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A Trio of Dualities: Walls, Trees and Cascades

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Abstract:

We study the RG flow of $\mathcal{N}=1$ world-volume gauge theories of D3-brane probes on certain singular Calabi-Yau threefolds. Taking the gauge theories out of conformality by introducing fractional branes, we compute the NSVZ beta-function and follow the subsequent RG flow in the cascading manner of Klebanov-Strassler. We study the duality trees that blossom from various Seiberg dualities and encode possible cascades. We observe the appearance of duality walls, a finite limit energy scale in the UV beyond which the dualization cascade cannot proceed. Diophantine equations of the Markov type characterize the dual phases of these theories. We discuss how the classification of Markov equations for different geometries into families relates the RG flows of the corresponding gauge theories.

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1 Introduction and Conclusions

D3-brane probes on singular Calabi-Yau threefolds have been widely used both as a means to study (potentially realistic) four dimensional, $\mathcal{N}=1$ supersymmetric gauge theories and as a tool for the investigations of many beautiful mathematical phenomena (for a review, cf. e.g. [1]). A central phenomenon in these theories is the the existence of Seiberg duality [2]. This corresponds to an exact equivalence between different gauge theories in the IR limit. Armed with various technologies, such as the Inverse Algorithm (which

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computes the gauge theories on the world-volume of D3-branes probing toric singularities), (p, q)-webs, exceptional collections and the derived category, a host of gauge theories have been constructed for a plethora of geometries that constitute non-trivial horizons for the bulk AdS theory. Of particular interest are the various guises and generalization of Seiberg duality, called "Toric Duality" in [3, 4] (for a short introduction, cf. e.g. [5]), that emerge naturally when realizing the gauge theories geometrically in String Theory.

The world-volume theories on the D-brane probes are quiver theories. By successive applications of dualities, it is possible to construct, for each probed geometry, an infinite web of dual phases. The constructions that represent the space of dual theories and the connections between them have been affectionately called **Duality Trees** and **Flowers** [6]. As we dualize upon a node in the quiver at each stage, a new branch blossoms.

A famous example which exploits Seiberg duality and which exhibits many intricacies of the AdS/CFT correspondence is the theory for the conifold [7]. With the addition of M fractional branes, the theory is taken out of conformality and there is a subsequent RG flow for the gauge couplings. This running can be identified with the radial variation of the fiveform flux in AdS. The gauge theory interpretation of this flow states that when one of the couplings becomes strong, one, à la Seiberg [2], dualizes the theory and flow to the IR. This RG flow can be followed *ad infinitum* and the process was referred to as a **cascade**. Thus, every time one of the couplings becomes infinite, we switch to an alternative dual description in which the effective number of D3-branes is decreased by M.

When dealing with non-conformal theories, duality trees represent the possible paths or cascades that can be followed along an RG flow. In this context, the topology of the tree can be used to identify the qualitative features of the possible flows. For example, the existence of closed cycles would signify that certain dualities may be trapped within a group of theories, given rise to a cascade resembling the conifold one.

The generalization of the cascade phenomenon to other geometries is hindered by the fact that the conifold is really the only geometry for which we know the metric. Nevertheless nice extensions from the field theory side have been performed [9, 8, 10]. There has been some evidence that for other geometries the cascade is qualitatively different. An interesting possibility is the existence of **Duality Walls**. This concept was introduced in [8], where it appeared as fundamental limitation on the UV scale of the theory when trying to accommodate the Standard Model at the IR limit of a duality cascade.

Inspired by the myriad of dual phases of theories constructed for the immediate and most important generalizations of the conifold, namely the del Pezzo surfaces [3], we study their cascades. We focus on the illustrative example of F_0, the zeroth Hirzebruch surface. We find that there is indeed a duality wall past which Seiberg duality can no longer be performed. To this claim we shall soon supplant with analytic proof [18] and also demonstrate some fascinating chaotic behavior. In due course of our investigations, we shall also delve into such elegant mathematics as resolution of singularities and the emergence of certain Diophantine equations of Markov type which characterize the duality tree. We will discuss how the connection between Markov equations for different singularities can be exploited to relate RG cascades for these geometries. We will see intricate inter-relations between D-brane gauge theories, algebraic singularities, AdS holographic duals, finite graphs and Diophantine equations.

2 The Beta Function

Let us study the quiver theory arriving from a stack of N parallel coincident D3-branes on a singular Calabi-Yau threefold. This is an $\mathcal{N}=1$ conformal 4D theory with gauge group $\prod_i U(N_i)$ and chiral bifundamental multiplets with multiplicities A_{ij} between the i-th and j-th gauge factors (the matrix A_{ij} is the adjacency matrix of the quiver). Anomaly freedom requires that the ranks N_i obey

$$(A - A^T)_{ij} \cdot \vec{N} = 0 \ . \tag{1}$$

We can add M fractional branes, whereby modifying the ranks N_i, in such a way that (1) is still satisfied. These new ranks $n_i = N_i + M_i$ take the theory out of conformality, inducing a nontrivial RG flow. This flow is governed by the NSVZ beta-function [11] that for each gauge group takes the following form

$$\beta_i = \frac{d(8\pi^2/g_i^2)}{d\ln\mu} = \frac{3T(G) - \sum_i T(r_i)(1-\gamma_i)}{1 - \frac{g_i^2}{8\pi^2}T(G)} := \frac{dx_i}{dt} \tag{2}$$

where μ is the energy scale; we have defined x_i to be the inverse square gauge coupling and t to be the log of the energy. For an $SU(N_c)$ gauge group, $T(G) = N_c$ and $T(fund) = 1/2$.

An interesting limit in which computations can be carried on easily, is the one in which the number of fractional branes is much smaller than the number of probe branes. In this regime, the beta functions can be computed as follows [13, 17, 6]. Let γ_{ij} be the anomalous dimensions of the field A_{ij}. The beta functions for the gauge and superpotential couplings, can be written as

$$\begin{aligned}
\beta_{i\in\text{nodes}} &= 3n_i - \frac{1}{2}\sum_{j=1}^{k}(A_{ij}+A_{ji})n_j + \frac{1}{2}\sum_{j=1}^{k}(A_{ij}\gamma_{ij}+A_{ji}\gamma_{ji})n_j \\
\beta_{h\in\text{loops}} &= \text{length}(loop) - 3 + \frac{1}{2}\sum_h \gamma_{h_i h_j} \ ;
\end{aligned} \tag{3}$$

where we consider one h coupling for each of the gauge invariant terms in the superpotential, which are represented by closed loops in the quiver.[1] We then solve for γ_{ij} by setting (3) to zero at the conformal point where $n_i = N_i$, and fix the remaining freedom using maximization of the central charge [13]

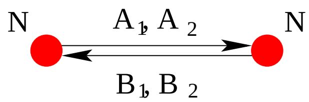
$$a = \frac{3}{32}(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R) = \frac{3}{32}\left[2\sum_i N_i^2 + \sum_{i<j} A_{ij}N_iN_j\left[3(R_{ij}-1)^3 - (R_{ij}-1)\right]\right] \tag{4}$$

where the anomalous dimensions are related to the corresponding R charges by $\gamma_{ij} = 3R_{ij} - 2$. With the solutions, back-substitute into the beta-functions and proceed out of conformality by taking the ranks $n_i = N_i + M_i$. We remark that the beta function thus computed is of leading order and we assume that the next order is $\mathcal{O}(M/N)^2$ while $\mathcal{O}(M/N)$ vanishes.

[1]The number of independent superpotential couplings can be reduced in the presence of symmetries.

3 The Conifold Cascade

It is instructive to review our methodology by applying it to the classic Klebanov-Strassler conifold theory. In this case, we have an $SU(N) \times SU(N)$ gauge theory with four bifundamental chiral multiplets $A_{1,2}$ and $B_{1,2}$ and superpotential as given in (5)



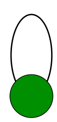
$$W = \epsilon^{ij}\epsilon^{kl} \operatorname{Tr} A_i B_k A_j B_l \ . \tag{5}$$

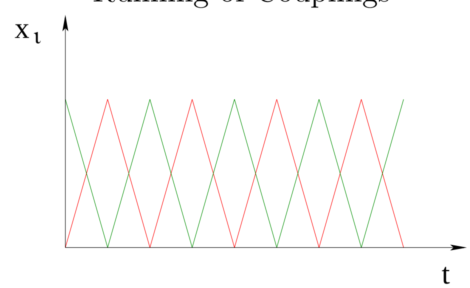
Adding fractional branes in accordance with (1), we have an $SU(N+M) \times SU(N)$ theory and the beta functions, using the techniques outlined above, are given in (6). Indeed by the $\mathbb{Z}_2$ symmetry of the quiver, the $\mathcal{O}(M/N)$ corrections to the beta functions are zero. The subsequent duality tree consists of a single node with a single closed cycle taking the theory (up to permutation of gauge groups) back to itself. The running of the two beta-functions cascade alternatingly in weave pattern:

Duality Tree

Running of Couplings

$$\begin{aligned} SU(N+M) : \beta_{g_1} &= 3M \\ SU(N) : \beta_{g_2} &= -3M \ ; \end{aligned}$$



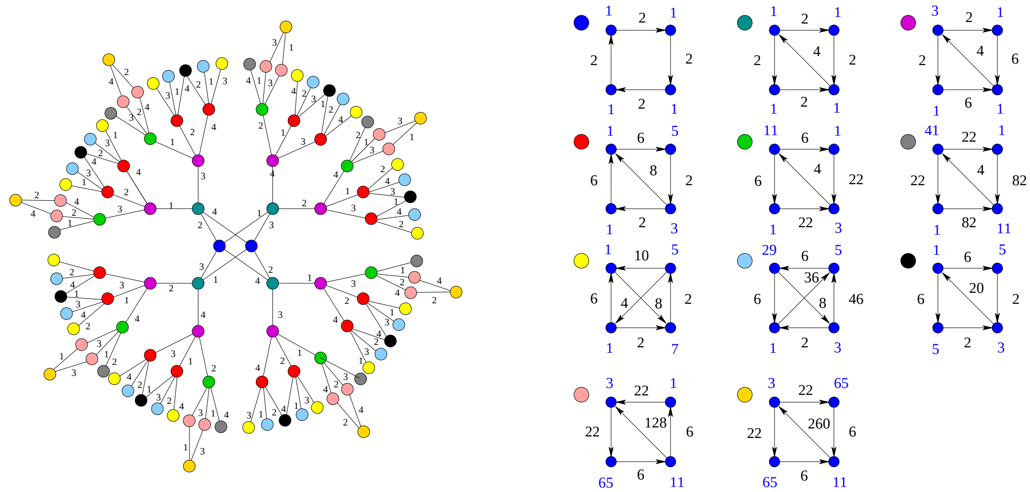


(6)

We see that the t interval between consecutive dualizations is constant along the cascade. The ranks of the gauge group vary linearly with the step, decreasing towards the IR, and thus diverging as the logarithm of the energy in the UV.

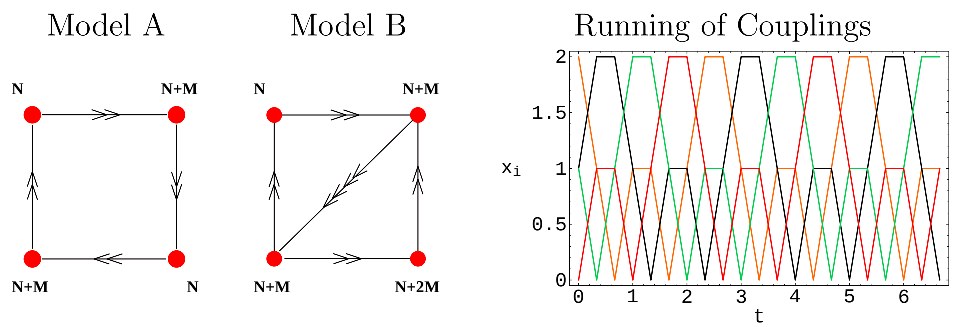
4 Cascade for F_0, the Zeroth Hirzebruch

We can now move on to the theory of our principal concern, the gauge theory corresponding to the probe on a complex cone over F_0, the zeroth Hirzebruch surface, or $\mathbb{P}^1 \times \mathbb{P}^1$. Interesting phenomena arise in this case. The duality tree is much richer; it resembles a flower which we call *flos Hirzebruchiensis*. This is shown, together with some dual phases of the quivers in plot (7)



(7)

There is a conifold-type cascade, given by the alternation between 2 models, which we will call A and B. Their quivers and the evolution of the four inverse gauge couplings x_i are given in (8):



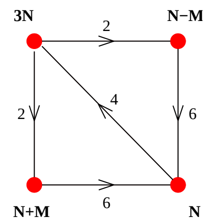
(8)

This periodic RG cascade is represented as a closed cycle at the center of the duality tree in (8).

It is important to clarify at this point the philosophy that will be used along the rest of the paper. We are interested in finding duality walls. As we described in the introduction, these phenomenon appears when reconstructing a duality cascade that has a given theory at its IR bottom. Thus, we will proceed to derive one possible RG trajectory in the UV such that, when we consider the flow towards the IR using Seiberg duality in a cascading manner, we arrive at the desired model. The reader should keep in mind that this traditional perspective is considered although, for simplicity, we will number cascade steps starting from the IR theory and increasing towards the UV.

4.1 The Duality Wall

On the other hand, if we started model C (a Seiberg dual of models A and B), given in (9), with associated beta-functions as shown,



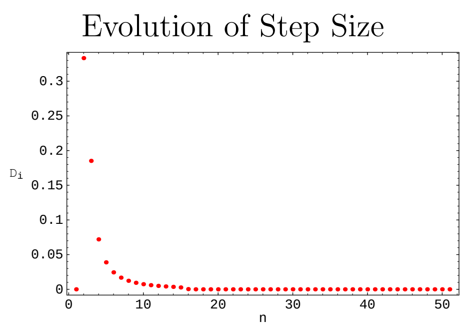
$$
\begin{aligned}
\beta_1 &= 0 \\
\beta_2 &= -3M \\
\beta_3 &= 0 \\
\beta_4 &= 3M \ .
\end{aligned}
\tag{9}
$$

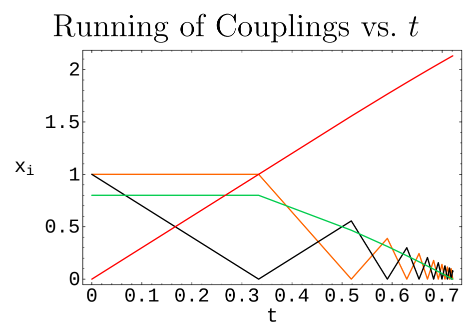
a significantly different behavior emerges.

The inverse gauge couplings x_i, by definition (2), evolve with β_i as

$$
x_i(k) = x_i(k-1) + \beta_i(k-1)\Delta(k) \tag{10}
$$

for the k-th step in which the linear piece of the most negative beta-function makes the corresponding x step by $\Delta(k)$. Let us, for concreteness, consider initial conditions $(x_1, x_2, x_3, x_4) = (1, 1, 4/5, 0)$. Then, the step $\Delta(k)$ decreases at each dualization and the couplings x_i evolve to a finite value:





(11)

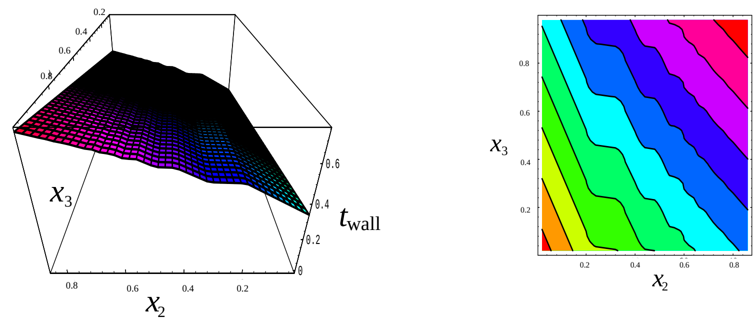
This is markedly different from the Klebanov-Strassler case. Something very drastic occurs after the third node gets dualized, which produces an explosive growth of the number of chiral and vector multiplets in the quiver. After node 3 is dualized, the subsequent quivers have all their intersection numbers greater than 2 and become of hyperbolic type. The cascade then is characterized by a flow of the dualization scales towards an UV *accumulation point* with a divergence in the number of degrees of freedom. This phenomenon is called a *duality wall* [2].

A $\mathbb{Z}_2$ symmetry as T-Duality Let us study how, starting from Model C, the RG flow continues further into the IR. This can be determined from our previous results by simply changing the signs of the beta functions and replacing the log-scale t by $-t$. The later transformation, in the holographic dual, resembles a T-duality-like action since the scale μ is associated with the radial coordinate. However, changing the sign of the beta function amounts to changing M to $-M$, which we observe to be a $\mathbb{Z}_2$ reflection of the quiver along the (13) axis of the quiver in (9). Therefore, the cascade to the IR from model C is simply the cascade that flows into it from the UV after reflection of the quiver.

Sensitivity to initial conditions Let us briefly examine the sensitivity of the location of the duality wall to the initial inverse couplings. This problem was studied in [6], and a detailed analytical study will appear in [18]. Let our initial inverse gauge couplings be of the form $(1, x_2, x_3, 0)$, with $0 < x_2, x_3 < 1$. We study the running of the beta functions,

[2] A similar phenomenon has been observed in [12], in the context of supergravity.

and determine the position of the duality wall, t_{wall}, for various initial values. We plot in (12), the position of the duality wall against the initial values x_2 and x_3, both as a three-dimensional plot in I and as a contour plot in II. We see that the position is a step-wise function. A similar behavior has been already observed in [10].



(I) (II) $$(12)$$

5 Generating Cascades from Families of Diophatine Equations

Having expounded upon the intricacies of duality cascades and flows, let us return to some discussion on the generic structure of duality trees. The connection between Seiberg Duality and Picard-Lefschetz (PL) monodromy transformations (alternatively, mutations of exceptional collections) has been explored in [14, 5]. When studying different dual gauge theories that correspond to D-branes over the same geometry, a quantity that remains invariant under PL transformations is the trace of the total monodromy K [15]. For the cases under study we have

$$\mathrm{Tr}K = 2 \tag{13}$$

This equation, when expressed in terms of the intersection numbers, gives rise to a *Markov type Diophantine equation*. Such Diophatine equations have been derived for three-block exceptional collections over del Pezzos by algebraic geometers [16]. In [14, 17], the equations were shown to coincide. For a given singularity, there may be more than one such equations. For example, exceptional collections with different number of blocks are classified by different Diophantine equations. In general, there may be more than one equation even for a given number of blocks. Each of these equations encode an infinite set of all the gauge theories that can be obtained for a given singularity using PL transformations, and hence a subset of all Seiberg duals. Remarkably, the set of 3-block equations for the del Pezzo surfaces is shown to be finite and all the equations can be organized into four families [16] as follows (the superscripts apply to those which admit more than one Diophantine equation):

$$\begin{array}{rlrl} \textbf{Family I:} & dP_0, dP_6^I, dP_8^I & \textbf{Family II:} & F_0, dP_5, dP_7^I, dP_7^{II}, dP_8^{II} \\ \textbf{Family III:} & dP_3, dP_6^{II}, dP_7^{III}, dP_8^{III} & \textbf{Family IV:} & dP_4, dP_8^{IV}\ . \end{array} \tag{14}$$

Within each family, the equations are the same, up to a mere change of variables. In this way, we can start from a Diophantine equation and its solutions for a given del Pezzo, and generate the equation and solutions for other geometries. In other words, the duality trees for different singularities may have subtrees that coincide, and the map between them can be derived from the corresponding equations.

Indeed, the physical content of this connection between Diophantine equations makes it more than a quiver generating tool along a tree for a given singularity. It is possible to go beyond and use it to, starting from an RG flow containing solutions to one of the members, generate entire cascades for other theories in the same family. Let us mention an explicit example to illustrate this statement. We have described in Section 4 a KS type cascade for $F0$ with a flow involving a 3-block quiver (Model B). We can use (14) to conclude that there are analogous flows for dP_5, dP_7 and dP_8, the other members of Family II. Furthermore the correspondence between theories in a given family indicates how to choose fractional branes in order to obtain the related cascades. We refer the reader to [18], where these cascades will be analyzed in detail. It is natural to conjecture that this reasoning can be extended to exceptional collections with a larger number of blocks and their associated Diophantine equations and cascades.

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