixed Average Contribution Rate the asset allocation with the lowest Mean Shortfall risk is along QR.

Hence, only the asset allocations along QR are efficient. Thus QR is the efficient frontier under Mean Shortfall and Average Contribution Rate for a given Normal contribution rate.

We observe that the asset allocations along the efficient frontier QR coincide with the asset allocations along the efficient frontier, BC, for the case of the Mean Shortfall risk and Excess Contribution Rate risk. This is similar to the result in Section 6.4

where it was shown that the asset allocations in the efficient region minimize the Mean Shortfall risk, Excess Contribution Rate risk and the Average Contribution Rate.

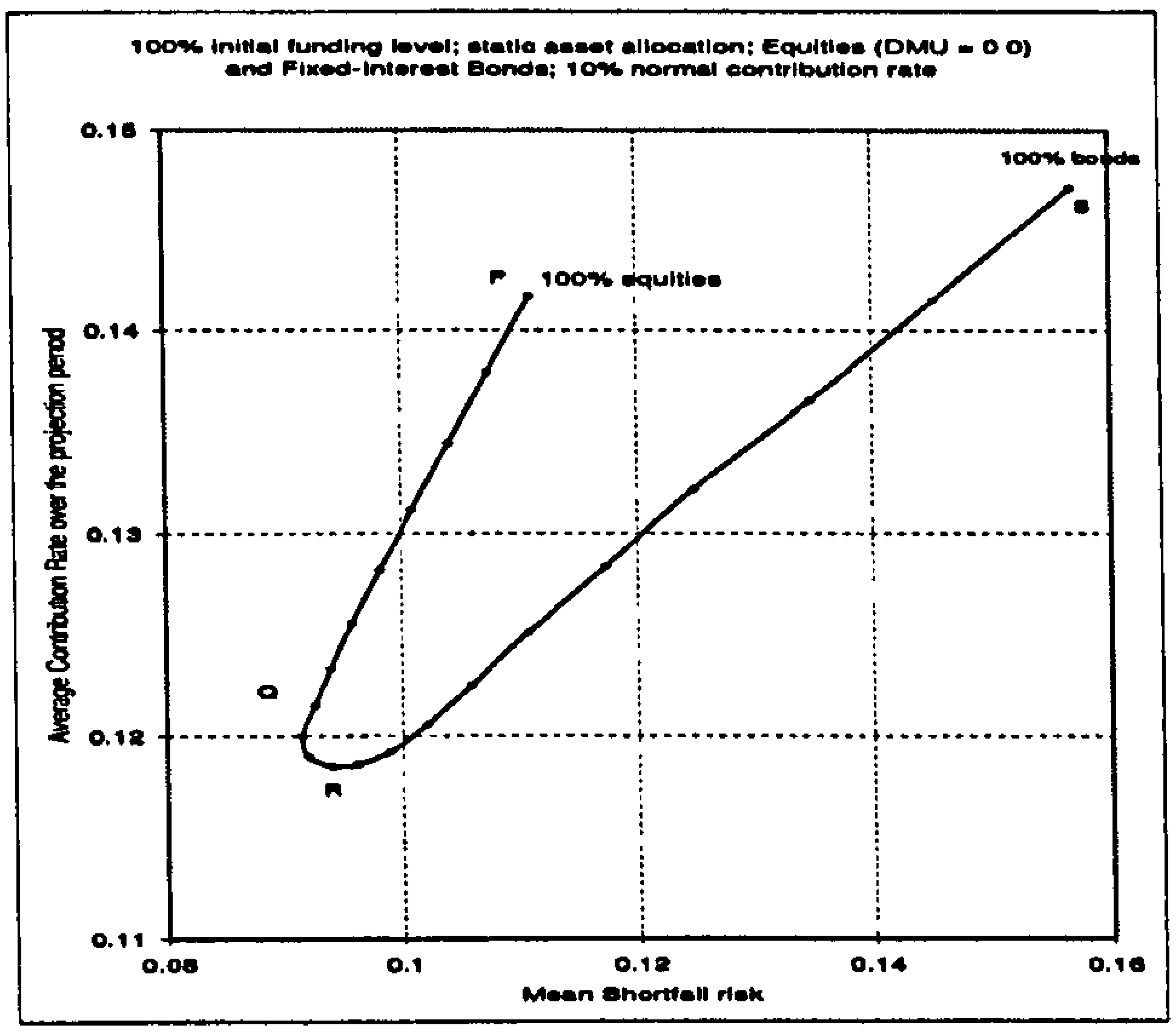

Figure 6.15: Mean Shortfall risk and Average Contribution Rate trade-off at the end of 15 years for 10% Normal contribution rate.

6.8.2 The Normal contribution rate as a free variable

Decision Making

We now consider the case where the Normal contribution rate is a free variable. We will consider four different Normal contribution rates: 6%, 10%, 14% and 18%. In each case, stochastic projections are carried out over 15 years. We will assume that the pension fund is invested in equities and fixed-interest bonds.

Figure 6.16 shows the curves for the four selected Normal contribution rates. A similar analysis as above could be undertaken on each curve to find the efficient frontiers for a given Normal contribution rate.

We observe that the top curve is for the lowest Normal contribution rate (i.e. 6%) whilst the lowest curve is for the highest Normal contribution rate (i.e. 18%). This result is intuitive: all things equal we would expect a low (high) Normal contribution rate to lead to a high (low) Mean Shortfall risk and a high (low) Excess Contribution Rate risk.

Performance Measure: Average Contribution Rate

Figure 6.17 shows the curves for 3 Normal contribution rates: 6%, 10% and 14%. As before we plot the Mean Shortfall risk on the x-axis and Average Contribution Rate

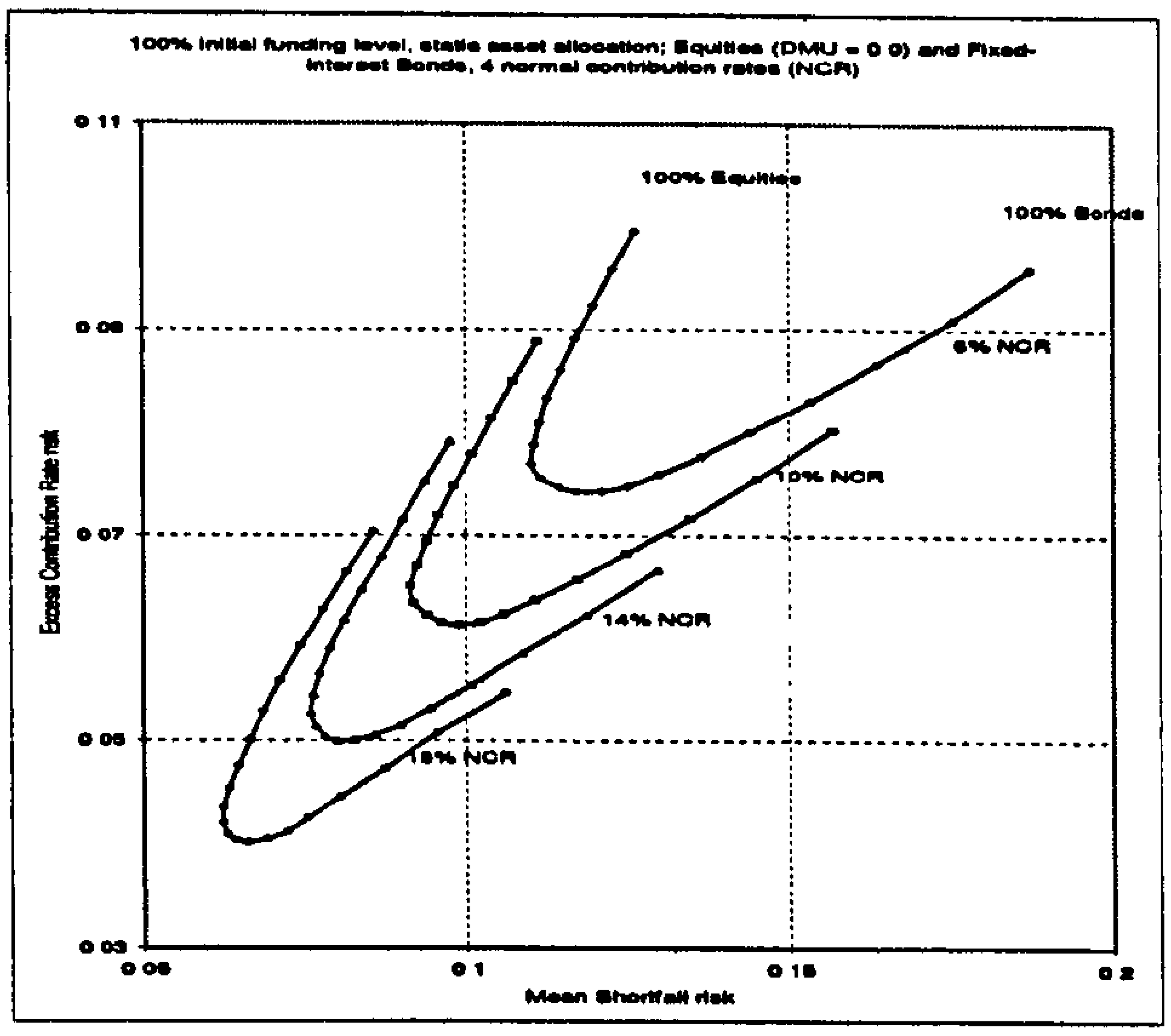

Figure 6.16: Mean Shortfall risk and Excess Contribution Rate risk trade-off at the end of 15 years for various Normal contribution rates

on the y-axis. The highest curve is for a Normal contribution rate of 14% whilst the lowest curve is for the lowest Normal contribution of 6%.

As in the previous case of Mean Shortfall and Excess Contribution Rate risk, this result is intuitive: all things equal we would expect a high Normal contribution rate to lead to a low Mean Shortfall risk and high Average Contribution Rate; whilst a low Normal contribution rate would be expected to lead to a high Mean Shortfall and low Average Contribution Rate.

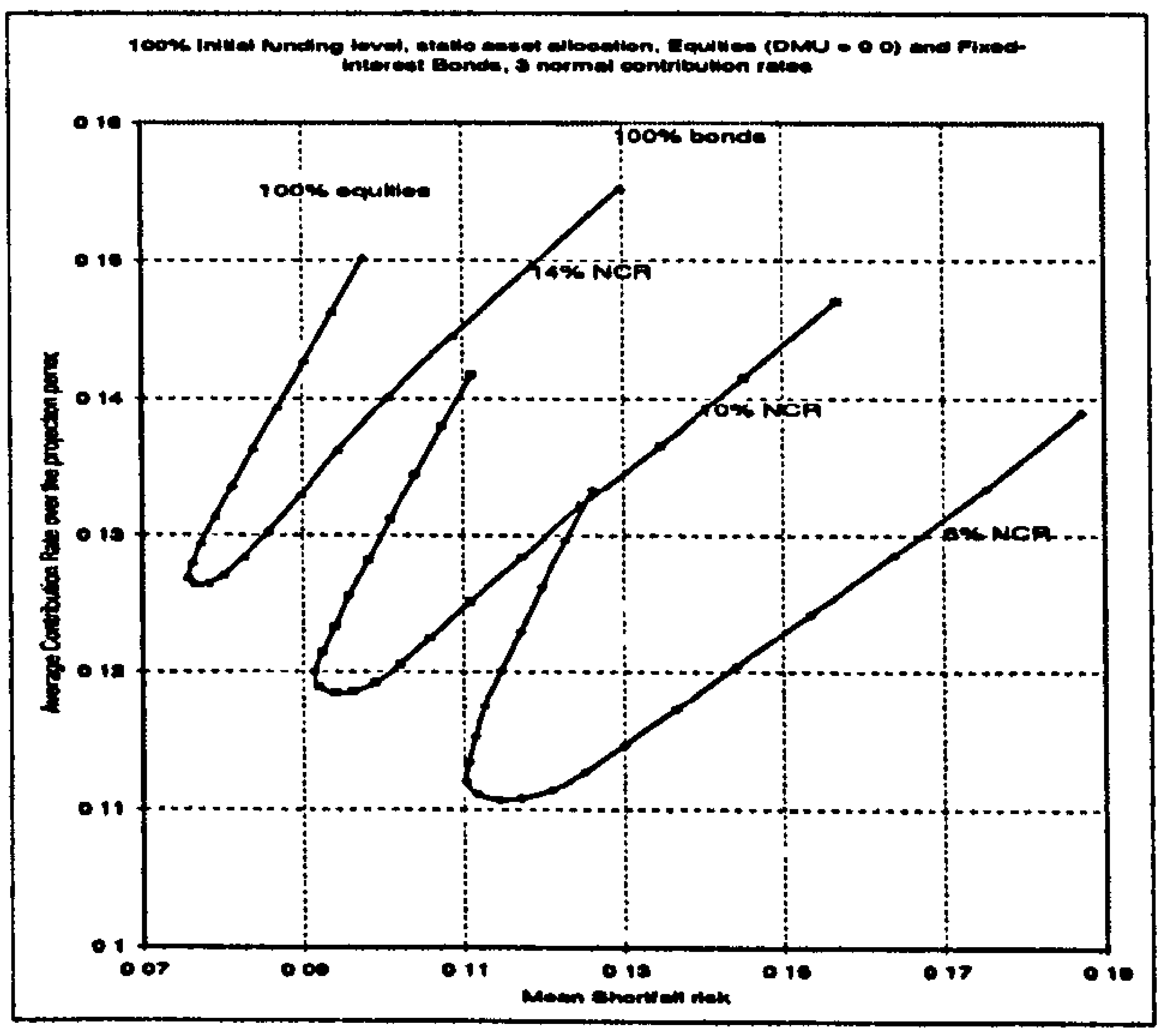

Figure 6.17: Mean Shortfall risk and Average Contribution Rate trade-off at the end of 15 for various Normal contribution rates

6.8.3 Investing in Index-Linked Gilts

We now consider the case where the pension fund is invested in Index-Linked Gilts. In the previous analysis, we assumed that the fund is invested only in Equities and Fixed-Interest Bonds. We now consider two further investment policies. In the first policy, the fund is invested only in Equities and Index-Linked Bonds while, in the second policy, the fund is invested only in Index-Linked Bonds and Fixed-Interest Bonds.

As before, we assume that the Normal contribution rate is fixed at 10% and that stochastic projections are carried out over a period of 15 years.

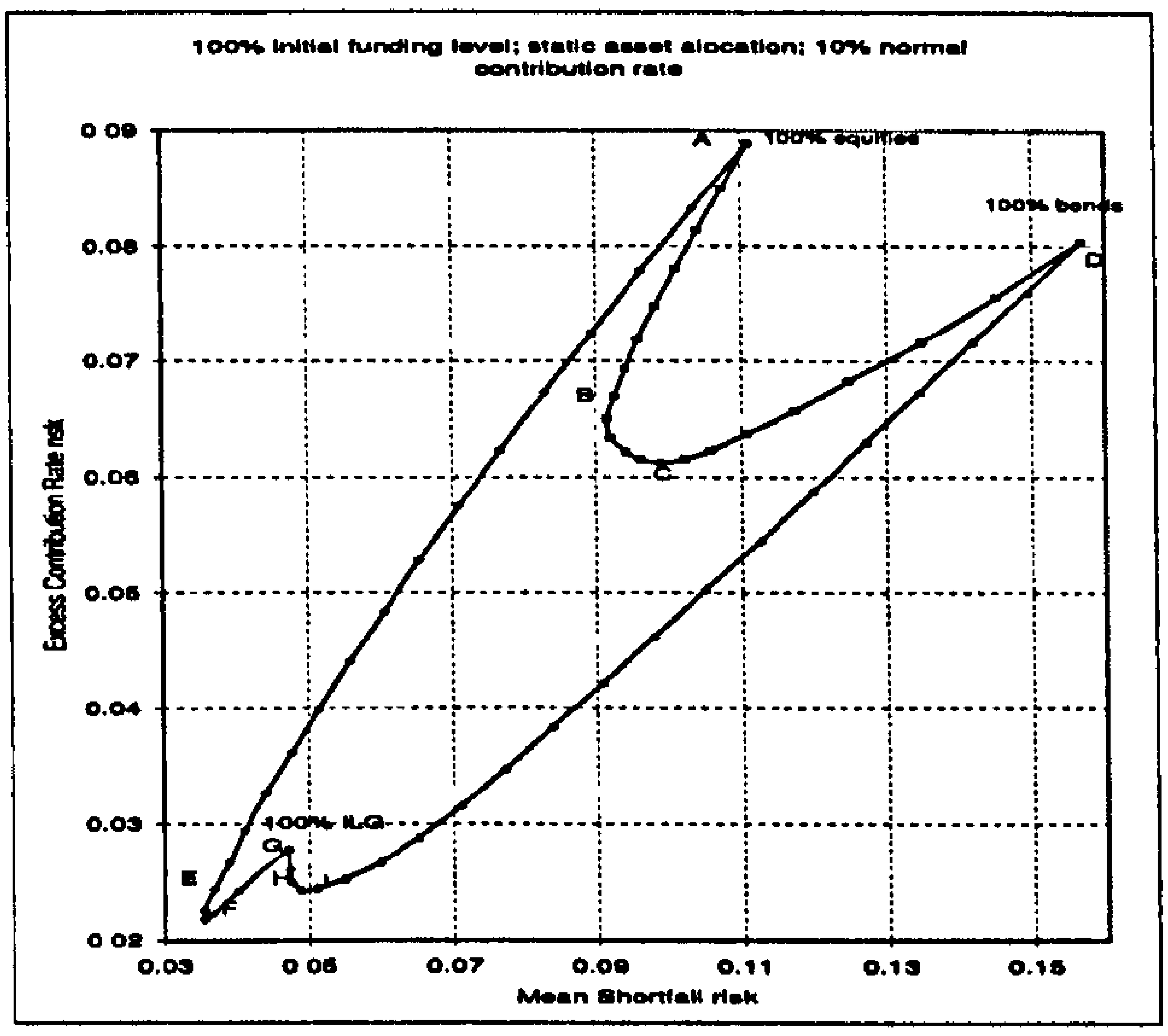

Figure 6.18: Mean Shortfall risk and Average Contribution Rate trade-off at the end of 15 for different asset allocation policies

In Figure 6.18, we plot curves for the Mean Shortfall risk and the Excess Contribution Rate risk for different initial asset allocations for the three policies. Curve ABCD represents the case where the fund is invested in equities and fixed-interest bonds (i.e. the case discussed previously). Curve AEFG represents the case where the fund is invested in equities and index-linked bonds, while curve GHJD represents the case where the fund is invested in index-linked bonds and fixed-interest bonds.

As shown before, the efficient frontier would be BC in the equities and fixed-interest bonds case, EF in the equities and index-linked bonds case, and HJ in the index-linked and fixed-interest bonds case. However, we observe that the efficient frontier EF includes a very small range of asset allocations around 15% initial allocation in equities; whilst the other efficient regions, BC and HJ, include a wider range of asset allocations. This result agrees with our previous analysis where we observed that the efficient region for equities and index-linked bonds was more tightly defined than the efficient region for equities and fixed-interest bonds.

6.9 Concluding Remarks

In this chapter we have presented a case study of the risk management approach considered in Chapter 5. We have illustrated how the approach can be applied to a wide range of issues affecting the management of defined benefit pension schemes.

We have shown how an indifference curve analysis and an efficient frontier analysis can be used in the choice of asset allocation strategies and Normal contribution rates. One of the main results arising from this study is that asset allocation strategies for a pension scheme can be improved by choosing those strategies that lie within an efficient region. Alternatively, one could consider Pareto optimal strategies which lie along an efficient frontier. Such strategies have been shown not only to be optimal in terms of reducing the solvency and contribution risks but also in terms of reducing the cost to the employer.

Several general results have also been shown to hold in the risk management setting. Firstly, counter-intuitive strategies, which several actuarial researchers and our analysis in Chapter 4 have shown to be optimal, have also been shown to be optimal in our approach. Furthermore, just as in the dynamic programming problem of Chapter 4, we have shown that for long projection horizons we get a convergence in the optimal asset allocations for different initial funding levels.

It can be argued that our results depend on the chosen investment model and its parameter values and also on the liability structure of the scheme. Hence, in the appendices we undertake extensive sensitivity analysis of the various crucial parameters in the investment model. We also consider a completely different investment model in Chapter 7. The issue of how the strength of the employer could affect the optimal asset allocation strategies is considered in Chapter 8. Meanwhile, the problem of different liability structures is left as a future extension of this work.

Chapter 7

Incorporating Model Risk

7.1 Introduction

In this chapter we investigate the effect of asset model risk on decision making in defined benefit pension schemes. Parameter uncertainty and model risk occurs naturally in all actuarial modelling problems.

Parameter uncertainty arises from the uncertainty in estimating the parameter values in a chosen asset model. In our decision making framework we investigate parameter uncertainty and present the results in Appendices D, E and F.

Model risk arises from employing an inappropriate asset model in the decision making process. In the discussion of Parker (1997), Andrew J.G. Cairns observes that "...the models themselves are not known with certainty and indeed are...only approximations to a much more complex world..." (p. 73).

In this chapter we follow Cairns (2000*a*) in reserving the term model risk to "...circumstances where the results and decisions emerging from an analysis are sensitive to the choice of model..." (p. 314). We investigate whether our results hold under a different asset model. And in cases where the results hold, we assess the sensitivity of the decisions to changes in the asset model.

This chapter is organized as follows: in Section 7.2 we present some background ideas for modern asset pricing. And in Section 7.3 we consider the formulation of modern asset pricing models. In Section 7.4 we present a stochastic asset model suggested by Smith and Southall (2001). For simplicity we will refer to this model as the "Smith-Southall model". Whilst in Section 7.5 we present and discuss results of a case study using the Smith-Southall model. Lastly, in Section 7.6 we present some concluding remarks.

7.2 Background Information

In this section we present some background information necessary for understanding modern asset pricing formulations. We firstly consider the idea of deflators and then we discuss the idea of change of numéraire.

7.2.1 Deflators

Definition 7.1 *Deflator* A deflator (also referred to as pricing kernel, stochastic discount factor etc) is a stochastic process, ζ_t, such that

$$1 = E_t\Big[\zeta_{t+1}\, X_{t+1}\Big] \tag{7.1}$$

where X_{t+1} is the return on an asset in the period t to $t+1$ and $E_t[\ldots]$ refers to the expectation given information up to time t.

We shall use the names deflator, stochastic discount factor, pricing kernel and state-price density interchangeably.

Deflators as Stochastic Discount Factors

The idea behind state-price deflators is fairly simple if we think in terms of stochastic discount factors. Here we combine the approaches of Duffie (2001, pp. 3-11) and Cochrane (2001, p.9) to discuss deflators in an intuitive manner.

It is well-known that the price of an uncertain future cash flow can be established by considering the expected discounted future value. Suppose we have a finite number of future states of the world, $\{1, \cdots, S\}$. Consider a security whose payoff can be described by a vector $X = \Big(X_s\Big)$, where X_s is the payoff in state s. Let $p = \Big(p_s\Big)$ be the vector of (actual) probabilities of the states. That is, p_s is the probability of state s. Furthermore, suppose that instead of a single discount factor we have stochastic discount factors for each state of the world $\zeta = \Big(\zeta_s\Big)$.

Then the discounted (or deflated) payoff in state s is $\zeta_s\, X_s$ and this has a probability of p_s. Thus the price can be established by taking the expectation over all states:

$$\begin{aligned} \text{Price} &= \sum_{i=1}^{n} p_i\, \zeta_i\, X_i \\ &= E\Big[\zeta\, X\Big] \end{aligned} \tag{7.2}$$

In other words, $1 = E\Big[\zeta\, R\Big]$ where R is the gross return.

Deflators as the marginal rate of substitution

In a so-called representative agent economy, the pricing kernel is formulated as the intertemporal marginal rate of substitution. This is the approach followed by Sargent (1987) and Cochrane (2001) (also see Abel (1990)). The idea is to consider a representative agent's consumption problem, derive a first-order condition for the optimal consumption and set the marginal rate of substitution as the pricing kernel.

Let $U(C_t)$ be the agent's utility function where C_t is the agent's consumption at time t. Suppose the agent's goal is to find the optimal consumption policy to maximize

$$\max E_t \sum_{i=0}^{\infty} \beta^i \, U(C_{t+i}) \tag{7.3}$$

subject to some wealth constraints.

Then using the technique of dynamic programming (see Chapter 4) it can be shown that the first-order condition is:

$$U'(C_t) = E_t\Big[\beta \, U'(C_{t+1}) \, X_{t+1}\Big] \tag{7.4}$$

where X_{t+1} is the gross return on investments.

In this case the pricing kernel is

$$\zeta_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)}. \tag{7.5}$$

Several articles have appeared in the UK actuarial literature on the idea of deflators. The most notable amongst these is arguably Jarvis et al. (2001). Most of these papers consider how actuaries can use deflators in the valuation problems.

7.2.2 Numéraire

Definition 7.2 *Numéraire* In the financial mathematics literature, a numéraire is defined as an asset with a positive price which is taken as a unit of denomination for any other asset

The technique of numéraire change has become very central in financial problems where a change in the currency of denomination can lead to an elegant solution. Rogers (1997) considers the change of numéraire in the case of countries with freely exchangeable assets: country i with state-price density ζ^i and country j with state-price density ζ^j.

He shows that "*...at a time t a unit of currency j can be exchanged for $Y_0^{ij}\,\zeta_t^j/\zeta_t^i$ units of currency i...*" (p. 174), where Y_0^{ij} is the exchange rate at time 0.

For similar approaches see Geman et al. (1995) and Schroder (1999). This technique is crucial to the idea of 'gauge transforms' described below.

7.3 Asset pricing model formulation

Cochrane (2001) refers to Equation 7.1 as "*the* central asset pricing formula". This is due to the fact that pricing models for different asset classes can be constructed in a similar way, as shown in the examples below for bonds and equities.

No-arbitrage

The existence of the pricing kernel is assured due to the absence of arbitrage opportunities. Duffie (2001) shows that a pricing kernel exists if and only if there is no arbitrage. Thus in the examples below we consider buying at time t and selling at time $t+1$ with the knowledge that there can be no arbitrage.

We now consider the pricing of bonds and equities. For more details and other asset classes see Cochrane (2001, pp. 10-12).

pricing of bonds

Let $P_{t:t+n}$ be the price at time t of a zero coupon bond paying 1 at time $t+n$. At time $t+1$ this becomes a zero coupon bond of term $n-1$. We can thus sell it at a price of $P_{t+1:t+n}$. Then, by writing the return X_{t+1} as $P_{t+1:t+n}/P_{t:t+n}$ and using Equation 7.1, to price bonds we need the solution for the equation:

$$\begin{aligned} 1 &= E_t\left[\zeta_{t+1}\,\frac{P_{t+1:t+n}}{P_{t:t+n}}\right] \\ \Rightarrow P_{t:t+n} &= E_t\left[\zeta_{t+1}\,P_{t+1:t+n}\right] \end{aligned} \tag{7.6}$$

pricing of equities

Let P_t be the ex-dividend equity price and D_t be the dividend at time t. We can buy this share ex-dividend at time t for P_t. At time $t+1$ we receive a dividend of D_{t+1} and sell the share ex-dividend for P_{t+1}. Thus we can set the gross return X_{t+1} as $\big(P_{t+1}+D_{t+1}\big)/P_t$. Then, using Equation 7.1, to price equities we need the solution for the equation:

$$\begin{aligned} 1 &= E_t\left[\zeta_{t+1}\,\frac{P_{t+1}+D_{t+1}}{P_t}\right] \\ \Rightarrow P_t &= E_t\left[\zeta_{t+1}\,(P_{t+1}+D_{t+1})\right]. \end{aligned} \tag{7.7}$$

Model Formulation

The central pricing formula has revolutionized the way financial economists formulate asset pricing models. Early models involved specifying the dynamics of the spot rate or the forward rate processes from which bond prices could be derived. This is the approach taken by such famous models as Vasicek and Cox-Ingersoll-Ross (see

Vasicek (1977) and Cox et al. (1985)).

These approaches have recently come into criticism. Rogers (1996) observes that these models can lead to negative interest rates. He argues that this possibility is undesirable in certain instances like the pricing of derivatives and long bonds. He shows that in cases like these, negative interest rates can lead to exponentially large discrepancies.

To address these problems, Flesaker and Hughston (1996) introduce a pricing approach where zero-coupon bond price processes are the basic building blocks. Thus the positivity property of interest rates is introduced in the construction of these bond price processes.

They treat the system of bond price processes as being, by assumption, complete and arbitrage-free. This implies that every contingent claim can be uniquely priced as a linear combination of bond prices.

They state the condition for interest rates to remain positive (Positive Interest Rate Property) as the requirement that at any given time the prices of zero-coupon bonds decrease as the term to maturity increases. In other words, $P_{t:t+n_0} > P_{t:t+n_1}$ for $n_0 < n_1$.

Potential approach

Rogers (1997) introduces an approach to pricing model formulation which has as the starting-point the specification of the state-price density (also see Leippold and Wu (1999)). He observes that there are several advantages of this approach (which he refers to as the potential approach).

Amongst these advantages are that with the potential approach it is easy to deal with yield curves and exchange rates in different countries. Furthermore, one can achieve the same tractability as in the spot rate models of, for instance, Vasicek whilst ensuring that the interest rates remain non-negative.

Example: Three-Factor Pricing model

As an example of the potential approach, Bekaert and Grenadier (2001) suggest a three-factor asset pricing model in which the pricing kernel is driven by inflation, dividend growth rate and an unobservable factor. From this formulation of the pricing kernel, they derive models for pricing bonds and equities.

7.4 Smith-Southall Model

The approach of Smith and Southall (2001) can best be understood by considering the idea of gauge transforms as presented by Smith and Speed (1998).

Gauge Transforms

The idea of gauges is considered by Smith and Speed (1998). They suggest that the basic features of an asset be captured by a 'gauge'. They define a gauge as an ordered pair (ζ, P) comprising a deflator and a term structure.

They consider the deflator, ζ, as an indicator of the relative value of the gauge whilst the term structure, P, is a description of the price process of zero-coupon bonds denominated in units of the gauge. Superscripts are used to denote gauges for different asset classes.

Intuitively, we can think of the transactions at any given time in terms of numéraire change as shown for different currencies in Section 7.2.2. Consider two gauges: $(\zeta^{asset}, P^{asset})$ a gauge denominated in units of *asset*; and (ζ^{cash}, P^{cash}) a gauge denominated in units of *cash*. At time t an investor could buy a zero coupon bond of term n for $P^{asset}_{t:t+n}$ in units of *asset* and receive 1 at time $t+n$ in units of *asset*.

Now suppose we want to express this cash flow in units of the other numéraire, *cash*. Using the change of numéraire technique described above, we would say that for a payment of $P^{asset}_{t:t+n}\zeta^{asset}_t/\zeta^{cash}_t$ at time t in *cash*, the investor would receive $\zeta^{asset}_{t+n}/\zeta^{cash}_{t+n}$ at time $t+n$ in *cash*.

The return from such a transaction can be analyzed as follows. The total return from t to $t+n$ in units of the numéraire, *cash*, can be written as:

$$\frac{\zeta^{asset}_{t+n}}{\zeta^{cash}_{t+n}} \div \frac{\zeta^{asset}_t P^{asset}_{t:t+n}}{\zeta^{cash}_t} = \frac{\zeta^{asset}_{t+n}}{\zeta^{cash}_{t+n}}\frac{\zeta^{cash}_t}{\zeta^{asset}_t P^{asset}_{t:t+n}}. \tag{7.8}$$

This can also be expressed in terms of zero-coupon bonds of term 1:

$$\begin{aligned}
\text{total return} &= \left[\frac{\zeta^{asset}_{t+1}}{\zeta^{cash}_{t+1}}\frac{\zeta^{cash}_t}{\zeta^{asset}_t P^{asset}_{t:t+1}}\right]\left[\frac{\zeta^{asset}_{t+2}}{\zeta^{cash}_{t+2}}\frac{\zeta^{cash}_{t+1}}{\zeta^{asset}_{t+1} P^{asset}_{t+1:t+2}}\right]\cdots\left[\frac{\zeta^{asset}_{t+n}}{\zeta^{cash}_{t+n}}\frac{\zeta^{cash}_{t+n-1}}{\zeta^{asset}_{t+n-1} P^{asset}_{t+n-1:t+n}}\right] \\
&= \frac{\zeta^{cash}_t}{\zeta^{asset}_t}\frac{1}{P^{asset}_{t:t+1}}\frac{1}{P^{asset}_{t+1:t+2}}\cdots\frac{1}{P^{asset}_{t+n-1:t+n}}\frac{\zeta^{asset}_{t+n}}{\zeta^{cash}_{t+n}} \\
&= \frac{\zeta^{cash}_t}{\zeta^{asset}_t}\frac{\zeta^{asset}_{t+n}}{\zeta^{cash}_{t+n}}\prod_{s=t}^{t+n-1}\left[P^{asset}_{s:s+1}\right]^{-1}.
\end{aligned} \tag{7.9}$$

For a dividend paying asset we may need to calculate the income return and capital return separately. For simplicity consider one term with dividend payable once at the end of the term.

At time t we have a deflator ζ_t^{asset} and zero-coupon bond $P_{t:t+1}^{asset}$ whilst at time $t+1$ we have ζ_{t+1}^{asset} and zero-coupon bond $P_{t+1:t+2}^{asset}$. As before we will use *cash* as the numéraire.

Investing $P_{t:t+1}^{asset}\zeta_t^{asset}/\zeta_t^{cash}$ at time t would mean that we receive $\zeta_{t+1}^{asset}/\zeta_{t+1}^{cash}$ at time $t+1$. The capital component of this is

$$\text{capital} = \frac{\zeta_{t+1}^{asset} P_{t+1:t+2}^{asset}}{\zeta_{t+1}^{cash}} \tag{7.10}$$

whilst the income component is

$$\begin{aligned} \text{income} &= \frac{\zeta_{t+1}^{asset}}{\zeta_{t+1}^{cash}} - \frac{\zeta_{t+1}^{asset} P_{t+1:t+2}^{asset}}{\zeta_{t+1}^{cash}} \\ &= \frac{\zeta_{t+1}^{asset}}{\zeta_{t+1}^{cash}}\left[1 - P_{t+1:t+2}^{asset}\right]. \end{aligned} \tag{7.11}$$

Thus the capital return from time t to time $t+1$ is

$$\begin{aligned} \text{capital return} &= \frac{\zeta_{t+1}^{asset} P_{t+1:t+2}^{asset}}{\zeta_{t+1}^{cash}} \div \frac{\zeta_t^{asset} P_{t:t+1}^{asset}}{\zeta_t^{cash}} \\ &= \frac{\zeta_{t+1}^{asset} P_{t+1:t+2}^{asset}}{\zeta_{t+1}^{cash}} \frac{\zeta_t^{cash}}{\zeta_t^{asset} P_{t:t+1}^{asset}}. \end{aligned} \tag{7.12}$$

Similarly, the income return is

$$\text{income return} = \frac{\zeta_{t+1}^{asset}}{\zeta_{t+1}^{cash}}\left[1 - P_{t+1:t+2}^{asset}\right] \frac{\zeta_t^{cash}}{\zeta_t^{asset} P_{t:t+1}^{asset}}. \tag{7.13}$$

Adding these two factors in Equations 7.12 and 7.13 leads to the formula for the total return from time t to time $t+1$ in Equation 7.8.

The Smith-Southall model employs the potential approach where a pricing kernel is suggested from which the prices of different asset classes are derived. Smith and Southall (2001) suggest the following formula for calculating deflators and zero-coupon bond prices:

$$\begin{aligned} \zeta_{t_0}\left(P_{t_0:b} - P_{t_0:b+1}\right) &= \frac{const}{(1+f)^b} \times \\ & Exp\Big\{\boldsymbol{\sigma} \cdot \left(\mathbf{C}_{t_0} - \mathbf{C}_{\min\{t_0, b-\tau\}}\right) + \boldsymbol{\lambda} \cdot \mathbf{B}_{t_0} \\ & + \min\{b - t_0, \tau\}\, \boldsymbol{\sigma} \cdot \mathbf{B}_{t_0} - \frac{1}{2}(\boldsymbol{\lambda} + \boldsymbol{\sigma}\tau)^2\, t_0 \\ & + \frac{1}{12} \max\{0, \tau - b + t_0\}\, (\tau + 1 - b + t_0) \times \\ & \qquad \left[6\boldsymbol{\sigma} \cdot \boldsymbol{\lambda} + (4\tau + 2b - 2t_0 - 1)\, \boldsymbol{\sigma}^2\right]\Big\} \end{aligned} \tag{7.14}$$

where f is the long forward rate and τ is a positive integer whilst $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ are 5-dimensional vectors. Also $\mathbf{B}_{t_0}$ is a 5-dimensional Brownian motion and $\mathbf{C}_{t_0}$ is an accumulation of the history of $\mathbf{B}_{t_0}$, that is,

$$\mathbf{C}_{t_0+1} = \mathbf{C}_{t_0} + \mathbf{B}_{t_0}. \tag{7.15}$$

They consider three asset classes (Cash, Inflation and Equities) and suggest the parameter values shown in Table 7.1. The fact that this model is given without derivation opens it up to criticism. However, as shown by Smith and Southall (2001, p.3) some important properties are satisfied: no-arbitrage, positive interest rates and analytical tractability.

Cash		Inflation		Equity	
$\tau = 20$		$\tau = 20$		$\tau = 20$	
f = 5.13%		f = 1.57%		f = 2.08%	
λ	σ	λ	σ	λ	σ
0.366665	-0.006490	0.236537	0.002906	0.302127	-0.009461
0	0.007547	0.031203	0.0003599	0.106377	-0.009761
0	0	0.118099	-0.006942	-0.003413	0.000898
0	0	0	0.000290	0.000840	-0.000291
0	0	0	0	0.066525	-0.004100

Table 7.1: Suggested parameter values for the Smith-Southall model

7.5 Case Study

In this section we apply the Smith-Southall model to the problem of risk management in defined benefit pension schemes. We will follow the framework presented in Chapter 5.

The main aim of this chapter is to consider the effect of model risk on the optimal investment and funding strategies. Hence we replace the Wilkie model in order to assess the extent of the effect of the chosen model on the results. We thus let all aspects of the economic uncertainty surrounding a pension scheme to be governed by the chosen investment model.

7.5.1 Applying the Smith-Southall model

In applying an investment model to the stochastic modelling of a defined benefit pension scheme, there are several aspects that ought to be considered.

Firstly, we need to decide on how inflation, both price and wage, will be treated. Our approach is to treat inflation as a stochastic process. The Smith-Southall model generates output for the Retail Price Index, $Q(t)$. Thus we set the force of price inflation at time t, $I(t)$, as

$$I(t) = log\Big(Q(t)/Q(t-1)\Big). \tag{7.16}$$

However, the Smith-Southall model does not generate output for wage inflation. We circumvent this problem by setting the average earnings index at time t, $W(t)$, as

$$W(t) = \alpha_{wage} I(t) + \mu_{wage} \tag{7.17}$$

where α_{wage} and μ_{wage} are non-negative parameters.

This is a fairly basic approach to modelling wage inflation which is based on the fact that generally wages tend to increase in line with prices. We adopt this approach in order not to disturb the general structure of the Smith-Southall model.

We set α_{wage} and μ_{wage} equal to 1.0 and 1.5%, respectively. The estimate for μ_{wage} comes from a consideration of the long-term mean parameter for wage inflation (WMU) under the Wilkie model. Wilkie (1995) suggests an estimated value of WMU of 1.6%. We note that setting α_{wage} equal to zero would be similar to assuming that wages grow at a constant rate.

The second aspect that ought to be considered in stochastic modelling is how to treat salary-related liabilities. The basic problem that arises in the treatment of these liabilities is that they are not marketable and hence we can not attach a unique market price. For more on this see Appendix A of Haberman et al. (2003a).

Our approach in valuing liabilities is to assume a discontinuance basis and use the current real yield on long-term index-linked bonds as the valuation rate of interest. Thus in the Smith-Southall model we consider the spot yield on inflation-indexed bonds.

As shown by Smith and Southall (2001, p.4), let $P^{ILG}_{t:t+u}$ be the price at time t of an index-linked zero-coupon bond of term u. Then the real spot yield at time t is

$$\frac{1}{\left(P^{ILG}_{t:t+u}\right)^{1/u}} - 1. \tag{7.18}$$

In our analysis we consider a 20-year bond. This was chosen in order to correspond with our longest projection horizon.

The third aspect in stochastic modelling concerns the generation of returns from different asset classes. In our case we only consider equities and bonds. The Smith-Southall model generates output for returns for both of these asset classes.

We recall that in the Wilkie model we use the 'rolled-up' equity index at time t, $PR(t)$, and at time $t+1$, $PR(t+1)$. In this case the return on equities between t and $t+1$ is calculated as $PR(t+1)/PR(t)-1$.

The equivalent of this in the Smith-Southall model is the formula we illustrate in Equation 7.8 and by Smith and Southall (2001, p.4). Thus we set

$$\frac{PR(t+1)}{PR(t)} = \frac{\zeta_{t+1}^{equity}}{\zeta_{t+1}^{cash}} \frac{\zeta_t^{cash}}{\zeta_t^{equity} P_{t:t+1}^{equity}}. \tag{7.19}$$

In the case of the return on bonds, we consider zero-coupon bonds of a 20-year term. A more rigorous approach would be to construct fixed-interest bonds by considering one-year zero-coupon bonds.

The return between time t and time $t+1$ on the 20-year zero-coupon bond can be established as follows. At time t the price of the 20-year bond is $P_{t:t+20}^{cash}$. At time $t+1$, assuming no arbitrage opportunities, this bond should have a price of $P_{t+1:t+19}^{cash}$.

Hence, as shown by Smith and Southall (2001, p.4), the return from t to $t+1$ is

$$\frac{P_{t+1:t+19}^{cash}}{P_{t:t+20}^{cash}}. \tag{7.20}$$

7.5.2 Results of Stochastic Projections

We consider stochastic projections over two projection periods: a medium-term horizon of 6 years and a long-term horizon of 15 years. In each case we let the Normal contribution rate and the asset allocation to be free variables. These are fixed at the outset (that is, at time 0) by the decision-maker.

The asset allocation is re-balanced annually until the end of the projection horizon. Meanwhile, the recommended contribution rate is fixed for a three-year period. At the end of the three years a new contribution rate is recommended depending on the gains or losses in the scheme.

We calculate the Mean Shortfall risk and the Excess Contribution rate risk for various combinations of Normal contribution rate and asset allocation. This leads to the idea of indifference curves showing combinations which lead to a similar level of risk.

Projections over 6 years

In Figure 7.1 we show the indifference curves for the Mean Shortfall risk and Excess Contribution rate risk for the Smith-Southall model (top left and bottom left diagrams) and the Wilkie model (top right and bottom right diagrams) at the end of 6 years.

As explained in Chapter 6, the lines PQ and LM pass through the efficient asset allocations under the Mean Shortfall risk and the Excess Contribution rate risk, respectively.

Firstly, we observe that changing asset models does not affect the general results. For a given asset model and risk measure we get a set of efficient asset allocations which lead to the lowest risk for a given Normal contribution rate. Also, for a given asset allocation, both the Mean Shortfall risk and the Excess Contribution rate risk decrease as the Normal contribution rate increases, and vice versa. These general result holds under the Wilkie model and under the Smith-Southall model.

Secondly, we observe that the Smith-Southall model leads to efficient asset allocations with a significantly higher weighting in bonds than the Wilkie model. Thus, for example, the efficient asset allocations under the Mean Shortfall risk are in the region of 20% equities and 80% bonds for the Smith-Southall model. Meanwhile, for the Wilkie model the efficient asset allocations under the Mean Shortfall risk are in the region of 50% equities and 50% bonds. A similar result is obtained under the Excess Contribution rate risk although the weighting in bonds is not significantly different under the two asset models.

A further observation concerns the differences in the efficient asset allocations under the Mean Shortfall risk and the Excess Contribution rate risk. For this we consider the lines PQ and LM. The difference in the location of PQ and LM is much wider under the Wilkie model than under the Smith-Southall model.

This implies that we get a lower trade-off region in the analysis of asset allocations that are efficient under both risks in the case of the Smith-Southall model than in the case of the Wilkie model.

Projections over 15 years

The general results obtained for projections over 6 years also hold for projections over a longer period. But an additional result is that the weighting in equities in the efficient asset allocations increases for both asset models.

Figure 7.2 shows the efficient regions for projections over 15 years for the two asset

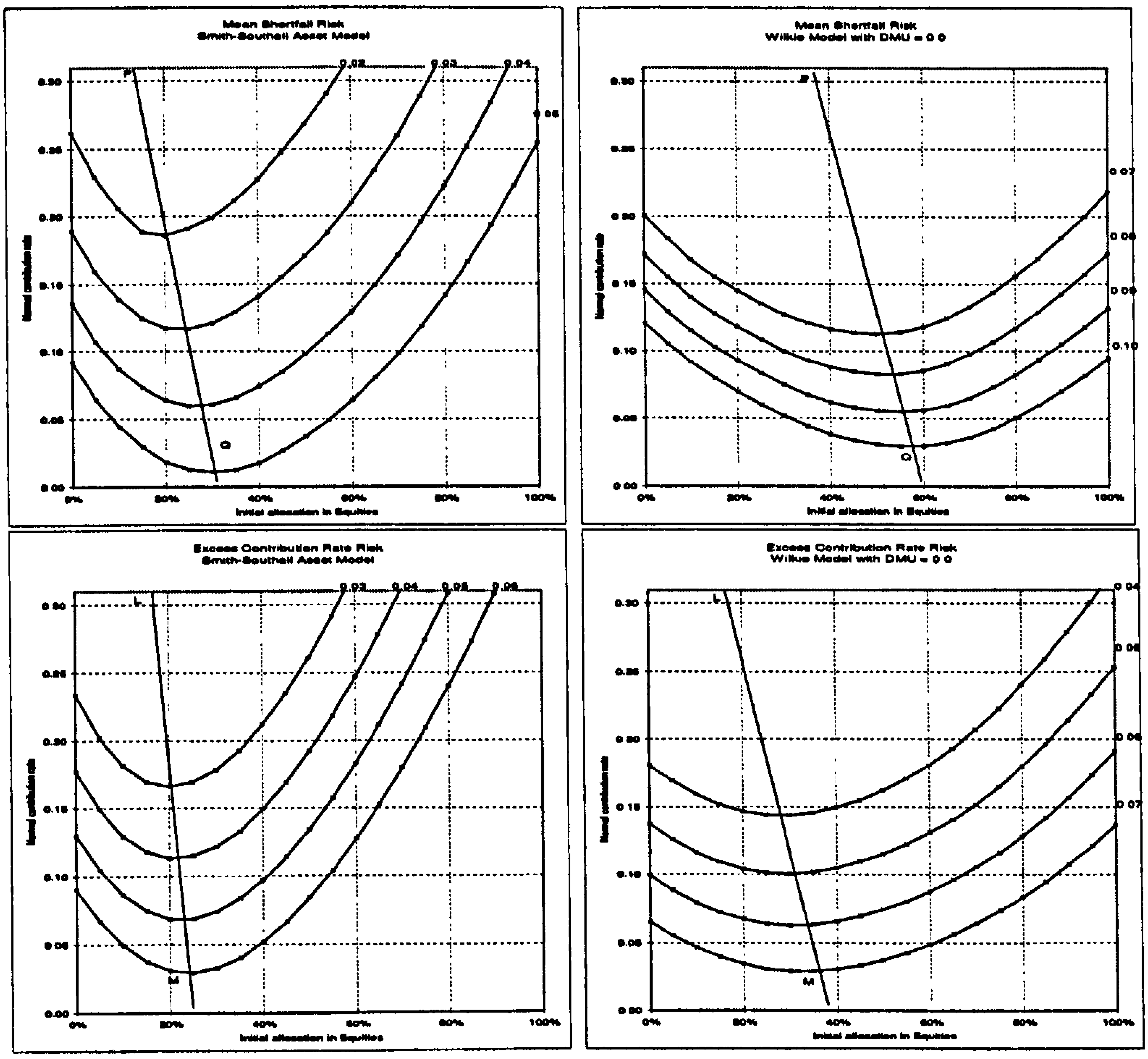

Figure 7.1: Mean Shortfall and Excess Contribution Rate risk levels at the end of 6 years for the Smith-Southall model and the Wilkie model

models. The idea of an efficient region was introduced and discussed in Chapter 6.

Firstly, we observe that for both asset models the lines LM and PQ shift towards higher allocations in equities for 15-year projections than for 6-year projections. Thus the general result of investing more in equities as the projection horizon increases holds under both asset models.

However, as for the 6-year projections, the efficient allocation in bonds is higher under the Smith-Southall model than under the Wilkie model. Also the efficient region is more tightly defined under the Smith-Southall model leading to higher definiteness in the choice of the efficient asset allocations.

7.6 Effect of Model Uncertainty

We can conclude from the results in the previous section that the choice of the asset model has no effect on the general results. That is, for both asset models we get efficient asset allocations and efficient regions. Secondly, as the Normal contribution

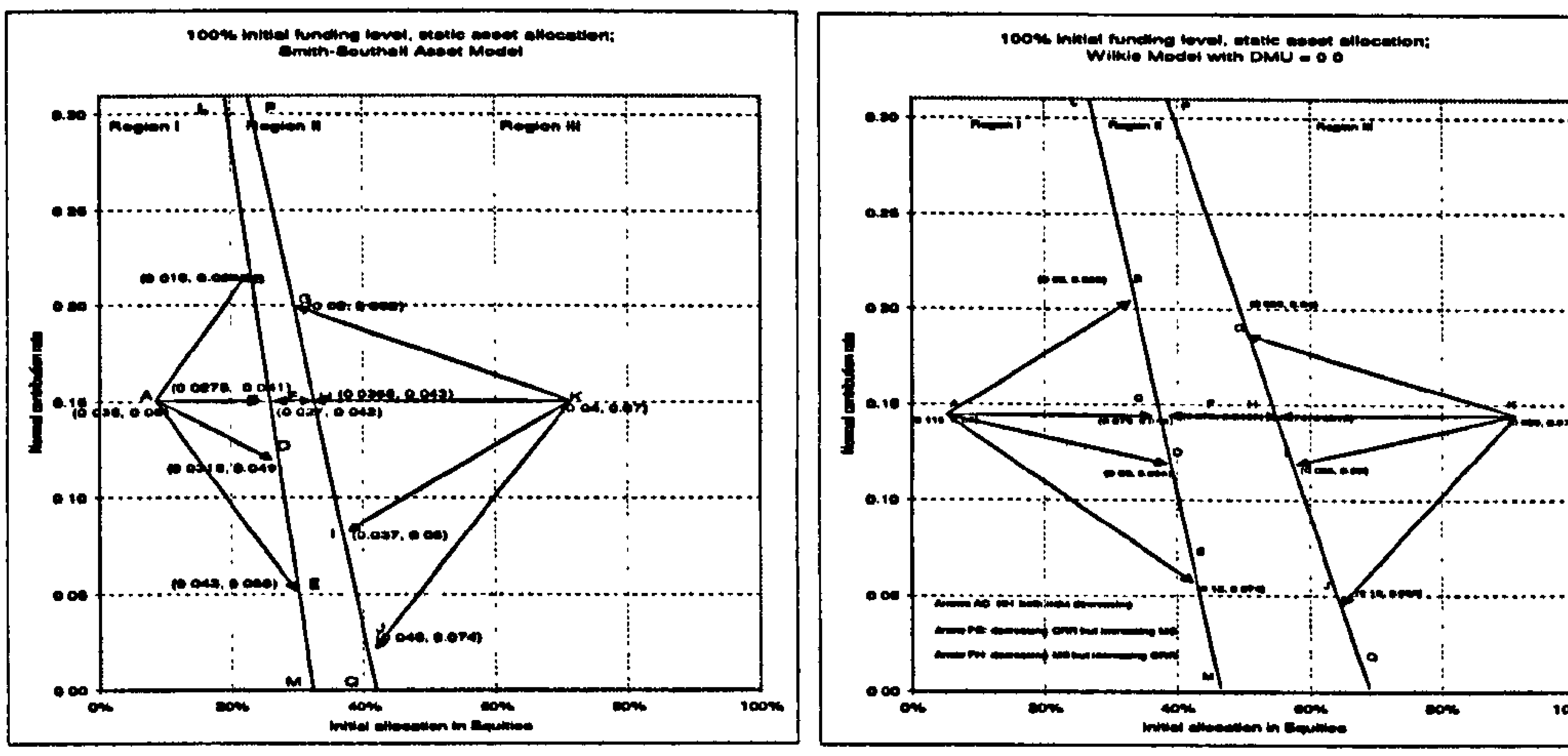

Figure 7.2: Efficient regions for projections over 15 years for the Smith-Southall model and the Wilkie model

rate increases the risk (Mean Shortfall or Excess Contribution rate) decreases. And furthermore, the weighting of equities in the efficient asset allocations increases as the projection horizon increases.

However, for a given projection horizon, the asset model has an effect on the location of the efficient regions. These depend on the characteristics of the asset model. The higher bond weighting for the Smith-Southall model could also be understood in terms of the calibration. This model was calibrated during an economic downturn. The effect of this could be assessed by considering the results obtained when the model is calibrated during an economic upturn.

The shift in the efficient regions could also be explained by considering the volatility in the total returns under the two models. The volatility of the total equity return is 20% under the Smith-Southall model (see Smith and Southall (2001, p.7)). However, under the Wilkie model the volatility reduces with increasing term (see Wilkie (1995, p.904)). Hence, even though the volatility of the return on bonds also reduces with term, equities will tend to be favoured under the Wilkie model for long-term projections.

Another factor that could be considered is the equity risk premium under the two models. This is approximately 3.5% under the Smith-Southall model. However, our choice of parameter values for the Wilkie model (i.e. DMU = 0) implies that the equity risk premium is not significantly different from that under the Smith-Southall model[1]. Thus the differences in the results for the two models could be due to the equity risk premium but not to a large extent.

[1]See Appendix D for a further analysis of the effect of the equity risk premium in the Wilkie model on the main results

Chapter 8

Risk Management: Allowing for Probability of Default

8.1 Introduction

Asset and Liability modelling in defined benefit pension schemes has been the focus of extensive actuarial research over the past decade as noted in Chapter 3. This has involved modelling asset returns and liability growth as stochastic processes. Thus various investment models have been suggested in the actuarial literature and employed in the choice of optimal asset allocations.

In most cases such asset allocations have been more weighted towards equities than fixed-interest bonds. This could be attributed to the long-term nature of the liabilities and the need to invest in assets that are a good match for salary-linked liabilities.

In recent times there has been a considerable emphasis on the suitability of a corporate finance-based approach to the asset allocation for defined-benefit scheme funds. One of the main ideas proposed is that equities are a sub-optimal asset class when it comes to pension fund investment. Several reasons are cited to back up this idea: Modigliani-Miller irrelevance proposition; failure to prove the equity-salary link; and, in some cases, the idea that it is not necessary to consider future salary growth rates since sponsoring employers do not make such an allowance in their corporate balance sheets.

It has been argued that pension fund modelling ought to address the financial circumstances of a pension scheme's sponsoring employer by recognising the financial strengths or weaknesses of the sponsor in relation to equity markets. The underlying argument here is that periods of weak equity markets could coincide (or lead to) large pension fund deficits and hence increased scheme contributions at a time when the sponsor might be financially weakest. And thus incorporating probability of sponsor default explicitly in the risk measurement could lead to optimal asset allocations with low (or even zero) weighting in equities.

In particular, it is suggested that optimizing pension fund asset allocation using stochastic asset-liability modelling to project future long-term cash flows underestimates the risks inherent in equity investment as a result of the increased volatility of return. It is suggested that by ignoring the risk of bankruptcy of the sponsoring employer, stochastic asset-liability modelling leads to bias towards higher-risk equity investment when determining asset allocations.

For a defined-benefit pension fund invested largely in equities, consideration of the risk of bankruptcy of the sponsoring employer is therefore important as the employer may well be required to make large additional contributions (to reduce a funding deficit caused by poor asset returns) at the very time when he is financially least able to do so.

These arguments and other factors (including the prolonged downturn in global equity markets) have led to some UK pension funds shifting from the traditional largely equity-backed asset allocations to asset allocations which are increasingly weighted towards bonds.

As a consequence, some pensions actuaries now advocate bond-only investment strategies, asserting that any additional expected return that the shareholders of the sponsoring employer could have achieved from exposure to equities should be gained directly by the shareholder through re-arranging his/her portfolio.

In Chapter 5 we presented a framework for risk management and measurement for defined benefit pension schemes. This framework involved three main areas: stochastic projections of the pension fund over different horizons; simultaneous analysis of funding and investment strategies - thus treating both the contribution rate and the asset allocation as free variables; and the use of downside risk measures at key decision points. In so doing we have assumed the survival of the firm up to these decision points and hence ignore the effect of intertemporal probability of firm bankruptcy (and hence default).

The need to incorporate market information in a risk measure is well recognized in the financial literature. For instance, in their criticism of the VAR (value-at-risk), Aït-Sahalia and Lo (2000) observe that a measure of risk ought to include the fact that "...a one dollar loss is not always worth the same, and that circumstances surrounding the loss can affect its economic valuation, something that is completely ignored by purely statistical measures of risk [like the VAR]..."

In this chapter we extend the framework of Chapter 5 in order to assimilate some of these ideas. Optimal funding and asset allocation strategies, over a range of projection horizons, are determined by taking into account the probability of default by the sponsoring employer.

This chapter is organized as follows: In Section 8.2 we review the first Modigliani-Miller proposition and its early and recent applications to defined benefit pension schemes. Then in Section 8.3 we introduce a framework for risk management in defined benefit pension schemes which allows for probabilities of sponsor default. In Section 8.4 we present a case study and the main results for three projection periods. In Section 8.5 we summarize our main findings. In Appendix F we investigate the effect on our results of the frequency of pension scheme valuations and the period over which gains and losses are spread.

8.2 Modigliani-Miller Propositions

8.2.1 Proposition 1

The Modigliani-Miller propositions were originally set out in a 1958 paper entitled "The Cost of Capital, Corporation Finance and the Theory of Investment". Modigliani and Miller (1958) developed a theory for the market value analysis of a firm's cost of capital problem. To this end they consider a firm's financial structure and propose that, within a framework of partial equilibrium, "...the market value of any firm is independent of [the firm's] capital structure..." In other words, a firm's market value is independent of whether the firm is financed by bonds or equities.

Modigliani and Miller (1958) use arbitrage arguments to prove this proposition under the assumption that firms can be divided into risk classes in such a way that within each class the firms' cash flows are "perfectly correlated" and hence "perfect substitutes". Furthermore they assume that bonds and equities are traded in perfect markets where perfect substitutes have the same price.

Stiglitz (1969) developed this proposition further within a general equilibrium framework. He argued that some of the original assumptions, including the division of firms into risk classes, could be relaxed. He thus showed that the Modigliani-Miller proposition holds as long as there is no probability of bankruptcy by the firm and that individuals and firms can borrow at the same rates.

The Modigliani-Miller propositions and the underlying assumptions have been subjected to intense criticisms ever since they were first published. However, as noted by Stiglitz (1988, p. 122), some of these criticisms have lead to more "productive responses" to the propositions.

Stiglitz (1988) lists a number of instances, for example information asymmetries, where the assumptions might lead to a failure of the Modigliani-Miller proposition. Cuthbertson (1996, p. 186) observes that the proposition might fail if some market participants are "noise" traders. Eichberger and Harper (1997, p. 163) also observe that the proposition might fail if there is a probability of bankruptcy and the firm is a limited company.

On the empirical side, Miller (1988, p. 100) observes that the "empirical significance" of the proposition is less clear in corporate finance.

8.2.2 Application to Corporate Pension Funding

Tepper and Affleck (1974) is one of the earliest applications of the Modigliani-Miller proposition to corporate pension funding. They develop analogies between the Modigliani-Miller analysis of a firm's capital structure and a firm's funding of the corporate pension plan. One of their main assumptions is that bonds and equities are taxed at similar personal rates. In this framework bankruptcy costs are not introduced.

Tepper (1981) observes that such assumptions present a 'troubling framework' not just because in such a scenario it is irrelevant to consider pension fund investment as being financed by equities, but also because "...the limitations of the [Modigliani-Miller] framework as a robust theory of capital structure..." imply that "...any conclusions regarding corporate pension policy which are based upon it should be suspect..."

Thus Tepper (1981) applies a general equilibrium analysis to show the tax advantages of investing pension fund in bonds rather than equities, if bonds are taxed at a higher personal rate than equities. His main assumption is that the pension fund can be considered as part of the personal portfolio of a shareholder of the sponsoring company.

Black (1980) also employs taxation arguments, albeit at a corporate level, to show that it is advantageous to invest the pension fund in bonds rather than equities. He presents a two-part idea where pension fund investment is shifted from equities to bonds coupled with an issuing of bonds and share buy-backs by the corporation.

Smith (1996) applied the Modigliani-Miller proposition in the context of UK actuarial asset-liability modelling. Smith (1996, p. 1073) observes that investment strategies determined through actuarial asset-liability studies cannot add value to a firm's shareholders except possibly in the circumstances where the assumptions underlying the proposition do not hold. To this end he considers the effect of transaction costs.

He observes that "...in a Modigliani Miller world with no transaction costs, all markets have the same expected return after adjusting for risk, and hence there is no merit from a shareholder value perspective in participating in one market rather than in another..." (p.1092).

He concludes that in a world with transaction costs several complications arise. For instance, the concept of a shareholder value may be hard to define. Furthermore, it may no longer be possible to determine asset value by the market price but rather a range of prices in the bid-ask spread.

Exley and Mehta (1996) apply the Modigliani-Miller proposition to the context of defined benefit pension schemes. They argue that "...in the context of pension fund investment, [the Modigliani-Miller proposition] can be verified by analogy with an investor holding a short position in a corporate bond, which we equate with a pension liability...[T]he value of this short position (superficially similar to the market cost of the liability) can [not] be reduced or increased by the investor reallocating his portfolio of assets to or from equities. If assets are traded at market value then, (absent transaction costs), portfolio value at any point in time...must be the same, regardless of the choice of allocation [in equities or bonds]..." (p.641)

Hence, Exley and Mehta (1996) consider the problem of finding optimal asset allocation and optimal funding strategies for a model pension scheme taking into account the possibility of corporate bankruptcy. They assume that bankruptcy risk is only a function of Poisson jumps in the equity market and that such jumps occur "...on average once in every five years..." Bankruptcy risk is calculated by taking into consideration scenarios where the value of the firm falls below a given threshold and also where the pension fund falls below a statutory minimum funding threshold. Pension funding is treated as a "zero-sum-game" between the various stakeholders (in this case shareholders and trustees) and the optimal investment strategy from the perspective of the shareholder is taken to be one that minimizes the shareholder's loss in the event of corporate bankruptcy.

They conclude that under their model the optimal asset allocation in equities would be a function of four factors: the value of the firm and the threshold for the firm, the firm's beta (or sensitivity to market returns), and the initial value of the pension fund and its statutory threshold.

The application of the Modigliani-Miller proposition to defined benefit pension schemes is explored further in Exley et al. (1997). They argue that in a world without transaction costs the economic cost of the pension to the sponsor does not depend on the pension fund's investment strategy. They class this as a "first order" result and they advocate that "second order effects" ought to be considered in order to determine optimal funding or asset allocation strategies. These second order effects include the sponsor's credit risk, the effect of future benefit improvement, taxation effects, and corporate bankruptcy risk.

They conclude, for instance, that if one considers the sponsor's credit risk then the pension cost would be affected by the investment strategy. One of the reasons being that if the pension fund is invested in the sponsor's assets (i.e. self-invested) then in the event of the sponsor becoming bankrupt members of the scheme would suffer loss. Thus a good investment strategy would be to avoid self-investment.

Chapman et al. (2001) focus, in the light of the Modigliani-Miller proposition, on the interactions between the various stakeholders in a defined benefit scheme setting. They model the pension scheme not as a single-entity but as part of the sponsor's balance sheet and a "closed economic system" with several stakeholders. Their methodology centres on assessing the effects of pension funding decisions on the stakeholders.

They conclude, for example, that shifting the investment strategy of the pension fund to a bond-backed strategy would have the effect of reducing the chances that the sponsor would become insolvent by as much as half in their proposed model. Thus such a strategy would be beneficial to, among others, the shareholder.

These ideas are also considered by Whelan et al. (2002) and Day (2003).

8.3 Framework for stochastic projections with probabilities of sponsor default

We propose a framework for stochastic pension fund projections that takes into consideration the probability of sponsor default. Let T be the period (in years) over which stochastic projections of a pension fund $f(t)$ are to be undertaken. In the approach presented in Chapter 5 and in Haberman et al. (2003a) we implicitly assume that the sponsor does not default before the end of the projection period i.e. at any time t, $t < T$. In this chapter we extend this approach by assuming that there is a chance that the sponsor could default at the beginning of each year just before the next contribution is due.

8.3.1 Formulation of the Probability of Default

Our approach is in some ways similar to that of Boyce and Ippolito (2002). They investigate the cost of pension insurance under the US PBGC. Since under the

PBGC a claim only arises in the event of sponsor bankruptcy, they propose that the probability of bankruptcy for a given company be modelled as:

$$q_t = 1/\left\{1 + \exp(-a - \sum_{j=1}^{J} b_j\, x_{j,t-1})\right\} \tag{8.1}$$

where the factors $x_{j,t}$ include the company's leverage ratio, the cash-flow-to-asset-ratio, the pension funding ratio, the level of employment and dummy variables dependent on the company's industrial sector. (also see Hardy (1996) for a formulation of the probability of insolvency for a life insurer).

The underlying idea in our analysis is that the state of the economy could have a significant effect on the strength of the sponsoring employer and the likelihood of honouring the scheme contributions. That is, weak financial markets could lead to large pension scheme deficits at a time when the sponsor is financially weak and thus unable to meet with large additional scheme contributions.

We thus use the movements in the financial markets as a signal of the strength of the scheme sponsor and the prospect of default.

Cochrane (2001, p. 150) observes that the return on a broad-based portfolio can be used to measure the state of the economy. He further suggests that, in general, any factor that is related to consumption can be used as a measure.

This implies that the probability of default ought to have the form:

$$\mathsf{Prob[default]} = \frac{1}{[1 + p\, e^{\alpha r_b}]} \tag{8.2}$$

where p is a given constant, α is a risk aversion coefficient and r_b is the force of return on a broad-based portfolio.

Equation 8.2 has the desirable property that the probability of default tends to one in bad times (i.e. when the return on the market portfolio is very poor); whilst the probability tends to zero in good times.

In our analysis we will approximate the broad-based portfolio by considering the market portfolio consisting of equities and bonds. In a Markowitz mean-variance framework, the market portfolio (or tangency portfolio) is defined simply as the point of tangency between the Markowitz efficient frontier and the line from the risk-free rate (also known as the capital market line).

We determine the constant p by looking at the possible values of the probability of default when the force of return is equal to its expected (or mean) value. Meanwhile α could be determined by assuming that we have an investor with an exponential utility function with absolute risk aversion coefficient α and that the asset returns are normally distributed.

Cochrane (2001, pp. 154-155) uses these assumptions to derive the Capital Asset Pricing Model (CAPM). He shows that under this version of the CAPM we obtain a link between the coefficient of absolute risk aversion and the market price of risk (i.e. the price of systematic risk):

$$\alpha\,\sigma^2(r_m) = E(r_m) - \delta_f \tag{8.3}$$

where r_m is the force of return on the market portfolio, $\sigma^2(r_m)$ is the variance of the force of return and $E(r_m) - \delta_f$ is the price of systematic risk with δ_f as the risk-free force of return.

We thus propose the following formulation for the probability of default over one year, given that the force of return on the market portfolio is r_m:

$$q(r_m) = \frac{1}{1 + p_0 + \exp\{p_1 + \alpha\,(r_m - \delta_m)\}} \tag{8.4}$$

where

- r_m is the force of return on the market portfolio;
- $\delta_m = E(r_m)$, the expected force of return on the market portfolio;
- p_0 is a non-market-linked component to do with risk specific to the sponsor's business;
- p_1 is such that if r_m is equal to its expected value then the default probability is just $\frac{1}{1+p_0+\exp\{p_1\}}$; and
- α is as defined in Equation 8.3.

8.3.2 Parameter Estimates for the Probability of Default

Market Portfolio

In our analysis we assume that the investment returns are as generated under the Wilkie (1995) model and we adopt Wilkie's recommended parameter estimates and 'neutral' initial conditions. We assume that the pension fund can only be allocated in two asset classes: equities and fixed-interest bonds. As mentioned in a previous section, we assume that the market portfolio only contains these two asset classes.

We estimate the market portfolio over the first year and, for simplicity, we use the same market portfolio for all other years. We assume a risk-free rate of interest of 6.16%. This is equivalent to the mean nominal return on cash over one year in the

Wilkie Model. We estimate the mean nominal returns on equities and bonds over one year as 12.64% and 7.641%, respectively. Then the market portfolio comprises 56% equities and 44% bonds, with an expected rate of return of 10.44% and standard deviation of 0.128.

Estimation of α

To estimate α in Equation 8.3 we need the expected value and the variance of the force of return on the market portfolio. We thus use a lognormal approximation to obtain:

$$\alpha = \frac{0.0927 - \ln 1.0616}{0.01325} \approx 2.49. \tag{8.5}$$

General Conditions on $q(r_m)$

As observed before, our formulation of the probability of default in Equation 8.4 has the desirable property that the probability of default tends to one in poor economic times and tends to zero in good times. Furthermore by setting the force of return equal to its expected value we get

$$q(\delta_m) = \frac{1}{1 + p_0 + \exp\{p_1\}}. \tag{8.6}$$

We then use Equation 8.6 to estimate the values of p_0 and p_1. In this paper we only consider the case where the probability of default has no non-market-linked component i.e. $p_0 = 0$. The effect of the non-market-linked component will be investigated in a future paper.

In referring to general company debt, Crosbie and Bohn (2002) observe that "...[t]he typical firm has a default probability of around 2% [0.02] in any year. However, there is considerable variation in default probabilities across firms. For example, the odds of a firm with a AAA rating defaulting are only about 2 in 10,000 [0.0002] per annum. A single A-rated firm has odds of around 10 in 10,000 [0.001] per annum, five times higher than a AAA. At the bottom of the rating scale, a CCC-rated firm's odds of defaulting are 4 in 100 [0.04], 200 times the odds of a AAA-rated firm..."

We can infer from this observation that in our analysis the probability of default, when the force of return is equal to its expected value, should not be greater than 0.02.

Hence from Equation 8.6, with $p_0 = 0$, we set

$$\begin{aligned} q(\delta_m) &= 0.02 \\ \Rightarrow 1 + \exp\{p_1\} &= \frac{1}{0.02} \\ \Rightarrow p_1 &\approx 3.89. \end{aligned} \tag{8.7}$$

8.3.3 Effect of Market movements on Default Probabilities

As observed earlier the specification of $q(r_m)$ in Equation 8.4 means that the probability of default is an inverse function of the return on the market portfolio. Figure 8.1 shows the probabilities of default over the first year given the rate of return i on the market portfolio (using our parameter estimates). The force of return is calculated as $\ln(1+i)$.

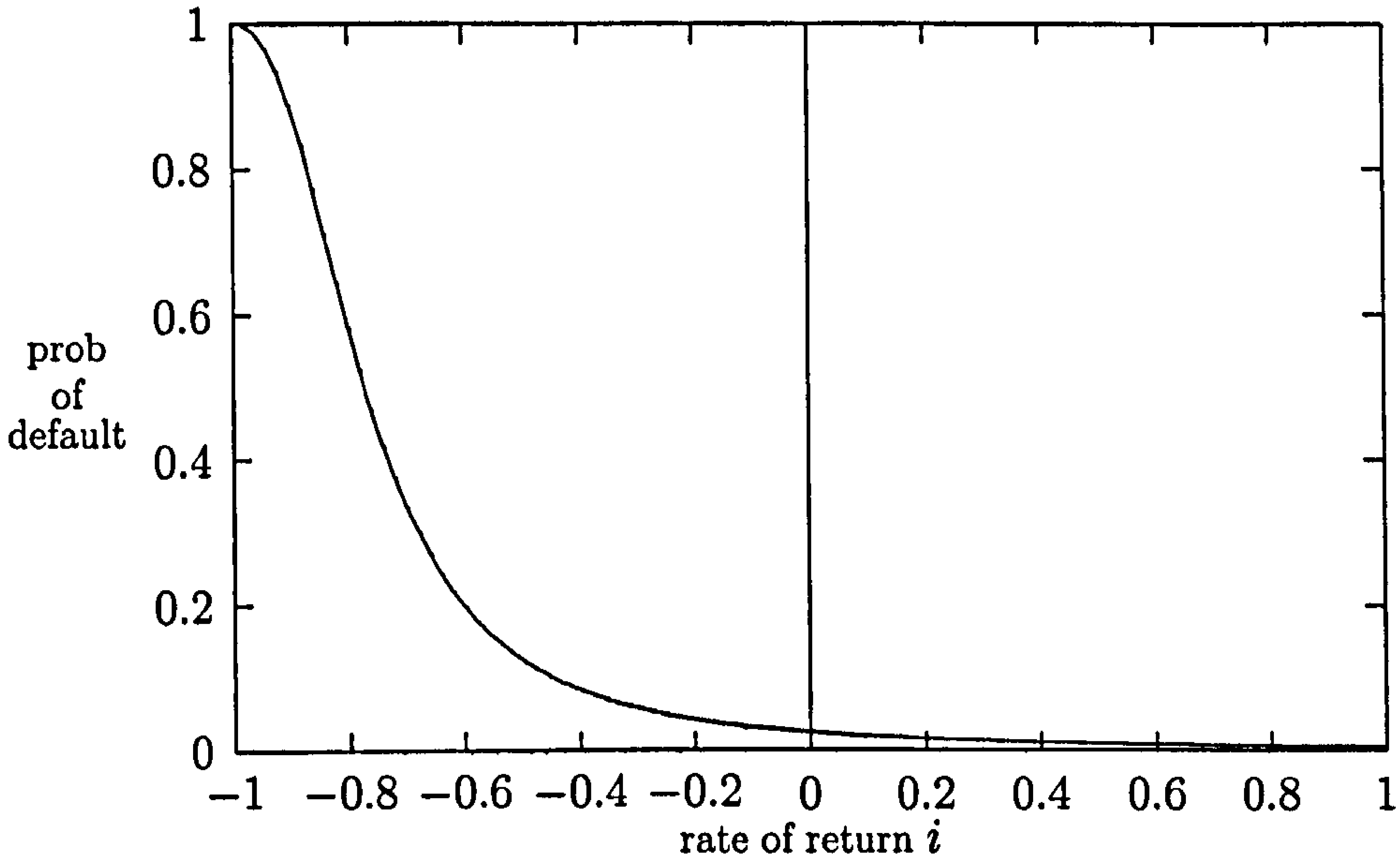

Figure 8.1: Probability of default over the first year for various rates of return i (force of return = $\ln(1+i)$)

This figure shows that if the market rises by 10% then the probability of default is approximately 0.02 and if the market falls by 20% then the probability of default is approximately 0.04. Meanwhile, if the market falls by 50% then the probability of default is approximately 0.13. And, if the market falls by 80% then the probability of default is approximately 0.60.

8.3.4 Risk Measurement

Mean Shortfall risk

This risk measure was introduced in Chapter 5 (also see Haberman et al. (2003a,b)). Under this risk measure we assume that the sponsor does not default before the end of the projection period. At the end of the projection period we calculate the average solvency deficit whilst treating all surpluses as zero.

Let $V(T)$ be the solvency deficit at the end of T years. Then

$$V(T) = \begin{cases} L(T) - f(T) & \text{if there is a deficit at } T \\ \\ 0 & \text{otherwise.} \end{cases} \tag{8.8}$$

The Mean Shortfall risk at the end of T years, $MS(T)$, is calculated as:

$$MS(T) = \frac{1}{f(0)} E\Big[V(T)\,\Big|\, f(0), L(0), c(0)\Big] \tag{8.9}$$

where $f(t)$ is the market value of the assets at time t, $L(t)$ is the discontinuance liability at time t and $c(t)$ is the contribution rate at time t. In this formula $\frac{1}{f(0)}$ acts as a scaling factor.

This risk measure is of the form

$$\text{Prob(Shortfall)} \times \text{E}\Big[\text{Shortfall}\,\Big|\,\text{Shortfall has occurred}\Big]. \tag{8.10}$$

In Chapter 5 (and in Haberman et al. (2003a,b)) we consider the Mean Shortfall risk in tandem with an Excess Contribution Rate risk measure. For a given projection period and normal contribution rate the Excess Contribution Rate risk is also a downside risk measure like the Mean Shortfall risk but is defined as the sum of discounted differences between the recommended contribution rate and the normal contribution rate, treating all negative excesses as zero.

However, in this chapter we focus only on optimal asset allocations for the Mean Shortfall risk and not on 'Efficient Regions'.

Shortfall & Default risk

The second risk measure we consider is the Shortfall & Default risk. We define the Shortfall & Default risk at the end of T years, $SD(T)$, as the expected amount of deficit defaulted by the end of the projection period and we calculate it as follows.

Consider a projection period of T years. Under the Mean Shortfall risk we would only concern ourselves with the discontinuance liabilities, $L(T)$, and the market value of assets, $f(T)$, at the end of the projection period. However under the Shortfall & Default risk we would take into consideration the possibility that the sponsor might default before the end of the projection period.

We assume that default events only arise at the end a given year, for example year $(t-1, t)$, if there is a shortfall (i.e. $L(t) - f(t) > 0$) and the probability of default $q(r_m^{(t)})$ is greater than some randomly generated uniform random variable $U(t)$. This approach is similar to that of Boyce and Ippolito (2002, p. 133).

If the sponsor defaults at time t_d, say, then we accumulate the shortfall defaulted to the end of the projection period using the risk-free force of interest. Thus at the end of the projection period the amount of shortfall defaulted would be $\left(L(t_d) - f(t_d)\right) e^{(T-t_d)\delta_f}$.

Let $D(T)$ be the amount of shortfall defaulted at the end of T years. Then

$$D(T) = \begin{cases} \left(L(t_d) - f(t_d)\right) e^{(T-t_d)\delta_f} & \text{if default event occurs at } t_d \leq T \\ \\ 0 & \text{otherwise.} \end{cases} \tag{8.11}$$

Clearly if the default event arises at the end of the projection period (i.e. $t_d = T$) then the amount of shortfall defaulted would be $L(T) - f(T)$. In particular if the projection period is one year then

$$D(1) = \begin{cases} L(1) - f(1) & \text{if default event occurs} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{8.12}$$

The Shortfall & Default risk at the end of T years is calculated as:

$$SD(T) = \frac{1}{f(0)} E\left[D(T) \,\middle|\, f(0), L(0), c(0)\right]. \tag{8.13}$$

This risk measure has a similar form to the Mean Shortfall risk:

$$\text{Prob(Default event)} \times \text{E}\left[\text{Shortfall Defaulted} \,\middle|\, \text{Default event has occurred}\right]. \tag{8.14}$$

8.4 Case Study

8.4.1 Liability Model

We consider an ongoing Pension Scheme with current pensioners. Further details of this model are set out in Appendix B of this thesis and also Appendix C of Haberman et al. (2003a). The discontinuance liabilities are evaluated using the current real yield on index-linked bonds generated by the asset model.

8.4.2 Asset Model

We assume that the investment returns are as generated by the Wilkie model. We use the parameter estimates and 'neutral' initial conditions as set out in Wilkie (1995). We assume that the pension scheme is initially fully funded. The fund is allocated in equities and bonds and annually rebalanced. We consider different combinations, in steps of 5%, of the two asset classes (no short-selling is allowed).

8.4.3 Projections

We consider stochastic projections over 1 year, 6 years and 15 years. We choose these projection periods to stand for the short-term, medium term and long-term, respectively. We carry out 20,000 simulations and at the end of every projection period the Mean Shortfall and Shortfall & Default risks are calculated as explained in a previous section. The Shortfall & Default risk is calculated only in the case where the non-market-linked component of the probability of default is zero.

We consider 17 different choices of Normal contribution rate: 0, 2%, 4%,...,32%. Indifference curves are calculated for given risk levels and in some cases interpolations are necessary. The indifference curves show all the combinations of asset allocation and Normal contribution rate that lead to a similar level of risk. Following Chapter 5 and Haberman et al. (2003a,b) we identify efficient asset allocations by considering the minimum points of the indifference curves.

Short-term Projections

Figure 8.2 shows the Mean Shortfall $MS(1)$ and Shortfall & Default $SD(1)$ risk indifference curves at the end of one year.

We observe that the indifference curves for the Mean Shortfall risk (left diagram in Figure 8.2) have their minimum points located in the region of 15% equities (85% bonds). This implies that under the Mean Shortfall risk, for projections over one year, the efficient asset allocation is in the region of 15% equities and 85% bonds.

We further observe that the indifference curves for the Shortfall & Default risk (right diagram in Figure 8.2) have their minimum points located around 0% equities. This implies that under the Shortfall & Default risk, for projections over one year, the optimal asset allocation is 100% bonds.

We can thus conclude that for projections over one year the inclusion of probabilities of sponsor default has a major effect on the indifference curves. We have observed that the efficient asset allocations shift from around 85% bonds for the case where probabilities of default are not included (i.e. Mean Shortfall risk) to 100% bonds for the case where the default probabilities are included and are wholly market-linked.

These results support the argument that the inclusion of probabilities of default should have the effect of reducing optimal equity allocations. As shown in Section 8.3.4, in our framework default events are triggered by poor equity performance over the one-year projection period. Our indicator of the state of the economy is a broad-based portfolio which has a higher weighting in equities than in bonds. Thus poor performances by equities tend to contribute more to the probability of default.

This explains why to reduce the amount of shortfall defaulted we initially have to invest more in bonds.

For the case where the default probability comprises of a non-market-linked component and a market-linked component we would expect that, as long as the parameter estimates for the two components are comparable, the efficient asset allocation would be located between 85% and 100% bonds.

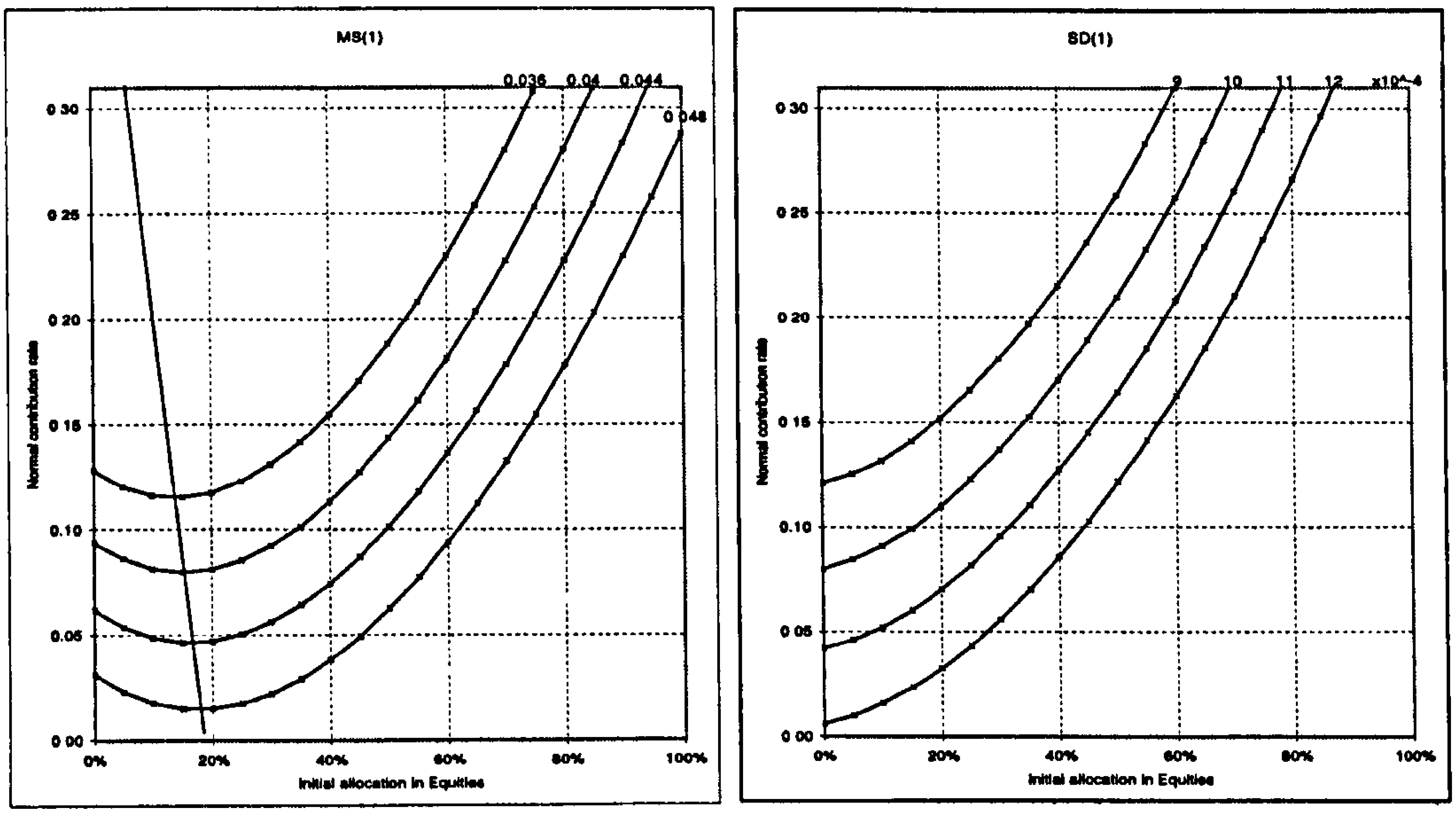

Figure 8.2: Mean Shortfall and Shortfall & Default risk levels at the end of 1 year

Medium-term Projections

Figure 8.3 shows the Mean Shortfall and Shortfall & Default risk levels at the end of a six-year projection period. As illustrated in a previous section the Shortfall & Default risk includes scenarios where the sponsor defaulted before the end of the projection period.

We observe that the minimum points for the indifference curves shift towards higher bond allocation in the cases where we include probabilities of default. For example, the optimal asset allocation under the Mean Shortfall risk are located in the region of 60% equities (40% bonds) whilst under the Shortfall & Default risk we get optimal asset allocations in the region of 40% equities (60% bonds).

Thus as for the short-term projections we observe that to minimize the expected amount of deficit defaulted we would require a higher allocation in bonds than would be required to minimize the expected shortfall. In general the optimal asset allocation for the Shortfall & Default risk contains approximately 50% more weighting in bonds than for the Mean Shortfall risk.

Therefore these medium-term results also support the argument that the inclusion of probabilities of default would lead to a reduction in the proportion of equities in the optimal asset allocation. However, compared to the short-term projections, we do not get a full weighting in bonds. This is probably due to a combination of several factors.

The first factor is that better performance by equities would lead to lower average deficits in the pension scheme. This would thus imply that we get fewer default events and hence lower Shortfall & Default risk. The full effect of this would not be appreciated for the one-year projection period due to the short time horizon. However, for medium term projections there would be sufficient time for a reduction in the average deficits. Thus equities would be seen as an optimal asset class for the pension fund.

The second factor is that shortfalls defaulted before the end of the projection period are accumulated at the risk-free rate of interest (see Section 8.3.4). We would expect a higher rate of accumulation to have an effect on the optimal asset allocations.

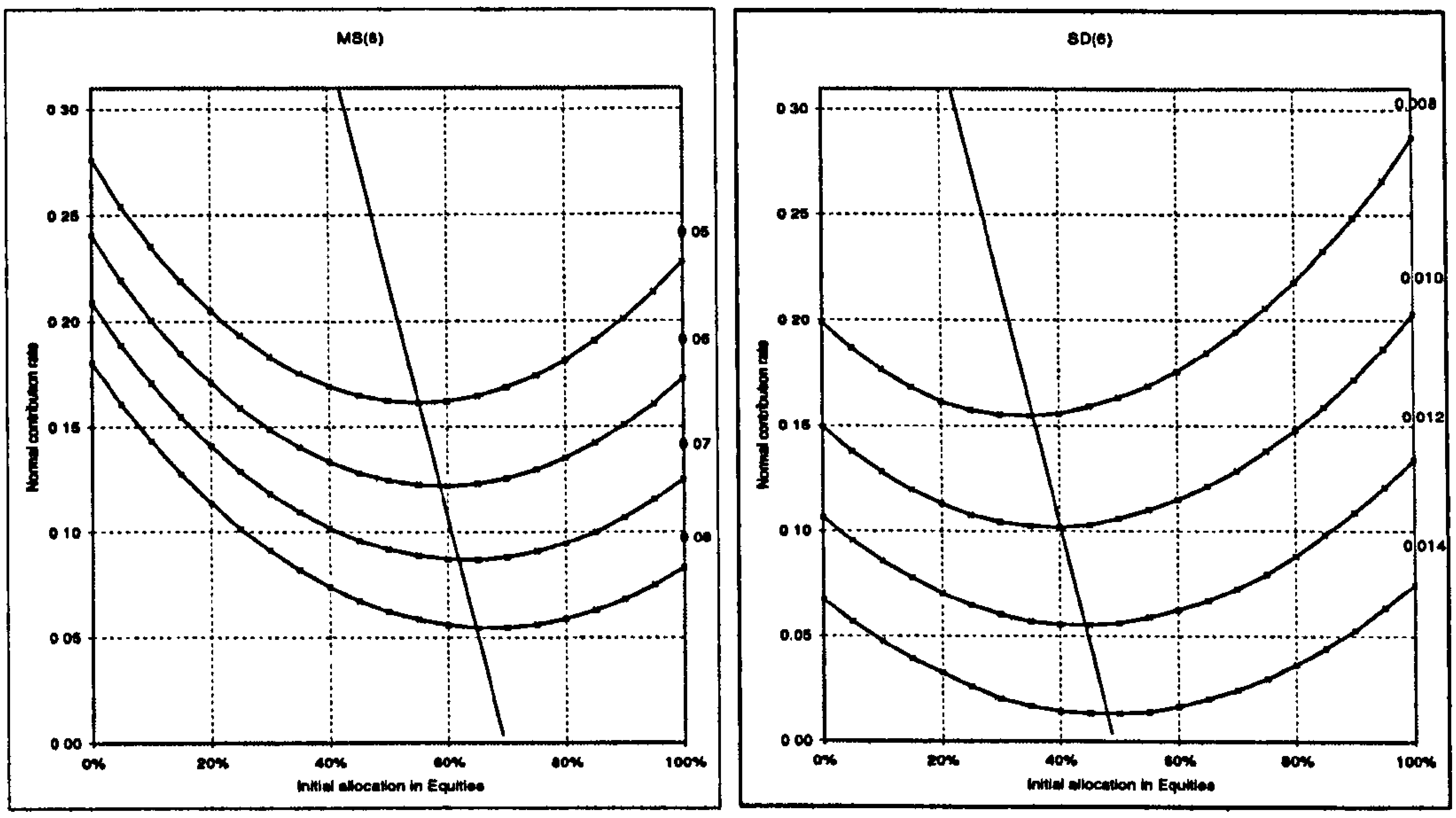

Figure 8.3: Mean Shortfall and Shortfall & Default risk levels at the end of 6 years

Long-term Projections

Figure 8.4 shows the Mean Shortfall and Shortfall & Default risk levels at the end of a 15-year projection period. The optimal asset allocation for the Mean Shortfall risk is in the region of 60% to 80% equities. Meanwhile for the Shortfall & Default risk the optimal asset allocation is in the region of 45% to 65% equities.

As for the medium-term, the long-term results also partially support the arguments about the effect of probabilities of default on the investment strategy. We get an increase in the bond-weighting under the Shortfall & Default risk; however, we do not get full weighting in bonds as was the case for the short-term results.

This can be explained, as in the case of medium-term projections, by considering the performance of equities over the projection horizon. Over the long-term the performance of equities would be expected to lead to lower deficits in the scheme. In our framework default events depend on both default probabilities and fund deficits. Thus in the long-term equities would lead to lower Mean Shortfall risk and also lower Default risk.

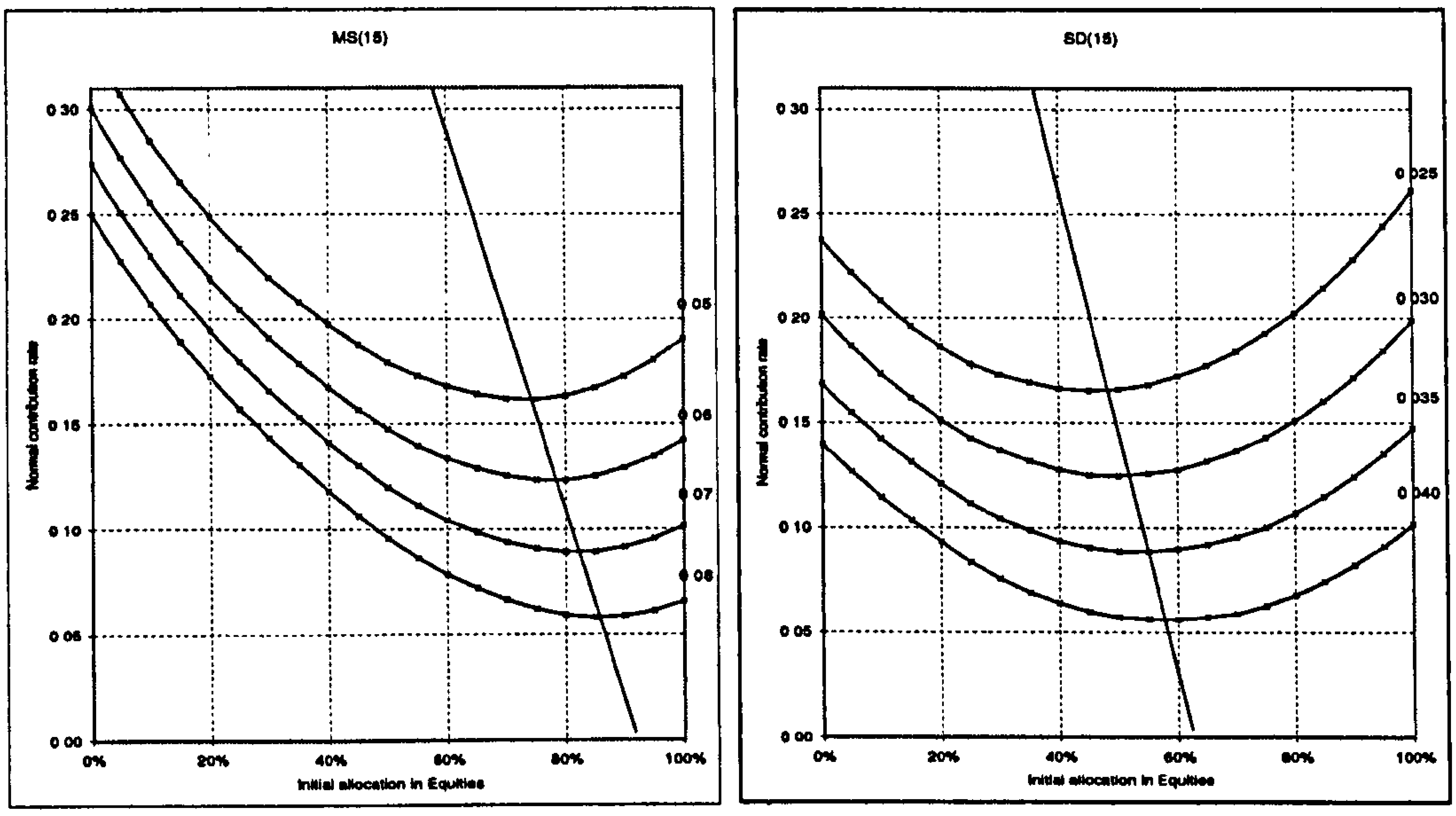

Figure 8.4: Mean Shortfall and Shortfall & Default risk levels at the end of 15 years

8.5 Concluding Remarks

In this chapter we have presented a framework showing how probabilities of sponsor default can be included in stochastic projections of a corporate pension scheme. This inclusion has lead to the development of a new risk measure: the Shortfall & Default risk measure. This risk measure is in a sense similar to the Mean Shortfall risk measure but differs significantly since it also includes instances where the sponsor defaults before the end of the projection period.

It is a standard result in corporate finance that possibility of bankruptcy by the firm could lead to the failure of the Modigliani-Miller proposition. Thus the chance of bankruptcy could have an effect on a firm's capital structure in the Modigliani-Miller world.

Our analysis shows that for short-term projections the inclusion of default probabilities leads to an optimal asset allocation of 100% bonds whilst if the default probabilities are not included we get optimal asset allocations with a lower bond weighting.

In the case of medium to long-term projection periods we have shown that the inclusion of probabilities of default also leads to optimal asset allocations with a higher bond weighting than under the Mean Shortfall risk. However, we do not get full (100%) bond investment as in the short-term case.

In the short-term, our results confirm the argument that the inclusion of probabilities of default could lead to bond-only optimal asset allocations. However, in the long-term equity-backed asset allocations are optimal regardless of whether or not probabilities of default are included in the stochastic projections.

In future we hope to extend this work in several ways. Our formulation of the probability of default includes a non-market-linked component. We hope to consider the effect of a non-zero value for such a component. An analysis of the effect of the various parameters in the asset model is also an area that we hope to consider. For example, the inflation parameters could be changed to a low and stable inflation scenario. Furthermore we hope to investigate the effect on the investment strategies of changing the liability model. For instance, the benefit structure in the current model could be changed or a closed and mature scheme could be considered.

Another interesting area that could be further investigated concerns the interest rate for accumulating the defaulted amount. In our analysis we have assumed that in the event of a default before the end of the projection period, the defaulted amount is accumulated to the end of the projection period at the risk-free rate of interest. A different rate of interest could be used in such a scenario. This could affect the results for the medium-term and long-term projections.

Chapter 9

Conclusion

In this thesis we have presented a stochastic approach to decision making in defined benefit pension schemes. We have shown how existing decision-making methods can be extended in order to fully address the objectives of the various stakeholders.

In Chapter 2 we have discussed the traditional decision making process in defined benefit pension schemes. We have shown how actuaries have traditionally addressed the main concerns of the stakeholders. We noted that the concerns for security and stability are met through, firstly, the establishment of a fund and, secondly, the choice of funding method.

The establishment of a fund ensures that the cost of provision is met during the member's working life. It also ensures the separation of the pension scheme from the sponsoring employer's business. We have observed that actuaries have traditionally tended to choose the funding method by taking into account several fundamental criteria, including security and stability.

The security criterion implies that the funding method should ensure that at any given time the value of the fund should cover the value of the accrued benefits. Meanwhile, the stability criterion implies that the funding method should lead to a stable contribution rate. This would then mean that the sponsoring employer would not face an uncertain future financial outflow.

We have also noted that in the traditional framework more emphasis is placed on funding methods and very little on the investment strategy. Even though the asset allocation strategies of pension funds have generally been responsive to changing market conditions, the funding and investment decisions have been treated separately.

We have also noted that some researchers have suggested that the tradition funding methods do not give a 'degree of confidence' to the likelihood of meeting the promised benefits. Hence, we have argued that the objective of the stakeholders are not properly addressed.

The traditional decision making techniques are also a deterministic framework. This, we have observed, implies that the stochastic nature of the environment in which pension schemes operate is not explicitly taken into account. We have thus argued for the use of stochastic approaches that can ensure that a full picture of the pension scheme and the interaction between various crucial factors is established.

In Chapter 3 we have discussed the Asset and Liability Modelling (ALM) approach to decision making in defined benefit pension schemes. This approach entails the use of stochastic models to represent the cash flows in the pension scheme.

The most common objective criterion for this approach has been the probability of failing to meet some target. We have noted that this can be extended by including the severity of the consequence of failing to meet the target.

Also, presentational problems arise in ALM. This is due to a number of factors. Firstly, the recipients of the results of a study might require that the results be depicted in a simplified form. This might imply the discarding of crucial information when presenting results through, for example, percentiles.

We have further noted that although the ALM approach goes a long way to address the problem of finding investment strategies for the pension scheme, such a decision is nevertheless treated separately from the funding decision. We have concluded that this need not be the case since these two crucial decisions can, and do, affect each other.

We have further observed that stochastic asset and liability modelling has also been criticized for failing to explicitly take into account the sponsoring employer's financial ability to maintain contributing into the scheme. Thus we argued for a framework that addresses and assesses these criticisms.

In Chapter 4 we have presented a dynamic programming approach to the problem of pension funding and investment. We have considered a criterion with quadratic and linear factors. We have shown that such a criterion leads to higher funding levels and lower contribution levels than a purely-quadratic criterion.

Although we have considered the asset allocation as the only control variable, this approach can easily be extended to include that contribution policy as a second control variable. In our analysis we have assumed that the fund is invested in a risk-free asset and a risky asset. We showed that the optimal asset allocation strategy is counter-intuitive: that is, it involves investing more in the risky asset as the pension fund deficit increases and investing more in the risk-free asset as the pension fund surplus increases.

This approach can be extended by considering more risky assets or, indeed, considering more realistic distributions for the return on the risky asset. Additionally, we could have considered a continuous-time approach employing stochastic differential equations and the Hamilton-Jacobi-Bellman equation rather than the discrete-time approach.

Lastly, we have argued in that chapter that for analytically-tractable criteria the actuary's understanding of the effect of crucial variables can be greatly enhanced. However, the simplifications involved imply that in most cases realistic applications of this approach might be limited.

In Chapter 5 we have presented a stochastic approach to decision making. Under this approach the funding and investment strategies have been considered as simultaneous decisions. We have presented downside risk measures for solvency risk and contribution rate risk.

The solvency risk measure is an end-of-horizon risk measure that treats pension fund surpluses differently from pension fund deficits. The risk measure incorporates both the probability of fund deficit and the severity of the fund deficit should it occur.

The contribution rate risk measure deals with contribution rate excesses: positive excesses are treated differently from negative excesses. To further differentiate this from the solvency risk measure we set up the contribution rate risk measure to incorporate all the expected excesses over the given horizon.

Furthermore, we have considered a performance measure for the cost of pension provision to the sponsoring employer. This performance measure is the average contribution rate over the given horizon.

These three risk and performance measures should, in tandem, address all the main objectives of the stakeholders: these being security of the accrued benefits, stability of the contribution and minimization of the cost.

We also showed how we can offer choices in the setting of the 'Normal contribution rate' (that is, the fixed part of the total contribution rate). This enabled us to consider a large spectrum of possible funding strategies. Furthermore, each of these choices was combined with a choice for the asset allocation strategy.

These combinations were then analysed by considering their effect on the risk and performance measures over pre-set projection horizons. The choice of different projection horizons also offered the decision maker flexibility to tailor the analysis to the needs or specifications of the stakeholders.

We suggested two approaches to choosing efficient combinations of the asset allocation and funding strategies. Firstly, there is the indifference curve approach. In this approach we showed that combinations which lead to similar level of risk (or cost) lie along the same indifference curve. And thus, for each choice of Normal contribution rate, the optimal asset allocation was determined by considering the point of intersection with some indifference curve.

The second approach involved recasting the pension fund problem as a stochastic multi-objective problem. To simplify the analysis we showed that the main problem could be split into two sub-problems: firstly, by considering only the risk measures, i.e. solvency risk and contribution rate risk; and secondly, by considering the solvency risk and the performance measure (that is, the average contribution rate).

In both cases the stochastic multi-objective formulation led to Pareto optimal solutions. These were solutions such that any other solution could not reduce one of the measures without increasing the other measure.

The stochastic approach described in that chapter presented a significant step forward in the management of defined benefit pension schemes. Firstly, the integration of the funding and investment decisions implied, at the very least, that separate exercises in the form of valuations and asset and liability modelling would not be necessary.

Secondly, the needs of all the stakeholders were met in a more satisfactory way: through the risk and performance measures we address security, stability and cost. And also through different choices of the Normal contribution rate we provided the sponsoring employer with flexibility in the running process. Furthermore, through different choices of the projection horizons the sponsor's financial strength could be addressed.

In Chapter 6 we presented a case study illustrating the stochastic approach to decision making. We showed that there was a conflict between strategies that were optimal under the solvency risk only and those strategies that were optimal under the contribute rate risk only. We argued that a way to resolve this conflict was to combine the indifference curves for the solvency risk with those for the contribution rate risk.

In so doing we obtained strategies that were efficient under both the solvency risk and the contribute rate risk. These were the kind of strategies that the decision maker could not improve: that is, choosing different strategies would involve increasing at least one of the risks.

We also showed that, interestingly, such efficient strategies were also efficient under the cost performance measure. This was because such strategies led to the lowest average contribution rate for a given Normal Contribution rate.

We also investigated the effect of different amortization periods. We showed that if deficits were spread over a short period the scheme attains full-funding quickly. This was demonstrated by better solvency risk positions when the spread period was short than when the spread period was long.

However, we also showed that there was a penalty to this since we got a higher contribution rate risk for the case when the spread period was short than when the spread period was long.

In that chapter we also demonstrated how the investment strategy could be changed to incorporate dynamic strategies. These were investment strategies that were changed dynamically depending on the funding position of the pension scheme.

We showed that counter-intuitive strategies were optimal under the stochastic approach. Such counter-intuitive strategies entailed shifting the fund into risky assets as the funding level deteriorated; and shifting the fund into risk-free or less risky assets as the funding level improved. Thus the stochastic approach results agreed with and reinforced the theoretical results from the dynamic programming approach in Chapter 4.

The case study also showed that for a scheme with different initial funding levels we got a convergence in the efficient strategies for long projection horizons. This result also agreed with a result obtained under the theoretical model.

We also illustrated the stochastic multi-objective approach. We showed that Pareto optimal solutions were located along an efficient frontier for a given Normal contribution rate. We also showed that flexibility in the choice of Normal contribution rate could be introduced as a natural extension.

In Chapter 7 we considered the incorporation of asset model risk in the decision making process. We observed that model risk could arise from using an inappropriate model and also from the fact that an asset model is essentially a *model* of a complex, real investment world.

In our analysis we considered an asset model built on modern asset pricing theory using the idea of pricing kernels. This was done in order to compare the results to those obtained under the Wilkie model which is a statistical model with parameters estimated from historical data.

We showed, interestingly, that the general principles underlying the stochastic approach held regardless of the asset model. Thus, for example, the results showed that indifference curve analysis could be used to establish efficient strategies under both asset models.

In Chapter 8 we extended the stochastic approach to decision making by considering the probability that the sponsoring employer might default on the pension obligations. Our approach was based on the argument that periods of weak equity markets could coincide with and/or lead to large pension fund deficits and hence increased sponsor contributions at a time when the sponsor might be financially weak and thus most likely to default.

Using this argument we considered the proposal that incorporating the probability of sponsor default in the stochastic approach to decision making could lead to optimal asset allocations with low (or zero) weighting in equities.

We designed the probability of default by considering the return on a tangency portfolio. The sponsor was taken to have defaulted if the probability of default was higher than some randomly generated uniform random variable and if the pension scheme had a deficit. We analysed this framework by considering different projection horizons. We assumed that the sponsor could default at the end of the year and if a default occurs, the defaulted deficit was accumulated to the end of the project horizon. And we introduced a default risk measure, which took into account the probability of default event and the amount of defaulted deficit should a default occur.

Our results showed that for short-term projections investment strategies with full-weighting in bonds were optimal under the default risk. Meanwhile, for medium-term and long-term projections the optimal investment strategies under the default risk were not significantly different from the optimal strategies under the solvency risk. These results implied that the arguments for the inclusion of sponsor default hold for the short-term projections. However, this was not the case for other projection horizons.

This thesis could be extended in several ways. Firstly, possible extensions for the dynamic progressing problem have already been noted above. Secondly, the stochastic approach to decision making could be extended by considering other ways of treating the funding strategy. Other asset classes could also be considered: in this thesis we have only looked at equities, fixed-interest bonds and index-linked gilts. But we could also have considered overseas equities, cash and property.

Although we have considered two asset models, other models could also be considered. For instance, the ARCH model could be used for inflation. On the other hand, other complete asset models could also be employed.

Different risk measures could also be considered for the solvency and contribution. Furthermore, it is possible to apply the stochastic approach to other actuarial areas. For instance, problems involving with-profits guarantees in life insurance could be analysed using the principles presented in this thesis.

Another interesting area that could be further investigated concerns the approach to incorporating default risk. Firstly, a different rate of interest could be used to accumulate the defaulted deficits. This could affect the results for the medium-term and long-term projections. Secondly, a different formulation for the probability of deficit could be used. For instance, the sponsor's financial strength could be incorporated directly in the probability of default.

Appendix A

Derivation of Coefficients for the Optimal Cost

In this appendix we consider the dynamic programming problem of Chapter 4. We show that the optimal cost is quadratic in $f(t)$ and we derive its coefficients.

From Equations 4.30 on page 59

$$\begin{aligned}
J \;=\;& \alpha_1 \Big(\frac{c(t)}{NC(t)} - 1\Big)^2 + \alpha_3 \Big(\frac{c(t)}{NC(t)} - 1\Big) + \nu\, R_{t+1} + (\alpha_2 + \alpha_4)\,\nu \\
& + \nu\, X_{t+1}\, AL(t+1)^{-1}\,(1+\mu_2)\big(f(t) + c(t) - B(t)\big) \\
& + \nu\, X_{t+1}\, AL(t+1)^{-1}\,(\mu_1 - \mu_2)\pi(t)\big(f(t) + c(t) - B(t)\big) \\
& + \nu\, Y_{t+1}\, AL(t+1)^{-2}\,\big(f(t) + c(t) - B(t)\big)^2 \times \\
& \qquad \Big\{\pi(t)^2 \sigma_1^2 + \big(1 + \mu_2 + (\mu_1 - \mu_2)\pi(t)\big)^2\Big\}.
\end{aligned} \tag{A.1}$$

and from Equation 4.34 on page 59

$$\pi(t) \;=\; \Omega_1^{-1}\,\Omega_3 - \frac{1}{2}(\mu_1 - \mu_2)\,\Omega_1^{-1}\, AL(t+1)\, X_{t+1}\, Y_{t+1}^{-1}\, \big(f(t) + c(t) - B(t)\big)^{-1}. \tag{A.2}$$

We can write

$$\begin{aligned}
\frac{c(t)}{NC(t)} - 1 \;&=\; \frac{k}{NC(t)}\,\big[AL(t) - f(t)\big] \\
&=\; k\,\frac{AL(t)}{NC(t)} - \frac{k}{NC(t)}\, f(t)
\end{aligned} \tag{A.3}$$

and

$$\Big(\frac{c(t)}{NC(t)} - 1\Big)^2 \;=\; k^2\,\frac{AL(t)^2}{NC(t)^2} - 2\,k^2\,\frac{AL(t)}{NC(t)^2}\, f(t) + \frac{k^2}{NC(t)^2}\, f(t)^2. \tag{A.4}$$

Thus

$$\alpha_1\Big(\frac{c(t)}{NC(t)}-1\Big)^2+\alpha_3\Big(\frac{c(t)}{NC(t)}-1\Big) = \alpha_1 k^2\frac{AL(t)^2}{NC(t)^2}+\alpha_3 k\frac{AL(t)}{NC(t)} -\Big[2k^2\frac{AL(t)}{NC(t)^2}\alpha_1+\frac{k}{NC(t)}\alpha_3\Big]f(t) +\alpha_1\frac{k^2}{NC(t)^2}f(t)^2. \tag{A.5}$$

Using the equation for the contribution at time t, $c(t)$, we can write

$$f(t)+c(t)-B(t)=(1-k)\,f(t)+NC(t)+k\,AL(t)-B(t). \tag{A.6}$$

Thus

$$\begin{aligned}&\nu X_{t+1}AL(t+1)^{-1}(1+\mu_2)\big(f(t)+c(t)-B(t)\big)\\ &\quad=\nu X_{t+1}AL(t+1)^{-1}(1+\mu_2)(1-k)\,f(t)\\ &\qquad+\nu X_{t+1}AL(t+1)^{-1}(1+\mu_2)(1-k)\big[NC(t)+k\,AL(t)-B(t)\big].\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\nu X_{t+1}AL(t+1)^{-1}(\mu_1-\mu_2)\pi(t)\big(f(t)+c(t)-B(t)\big)\\ &\quad=\nu X_{t+1}AL(t+1)^{-1}(\mu_1-\mu_2)\,\Omega_1^{-1}\Omega_3\big(f(t)+c(t)-B(t)\big)\\ &\qquad-\frac{1}{2}\nu X_{t+1}^2\,Y_{t+1}^{-1}(\mu_1-\mu_2)^2\,\Omega_1^{-1}\\ &\quad=\nu X_{t+1}AL(t+1)^{-1}(\mu_1-\mu_2)\,\Omega_1^{-1}\Omega_3(1-k)\,f(t)\\ &\qquad+\nu X_{t+1}AL(t+1)^{-1}(\mu_1-\mu_2)\,\Omega_1^{-1}\Omega_3\big[NC(t)+k\,AL(t)-B(t)\big]\\ &\qquad-\frac{1}{2}\nu X_{t+1}^2\,Y_{t+1}^{-1}(\mu_1-\mu_2)^2\,\Omega_1^{-1}.\end{aligned} \tag{A.8}$$

$$\begin{aligned}&\nu Y_{t+1}AL(t+1)^{-2}\big(f(t)+c(t)-B(t)\big)^2\Big\{\pi(t)^2\sigma_1^2+\big(1+\mu_2+(\mu_1-\mu_2)\pi(t)\big)^2\Big\}\\ &\quad=\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)^2\big(f(t)+c(t)-B(t)\big)^2\\ &\qquad+2\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)(\mu_1-\mu_2)\,\pi(t)\big(f(t)+c(t)-B(t)\big)^2\\ &\qquad+\nu Y_{t+1}AL(t+1)^{-2}\big(\sigma_1^2+(\mu_1-\mu_2)^2\big)\,\pi(t)^2\big(f(t)+c(t)-B(t)\big)^2\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}&\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)^2\big(f(t)+c(t)-B(t)\big)^2\\ &\quad=\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)^2(1-k)^2f(t)^2\\ &\qquad+2\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)^2(1-k)\big[NC(t)+k\,AL(t)-B(t)\big]f(t)\\ &\qquad+\nu Y_{t+1}AL(t+1)^{-2}(1+\mu_2)^2\big[NC(t)+k\,AL(t)-B(t)\big]^2\end{aligned} \tag{A.10}$$

$$
\begin{aligned}
&2\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)(\mu_1-\mu_2)\,\pi(t)\,\Big(f(t)+c(t)-B(t)\Big)^2\\
&\quad= 2\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,(1-k)^2\, f(t)^2\\
&\qquad +4\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,(1-k)\,\Big[NC(t)+k\,AL(t)-B(t)\Big]\, f(t)\\
&\qquad\quad +2\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,\Big[NC(t)+k\,AL(t)-B(t)\Big]^2\\
&\qquad\quad\quad -\nu\, X_{t+1}\, AL(t+1)^{-1}\,(1+\mu_2)(\mu_1-\mu_2)^2\,\Omega_1^{-1}\,(1-k)\, f(t)\\
&\qquad\quad\quad\quad -\nu\, X_{t+1}\, AL(t+1)^{-1}\,(1+\mu_2)(\mu_1-\mu_2)^2\,\Omega_1^{-1}\,\Big[NC(t)+k\,AL(t)-B(t)\Big]
\end{aligned}
\tag{A.11}
$$

Also

$$
\begin{aligned}
\pi(t)^2 \;=\;& \Omega_1^{-2}\,\Omega_3^2-(\mu_1-\mu_2)\,\Omega_1^{-2}\,\Omega_3\, AL(t+1)\, X_{t+1}\, Y_{t+1}^{-1}\,\Big(f(t)+c(t)-B(t)\Big)^{-1}\\
&+\frac{1}{4}(\mu_1-\mu_2)^2\,\Omega_1^{-2}\, AL(t+1)^2\, X_{t+1}^2\, Y_{t+1}^{-2}\,\Big(f(t)+c(t)-B(t)\Big)^{-2}.
\end{aligned}
\tag{A.12}
$$

Hence

$$
\begin{aligned}
&\nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_1\,\pi(t)^2\,\Big(f(t)+c(t)-B(t)\Big)^2\\
&\quad= \nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_1^{-1}\,\Omega_3^2\,(1-k)^2\, f(t)^2\\
&\qquad +2\nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_1^{-1}\,\Omega_3^2\,(1-k)\,\Big(NC(t)+k\,AL(t)-B(t)\Big)\, f(t)\\
&\qquad +\nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_1^{-1}\,\Omega_3^2\,\Big(NC(t)+k\,AL(t)-B(t)\Big)^2\\
&\qquad\quad -\nu\, X_{t+1}\, AL(t+1)^{-1}\,(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,(1-k)\, f(t)\\
&\qquad\quad -\nu\, X_{t+1}\, AL(t+1)^{-1}\,(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,\Big(NC(t)+k\,AL(t)-B(t)\Big)\\
&\qquad\quad +\frac{1}{4}\,\nu\, X_{t+1}^2\, Y_{t+1}^{-1}\,(\mu_1-\mu_2)^2\,\Omega_1^{-1}
\end{aligned}
\tag{A.13}
$$

We can thus write J as a quadratic in $f(t)$. The coefficient for $f(t)^2$ can be obtained from equations A.5, A.10, A.11 and A.13. This gives

$$
\begin{aligned}
&\alpha_1\,\frac{k^2}{NC(t)^2}+\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)^2\,(1-k)^2\\
&\quad +2\nu\, Y_{t+1}\, AL(t+1)^{-2}\,(1+\mu_2)\,(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3\,(1-k)^2\\
&\qquad +\nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_1^{-1}\,\Omega_3^2\,(1-k)^2\\
&=\\
&\qquad \alpha_1\,\frac{k^2}{NC(t)^2}+\nu\, Y_{t+1}\, AL(t+1)^{-2}\,\Omega_0\,\Omega_1^{-1}\,(1-k)^2\,(1+\mu_2).
\end{aligned}
\tag{A.14}
$$

The coefficient for $f(t)$ can be obtained from equations A.5, A.7, A.8, A.10, A.11 and A.13. This gives

$$
\begin{aligned}
&-\Big[2\,k^2\,\frac{AL(t)}{NC(t)^2}\,\alpha_1+\frac{k}{NC(t)}\,\alpha_3\Big]\\
&+\nu\,X_{t+1}\,AL(t+1)^{-1}\,(1+\mu_2)\,(1-k)\,\Big[1-(\mu_1-\mu_2)^2\,\Omega_1^{-1}\Big]\\
&+2\,\nu\,Y_{t+1}\,AL(t+1)^{-2}\,(1-k)\,\Big[NC(t)+k\,AL(t)-B(t)\Big]\,\times\\
&\qquad\Big\{(1+\mu_2)^2+2(1+\mu_2)\,(\mu_1-\mu_2)\,\Omega_1^{-1}\,\Omega_3+\Omega_1^{-1}\Omega_3^2\Big\}.
\end{aligned}
\tag{A.15}
$$

Appendix B

Modelling Framework

In this appendix we outline the modelling framework to illustrate the decision-making approach presented in Chapter 5.

B.1 Model Pension Scheme

B.1.1 Benefits Provided

We assume that the model pension has the following benefit structure:

Normal Retirement: a pension on normal retirement age of 65 of 1/60th of final pensionable salary at the date of retirement for each year of pensionable service;

Pensions in payment: pensions in payment are assumed to be increased in line with a Limited Price Indexation;

Withdrawal: on withdrawal a deferred pension, revalued up to normal retirement age, is payable;

Death in service: no benefit is provided.

B.1.2 Membership Profile

We assume that the scheme membership is stable over time with respect to age, pensionable salary in real terms and past pensionable service. Thus, the number of active members at each age prior to normal retirement age is based on Table B.1.

We assume that the youngest age is 25. Members retiring at the normal retirement age and deaths amongst active members are replaced by new entrants at the youngest age. Meanwhile, withdrawals prior to normal retirement age are replaced by new entrants at the same age.

At each age the past service is calculated recursively as a weighted average of the past pensionable service for those active members remaining in the scheme from the previous age and the zero past pensionable service assumed for the new entrants at the current age.

B.2 Asset Model

The choice of asset model is crucial to the stochastic approach described in this thesis. This is due to the fact that the results obtained will depend on the underlying asset model.

We assume that all the economic factors are as generated by the Wilkie (1995) model. We use Wilkie's recommended parameters and neutral initial conditions.

B.3 Funding Strategy

We assume that the contribution rate in the year $(t, t+1)$, payable at time t, is given by

$$c(t) = NC + \frac{1}{\ddot{a}_{\overline{m}|}} \big(AL(t) - f(t)\big) \tag{B.1}$$

where $AL(t)$ is the discontinuance liability at time t and $f(t)$ is the market value of the fund at time t.

At time 0 the decision maker sets the spread period m for regular amortization of gains and losses. As observed in Chapter 2 this method is referred to as the Spread Method.

To ensure flexibility in the funding strategy, we let the decision maker set the Normal contribution rate, NC (see Definition 5.1 on page 76). Thus we will consider a range of values for NC from 0% to 32% - this range could be changed depending on the circumstances of the decision-maker.

B.3.1 Investment Strategy

In most cases we have assumed that the fund is invested in Equities and Fixed-Interest Bonds. However, strategies involving Equities and Index-Linked Bonds are also considered.

We assume, unless otherwise stated, that the pension fund follows a fixed (or static) investment strategy with annual rebalancing. This implies that the proportions to be invested in each asset class are set at time 0. These proportions are then annually rebalanced.

For the sake of comparison, an analysis involving dynamic strategies is also presented. In these strategies the proportions invested in each asset class are changed over time depending on some pre-determined rule.

B.4 Cash Flows

B.4.1 Fund Growth

Let $f(t)$ be the fund at time t, $C(t)$ be the contribution for the year $(t, t+1)$ assumed payable at the beginning of the year, and $B(t)$ be the benefit outgo at time t. We also assume that the fund is to be invested in 2 assets. Let the return on the ith asset be $r_i(t)$ and the proportion invested in the ith asset be $x_i(t)$. These proportions sum to one: that is, $x_1(t) + x_2(t) = 1$.

Then the fund at time $t+1$ is given by

$$f(t+1) = \Big[\sum_{i=1}^{2} x_i(t)\big(1 + r_i(t)\big)\Big]\Big[f(t) + C(t) - B(t)\Big] \tag{B.2}$$

where $f(0) = f_0$ is given.

In our analysis we assume that the fund is invested in either equities and fixed-interest bonds or equities and index-linked bonds. In the Wilkie model we use the total nominal return series: $PR(t)$, $CR(t)$ and $RR(t)$ for equities, fixed-interest bonds and index-linked bonds, respectively (see Wilkie (1995, p.901)).

Thus, if the fund is invested in equities and fixed-interest bonds, we get:

$$f(t+1) = \Big[x_1(t)\,\frac{PR(t+1)}{PR(t)} + x_1(t)\,\frac{CR(t+1)}{CR(t)}\Big]\Big[f(t) + C(t) - B(t)\Big] \tag{B.3}$$

and, if the fund is invested in equities and index-linked bonds, we get:

$$f(t+1) = \Big[x_1(t)\,\frac{PR(t+1)}{PR(t)} + x_1(t)\,\frac{RR(t+1)}{RR(t)}\Big]\Big[f(t) + C(t) - B(t)\Big]. \tag{B.4}$$

In both cases for the static asset allocation strategy we have $x_1(t) = x_1$ and $x_2(t) = 1 - x_1$, but annually rebalanced.

The contribution amount, $C(t)$, at time t is given by

$$C(t) = \frac{1}{100} \times c(t) \times TS(t) \tag{B.5}$$

where $TS(t)$ is the total salary roll at time t.

B.4.2 Liability Growth

The total discontinuance liability for the scheme is calculated as the sum of the total discontinuance liability for the active members and the total discontinuance liability for the current pensioners. The total liability for the active members is calculated as the sum of the individual discontinuance liabilities. These are calculated, for a member aged x at time t, as the product of the accrual rate ($\frac{1}{f}$), the average past service, the salary at time t, and a deferred annuity, $({}_{NRA-x|}\ddot{a}_x(t))$, evaluated using the current real yield on index-linked bonds, $R(t)$.

Meanwhile, the total discontinuance liability for the current pensioners is calculated as the sum of the individual liabilities for the pensioners. For a pensioner aged x at time t, the liability is given by the product of the benefit amount at time t, $B(x,t)$, and an immediate annuity, $\ddot{a}_x(t)$, calculated using the current real yield on index-linked bonds.

The benefit amounts are assumed to increase in line with the LPI. Thus

$$B(x,t) = B(x-1,t-1)\,Max\Big[1,\ Min\Big(exp\big(I(t)\big), LPI_{ceil}\Big)\Big] \tag{B.6}$$

where $I(t)$ is the force of inflation at time t and LPI_{ceil} is the ceiling for the LPI assumed to 5% (in current legislation LPI_{ceil} has been reduced to 2.5%).

B.5 Simulations

The problem outlined in Chapter 5 is not amenable to analytical techniques. Hence, we use Monte Carlo simulations.

We have mostly assumed that 1,000 simulations are used to project the capital values and scheme liabilities. However, in cases where we are considering the probability of default we have used 20,000 simulations (see Chapter 8).

We use the so-called *path approach* where for N simulations we get N projected scenarios at any given horizon (see Booth and Ong (1994, p. 231)).

Table B.1: Service table and Salary scale

Age, x	l_x	w_x	d_x	i_x	r_x	s_x
16	1000000	100000	500	0	0	1.000
17	899500	89950	450	0	0	1.200
18	809100	80910	405	0	0	1.404
19	727786	72779	364	0	0	1.606
20	654643	65464	327	65	0	1.810
21	588786	58879	236	118	0	2.013
22	529554	52955	212	106	0	2.218
23	476281	47628	191	95	0	2.421
24	428367	40695	171	86	0	2.626
25	387415	34867	155	116	0	2.828
26	352277	29944	106	106	0	3.026
27	322122	25770	97	97	0	3.226
28	296159	22212	89	118	0	3.433
29	273739	19162	82	109	0	3.635
30	254386	16789	76	102	0	3.845
31	237418	14720	95	119	0	4.043
32	222485	12904	89	133	0	4.249
33	209358	11305	84	147	0	4.454
34	197823	9891	79	178	0	4.654
35	187674	8633	94	206	0	4.848
36	178741	7507	107	214	0	5.041
37	170912	6495	120	222	0	5.226
38	164076	5579	115	246	0	5.398
39	158136	4744	127	269	0	5.553
40	152997	4131	138	306	0	5.692

Age, x	l_x	w_x	d_x	i_x	r_x	s_x
41	148422	3562	134	341	0	5.813
42	144385	3032	144	375	0	5.922
43	140833	2535	155	408	0	6.011
44	137735	2066	165	441	0	6.101
45	135062	1621	176	473	0	6.192
46	132793	1195	186	505	0	6.285
47	130908	785	223	550	0	6.387
48	129350	517	259	621	0	6.482
49	127953	256	281	717	0	6.579
50	126699	0	317	824	0	6.676
51	125559	0	352	942	0	6.784
52	124265	0	385	1081	0	6.885
53	122799	0	430	1253	0	6.986
54	121117	0	472	1453	0	7.099
55	119191	0	513	1681	0	7.203
56	116998	0	550	1930	0	7.309
57	114517	0	584	2199	0	7.426
58	111734	0	626	2481	0	7.535
59	108628	0	673	2781	0	7.656
60	105174	0	726	3103	36811	7.768
61	64535	0	497	2194	5163	7.881
62	56681	0	493	2228	4534	8.006
63	49426	0	484	2244	3954	8.123
64	42743	0	474	2248	3419	8.252
65	36601	0	0	0	36601	8.371

Appendix C

Analysis of various asset classes

C.1 Introduction

In this section we investigate the allocation of the pension fund in different asset classes. We consider three asset classes: Equities, Fixed-Interest Bonds and Index-Linked Bonds. We could also have considered Cash, Property and Overseas Equities but we have left that as a future extension of this work.

Allocating the fund in Equities and Fixed-Interest Bonds is the asset allocation policy we firstly considered in Chapter 6. However, in that chapter we employed Wilkie's (1995) "neutral" estimates for all parameters. One such estimate is for the long-term mean dividend growth (DMU) and is set at 0.016. By reducing this estimate to zero it means that effectively we have reduced the equity risk premium. The general effect of various values of DMU is investigated in depth in Appendix D. In this section we only deal with the case where DMU is zero in order to analyze an economic scenario where the equity risk premium is very low.

As a further investment problem, we consider a third asset class: Index-Linked Bonds. These type of bonds were introduced in the UK in 1981 mainly for investors who need to hedge against the effect of price inflation. Index-linked bonds are thus a crucial asset class for defined benefit pension schemes due to the nature of pension liabilities (that is, they are real rather than nominal) and, indeed, the first issues were only limited to pension schemes (Lofthouse (2001, p. 331)).

In our case we also have the assumption that the valuation rate of interest is equal to the current real yield on index-linked bonds. Hence we would expect index-linked bonds to be a good match for our liabilities.

Thus allocating the pension fund in index-linked bonds and equities is the second asset allocation policy we consider in the second part of this section. There is a similarity in these asset classes in the sense that, just like index-linked bonds, equities offer a real return. Hence we are interested in the location of the efficient region.

C.2 Modelling Framework

We employ the same framework as in Chapter 6. We assume that the investment returns are generated under the Wilkie Model as set out in Wilkie (1995) but with DMU set to zero. We consider a range of possible combinations of Normal contribution rates and asset allocations.

For each combination we carry out stochastic projections of the pension scheme over 6 years and 15 years. At the end of each projection period we calculate the Mean Shortfall risk and the Excess Contribution Rate risk. We analyze the results using the idea of indifference curves that was introduced in Chapter 6. Although crucial to the decision-making process, we do not show the results for the cost measure due to space constraints.

C.3 Investing in Equities and Fixed-Interest Bonds

We firstly consider the policy of investing the pension fund in equities and fixed-interest bonds. As noted above, we assume that the long-term mean dividend growth estimate is zero in the Wilkie model. However, for fixed-interest bonds we use the recommended "neutral" estimates.

We firstly show the results for the Mean Shortfall risk.

C.3.1 Mean Shortfall risk

Figure C.1 shows the indifference curves for the Mean Shortfall risk at the end of 6 years. Each curve shows the combinations of Normal contribution rate and asset allocation that lead to the same Mean Shortfall risk. For instance, the top curve shows that by setting the Normal contribution rate equal to 18% and allocating 5% of the fund in equities we obtain the same Mean Shortfall risk of 0.07 as in the case where we set Normal contribution rate equal to 12% and allocate 60% of the fund in equities.

As shown in Section 6.2.1, for the case of Mean Shortfall risk, the efficient combinations of Normal contribution rate and asset allocation can only lie along the best-fit line PQ. This is due to the fact that for a given Normal contribution rate one can reduce the Mean Shortfall risk by shifting horizontally towards an asset allocation on PQ.

For instance, for a Normal contribution rate of 18%, the efficient asset allocation under Mean Shortfall risk is approximately 45% in equities whilst for a Normal contribution rate of 6% the efficient asset allocation is approximately 55% in equities.

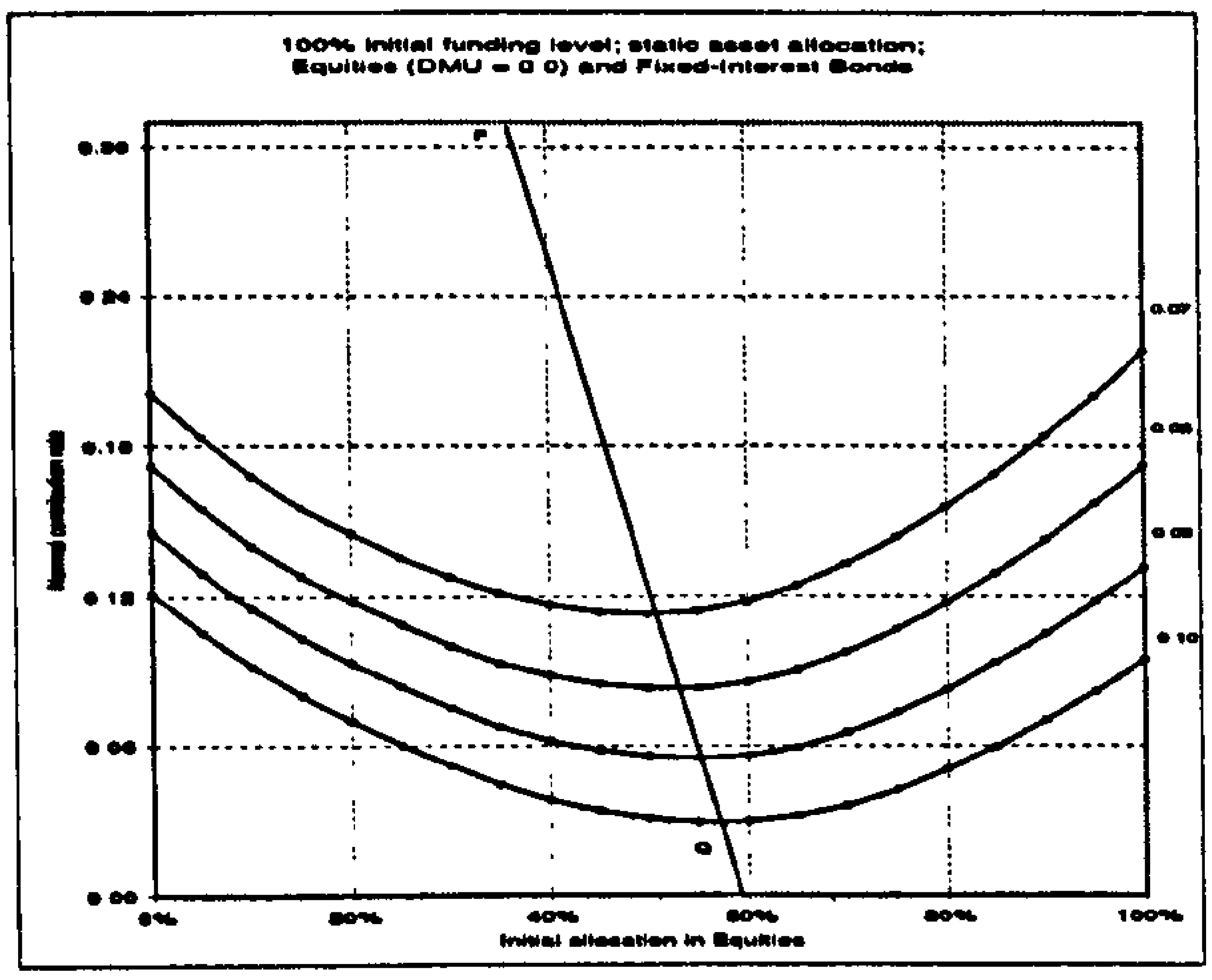

Figure C.1: Mean Shortfall risk indifference curves at the end of 6 years

Now we turn to the results for the Excess Contribution Rate risk.

C.3.2 Excess Contribution Rate risk

Figure C.2 shows the indifference curves for the Excess Contribution Rate risk at the end of 6 years. As in the Mean Shortfall risk case, each curve shows the combinations of Normal contribution rate and asset allocation that lead to the same Excess Contribution Rate risk. For instance, the top curve shows all the combinations of Normal contribution rate and asset allocation that lead to an Excess Contribution Rate risk of 0.04.

The best-fit line LM represents the efficient asset allocations for the case of Excess Contribution Rate risk. For instance, for a Normal contribution rate of 18% the efficient asset allocation is approximately 25% in equities; whilst for a Normal contribution rate of 6% the efficient asset allocation is about 33% in equities.

As in Chapter 6, this analysis indicates a potential conflict between efficiency for the case of the Mean Shortfall risk and efficiency for the case of the Excess Contribution Rate risk. For example, we have shown that with a Normal contribution rate of 18%, allocating 45% in equities is efficient under Mean Shortfall risk; however under Excess Contribution Rate risk 25% in equities is efficient. Indeed, the two risks will conflict whenever the best-fit lines PQ and LM do not overlap. In the next section, we show how this conflict can be resolved in the case of stochastic projections over 15 years.

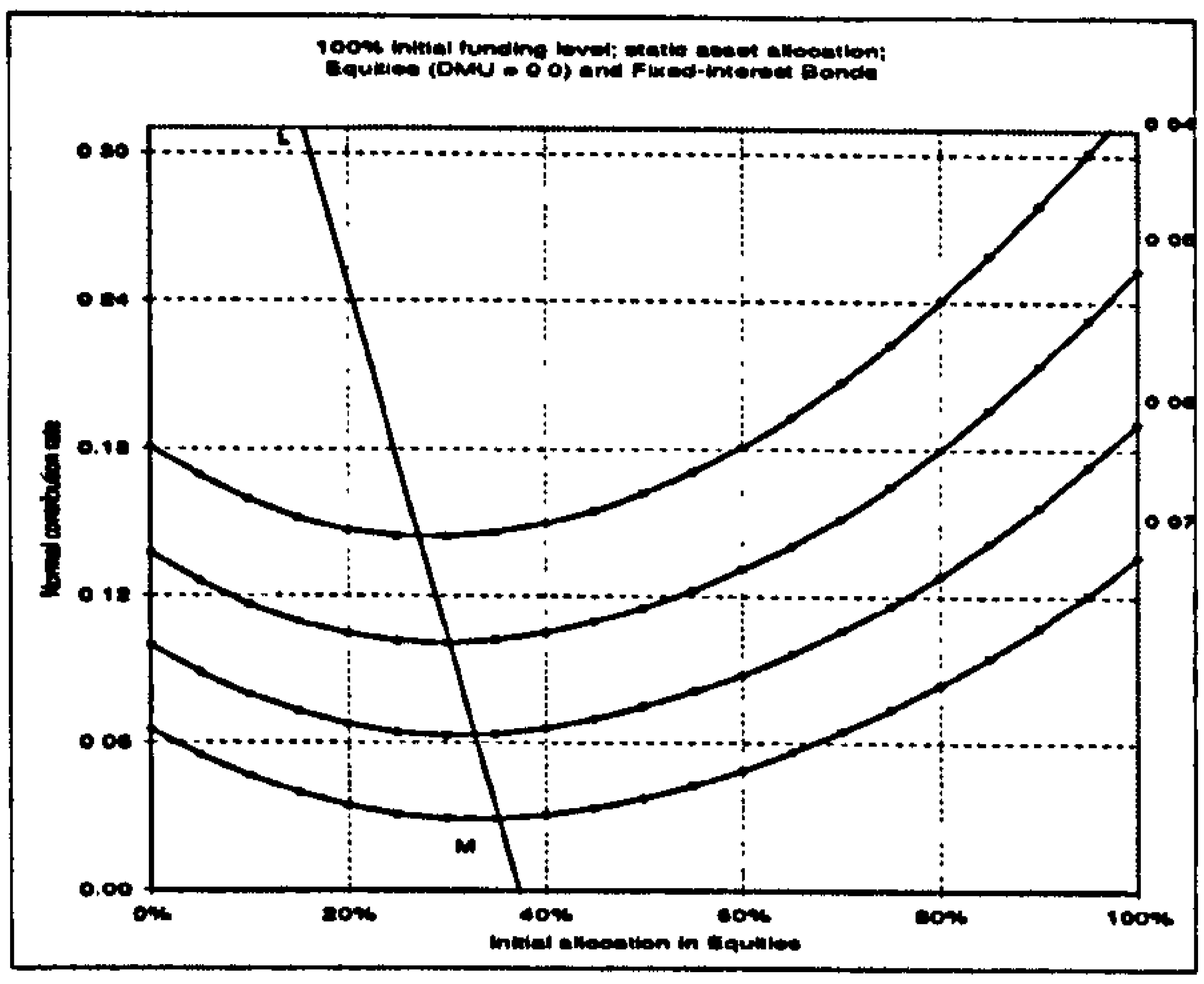

Figure C.2: Excess Contribution Rate risk indifference curves at the end of 6 years

C.3.3 Efficient region

In the last section, we have demonstrated that, by considering the Mean Shortfall and the Excess Contribution Rate risks separately, we obtain a conflict since asset allocations that are efficient under Mean Shortfall risk are not necessarily efficient under Excess Contribution Rate risk. In this section we show that, by combining the indifference curves for the two risks, the conflict can be resolved.

In Figure C.3, we combine the indifference curves for the Mean Shortfall and Excess Contribution Rate risks at the end of 15 years. The lines LM and PQ are the best-fit lines for the Excess Contribution Rate risk and the Mean Shortfall risk, respectively, as in the previous section. By combining the indifference curves we identify three regions: Region I, Region II and Region III as shown in Figure C.3.

Regions I and III

The asset allocations in Region I and Region III are inefficient since for a given Normal contribution rate we can decrease both risks by shifting towards lines LM and PQ, respectively. For instance, consider the arrow AC in Region I and the arrow KH in Region III.

At point A, the Mean Shortfall risk is 0.115 and the Excess Contribution Rate risk is 0.06; however, if we shift to point C the risks decrease to 0.079 and 0.048, respectively. Meanwhile, at point K the Mean Shortfall risk is 0.088 and the Excess Contribution Rate risk is 0.07; however if we shift to point H the risks decrease to 0.074 and 0.0515, respectively.

Region II

The asset allocations in Region II are efficient since, for a given Normal contribution rate, by shifting towards LM or PQ we can only reduce one of the risks whilst increasing the other risk.

For instance, consider the arrows FC and FH. At point F the Mean Shortfall risk is 0.076 and the Excess Contribution Rate risk is 0.0495. Thus shifting to point C leads to an increase of the Mean Shortfall risk to 0.079 and a decrease of the Excess Contribution Rate to 0.048. Meanwhile, shifting from point F to point H leads to a decrease of the Mean Shortfall risk to 0.074 and an increase of the Excess Contribution Rate to 0.0515.

Thus, Region II (LMPQ) is the efficient region.

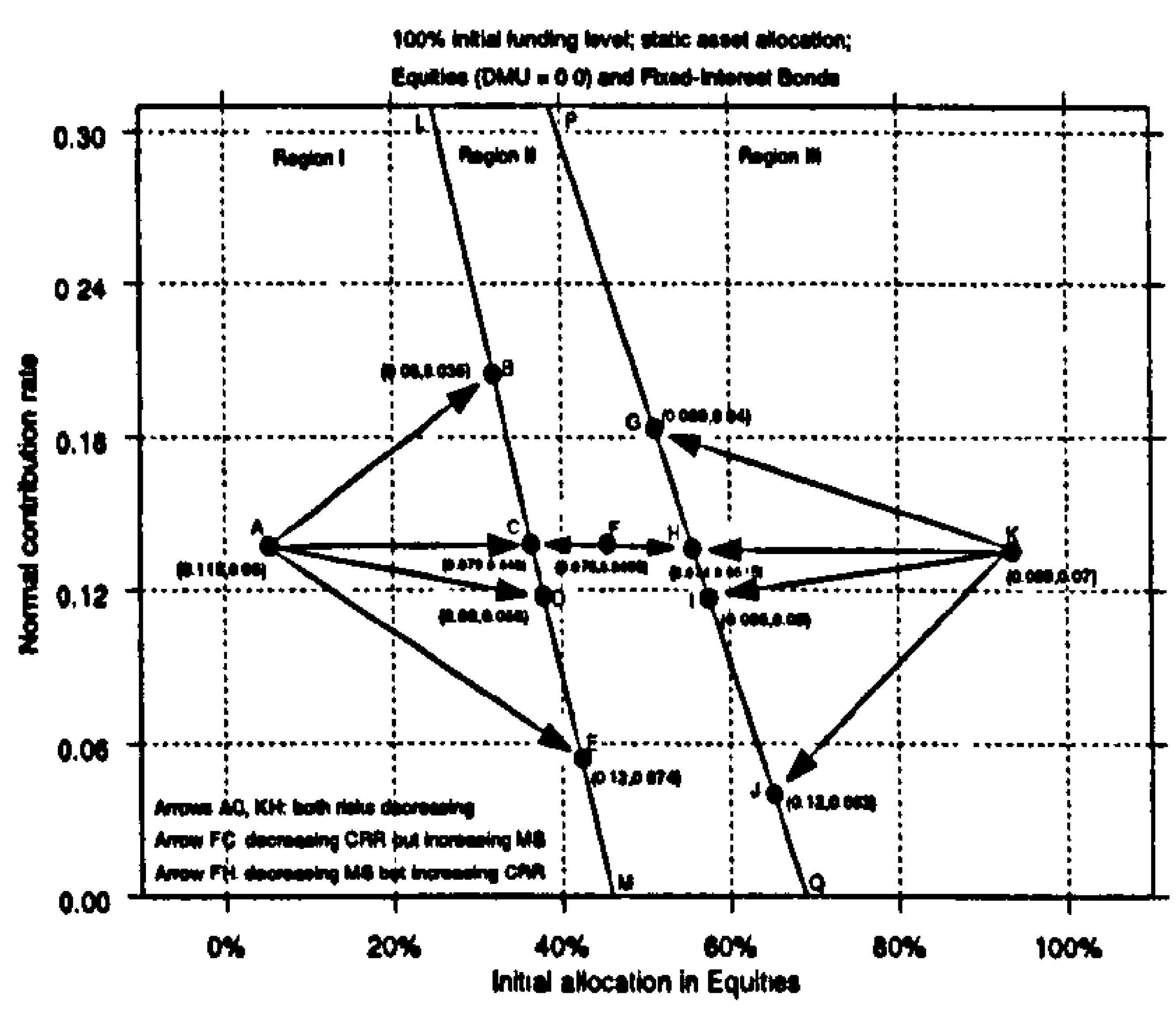

Figure C.3: The efficient region at the end of 15 years

C.3.4 Summary of Main Results

There are several results that have been shown to hold in both the high equity risk premium scenario of Chapter 6 and in the low equity risk premium of this section. Firstly, we obtain indifference curves for both risk measures and in both scenarios. Secondly, an efficient region still exists for the optimal asset allocations under the two risks. Interestingly, the efficient region is centred in the region of 50% equities and 50% bonds for projections over 15 years in the low equity risk premium. Thus, even though we have set the long-term mean dividend growth to zero, the optimal allocation in Equities is still high for long-term projections.

C.4 Investing in Equities and Index-Linked Bonds

In this section, we consider the second asset allocation policy: that is, investing the pension fund in equities and index-linked bonds. As noted above, since the pension liabilities are discounted using the current real yield on index-linked bonds and the pensions-in-payment are linked to the retail price index, we would expect that index-linked bonds would be a better match for the liabilities than either the fixed-interest bonds or equities.

Hence, intuitively, we would expect that, firstly, for a given Normal contribution rate and asset allocation in equities, both the Mean Shortfall risk and the Excess Contribution Rate risk would be lower in the case where we invest in equities and index-linked bonds than in the case where we invest in equities and fixed-interest bonds.

And secondly, for a given Normal contribution the efficient asset allocation would lie in a lower equities region if we invest in equities and index-linked bonds than if we invest in equities and fixed-interest bonds.

As in the first part of this appendix, we employ the framework summarized in Section C.2. Firstly, we consider the Mean Shortfall risk.

C.4.1 Mean Shortfall risk

Figure C.4 shows the indifference curves for the Mean Shortfall risk at the end of 6 years. We observe that, for efficient asset allocations for the case of the Mean Shortfall risk, we would invest less in equities if we allocate the assets in equities and index-linked bonds than in the case where we allocate the assets in equities and fixed-interest bonds.

This is clearly illustrated by considering the best-fit line PQ. PQ lies in the region of a 10% to 20% initial allocation in equities if we allocate assets in equities and index-linked bonds; whilst in the equities and fixed-interest bonds case the best-fit line lies in the region of a 40% to 60% initial allocation in equities (see Figure C.1).

For example, consider Normal contribution rates of 6% and 12%. Firstly, for a 6% Normal contribution rate, in the case where we invest in equities and index-linked bonds the efficient asset allocation would be approximately 15% in equities; whilst in the case where we invest in equities and fixed-interest bonds the efficient asset allocation would be approximately 55% in equities.

And secondly, for a 12% Normal contribution rate, in the case where we invest in equities and index-linked bonds the efficient asset allocation would be approximately

15% in equities; whilst in the case where we invest in equities and fixed-interest bonds the efficient asset allocation would be approximately 50% in equities.

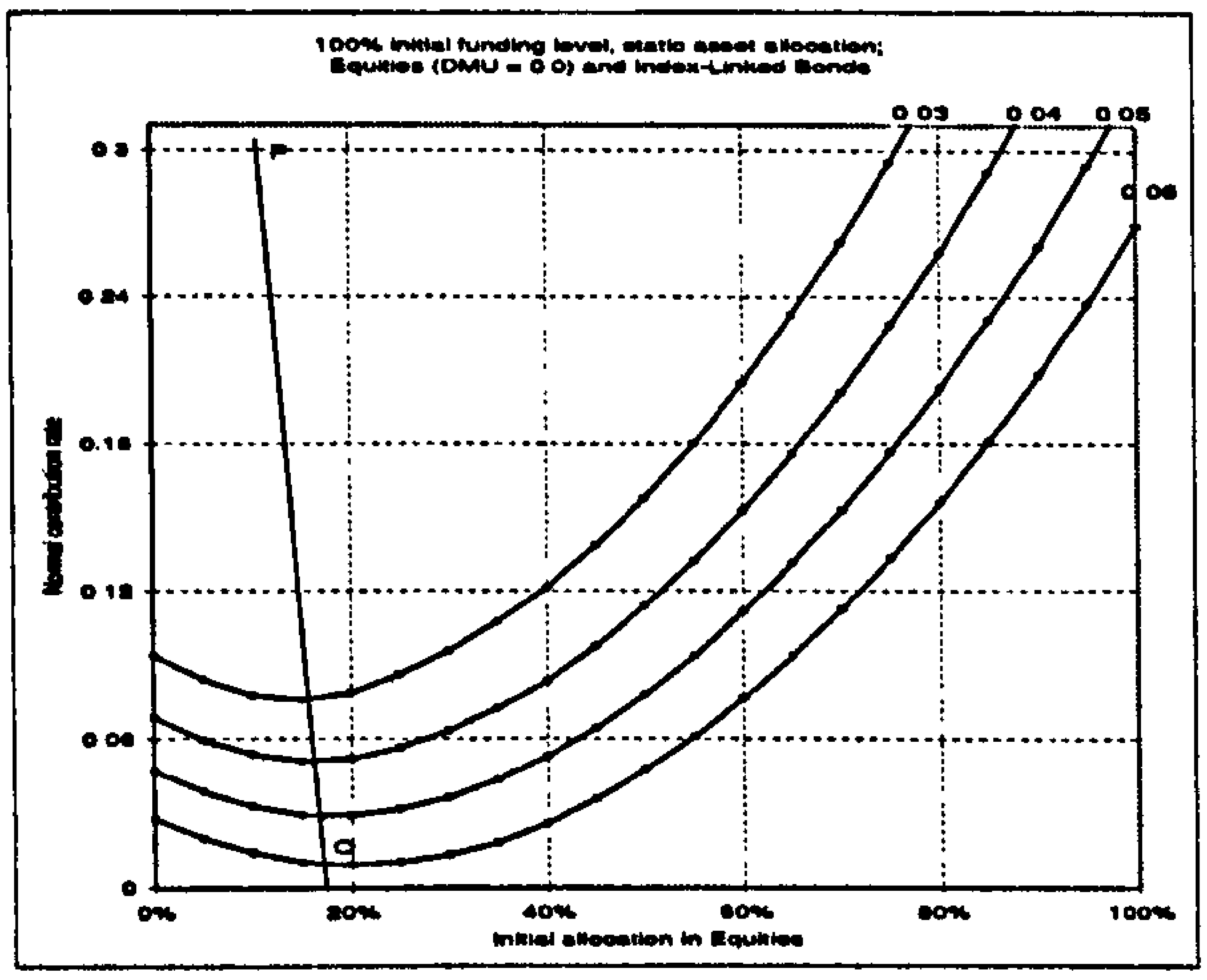

Figure C.4: Mean Shortfall risk indifference curves at the end of 6 years

We now consider the Excess Contribution rate risk.

C.4.2 Excess Contribution Rate risk

Figure C.5 shows the indifference curves for the Excess Contribution Rate risk at the end of 6 years. As in the Mean Shortfall risk case, the efficient asset allocations for the case of the Excess Contribution Rate risk have lower allocation in equities if we invest in equities and index-linked bonds than if we invest in equities and fixed-interest bonds.

As in the previous section, the best-fit line LM illustrates this point. In the case where we invest in equities and index-linked bonds LM lies in the region of a 5% to 15% initial allocation in equities; whilst in the case where we invest in equities and fixed-interest bonds LM lies in the region of a 20% to 35% initial allocation in equities (see Figure C.2).

For example, as for the Mean Shortfall risk, consider Normal contribution rates of 6% and 12%. Firstly, for a 6% Normal contribution rate, the efficient asset allocation for the case of Excess Contribution risk would be approximately 12% allocation in equities in the case where we invest in equities and index-linked bonds; whilst in the case where we invest in equities and fixed-interest bonds the efficient asset allocation for the case of Excess Contribution Rate risk would be approximately 37% allocation in equities.

And secondly, for a 12% Normal contribution rate, the efficient asset allocation for the case of Excess Contribution risk would be approximately 10% allocation in equities in the case where we invest in equities and index-linked bonds; whilst in the case where we invest in equities and fixed-interest bonds the efficient asset allocation for the case of Excess Contribution Rate risk would be approximately 30% allocation in equities.

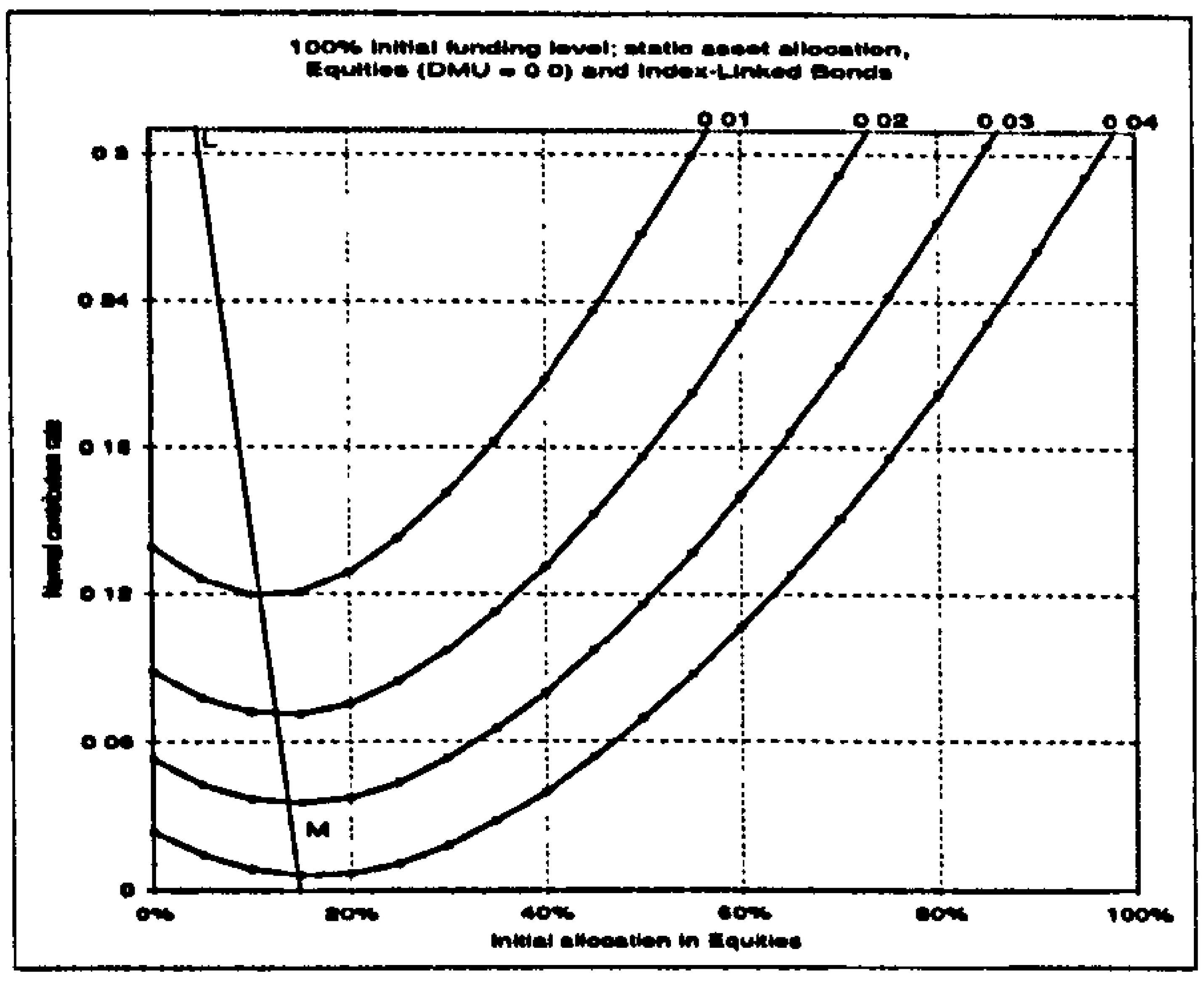

Figure C.5: Excess Contribution Rate risk indifference curves at the end of 6 years

C.4.3 The efficient region for equities and index-linked bonds

As shown in Chapter 6, efficient asset allocations can only be determined by combining the indifference curves for the Mean Shortfall risk and the Excess Contribution Rate risk to form an Efficient Region. Figure C.6 shows the efficient region LMPQ for stochastic projections over 15 years in the case where we invest in equities and index-linked bonds.

We observe that, as expected, the efficient asset allocations would have lower allocations in equities in the case where we invest in equities and index-linked bonds than in the case where we invest in equities and fixed-interest bonds (see Figure C.3 for comparison).

We also observe that, although for our pension scheme index-linked bonds could be considered as the asset that best matches the liabilities, the efficient region does not include 100% asset allocation in index-linked bonds for our chosen range of Normal contribution rates. This illustrates the benefits of diversification. Thus, assuming as we have done that index-linked bonds were sufficiently available, allocating our entire pension fund in index-linked bonds would not necessarily be an efficient static asset allocation strategy.

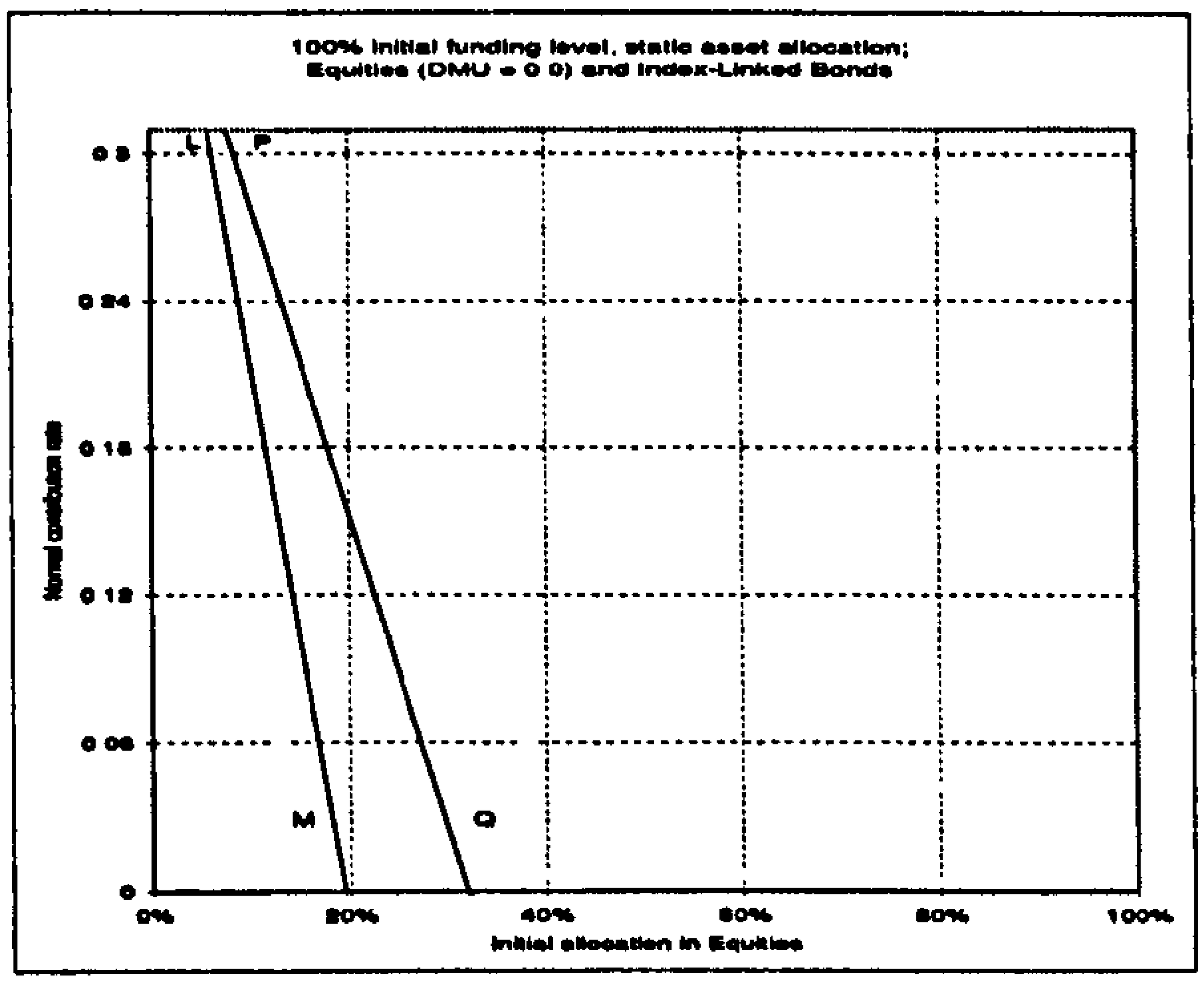

Figure C.6: The efficient region at the end of 15 years

Appendix D

The Effect of the Equity Risk Premium

D.1 Introduction

In this appendix we investigate the sensitivity of our results to changes in the Equity Risk Premium. This investigation is carried out by considering three different values of the long-term mean rate of real dividend growth[1] in the Wilkie Model. Wilkie (1995, p. 845) comments that using his original model "a 95% confidence interval for [the mean rate of real dividend growth] could be from about -0.9% to about +4.0%". We thus consider the following dividend growth rates: 0, 0.02, and 0.04.

Intuitively, we would expect the following results. Firstly, for a given normal contribution rate and initial equities allocation we would expect the Mean Shortfall risk, the Excess Contribution rate risk and the Average contribution rate to be lower in the high mean real dividend growth rate case than in the low (or zero) dividend growth case.

Secondly, if we initially allocate 0% to equities (i.e. 100% to gilts), then reducing (or increasing) the mean real dividend growth rate has no effect on both the risk levels and the Average contribution rate.

And thirdly, if the mean real dividend growth rate were high, then we would expect the decision maker to opt for a higher initial equities allocation. This is due to the fact that we would expect the efficient region to shift towards higher initial equities allocation in the high dividend growth case.

In the stochastic projections we consider a range of possible combinations of Normal contribution rates and asset allocations. Due to space constraints we only consider a

[1] We adopt Wilkie's notation and signify the long-term mean rate of real dividend growth as DMU in Figures D.1 and D.2

projection horizon of 15 years. At the end of the projection horizon we calculate the Mean Shortfall risk, the Excess Contribution Rate risk and the Average Contribution rate. We analyze the results using the idea of indifference curves that was introduced in Chapter 6.

D.2 Mean Shortfall and Excess Contribution rate risks

We firstly consider the results for the Mean Shortfall risk and the Excess Contribution rate risk. Figure D.1 shows risk levels at the end of 15 years.

As expected, for a given normal contribution rate and initial allocation in Equities, a higher mean real dividend growth rate leads to lower risk (except at 0% equities). Furthermore, the reduction in the risk level is very considerable since, for instance, by doubling the dividend growth rate the Mean Shortfall risk is almost halved. We illustrate this finding in the following examples.

Example 1

To illustrate the results, consider a normal contribution rate of 0.18 and 50% initial allocation in equities. This stands for a balanced asset portfolio. We observe that the Mean Shortfall risk is 0.062 for a dividend growth rate of 0; but the risk is almost halved to 0.037 for a dividend growth rate of 0.02; and the risk is further reduced to 0.018 for a dividend growth rate of 0.04.

Meanwhile, the Excess Contribution rate risk is 0.042 for a dividend growth rate of 0; but reduces to 0.029 for dividend growth rate of 0.02; and is further reduced to 0.019 for dividend growth rate of 0.04.

Example 2

As a second example of the effect of the dividend growth parameter we consider a normal contribution rate of 0.12 and 100% initial allocation in equities. This represents an asset portfolio with the whole fund invested in equities.

We observe that the Mean Shortfall risk is 0.104 for dividend growth rate of 0; but is more than halved to 0.048 for dividend growth rate of 0.02; and is further greatly reduced to 0.020 for dividend growth rate of 0.04.

Whilst the Excess Contribution rate risk is 0.084 for dividend growth rate of 0; it reduces to 0.054 for dividend growth rate of 0.02; and is further reduced to 0.032

for dividend growth rate of 0.04.

Thus both the Mean Shortfall risk and the Excess Contribution rate risk levels are very sensitive to the changes in the dividend growth rate. In the next section we deal with the effect on the efficient region.

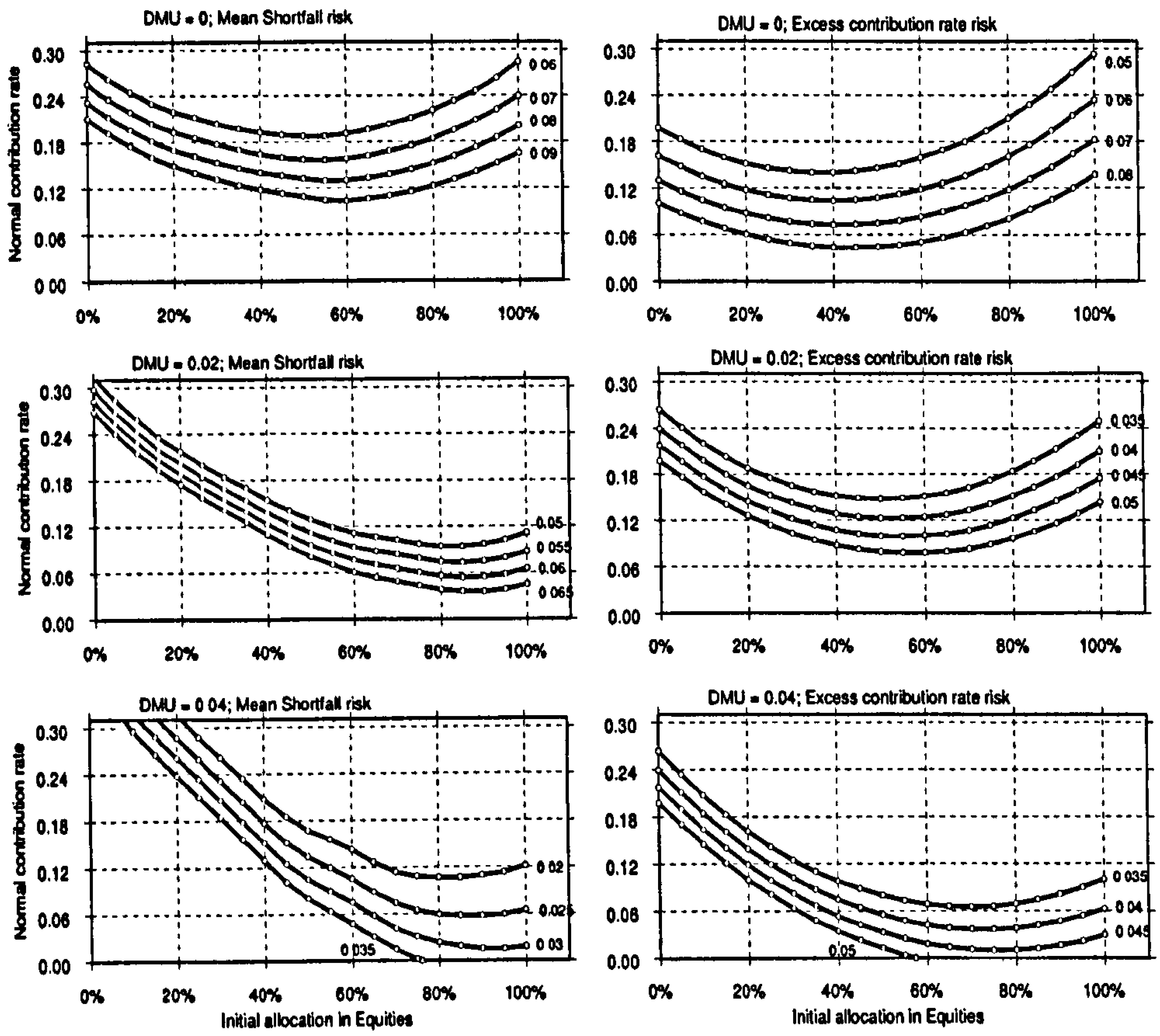

Figure D.1: Mean Shortfall and Excess contribution rate risk levels at the end of 15 years (static asset allocation)

D.3 The Efficient Regions

Figure D.2 shows the efficient regions at the end of 15 years for different estimates of the mean real dividend growth rate. ABCD is the efficient region if the mean real dividend growth rate is 0; EFGH is the efficient region if the mean real dividend growth rate is 0.02; and JKLM is the efficient region if the mean real dividend growth rate is 0.04.

As expected, the efficient region shifts towards higher initial equities allocation as the mean real dividend growth rate increases. For instance, the minimum points of of the Mean Shortfall risk shift from BC (when DMU is 0) to KL (when DMU is

0.04). Whilst the minimum points for the Excess Contribution rate risk shift from AD to JM. And the efficient regions ABCD and JKLM do not overlap.

Furthermore, even though the efficient regions when DMU is 0 and when DMU is 0.02 (that is, ABCD and EFGH) overlap, their region of intersection (EBCH) is not significantly large in comparison to the efficient regions.

This means that the optimal asset allocations are very sensitive to changes in the dividend growth parameter. This is especially the case when we change the parameter from 0 to either 0.02 or 0.04.

However, in changing the parameter from 0.02 to 0.04, we might say that the optimal asset allocations are not very sensitive. Firstly, the region of intersection (JFGM) is comparable to the size of the efficient regions. And secondly, the shifts in the minimum points are not very large. For example, for the Mean Shortfall risk we get a shift from FG to KL which is not as large as in the case of BC to FG or BC to KL.

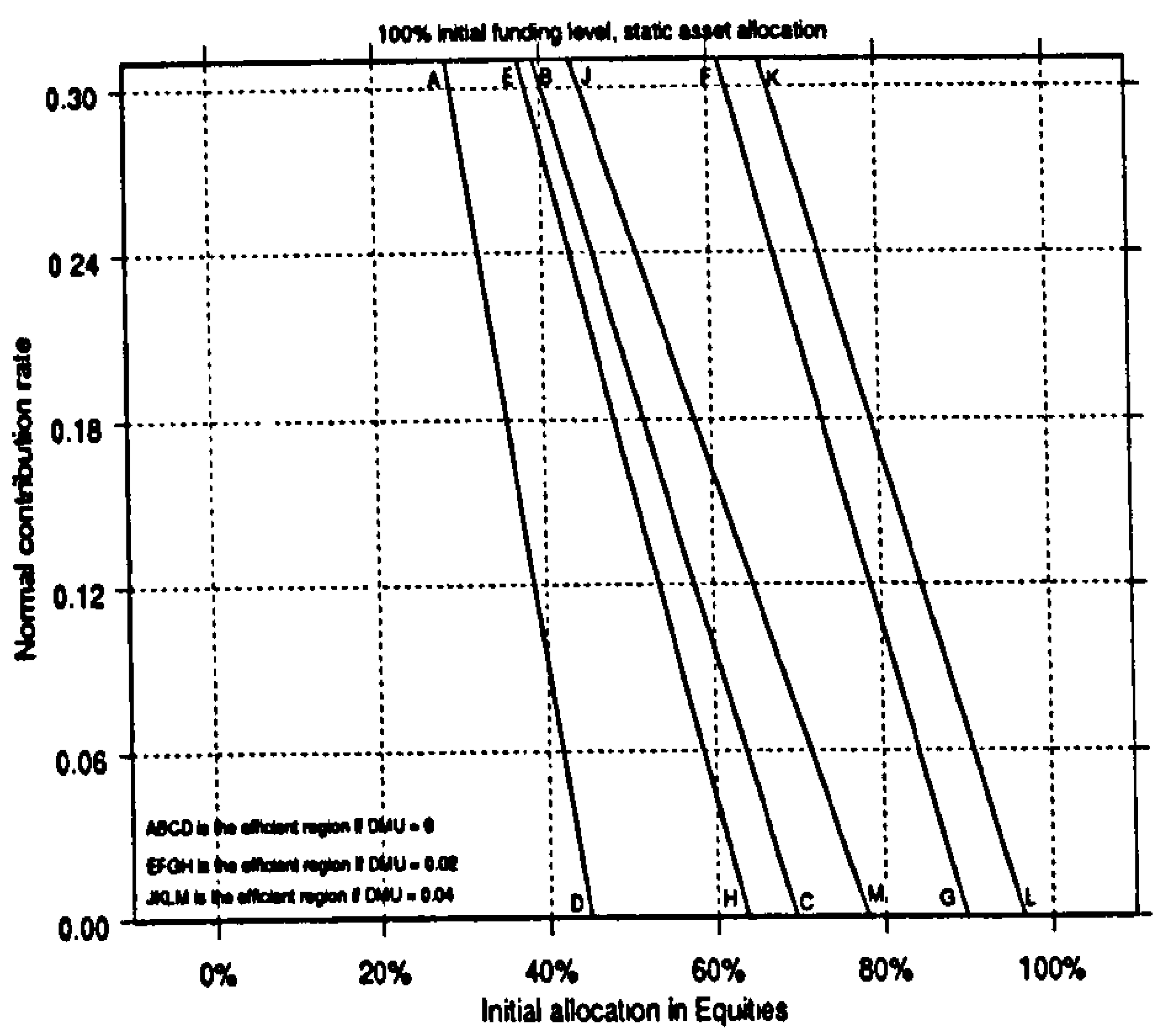

Figure D.2: The Efficient regions for a 15-year projection period for different estimates of the dividend growth rate (DMU)

D.4 Average contribution rate levels

We do not show the results for the Average contribution rate due to space constraints. However, in a similar way to the Mean Shortfall risk and the Excess Contribution rate risk, the results for the Average contribution rate show a significant sensitivity to changes in the dividend growth rate. Furthermore, as expected, the Average contribution rate decreases as the mean real dividend growth rate increases.

D.5 Concluding Remarks

We have shown that the Mean Shortfall risk, the Excess Contribution rate risk and the Average contribution rate are sensitive to the risk premium on equities. In the low equity risk premium scenario, optimal asset allocations have a low proportion in equities and both the Mean Shortfall risk and the Excess Contribution rate risk are high. Furthermore, the cost as measured by the Average contribution rate is high.

On the other hand, in the high equity risk premium scenario, optimal asset allocations have a high proportion in equities and the risks as well as the cost are low.

Appendix E

The Effect of Low and Stable Inflation

E.1 Introduction

In this section we consider the effect of low and stable inflation on the indifference curves. Instead of considering the effect of an entirely different investment model, we have decided to consider only different estimates for the parameters in Wilkie's autoregressive inflation model.

Wilkie (1995) estimated the parameters by considering the economic data for the period 1923-1994. The parameter estimates for the long-term inflation mean and the inflation volatility are 0.047 and 0.0425, respectively[1].

In this sensitivity analysis we consider a lower long-term inflation mean of 0.025 and a lower inflation volatility of 0.01548. The 0.01548 inflation volatility estimate is based on economic data for the period 1982-1994 (see Khorasanee (1999)).

The need for caution in setting the values of the mean parameters in asset allocation problems is accentuated by Chopra and Ziemba (1993). They study the effect on optimal asset allocation in a mean-variance framework of errors in the estimation of the values for the mean, variances and covariances of asset distribution.

They observe that "...misspecification of the parameters of the return distribution...does make a significant difference. Specifically, errors in means are about ten times as important as errors in variances and covariances..." (p. 7).

[1]We adopt Wilkie's notation and refer to the long-term inflation mean as QMU and the long-term inflation volatility as QSD

E.2 Modelling Framework

We employ the same framework as in Chapter 6. We assume that the investment returns are generated under the Wilkie Model as set out in Wilkie (1995) with various values for QMU and QSD. We consider a range of possible combinations of Normal contribution rates and asset allocations.

For each combination we carry out stochastic projections of the pension scheme over 3 years and 15 years. At the end of 3 years we only calculate the Mean Shortfall risk; whilst at the end of 15 years we calculate the Mean Shortfall risk, the Excess Contribution Rate risk and the Average contribution rate.

For comparison purposes we split the inflation scenarios we are considering into four cases:

Case 1 : QMU = 0.047 and QSD = 0.0425 (standard case)

Case 2 : QMU = 0.047 and QSD = 0.01548 (intermediate case)

Case 3 : QMU = 0.025 and QSD = 0.0425 (intermediate case)

Case 4 : QMU = 0.025 and QSD = 0.01548 (low and stable inflation case)

In Cases 1 to 4, all other factors are kept the same.

In Case 1 versus Case 2 and Case 3 versus Case 4, we leave the long-term inflation mean unchanged whilst the long-term inflation volatility is reduced from 0.0425 to 0.01548. Lower inflation volatility implies lower uncertainty in the inflation. Thus, we would expect lower Mean Shortfall risk and lower Excess Contribution rate risk in Cases 2 and 4 than in Cases 1 and 3, respectively. Furthermore, in the cases where inflation volatility is low, we would expect fixed-interest bonds to provide a better match for the liabilities than equities. Hence, we would expect lower initial allocation in equities in Cases 2 and 4 than in Cases 1 and 3, respectively. (In other words, the efficient region would shift towards a lower equities allocation in Cases 2 and 4).

In Case 1 versus Case 3 and Case 2 versus Case 4, we leave the long-term inflation volatility unchanged whilst the long-term inflation mean is reduced from 0.047 to 0.025. In this situation, we would expect our results to be complicated because of the definition of Limited Price Indexation (LPI). In the model, we have assumed that pensions-in-payment are increased at the lower of 5% and the Retail Price Index (RPI) with a lower bound of 0. In the cases where the inflation mean is low and the inflation volatility is high, for instance Case 3, we would expect to obtain more scenarios of negative inflation. With the current definition of the LPI, the scheme would not 'benefit' from negative inflation since the pensions-in-payment would be kept constant whilst the inflation is negative. However, this could be a short-term problem only, since over the long-term we expect the inflation to revert towards the mean (given the properties of the Wilkie model).

E.3 Results of Stochastic Projections

We firstly present the results for the Mean Shortfall risk.

E.3.1 Mean Shortfall Risk

Projections over 3 years

Figure E.1 shows the Mean Shortfall risk indifference curves at the end of 3 years. For the intermediate cases, we observe that firstly as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that the low inflation volatility leads to a lower Mean Shortfall Risk (except for higher equities allocation).

And secondly, leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies, surprisingly, that the low long-term inflation mean leads to higher Mean Shortfall risk.

For Cases 1 and 4, we observe that, for a given normal contribution rate and asset allocation, the low and stable inflation case (Case 4) leads to lower Mean Shortfall risk than in the standard inflation case (Case 1). However, the differences in the risk levels are not very considerable. This implies that, for 3-year projections, the risk levels are not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

Projections over 15 years

The results for the Mean Shortfall risk for projections over 15 years are shown in Figure E.2. For the intermediate cases we observe that, as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that low inflation volatility leads to a lower Mean Shortfall risk (this is more obvious for lower initial allocations in equities).

Meanwhile, leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low long-term inflation mean leads to a lower Mean Shortfall risk (this is more

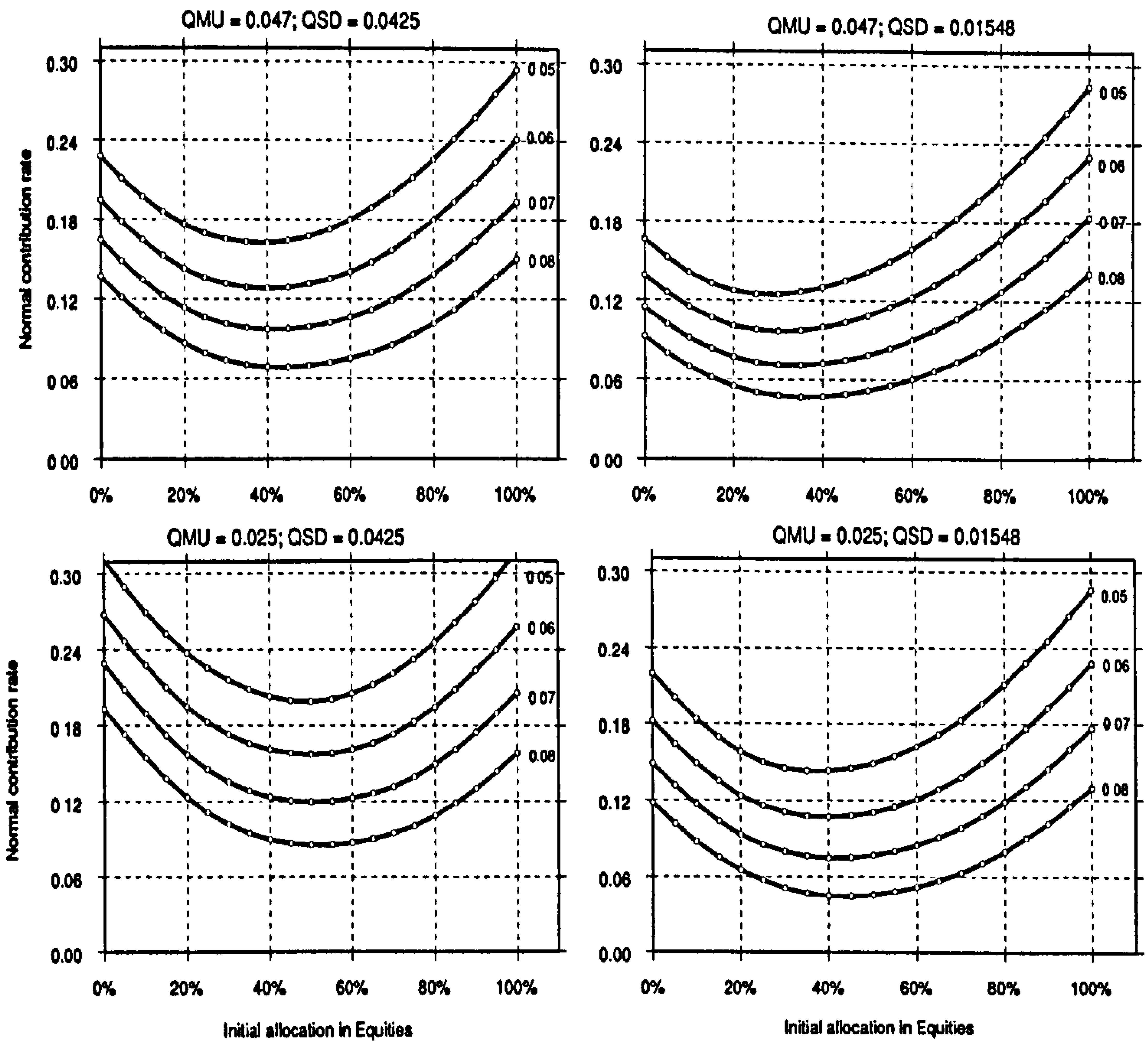

Figure E.1: Mean Shortfall risk levels at the end of 3 years

evident for higher initial allocations in equities).

And for Cases 1 and 4, we observe that, as for the 3-year projections, the low and stable inflation case leads to lower Mean Shortfall risk than in the standard inflation case. Furthermore, there are substantial differences in the risk levels. This means that, for projections over 15 years, the risk levels are fairly sensitive to the change from the standard inflation case to the low and stable inflation scenario.

E.3.2 Excess Contribution Rate Risk

Figure E.3 shows the Excess Contribution rate risk indifference curves at the end of 15 years. For the intermediate cases, we observe that, as expected, leaving the long-term inflation mean unchanged whilst decreasing the volatility (i.e. Case 1 versus Case 2 and Case 3 versus Case 4) implies that low inflation volatility leads to a lower Excess Contribution Rate Risk.

Also, leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low

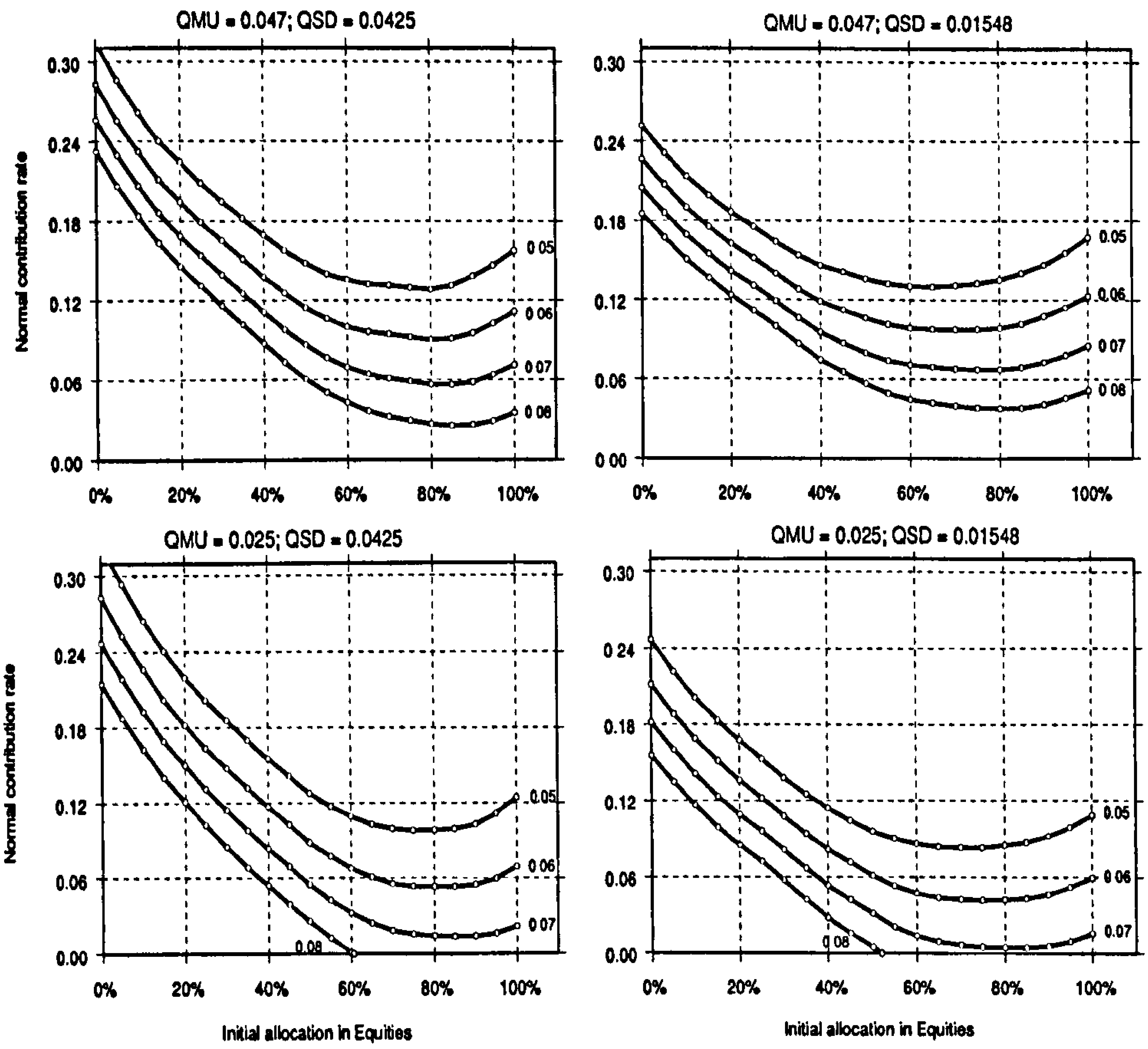

Figure E.2: Mean Shortfall risk levels at the end of 15 years

long-term inflation mean leads to a higher Excess Contribution rate risk.

And for Cases 1 and 4, we observe that the low and stable inflation case leads to higher Excess Contribution rate risk than in the standard inflation case. Nevertheless, the differences in the risk levels are not very considerable. This implies that, compared to the Mean Shortfall risk, the Excess Contribution rate risk levels are not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

E.3.3 Average contribution rate levels

The results for the Average contribution rate for projections over 15 years are illustrated in Figure E.4. For the intermediate cases, we observe that leaving the long-term inflation mean unchanged at 0.047 whilst decreasing the volatility (i.e. Case 1 versus Case 2) implies that low volatility leads to slightly higher average contribution rate.

However, leaving the long-term inflation mean unchanged at 0.025 whilst decreas-

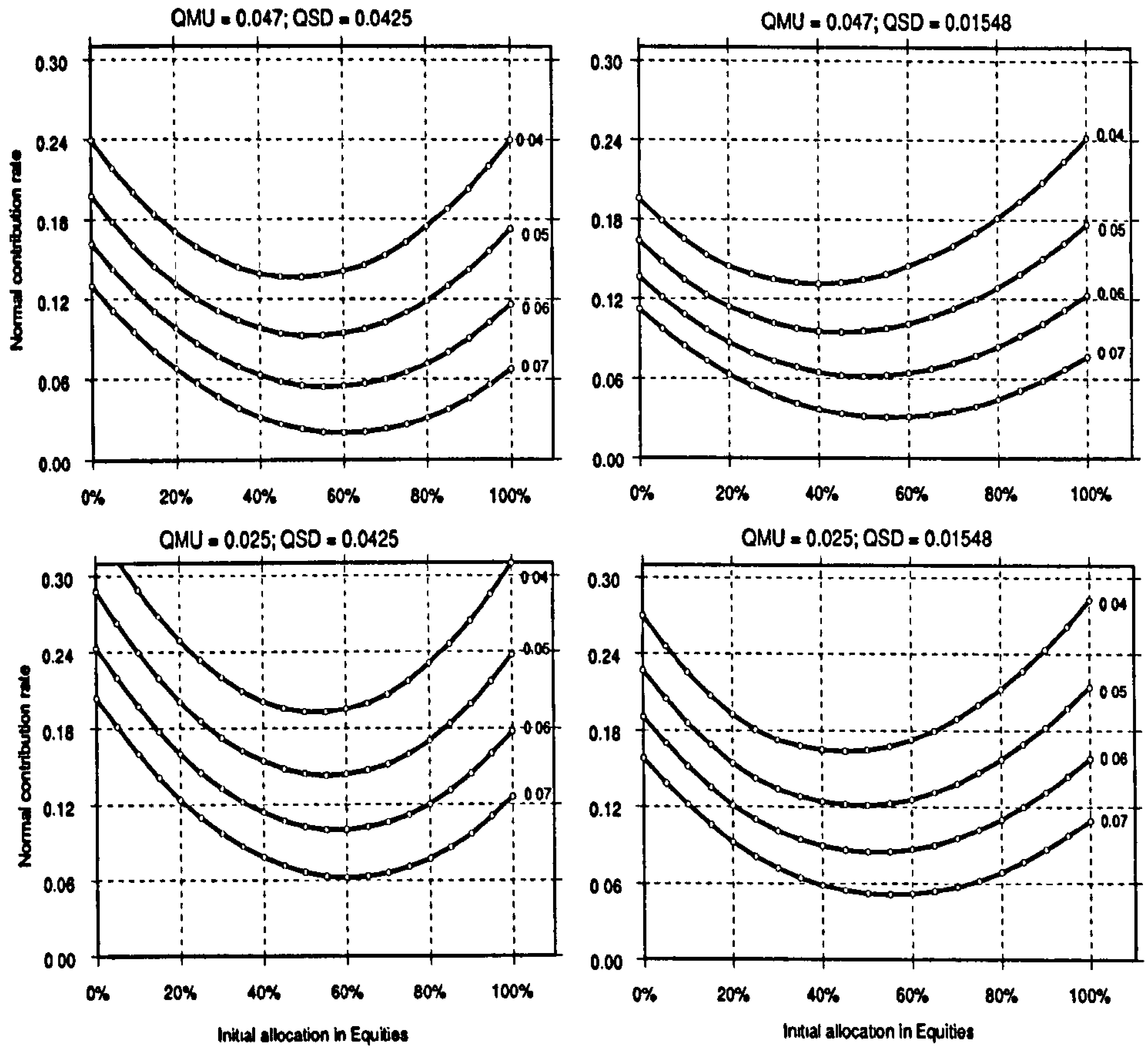

Figure E.3: Excess Contribution rate risk levels at the end of 15 years

ing the volatility (i.e. Case 3 versus Case 4) implies that low volatility leads to a lower average contribution rate. And furthermore, leaving the inflation volatility unchanged whilst decreasing the long-term inflation mean (i.e. Case 1 versus Case 3 and Case 2 versus Case 4) implies that low long-term inflation mean leads to a higher average contribution rate.

Lastly, for Cases 1 and 4, we observe that the low and stable inflation case leads to a higher average contribution rate than in the standard inflation case. Nevertheless, the differences in the average contribution rate levels in the two cases are not significant. This implies that the average contribution rate levels are not very sensitive to changes in the inflation scenario.

E.3.4 The Efficient Regions

Figure E.5 shows the efficient regions for projections over 15 years for the standard case (Casel) and the low and stable inflation case (Case 4). LMPQ is the efficient region for the standard case whilst ABCD is the efficient region for the low and stable inflation case.

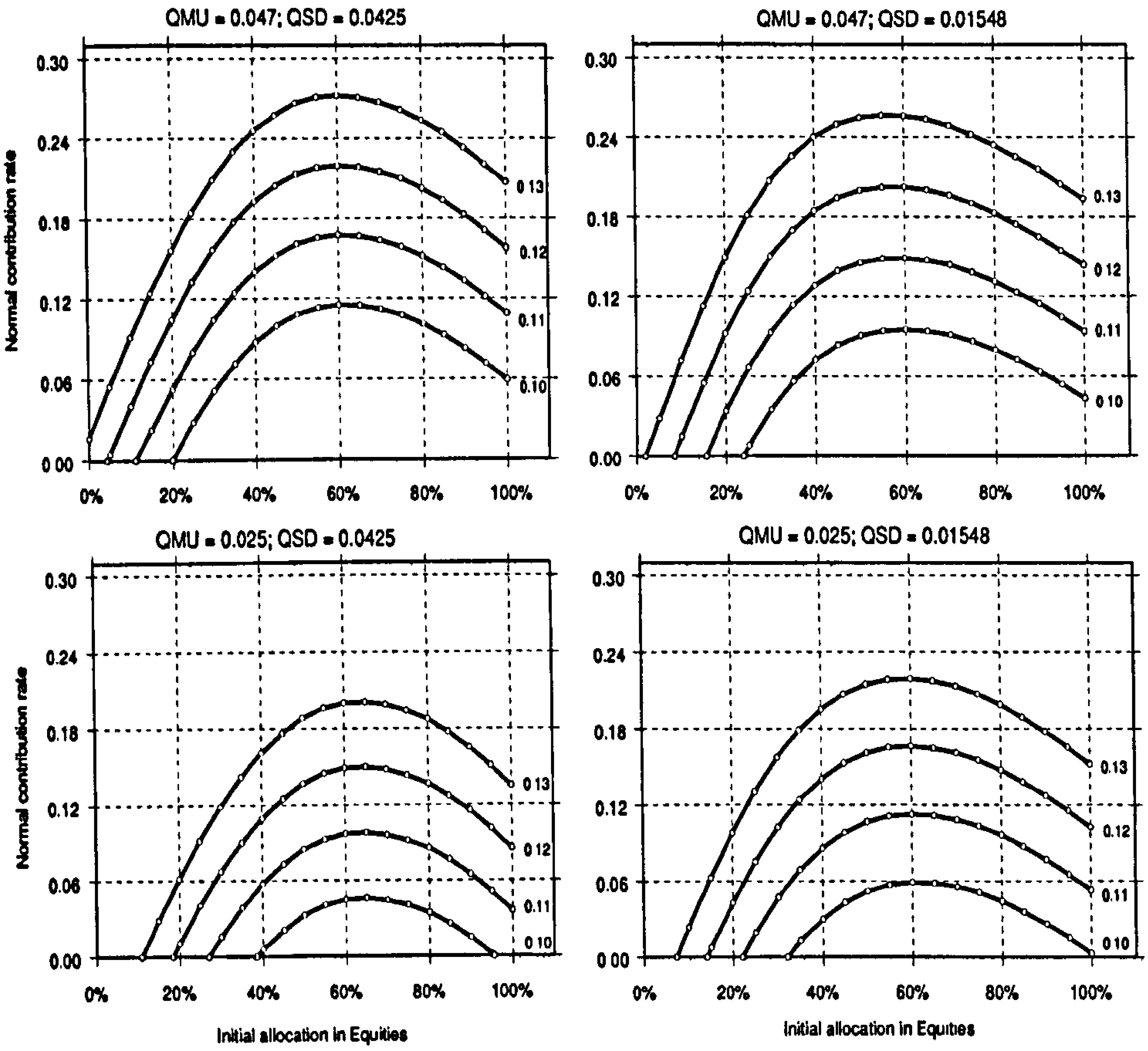

Figure E.4: Average contribution rate levels at the end of 15 years

We observe that in a low and stable inflation scenario the efficient region shifts towards less initial allocation in equities as compared to the standard inflation scenario. Thus, as expected, in the low and stable inflation case we obtain a shift in asset allocation towards fixed-interest bonds. Furthermore, in the low and stable inflation case, we get a 'smaller' efficient region (i.e. the efficient region 'shrinks' slightly). This result follows from the fact that there is less uncertainty when the inflation is low and stable.

However, the difference in the positions of the efficient regions is not substantial. This implies that the asset allocation decisions will be similar in the two inflation scenarios. Thus, the efficient region is not very sensitive to the change from the standard inflation case to the low and stable inflation scenario.

In the intermediate cases (Cases 2 and 3), we also conclude that the efficient regions are not very sensitive. This is because Figure E.2 shows that the Mean Shortfall risk minimum points are located in a similar region. Furthermore, Figure E.3 shows that the minimum points for the Excess Contribution rate risk curves are also in a similar region.

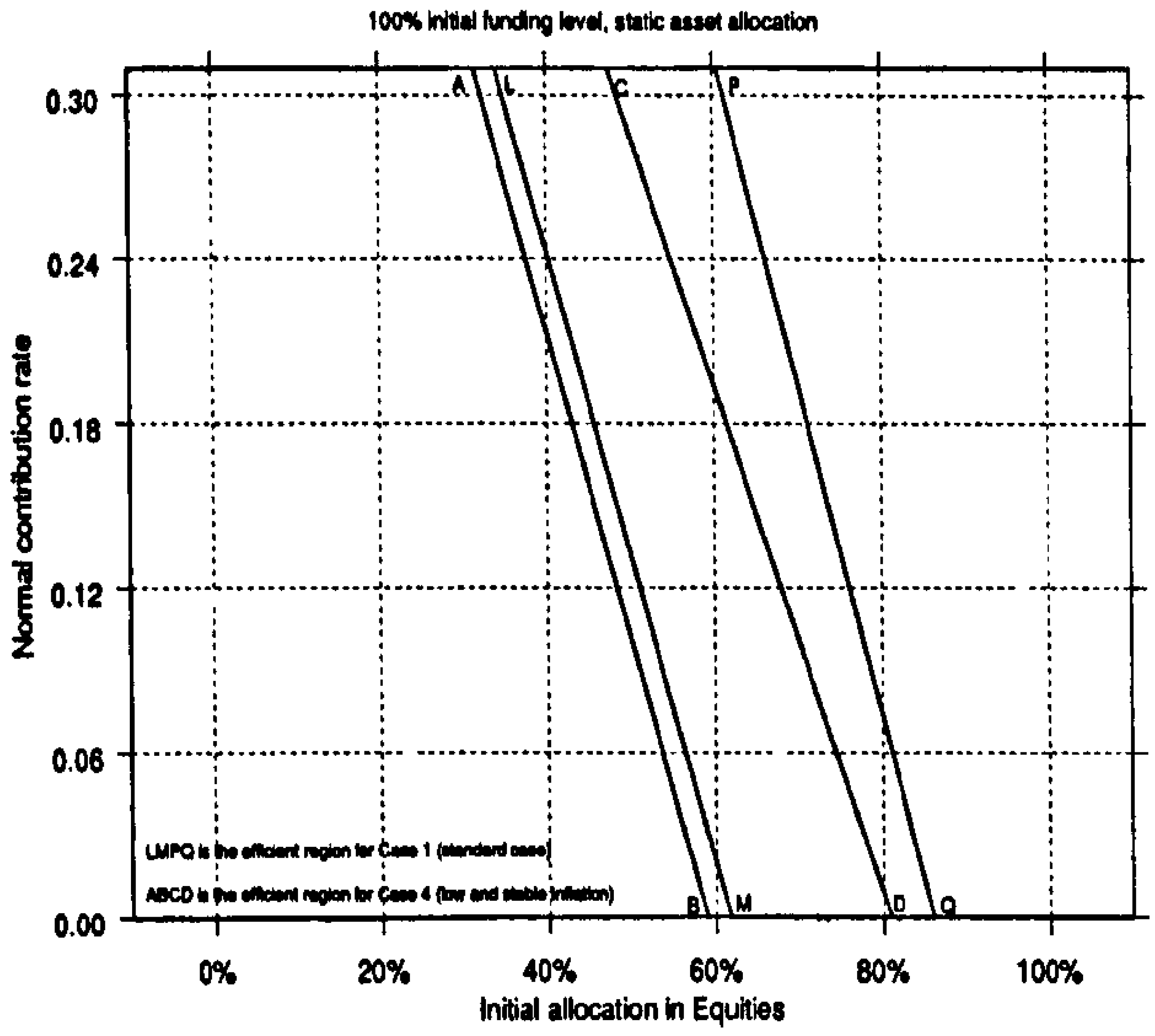

Figure E.5: The efficient regions for projections over 15 years for Case 1 (standard case) and Case 4 (low and stable inflation)

E.3.5 Pension increases at Retail Price Index (RPI)

Due to space constraints, we have not endeavoured to include a detailed section on the sensitivity analysis of the results to changes in the definition of the LPI. Thus, we will only summarize the results which we have obtained when pensions-in-payment are increased at the RPI.

We compare results from two new cases: in the first case we set the inflation mean and volatility at 0.047 and 0.0425, respectively; whilst in the second case we set the inflation mean and volatility at 0.025 and 0.0425, respectively. In both cases pensions-in-payment are increased at the RPI only.

The following results were obtained. Firstly, for projections over 3 years, the Mean Shortfall risk is higher in the second case (the low inflation mean case) than in the first case only for low initial allocation in equities (i.e. for equities allocations less than approximately 40%); whilst the Mean Shortfall risk is lower in the low inflation mean case (second case) than in the first case for initial allocation in equities greater than approximately 40%.

Meanwhile, for projections over 15 years, the Mean Shortfall risk is lower in the low inflation mean case than in the high mean case; whilst both the Excess Contribution rate risk and the Average contribution rate are higher in the low inflation mean case than in the high mean case.

These results shows that our conclusions are similar in the RPI case and LPI case except for projections over 3 years. Thus, in the 3-year projections the LPI complicates the results. However, these results also show that, as observed above, this problem occurs in the short-term only.

E.4 Concluding Remarks

For projections over 3 years, we get the surprising result that reducing only the long-term inflation mean leads to a higher Mean Shortfall risk. This result is due to the limited price index (LPI). In Section E.3.5 we have shown a further sensitivity analysis of the effect of the LPI by increasing pensions in payment at only the Retail Price Index. This analysis shows that keeping the inflation volatility unchanged at 0.0425 whilst decreasing the long-term inflation mean from 0.047 to 0.025 leads to a lower Mean Shortfall risk (except for a very low initial allocation in equities).

For projections over 15 years, the low and stable inflation scenario leads to a lower Mean Shortfall risk than in the standard inflation scenario; the Excess Contribution rate risk is higher in the low and stable inflation case than in the standard inflation case. Also, the Average contribution rate is higher in the low and stable inflation scenario. However, although the Mean Shortfall risk is fairly sensitive to the inflation scenarios, the Excess Contribution rate risk and the Average contribution rate are not very sensitive. Furthermore, the asset allocation decisions are not very sensitive to the inflation scenarios because of the lack of sensitivity of the efficient regions.

Appendix F

Effect of Valuations on the Shortfall & Default Risk

F.1 Introduction

In Chapter 8 we considered risk management in the presence of sponsor default. We presented optimal investment strategies under the Mean Shortfall risk which assumes that the sponsor does not default during the projection period and under the Shortfall & Default risk where we seek to minimize defaulted shortfalls.

In this section we conduct a sensitivity analysis of the results in Chapter 8 to changes in the frequency of pension scheme valuations and the period over which gains or losses are spread. In Chapter 8 we assumed that a valuation of the pension scheme is undertaken every three years and that all gains or losses arising at the valuation date are spread over a three year period. Thus at the valuation date we recommend a contribution rate by adjusting the normal contribution rate depending on whether gains or losses have arisen in the scheme. The recommended contribution rate is kept constant during the intervaluation period.

In this section we consider two further scenarios. In the first scenario triennial valuations are carried out but the arising gains or losses are spread over a period of 12 years. In the second scenario annual valuations are undertaken and the arising gains or losses are spread over one year.

Thus the first scenario is similar to our standard scenario in that in both scenarios a recommended contribution rate is established at the triennial valuation and is then kept constant during the intervaluation period. However, all things equal, the recommended contribution rate will defer in the two scenarios due to the spread period.

Meanwhile, in the second scenario a recommended contribution rate is set annually and is thus effective for only one year.

The first scenario was considered in Chapter 6 and Haberman et al. (2003a) for the Mean Shortfall risk in tandem with the Excess Contribution Rate risk. Thus the results were interpreted in terms of the effect of the spread period on the efficient region for the solvency and contribution rate risks.

In this section we consider the effect of the frequency of valuations and spread periods on the Mean Shortfall risk and the Shortfall & Default risk and on the optimal asset allocations under these two risk measures.

F.2 Intuitive Results

F.2.1 Mean Shortfall risk

As shown in Chapter 6 and in Haberman et al. (2003a), we would expect that for triennial valuations, all other things equal, for a given Normal contribution rate and initial asset allocation spreading gains or losses over three years (standard scenario) would lead to lower Mean Shortfall risk than when the spread period is twelve years.

We would further expect that, all other things equal, undertaking annual valuations and spreading gains or losses over one year would lead to lower Mean Shortfall risk than in either of the triennial valuation scenarios.

As shown in Equation 8.10 the Mean Shortfall risk depends on the probability of shortfall and on the expected amount of shortfall in the event that a shortfall has occurred. The frequency of valuations and the spread period would have an effect on both components of the Mean Shortfall risk.

For instance, all other things equal, if the frequency of valuations is high any arising losses will tend to be recouped quicker than when the frequency is low. Thus both the probability of shortfall and the amount of shortfall would decrease.

Furthermore, for a given frequency of valuation (for instance, triennial) spreading losses over the intervaluation period will lead to losses being recouped quicker than when losses are spread over a period longer than the intervaluation period.

F.2.2 Shortfall & Default risk

The Shortfall & Default risk is composed, like the Mean Shortfall risk, of two components: the probability of default events and the expected amount of shortfall defaulted given that a default event has taken place. Furthermore, the probability of a default event is dependent on the performance of the market portfolio and the probability of a shortfall.

The performance of the market portfolio is independent of the frequency of valuations and spread period. However, the frequency of valuation and the spread period would affect the probability of shortfall and the defaulted amount in the same way as for the Mean Shortfall risk.

Thus we would expect, all other things equal, that the Shortfall & Default risk would be lower in the annual valuations scenario than in the triennial valuations scenario. Furthermore, for the triennial valuations scenario we would expect that the Shortfall & Default risk would be lower in the case where the spread period is equal to the intervaluation period than in the case where the spread period is much longer (i.e. twelve years).

F.3 Stochastic Projections results

Figures F.1 and F.2 show the risk levels at the end of 6 and 15 years, respectively, for all the three scenarios. We do not show the results at the end of one year since we assume that our pension scheme is initially fully-funded. Hence, differences in the initial valuations would have no effect on the financial position of the scheme at the end of the first year.

F.3.1 Risk Levels

The results in the two figures confirm our initial expectations. For the triennial valuations, for a given Normal contribution rate and asset allocation, both the Mean Shortfall risk and the Shortfall & Default risk increase as the spread period is increased from three to twelve years. This result holds for both projection periods.

Furthermore, as expected, our results show that the annual valuations scenario leads to lower Mean Shortfall risk and lower Shortfall & Default risk than either of the triennial valuations scenarios.

On the other hand, our results do not show the effect of the frequency of valuations and spread period on the contribution rate risk and the average contribution rate. We can however infer from Chapter 6 and Haberman et al. (2003a) that for those cases where the Mean Shortfall risk decreases the 'penalty' would be a high Excess Contribution Rate risk.

F.3.2 Asset allocation

The results also show that the optimal asset allocations under the two risk measures do not change massively as the frequency of valuation and spread period are varied.

However, a close comparison of the standard scenario (triennial valuations with 3-year spread period) and the annual valuations scenario shows that the optimal asset allocations change differently under the two risk measures. Although these differences are not very significant further investigation would be necessary to check whether this observation holds in general.

We observe that, for both projection periods, changing from the standard scenario to the annual valuations scenario leads to slightly higher weighting in equities under the Mean Shortfall risk. Meanwhile, under the Shortfall & Default risk changing from the standard scenario to the annual valuations scenario leads to slightly lower weighting in equities.

This implies that the difference in the optimal asset allocation under the two risk measures is much less in the triennial valuations (with three-year spread period) scenario than the difference in the optimal asset allocation under the two risk measures in the annual valuations scenario.

On the other hand, the optimal asset allocations exhibit different variability as the spread period is changed. The optimal asset allocations obtained when we assume triennial valuations and a twelve year spread period show the greatest variability. Meanwhile the optimal asset allocations in the case of annual valuations show the least variability.

F.4 Concluding Remarks

In conclusion, our results show that both the optimal asset allocations and risk levels are sensitive to choices for the frequency of valuation and spread period. Although the optimal asset allocations are much less sensitive than the risk levels, it is evident that the two risk measures lead to different optimal asset allocations as the frequency of valuation is varied.

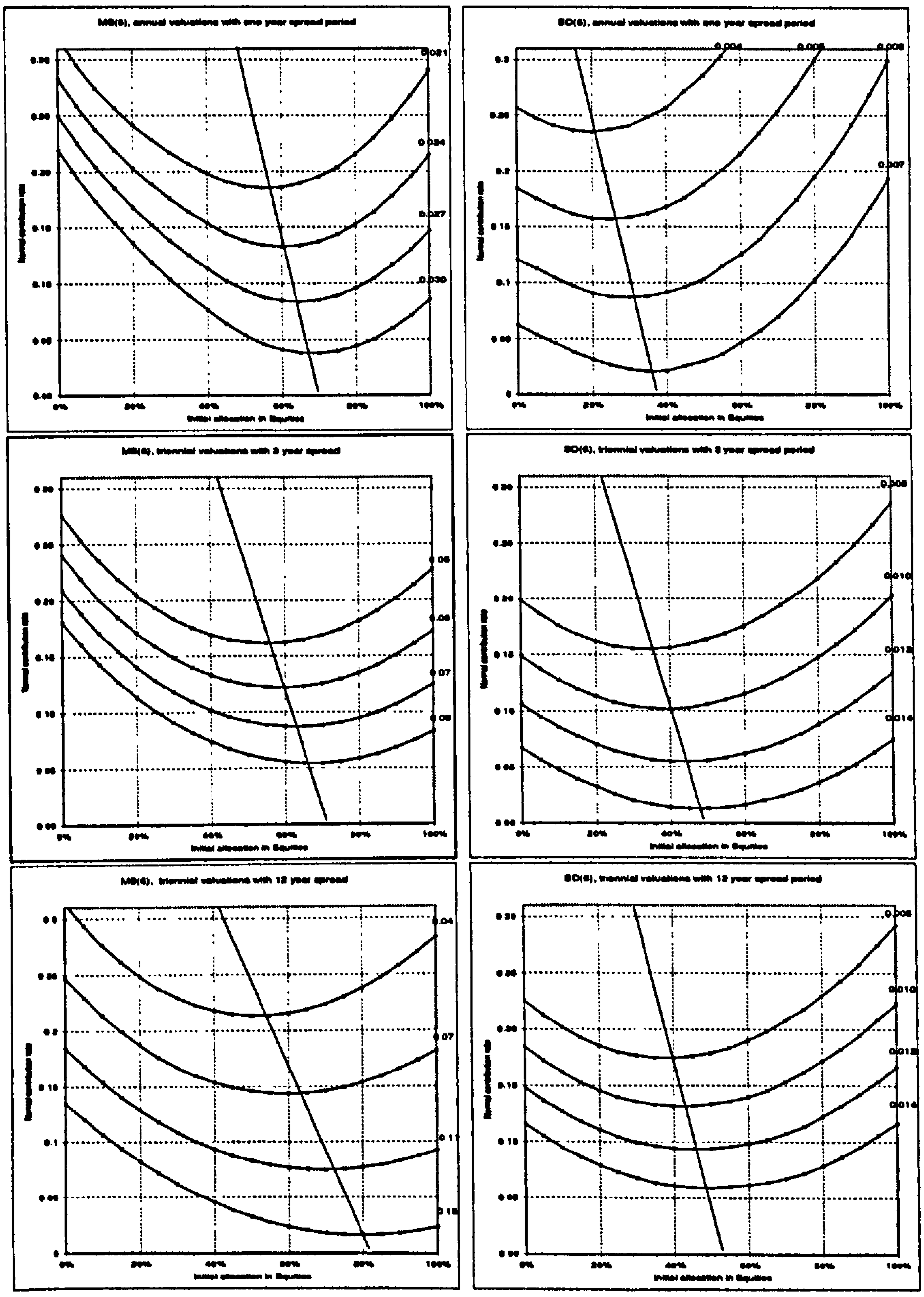

Figure F.1: Mean Shortfall and Shortfall & Default risk levels at the end of 6 years

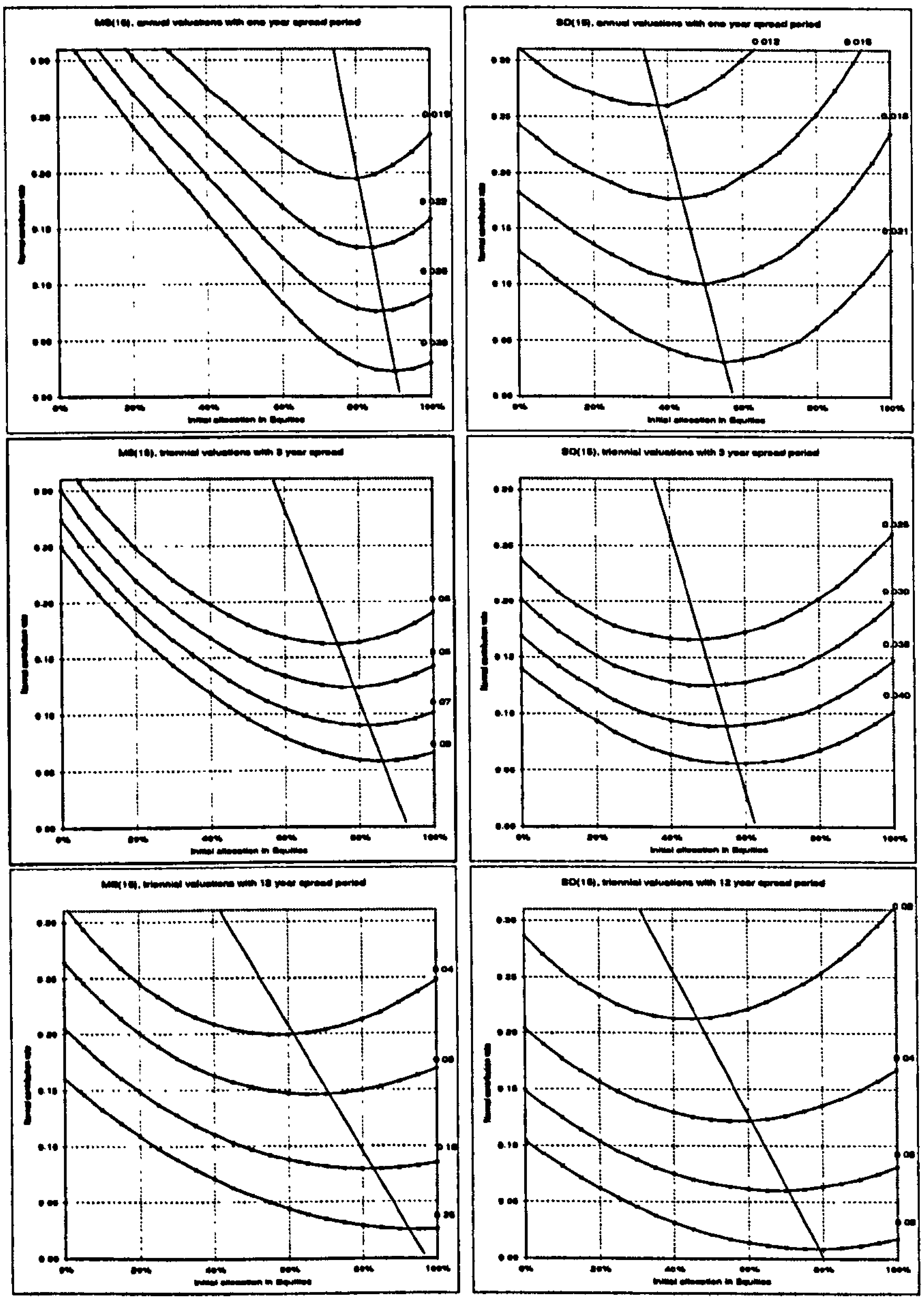

Figure F.2: Mean Shortfall and Shortfall & Default risk levels at the end of 15 years

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