Dirichlet bridge sampling for the Variance Gamma process: pricing path-dependent options

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The authors develop a new Monte Carlo based method for pricing path-dependent options under the variance gamma (VG) model. The gamma bridge sampling method due to Avramidis et al. (2003) and Ribeiro and Webber (2004), is generalized to a multivariate (Dirichlet) construction, bridging 'simultaneously' over all time partition points of the trajectory of a gamma process. The generation of the increments of the gamma process, given its value at the terminal point, is interpreted as a Dirichlet partition of the unit interval. The increments are generated in a decreasing stochastic order and, under the Kingman limit, have a known distribution. Thus, simulation of a trajectory from the gamma process requires generating only a small number of uniforms, avoiding the expensive simulation of beta variates via numerical probability integral inversion. The proposed method is then applied in simulating the trajectory of a VG process using its difference-of-gammas representation. It has been implemented both in plain Monte Carlo and Quasi-Monte Carlo environments. It is tested in pricing lookback, barrier and Asian options and shown to provide consistent efficiency gains, compared to the sequential method and the difference-of-gammas bridge sampling due to Avramidis and L'Ecuyer (2006).

Key words: option pricing; gamma bridge; Variance Gamma process; Dirichlet partitions; quasi-Monte Carlo; Kingman limit

1. Introduction

Pricing path-dependent options whose underlying financial asset is driven by the so-called Variance Gamma (VG) process, introduced by Madan and Seneta (1990), has recently been considered by Ribeiro and Webber (2004), Avramidis et al. (2003) and Avramidis and L'Ecuyer (2006). These authors develop bridge methods for sampling from Gamma and Variance Gamma processes in Monte Carlo (MC) and Randomized Quasi-Monte Carlo (RQMC) environments, which demonstrate very good efficiency in estimating exotic option values. Developing such methods and improv-

ing further their efficiency is of considerable practical importance, since different types of new and existing exotic derivatives are actively traded in the over-the-counter market and their fast and accurate pricing under the VG model has proved to be crucial. There are several advantages in assuming that a VG process is the driver of the underlying asset price. The VG process represents a pure jump Lévy process, constructed by randomly changing the time in a Brownian motion, following a gamma process with unit mean rate and certain variance rate. Such a random time change allows for modelling the flow of economically relevant time, reflecting the random speedups and slowdowns in real time economic and business activity. Choosing unit mean rate of the gamma subordinator guarantees the unbiasness of the random transformation of the time unit.

Another advantage of the VG process, pointed out by Madan et al. (1998), is that it offers much more flexibility in modelling skewness and kurtosis of the asset returns, compared to Brownian motion. As shown by the authors, once calibrated to market prices, the VG model captures volatility smile and fat-tailness of the asset return distribution. The modelling power and flexibility of the VG process has recently been emphasized by Carr et al. (2007). As they point out, the random change in time of the rate at which business news on stocks arrive, has a direct impact on the movement of their prices, hence on the volatility of the related option prices. Carr et al. (2007) highlight the ability of the VG process to successfully capture upward and downward jumps as well as infinitesimally small movements (jitters) in the underlying stock price.

The VG model has been reported to perform better than the geometric Brownian motion in a number of empirical studies, such as those by Daal and Madan (2005), in pricing foreign currency options, by Fiorani (2004) in pricing European and American options on S&P 500, and by Fiorani and Luciano (2006) in credit risk modelling under VG log-asset values. The latter study establishes that VG jumps in the company's asset value give much smaller prediction errors in the credit default swap spreads than diffusion-based structural models. Moreover the VG model is reported to adequately solve the problem of the deep understatement of credit spreads produced by diffusion-based models. These findings, and the recent paper by Hurd (2007), clearly indicate the very good potential of applying the VG process in the credit risk modelling paradigm.

The nice properties of the VG model have led to its recent implementation in the Bloomberg system, through the function SKEW. It allows for contrasting VG against Black-Scholes in pricing options, based on market data. No doubt, this important step will boost further the popularity of the VG model among financial analysts, traders and other practitioners. The importance of the VG model has been highlighted also in the growing number of publications devoted to its empirical and theoretical properties, computational methods and various financial applications. Among these are recent contributions by Yor (2007), Fu (2007), Carr et al. (2007), Daal and Madan (2005), Carr et al. (2002), to name only a few. All this suggests that exploring the VG model further and developing efficient methods for option pricing under VG becomes more and more relevant both from the practical and theoretical point of view.

Along with the above mentioned advantages, a difficulty in using a VG process, and more generaly a Lévy process, as driver of the price of the underlying asset is that they require more sophisticated stochastic analysis and in the case of path-dependent option pricing do not lead to closed-form solutions. Thus, in the latter case it has proved essential to develop efficient Monte Carlo based valuation methods. However, in general, a well known drawback of the plain Monte Carlo methods is their slow convergence, which can make the estimation process very time consuming if a precise estimator is required. Various techniques, among which control and antithetic variates, stratified sampling and QMC methods, have recently been used for efficiency improvement over the plain Monte Carlo. A detailed account on these techniques is to be found in Glasserman (2004). For a recent overview of Monte Carlo methods for sampling from the VG process see Fu (2007).

As has recently been demonstrated by Avramidis et al. (2003), using low discrepancy sequences is a very promising approach to variance reduction of the estimated payoff of a path-dependent option. The authors have combined this approach with the so-called bridge sampling (see Moskowitz and Caflisch 1996) and have developed a method for sampling from the gamma process. It has been applied by Avramidis et al. (2003) to the gamma process used to randomize the time in the Brownian motion representation of the VG process, a method, named Brownian Gamma Bridge

Sampling (BGBS). The same bridge technique for sampling from the VG process has been independently developed by Ribeiro and Webber (2004). Avramidis et al. (2003) and Avramidis and L'Ecuyer (2006) have also applied gamma bridge sampling to develop the Difference-of-Gamma Bridge Sampling (DGBS) method, based on the representation of the VG process as a difference of two independent gamma processes (see Madan et al. 1998). Both the BGBS and the DGBS methods utilize the fact that the gamma bridge sampling improves the efficiency of the QMC by concentrating the variance at the first few simulated random numbers, thus reducing the effective dimension of the valuation problem. However, these methods involve generating computationally expensive beta random variables via numerical probability integral inversion.

Our aim in this paper is to develop a new efficient Monte Carlo method for pricing path-dependent options, when the underlying asset is driven by a Variance Gamma process, which requires generation of uniform variates only. It is called Dirichlet bridge sampling (DirBS) and incorporates a multivariate (Dirichlet) generalization of the gamma bridge sampling. Our approach can be interpreted as a multivariate bridging 'simultaneously' over all the partition points of the trajectory of a gamma process. It is based on the fact that the beta distribution defined by the two increments of the gamma bridge, generalizes to a Dirichlet distribution of the increments at all time partition points, of the gamma process, given its terminal value (see e.g. Wilks 1962 and Kotz et al. 2000, chapter 49). Furthermore, we view this Dirichlet distributed random vector of increments as a Dirichlet partition of the unit interval. Hence, the simulation of a trajectory of a gamma process is interpreted as a Dirichlet division of the unit interval. This important observation has allowed us to exploit the properties of the so-called size-biased permutation of the Dirichlet fragments of the unit interval, which represents a size-biased Dirichlet fragment selection procedure (see section 3.2). Thus, following this procedure, we generate the increments of a gamma process in a decreasing stochastic order.

To achieve sufficient accuracy of the estimated price of a path-dependent option, whose payoff is a function of the entire continuous-time sample path, the number of points in the time partition needs to be sufficiently large which causes the mesh size of the time partition to decrease. In the

limit, as the number of time points goes to infinity, the mesh size goes to zero, but their product remains constant. We use the fact that under such a limit, called Kingman limit (see section 3.2), for large enough number of partition points, the stochastically ordered Dirichlet fragments (i.e., the increments of the gamma process) have the so-called Griffiths-Engen-McCloskey or $GEM(\beta)$ distribution (see e.g. Johnson et al. 1997, p.237), where the parameter β is defined as the terminal time divided by the variance rate of the underlying gamma process. The considerable advantage which these asymptotic considerations bring to bear is that, as shown in section 3.2, the simulation of Dirichlet fragments (and hence of the increments of the gamma process) is reduced to the problem of simulating $Beta(1,\beta)$ variates, and therefore, to simulating only uniforms (since the cdf of $Beta(1,\beta)$ is analytically invertible). Thus, the expensive simulation of beta variates via inversion, required by the BGBS and the DGBS methods, is eliminated. A further speed up of the DirBS method is achieved by noting that, since we generate the increments in a decreasing stochastic order, we can take only the first few increments which are significantly different from zero, i.e. influence significantly the gamma trajectory, hence the estimated option price. This allows to considerably reduce the number of uniforms required to sample the gamma (and hence the VG) process. For example, only 40 uniforms were required to produce a gamma trajectory in the numerical examples considered in section 4. In conclusion, all these considerations, described in details in section 3 make DirBS a competitive (multivariate) bridge sampling method (see section 4 for the efficiency gains achieved by DirBS in comparison with existing methods). It has to be noted that in the case of a contingent claim whose payoff is a function of the path at a finite number of (discrete) observation times, the basic (non-asymptotic) version of the DirBS method described in section 3.1 can be applied as a competitive alternative to existing methods, e.g. BGBS and DGBS.

The structure of the paper is as follows. In the next section, we define the problem of contingent claims pricing under the VG model and review the existing bridge sampling methods. In section 3, we develop the Dirichlet bridge sampling scheme and provide the necessary theoretical background. In section 4, the proposed method is implemented in MC and RQMC environment and compared with the sequential sampling and the (symmetrical) DGBS method of Avramidis and

L'Ecuyer (2006), in terms of efficiency in pricing Asian, barrier and lookback options. Comments and conclusions are provided in section 5.

2. Pricing path-dependent options under the Variance Gamma model

2.1. Background

Assume the dynamics of the price of a financial asset is described by the risk neutral asset price process $\{S(t), t \geq 0\}$,

$$S(t) = S(0) \exp\left\{(\omega + r)\, t + X(t)\right\}, \tag{1}$$

where $X(t) = X(t; \vartheta, \sigma, \kappa)$, $t \geq 0$ is a Variance Gamma (VG) process, $r > 0$ is the risk free rate of interest and the constant $\omega = \left[\ln\left(1 - \vartheta\kappa - \left(\sigma^2\kappa\right)/2\right)\right]/\kappa$ is chosen so that $\mathbb{E}\left(S(t)\right) = S(0)\exp\{rt\}$, i.e. the process $\exp\{X(t)\}$ is a martingale, which imposes the requirement $\left(\vartheta + \sigma^2/2\right)\kappa < 1$ on the parameters of the VG process.

The VG process was first introduced by Madan and Seneta (1990) and developed further by Madan and Milne (1991) and Madan et al. (1998). It represents a Brownian motion $W(t) = W(t; \vartheta, \sigma)$ with drift parameter $\vartheta \in \mathbb{R}$ and variance parameter $\sigma > 0$, in which the time variable is replaced by an independent gamma process $G(t; \alpha, \lambda)$ with parameters $\alpha > 0$ and $\lambda > 0$, and density at t given by

$$f_{G(t;\alpha,\lambda)}(x) = \frac{\lambda^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\lambda x}, \qquad x > 0,$$

where $\Gamma(\cdot)$ denotes the gamma function. The parameter α of a gamma process $G(t; \alpha, \lambda)$ controls the intensity of the jumps of all sizes simultaneously, whilst λ captures the decay rate of big jumps. In the case of the VG process $\alpha = \lambda = 1/\kappa$, so that the variance rate of the gamma subordinator is $\kappa > 0$ and its mean rate is unity.

It can be shown (see Madan et al. 1998) that the VG process can also be expressed as a difference of two independent gamma processes

$$X(t) = G_0(t; \alpha, \lambda_0) - G_1(t; \alpha, \lambda_1), \tag{2}$$

with a common shape parameter $\alpha = 1/\kappa$ and scale parameters

$$\lambda_0 = \frac{2}{\kappa\left(\sqrt{\vartheta^2 + \frac{2\sigma^2}{\kappa}} + \vartheta\right)},$$

$$\lambda_1 = \frac{2}{\kappa\left(\sqrt{\vartheta^2 + \frac{2\sigma^2}{\kappa}} - \vartheta\right)}.$$

A recent overview of the properties of gamma processes, which appear as building blocks for the VG process, can be found in Yor (2007). In the sequel, we assume that a Variance Gamma model has been fitted to financial data and therefore, the values of the parameters of the VG process, $X(t;\vartheta,\sigma,\kappa)$, have been determined (see e.g. Chan 1999, Seneta 2004 and Madan et al. 1998 for further details and ideas on how this can be done).

Now, consider the problem of pricing a contingent claim, such as an option contract, with payoff at maturity, T, given by

$$C_T = f\left(\{S(t), 0 \le t \le T\}\right),$$

where f is some function of the stock price process (1). Then, the price at inception of the contract is

$$C_0 = \hat{\mathbb{E}}\left(\tilde{C}_T\right),$$

where $\hat{\mathbb{E}}$ denotes the expectation under some risk neutral martingale measure $\hat{\mathbb{P}}$, and $\tilde{C}_T$ denotes the terminal payoff discounted at the (possibly stochastic) risk free interest rate.

In reality, the process $S(t)$ is observed at some fixed points in time $0 = t_0 < t_1 < \cdots < t_N = T$, (or the payoff may depend on the value of $S(t)$ at a finite number of time epochs), therefore

$$C_T = f\left(S(t_1), \ldots, S(t_N)\right).$$

In order to estimate C_0, the process $S(t)$ is sampled at the time points $0 = t_0 < t_1 < \cdots < t_N = T$, using Monte Carlo methods. By generating $M > 0$ sample paths, M values of the discounted payoff function, $\tilde{c}_T^m$, $m = 1, \ldots, M$, can be calculated. The Monte Carlo estimate, c_0, of the contract price can then be obtained as

$$c_0 = \frac{\sum_{m=1}^{M} \tilde{c}_T^m}{M}. \tag{3}$$

In the next section, for the case of a VG economy, we present efficient and accurate Monte Carlo algorithms for pricing of exotic contracts, whose terminal payoffs highly depend on the trajectory followed by the underlying since inception.

2.2. Bridge sampling of Gamma and VG processes

As described in the introduction, there are two bridge approaches to the construction of a sample trajectory $X(t_1),\ldots,X(t_N)$ of the VG process $X(t)$, namely the BGBS method, which is based on the representation of the process as a subordinated Brownian motion, and the DGBS method, which utilizes the representation (2) of the VG process. The core part of both approaches is the bridge sampling of a gamma process $G(t;\alpha,\lambda)$ which can be summarized as follows.

Given any three time points $0 \le t_i < t_j < t_k \le T$ and the values of the gamma process, $G(t_i;\alpha,\lambda)$ and $G(t_k;\alpha,\lambda)$, at t_i and t_k, respectively, consider the problem of generating $G(t_j;\alpha,\lambda)$. Let $Y_1 = G(t_j;\alpha,\lambda) - G(t_i;\alpha,\lambda)$ and $Y_2 = G(t_k;\alpha,\lambda) - G(t_j;\alpha,\lambda)$. Hence, Y_1 and Y_2 are mutually independent, and $Y_1 \sim Gamma(g_1,\lambda)$, $Y_2 \sim Gamma(g_2,\lambda)$, where $g_1 = (t_j - t_i)\,\alpha$ and $g_2 = (t_k - t_j)\,\alpha$. Furthermore, $Z = Y_1 + Y_2 = G(t_k;\alpha,\lambda) - G(t_i;\alpha,\lambda)$ is $Gamma(g_Z,\lambda)$ distributed with $g_Z = (t_k - t_i)\,\alpha$. It can be shown that the conditional density of Y_1, given $Z = z$, is

$$f_{Y_1|Z}(y_1|z) = \frac{1}{B(g_1,g_2)} \left(\frac{y_1}{z}\right)^{g_1-1} \left(1 - \frac{y_1}{z}\right)^{g_2-1} z^{-1},$$

which implies that

$$G(t_j;\alpha,\lambda) = G(t_i;\alpha,\lambda) + b_{t_j} z,$$

where $b_{t_j} \sim Beta(g_1,g_2)$ and $B(a,b)$ denotes the beta function.

Ribeiro and Webber (2004) show that the BGBS method, when combined with stratification at certain time points, leads to substantial efficiency gains, relative to plain Monte Carlo, despite the time-consuming generation of the beta random variables $b_{t_j} \sim Beta(g_1,g_2)$ using the inverse transform method. Avramidis et al. (2003) compare different algorithms for sampling from the VG process in Monte Carlo (MC) and Quasi-Monte Carlo (QMC) environment and show that generally, DGBS gives the maximal variance reduction, but BGBS often leads to higher efficiency gains. The latter is again due to the expensive simulation of $b_{t_j} \sim Beta(g_1,g_2)$, based on the inverse transform method, compared to generating normal or gamma variates.

Recently, Avramidis and L'Ecuyer (2006) enhanced further the DGBS method by using dyadic partition for the time points $0 \le t_i < t_j < t_k \le T$ and hence, by generating symmetrical beta random

variables $b_{t_j} \sim Beta(g,g)$ with the Fast Beta Generator of L'Ecuyer and Simard (2006). They also find a lower and an upper bound for the resulting estimator and illustrate how in an RQMC environment the sampling procedure can be truncated and combined with bias extrapolation. The numerical results presented in Avramidis et al. (2003) and Avramidis and L'Ecuyer (2006) show that the DGBS method is very competitive for the pricing of Asian, barrier and lookback options.

3. Dirichlet Bridge Sampling for Gamma and VG processes

As described in the previous section, the gamma bridge involves three time points from the chosen time partition of $[0,T]$. Given the values of the process at the two end points, it samples the gamma process at the intermediate (bridge) point by generating a $Beta(g_1,g_2)$ random variable. This bridge technique is applied sequentially until all the points in the partition are exhausted. A possible way to generalize this bridge construction is to base the bridge on all intermediate time points in the partition of $[0,T]$ simultaneously and observe that the joint distribution of the increments of the gamma process at the bridge points generalizes from $Beta(g_1,g_2)$ to the Dirichlet distribution, $\mathcal{D}(g_1,\ldots,g_N)$ (see, e.g., Wilks 1962 or Kotz et al. 2000, chapter 49).

A significant enhancement of this Dirichlet generalization of the gamma bridge, is achieved by considering its large sample (asymptotic) properties. This asymptotic approach is well justified, noting that accurate valuation of path-dependent contracts, such as Asian, barrier and lookback options, requires more frequent monitoring of the trajectory of the underlying asset and hence a larger number of observations, N (see, e.g. Fu 2007). However, it should be mentioned that such frequent price monitoring is not always needed and in such cases, it is more appropriate to use the basic Dirichlet bridge sampling algorithm described in section 3.1.

3.1. The Dirichlet Bridge

The generalized bridge sampling method, which we introduce in this section, is applied to generate trajectories of the VG process, $X(t)$, using its representation as a difference of two independent gamma processes, $X(t) = G_0(t;\alpha,\lambda_0) - G_1(t;\alpha,\lambda_1)$, given by (2).

The following proposition establishes that the appropriately normalized increments of $G_i(t;\alpha,\lambda_i)$ at a set of points $0 = t_0 < \cdots < t_N = T$, given that the process has taken value $G_i(t_N;\alpha,\lambda_i)$ at t_N,

have a joint Dirichlet distribution. We recall that the random variables $\theta_1,\ldots,\theta_n$ have a Dirichlet distribution $\mathcal{D}(g_1,\ldots,g_n)$ with (real) parameters $g_1>0,\ldots,g_n>0$, i.e., $(\theta_1,\ldots,\theta_n)\sim\mathcal{D}(g_1,\ldots,g_n)$, if $\theta_n=1-\sum_{j=1}^{n-1}\theta_j$ and the joint probability density of $\theta_1,\ldots,\theta_n$ with respect to the Lebesgue measure is

$$f_{\theta_1,\ldots,\theta_n}(y_1,\ldots,y_{n-1}) = \begin{cases} \frac{\Gamma(g_1+\cdots+g_n)}{\prod_{i=1}^{n}\Gamma(g_i)} y_1^{g_1-1}\cdots y_{n-1}^{g_{n-1}-1}(1-y_1-\cdots-y_{n-1})^{g_n-1}, & \text{if } y_i\geq 0, \sum_{i=1}^{n-1} y_i\leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 1. *Define the random variables* $\theta_j=Y_j/Z$, $j=1,\ldots,N$, *where* $Y_j=G_i(t_j;\alpha,\lambda_i)-G_i(t_{j-1};\alpha,\lambda_i)$, $j=1,\ldots,N$ *are the increments of* $G_i(t;\alpha,\lambda_i)$, *at the points* $0<t_1<\cdots<t_N=T$, *with* $G_i(t_0;\alpha,\lambda_i)=0$ *and* $Z=G_i(t_N;\alpha,\lambda_i)=\sum_{j=1}^{N}Y_j$, *i.e.* Z *is the value of the process at the terminal time* T. *The joint distribution of the random variables* $(\theta_1,\ldots,\theta_N)$ *is Dirichlet with parameters* $g_j=(t_j-t_{j-1})\alpha$, $g_j>0$, $j=1,\ldots,N$, *i.e.* $(\theta_1,\ldots,\theta_N)\sim\mathcal{D}(g_1,\ldots,g_N)$.

Proof of Proposition 1. Since the increments Y_j of $G_i(t_j;\alpha,\lambda_i)$ are independent and $Y_j\sim Gamma(g_j,\lambda_i)$, $j=1,\ldots,N$, we can apply a well known result according to which the r.v.s $Y_j/\sum_{j=1}^{N}Y_j$ have the stated joint Dirichlet distribution (see e.g. Wilks 1962). □

Although the result of Proposition 1 has first been noticed in the probabilistic literature by Kingman (1975), somewhat surprisingly, to the best of our knowledge, it has not been exploited for the development of simulation methods for the Gamma process based on the Dirichlet distribution and its (asymptotic) properties. Proposition 1 motivates the following straightforward algorithm for generating a sample path from $G_i(t;\alpha,\lambda_i)$, $i=0,1$.

1. Generate $Z\sim Gamma(\alpha t_N,\lambda_i)$, which represents a sample value of the process $G_i(t,\alpha,\lambda_i)$ at the terminal time $t_N=T$.
2. Generate $Y_j\sim Gamma(g_j,1)$, $j=1,\ldots,N$
3. Calculate the value of the process $G_i(t_j;\alpha,\lambda_i)$ at t_j as

$$G_i(t_j;\alpha,\lambda_i)=G_i(t_{j-1};\alpha,\lambda_i)+\frac{Y_j}{\sum_{j=1}^{N}Y_j}Z, \qquad j=1,..,N.$$

Note that, at step 2, there is no need to generate $Y_j \sim Gamma\,(g_j, \lambda_i)$, $j = 1, \ldots, N$ since when the ratio $Y_j / \sum_{j=1}^{N} Y_j$ is evaluated, the parameter λ_i cancels out, due to the rescaling property of the gamma distribution.

The above algorithm for Dirichlet bridge sampling of the trajectory of a gamma process is a generalization of the gamma bridge sampling method developed by Ribeiro and Webber (2004) and Avramidis et al. (2003). It is easy to implement, given that a fast and reliable gamma generator is available. Unfortunately, standard fast gamma generators, e.g. those of Ahrens and Dieter (1974) and Best (1983), are unstable for (very) small values of g_j, which is the case in financial applications as the ones considered in this paper. Therefore, one has to use the inverse transform method based on the Newton algorithm when simulating the increments Y_j. Our numerical experiments show that the Dirichlet bridge algorithm based on Proposition 1 does not provide substantial efficiency gain in simulating a gamma process, compared to the existing methods.

3.2. Asymptotic Dirichlet Bridge: the DirBS method

Despite the fact that Proposition 1 does not directly lead to substantial efficiency gains, it is an important generalization which, in conjunction with some asymptotic arguments with respect to the number of time points N, allows us to enhance the Dirichlet bridge sampling technique and refer to it as the DirBS method. For the purpose of developing DirBS, let us consider equidistant points, $0 = t_0 < t_1 < \cdots < t_N = T$, with constant mesh size $\Delta = (t_j - t_{j-1}) = T/N$, $j = 1, \ldots, N$, as has also been assumed in other bridge sampling methods (see e.g. Avramidis and L'Ecuyer 2006). In this case, $g_1 = \cdots = g_N = g = \Delta\alpha = \Delta/\kappa$ and the distribution of $(\theta_1, \ldots, \theta_N)$ simplifies to an exchangeable Dirichlet, $\mathcal{D}_N\,(g)$. Therefore, the random vector $(\theta_1, \ldots, \theta_N) \sim \mathcal{D}_N\,(g)$ has density on the simplex

$$f_{\theta_1,\ldots,\theta_N}\,(y_1, \ldots, y_N) = \frac{\Gamma\,(Ng)}{\Gamma\,(g)^N} \prod_{j=1}^{N} y_j^{g-1} \mathbf{1}_{\{y_i \geq 0, \sum_{i=1}^{N} y_i = 1\}},$$

where $\mathbf{1}_{\{A\}}$ is the indicator function of the event A.

Note that $(\theta_1, \ldots, \theta_N) \sim \mathcal{D}_N\,(g)$ can be interpreted as N fragments of a partition of the interval $[0, 1]$. For our purpose of developing the DirBS method, it will be instructive to gain some insight

into the way in which the probability mass of the Dirichlet distribution redistributes as the parameter g varies. Thus, the larger the parameter g, the more homogeneous are the fragments' sizes θ_j, $j=1,\ldots,N$. In this case, the exchangeable Dirichlet distribution $\mathcal{D}_N(g)$ concentrates mass at the center of the simplex $\left\{y_i \geq 0, \sum_{i=1}^N y_i = 1\right\}$. On the other hand, the smaller the value of g, the more disparate are the sizes of the fragments θ_j, $j=1,\ldots,N$ and the difference between the largest and the smallest fragments increases as g decreases. In the latter case, the distribution $\mathcal{D}_N(g)$ concentrates closer to the boundaries of the simplex $\left\{y_i \geq 0, \sum_{i=1}^N y_i = 1\right\}$.

In order to achieve sufficient accuracy in the Monte Carlo estimation of the price of a contingent claim, such as a general path-dependent option, whose payoff is a function of the entire continuous-time sample path, the number of points, N, should be sufficiently large, which causes the mesh size $\Delta = T/N$ to decrease. In the limit, Δ, and hence g, converge to zero as N goes to infinity, while at the same time the product

$$gN = \frac{\Delta}{\kappa}N = \frac{T}{\kappa} = \beta > 0$$

remains constant. This important observation motivates the next stage in developing the DirBS algorithm at which we take an asymptotic point of view. More precisely, we look at the asymptotic distribution of the random vector $(\theta_1,\ldots,\theta_N) \sim \mathcal{D}_N(g)$ as $N \uparrow \infty$, $g \downarrow 0$ while $gN = \beta > 0$. As it has first been noted by Kingman (1975), this asymptotic distribution has a degenerate weak limit (with each θ_j converging to zero almost surely, as $N \uparrow \infty$), which implies that its direct manipulation is impossible. However, useful results have been obtained for the decreasing order statistics $\left(\theta_{(1)},\ldots,\theta_{(N)}\right)$ of $(\theta_1,\ldots,\theta_N)$, in the limit $N \uparrow \infty$, $g \downarrow 0$ while $gN = \beta > 0$, known as the Kingman limit. In particular, in the Kingman limit, $\left(\theta_{(1)},\ldots,\theta_{(N)}\right)$ converges in law to a distribution known as the Poisson-Dirichlet distribution (see Kingman 1993, Section 9.3, pages 93-94). We will exploit this fact in developing the DirBS algorithm. In order to do so, it is useful to recall once again that, given the value, Z, of the process at the terminal time T, the vector $(\theta_1,\ldots,\theta_N)$ completely describes a sample path from the Gamma subordinator $G_i(t;\alpha,\lambda_i)$. On the other hand, following Proposition 1, it is also clear that $(\theta_1,\ldots,\theta_N)$ comprises a random Dirichlet

partition of the unit interval $[0,1]$. Thus, given Z, one can view the simulation of a path from $G_i(t;\alpha,\lambda_i)$ as a Dirichlet division of the unit interval and utilize the properties of the so-called size-biased permutation of Dirichlet fragments. The size bias comes from the fact that the Dirichlet fragments are picked up with probabilities proportional to their sizes. This size-biased permutation equivalently arises also from partitioning the unit interval and the subsequent residual subintervals, following an appropriate beta distribution. This partitioning scheme, known as Residual Allocation Model (RAM), generates the stochastically ordered Dirichlet fragments which arise in the size-biased permutation procedure. In order to use this interpretation, we need to introduce some additional notation.

Denote $V_1,\ldots,V_{N-1}$ a sample of independent beta random variables with $V_m \sim Beta(1+g,(N-m)g)$, $m=1,\ldots,N-1$. Let

$$L_m = \prod_{j=1}^{m-1}(1-V_j)\,V_m, \quad m=1,\ldots,N-1, \tag{4}$$

$$L_N = 1-\sum_{m=1}^{N-1} L_m = \prod_{j=1}^{N-1}(1-V_j). \tag{5}$$

The random variables, L_m, $m=1,\ldots,N$, are stochastically ordered, i.e. $L_1 \succcurlyeq \cdots \succcurlyeq L_N$, and correspond to a scheme of sequential partitioning of the unit interval, known as RAM. The random variables L_m, $m=1,\ldots,N$, arising from RAM, naturally arise also as a result of the following size-biased sampling scheme of Dirichlet partitions $(\theta_1,\ldots,\theta_N)\sim\mathcal{D}_N(g)$. Given a random Dirichlet partition $(\theta_1,\ldots,\theta_N)$ of the interval $[0,1]$, select a fragment θ_{I_1} at random so that the probability $\mathbb{P}(I_1=i_1|\theta_1,\ldots,\theta_N)=\theta_{i_1}$, and denote the size of the fragment chosen first as $L_1:=\theta_{I_1}$, noting that, $L_1\sim Beta(1+g,(N-1)g)$. Rearrange the fragments as $(L_1,\theta_1,\ldots,\theta_{I_1-1},\theta_{I_1+1},\ldots,\theta_N)$, which may equivalently be written as $\left(L_1,(1-L_1)(\theta_1^{(1)},\ldots,\theta_{I_1-1}^{(1)},\theta_{I_1+1}^{(1)},\ldots,\theta_N^{(1)})\right)$, where

$$\left(\theta_1^{(1)},\ldots,\theta_{I_1-1}^{(1)},\theta_{I_1+1}^{(1)},\ldots,\theta_N^{(1)}\right) = (\theta_1,\ldots,\theta_{I_1-1},\theta_{I_1+1},\ldots,\theta_N)/(1-L_1)$$

is $\mathcal{D}_{N-1}(g)$ distributed, independent of $(1-L_1)$ (see e.g., Wilks 1962). Next, select at random a fragment from $\left(\theta_1^{(1)},\ldots,\theta_{I_1-1}^{(1)},\theta_{I_1+1}^{(1)},\ldots,\theta_N^{(1)}\right)$ with a probability equal to its size, denote its length

as V_2 and note that $V_2 \sim Beta(1+g,(N-2)g)$. The size of the second fragment is $L_2 = (1-V_1)V_2$, where $V_1 = L_1$. Iterating until all fragments have been picked up avoiding already selected ones, will finally yield the size biased permutation of fragments $(L_1,\ldots,L_N)$. For further details on size biased permutation, see (Kingman 1993, Section 9.6).

Denote by $(\pi_1,\pi_2,\ldots,\pi_N)$ one of the $N!$ equally probable random permutations of the numbers $(1,2,\ldots,N)$. The following proposition is a direct consequence of the exchangeability of the Dirichlet distribution.

PROPOSITION 2. *The random vectors* $(L_{\pi_1},\ldots,L_{\pi_N})$ *and* $(\theta_1,\ldots,\theta_N)$ *coincide in distribution, i.e.* $(L_{\pi_1},\ldots,L_{\pi_N}) \stackrel{d}{=} (\theta_1,\ldots,\theta_N) \sim \mathcal{D}_N(g)$.

Proof of Proposition 2. Follows by noting that $(L_1,\ldots,L_N)$ are obtained as a result of the size-biased permutation of the exchangeable Dirichlet partitions $(\theta_1,\ldots,\theta_N)$. □

REMARK 1. Note that, in the Kingman limit, Proposition 2 does not hold since, as noted by Kingman (1975), there is no exchangeable distribution on the infinite dimensional simplex $\{y_i \geq 0, \sum_{i=1}^{\infty} y_i = 1\}$. However, the following proposition (see Kingman 1993, Chapter 9) establishes the Kingman asymptotics of the variables L_m, $m = 1,\ldots,N$, which are central in developing the DirBS algorithm.

PROPOSITION 3. *When* $m = o(N)$, *in the Kingman limit,* $(L_1,\ldots,L_N) \stackrel{d}{\to} \left(\tilde{L}_1,\ldots,\tilde{L}_m,\ldots\right)$, *where*

$$\tilde{L}_m = \prod_{j=1}^{m-1}\left(1-\tilde{V}_j\right)\tilde{V}_m, \quad m = 1,2,\ldots \tag{6}$$

and $\tilde{V}_m$ *are independent random variables,* $\tilde{V}_m \sim Beta(1,\beta)$, *with* $\beta = gN$.

Proof of Proposition 3. Follows from equations (4) and (5), noting that, in the Kingman limit, $V_m \stackrel{d}{\to} \tilde{V}_m \sim Beta(1,\beta)$. □

Note that $\tilde{V}_m$, $m = 1,2,\ldots$ are independent, identically distributed random variables with generic distribution $\tilde{V}_m \sim Beta(1,\beta)$, hence the asymptotic distribution of $(L_1,\ldots,L_N)$ no longer depends on the parameter g and hence, on the mesh size, Δ, and the number of time partition points,

N. Furthermore, the variables $\tilde{L}_1, \ldots, \tilde{L}_m, \ldots$, are stochastically ordered, i.e., $\tilde{L}_1 \succcurlyeq \cdots \succcurlyeq \tilde{L}_m \succcurlyeq \ldots$, which is due to the size-biased character of their underlying RAM, given by (6). Their distribution is known as the Griffiths-Engen-McCloskey or $GEM(\beta)$ distribution.

Proposition 3 suggests that, under the Kingman limit, for large enough fixed N, the distribution of $(L_1, \ldots, L_N)$ can be well approximated with that of the random vector $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$. However, from Proposition 2, by randomly permuting the elements of $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$, one obtains a random vector which is approximately Dirichlet distributed, i.e.,

$$\left(\tilde{L}_{\pi_1}, \ldots, \tilde{L}_{\pi_N}\right) \stackrel{d}{\simeq} (\theta_1, \ldots, \theta_N) \sim \mathcal{D}_N(g).$$

In the latter approximate equality, the higher the value N, the better the quality of the approximation. Based on this, we conclude that, in order to simulate from $(\theta_1, \ldots, \theta_N) \sim \mathcal{D}_N(g)$, it is sufficient to simulate N variates $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$, from the $GEM(\beta)$ distribution, following the RAM given by (6), and then randomly permute the elements of $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$, in order to obtain the required (approximately) $\mathcal{D}_N(g)$ distributed random vector $\left(\tilde{L}_{\pi_1}, \ldots, \tilde{L}_{\pi_N}\right)$. The considerable advantage of this approach over the algorithm which follows from Proposition 1, is that generating $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$ requires generation of the $Beta(1, \beta)$ variates, $\tilde{V}_m$, for which the inverse distribution function is $F^{-1}_{Beta(1,\beta)}(u) = 1 - u^{1/\beta}$. Therefore, one needs only to generate uniform variates, $U(0, 1)$, and raise them to the power $1/\beta$ in order to sample from a Gamma process, which considerably speeds up the simulation procedure.

To summarize, instead of simulating $(\theta_1, \ldots, \theta_N) \sim \mathcal{D}_N(g)$, the elements of the random vector $\left(\tilde{L}_1, \ldots, \tilde{L}_N\right)$ can be simulated and then randomly permuted. It has to be noted that the mean value of any of the fragments $(\theta_1, \ldots, \theta_N)$ is $\mathbb{E}[\theta_j] = 1/N$, $j = 1, \ldots, N$, but it can be proved that asymptotically the smallest one decreases like $N^{-(g+1)/g}$ while the largest is of order

$$\frac{1}{Ng} \ln\left(N(\ln N)^{g-1}\right)$$

(see e.g. Barrera et al. 2005). As revealed by the RAM in (6), $\tilde{L}_1 \succcurlyeq \cdots \succcurlyeq \tilde{L}_N$ forms a scheme of sequential partitioning of residual subintervals of the unit interval, which suggests that only the

first, say, k^* elements in the list $\left(\tilde{L}_1,\ldots,\tilde{L}_N\right)$ are significantly different from zero. Therefore, one may generate only the first k^*-largest increments which affect the price of the underlying contingent payment, and need not generate the insignificant, nearly zero increments, whose generation substantially increases the computational burden without improving the quality of the price estimate. This is more formally described in the following section.

3.3. Refining the DirBS method: cutting out 'small' jumps

Formally, we are interested in determining the value $k^* \in \mathbb{N}$ such that $\mathbb{P}\left(Z\sum_{m=k^*+1}^{\infty}\tilde{L}_m \le \epsilon\right) \ge p$, for some p close to 1, where $Z \sim Gamma(\alpha t_N, \lambda_i)$ and $\epsilon > 0$ is a preliminary fixed small number. Obviously, the random variable $V = Z\sum_{m=k^*+1}^{\infty}\tilde{L}_m$ represents the remaining fraction of the total increase $G_i(T;\alpha,\lambda_i) - G_i(0;\alpha,\lambda_i)$, i.e. the sum of the increments of size 'nearly zero' which need not be generated. In order to find k^*, one needs to know the distribution of the random variable V and in particular its cumulative distribution function, $F_V(x)$, and find the solution of

$$\min_{k\in\mathbb{N}} F_V(k;\epsilon) \ge p. \tag{7}$$

Next we give a proposition which allows for the exact calculation of k^* and also present two alternative simpler ways of estimating it. This gives rise to a further refinement of the DirBS method which makes it an elegant, simple and efficient algorithm for simulating a sample path from a Gamma process $G_i(t;\alpha,\lambda_i)$ and hence, from a VG process.

In the following proposition, an explicit expression for $F_V(x)$ is derived and as a byproduct, the distribution of the sum of the remaining normalized increments, $W = \sum_{m=k^*+1}^{\infty}\tilde{L}_m$, is also obtained.

PROPOSITION 4. *Let* $\left(\tilde{L}_1,\tilde{L}_2,\ldots\right)$ *be defined as in (6) and* $Z \sim Gamma(\alpha t_N,\lambda_i)$. *The cumulative distribution function,* $F_V(x)$, *of the random variable* $V = Z\sum_{m=k^*+1}^{\infty}\tilde{L}_m$ *is given by*

$$F_V(x) = \frac{{\lambda_i}^{\beta}}{\Gamma(\beta)}\frac{\beta^{k^*}}{\Gamma(k^*)}\int_0^x v^{\beta-1}\int_v^{\infty}\frac{e^{-\lambda_i u}}{u}\left(\ln\frac{u}{v}\right)^{k^*-1}dudv. \tag{8}$$

Proof of Proposition 4. Consider the random variable

$$W = \sum_{m=k^*+1}^{\infty}\tilde{L}_m = 1 - \sum_{m=1}^{k^*}\tilde{L}_m = (1-\tilde{V}_1)(1-\tilde{V}_2)\ldots(1-\tilde{V}_{k^*}),$$

where, from (6), $(1-\tilde{V}_m)$ are independent $Beta(\beta,1)$ distributed random variables. Hence,

$$\ln W = \sum_{m=1}^{k^*} \ln(1-\tilde{V}_m) = -\sum_{m=1}^{k^*} \left(-\ln(1-\tilde{V}_m)\right)$$

and it is not difficult to see that $\left(-\ln(1-\tilde{V}_m)\right) \sim Exp(\beta)$ and that $(-\ln W) \sim Gamma(k^*,\beta)$ as a sum of k^* exponentially distributed random variables. Therefore, $W \stackrel{d}{=} e^{-\tilde{W}}$, where $\tilde{W} \sim Gamma(k^*,\beta)$, and $F_W(w) = 1 - F_{\tilde{W}}(-\ln w) = \sum_{j=0}^{k^*-1} w^{\beta} \frac{(-\beta \ln w)^j}{j!}$, for $w>0$ and $k^* = 1,2,\ldots$. Now, note that $\alpha t_N = gN = \beta$ and that $Z \sim Gamma(\alpha t_N, \lambda_i)$ and $\tilde{W}$ are independent random variables with a joint density function $f_Z(z) f_{\tilde{W}}(\tilde{w})$. Performing the change of variables, $V = Ze^{-\tilde{W}}$ and $U = Z$, for the cumulative distribution function, $F_V(x)$, $x>0$, we obtain

$$F_V(x) = \int_0^x \int_v^{\infty} f_Z(u) f_{\tilde{W}}(-\ln \frac{v}{u}) \left| \det \begin{pmatrix} -\frac{1}{v} & \frac{1}{u} \\ 0 & 1 \end{pmatrix} \right| dudv$$

$$= \frac{{\lambda_i}^{\beta}}{\Gamma(\beta)} \frac{\beta^{k^*}}{\Gamma(k^*)} \int_0^x v^{\beta-1} \int_v^{\infty} \frac{e^{-\lambda_i u}}{u} \left(\ln \frac{u}{v}\right)^{k^*-1} dudv,$$

which completes the proof. □

Although expression (8) does not facilitate the analytical solution of problem (7), the latter can be solved numerically by using an appropriate numerical method (see e.g. the built-in function FindRoot in *Mathematica* which takes a couple of seconds to find a solution). However, a simpler way of estimating k^*, which avoids the evaluation of (8) and may serve as a practical alternative to it, as demonstrated in section 4, is given next.

Using the fact that

$$\mathbb{E}[\tilde{L}_m] = \left(\frac{\beta}{1+\beta}\right)^{m-1} \frac{1}{1+\beta}$$

(see for example Barrera et al. 2005) with $\sum_{m=1}^{\infty} \mathbb{E}[\tilde{L}_m] = 1$ as the sum of a geometric series, one can find, $\hat{k}^*$, such that

$$z_p \sum_{m=\hat{k}^*+1}^{\infty} \mathbb{E}[\tilde{L}_m] = z_p \left(1 - \sum_{m=1}^{\hat{k}^*} \mathbb{E}[\tilde{L}_m]\right) = z_p \left(\frac{\beta}{1+\beta}\right)^{\hat{k}^*} \leq \epsilon,$$

where z_p is the $100p$-th percentile of the $Gamma(\beta,\lambda_i)$ distribution, i.e. $z_p = F^{-1}_{Gamma(\beta,\lambda_i)}(p)$. From the last inequality, it follows that

$$\hat{k}^* \geq \frac{\ln(z_p/\epsilon)}{\ln(1/\beta+1)}. \tag{9}$$

Clearly, $\hat{k}^*$ gives an estimate of the number of stochastically ordered elements in the list $\left(\tilde{L}_1, \tilde{L}_2, \dots\right)$ which determine the 'large' increments in the path of $G_i(t;\alpha,\lambda_i)$. A slightly different approach of estimating k^*, can be described as follows. Assume $\epsilon > 0$ and consider the number, k', of increments of $G_i(t;\alpha,\lambda_i)$ such that $z_p\tilde{L}_m > \epsilon$, $m = 1,\dots,k'$, i.e. $\tilde{L}_m > \epsilon'$, $m = 1,\dots,k'$, $\epsilon' = \epsilon/z_p$. Denote by $M_{\epsilon'} = \sum_{m=1}^{\infty} \mathbf{1}_{\{\tilde{L}_m > \epsilon'\}}$ the random number of fragments $\tilde{L}_m$ greater than ϵ'. Following Hirth (1997), the random variable $M_{\epsilon'}$ is approximately $Poisson(\delta)$ distributed, asymptotically for $\epsilon' \downarrow 0$, where $\delta = \mathbb{E}[M_{\epsilon'}] = \int_{\epsilon'}^{1} \frac{\beta}{y}(1-y)^{\beta-1}dy$. Therefore, having fixed $\epsilon > 0$, one may calculate $\delta = \mathbb{E}[M_{\epsilon'}]$ and the estimate $\hat{k}'$ can be set to be (greater then or) equal to the $100p$-th percentile of $Poisson(\delta)$ distribution, i.e.

$$\hat{k}' \geq F^{-1}_{Poisson(\delta)}(p). \tag{10}$$

Obviously, as an estimate of k^*, it is natural to take the smallest integer which satisfies either (9) or (10). Our empirical tests show that both (9) and (10) yield reasonably close estimates of k^*, (10) giving slightly higher values than (9) (see section 4). Thus, for a fixed number of time points N, both $N - \hat{k}^*$ and $N - \hat{k}'$ represent good approximations to the number of jumps, $N - k^*$, of the Gamma processes $G_i(t;\alpha,\lambda_i)$, $i = 0, 1$, which need not be generated as being 'nearly zero', without this having an effect on the final estimated price. Proposition 4 (or its alternatives given by (9) and (10)), lead to a refinement of the DirBS method which results in its further substantial speed up as will be shown in the next section. The pseudocode of the proposed Dirichlet Bridge method for sampling of a VG process is given in Figure 1. Note that, in permuting the elements of the random vector $\left(\tilde{L}_1,\dots,\tilde{L}_{k^*},0,\dots,0\right)$ in order to obtain $\left(\tilde{L}_{\pi_1},\dots,\tilde{L}_{\pi_N}\right)$, there is no need to permute the zero values. Different methods, which could be used in implementing this random permutation, are to be found in Devroye (1986), Chapter 12, and Knuth (1997), p. 145 and p. 148.

4. Numerical study

In order to illustrate the performance of the DirBS method, we have applied it to pricing Asian, barrier and lookback options. In this pricing exercise, the DirBS has been compared with the (symmetrical) DGBS method of Avramidis and L'Ecuyer (2006), and with sequential Monte Carlo

set ϵ, p; $\beta \leftarrow T/\kappa$; $k^* \leftarrow \min_{k\in\mathbb{N}}\{F_V(k;\epsilon) \geq p\}$;
$G_+(0) \leftarrow 0$; $G_-(0) \leftarrow 0$; $X(0) \leftarrow 0$;
Generate $G_+(T) \sim Gamma(\beta, \lambda_0)$;
Generate $G_-(T) \sim Gamma(\beta, \lambda_1)$;
For $m = 1$ to $min\,(N-1,\, k^*)$ $\Big\{$

Generate $U_+ \sim U(0,1)$; $V_{+,m} \leftarrow 1 - U_+^{1/\beta}$;

$$\tilde{L}_{+,m} \leftarrow \prod_{j=1}^{m-1}\left(1-\tilde{V}_{+,j}\right)\tilde{V}_{+,m};$$

Generate $U_- \sim U(0,1)$; $V_{-,m} \leftarrow 1 - U_-^{1/\beta}$;

$$\tilde{L}_{-,m} \leftarrow \prod_{j=1}^{m-1}\left(1-\tilde{V}_{-,j}\right)\tilde{V}_{-,m};\Big\}$$

$\left(\tilde{L}_{+,k^*+1},\ ...,\ \tilde{L}_{+,N}\right) \leftarrow (0,\ ...,\ 0)$;
$\left(\tilde{L}_{-,k^*+1},\ ...,\ \tilde{L}_{-,N}\right) \leftarrow (0,\ ...,\ 0)$;
$\left(\tilde{L}_{+,\pi_1},\ ...,\ \tilde{L}_{+,\pi_N}\right) \leftarrow \text{permute}\left(\tilde{L}_{+,1},\ ...,\ \tilde{L}_{+,N}\right)$;
$\left(\tilde{L}_{-,\pi_1},\ ...,\ \tilde{L}_{-,\pi_N}\right) \leftarrow \text{permute}\left(\tilde{L}_{-,1},\ ...,\ \tilde{L}_{-,N}\right)$;
For $m = 1$ to $N-1$ $\Big\{$

$G_+(t_m) \leftarrow G_+(t_{m-1}) + G_+(T)\,\tilde{L}_{+,\pi_m}$;
$G_-(t_m) \leftarrow G_-(t_{m-1}) + G_-(T)\,\tilde{L}_{-,\pi_m}$;
$X(t_m) \leftarrow G_+(t_m) - G_-(t_m)$;$\Big\}$

Figure 1 **Dirichlet Bridge Sampling of a VG Process $X(t)$ with parameters $(1,\kappa,\vartheta,\sigma)$ at a (finite) sequence of equidistant time points $0 = t_0 < t_1 < \cdots < t_N = T$, (all generated variates are independent).**

sampling of VG paths, based on Gamma Sequential Sampling (GSS). The reference set of parameters, we use throughout this section, $S(0) = 100$, $\vartheta = -0.2859$, $\sigma = 0.1927$, $\kappa = 0.2505$, $T = 0.40504$, $r = 0.0548$, is taken from Hirsa and Madan (2004) and is the one used also by Avramidis and L'Ecuyer (2006).

For the purpose of this comparison, the DirBS and the DGBS methods have been implemented both in the plain MC and Randomized QMC environment. It is known, (see e.g., Moskowitz and Caflisch 1996, L'Ecuyer and Lemieux 2002, Glasserman 2004, Avramidis and L'Ecuyer 2006), that RQMC allows for a considerable variance reduction, compared to plain MC. This is achieved by replacing the vectors of independent uniforms, generated in the MC simulation process, by so-called low discrepancy sequences which cover the unit hypercube much more evenly than purely random points. We have chosen to work with a d-dimensional Sobol' sequence. The dimension of the problem, d, is defined as the upper bound on the number of uniforms required to produce a

simulation estimate. As is well known, methods which are characterized by a lower dimension are more RQMC friendly in the sense that generally, they result in smaller errors in the estimate.

The dimension of the problem under the DGBS method is equal to twice the number of time partition points at which values of VG paths are generated, i.e. $d = 2N$. However, the effective dimension is relatively low, i.e. DGBS is highly QMC friendly, due to the fact that it concentrates the variance on the first few sites of the dyadic partition, used by Avramidis and L'Ecuyer (2006) to produce the gamma bridge points (for a formal definition of effective dimension and further discussions see e.g. Caflisch et al. 1997).

In order to estimate the problem dimension under the DirBS method, we shall use (8) to find the number of increments which need to be generated for each of the two gamma processes, $G_i(t;\alpha,\lambda_i)$, $i = 0, 1$, and therefore, the number of uniforms needed to simulate a trajectory from $G_i(t;\alpha,\lambda_i)$, $i = 0, 1$, (i.e., the r.v.s, $\tilde{V}_m$ in (6)). For the chosen parameter set $(\kappa, \vartheta, \sigma)$, $\epsilon = 10^{-6}$ and $p = 0.99998$, we have solved problem (7) and obtained $k^* = \lceil 38.7357 \rceil = 39$. One can check that the alternative estimates (9) and (10) give $\hat{k}^* = 39$ and $\hat{k}' = 42$ for $\epsilon = 10^{-6}$ and $p = 0.99$, and for $p = 0.99998$, (9) and (10) give 41 and 55, respectively. Bearing in mind that there are two gamma trajectories per one VG path, each of them requiring 39 uniforms in order to generate the k^*-largest jumps, plus another 39 uniforms to randomly locate them over the N time partition points, plus one uniform used to simulate the terminal value of each gamma process, we arrive at $d = 158$ (or less if $N < 39$).

The uniform variates, required for the GSS method and in the plain MC versions of DirBS and DGBS, have been generated using the 64-bit universal random number generator of Marsaglia and Tsang (2004). This generator provides numbers with a 10^{61} period which pass all the tests developed by Marsaglia and Tsang (2002). We use the modified version of Joe and Kuo (2003) of the algorithm of Bratley and Fox (1988) to generate Sobol' sequences up to dimension $d = 1111$ which satisfy Sobol's so-called Property A (a C++ code written by John Burkardt is downloadable from `http://people.scs.fsu.edu/~burkardt/cpp_src/sobol/sobol.html`). For the RQMC implementations of DirBS and DGBS, we randomize the d-dimensional Sobol' sequence by applying a random shift modulo 1 (see e.g. Boyle et al. 1997 and Ökten and Eastman 2004).

The gamma random variables, required to generate the terminal value of the VG process, have been generated by inversion, based on the Newton algorithm. In the implementation of the DGBS method, the fast generator of symmetrical beta random variables developed by L'Ecuyer and Simard (2006) is employed.

In what follows, the DirBS method is tested and compared with the GSS and the DGBS methods in terms of efficiency. The efficiency ratio is defined as

$$E_{A|B} = \frac{t_B \sigma_B^2}{t_A \sigma_A^2}, \tag{11}$$

where σ^2 is the variance of the estimate obtained in time t for the corresponding method. When $E_{A|B} > 1$ we say that method A is more efficient than (or should be preferred over) method B and vice versa if $E_{A|B} < 1$ (see e.g., Hammersley and Handscomb 1964). Since in our case there is an estimation bias, in computing the efficiency ratio we have replaced variance by mean square error. For a deeper study of efficiency we refer the interested reader to Glynn and Whitt (1992).

As in Avramidis and L'Ecuyer (2006), we consider the following three options. A floating strike lookback call option with payoff

$$C_T = \left[S(T) - \inf_{0 \le t \le T} S(t) \right].$$

A barrier option of the type *up-and-in call* with a payoff

$$C_T = (S(T) - K)^+ \mathbf{1}_{\left\{\sup_{0 \le t \le T} S(t) > b\right\}},$$

where $b > S(0)$ is the activating barrier and K is the strike price of the European call option underlying the barrier feature. Specifically, we fix $K = S(0) = 100$ and $b = 120$. And an Asian option with a payoff

$$C_T = \left(\frac{1}{T} \int_0^T S(t)\, dt - K \right)^+,$$

where K is the given strike price. In this case, we consider $K = S(0) = 100$.

The three options are valued for different number of equally spaced time partitions, $N = 2^6, \ldots, 2^{12}$, of the interval $[0, T]$. For each of the options, in order to obtain an estimate of the price,

following (3), and an estimate of its variance, we run 100 independent replications (randomizations of the corresponding d-dimensional Sobol' point sets) with $M = 250,000$ sample paths.

Both DirBS and DGBS are tested in plain MC environment and also with different number of time points, $n \leq N$, where stratification is applied using a QMC sequence. The results presented here are for the following three cases: $n = 0$, i.e. the plain MC case; $n = 2$, when a (randomized) 2-dimensional Sobol' sequence is used to stratify the terminal values of the two gamma processes, $G_i(t_N; \alpha, \lambda_i)$, $i = 0, 1$; and $n = n_{max}$, when a (randomized) n_{max}-dimensional Sobol' sequence is used for all (possible) uniforms. This means that, in the case of DGBS, $n_{max} = 2min(N, 2^9)$, due to the upper bound of 1111 for the dimension of the Sobol' sequence used, and in the case of DirBS, $n_{max} = 158$.

In Figures 2, 3 and 4, the estimated absolute bias for different number of time points, for the lookback, the barrier and the Asian options, respectively is presented in log-log scale. To calculate the bias, we use the estimated exact values, with 95% confidence, of 9.39805 ± 0.00015, 2.1575 ± 0.0010 and 3.68538 ± 0.000048 for the lookback, the barrier and the Asian options, respectively, obtained by Avramidis and L'Ecuyer (2006), using extrapolation. As can be seen from the left panels of Figures 2, 3 and 4, without stratification the bias of the DirBS method is greater than that for the GSS and the DGBS methods for small number of time points, e.g. $N = 2^6, 2^7, 2^8$. However, for more refined partitions of $[0, T]$, e.g. $N = 2^{10}, 2^{11}, 2^{12}$, DirBS has smaller bias which can be explained with the asymptotic nature of the method. In the case of full stratification, the bias decreases with N at the same rate for both DirBS and DGBS, with DirBS behaving more stably, as illustrated in the right panels of Figures 2, 3 and 4.

The variance reduction achieved by using the DirBS method compared to the GSS and the DGBS method is illustrated in Figures 5, 6 and 7 in log-log scale for the lookback, the barrier and the Asian options, respectively. We see that stratifying the DirBS and the DGBS method at the terminal time, $t_N = T$, only, leads to similar reduction in the variance for both estimators, with DirBS being slightly better and more stable. However, the fully stratified version of DGBS leads to slightly greater reduction in the variance, compared to the fully stratified DirBS method. One

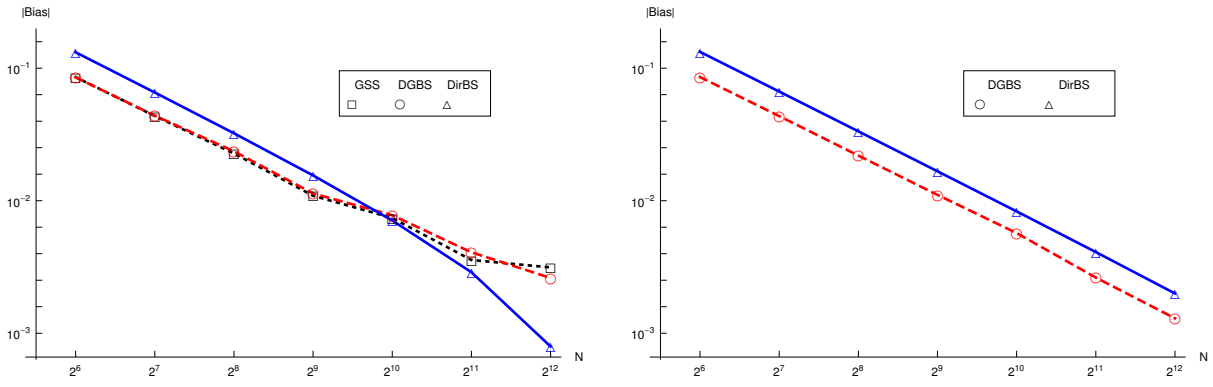

Figure 2 **Lookback option example. Estimated absolute bias: $n = 0$, plain MC (left panel); $n = n_{max}$, RQMC points used for all uniforms (right panel).**

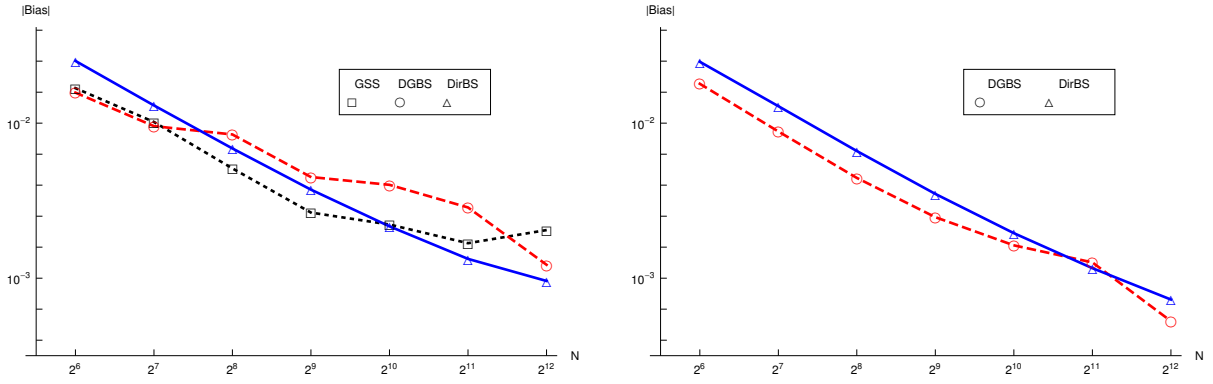

Figure 3 **Barrier option example. Estimated absolute bias: $n = 0$, plain MC (left panel); $n = n_{max}$, RQMC points used for all uniforms (right panel).**

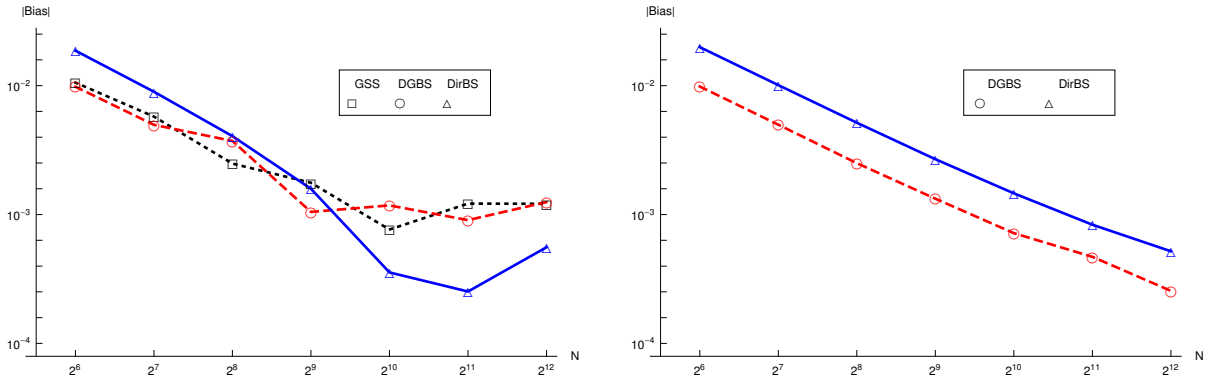

Figure 4 **Asian option example. Estimated absolute bias: $n = 0$, plain MC (left panel); $n = n_{max}$, RQMC points used for all uniforms (right panel).**

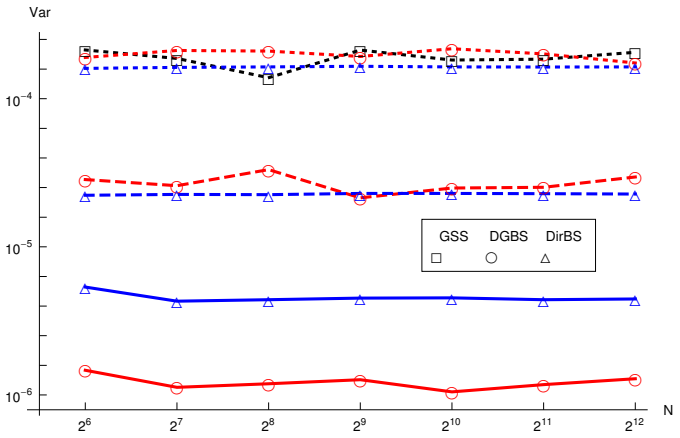

Figure 5 **Lookback option example. The variance of the GSS, DGBS and DirBS estimators: $n=0$, plain MC (dotted lines); $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ (dashed lines); $n=n_{max}$, RQMC points used for all uniforms (solid lines).**

reason for this is because, in DGBS, stratification is applied at a larger number of points than in DirBS. Thus, in the case of DGBS, the values of the two gamma processes $G_i(t;\alpha,\lambda_i)$, $i=0,1$ are stratified at the first $n_{max}=2min(N,2^9)$ time points of the dyadic partition used. Whereas, for DirBS stratification is applied only in generating $G_i(t_N;\alpha,\lambda_i)$, $i=0,1$, and the sizes and the positions of the $\hat{k}^*=39$-largest increments of each of the two processes $G_i(t;\alpha,\lambda_i)$, $i=0,1$, which leads to $n_{max}=158$. However, it has to be noted that using a different low discrepancy sequence, for example Korobov lattice rules, may lead to different variance reduction factors for the DirBS and DGBS methods. For the DGBS method this has been illustrated in Table 1 of Avramidis et al. (2003).

The efficiency gains for the three methods, compared with one another, are presented in Tables 1, 2 and 3. As can be seen, for all three option pricing examples presented here, the two bridge methods, DirBS and DGBS, provide significant efficiency gains compared to the plain MC sequential gamma sampling method, GSS. Similar results have been observed also by Ribeiro and Webber (2004) and Avramidis et al. (2003). The efficiency gain of the DirBS method compared to DGBS ranges between 3.2 and 31.2 for the lookback example, 4.0 and 32.5 for the barrier option, and 1.8 and 27.3 for the Asian option example.

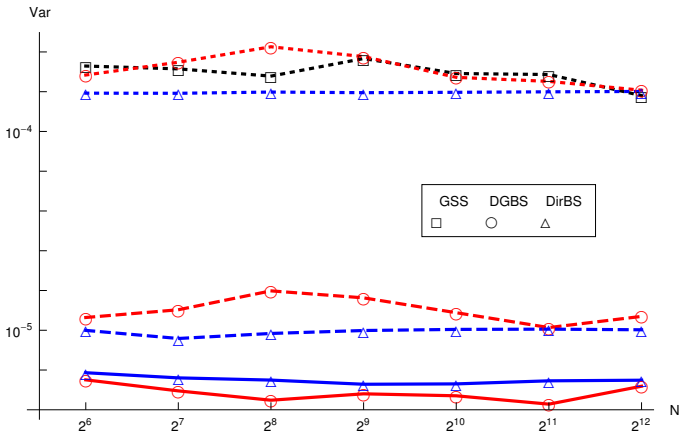

Figure 6 **Barrier option example. The variance of the GSS, DGBS and DirBS estimators: $n=0$, plain MC (dotted lines); $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ (dashed lines); $n=n_{max}$, RQMC points used for all uniforms (solid lines).**

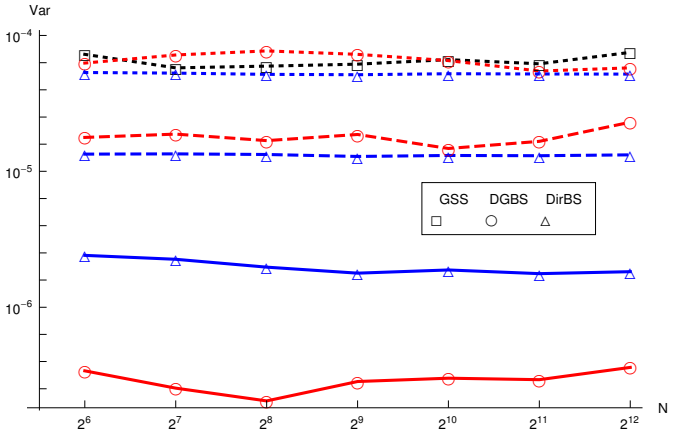

Figure 7 **Asian option example. The variance of the GSS, DGBS and DirBS estimators: $n=0$, plain MC (dotted lines); $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ (dashed lines); $n=n_{max}$, RQMC points used for all uniforms (solid lines).**

In Figure 8, we give the time for a single run with $M=250,000$ sample paths for the lookback option example with full stratification for DirBS and DGBS (the PC used for the calculations has AMD Athlon64FX–55 processor, 2.61GHz and 2.00GB RAM). The computation times for the other two examples are very similar and therefore, are omitted. As can be seen from Figure 8, DirBS is much faster compared to GSS and DGBS. In particular, for $N=2^6$, it finishes in 12.27 sec, and is 3.68 times faster than GSS and 7.74 times faster than DGBS; for $N=2^{12}$, its computation

Table 1 **Lookback option example. Efficiency gains: (a) $n=0$, plain MC; (b) $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ for DirBS and DGBS; (c) $n=n_{max}$, RQMC points used for all uniforms for DirBS and DGBS.**

	$N=$	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
	(a)	0.5	0.5	0.4	0.5	0.5	0.5	0.7
$E_{\text{DGBS\|GSS}}$	(b)	0.5	0.5	0.6	1.3	2.5	3.6	4.0
	(c)	0.5	0.5	0.7	1.4	4.0	14.3	44.1
	(a)	1.6	2.5	4.1	8.2	13.6	16.0	19.7
$E_{\text{DirBS\|GSS}}$	(b)	1.6	2.5	4.2	10.7	30.4	65.3	116.7
	(c)	1.6	2.5	4.5	12.1	40.4	133.2	393.0
	(a)	3.3	5.4	9.8	15.2	29.2	31.2	28.4
$E_{\text{DirBS\|DGBS}}$	(b)	3.3	4.9	6.7	8.3	12.0	18.2	29.0
	(c)	3.2	4.8	6.5	8.4	10.1	9.3	8.9

Table 2 **Barrier option example. Efficiency gains: (a) $n=0$, plain MC; (b) $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ for DirBS and DGBS; (c) $n=n_{max}$, RQMC points used for all uniforms for DirBS and DGBS.**

	$N=$	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
	(a)	0.5	0.5	0.3	0.5	0.5	0.6	0.6
$E_{\text{DGBS\|GSS}}$	(b)	0.7	1.5	2.1	5.1	7.2	9.7	7.2
	(c)	0.7	1.8	4.5	11.8	15.5	19.8	17.3
	(a)	2.3	5.0	7.8	14.2	15.1	16.4	13.2
$E_{\text{DirBS\|GSS}}$	(b)	3.1	10.4	35.2	124.7	207.5	262.7	232.4
	(c)	2.9	9.6	32.1	136.3	271.6	398.8	382.3
	(a)	4.3	10.5	24.9	29.2	28.1	27.8	23.8
$E_{\text{DirBS\|DGBS}}$	(b)	4.3	6.9	16.6	24.5	28.8	27.1	32.5
	(c)	4.0	5.2	7.1	11.5	17.5	20.1	22.1

time is 115.95 sec, and it is 15.41 and 25.28 times faster than GSS and DGBS, respectively. The method becomes more time efficient as the number of time partition points increases. This makes it especially suitable for pricing contingent claims when observations over large (possibly infinite) number of time points are needed.

5. Comments and conclusions

It is not difficult to see that the proposed Dirichlet bridge method for sampling the gamma process can be readily applied to compute the upper and lower bounds, $L_m(t)$ and $U_m(t)$, on the path of the

Table 3 **Asian option example. Efficiency gains: (a) $n=0$, plain MC; (b) $n=2$, RQMC points used to generate $G_i(T,\alpha,\lambda_i)$, $i=0,1$ for DirBS and DGBS; (c) $n=n_{max}$, RQMC points used for all uniforms for DirBS and DGBS.**

	$N=$	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
$E_{\text{DGBS}\|\text{GSS}}$	(a)	0.6	0.5	0.4	0.5	0.6	0.7	0.8
	(b)	0.8	1.2	1.3	1.8	2.6	2.4	2.2
	(c)	0.9	1.7	5.0	16.7	45.6	71.3	107.7
$E_{\text{DirBS}\|\text{GSS}}$	(a)	1.7	3.7	7.3	12.3	15.2	16.5	21.2
	(b)	1.8	5.0	14.8	39.4	58.7	67.3	84.3
	(c)	1.7	4.5	16.9	71.8	203.4	353.4	537.0
$E_{\text{DirBS}\|\text{DGBS}}$	(a)	3.0	8.0	19.3	24.6	25.3	23.1	25.3
	(b)	2.4	4.2	11.2	21.8	23.0	28.4	39.0
	(c)	1.9	2.6	3.4	4.3	4.5	5.0	5.0

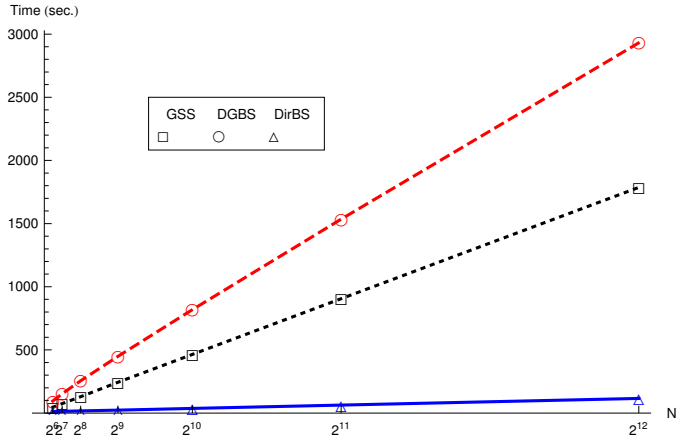

Figure 8 **Lookback option example. Computation time for one estimate of the price based on $M=100,000$ sample paths.**

process $S(t)$, developed by Avramidis and L'Ecuyer (2006) and calculate option price estimators, based on truncated DirBS and (Richardson) extrapolation (see section 4 of Avramidis and L'Ecuyer 2006). As noted by these authors, this is a valid approach only for options which obey a certain monotonicity condition imposed on the payoff as a function of the path of $S(t)$. Examining the properties and the behavior of such estimators involving the Dirichlet bridge are beyond the scope of this paper. Here, our purpose has been to introduce the Dirichlet bridge sampling of a gamma process as a multivariate generalization of the gamma bridge sampling technique and examine its relative performance.

An interesting insight, which results from our work, is the enlightening interpretation of the generation of the increments of a gamma process at equally spaced time points, given its value at the terminal time, as partitioning of the unit interval into fragments, whose distribution, under the Kingman limit, is $GEM(\beta)$. Based on this interpretation, we have developed a very efficient method, named DirBS, for simulating trajectories from a VG process, and estimating the price of path-dependent options under the VG model.

A nice feature of the DirBS method is that the related dimension is relatively low and does not increase with the number of time partition points, N. This and the observation that we can simulate $\tilde{L}_m$ by inverting analytically the cdf of $\tilde{V}_m$ (see (6) and Figure 1) makes DirBS very fast and leads to consistent efficiency gains. The latter is due more to considerable computation time saving (of growing relevance as N increase) rather than to greater variance reduction.

A further direction of research would be to explore the performance of DirBS under alternative choices of low discrepancy sequences (e.g. Korobov lattice rules) and alternative ways of randomizing the QMC (e.g. affine linear scrambling). It would not be surprising to find out that there might be a combination of these alternatives which is more favorable to the performance of DirBS than the chosen Sobol' sequence and a random shift as the QMC randomization algorithm.

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