PHD THESIS

Modelling Portfolios of Credit Securities

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Abstract

The study of credit derivatives is one of the most popular and controversial issues that concerns the entire financial industry. Increases of defaults and bankruptcies during the recent credit crunch has stipulated a heated debate about the adequacy of the existing pricing and hedging methodologies for credit derivatives portfolios. The main objective of this thesis is to propose and evaluate a treatable framework that addresses many of the deferences of the standard market model for portfolios of credit instruments.

After review and product introductions in **CHAPTER 1** we first summarize the common simulation methods for pricing portfolio credit derivatives, then we propose an alternative methodology that is based on an economical sense of the models and market observables in **CHAPTER** 2. Such simulation method provides a testing environment which houses the asset value based models with reliable assumptions. Meanwhile, a PCA analysis on CDO market spreads is performed on market data in **CHAPTER** 3. In **CHAPTER 4**, we develop an old school dynamic model for credit derivative valuation, it match the market needs, fit quoted spreads while providing time evolution using historical market observable measure. Finally, combining together the model and simulation framework, we are able to construct hedging strategies based on simulation results in **CHAPTER** 5. We mainly focus on the utilization of default probabilities in pricing techniques and a close-form formula is provided to calculate probability of default from the proposed growth rate factor.

Acknowledgments

Early in the process of completing my research, it became quite clear to me that a researcher cannot complete a Ph.D. alone. This thesis would not have been possible without help and support from many people, although the list of individuals I wish to thank extends beyond the limits of this format, I would like to express how grateful I am for their dedication, prayers, and support:

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Chapter 1

Overview of Credit Derivatives

Credit risk is one of today's heated topics in finance. Being in the center of a financial crisis which is described as 'once in a hundred year time.' by Mr. Alan Greenspan (formal Chairman of the Federal Reserve), banks are increasing their attention on capital management and are hoping to adopt a refined approach towards credit risk. In the meantime, insurance companies, reinsurance companies and hedge funds are all big players in this market.

1.1 Introduction

Allowing protection buyers to hedge default and recovery risks, credit derivatives give the protection sellers the flexibility to define credit exposure by different risk appetites. The payoffs of these financial products depend on the occurrence of certain credit events defined in the contract, and normally this occurrence is described as a *default* of the referenced entity. Generally, a 'default' is meant to be a failure to make payments on due dates either through inability to maintain the interest servicing or because of bankruptcy which leads to inability to repay the principal received from investors, other credit events such as a change in credit ratings are also widely used in the industry. In the year 1999, the International Swaps and Derivatives Association (ISDA) released the standard documentation which defined the credit obligations and events by six main categories as listed below:

- Bankruptcy.
- Failure to Pay.
- Restructuring.
- Obligation Acceleration.
- Obligation Default.
- Repudiation/Moratorium.

Meanwhile, it is quite possible that during the credit event, the investor does not lose the whole amount invested, instead, the loss is minimized by the recovery amount, hence the reviewed and proposed models here will only focus on the models that give the payoff of the financial product as a function of time of default and recovery rates. And in most cases, we adopt the simplicity of assuming a fixed and constant recovery rate and keep focusing on modelling the distribution of the times of default(s).

On the modelling aspect of the products, there are basically two types of models involved: the structural approach and reduced form approach. The former aims to model micro finance structure of the firm and define the trigger of default using the asset value and outstanding debt level, while the reduced form models try to deal with the common default probability using Poisson-type processes driven by market factors like LIBOR and in this case the default is not linked together with inner company issues.

In the following sections we will review the main types of credit derivatives starting with the very basic *default-able corporate bonds*, then the *credit default swaps* (CDS) and the *collateralized debt obligations* (CDO). Related products such as the CDS index, are also included to help understanding the working mechanics of the other complex credit products.

1.2 Review of Credit Derivatives

In this section we will start from the underlying default-able bonds which is the key credit-risk-embedded financial instrument, then move on to building bricks of the whole credit derivatives world, through credit default swaps, including their modelling and valuation, to finally conclude with a review of tranched portfolio products. Simple examples will be given for each of the products to explain their working mechanics in detail but for complex products like CDOs, the pricing details will be shown in later chapters.

1.2.1 Credit-risky Corporate Bonds

A *Corporate Bond* is a bond issued by a corporation. The term is usually applied to long maturity debt contracts. Sometimes, the term is used to include all bonds except those issued by governments in their own currencies, however, in this thesis the term *corporate bond* only applies to those issued by corporations.

Corporate bonds are often listed on major exchanges and the coupon is usually taxable. Bonds with zero coupon but higher redemption value are called zero coupon corporate bonds, there is no difference in pricing principle, i.e. the time-value discounting structure. However, despite being listed on exchanges, the vast majority of trading volume in corporate bonds in most developed markets takes place in decentralized, dealer-based, over-the-counter markets.

Comparing to government bonds, corporate bonds comes with certain default risk of the issuing company, thus corporate bonds generally have a higher risk of default. This risk depends on the particular bond issuing corporation, the current market/rating/ranking conditions and the governments to which the bond issuer is being compared. Corporate bond holders are compensated for this risk by receiving a higher yield than government bonds. To hedge the credit risk exposure of investing in a corporate bond, one can obtain an unfunded synthetic exposure matching the bond issuing company via credit default swaps.

The reverse also holds, i.e. we can use corporate bonds as hedging instruments to cover credit exposure of selling CDS contracts. The bond used as hedging instrument is the referenced underlying bond of the credit derivative. One of the main attractions of choosing the bond hedging route is that, in case of a physical CDS settlement, one may have the bonds delivered as payoff of CDS claim instead of paying cash.

On the other hand, the hazard rate or the probability of default for the CDS contract and the risky bond is the same as they refer to the same entity, in other words, both securities move in parallel. So it is natural to have the referenced bonds as the first choice of a hedging instrument. One disadvantage is that strategies built with a long position of bonds might be expensive as the initial payment of bond could be large. However, very similar to the use of options in the stock market, this expense could be covered using options or futures on the underlying bonds.

Defaultable Bond Pricing

We know that the price of a CDS is given in the form of basis points standing for a percentage of the referenced defaultable bond price. Consider a defaultable zero coupon bond with face value £1 at T if it survives until maturity, in case there is a default before T, we suppose the recovery rate is no larger than one and this rate is greater than zero. We denote this recovery rate by $\hat{R} \in (0, 1)$. With a constant interest rate and recovery rate r, the difference of bond yield to maturity (YTM, hereafter) and the risk free interest rate is denoted by a where $r + a =$ YTM, the defaultable bond price B' at time zero[1] is equal to the difference between a risk-free bond and the time zero value of the loss at default:

$$B' = e^{-rT} - e^{-rT}(1 - \hat{R})(1 - e^{-aT}) = e^{-(r+a)T} + e^{-rT}\hat{R}(1 - e^{-aT})$$

In case there is a loss of total investment, or say, zero recovery, we obtain:

$$B' = e^{-(r+a)T} \tag{1.1}$$

Relaxing the constant interest rates, recovery rates assumption, the prices of default-free and zero recovery defaultable bonds are given by: (see Schonbucher (1998)):

$$B = E\left[e^{-\int_0^T r(s)\,ds}\right] \text{ and } B' = E\left[e^{-\int_0^T r(s)+a(s)\,ds}\right].$$

We illustrate below the case of having an offset cash-flow using a simple example with a single name Credit Defautl Swap (CDS, hereafter) and its underlying

[1]Normally, quantities of defaultable bonds are denoted by the same notation for a risk-free bond plus a dash(').

default-able bond.

Say we have a T-year corporate bond with face value of N_b, coupon is paid semi-annually with rate c, and the present value of this risky bond is V_b. The CDS contract has a notional of N_{cds} and the premium spread of this CDS contract is set to be s over the life time of the contract. The interest rate is considered constant and denoted with r. The recovery rate R is set the same for both the CDS and the risky bond[2].

Consider the cash-flow in the trading book, the example is demonstrated in the following figure:

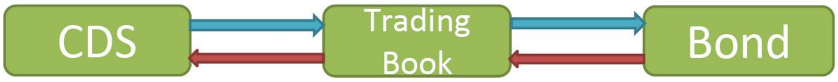

Figure 1.1: *Hedging a single name CDS with the underlying bond.*

In case there is no default until maturity, the value of portfolio, ΔV, is given by:

$$\Delta V = -V_b + N_b e^{-rT} + \sum_{0}^{2T} (cN_b - sN_{cds}) e^{-rt_i}$$

If the default arrive prior to the maturity of time T, the value of portfolio over the whole time interval is given by:

$$\Delta V = -V_b + (RN_b + (1-R)N_{cds}) e^{-r\tau} + \sum_{0}^{\tau} (cN_b - sN_{cds}) e^{-rt_i}$$

Here τ stands for the time of default in the above equation. The optimal hedging aims to achieve a portfolio value which satisfies: $E[\Delta V] = 0$, by changing the

[2]This is always true even in practice, as the recovery rate refers to the proportional value recovery of the referenced bond, details can be found in Rajan, McDermott & Roy (2006).

bond notional against the CDS notional. In other words, the portfolio manager may need to regularly adjust the position of longing bonds by active bond trading. Note that for portfolio product hedging such as hedging strategy for CDOs, this could be expensive if hedging is the only motive as there is a name match between the bond portfolio and the CDO reference pool.

1.2.2 Credit Default Swaps and Baskets

Credit Default Swaps in the form of a single-name contract, simply allows an investor to gain or sell risk exposure to a reference asset without using any funding (which means we don't have to buy or sell any underlying bond or loans of this entity). Most of the CDSs are contracted on defaultable bonds such as corporate bonds, and normally we can consider the default of the reference asset as the default of a defaultable bond issued by the reference company.

Very similar to insurance contracts, a credit default swap provides protection against the risk of one event of default of a reference company. We give the key word explanations as following:

Reference entity**:** the particular company with the risk of default mentioned in the financial agreement.

Credit event**:** the default of the reference entity.

For example, assume the protection buyer (or the insured) holds a bond issued by the reference entity with a par value at maturity time T. If the company *fail to pay*[3] the par value at time T, the protection buyer (CDS holder) has the right to

[3]Other defaults like bankruptcy or sudden lose in stock market may be contracted differently

sell[4] the bond for its par value to the protection seller (or say, insurer) and in case for a plain vanilla CDS, once it is exercised, it is knocked out; in the meantime, the insured has to pay for the protection he got for the valid period of the contract and a final accrual fee might be needed for the settlement (See Wilmott (1998), Jarrow & Turnbull (2000) and Kolb (2003) for more about credit derivatives). We further define three more key words below:

***Reference obligation*:** the bond.

***Notional principal*:** agreed total par value of the bond that can be sold in the swap.

***Basis points*:** the *fee rate*[5] applied in the CDS, the payment is the product of *Basis points* and *Notional principal*.

Continuing the example, we add more details to illustrate how a typical CDS works and settled. Assume that a three-year CDS starts Jan 1st 2006 with notional principal of £100 and the annual fee rate is 100BP. Then, the insurer receives $\pounds 100 \times 1\% \times 1\text{year} = \pounds 1$ on Jan 1st of each of the years 2007, 2008 and 2009 if there is no default. In case there is a default on 1st Jun 2008, the CDS can be settled either in physical or cash terms depending on the contract. The protection buyer has the right to sel the bond for its par value to the insurer in physical method of settlement. If cash settlement is required, then the cash payoff will be the difference between the par value and the market price of the bond, that is to say, if the value of the bond is £35 at default, the total difference is then £65

in different CDSs.

[4]Mainly there are two different ways of settlement when a credit event occurs, details will be given later in this section.

[5]Basis Points (BP) rate, 100BP=1%.

which is equal to the cash payoff in this case. Meanwhile, the final accrual fee is calculated to be:

$$\pounds 100 \times 1\% \times \frac{1}{2}\text{year} = \pounds 0.5$$

which is the insurance fee for half a year.

Of all standard credit default swaps, the diversity exists mainly in the contract terms. And here we give a short list of main differences of the most mentioned standard CDSs:

- *Binary*: the indemnity payoff at default can be contracted to any amount.
- *Basket*: there is a list of specified reference entities in the contract and the default payment is made only when the first name in the premised list defaults.
- *Contingent*: in this case, there is a additional condition to trigger the default payment. This trigger can be stated in the contract as a default of another reference entity or change of certain market variable values.
- *Dynamic*: the notional principal is set to be related to the market value of a swap portfolio.

In the next sections, we will present first the basic valuation method of pricing a regular credit default swap, and then we will extend the methodology for more complex CDS contracts such as forward CDSs.

The Valuation of CDS

A corporate bond has either a higher interest rate than a similar Treasury bond or it sells at a cheaper price. If we simply assume this is happening only because of the probability of default, we can easily get that the present value of the cost of default as the difference between the present value of the two bonds. Following this idea, we may want to know something about the default probability before we buy the bond or start thinking about the related CDS price.

The Default Probability

We start with a simple example. Say that the 'credit-risk free' bond rate of a three year Treasury bond with par value £1 is 2% and of a zero coupon corporate bond of the same face value and maturity date with zero recovery is 2.5%. The present value of both bonds is $e^{-0.02\times 3}$ and $e^{-0.025\times 3}$ respectively, and the value of default cost at present is therefore: $e^{-0.02\times 3} - e^{-0.025\times 3} = 0.0141$. (Here we assume that the corporate bond can only default at maturity.)

Here we use p to denote the risk-neutral probability of default. In case of a default under our assumed situation, it will cause a loss of full face value of £1 at maturity, so the risk-neutral expected loss from this default is then: $pe^{-0.02\times 3}$, in other words:

$$pe^{-0.02\times 3} = 0.0141 = e^{-0.02\times 3} - e^{-0.025\times 3}$$

so we have $p = 1.49\%$. In general, let r_B denote the cooperate bond rate while r is the risk-free rate and denote the maturity by T . From the above equation, we get:

$$p = \frac{e^{-rT} - e^{-r_B T}}{e^{-rT}}$$

which follows that:

$$p = 1 - e^{-(r_B - r)T} \tag{1.2}$$

(See Hull & White (2000) for more details.)

Extending to More Complicated Cases:

In the case the bond defaults not only at maturity but at any time during its life time, the calculation of default probability becomes more complex and even if we get it from the approach above the result will involve more variables.

Denote the credit-risky bond yield to maturity by Y, the difference between the bond YTM and the risk-free interest rate r is calculated by $(Y-r)$ and denoted by a. With a slight abuse of terms, we name a as the risky YTM of a credit-risky bond. Note that the corporate bond always has higher interest rates than Treasury bonds[6] with similar maturity, so the risky YTM of our bond, a, is nonnegative. Now as the bond can default at any time during the life of the contract, we can simply imagine it is a combination of infinitely many bonds with different maturity dates which are only default-able on their maturity. As the bond yield a and the non-default contract time of the bond are independent, it follows that the bond yield rates of the group of bonds with maturity dates in the time interval $[0, T]$ form a stochastic process. It is not hard to get that the default is more likely to happen on maturity dates of bonds with higher yield to maturity, and as the outcomes at any time during the bond maturity is just default or not, the default dates form a nonhomogeneous Poisson process[7] in the interval $[0, T]$ which is characterized by its stochastic intensity a.

Define the default probability by $q(t)$, so the default probability between time $t > 0$ and $s > t$ seen from time t is $q(t) \times (s-t)$. Meanwhile, define the intensity at time t as the hazard rate of time t which is denoted by $a(t)$. Applying the non homogeneous density function of Poisson process, the default intensity for time

[6]Cheaper selling prices can be converted into higher interest rates easily.

[7]Known as the default counting process or a *Cox process*. For more details or formal definition, see Lando (1999).

interval $[0, t]$ is given by $\int_0^t a(s)\, ds$, for $s \in [0, t]$, now suppose the default took place on date τ, the related default probability for the default to take place within time interval $[0, t]$, i.e. $0 \leq \tau \leq t$, is:

$$q(t) = e^{-\int_0^t a(s)\, ds} \tag{1.3}$$

which lead us to the survival probability (no default over time interval, i.e. $\tau \notin [0, T]$) as following:

$$1 - e^{-\int_0^T a(s)\, ds} \tag{1.4}$$

(Similar results and details can be found in Bielecki & Rutkowski (2002).)

The expected risk-neutral survival probability over time interval $[0, T]$ conditioning on the hazard process $a(t)$ is:

$$\int_0^T (1 - q(t))\, ds = \int_0^T (1 - e^{-\int_0^t a(s)\, ds})\, dt.$$

CDS Pricing

Before we sign a CDS contract or an insurance contract, we are interested in the compensation we can get in comparison we need to pay. So, looking to the simple zero coupon bond example we had before equation (1.2) earlier in this section, knowing that under the principle of no-arbitrage what we can get from the indemnity is no more than the amount we lose. Thus, as our loss is 0.0141 in the example, our best claim is exactly the same number 0.0141, and so is the price of a CDS referencing the bond.

Suppose the CDS we mentioned above allows the bond to default at any time t before T, but the indemnity payment is remained to be made at time T and there is only one fee payment made by the CDS buyer. For no-arbitrage reasons, the fair price of the fee payment should be the same as the discounted claim payment, this gives us the so called European-style zero recovery default put (assuming a par value of £1). Denoting the value of such a claim by C, and the default probability by $q(t)$. On default, the value of claim is the discounted value of the underlying, so the value of the default put can be easily found:

$$\begin{aligned} C &= E\left[1 - q(t)\right] e^{-\int_0^T r(s)\, ds} = E\left[e^{-\int_0^T r(s)\, ds} - e^{-\int_0^T r(s)+a(s)\, ds}\right] \\ &= E\left[e^{-\int_0^T r(s)\, ds}\right] - E\left[e^{-\int_0^T r(s)+a(s)\, ds}\right] = B - B' \end{aligned}$$

For a general but zero recovery CDS, the payment is made at default and the claim at default will be the expected bond value at time t, calculated as of at time 0:

$$E\left[e^{-\int_0^T r(s)\, ds} E\left[1 - e^{-\int_0^T a(s)\, ds} | a\right]\right] = E\left[\int_0^T a(t) e^{-\int_0^t a(s)\, ds} e^{-\int_0^T r(s)\, ds}\, dt\right]$$

which can be rewritten in the form:

$$\begin{aligned} \int_0^T E\left[a(t) e^{-\int_0^t a(s)\, ds} e^{-\int_0^T r(s)\, ds}\right] dt &= \int_0^T E\left[a(t) e^{-\int_0^t a(s)\, ds}\right] B\, dt \\ &= \int_0^T E\left[a(t)\right] B'\, dt \end{aligned}$$

where $E\left[a(t)\right]$ is the associated *forward rate* of the spreads.

The cash flow in a CDS consists of two payment legs: *fixed* and *floating*. The cash flow of the *floating* leg is the indemnity payment from the insurer while the cash flow of the *fixed* leg is the payment fees for buying the CDS, and the principle of CDS pricing is the equality of these two legs. The *floating* leg is the loss at default we calculated above and the *fixed* leg is the sum of the periodical fee payments until maturity if there is no default.

The periodical fee payment of each time length dt from time t to $t + dt$, is the product of: the CDS spread rate $\bar{s}$; and the value of the underlying bond over the specified time interval $[t, t + dt]$, which can be calculated by $B'_{t,t+dt} dt$. Hence the fee payment for this specified time period is $\bar{s} \cdot B'_{t,t+dt}dt$. So the *fixed* leg over the whole time interval is given by

$$\int_0^T \bar{s}B'\,dt = \int_0^T \bar{s}E\left[e^{-\int_0^T r(s)+a(s)\,ds}\right]\,dt = \bar{s}\int_0^T E\left[e^{-\int_0^T r(s)+a(s)\,ds}\right]\,dt$$

Now by applying the equality of the two legs, we have:

$$\int_0^T E\left[a(t)e^{-\int_0^t a(s)\,ds}\right]B\,dt = \bar{s}\int_0^T E\left[e^{-\int_0^T r(s)+a(s)\,ds}\right]\,dt$$

Thus, the CDS spread $\bar{s}$ is easily found as:

$$\bar{s} = \frac{\int_0^T E\left[a(t)e^{-\int_0^t a(s)\,ds}\right]B\,dt}{\int_0^T E\left[e^{-\int_0^T r(s)+a(s)\,ds}\right]\,dt} = \frac{\int_0^T E\left[a(t)\right]B'\,dt}{\int_0^T B'\,dt}$$

In the case that the recovery rate is $\hat{R}$, the new loss at default should be multiplied[8] by $(1-\hat{R})$, which gives us:

$$\bar{s} = (1-\hat{R})\frac{\int_0^T E\left[a(t)\right]B'\,dt}{\int_0^T B'\,dt} \tag{1.5}$$

Here notice that: as the default swap is knocked out at default, the fixed leg is calculated with the defaultable bond price B', but if the swap is not terminated at default, the fixed leg value should be calculated using the default-free bond price B. So, with different claim and stop strategies of the swap, the CDS spread can vary.

[8]Note that there are different claim strategies resulting in different values in the pricing equation. An alternative claim strategy given in Hull & White (2000) is when the recovery rate is effecting both the value of the bond and a final accrual payment A, that is to say the default should be multiplied by $1-\hat{R}(1+A)$ under this strategy.

1.2.3 CDS Index Tranches

Standard CDS indices are traded similar to a stock index on the request of liquidity on a diversified set of credit products. But there is quite a difference for an index with a tranched structure, as they are considered more as a class of Collateralized Debt Obligations (CDOs). Typically, CDS index tranches are taken as a proxy for common CDOs in both academic and business research.

There are two basic CDS index families, CDX for the North American market and iTraxx for European market with subindices covering Japan and the rest of Asia. The main indices are named CDX NA IG and iTraxx Europe and they contain 125 investment grade[9] underlying names each, and the underlying companies are equally weighted with same CDS notional. The number of tranches on trade is 5 for each index with a slight difference in the attach and detach point setting, and the maturity of contracts is optional ranging between 3, 5, 7 or 10 years.

The settlement method on default and fair spread pricing for a CDS index tranche is the same as in single name CDS contracts.

1.2.4 Collateralized Debt Obligations

The Collateralized Debt Obligations (CDO) is one of the most complex derivative structures in the credit derivative family. The first trading of this product took place in 1986 in the U.S. market and then heated in the 1990s.

The name CDO covers a wide range of products as a CDO can hold a variety

[9] i.e. with credit ratings above BBB in S&P ranking structure.

of underlying assets such as bonds, loans, debts, asset backed securities (ABS) or mortgage backed securities (MBS) and much more. These different classes of financial products is called the 'pool' of the CDO, and a pool can be comprised of more than one class of underlying assets and have multiple names under one class of the assets.

In fact, the CDO[10] itself is also an asset backed security according to the underlying pool, and its name also differs for different classes of underlying assets. For example, if the underlying is a bank debt then it is of course a collateralized *debt* obligation - CDO, but for corporate or emerging market bonds, it is then called a collateralized *bond* obligation - CBO, and similarly for a underlying pool of bank loans, it is referred to as a collateralized *loan* obligation, or a CLO.

Based on the type of underlying assets, CDOs are commonly classified into either *synthetic* or *cash* CDOs, where the reference pool of a cash CDO is made up of cash assets, i.e, corporate bonds or loans, while in a synthetic CDO the referenced portfolio consists of credit default swaps and the portfolio is not managed. Meanwhile, the synthetic CDOs are unfunded transactions and the word 'synthetic' refers to the fact that exposure to credit risk is gained synthetically via credit default swaps without buying any defaultable assets.

This main difference affects the origination of the CDO and can be described as follows: for a cash CDO, the reference assets are transferred to a Special Purpose Vehicle (SPV) and this SPV then issues tranches to meet the needs of investors. It is the proceeds from the issuance of the tranches that are used to fund the purchase of the collateral for the SPV. In case of a synthetic CDO, the roll of SPV is not

[10] For more details about the family of CDO products, please refer to Choudhry (2000), JPMorgan (2001), Gibson (2004) and Choudhry (2005).

needed, which means the originator is acting as the protection seller. In the rest of this paper, we will focus on modelling and pricing techniques of synthetic CDOs for its simplicity in cash flow (only between the seller and buyer) and the static character after issuance.

Upon creation of a CDO, the purposes of the issuing and choosing the CDO and its assets are distinguished in two ways: If the holder of certain assets wants to sell or transfer the risk associated with such assets, and a CDO is then created to accomplish the goal or, in other cases, to reduce the regulatory/economic capital requirements, this type of CDO is classified as the *balance sheet* CDOs. In contrast, if the goal is to achieve a leveraged return, such as the spread income when equity trap excess interest proceeds[11], these CDOs are referred to as *arbitrage* CDOs. Another main difference between the two kinds of CDOs is how they are going to be accomplished, the key, at this point, is the how to close the Equity tranche - the most risky portion. Normally, a balance sheet CDO is more likely to close comparing to an arbitrage CDO, as balance sheet CDOs often has a prepackaged investor for most of the equity tranches while asset manager for arbitrage CDOs is only partly committed.

The defining feature of a CDO structure is the *tranching* of credit risk, by dividing the risk of loss on the reference pool of assets into small tranchelets, different classes of securities are created to meet specified risk exposure and return requirements of different investors.

According to the increasing seniority from the most vulnerable tranche to the tranche which is rated highest against risk but with lowest return, the loss from

[11]Here, the term 'equity trap' is used to describe the fact that equity holders of an arbitrage CDOs hope to capture the difference between the after-default yield on the assets and the financing cost due debt tranches - normally the Equity tranche.

credit events will affect and consume each of the tranches one by one. In most CDOs, the most senior tranche provides the majority of the financing of the product, while other tranches are sized around 5% to 10%.

For a synthetic CDO, its tranche structure is defined by the attachment and detachment points, K_a and K_d, of the tranche, respectively. Let the final maturity date be T and consider the following example: the total notional of the portfolio is 100 million, consisting of 100 CDSs with 1 million notional each and the tranches are sized 0% - 3% for the equity tranche, 3% - 10% for mezzanine and the rest are held as senior. Then the notional for these three tranches is 3 million, 7 million and 90 million respectively, as shown below:

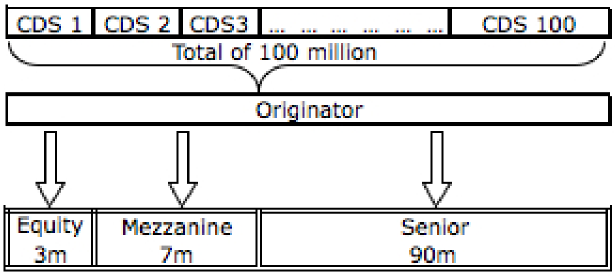

Figure 1.2: *Example of synthetic CDO transaction. 100 underlying CDSs with a total of £100m. And the three tranches are sized 0% $\sim$ 3%, 3% $\sim$ 10% and 10% $\sim$ 100%.*

When a credit event occurs, the first subordinating tranche is the so called *equity* tranche, and the second tranche - the *mezzanine* tranche is only affected if the size of total loss exceeds the size of the equity tranche, and similarly, the *senior* tranche is affected only if the first two tranches are all lost in the credit event.

In contrast, when it comes to payment, the investors on the senior tranche will be paid first according to an agreed rate (it is known as the 'coupon' rate of a CDO) on the current notional of the tranche, then the mezzanine tranche and those who invest in the equity tranche will get paid only if the senior tranche is not ‘worn out’.

It is easy to see how risky the equity tranche is and how the senior tranche is protected under such settings. It is common that a CDO is structured into more than three tranches, for instance, a CDO could have two mezzanine tranches sized $3\% \sim 6\%$ and $6\% \sim 9\%$ and a senior tranche from $9\% \sim 12\%$ then its first and second super senior tranches from $12\% \sim 22\%$ and $22\% \sim 100\%$ respectively.

After the tranche structure is created for a CDO, the tranches are normally rated for their protecting ability against the risks, and according to the inter-tranche working procedure described above. Before the year 2007, according to the S&P rating system, a senior tranche is normally rated above ‘Single - A’ and no lower than ‘Double - B’ for the mezzanine tranche, but as the equity is always facing the risk directly, this tranche is not rated. With reference to the structure in Figure (1.2) the following stylized CDO example is given in the Table below:

Table 1.1: **A stylized CDO example** (Spreads in basis points)

Tranche	Attach/Detach Points	Notional	Credit Rating	Spread
Equity	0% - 3%	£3m	Not Rated	1000
Mezzanine	3% - 10%	£7m	A	300
Senior	10% - 100%	£90m	AAA	10
Whole Portfolio	0% - 100%	£100m	A	60

In order to lower the risk for itself and to offer better protection to the other more senior debt tranches, the equity tranche typically holds stock or income notes as its underlying asset, hence the term 'equity'. We can extend the example above to see the possible cash flows that may occur in a CDO transaction.

Table 1.2: **CDO notional and coupon payments**

Year(End)	Loss	Notional			Coupon		
		Equity	Mezzanine	Senior	Equity	Mezzanine	Senior
0	0	3m	7m	90m	0	0	0
1	2m	1m	7m	90m	0.3m	0.21m	0.09m
2	3m	0	5m	90m	0.1m	0.21m	0.09m
3	0	0	5m	90m	0	0.15m	0.09m

Mathematically, the calculations in Table (1.2) are straight forward. Denote the total loss until time t as L_t, then the loss incurred on a tranche is given by:

$$L_t^{trch} = (L_t - K_a)^+ - (L_t - K_d)^+$$

where K_a and K_d are the characterizing attachment/detachment points. The periodic premium payments from buyer to the seller is the product of the fixed premium spread and the up-to-date outstanding tranche notional. If we denote the original tranche notional by N^{trch}, then the time t outstanding tranche notional is:

$$N^{trch} - L_t^{trch}$$

As shown in Table (1.2), by the end of second year, the notional of mezzanine tranche is written down to 5m, which is the result of:

$$7 - [(5m - 3m)^+ - (5m - 10m)^+] = 7 - [2 - 0]$$

In other words, 5m is the total loss of the first two years. And thereby the amount paid by the end of the third year is $3\% \times 5m = 0.15m$.

It is clear from the above example that the key problem in CDO pricing is to find the spread rate which in turn is related with both the tranche settings and the cumulative loss. The latter is associated with the cumulative joint default times of all the underlying assets (CDSs for a synthetic CDO) that make up the referenced portfolio. We will highlight the modelling details in later sections.

1.3 Review of Credit Default Models

Credit default models typically fall under two categories: *structural* and *reduced form*. In this section, we provide a brief review of both of these two approaches and give details of the copula models as well as the growth rate type of model.

The direction of mainstream model development by both academic researchers and industry practitioners is moving from static models to dynamic ones. This is due to the increasing demand for hedging techniques on credit derivatives which prerequisite a time dependent framework. In the later chapters of this thesis we will develop a dynamic model that fits well the market data, and tracks the credit risk change of a structured portfolio over multiple time periods.

1.3.1 Main Approaches of Credit Default Models

Structural Models

This approach is based on the credit model first proposed by Black & Scholes (1973), Merton (1974) and then extended by Black & Cox (1976). Recently, Hull, Predescu & White (2005) offered a further extension to price the default correlation in tranches of structured securities. Common to all models of this type, the basic idea is that the asset value of the company must stay above its debt, and for this reason, structural models are also called *firm-value models*. Under the assumption of stochastic company asset value and a default barrier of the minimum asset value, a credit event is then defined to be triggered when the asset value breaks the barrier.

Originally, Merton assumed that if the asset value exceeds the value of debt, then it is always possible for the company to sell its value to make payments. In such a model, a company can only default on the maturity of the debt and the relationship of asset, debt and equity is simply described by: Asset = Debt + Equity. As no extra payment will be demanded from the equity holders if the asset value is insufficient to pay the debt in the case of bankruptcy, the equity value can never go negative, so the value of equity at maturity T is defined to be the residual left after the “asset” is consumed to pay off the debt: $E_T = (A_T - D)^+$ where A_T is the asset value at T and D is the value of debt.

Then, the asset value is: $A_T = D_T + (A_T - D)^+$ which is assumed to follow a log normal stochastic process:

$$dA(t) = \mu A(t)dt + \sigma A(t)dX_t \quad (1.6)$$

where X_t follows a Brownian motion, and μ is the drift parameter of expected growth rate of asset value of the ith company and σ is its volatility. Then it is always possible to value this option style payment using the Black-Scholes option pricing formula. The default is assumed to occur only on maturity if the asset value lies below the face value of debt. If we assume a single period, the probability of default at maturity is:

$$p = \int_{-\infty}^{D} \phi[A(T)]dA[T] = 1 - N(d_2) \tag{1.7}$$

where $\phi(\cdot)$ is the log normal density function and $N(\cdot)$ is the cumulative normal probability function, the same as defined in Black-Scholes option pricing formula:

$$d_2 = \frac{\ln A(0) - \ln D + (\mu - \sigma^2/2)}{\sigma\sqrt{T}}$$

This probability is also the survival probability as it implies the in-the-money probability in case for an option. Then the current value of the debt equals to the value of a call option:

$$D(0) = A(0)[1 - N(d_1)] + e^{-\mu T} DN(d_2)$$

where $d_1 = d_2 + \sigma\sqrt{T}$.

Market experience of recent years[12] showed that two assumptions in the model are not realistic: first, it is hard to trade the assets with its present value any time we need the cash; second, it is almost impossible to find the exact present value of the asset at any time as the estimation of this value is based on the balance sheet data which is not updated in real-time.

In the extended model by Hull et al. (2005), the number of companies is defined by N, A_i is the value of the ith (for $1 \leq i \leq N$) company at a given time t, $0 \leq t \leq T$

[12]See Deacon (2003) and Caselli & Gatti (2005). For empirical of term structures, see Fons (1994) and Helwege & Turner (1997).

with T, the maturity. For simplicity of notations, we omit the time identifier of t in the following equations. The asset value still follows the Geomitric Brownian Motion process:

$$dA_i = \mu_i A_i dt + \sigma_i A_i dX_i$$

where X_i follows a Brownian motion, and μ_i is the drift parameter of expected growth rate of asset value of the ith company and σ_i is the volatility.

If we define the default barrier for this company by C_i, then the default is triggered when the asset value falls below the barrier, i.e, company i defaults if $A_i < C_i$.

Solving the above equation for A_i by applying Ito formula, we obtain:

$$A_i = A(0) \cdot \exp[(\mu_i - \frac{\sigma_i^2}{2})t + \sigma_i X_i] \tag{1.8}$$

Since the asset value A_i is a log-normal process, it is easy to get:

$$X_i = \frac{\ln(\frac{A_i}{A(0)}) - (\mu_i - \frac{\sigma_i^2}{2}) \cdot t}{\sigma_i} \tag{1.9}$$

where $A(0)$ is the initial value of the process.

Further, if we substitute the default threshold C_i into the asset value process, the equation we get is:

$$dC_i = \mu_i C_i dt + \sigma_i C_i dX_i^*$$

solving the equation for X_i^* we get:

$$X_i^* = \frac{\ln(\frac{C_i}{C(0)}) - (\mu_i - \frac{\sigma_i^2}{2}) \cdot t}{\sigma_i} \tag{1.10}$$

Here $C(0) = A(0)$ as both of them refer to the initial asset value, and the probability of company i to default at maturity T is:

$$P[A_i(T) < D] \quad = \quad P[A(0) \cdot \exp[(\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i(X_i(T) - X_i(0))] < C_i]$$

$$
\begin{aligned}
&= P(X_i(T) - X_i(0) \leq X_i^*) \\
&= P(X_i(T) - X_i(0) \leq \frac{\ln(\frac{C_i}{A(0)}) - (\mu_i - \frac{\sigma_i^2}{2})T}{\sigma_i}) \\
&= \Phi\left(\frac{\ln(\frac{C_i}{A(0)}) - (\mu_i - \frac{\sigma_i^2}{2})T}{\sigma_i\sqrt{T}}\right) \qquad (1.11)
\end{aligned}
$$

where the function $\Phi(\cdot)$ is the cumulative standard normal distribution function.

To model the default correlation, Hull et al. (2005) assumed that each Brownian motion process X_i follows a two-component process which includes a common Wiener process M for all N companies and a idiosyncratic Wiener process Z_i for each of the N companies:

$$
dX_i = a_i dM + \sqrt{1 - a_i^2} dZ_i, \qquad (1.12)
$$

where a_i is used to define the weight of the two i.i.d components. And the default correlation between two companies i and j is given by: $a_i \cdot a_j$.

Early Extensions to the Black-Scholes-Merton (BSM) model include the Black & Cox (1976)version which views default as a knockout option when the asset value falls down to the barrier and the Geske compound option model (see Geske & Johnson (1984)) which considers defaults on a series of contingent events.

Some of the recent structural models have addressed many of the limitations and assumptions of the original BSM model: Longstaff & Schwartz (1995) examined the assumption of stochastic interest rates. Forward price based firm value models were introduced in Briys & de Varenne (1997) and the study of incorporating jumps into the Longstaff & Schwartz (1995) model was carried out by Zhou (2001).

The typical criticism of structural models is that they are difficult to calibrate and are very computationally intensive. When it comes to a portfolio or basket of credit securities, slight change of setting could result in a great amount of required simultaneous valuation work.

Compare to the market standard model, the *One Factor Gaussian Model* first proposed in Li (2000), which is considered a benchmark of other models, the structural model clearly has a better definition for the relationships of variables in an economic sense, and more importantly, it is a dynamic model. However, the major disadvantage with the structural models is, that in each case the asset value process is a continuous process which means that for short time intervals as $[0, t] \rightarrow 0$, the probability of default also moves close to zero, so sudden exposure to credit events is unrealistically missed. And for this reason, models with jump diffusion[13] asset value process were introduced as in this type of models there is always the possibility that the asset value may drop below the default barrier at any time.

Reduced Form Models

Reduced form models directly model the default probability using Poisson type processes and generally do not link the default with a generating process. This approach was pioneered by many researchers including Jarrow & Turnbull (1995), Lando (1999) and Duffie & Singleton (1999). The main idea of this approach is that the risky bond value is the expected sum of value at default and the value if there is no default.

A standard Poisson process is a right continuous integer valued stochastic counting process N_t, for $t \geq 0$ with independent increments: $N_t - N_s, t \geq s$. The

[13] See Zhou (1997) and Zhou (2001).

distribution parameter of a Poisson distribution is $\lambda(t-s)$ where λ is used to describe the intensity of the process. The occurrence of default is set to be the first jump of the process; whenever $N_t - N_s > 0$ the modeled firm defaults. It is easy to see that the default times are totally unpredictable stopping times in reduced form models.

The survival probability which indicates the probability of no jump in the Poisson process during time interval $[s,t],$ for $s \leq t$, is given by:

$$P[(N_t - N_s) = 0] = e^{-\int_s^t \lambda(u)\, du}$$

As the parameter $\lambda(t)$ is a time dependent parameter, the process is a time inhomogeneous Poisson process. In order to simulate its first jump time, for a standard uniform random variable, $V \in [0,1]$, we have:

$$\tau = inf\{t : e^{-\int_0^t \lambda(u)\, du} \leq V\}$$

Moreover, if the intensity parameter λ is also defined by another stochastic process (which refers to the hazard process), then λ becomes a doubly stochastic process depending on time and adapted to a certain filtration $\mathcal{F}_t$ under a probability measure. This is the so called Cox process, and the default time (first jump time of the process) is now:

$$\tau = inf\{t : e^{-\int_0^t \lambda(u)\, du} \leq V|\mathcal{F}_t\}$$

Thus the survival probability is:

$$P[N_t - N_s = 0|\mathcal{F}_t] = e^{-\int_s^t \lambda(u)\, du},$$

which means the probability that the survival time of an underlying name until time t is: $P[\tau > t|\mathcal{F}_t] = E[e^{-\int_s^t \lambda(u)\, du}]$. This result adapts to both interest and credit spread models.

As we did mention in Section 1.2.2, the price of a defaultable bond is:

$$B' = e^{-rT} - e^{-rT}(1-\hat{R})(1-e^{-aT}) = e^{-(r+a)T} + e^{-rT}\hat{R}(1-e^{-aT})$$

If we now assume that both the rates r and a depend on time t, then the risky bond price can be expressed as:

$$B' = E[e^{-\int_0^T r(u)+a(u)du} + e^{-\int_0^T r(u)\,du} \cdot \int_0^T \hat{R}a(t)e^{-\int_0^t a(u)\,du}\,dt],$$

where $r(t)$ is the compounded spot interest rate process, $a(t)$ is the intensity and $\hat{R}$ is the recovery rate. From previous discussion, the default probability density of default time τ is given by:

$$a(t)e^{-\int_0^t a(u)\,du},$$

and the default time distribution is:

$$1 - e^{-\int_0^T a(u)\,du} \text{ for } 0 \leq \tau \leq T$$

Hence the probability of no default until time T is $P[\tau \geq T] = E[e^{-\int_0^t a(u)\,du}]$ and the probability of default during the time interval $[t, t+dt]$ is: $P[t \leq \tau \leq t+dt] = E[a(t)e^{-\int_0^t a(u)\,du}dt]$.

Reduced form modelling can be traced back to Jarrow & Turnbull (1995) in which they used a constant intensity and fixed recovery rate at maturity. With this assumption, the modeler does not need to consider any dependency between the bond price and the conditional default probability and enjoys the advantage of a closed-form solution for the bond price. Yet this assumption is too far from reality[14] as: first, the recovery takes place soon after default and second, the recovery rates randomly change over time. This is what Duffie & Singleton (1999) tried to

[14]Detailed spread analysis can be found in Jarrow, Lando & Turnbull (1997) and Chen & Huang (2001).

address. They assumed that the recovery could appear at any time[15] during life of the bond but the recovered amount at default is only a proportion of the non-defaultable bond value at the same time.

One of the drawbacks is that the recovery rate of a given default must be specified exogenously, and this is the reason that most reduced form models mainly distinguish from each other by different definitions of recovery rate and intensity of the stochastic process. For instance, a significant difference lies between the Duffie Singleton (DS) and the Jarrow Turnbull (JT) models, in the JT model, the recovery assumption is separate from the default probability, but in the DS model, the recovery and the default probability together form an instantaneous spread.

Stochastic Loss Distribution Approach

The Stochastic Loss Distribution Approach was originated by Heath, Jarrow & Turnbull (1992), Jarrow & Turnbull (1995) and Jarrow et al. (1997). Distinguishing this type of models as an individual category is controversial[16] because these models focus on the probability of the portfolio losses to take place or reach some level in the future. Thus this approach is also referred to as the "top down" approach.

Recent research including extension of Heath et al. (1992) with a loss deduction assumption is provided by Sidenius, Piterbarg & Andersen (2004). Bennani (2006) assumed that the instantaneous loss is a percentage of the remaining principal. Errais, Gieseke & Goldberg (2007) suggest a model of default probability with jumps while Longstaff & Rajan (2006) suggested that it is the loss that fol-

[15]Although when it comes to implementations, both default and recovery are often assumed to occur at coupon times only.

[16]See Anson, Fabozzi, Choudhry & Chen (2004) and Choudhry (2005).

lows a jump process and different types of jumps are tested. Schonbucher (2005) and Walker (2007) considered the evolution of loss distribution in a Markov loss model.

1.3.2 Static Copula Models

The current market model, also called the one-factor-Gaussian copula model. Was approach is originally introduced by Vasicek (1986), Li (2000) and recently developed by Gregory & Laurent (2005). The working assumption is that the default probability over the whole life of the contract is determined by the normally distributed asset value, in which case, if the asset value is high then the probability of default is low, and vice versa. *Default* is defined as the first time the asset value crosses a predefined value barrier[17].

The model assumes a constant hazard rate and ignores the change of probability of default (PD) over the whole time period. Considering only the loss distribution, many alternative copula and distribution functions to Gaussian copula have been suggested[18], including: the student t-copula, the double t-copula, the Archimedian copula, the Clayton copula, the Marshall Olkin copula, and distributions like Normal Inverse Gaussian and Variance Gamma.

The main takeaway from the model is the correlation does not change the expected loss in the portfolio - it effectively only changes the shape of the loss distribution. When correlation changes, it does not affect the asset but only the liabilities. Since the assets are the same, the liabilities in total are unchanged as well. This means

[17]Normally the debt value of the company, or the face value in case of a bond.

[18]See Hull & White (2004), Burtschell, Gregory & Laurent (2005), Guegan & Houdain (2005) and Kalemanova, Schmid & Werner (2005) for detailed discussions on copula functions.

that when correlation changes, it essentially moves the value between the tranches - some tranches lose value some tranches gain value. However, the total value is unchanged - because the assets have not changed.

This type of model is, in general, static as it is not able to describe how the default environment evolves. But the following two points are worth considering when building asset value based or barrier credit default models:

1. The probability of default (PD) , which is conditioned on a market momentum of Y, is related to the default barrier K with a normal distribution function, i.e.:
$$p(Y) = \Phi\left(\frac{K - \rho \cdot Y}{\sqrt{1-\rho^2}}\right) \tag{1.13}$$
2. The relationship between the default probability and the asset value is expressed as:
$$A = \Phi^{-1}(1-p) \tag{1.14}$$

We will move on from here to dynamic models and further discussion will be given in the next chapter.

1.3.3 Duffie-Singleton Discount Rate Approach

Duffie & Singleton (1999) proposed a Reduced Form model characterizing the default exogenously by a jump process. The event of default is led by a hazard rate and the losses at default are parameterized as a fractional reduction in pre-default market value.

Suppose we have a corporate bond paying X at maturity time T, denote the hazard rate at time $0 \leq t \leq T$ by h_t and the expected fractional loss at time t by L_t. Under a risk-neutral environment, $h_t L_t$ stands for the 'mean-loss rate', thus if the risk free interest rate r is replaced by an adjusted short rate R, where $R = r + h_t L_t$, then the market value of this bond at time zero is given by:

$$V_0 = E_0^Q \left[e^{-\int_0^T R_t dt} X \right], \tag{1.15}$$

where Q is the risk-neutral martingale measure.

As the mean-loss rate $h_t L_t$ does not depend on the bond value, if R is chosen carefully, standard term-structure default-free debt models are directly applicable to defaultable debts by replacing the risk-free rate r by R.

Under this set up, denote the unit recovery at time $t+1$ by φ_{t+1}. It is natural that the contract value at time t consists of two parts: $h_t \cdot e^{-rt} \cdot E_t^Q(\varphi_{t+1})$ for the event of default and $(1-h_t) \cdot e^{-rt} \cdot E_t^Q(V_{t+1})$ as the bond value continuous to evolve to its value at time $t+1$ in case of no default. Mathematically, the bond value at time t is expressed as:

$$V_t = h_t e^{-rt} E_t^Q(\varphi_{t+1}) + (1-h_t) e^{-rt} E_t^Q(V_{t+1}) \tag{1.16}$$

Meanwhile, the unit recovery of market value at time $t+1$ is the difference between real market value at time $t+1$ and the expected remaining value at time t, i.e.

$$E_t^Q(\varphi_{t+1}) = (1-L_t) E_t^Q(V_{t+1}) \tag{1.17}$$

Substituting equation (1.17) into equation (1.16), we can rewrite equation (1.16) as:

$$V_t = (1-h_t L_t) e^{-rt} E_t^Q(V_{t+1}) \tag{1.18}$$

In the default adjusted structure, the adjusted discount factor e^{-Rt} at time t is given by:

$$e^{-Rt} = (1 - h_t L_t) e^{-rt} \tag{1.19}$$

It is known that for a small number ϵ, an estimation of e^{ϵ} is $1 + \epsilon$, similarly if the contract time is observed in small length, we can approximately have: $R_t \cong r_t + h_t L_t$. Now if one recursively solve equation (1.18) for the whole time interval, it is easy to show that:

$$V_t = E_t^Q (e^{-\sum_{i=t}^{T-1} R_i} X) \tag{1.20}$$

And thus the financial contract is priced.

The authors derived the fair prices of securities which are embedded with default risk. This approach provides a default-able version of the Heath et al. (1992) model, and the authors concluded that this default adjusted approach is not suitable when dealing with non-callable bonds, because h_t and L_t must work together as the 'mean-loss rate' and cannot be identified separately from data of default-able bond prices alone.

We will follow the Duffie & Singleton (1999) approach and take their work as a pre-cursor of our model as a main advantage is that it is possible to directly calibrate model variables to market observable instrument prices such as corporate bonds. Further, it is possible to parameterize R directly as it is exogenous. However, the downside of this type of model is, as the loss is priced with a default adjusted 'risk free' rate, that the final value is expressed as an exponential function and thus the model is not suitable for pricing financial contracts like CDS which have no payoff at maturity.

In the next chapter, we will model the evolving environment of the underlying asset value using a default adjusted yield, in the model, default probability is considered against the risk-free appreciation of asset value along time to maturity, after presenting the close-form formula of default probability, we will move on to find the growth rate of asset value as well as the expected loss according to the default adjusted yield.

1.3.4 Market Model

This approach was originally introduced in Li (2000). The general copula methodology uses a copula function to specify the joint distribution of survival times. Copulas are handy because for any multivariate distribution, the univariate margins are defined independently.

Denote τ_i as the default time for the ith company, and $F_i(\tau_i)$ as the probability function of default at time τ_i for the same company. The joint distribution of default times is given by:

$$p(\tau_1, \tau_2, \ldots, \tau_n) = C(F_1(\tau_1), F_2(\tau_2), \ldots, F_n(\tau_n)),$$

where $C(\cdot)$ is the copula function. (See Li (2000), Schlogl (2003) and Luescher (2005) for details on copula and implementations.)

The copula approach does restrict the possible types of copula functions, so there are many versions[19] of it which involve various types of copula functions. But copula approaches on bulk price the CDOs with the generated portfolio loss distribution.

[19] See Kalemanova et al. (2005) and Luescher (2005) for implementations with normal inverse copulas, and Melchiori (2003) for further discussion on copula functions.

Suppose there are n companies in the portfolio, each has its asset value process with two controlling variables, M and X_i, the market momentum and a stochastic factor, respectively. Further, it is assumed that the individual default probability for each company is exponentially distributed over time, so we have:

$$p_i = e^{-h_i t}$$

The probability of not defaulting until time t in the equation above is: $1 - p_i$, for $1 \leq i \leq n$. Then the asset value of each firm in the portfolio is:

$$A_i = \rho_i \cdot M + \sqrt{1 - \rho_i^2} \cdot X_i \tag{1.21}$$

A closed-form solution of the default probability conditional on a value Y of the market momentum can be given by:

$$\begin{aligned} p_i(Y) &= P[A_i < C_i | M = Y] \\ &= P[\rho_i \cdot M + \sqrt{1 - \rho_i^2} \cdot X_i < C_i | M = Y] \\ &= P[X_i < \frac{C_i - \rho_i M}{\sqrt{1 - \rho_i^2}} | M = Y] \\ &= \Phi \left(\frac{C_i - \rho_i Y}{\sqrt{1 - \rho_i^2}} \right) \end{aligned} \tag{1.22}$$

The default in this model is defined to be the first time an asset value falls through a default threshold, denoted C_i for the ith company, and this threshold is related to the probability of default (PD) with a normal distribution function, i.e. $C_i = \Phi_N^{-1}(p_i)$.

1.4 Introduction to Hedging Sensitivities

Investing in a simple credit product such as the credit default swap or the more complex structured portfolio products is like providing insurance on the financial activities of the referenced counter-party. It is normal practice that single name Credit Default Swaps (CDSs) are hedged using underlying bonds, but things become more complicated when dealing with multi-underlying name contracts.

Before the market squeeze in early 2007, most players in the credit market took the senior or super senior CDO tranches as a save investment that generates high return with low risk. However everyone finds it unsecured in current market as investors are losing money from these ‘safe’ investments. Practitioners find it hard to capture market movements and effectively carry out an optimal hedging strategy. Part of the reason is that the market copula model relies on a certain type of distribution and is not able to dynamically model the ‘surprises’ in the portfolio. Another reason is that it is hard to make sure that the credit models are dynamically congruent with the pricing models of the hedging instruments.

In this section we review dynamic strategies that hedge the embedded credit risk within a portfolio of credit products; in other words, our aim is to find offsetting cash-flows from related derivatives that covers the payouts of protection on the underlying reference entities. We first introduce several sensitivity measures on the credit instruments.

1.4.1 DV01

DV01 is also known as the *Dollar Value per basis point*, and is defined as the value change of a credit derivative when there is a shift with one basis point in the credit curve. In principal, DV01 is the same as the *Present Value per basis point* (PV01) for a bond, where PV01 describes the bond value change against 1bps change in the interest rate, DV01 is the value change of the credit derivative when the credit spread moves 1bps. DV01 for a default swap quantifies the credit risk of an investor in the CDS market, similarly, the DV01 for a default swaption or risky bond measures the security value change according to single basis point shift.

To actually carry out the calculation of CDS DV01, one needs the following market observables as inputs: the Recovery Rate R, market CDS spread s, contract term T, notional N and the risk free interest rate r.

Assuming both the spread and interest rate curve is reasonably flat for a default swap. The value of the swap is the difference of its two payment legs: premium and protection. Suppose default only happens at maturity and no recovery, we have value of the two legs as:

$$\text{premium} = e^{-(r+h)T} N$$

$$\text{protection} = N$$

For unit value DV01 calculation, N is set to be 1, and value of the swap is thus: $1 - e^{-(r+h)T}$. By linear interpolation on spread curve interval $[0, r+h]$, the DV01 of a default swap under a static model[20] is given by the equation below:

$$\text{DV01} = \frac{1 - e^{-(r+h)T}}{r+h} \tag{1.23}$$

In practice, the hazard rate h is replaced by using an approximation which is given

[20]Here 'static' comments only on the related hazard rate used in the equation, dynamic practice will be illustrated in later section of this work.

by:

$$h = \frac{\text{spread} \times 365}{(1-R) \times 360}$$

Similar results can be found in Chaplin (2005).

Also, it is important to mention that in case of a credit swaption, the payoffs depend on the DV01 of the referenced CDS as the option strike is in terms of spread rather than price. So the swaption DV01 is converted into dollar value from spread term upon expiry of option. Details and examples will be discussed in next section.

DV01 - Example

Suppose we have a CDS with contracted maturity of 3 years, the market spread on the observed day is 1500 bps, with interest rate at 5% and recovery rate 40%.

Applying equation (1.23) we have the DV01 for our CDS is £1.97, which means that a 1 bps shift will cause a money change of 1.97 in the present value of the contract. Similarly, DV01 for a CDS index or basket of CDS contracts can be calculated using a unified hazard rate and the index average spread.

1.4.2 Credit Delta

Compare to DV01, the delta factor is more like a value change-per-default measurement for a portfolio of credit embedded contracts. While DV01 measure the value change according to 1bps move of spread curve, credit delta of portfolio credit derivatives is normally marked to actual portfolio loss when a credit event is triggered for one of the contracts in the portfolio.

The principle is to examine the value change when a default takes place at an observed time, thus calculation of the delta closely depends on the pricing techniques chosen for the product valuation. In all cases, the delta is simply the difference between the two values of the product: with or with out the default.

Delta - Example

Assume we have a 5 year CDS index traded at the average spread of 220 bps, and the present value of index is £4400. Meanwhile, the present value of the same index with an extra simulated default is £4000. We can then calculate the delta factor of the CDS index as 400.

Note that the value of a swap-like contract with two cash-flow legs, is given by the difference between the two cash flows. For instance, a CDS contract value is obtained by subtracting the protection leg value from the value of premium leg[21].

1.4.3 Credit Gamma

Same as the Gamma factor of options and other financial derivatives, credit Gamma describes the pace of change in credit Delta, thus it gives detail of changes in Delta sensitivity from a more 'micro' scope.

Consider a binomial tree-type model for the simplest illustration of the idea. As shown in Figure (1.3), to examine $\Gamma(1, t+1)$, the change of delta for time $t+1$, both $\Delta(1, t+2)$ and $\Delta(2, t+2)$ is needed, thus we need to obtain and perform a

[21] Here we assume that all the concerned contracts in this work are 'old school vanilla'; For more details on the valuation of CDS and forward CDS contracts, see Jabbour, El-Masri & Young (2008).

backward observation from the $t+2$ of the considered step t. S here in the figure stands for the credit spread.

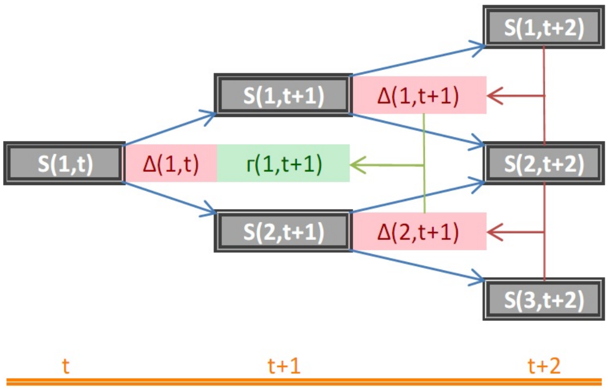

Figure 1.3: *Credit Gamma Calculation for Binomial Tree Model.*

Both concepts of Delta and Gamma are born naturally within every traded financial contract, in this thesis we consider both factors for the primary CDO against the hedging instrument. In our Monte Carlo simulation framework, the portfolio loss and default of one name in the CDO asset pool is handled together through all the time steps till maturity. This way, the time dependent movements of CDO spread, is illustrated according to time step and simulated defaults.

1.5 Summary

In this chapter, we briefly reviewed the most well known members of the credit derivative family, the mainstream models of single/multi underlying credit risky contracts and the related Greek factors. The first objective of this thesis is to develop a dynamic structural model for credit risk embedded financial instruments, especially structured portfolios such as CDO-type contracts, then, we will demonstrate calculations to obtain hedging factors based on the dynamic model.

For the rest of this thesis, we will define a Multi-step Monte Carlo simulation framework which handles the structural model in **CHAPTER** 2, the framework is designed to overcome static nature and default time conflict[22] of copula models. Then we examine the market data collected from before and after the current credit crisis with Principal Component Analysis in **CHAPTER** 3, as a result we find the current market model is under performing in intense markets. Later in **CHAPTER** 4 we introduce the Dynamic Growth Rate Model, together with numerical results and possible extensions for further developments. We will also apply the simulation framework with the model and try to estimate the market CDO spread curve from our model. Finally in **CHAPTER** 5, we will illustrate simulation and calculation of credit sensitivities starting from raw market observables to the final results of Greek factors, thus, application various hedging strategies are made possible for structured portfolio of credit securities according to different underlying and hedging instruments.

[22]As the copula models focus on the occurrence of default time τ, the simulation techniques employed is not suitable to describe the evolve of default probability over life time of the contract.

Chapter 2

Multi-step Monte Carlo Framework

In this chapter we present the conditional independence framework for pricing structured synthetic CDO products. The advantage of such an approach is that it allows semi-analytical pricing methodologies which are considerably more accurate and faster than the traditional Monte Carlo method used for structural models. Our first focus is the one factor Gaussian copula model. This model is generally regarded as the market standard model for structured credit risk products. We try to extend this model with a correlated stochastic process to describe the change in the debt value over the life time of the contract. This type of extended model will be also able to describe the multi state case for the underlyings. Our main contributions are: First, we obtain a default threshold, from a model-free probability of default for each name in the portfolio. Second, we develop a pricing methodology for portfolio credit derivatives using a combination of the conditional independence approach with stochastic control variables and Monte Carlo simulation.

2.1 Notations

Unless specified, these notations refer to the rest of this thesis.

T : contract maturity.

$0 \leq t \leq T$: a general time before maturity.

N : number of referenced companies.

$0 \leq i \leq N$: i^{th} company in the portfolio.

$n(t)$: number of cumulative defaults at time t.

L_n^t : cumulative loss on the portfolio at time t.

r : risk-free interest rate.

R_i : default-adjusted asset growth rate for company i.

c_i : coupon rate of company i. Also defined as c for all underlying bonds in a homogeneous portfolio.

e^{Rt} : default-adjusted unit asset value at time t.

$p(t)_i$: default probability of company i at time t.

τ_i : time of default for company i.

s : premium spread.

$A(t)_i$: asset value of company i at time t.

K_i : default threshold for company i.

$x(t)_i$: stochastic growth rate of company i at time t.

2.2 Fair Spread for Tranched Portfolio Products

In the following, we consider a pure CDS index or synthetic CDOs with reference underlyings consisting of CDSs only.

The seller of a tranche of such a product receives periodic spread payments from the protection buyer on preset payment dates, and in the case that the loss exceeds the tranched notional, compensation payments are made to buyers from the protection sellers.

As described in Chapter 1, the cash flow is divided in two legs, same as in the pricing of credit default swaps: premium payment leg and the protection payment leg. We take a single tranche out of the whole as an example, under the assumption that the time set before maturity is divided into n payment dates with: $0 \leq t_0 < t_1 < \ldots < t_{n-1} < t_n = T$, where T is the maturity. Then the expected spread payments of the *premium leg* is given by:

$$\sum_{i=1}^{n} \Delta t_i \cdot \omega \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}}, \tag{2.1}$$

where ω is the *spread*[1] and $EL_{t_{i-1}}$ is the *Expected Loss* of this certain tranche at payment date t_{i-1}, and $B_{t_{i-1}}$ is the price of a share of the government bond with the same maturity.

On the other hand, the protection leg value is given by:

$$\sum_{i=1}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i} \tag{2.2}$$

Following the principle of no-arbitrage-pricing, the value of the two legs should be the same under the risk neutral assumption.

[1]The contracted rate of premium payment, in percentage form of the underlying notional.

Hence it is possible to solve for the *spread* value by equating the two cash-flows in equations (2.1) and (2.2):

$$\omega = \frac{\sum_{i=1}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i}}{\sum_{i=1}^{n} \Delta t_i \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}}} \tag{2.3}$$

The expected tranche loss at each time step for a given tranche is based on the discrete time loss distribution L_{t_i}. If we denote the notional of defaulted names by $N_{t_i}^d$ and the total notional at the same time by N_{t_i}, we can then define L_{t_i} as:

$$L_{t_i} = (1 - R)\frac{N_{t_i}^d}{N_{t_i}}$$

where R is the recovery rate which we can assume to be zero in the simplest case.

For the tranche starting with the attachment point K_a and detaching at point K_d, the expected tranche loss is calculated by:

$$\frac{1}{K_d - K_a} \sum_{j=1}^{m} (min(L_j, K_d) - K_a)^+ \cdot p_j,$$

where m is the number of underlying names and p_j indicates the default probability of the jth asset. It is not hard to rewrite the above equation in a continuous-time framework:

$$\frac{1}{K_d - K_a}\left(\int_{K_a}^{1} (x - K_1)\, dF(x) - \int_{K_d}^{1} (x - K_d)\, dF(x)\right) \tag{2.4}$$

Thus, the goal now is to find out the portfolio loss denoted by $F(x)$ in equation (2.4). (For a detailed exposition, see Kalemanova et al. (2005).)

Further, consider an explicit recovery rate R, which is set as a constant of 40% in many empirical applications[2]. The basic idea is as the loss of each default is now

[2]See Li (2000) and Wang, Rachev & Fabozzi (2006)

smaller, more defaults are needed to fulfill each tranche. But since copula functions give the loss using both the attachement/detachement levels and the default threshold, it is the same if we just enlarge the tranche points by $(1-R)$, so in case with a constant recovery, the 'new' tranche is now defined as $(\frac{K_a}{1-R}, \frac{K_d}{1-R})$.

2.3 The Gaussian Factor Copula Model

In the following sub-sections, we will provide the details of calibrating the above model within the standard market framework and we will also outline the mathematical method of its implementation.

2.3.1 Market Standard Model in Details

The appeal of the one factor Gaussian model is that it is simple to understand and easy to implement. Wang et al. (2006) discuss that the the various approximations of the market model are all about the correlation of defaults. In its "standard" version, the Factor Gaussian model is simplified by assuming the following:

- The recovery rate is fixed to 40%, i.e. $R = 40\%$.
- An average CDS spread[3] of names in the portfolio is used instead of single CDS spreads for each name. This means the default barrier is set to be the same for all companies, i.e. $C_i = C$.
- Same pairwise correlation for all the names in the portfolio, i.e., $\rho_{ij} = \rho$.

[3]Or credit spread/risky bond spread, which are all related to the default probability.

- Constant default intensity (hazard rate) over time, i.e., $h_i = h$.

Within a homogeneous portfolio under the above framework with a pool of n underlying names, the approximate asset value A is given by:

$$A_i = \rho \cdot M + \sqrt{1-\rho^2} \cdot X_i \tag{2.5}$$

Here M is the market common factor, X_i for all companies, $1 \leq i \leq n$, follows the standard normal Gaussian distribution and under the homogeneous portfolio assumption, we denote the uniform pairwise correlation between any two companies in the portfolio by ρ, i.e., $\rho_{ij} = \rho$ for $i, j = 1, ..., 125, i \neq j$.

The threshold C which indicates the default is implied by the market average default probability p, that is:

$$C = \Phi^{-1}(p) \tag{2.6}$$

Clearly, from equations (2.5) and (2.6), an underlying asset defaults if $A_i \leq C$. In turn, the default probability conditioned on the market factor $M = Y$ is:

$$p(Y) = \Phi\left(\frac{C - \rho \cdot Y}{\sqrt{1-\rho^2}}\right) \tag{2.7}$$

2.3.2 The large portfolio loss distribution approximation

Rearranging equation (2.7) we get:

$$Y = \frac{\Phi^{-1}(p(Y))\sqrt{1-\rho^2} - C}{\rho}$$

Then, the conditional probability of having exactly k defaults is given by the binomial distribution:

$$\binom{n}{k} p(Y)^k (1 - p(Y))^{n-k}$$

The unconditional probability of having k defaults is the integral over the market momentum:

$$p = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(\frac{C-\rho\cdot Y}{\sqrt{1-\rho^2}}\right)^k \left(1-\Phi(\frac{C-\rho\cdot Y}{\sqrt{1-\rho^2}})\right)^{n-k} d\Phi(Y) \tag{2.8}$$

Substituting Y in the integrator function we get:

$$p = \binom{n}{k} \int_{-\infty}^{\infty} \Phi\left(\frac{C-\rho\cdot Y}{\sqrt{1-\rho^2}}\right)^k \left(1-\Phi\left(\frac{C-\rho\cdot Y}{\sqrt{1-\rho^2}}\right)\right)^{n-k} \cdot d\Phi\left(\frac{\Phi^{-1}(p(Y))\sqrt{1-\rho^2}-C}{\rho}\right)$$

Further, according to the large portfolio limit approximation results given by Vasicek (1986) and Vasicek (1987), with the number of underling assets in the portfolio infinitely large (i.e., $n \to \infty$), the loss distribution in the above equation becomes:

$$F_n(x) = \lim_{n\to\infty} \sum_{k=0}^{[nx]} \int_{-\infty}^{\infty} \binom{n}{k} p(Y)^k (1-p(Y))^{n-k}\, d\Phi\left(\frac{\Phi^{-1}(p(Y))\sqrt{1-\rho^2}-C}{\rho}\right)$$

where

$$p(Y) = \Phi\left(\frac{C-\rho\cdot Y}{\sqrt{1-\rho^2}}\right)$$

Since

$$\lim_{n\to\infty} \sum_{k=0}^{[nx]} \binom{n}{k} p(Y)^k (1-p(Y))^{n-k} = \begin{cases} 0, & \text{if } x \le p(Y); \\ 1, & \text{if } x > p(Y). \end{cases}$$

we can easily calculate the cumulative portfolio loss distribution as:

$$F_\infty(x) = \Phi\left(\frac{\sqrt{1-\rho^2}\Phi^{-1}(x)-C}{\rho}\right) \tag{2.9}$$

Then the expected loss can be obtained by substituting the above equation into equation (2.4) or by applying the bivariate distribution copula function. The expected tranche loss is given analytically by:

$$EL = \frac{\Phi_B(-\Phi^{-1}(K_a), C, -\sqrt{1-\rho^2}) - \Phi_B(-\Phi^{-1}(K_d), C, -\sqrt{1-\rho^2})}{K_d - K_a} \tag{2.10}$$

where $\Phi_B(\cdot)$ stands for the binomial normal distribution function[4].

2.3.3 Model Implementations

Given the default probability in equation (2.7), the expected loss of the portfolio can be calculated either analytically with the large homogeneous portfolio approximation as shown above or via Monte Carlo simulation methods. We illustrate the latter in this section.

Hazard Rate Approach

According tot the standard assumptions listed above, the default times conditional on a Gaussian distributed market momentum Y for a certain default probability are normally distributed. It follows that the default probability can be calculated using the survival function: $S_i(t) = 1 - p_i = 1 - e^{-h_i t}$. We also know that the ith company in the referenced portfolio will not default until time τ_i for a given default probability $p_i = e^{-h_i \tau_i}$.

Hence, we have:

$$\Phi(A_i) = 1 - e^{-h_i \tau_i}$$

and after rearranging, the default time τ_i is given by:

$$\tau_i = -\frac{\ln(1 - \Phi(A_i))}{h_i} \tag{2.11}$$

[4] For more details, see Vasicek (1986) and Vasicek (1987).

For a five-year contract, the hazard rate h_i is calculated from the cumulative default probability for the next five years where this probability can be retrieved from the credit curve. Hence $h_i = -\ln p_i/5$.

For a portfolio with N referenced underlying companies, the simulation process can be described in the following steps:

1. Generate a normally distributed market momentum factor M for the whole portfolio, and a normally distributed[5] random factor X_i for each i of the N companies.
2. Generate the asset value for each of the underlying companies in the portfolio using equation (2.5).
3. Calculate the default times τ_i for each of the companies in the portfolio using equation (2.11).
4. Compare payment date and the default time τ_i for each company. If τ_i is smaller than any payment date $0 \leq j \leq T$, then the company is considered to have defaulted by time j.
5. Calculate the loss and the two cash flow legs for this trial at each time using equations (2.1) and (2.2).

- Steps 1-5 complete a single trial.

6. Repeat steps $1 \sim 5$ for a large number of times and take average value of the two cash flow legs.

[5]We adopt the normal distribution setting of both M and X because we intend to test the market standard model with our multi-step Monte Carlo framework; therefore, keeping the original setting, we focus on the simulation rather than a different modelling approach.

7. Finally, the air tranche spread for this structured product is calculated using equation (2.3).

To compare with the market value, one may consider the common case of making an upfront payment for the first tranche[6] and take into account that the yearly spread is set normally to 500bps for the equity tranche.

Cumulative Tranche Loss Approach

In this section we introduce the simulation method proposed by Loffler & Posch (2007) which is relatively straight forward and fast in computation time.

Suppose we have all the information given in the previous section, i.e., size of portfolio, the average single name default probability and a universal correlation, recovery rate and notional.

Default takes place if the asset value in equation (2.5) is smaller than or equal to the default barrier given by equation (2.6). Note that if the default probability is used to describe the probability to default for the period of whole contract life, the simulated loss in each trial is just the loss given default simulated from a single big time step. We shall present below the simulation details of a single step, by considering for simplicity just default at maturity.

1. Generate a normally distributed market momentum factor M for the whole portfolio, and a normally distributed random factor X_i for each i of the N companies.

2. Generate an asset value for each of the underlying companies in the portfo-

[6] This is done in order to comply with liquidity requirements.

lio using equation (2.5).

3. Calculate the default barrier using equation (2.6).

4. Compare the results for the asset value and the barrier. If a company defaults, calculate its loss at default using $(1 - R)$. Then calculate the total loss for this trial.

5. For each tranche, if the total loss exceeds its attachment point, the default probability of this tranche is increased by $\frac{1}{\text{number of trials}}$. On the other hand, the expected loss for this tranche is increased by $\frac{\text{percentage of loss on tranche}}{\text{number of trials}}$.

- Steps 1-5 construct a single trial.

6. Repeat steps $1 \sim 5$ for a large number of times and with the tranche default probability and expected loss, the tranche spread can be calculated as in case of a single name CDS.

So far we have introduced two Monte Carlo based approaches to get the structured portfolio tranche spread. The reason that simulation methods in general are more favored is because the large homogeneous portfolio assumption is based on the condition that the number of underlying assets goes to infinity. Whilst 125 underlying names of iTraxx Europe or CDX.NA.IG is large enough, most individually traded sub indices contain only about 20 names. Researchers are still debating about the accuracy of such pricing procedures for these subindices[7].

[7] See the extension in Greenberg, O'Kane & Schlogl (2004), Moosbrucker (2006), and for a discussion of limitations of large homogeneous portfolio assumption (LHP) see Schlogl (2004).

2.4 The Multi-step Monte Carlo Simulation Framework

In this section we demonstrate the implementation of Monte Carlo simulation within a discrete time step framework. The aim is to set up an all encompassing simulation environment for many existing models with assumptions based on market observables. Moreover, multi step implementations are straightforward to carry out and modify, thus important modifications such as *rating change*[8], *random recovery rate* and *random short term interest rate* can be accommodated and the various related problems could be fruitfully addressed.

2.4.1 PD, CPD and MPD

Recall that in the definition of a single name credit default swap (CDS), for a five-year contract, the probability of default (PD) over the whole five years is called the cumulative probability of default (CPD). Also note that with the help of a credit curve, we can calculate the default probability and future default probability for each year or each payment date over the whole period. This default probability is given at each time step conditioned on survival of previous time steps and is called the marginal default probability (MPD)[9].

In the rest of this section, we use PD to denote the probability of default seen today. The relationship between the variables is shown in Figure (2.1) below:

[8] Also called the *multi-state* problem which refers to change of credit rating during life time of the contract.

[9] For details please refer to Anson et al. (2004), Bluhm & Overbeck (2007) and Loffler & Posch (2007)

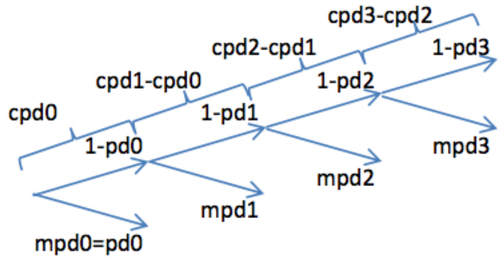

Figure 2.1: *PD seen from today, MPD and CPD*

For the case of a single time step, we have PD $=$ MPD $=$ CPD. In a two-step case, $\text{PD}_{(1,2)}$ is the PD seen at time 0 for period $(1, 2)$, and is equal to:

$$\text{PD}_{(1,2)} = \text{CPD}_{(0,2)} - \text{CPD}_{(0,1)} \tag{2.12}$$

$$\text{PD}_{(1,2)} = \text{MPD}_{(1,2)} \cdot (1 - \text{PD}_{(0,1)}) \tag{2.13}$$

Hence:

$$\text{CPD}_{(0,2)} = \text{CPD}_{(0,1)} + \text{MPD}_{(1,2)} \cdot (1 - \text{CPD}_{(0,1)}) \tag{2.14}$$

taking into account that $\text{CPD}_{(0,1)} = \text{PD}_{(0,1)}$.

2.4.2 Discrete Time Set-up

Using of the Law of Large Numbers[10], the probability of default can be written as:

$$p = \Lambda\left(\frac{\text{barrier} - \text{mean}}{\text{standard deviation}}\right) \tag{2.15}$$

where Λ is an arbitrary distribution function, *barrier* is the pre-set asset value of default, the expressions above refers to the *mean* and *standard deviation* of the asset value.

We will apply first the barrier within the market model together with the widely used structural models; we repeat the process for the barrier based model in Section 2.4.3.

Suppose we have a 3-month PD backed out from the credit curve and it is 0.02%. We assume that for every two steps, $\text{PD}_{(t_0,t_1)} = \text{PD}_{(t_1,t_2)}$. Using equations (2.12), (2.13) and (2.14), we can calculate the marginal default probability of the next 3 months as:

$$\text{MPD}_{(1,2)} = \frac{0.02\%}{1 - 0.02\%} = 0.020004\%$$
$$\text{MPD}_{(2,3)} = \frac{0.020004\%}{1 - 0.020004\%} = 0.020008\%$$

We can then obtain the MPD for the next 5 years with a 3 month step length. Figure (2.2) gives the 20 time step results for a 5-year period:

[10] The **Law of Large Numbers** states that:

"Given a sample of independent and identically distributed random variables with a finite population mean and variance, the average of these observations will eventually approach and stay close to the population mean."

Therefore, the probability of occurrence of a trial outcome is fixed according to the sample distribution. (See Jaynes (1996)) In our implementation, if the barrier value occurs with probability q, it means that the default probability for this company is q.

0.0002		0.0002004008016	
0.000200040008	0.0002002002002	0.0002004409701	0.0002006018054
0.000200080032	0.0002002402883	0.0002004811548	0.0002006420546
0.000200120072	0.0002002803925	0.0002005213555	0.0002006823199
0.0002001601281	0.0002003205128	0.0002005615724	0.0002007226014
	0.0002003606492		0.000200762899

Figure 2.2: *Step MPDs for a 5-year period.*

By omitting the difference in seven digits after zero, we can see that the discrete probabilities of default for each time step roughly remain the same over the whole period.

On the other hand, consider a single step CDS contract with unit value notional. The relationship between the probability of default and the CDS spread (fair price of CDS) is $spread = (1 - R) \cdot$ PD. Although this may be viewed as a very restrictive relationship between the two, we will relax it later.

Now, if we have a market-quoted 5-year CDS spread of 24 bps and recovery rate of 40%, this simply means that the PD for this underlying company is 0.4%. If we further consider that this CDS consists of twenty payment days, the marginal default probability can be considered as constant over time almost surely.

2.4.3 Monte Carlo Simulation

We have now obtained all the building blocks for our model implementation. We illustrate next the procedure of finding out the default barrier for each of the time steps using the marginal default probability. We observe the scenarios in time order, as the marginal default probabilities are conditioned on their own predecessors, i.e, tomorrow's PD is today's marginal PD.

The simulation process can be performed in the following steps:

1. Generate a normally distributed market momentum factor M_t for the whole portfolio at time t, for $0 \leq t \leq T$, and a normally distributed random factor $X_{i,t}$ for each ith of the N companies at the same time step.

2. Generate asset values for each of the underlying companies in the portfolio using equation (2.5). Repeat for all companies over the whole time period $[0, T]$.

3. Calculate the default barrier using equation (2.6) for each time step.

4. Compare the results for the asset value and the barrier. If a company defaults at a time step, it is also knocked out for the rest of the contract life. We can then calculate the cumulative loss at default using $(1 - R)$ for the whole portfolio over all time steps.

- Steps 1-4 carry out the simulation of the cumulative loss.

5. For each tranche, the expected loss at each time is given by the comparison between tranche size and the tranched loss. We can use equation (2.1) and equation (2.2) to get the value of premium and protection respectively.

6. Repeat steps $1 \sim 5$ for a large number of times and average the two cash flow legs with the number of simulation trials. Then, the final fair tranche spread can be calculated using equation (2.3).

The above procedure portrays the simulation process of a single factor Gaussian model; in fact the simulation process adapts many models and assumptions, for example the Merton (1974) structural model or the Hull and White structural model introduced in Hull et al. (2005). Meanwhile, in case the simulation process is preformed using Microsoft Excel and VBA, as in each trial we have a table of the number of underlying companies in rows and the number of time steps in columns, if assumptions such as fixed recovery rate need to be relaxed, we can accordingly simulate the change in recovery rates and 'input' the discrete rate into each cell of the table. Furthermore a random interest rate test can be preformed in the same way, but for the multi-state problems due to sharp change in credit ratings (which can be shown as change in CDS spreads), we need to have the default barrier modified to catch the rating change.

2.5 Numerical Results

In order to illustrate our approach, we use the most actively traded iTraxx Europe tranches on a five-year maturity basis. The number of underlying names is 125, the tranches are $0\% \sim 3\%$, $3\% \sim 6\%$, $6\% \sim 9\%$, $9\% \sim 12\%$ and $12\% \sim 22\%$. The payments are made quarterly and the recovery rate is set to 40%.

The market data is recorded on 12th April 2006 for iTraxx series four, the contract started at 20th March 2006 and ends on 20 June 2011. The average CDS spread quote on that day is 32 bps and the pairwise homogeneus correlation is 16% for all underlyings. The market spreads and implied correlations are all quoted from the *Reuters' CDS Views 3000*[11].

We aim to compare the market quotes with the results from the two versions of the market model as well as with the results from discrete Monte Carlo simulation of the Gaussian and the Structural model.

[11]Note that market quotes may be different if quoted from other contributers.

In each implementation, the annual interest rate is set to be 5%, the number of simulations is $100,000$, and the volatility in the structural model is 0.35[12]. The numerical results are presented in Table (2.1):

Table 2.1: **Numerical Results of Simulation Based approaches for iTraxx**

Tranche	**Market**	**Gaussian**	**Cum. Loss**	**Disc. Gaussian**	**Disc. Struc.**
0% - 3%	23,00%	23,92%	21,67%	23,29%	24,08%
3% - 6%	60 bps	156,72 bps	43,8 bps	41,1 bps	85,16 bps
6% - 9%	14 bps	32,07 bps	18 bps	9,2 bps	21,27 bps
9% - 12%	6 bps	7,2 bps	12 bps	2,1 bps	4,29 bps
12% - 22%	2.5 bps	1,25 bps	0.6 bps	0.8 bps	0.9 bps
Abs. error		117,24 bps	28,1 bps	29,3 bps	35,74 bps

We can see[13] from the Table (2.1) that the normal Gaussian hazard rate approach gives a considerably larger spread for the three tranches after equity, whilst the cumulative loss approach provides a better simulation overall, especially in the second tranche, albeit it produces a spread for the fourth tranche twice as much as the market quote.

As for the implementation results for the Gaussian and Structural models, with our discrete Monte Carlo approach, the absolute error[14] from both models is compara-

[12]Here the annual rate is the 5-year long term fixed rate, and the volatility of average underlying CDSs is used instead of risky bond volatilities. For more explaination of parameter setting, see Kalemanova et al. (2005), London (2006) and Loffler & Posch (2007).

[13]The absolute error cover the tranches except the equity tranche as it is normal practice that the first tranche spread is quoted from market, see Kalemanova et al. (2005).

[14]The error is the cumulative difference between the simulation outcomes and the market quotes over all tranches expect the equity.

ble. Note that although the discrete structural model is over estimating the second and third tranche spreads, it is providing better simulation results comparing to the results from the market standard model listed under column name ‘Gaussian’ in Table (2.1).

Here we conclude that comparing to the static copula method, the Gaussian model under a discrete setting is more effective in catching the market movements. And as simulated results from different models are by and large similar and reasonable, one could argue that the simulation framework is effectively housing assumptions from different models. As far as the cumulative loss approach is concerned, our results show that the method is powerful and fast in computation, yet this methodology has its limitations due to lack of economic sense in spread pricing when dealing with multi-state or multi-factor problems.

Chapter 3

Analyzing the Sub-prime Impact on Structured Credit Product Spreads with the Method of Principal Component

The main motivation of this chapter is to better understand the effect of the sub-prime financial crisis on structured credit product spreads. It is well known that the standard market model is a correlation driven copula model, which means that the trading of correlation - base and/or implied - plays a very important role in the movements of multi-name structured credit products. Since the market slide in 2007 and 2008 that resulted to a rounded loss of 3 trillion USD, one may easily come to the conclusion that correlation trading became probably the most expensive and dangerous game in finance - at least for the second half of the year 2007.

In the world of CDO modelling, researcher are observing various spread driving factors. Given a debt related collateralized pool, the market model uses only one input factor, the correlation, to describe the spread variation. The purpose of this chapter is to see if it is reliable to use an universal factor for all the tranches.

In the next Section we describe the data set. Then we introduce the empirical methodology used in our present study, followed by output results. We summarize our findings in the last Section of this Chapter.

3.1 Data Description

We have chosen to investigate the standard contracts of iTraxx credit default indices. The product selection is narrowed to the most actively traded 5-year *European* contracts with tranches: $0 \sim 3\%$, $3 \sim 6\%$, $6 \sim 9\%$, $9 \sim 12\%$ and $12 \sim 22\%$ percentage components of the whole 125 name index. The tranches in turn are named as *Equity*, *1st Mezzanian*, *2nd Mezzanian*, *Senior* and *Super Senior* respectively.

Data are obtained from *Reuter's Credit Index Viewer* and *Markit*. The data is recorded weekly in two different series: the first covers the period before the sub-prime crisis between the 09$^{\text{th}}$ Jun 2006 and 29$^{\text{th}}$ Dec 2006. The second spans the sub-prime period and is dated from 02$^{\text{th}}$ Nov 2007 to 23$^{\text{rd}}$ May 2008.

Table (3.1) below depicts the pre-subprime period:

	0%-3%	**3%-6%**	**6%-9%**	**9%-12%**	**12%-22%**
29/12/2006	1050	42.5	13	10	2.25
22/12/2006	1100	48	12.5	9.5	2
15/12/2006	1150	50.5	12.5	10.125	1.75

Continued on next page

Table 3.1 – continued from previous page

	0%-3%	3%-6%	6%-9%	9%-12%	12%-22%
08/12/2006	1187.5	55	12.625	11.5	1.875
01/12/2006	1150	58.5	13	12.625	1.625
24/11/2006	1175	57	13.375	10.875	1.5
17/11/2006	1300	55.75	14.125	11.125	1.5
10/11/2006	1200	54.5	14.625	10.5	1.625
03/11/2006	1150	55	15	9	1.5
27/10/2006	1425	60	15.125	9.5	1.375
20/10/2006	1612.5	62	15	9.125	1.25
13/10/2006	1600	70	14.875	8.75	1.125
06/10/2006	1875	76	14.9	7.75	1.375
29/09/2006	1912.5	74.5	15.125	7.125	1.625
22/09/2006	2000	73	15	6.875	1.5
15/09/2006	1550	47	15.125	6	1.375
08/09/2006	1450	53	15.75	6.25	1.5
01/09/2006	1675	50	15.5	7	1.75
25/08/2006	1700	55	15.625	7.625	1.625
18/08/2006	1710	56	17.625	8	1.75
11/08/2006	1912.5	64	19.375	8.5	2
04/08/2006	1975	69	21.125	9	1.875
28/07/2006	1912.5	78.5	23	9.625	2
21/07/2006	2150	77	23.125	10.375	2.125
14/07/2006	2175	80	23	11	2.375
07/07/2006	2000	71	22.5	10.875	2.625
30/06/2006	2512.5	83	22.75	10.5	2.5

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Table 3.1 – continued from previous page

	0%-3%	3%-6%	6%-9%	9%-12%	12%-22%
23/06/2006	2487.5	87.5	24	11	2.25
16/06/2006	2350	80.5	23.5	10.5	2.375
09/06/2006	2212.5	74.25	22.5	10	2.5

Table 3.1: **Data(in basis points) covers the period 09/06/2006 to 29/12/2006.**

In the table (3.2) below we continue the weekly data collected after the subprime crisis erupted from late 2007 to mid 2008:

	0%-3%	3%-6%	6%-9%	9%-12%	12%-22%
23/05/2008	3425	295	177	115	50
16/05/2008	2850	290	168	97	57
09/05/2008	3600	300	150.5	113	63
02/05/2008	3250	310	142	126	69
25/04/2008	3150	315	203	163	74
18/04/2008	3825	325	235	185	78
11/04/2008	3850	425	285	217	85
04/04/2008	3300	395	310	138	92
28/03/2008	3700	400	330	205	105
21/03/2008	5000	610	385	237	112
14/03/2008	5050	640	403	235	114
07/03/2008	4650	570	380	220	110
29/02/2008	3250	414	372	260	115

Continued on next page

Table 3.2 – continued from previous page

	0%-3%	3%-6%	6%-9%	9%-12%	12%-22%
22/02/2008	3300	470	360	280	109
15/02/2008	3200	485	290	235	97
08/02/2008	3100	395	220	190	85
01/02/2008	3300	295	197	145	78
25/01/2008	3950	320	215	131	66
18/01/2008	3400	326	194	115	58
11/01/2008	2650	250	173	100	57
04/01/2008	2550	197	152	92	55
28/12/2007	2425	195	130	80	50
21/12/2007	2350	190	109	71	47
14/12/2007	2325	184	88	62	44
07/12/2007	2412.5	180	85	56	43
30/11/2007	2587.5	178	78	50	41
23/11/2007	2950	172	70	41.75	40
16/11/2007	2675	168	63	35	37
09/11/2007	2350	155	55	28.25	35
02/11/2007	1950	144	49	21.5	32

Table 3.2: **Data(in basis points) covers the period 02/11/2007 to 23/05/2008.**

Looking at Table (3.2) it is worth mentioning that for the first three weeks, on dates: 02/11/2007, 09/11/2007, and 16/11/2007, the market spreads of the ‘high yield and safe' super senior 12%∼22% tranche is commanding a higher price than the 'not so safe' senior 9%∼12%tranche. A possible reason of this overtake is the larger tranche notional which is 10% in the super senior case, making this tranche more risky in a low credit market condition.

Also, from the movements of market spreads after the crisis, one can observe from Table (3.2) that there is a fatter tail effect in the spreads, especially in the last two tranches. In other words, the whole spread figure is 'raised' to match a certain distribution, in the case of the market standard model, the Gaussian one .

3.2 Principal Component Analysis

Principal Component Analysis (*PCA*) is a variable reduction procedure which allows for the development of a set of artificial variables - the so called principal components - out of observed data, with the aim that these *principal* variables will account for most of the variance in the original data.

This method of analysis is useful in our context because the aim is to further study the correlation within the same pool, and *PCA* is one of the most effective ways to examine the redundancy in a number of variables.

The principal component is a linear combination of optimal weight in the original data. Normally, it is possible to calculate a score for every sub-set or subject of the principle component, i.e. we are expecting 5 scores out from the analysis as we have five tranches in the data set[1].

[1]Detail and advanced studies of this procedure include extracting different numbers of components, orthogonal and oblique (also known as *uncorrelated* and *correlated*) solutions of the

The results given in the next section are produced using *MatLab*©, based on the data described in the previous section.

3.3 Output Results and Conclusion

In this section, we look into the details of results obtained using *MatLab*© built-in functions.

First of all we find the covariance matrix and correlation coefficients of the observed spreads.

Table 3.3: **Correlation Coefficients.**

02/11/2007	—	—	23/05/2008	
1.0000	0.6624	0.3380	0.2406	0.0641
0.6624	1.0000	0.5389	0.3399	0.2464
0.3380	0.5389	1.0000	0.5382	0.6614
0.2406	0.3399	0.5382	1.0000	0.5734
0.0641	0.2464	0.6614	0.5734	1.0000
09/06/2006	—	—	29/12/2006	
1.0000	0.6492	0.0663	0.1822	0.0153
0.6492	1.0000	0.3505	0.3236	-0.1062
0.0663	0.3505	1.0000	0.2807	0.0114
0.1822	0.3236	0.2807	1.0000	0.0775
0.0153	-0.1062	0.0114	0.0775	1.0000

principal components and number of item loading on each component, but technical advancement is not the aim of this study

From Table (3.3), comparing with the pre-subprime correlation shown in the second part of the table, we can see that the correlation between the 12%$\sim$22% *super senior* tranche and other four tranches are dramatically increased in the after-subprime 2007-2008 period. Meanwhile, the correlation score shows that there is an increase of correlation in between every two tranches within the period of the crisis comparing to scores from before the crisis.

Table 3.4: **Covariance Matrix.**

02/11/2007	—	—	23/05/2008	
2.7052	0.2216	0.0534	0.0372	0.0022
0.2216	0.0414	0.0105	0.0065	0.0010
0.0534	0.0105	0.0092	0.0049	0.0013
0.0372	0.0065	0.0049	0.0088	0.0011
0.0022	0.0010	0.0013	0.0011	0.0004
09/06/2006	—	—	29/12/2006	
3.1679	0.0830	0.0009	0.0025	0.0000
0.0830	0.0052	0.0002	0.0002	-0.0000
0.0009	0.0002	0.0001	0.0000	0.0000
0.0025	0.0002	0.0000	0.0001	0.0000
0.0000	-0.0000	0.0000	0.0000	0.0000

From the covariance matrix given in Table (3.4) it is clear that the convergence between tranches is slower in the period of crisis. For the period 09/06/2006-29/12/2006, take the second tranche for instance: the score decreased from 0.0830 to 0.0002 in the 9%$\sim$12% tranche and simply vanished in the super senior 12%$\sim$22% tranche. However, the comparable score from the more current data converge from 0.2216 to 0.0010 in the last tranche.

Meanwhile, one may notice that the magnitude of the two set of scores has a difference of 10 times, i.e., it is 1.0e+004 (or 10^4) for the period 09/06/2006-29/12/2006, and 1.0e+005 (or 10^5) for the more recent period. This finding provides additional support for the slow convergence pattern of the covariance matrix evidenced before.

Next, we list the principal components and *PCA* scores for the two data samples.

Table 3.5: **PC and Score for 09/06/2006-29/12/2006.**

		Principal	Components		
PC =	PC1	PC2	PC3	PC4	PC5
Tranche 1	-0.9997	0.0259	-0.0005	-0.0009	-0.0001
Tranche 2	-0.0260	-0.9979	0.0001	0.0587	0.0030
Tranche 3	-0.0003	-0.0497	-0.5289	-0.8463	0.0397
Tranche 4	-0.0008	-0.0309	0.8487	-0.5275	0.0221
Tranche 5	-0.0000	0.0057	0.0023	0.0451	0.0990
		PCA	Score		
score =	PC1	PC2	PC3	PC4	PC5
1.0e+004					
Tranche 1	-2.5179	0.0001	-0.0000	-0.0000	0
Tranche 2	0.5680	-0.0022	0.0000	0.0000	0
Tranche 3	0.6502	0.0006	-0.0000	-0.0000	0
Tranche 4	0.6486	0.0007	0.0000	-0.0000	0
Tranche 5	0.6510	0.0008	-0.0000	0.0000	0

Table 3.6: **PC and Score for 02/11/2007-23/05/2008.**

		Principal	Components		
PC =	PC1	PC2	PC3	PC4	PC5
Tranche 1	-0.9966	0.0808	0.0132	0.0027	-0.0014
Tranche 2	-0.0790	-0.9838	0.0831	-0.1372	-0.0085
Tranche 3	-0.0179	-0.1594	-0.5341	0.8234	0.1048
Tranche 4	-0.0121	0.0064	-0.8352	-0.5474	0.0518
Tranche 5	-0.0005	-0.0082	-0.1006	0.0595	-0.9931
		PCA	Score		
score =	PC1	PC2	PC3	PC4	PC5
1.0e+005					
Tranche 1	-2.1083	0.0012	0.0000	0.0000	0
Tranche 2	0.3824	-0.0155	0.0009	-0.0006	0
Tranche 3	0.5525	0.0015	-0.0018	-0.0031	0
Tranche 4	0.5690	0.0049	-0.0033	-0.0022	0
Tranche 5	0.6045	0.0079	0.0042	-0.0003	0

It is evident from the PCA score Tables (3.5) and (3.6), that the score for the period 02/11/2007-23/05/2008 is on bulk higher than the corresponding from the year 2006.

It is well known that PCA test results describe the distance of the simulated trend of movements from the original data. For example, the value -0.9997, the first entry in Table (3.5) indicates a simulated point *lower* almost *1* in distance from the real data point.

Similarly, we can see that the score for the crisis period in Table (3.6) is larger in absolute value than the score form the pre-crisis sample. This implies that the movements of the spreads in 07/08 have a much bigger variance while the spreads are bearing also a higher correlation.

Thus, we come to conclusion that although the market standard model is thought to be powerful, easy to implement and can be adjusted using the implied correlation to match fatter tail market situations, yet given extreme market condition, the market standard model becomes less attractive as it fails to capture the time dependent spread change. The reason is that whilst one may increase the implied correlation to obtain a better spread according to a certain (Gaussian) distribution, this is done at the expense of increased 'noise' that results from the fitting of a higher spread in turbulent credit market.

Chapter 4

Dynamic Growth Rate Model

4.1 A Dynamic Growth Rate Model

In this Chapter, we propose a dynamic credit risk model based on asset growth rate. The model can be used to analyze and dynamically price a wide spectrum of credit derivatives.Further, it is easy to calibrate and captures well both bullish and bearish market movements. Further, our growth rate framework depends only on an initial condition and extends the literature by covering zero face-value instruments as well. We study two specifications for default conditions and test our findings for robustness.

4.1.1 Model Setup

Suppose we have a portfolio of default-able zero coupon bonds which constitute the only debt of the referenced names. Denote r as the risk-free interest rate and c is the risky yield rate, assume that the portfolio is homogeneous, and each bond pays a risky yield-to-maturity of $(r + c)$, means the discrete asset growth factor over a short time interval is: $e^{(r+c)}$.

As the collateral value of the debt evolves until maturity T, the default-able bond losses its market value if its (time t) growth rate is lower than the risk-free *Treasury Bill* interest rate. In other words, the bond is downgraded to junk in this case, and a credit event is triggered.

We assume that the growth rate of a referenced company consists of two components: the risk-free rate r and a stochastic growth rate x. The latter follows an Ornstein-Uhlenbeck type process:

$$dx = -axdt + \sigma dz, \tag{4.1}$$

where a is the drift, σ is volatility and z is a Brownian Motion. Solving equation (4.1) we have:

$$x(t) = x(0)e^{-at} + \sigma \int_0^t e^{-a(t-u)} dw_u \tag{4.2}$$

Here dw_u in above equation is the first order derivative of stochastic process w_u for $t \in [0, \dots, t]$, in other words, it is the time dependent change of w_u.

Assume that the time zero value of the underlying bond is 1, then the asset value growth for time interval $(0, t)$ is: $e^{rt+x(t)}$ where $x(t)$ is given by the equation above. Then the value of default can be calculated and thus the probability of default at each time t before maturity is found.

We illustrate how to find the time t default probability $p(t)$ with a simple numerical example: Say that the yield of a risk-free, five-year, zero coupon Treasury bond paying £100 at maturity is 3%. Also, the yield of a zero coupon, zero recovery corporate bond with the same face value and maturity is 4%. So $r = 3\%$, $c = 1\%$, and as a result $r + c = 4\%$Then, at present, the Treasury bond worths: £$100e^{-0.03\times5} = 86.071$ and the corporate: £$100e^{-0.04\times5} = 81.873$. The value of default is their difference, £4.198.

In case of default, the corporate bond will cause a loss of full face value of £100 at maturity, so the risk-neutral expected loss from this default is simply: $100e^{-0.03\times5}p(0)$. Hence:

$$100e^{-0.03\times5}p(0) = 100e^{-0.03\times5} - 100e^{-0.04\times5}$$

and thus:

$$p(0) = \frac{e^{-0.03\times5} - e^{-0.04\times5}}{e^{-0.03\times5}} = 0.0488$$

At the end of the first year, the value of the Treasury bond is increased with the risk-free rate 3% to: £$86.071 \times e^{0.03} = 88.692$. As for the corporate bond, if the growth of first year is lower than the promised 4%,say, 3.5% (this can be seen as an addition of the 3% risk-free rate and $x(1) = 0.5\%$), the value of the bond is now: £$81.873 \times e^{0.035} = 84.789$.

Ideally, one would expect the corporate bond to grow with an average rate of 4% every year during the five years and make it to the par value of £100 at maturity, so the first year target should be: $81.873 \times e^{0.04} = 85.214$. Thus, with the value of £84.789, it is more difficult to reach £100 and therefore a larger probability of default (PD) is realized, calculated as:

$$100e^{-0.03\times4}p(1) \;=\; 100e^{-0.03\times4} - 100e^{-0.04\times5}e^{0.035}$$

$$\Rightarrow 88.692p(1) = 88.692 - 84.789$$
$$\Rightarrow p(1) = 0.044$$

where 0.04 in above calculation is the result of $r + c = 3\% + 1\% = 4\%$.

Mathematically, the general form equation of the probability of default is summarized as following:

$$p(t) = \frac{e^{-r(T-t)} - e^{-(r+c)T+(rt+x(t))}}{e^{-r(T-t)}} = 1 - e^{x(t)-cT} \tag{4.3}$$

where the term $e^{-(r+c)T+(rt+x(t))}$ is the one year compounding component with regard to the risky growth rate $rt + x(t)$, this is equivalent to the expanded product of two compounding terms $e^{-(r+c)T} \times e^{rt+x(t)}$.

It is easy to see from above equation (4.3) that for any time t, if $x(t) < ct$, then we will alway have a probability of default based on the performance of the bond for the first t years (as is 0.044 in our numerical example).

On the contrary, if $x(t) \geq ct$, the corporate bond is doing well for the period $(0, t]$, and to obtain the probability for the rest of (t, T) years, one may simply focus on the promised yield $r + c$ and the maturity time. So the probability of default is now defined as:

$$p(t) = \frac{e^{-r(T-t)} - e^{-(r+c)(T-t)}}{e^{-r(T-t)}} = 1 - e^{-c(T-t)} \quad \text{for} \quad x(t) \geq ct. \tag{4.4}$$

Continuing with the previous example, for any growth $x(t) \geq ct = 1\%$, for $t = 1$, by the end of the first year, we obtain the default probability of the corporate bond seen at time $t = 1$ for the remaining $(5 - 1) = 4$ years:

$$p(1) = \frac{88.692 - 85.214}{88.692} = 0.0392$$

Summarizing our findings, the default probability dependent on the stochastic growth rate $x(t)$, is given by:

$$p(t) = \begin{cases} x(t) \geq ct: & \frac{e^{-r(T-t)} - e^{-(r+c)(T-t)}}{e^{-r(T-t)}} = 1 - e^{-c(T-t)}; \\ x(t) < ct: & \frac{e^{-r(T-t)} - e^{-r(T-t)-cT+x(t)}}{e^{-r(T-t)}} = 1 - e^{x(t)-cT}; \end{cases} \tag{4.5}$$

4.1.2 Trigger of Default

In this section we analyze two alternative specifications of default. We shall compare the robustness of the two settings in the subsequent section.

Growth Rate Factor

Having obtained the default-adjusted asset growth rate from the previous section, we observe that the asset growth is limited by a lower rate of e^{rt}; in other words, the asset growth rate $x(t)$ has to stay above 0 for the company to survive until time t. In other words,having the company grown from time 0 to period t, its value is given by: $e^{-(r+c)T+(rt+0)} = e^{-(r+c)T}e^{rt}$.

So the default condition can be specified as: $x_i(t) > 0$ or $x(t) > 0$ for a homogeneous portfolio. In the context of the numerical example provided in the previous section, the probability of default at $x(1) = 0$ is:

$$p(1) = \frac{88.692 - 100e^{-0.04\times5+0.03}}{88.692} = 0.0488$$

for any $x(1) < 0$, the probability $p(1)$ becomes larger than 0.0488.

In this setting, a practitioner does not need to worry about the time dependent default probability given by equation (4.5), as the default is simply triggered when $x(t)$ falls below 0.

As for the calculation of the c value used in equation (4.5), it is easy to obtain it given the bond face value, but in case for a portfolio, we can have it from the average default probability using equation (4.5). Details of calibration together with numerical results will be given in a subsequent section.

Asset Value Approach

For the asset value approach, we may obtain c from market data, but, default instead is calculated according to the default barrier used in the standard market model.

As in Li (2000) and Kalemanova et al. (2005), default is triggered if the asset value falls below a threshold. The barrier is given by equation (1.14): $A = \Phi^{-1}(1 - p)$, using a static probability of default from the credit curve, i.e.:

$$K = \Phi^{-1}(1 - p_k) \tag{4.6}$$

where K is the value of default barrier and Φ is the normal Gaussian distribution function.

The variable p_k is defined as the probability of default over the whole time interval in the market model. Meanwhile, the threshold K is typically set to be a constant for CDO type of contracts[1].

Within our continuing example, with $x(t) = 0$, $p_k = 0.0488$, thus, $K = 1.6566$.

Similarly, in the time-dependent case, we can have a time t ‘asset value’, $A(t)$,

[1] For details see Anson et al. (2004), Bluhm & Overbeck (2007) and Loffler & Posch (2007).

using equation (1.14) as:

$$A(t) = \Phi^{-1}(1 - p(t)) \tag{4.7}$$

Use the same values of $x(1) = 0.5\%$ and $p(1) = 0.044$ we had earlier as example, $A(1) = 1.706$. So the asset survives until $A(t) < K = 1.6566$.

In this case, default is triggered if the value $A(t)$ falls below the default barrier K, i.e., $A(t) < K$ or equivalently:

$$\Phi^{-1}(1 - p(t)) < \Phi^{-1}(1 - p_k) \tag{4.8}$$

As the Φ^{-1} function value decreases with an increase of $p(t)$, to hold true that $A(t) < K$ one will require that $p(t) > p_k$. Subsequently, apply equation (4.3) and one finds that $x(t) < 0$ from the above equation (4.8).

4.1.3 Simulation Tests

Following the previous results, the two conditions of default are summarized in Figure (4.1):

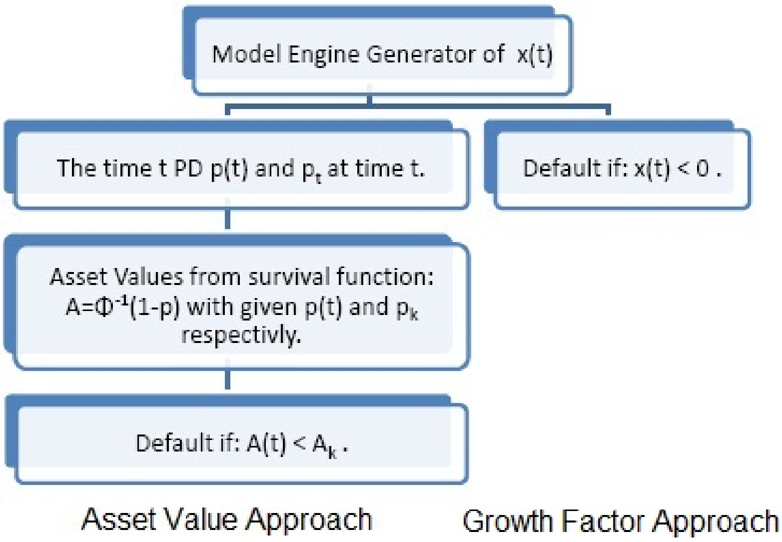

Figure 4.1: *Model calibration procedure of two approaches.*

As shown in the figure, we can see that the procedure of *Growth Factor* approach shown on the right side is easier to program and superior in computation time due to fewer calculations in case of simulation. Meanwhile, if the practitioner needs to observe the change in time-value and carry out portfolio or match to name hedging strategy based on asset value, the *Asset Value* approach is much more reliable and works well with a variety of asset value models, because with the simulation results in hand one may easily plug in a different barrier and obtain the estimated defaults of the simulated data.

In order to examine the convergence and robustness of the two approaches, we use a multi-step[2] Monte Carlo simulation using the same results generated by the random variable generating engine; the simulated default is shown in Figure (4.2) below:

[2]Traditional Monte Carlo is known as *static* because time does not play a part in the process. Multi-step Monte Carlo is suitable for dynamic discrete event simulation as it observes the behavior of individual entities in a system over a period of time.

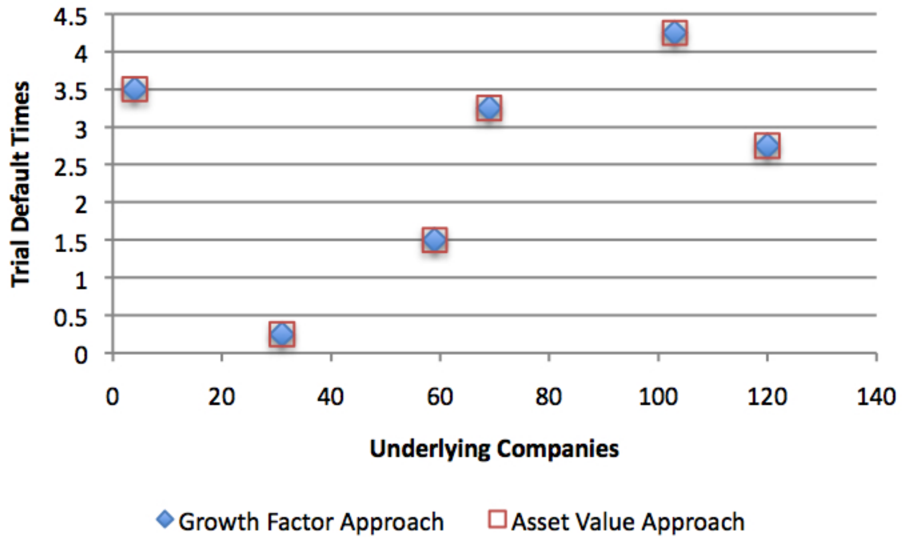

Figure 4.2: *Simulated trial default times.*

It is obvious from Figure (4.2) that both approaches provide identical results, in other words, the same company default/survive at the same time in both approaches:

$$\because A(t) < K \quad \text{i.e.} \quad \Phi^{-1}(1 - p(t)) < \Phi^{-1}(1 - p_k)$$

$$\therefore p(t) > p_k$$

$$\therefore \text{apply equation (4.5)} \quad 1 - e^{x(t) - cT} > 1 - e^{-cT}$$

$$\therefore x(t) < 0$$

The figure below shows the computation time[3] of the two default trigger approaches:

[3] The test is performed under Microsoft Excel VBA environment, the computer we used has Intel P4 3.6GHz CPU with 1G Memory.

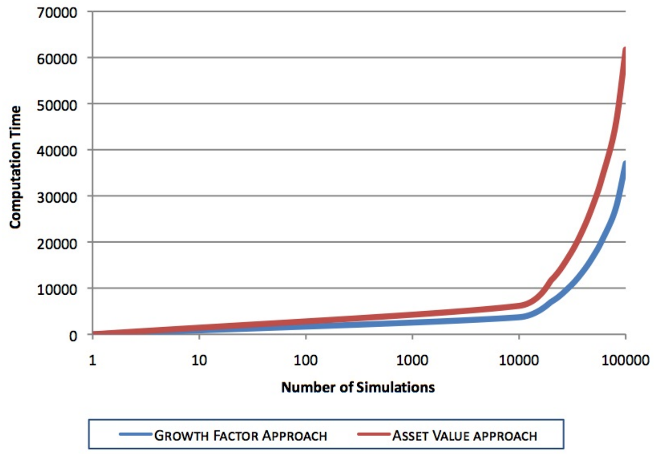

Figure 4.3: *Computation time(in seconds) vs. number of simulations.*

As shown above, the asset value approach is more time consuming; however, this approach is more suitable when considering a time dependent underlying asset value. Thus the additional time cost is bearable when calculating hedging parameters of the underlying portfolio.

We move on next to calibration and implementation using market data. The valuation process is carried out using multi time-step Monte Carlo method.

4.2 Model Implementation: Valuation of a CDO

In this section we focus on a Collateralized debt obligation (CDO) contract.

4.2.1 The Simulation Procedure

We follow the asset value approach in defining a credit event. The simulation process is broken down into the following steps:

1. Generate $x(t)$[4] using equation (4.1) and (4.2) for each underlying company over the entire contract time.
2. Calculate the default adjusted asset growth factor $e^{(r+c)t}$ for each time $0 \leq t \leq T$.
3. Calculate the time dependent default probability from equation (4.5) for all companies over the whole time period.
4. Calculate the expected time t asset value for all companies using the default probability from the above step.
5. Calculate the default barrier using equation (4.6).
6. Compare the expected asset value and the default barrier and determine the credit event using equation (4.8). Defaulted companies are knocked out for the remaining contract life.
7. Calculate the cumulative loss from default. The recovery rate is chosen to be 40% inline with market standards.

[4]To find the drift, we assume that the stochastic variable is 0, then the credit risky yield $c = \int_0^T \alpha dt$, as we have both c (the risky yield, 1% in previous example) and the contract maturity time T, the drift parameter α is then calculated using: $\alpha = \frac{c}{nT}$, where n is the number of payments per year ($n = 4$ in the numerical example as the payments are assumed quarterly.).

8. For each tranche, the loss at each time is given by the comparison between tranche size and the tranche loss. Fair spread is given by the component that equals the tranche loss and the tranche notional.

9. Repeat above steps for a large number of times and average the fair spread from all trials. Thus the fair spread is calculated for each tranche.

4.2.2 Numerical Results

The CDO-type contract we consider is the 5-year iTraxx Europe. Total underlying names is 125, and the six structured tranches are sized: $0 \sim 3\%$, $3 \sim 6\%$,$6 \sim 9\%$,$9 \sim 12\%$, $12 \sim 22\%$ and $22 \sim 100\%$. The payment days are set quarterly and the recovery rate is fixed to 40%.

Bearing in mind the on going tsunami in credit market we use two sets of market data, one *bearish* and the other *bullish*[5]: (i) the first data set is the iTraxx series five which started on 20th September 2006 and ends on 20th December 2011. The market quote[6] we use is recorded on the 31th of January of 2007, with compound spread 23bps. (ii) The second data set is the latest on-the-run iTraxx series eight version one with contract maturity the 20th December 2012. The quote date is 30st January 2008 with compound spread 123.75bps.

The interest rates from Bank of England (BoE) are 5.5% and 5% respectively, and the σ factors are 0.0031 and 0.0072, respectively, the latter are the available

[5]Define the term *bearish* for low credit market condition, thus high credit risk and high credit derivative spread, meanwhile *bullish* means high credit market and low derivative spread.

[6]All data is quoted from *Markit* and *Reuters' CDS Views*. Note that different contributors may submit different quotes.

volatilities of average CDO spreads in the database.

The numerical results are summarized in Table (4.1) and (4.2):

Tranche	**Market**	**Growth Rate**
0% - 3%	10.34%	17.03%
3% - 6%	41.59 bps	64.36 bps
6% - 9%	11.95 bps	20.14 bps
9% - 12%	5.6 bps	2.7 bps
12% - 22%	2 bps	0.85 bps
Absolute Error		26.91 bps

Table 4.1: **Numerical Results for iTraxx Tranches on 31th Jan 2007**

Tranche	**Market**	**Growth Rate**
0% - 3%	30.98%	37.03%
3% - 6%	316.9 bps	360.15 bps
6% - 9%	212.4 bps	247.31 bps
9% - 12%	140.0 bps	172.64 bps
12% - 22%	73.6 bps	85.28 bps
Absolute Error		122.48 bps

Table 4.2: **Numerical Results for iTraxx Tranches on 30th Jan 2008**

We can observe from Table (4.1) which depicts, a bearish market, that our simulation results differ by 50% or more compared with market data. Given that our absolute error is 26.91 bps, we believe that the main reason for the large percentage differences is that for the 9%$\sim$12% and 12%$\sim$22% tranches, the market spreads are low thus, 1.15bps difference in 12%$\sim$22% tranche between our model and the market results in 57.5% error. Note also that the overall absolute error[7] that

[7]The error is the sum of absolute difference between simulation outcome and market data

we experience for this period is reasonable comparing to result obtained from the market standard model in Table (2.1), the overall absolute error 122.48 is at the same level of pre-crisis error (117.24) given by the one factor Gaussian model, therefore, one may find the error acceptable for an intensive market.

As for the intense post sub-prime market, we can see from Table (4.2) that simulated spreads are within a the percentage difference of 15% compared to the market spreads. However, as evidenced in Table (4.1), the differences are larger for the last two tranches. The absolute error we observe here from each simulation outcome is about 5 times that of a good market, but if one bears in mind that the tranche spread level in the turbulent period compared with similar readings from a year before, for example, tranche 3%∼6% spread is about 8 times higher and the super senior tranche is now 36 times more expensive, the simulated results are still acceptable.

We may conclude that the growth rate model is useful in capturing both normal and intense market movements and the multi-step Monte Carlo simulation procedure that we implemented is effective when dealing with structural credit models.

4.2.3 Fitting Market Data

In this section, with the purpose of examining the "goodness" and capability of capturing market movement, we extend the previous numerical example to fitting more market data. As highlighted previously, the competition of static copula models is very heated for benign and easy markets, yet as observed by Bystrom

excluding the equity tranche, as according to market rules, the equity tranche spread is locked to 500bps.

(2009), almost everyone relics completely on market implied factors to better fit the spreads during the recent credit crunch, allowing for even 30% differences. For our present investigation, the data set is ambitiously selected to coincide with the climax of the credit crunch, covering the period from November 2007 to May 2008, including fall of Bear Stearns and the slide of the share price of Lehman Brothers.

The simulations are run in the context of two different approaches: (i) *Mark-to-date*, i.e., observe the data first and then update the model inputs according to latest market movements. For example, on the day of Christmas, we update the weekly volatility for our model and set the simulation running to forecast spreads on New Year's day. (ii) The *Dynamic*[8] approach, in which the inputs are only updated at the beginning of the simulation. In our examination, the variables are fixed from **02/11/2007** till **23/05/2008**. The purpose here is to test the dynamic ability to capture market movements, which is an impossible mission in static models. Fixing inputs at the beginning during the most volatile credit market ever experienced might be the last thing that any risk manager would do, and this is certainly not recommended by this research. However, we are still interested in the accuracy of estimating spreads for short periods, say for a three month term, which is the normal time period between two payments for a CDO contract.

The simulated results are shown below, the '*error*' column is the difference between simulated result and the market data, where positive error means a higher simulated result and negative error suggests a lower result than market spread. Detailed description and discussions are given in following subsections.

[8]Here *Dynamic* is the name of this approach as the model comes with dynamic growth rate, but the inputs are rather 'static as they are fixed since the beginning of simulation while the 'mark-to-date approach has the inputs updated weekly.

	Market Data	**Mark-to-Date**	*error*	**Dynamic**	*error*
23/05/2008	3425	3823.05	*398.04*	3228.62	*-196.38*
16/05/2008	2850	3633.07	*783.067*	3220.96	*370.96*
09/05/2008	3600	3507.81	*-92.19*	3232.98	*-367.02*
02/05/2008	3250	3768.79	*518.79*	3219.55	*-30.45*
25/04/2008	3150	4167.05	*1017.05*	3217.42	*67.42*
18/04/2008	3825	3800.09	*-24.91*	3218.11	*-606.89*
11/04/2008	3850	3967.92	*117.92*	3193.01	*-656.98*
04/04/2008	3300	3382.16	*82.16*	3164.14	*-135.864*
28/03/2008	3700	4290.92	*590.92*	3148.98	*-551.02*
21/03/2008	5000	5559.49	*559.49*	3130.47	*-1869.53*
14/03/2008	5050	5916.95	*866.95*	3035.83	*-2014.17*
07/03/2008	4650	4966.44	*316.44*	2928.55	*-1721.45*
29/02/2008	3250	4099.09	*849.09*	2830.99	*-419.01*
22/02/2008	3300	3176.07	*-123.94*	2804.11	*-495.89*
15/02/2008	3200	3125.04	*-74.96*	2777.56	*-422.44*
08/02/2008	3100	3401.51	*301.51*	2744.78	*-355.22*
01/02/2008	3300	3968.95	*668.95*	2713.01	*-586.99*
25/01/2008	3950	4129.44	*179.44*	2703.69	*-1246.3*
18/01/2008	3400	3734.58	*334.58*	2571.36	*-828.63*
11/01/2008	2650	3170.41	*520.41*	2488.11	*-161.88*
04/01/2008	2550	2672.04	*122.04*	2471.65	*-78.34*
28/12/2007	2425	2475.48	*50.48*	2461.27	*36.27*
21/12/2007	2350	2118.38	*-231.61*	2460.23	*110.23*
14/12/2007	2325	2805.44	*480.44*	2477.02	*152.02*

Continued on next page

Table 4.3 – continued from previous page

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
07/12/2007	2412.5	2515.41	*102.91*	2428.84	*16.34*
30/11/2007	2587.5	3341.35	*753.85*	2324.08	*-263.41*
23/11/2007	2950	3518.01	*568.01*	2671.83	*-278.16*
16/11/2007	2675	3466.59	*791.59*	2626.89	*-48.11*
09/11/2007	2350	2256.29	*-93.70*	2549.54	*199.54*
02/11/2007	1950	1688.13	*-261.86*	1688.13	*-261.86*

Table 4.3: **Fitting tranche spread 0-3% 02/11/2007-23/05/2008.**

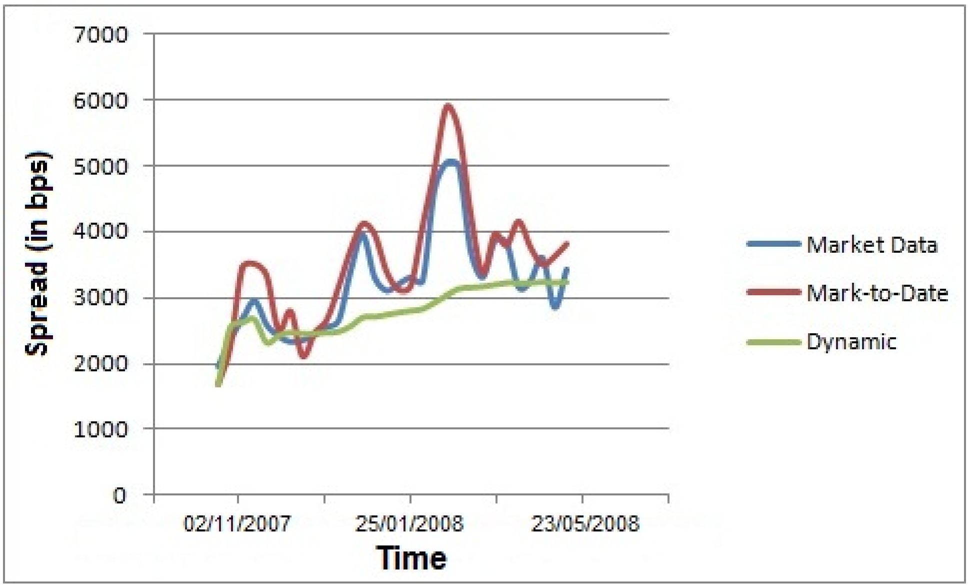

Figure 4.4: *Fitting tranche spread 0-3% 02/11/2007-23/05/2008.*

	Market Data	**Mark-to-Date**	*error*	**Dynamic**	*error*
23/05/2008	295	324.76	*29.76*	320.19	*25.19*
16/05/2008	290	279.05	*-10.95*	321.02	*31.02*
09/05/2008	300	382.58	*82.58*	322.14	*22.14*
02/05/2008	310	382.92	*72.92*	322.78	*12.78*
25/04/2008	315	334.83	*19.83*	323.17	*8.17*
18/04/2008	325	340.63	*15.63*	322.77	*-2.23*
11/04/2008	425	481.22	*56.22*	323.96	*-101.03*
04/04/2008	395	375.34	*-19.65*	318.80	*-76.19*
28/03/2008	400	411.04	*11.04*	312.38	*-87.61*
21/03/2008	610	432.95	*-177.04*	311.36	*-298.63*
14/03/2008	640	656.56	*16.56*	296.87	*-343.13*
07/03/2008	570	558.72	*-11.28*	278.77	*-291.23*
29/02/2008	414	365.27	*-48.72*	262.58	*-151.41*
22/02/2008	470	503.55	*33.55*	253.68	*-216.311*
15/02/2008	485	503.46	*18.46*	244.64	*-240.35*
08/02/2008	395	289.06	*-105.94*	224.43	*-170.56*
01/02/2008	295	351.57	*56.57*	211.08	*-83.91*
25/01/2008	320	347.61	*27.61*	205.33	*-114.66*
18/01/2008	326	357.81	*31.81*	196.86	*-129.14*
11/01/2008	250	242.32	*-7.67*	188.15	*-61.84*
04/01/2008	197	205.11	*8.11*	177.19	*-19.80*
28/12/2007	195	227.35	*32.35*	171.16	*-23.83*
21/12/2007	190	199.64	*9.64*	173.73	*-16.26*
14/12/2007	184	227.05	*43.05*	170.26	*-13.73*

Continued on next page

Table 4.4 – continued from previous page

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
07/12/2007	180	195.56	*15.56*	167.56	*-12.43*
30/11/2007	178	156.16	*-21.83*	165.17	*-12.82*
23/11/2007	172	232.61	*60.61*	163.50	*-8.49*
16/11/2007	168	158.63	*-9.36*	162.17	*-5.82*
09/11/2007	155	179.18	*24.18*	161.81	*6.81*
02/11/2007	144	161.53	*17.53*	161.53	*17.53*

Table 4.4: **Fitting tranche spread 3-6% 02/11/2007-23/05/2008.**

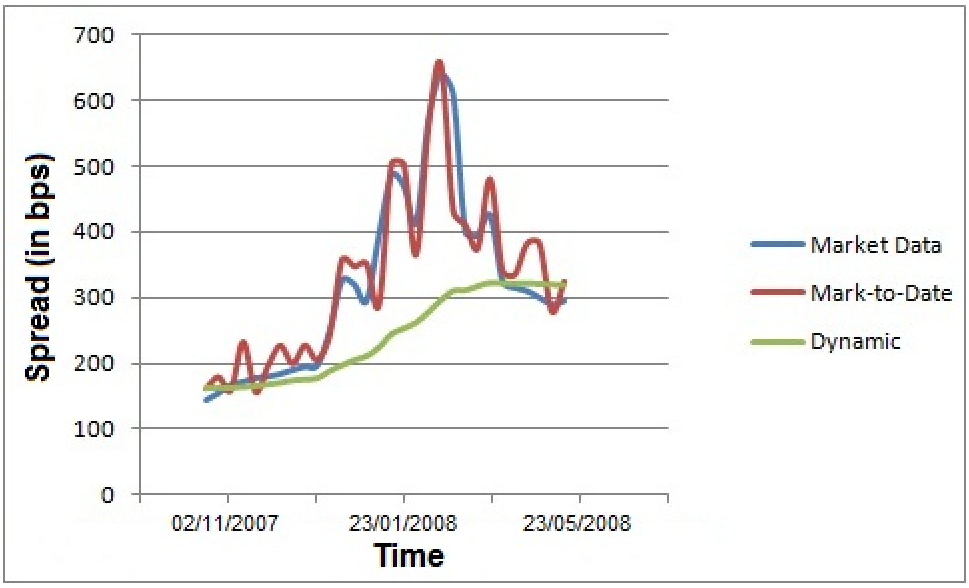

Figure 4.5: *Fitting tranche spread 3-6% 02/11/2007-23/05/2008.*

	Market Data	**Mark-to-Date**	*error*	**Dynamic**	*error*
23/05/2008	177	207.07	*30.07*	205.16	*28.16*
16/05/2008	168	180.27	*12.27*	203.97	*35.97*
09/05/2008	150.5	183.49	*32.99*	205.23	*54.73*
02/05/2008	142	173.17	*31.17*	207.20	*65.20*
25/04/2008	203	292.27	*89.27*	209.63	*6.63*
18/04/2008	235	304.69	*69.69*	209.74	*-25.25*
11/04/2008	285	295.01	*10.01*	208.50	*-76.49*
04/04/2008	310	260.49	*-49.50*	205.01	*-104.99*
28/03/2008	330	332.99	*2.99*	200.56	*-129.43*
21/03/2008	385	352.53	*-32.46*	194.38	*-190.61*
14/03/2008	403	387.97	*-15.02*	184.97	*-218.02*
07/03/2008	380	461.71	*81.71*	173.50	*-206.49*
29/02/2008	372	447.39	*75.39*	162.12	*-209.87*
22/02/2008	360	305.69	*-54.30*	150.59	*-209.40*
15/02/2008	290	365.47	*75.47*	136.38	*-153.61*
08/02/2008	220	256.79	*36.79*	125.83	*-94.16*
01/02/2008	197	226.45	*29.45*	118.85	*-78.14*
25/01/2008	215	209.55	*-5.44*	115.83	*-99.16*
18/01/2008	194	193.66	*-0.33*	105.66	*-88.33*
11/01/2008	173	196.78	*23.78*	96.87	*-76.12*
04/01/2008	152	163.33	*11.33*	91.17	*-60.82*
28/12/2007	130	166.72	*36.72*	82.76	*-47.23*
21/12/2007	109	135.57	*26.57*	75.61	*-33.38*
14/12/2007	88	71.70	*-16.29*	70.27	*-17.72*

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Table 4.5 – continued from previous page

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
07/12/2007	85	88.99	*3.99*	67.88	*-17.11*
30/11/2007	78	89.26	*11.26*	66.99	*-11.01*
23/11/2007	70	78.49	*8.49*	63.18	*-6.81*
16/11/2007	63	66.44	*3.44*	61.21	*-1.78*
09/11/2007	55	59.20	*4.20*	58.40	*3.40*
02/11/2007	49	44.71	*-4.28*	44.71	*-4.28*

Table 4.5: **Fitting tranche spread 6-9% 02/11/2007-23/05/2008.**

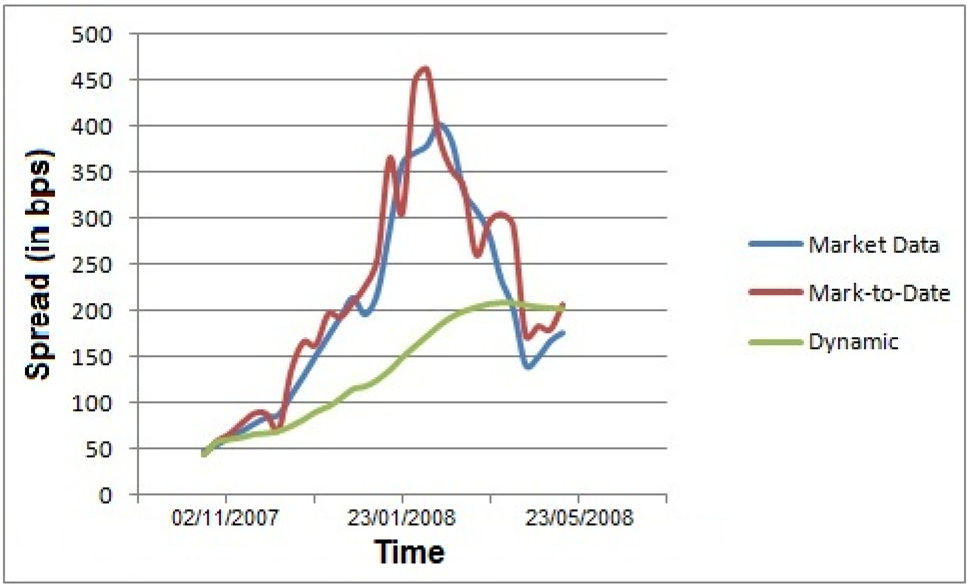

Figure 4.6: *Fitting tranche spread 6-9% 02/11/2007-23/05/2008.*

	Market Data	**Mark-to-Date**	*error*	**Dynamic**	*error*
23/05/2008	115	136.29	*21.29*	135.94	*20.94*
16/05/2008	97	111.45	*14.45*	136.11	*39.11*
09/05/2008	113	129.14	*16.14*	137.49	*24.49*
02/05/2008	126	153.98	*27.98*	138.37	*12.37*
25/04/2008	163	182.15	*19.15*	138.48	*-24.51*
18/04/2008	185	212.27	*27.27*	137.79	*-47.20*
11/04/2008	217	200.01	*-16.98*	135.84	*-81.15*
04/04/2008	138	159.01	*21.01*	132.25	*-5.74*
28/03/2008	205	250.59	*45.59*	131.68	*-73.31*
21/03/2008	237	242.25	*5.25*	128.39	*-108.60*
14/03/2008	235	275.11	*40.11*	122.94	*-112.05*
07/03/2008	220	278.75	*58.75*	116.79	*-103.20*
29/02/2008	260	310.53	*50.53*	110.67	*-149.32*
22/02/2008	280	268.95	*-11.04*	102.25	*-177.74*
15/02/2008	235	252.93	*17.93*	92.90	*-142.09*
08/02/2008	190	182.23	*-7.76*	84.18	*-105.81*
01/02/2008	145	153.09	*8.09*	73.70	*-71.29*
25/01/2008	131	158.50	*27.50*	68.61	*-62.38*
18/01/2008	115	132.91	*17.91*	62.89	*-52.10*
11/01/2008	100	89.85	*-10.14*	59.15	*-40.84*
04/01/2008	92	90.94	*-1.05*	54.43	*-37.56*
28/12/2007	80	99.47	*19.47*	50.40	*-29.59*
21/12/2007	71	65.94	*-5.05*	46.94	*-24.05*
14/12/2007	62	68.20	*6.20*	42.89	*-19.10*

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Table 4.6 – continued from previous page

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
07/12/2007	56	55.76	*-0.23*	40.97	*-15.02*
30/11/2007	50	46.55	*-3.44*	37.30	*-12.69*
23/11/2007	41.75	46.22	*4.47*	35.50	*-6.24*
16/11/2007	35	31.22	*-3.77*	32.29	*-2.70*
09/11/2007	28.25	36.84	*8.59*	32.22	*3.97*
02/11/2007	21.5	22.14	*0.64*	22.14	*0.64*

Table 4.6: **Fitting tranche spread 9-12% 02/11/2007-23/05/2008.**

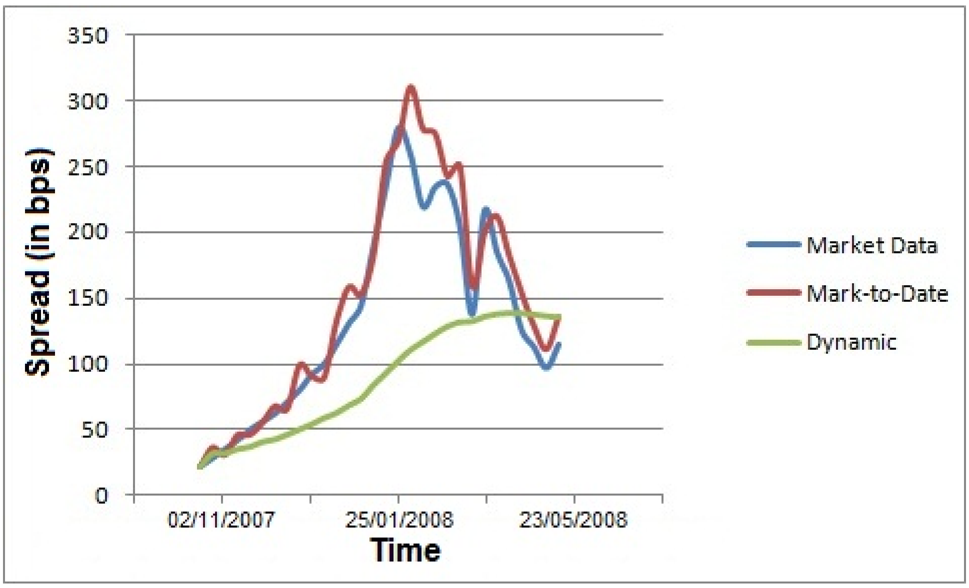

Figure 4.7: *Fitting tranche spread 9-12% 02/11/2007-23/05/2008.*

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
23/05/2008	50	56.69	*6.69*	70.31	*20.31*
16/05/2008	57	65.08	*8.08*	70.97	*13.97*
09/05/2008	63	65.41	*2.41*	71.47	*8.47*
02/05/2008	69	66.28	*-2.71*	71.76	*2.76*
25/04/2008	74	82.93	*8.93*	71.90	*-2.09*
18/04/2008	78	91.55	*13.55*	71.76	*-6.23*
11/04/2008	85	83.93	*-1.06*	71.48	*-13.51*
04/04/2008	92	91.88	*-0.11*	70.90	*-21.09*
28/03/2008	105	104.49	*-0.50*	69.94	*-35.05*
21/03/2008	112	128.92	*16.92*	68.27	*-43.72*
14/03/2008	114	128.51	*14.51*	66.03	*-47.96*
07/03/2008	110	111.14	*1.14*	63.37	*-46.62*
29/02/2008	115	123.77	*8.76*	60.93	*-54.06*
22/02/2008	109	118.62	*9.62*	57.63	*-51.36*
15/02/2008	97	91.90	*-5.09*	54.39	*-42.60*
08/02/2008	85	88.64	*3.64*	51.71	*-33.28*
01/02/2008	78	95.87	*17.87*	49.14	*-28.85*
25/01/2008	66	70.95	*4.95*	46.99	*-19.00*
18/01/2008	58	59.99	*1.99*	45.19	*-12.80*
11/01/2008	57	63.32	*6.32*	44.07	*-12.92*
04/01/2008	55	62.86	*7.86*	42.44	*-12.56*
28/12/2007	50	52.74	*2.74*	41.63	*-8.36*
21/12/2007	47	56.76	*9.76*	41.08	*-5.91*
14/12/2007	44	47.35	*3.35*	39.10	*-4.89*

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Table 4.7 – continued from previous page

	Market Data	Mark-to-Date	*error*	Dynamic	*error*
07/12/2007	43	45.93	*2.93*	38.61	*-4.38*
30/11/2007	41	45.62	*4.62*	37.40	*-3.59*
23/11/2007	40	40.29	*0.29*	38.24	*-1.75*
16/11/2007	37	32.56	*-4.43*	35.79	*-1.20*
09/11/2007	35	35.08	*0.08*	36.23	*1.23*
02/11/2007	32	39.14	*7.14*	39.14	*7.14*

Table 4.7: **Fitting tranche spread 12-22% 02/11/2007-23/05/2008.**

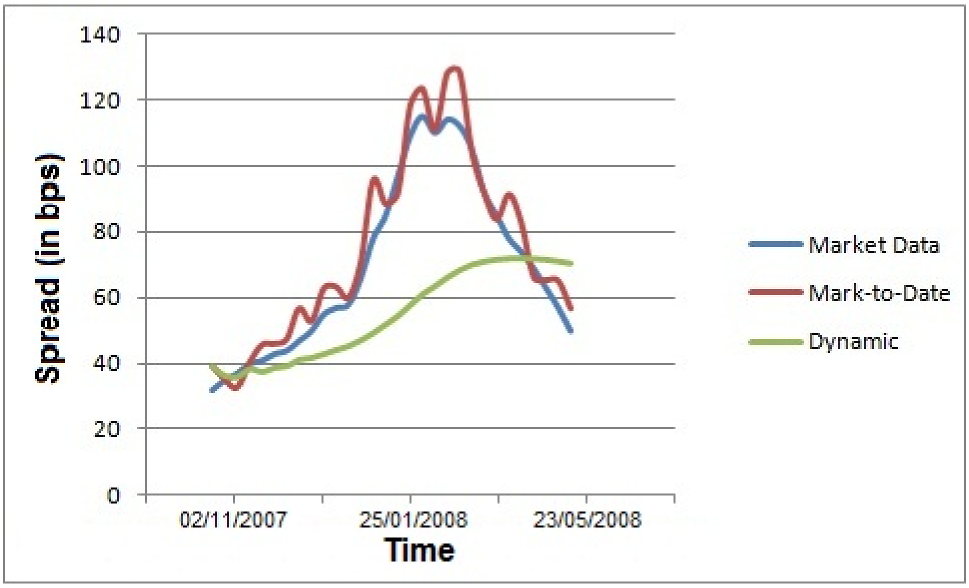

Figure 4.8: *Fitting tranche spread 12-22% 02/11/2007-23/05/2008.*

Long Term Fitting

It is evident from the Tables and Figures that the *Mark-to-Date* approach is giving results inline with the market data curve for all five tranches. The largest errors are occurring on dates: 25/04/2008 for the 0-3% tranche with error 1017.05 bps, 21/03/2008 for 3-6% tranche with error -177.04 bps, 25/04/2008 for the 6-9% tranche with error 89.27 bps, 07/03/2008 for 9-12% tranche with error 58.75 bps, and on the 01/02/2008 for the 12-22% tranche with error 17.87 bps.

Comparing with the market spread for the above dates, the percentage errors are: 32.29%, 29.02%, 43.97%, 26.71% and 22.91%, respectively. It is interesting to note that the spread changes between these 5 days and the previous correspounding ones are: 21.43%, 4.91%, 15.76%, 18.19% and 15.38%. Note also that the market practice is to take the Equity tranch spread directly from market. Finally, one has to take into account that the spread changes of all five tranches during the entire considered time period from November 2007 to May 2008 are: 2.17 times higher for the 0-3% Equity tranch, 4.44 times higher for the 3-6% Junior Mezzanine tranch, 8.23 times higher for the 6-9% Senior Mezzanine tranch, 13.02 times higher for the 9-12% Senior tranch and 3.59 times higher for the 12-22% Super Senior tranch. Bystrom (2009) concluds that the widly adopted copula models are ‘useless’ under extreme market condition, however, been granted a larger tollerance under extrame market conditions, *Mark-to-Date* approach distinguish itself from the group of traditional copula models.

On the other hand, the *Dynamic* approach, seems to have a very tough ride. As highlighted in the previous section, the main reason for the observed big gaps between simulation and market data is the fixed model input. As it is widely known already, the pre-crisis market had only low spreads and very flat curves across all traded tranches, thus the fixed volatility input will never result in a sky high spread like ten times the pre-crisis curve for the 9-12% tranche. Meanwhile, we see from the Tables that the dates in which the tranche largest errors took place for the dynamic approach are: 14/03/2008, 14/03/2008, 14/03/2008, 22/02/2008 and 29/02/2008. Incidentally, these dates are also the peaks for the market spread. We can see that all these extreme values took place on or around the date of 14th March 2008, the same day that the Wall Street giant Bear Stearns revealed the news that it financial position had "significantly deteriorated in the last 24 hours". On the same day, the Fed stepped in by arranging for a rival bank, JP Morgan Chase, to inject short-term capital for Bear Stearns rescue. Singling out the most troublesome part of the credit market in our sample, we have to admit that it is impossible for our model to comply with extreme events such as the default of the 5th largest investment bank of Wall Street.

Interestingly, none of the largest errors took place on the above dates according to the *Mark-to-Date* approach. This slightly suggests that by updating the latest market information really helps our model to capture the evolving spreads. To better gauge the strengths and weaknesses of the *Dynamic* approach we turn our attention next to examine its short term performance.

Short Term Fitting

Here we set the period of observation to a three-month interval, or 12 weeks (in order to take into account the Christmas off-trading days), i.e. we only look at the data from 02/11/2007 to 18/01/2008.

The 0-3% Equity tranch spread shows an error ranging from around 36 bps to

-828 bps, the average error is -52.48 bps for 11 among the 12 observed dates, excluding the 18/01/2008 where the numerous -828.63 caused by a jump of the market by 28.3%./ For the 3-6% tranche, the market showed a jump of 30.4% on 18/01/2008, while during the other 11 weeks starting from the 02/11/2007 it showed an average error of -13.69 bps. The 6-9% tranche jumped 12.13% from 173 bps to 194 bps on 18/01/2008, whilst the average error for the other 11 weeks was -24.81 bps. As for the 9-12% tranche, the average error in bps was found to be -16.65bps while the jump on 18/01/2008 was 15%. For the last 12-22% tranche, the average error before 18/01/2008 was -4.29 bps, while the jumps that the other tranches experienced on 18/01/2008 seem not to affect the super senior tranche.

In a nutshell, the dynamic simulated tranche spreads for all the five tranches are all underestimating the market volatility. This is due to the input from a considerably flat curve at the beginning of the sample period whilst thereafter the market spread was increasing to peak on the 14/03/2008.

Our results for a short-period indicate that the model works adequately in volatile market and thus it is possible to construct fairly reliable hedging strategies based on short term simulations. Our research will continue in the next Chapter by carrying our simulations over three-month periods to develop dynamic hedging strategies.

4.3 A Note on the Flexibility of our Framework

The assumptions that were utilized in the previous simulations were guided by the standard market model. However, as the model was implemented with a multi-step simulation, 'richer' assumptions can also be added in case one needs to analyze the market with the aid of more parameters, making the framework more consistent with reality.

Dynamic Recovery Rate

Although, the market practice is to set a uniform recovery rate R to all classes of credit derivatives, recent research outlines the benefits of stochastic recovery rates as a superior fit to reality.

Yu (2003) and Herkommer (2007) have shown that the recovery rate can not be disassociated from default probability. Moreover, Hu & Perraudin (2002), Carey & Gordy (2003) and Altman, Brady, Resti & Sironi (2005) suggest that there is a negative correlation between the default probability and the recovery rate.

In our dynamic asset growth rate model, for each observed time t, a negative correlation between the derived default probability and the recovery rate is implied by our default conditions. Thus, our model framework is compatible with recent empirical findings regarding the impact of the recovery rate.

Match to Name Correlation

According to reviews by Deacon (2003) and Anson et al. (2004) most distribution and copula based credit models are correlation centered, and as practitioners price the products with the market standard Gaussian copula model, discussions on pairwise correlation between underlying companies will continue to be popular in the future.

To examine the effects of correlation factors within our model, we follow the approach proposed by Hull et al. (2005). The correlation parameter ρ is set to be embedded in the Brownian Motion process z, so we now have z as:

$$dz_i = \rho_i dM + \sqrt{1 - \rho_i^2} dZ_i$$

Here i indicates the i^{th} company and M is a common Wiener process for all underlying names. The above equation accommodates idiosyncratic correlation assumptions for each of the individual companies, therefore one could extend it by setting correlation assumptions for the companies' growth rates.

Thus, the growth rate process is driven by the macro market momentum together with an individual process. In this way, one may apply the match to name correlation factors with our proposed dynamic structure model and observe the changes due to difference in correlations. Further, for time dependent simulation processes, our framework well houses the time dependent correlation assumption as asset growth rate for each time step is distinguished from the previous steps.

Time Dependent Derivative Pricing

Since the 2007 credit crisis, the issue of hedging credit risk with other derivatives is heated more than ever before. As a dynamic model, our approach, can be used to perform continuous time cash flow analysis as well as valuation of option-type securities.

Early structural models such as Merton (1974) and Black & Cox (1976) consider credit default products as exotic options in which the default trigger condition is set as the strike price which makes an option exercisable. In such a context, the asset value is assumed to follow a log-normal process and the derivative is priced as follows. The well known solution of a vanilla call option is:

$$C = A(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where $A(0)$ is the asset value at time 0, K the option strike, r the risk-free interest rate and T the maturity, with d_1 and d_2 defined by:

$$d_1 = \frac{\ln A(0) - \ln K + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

Having the time dependent asset value $A(t)$, one is able to derive the time t call and/or put price for an option-type contract on credit default-able assets. Furthermore, calculations of the *Greek* factors are straight forward and time dependent sensitivity analysis is made easy. Thus, time dependent hedging strategies can be developed on expected cash flow and sensitivity analysis from the *Greeks*.

For exotic options written on credit derivatives, the referenced value is then the credit spreads (or tranche spreads in the case of a structured credit portfolio). As our pricing model is capable of providing dynamic spreads, the option value can then be calculated using standard option pricing methodologies.

In this chapter we proposed a new dynamic approach for structural credit risk modelling and we developed a time dependent pricing technique that we believe is vital since the whole market is facing the challenge of actively managed and/or replicated credit portfolios.

The growth rate model that we suggest is easy to calibrate as inputs are either given directly or easily derived from market data. It turns out that our framework can accommodate both bearish and bullish credit markets and fits market quotes reasonably well.

Obviously, further extensions can be made to analyze the effects of many other factors which are exogenous to our model, such as the pairwise correlation and interest rates. Our philosophy though was to keep everything relatively simple and “old school”, since in the recent climate market practitioners are increasingly returning to basics when facing credit risk exposures.

Chapter 5

Dynamic Hedging Strategy

In this Chapter, we demonstrate how our *Dynamic Growth Rate Model* and the multi-step simulation procedure may be used to generate the dynamic evolution of the spreads for a CDO type product across tranches over the life time of the contract. A full discussion based on numerical analysis of the results between the CDO and basket CDS contract from the same pool of collateral is provided. Furthermore, we develop a formula for calculating the hedge ratio of a 'non-credit risk' hedge. Finally, for the first time in the literature, a time and default dependent portfolio loss ratio is derived to fit diverse hedging needs.

5.1 Default Model

The credit risk for structured CDO-type products is modeled using the growth rate factor model, where the default-adjusted rate $x(t)$ is defined to follow a generic short process:

$$dx = -axdt + \sigma dz \tag{5.1}$$

where a is the drift and σ the volatility, as introduced in Chapter 4.

The correlation is handled following the approach suggested by Hull et al. (2005), and we further define the Brownian Motion process z with correlation factor ρ:

$$dz_i = \rho_i dM + \sqrt{1-\rho_i^2} dZ_i \tag{5.2}$$

Here M is the generalized market momentum factor for all the names in the reference pool and Z_i is the idiosyncratic Wienner process marked to each of the i^{th} underlying names. Here the correlation between underlying company i and j is given by ρ_{ij}, if the assumption of a homogeneous portfolio is applied, i.e. $Z_i = Z$ and $\rho_i = \rho$, one may easily have $z_i = z$ with the pairwise correlation ρ.

Other factors involved in the model including the interest rate r and the credit risky coupon rate c. The model growth rate output for time t unit asset value is then: $e^{rt+x(t)}$, note that $x(t)$ is time dependent and so does the growth rate factor.

Finally, the time dependent default probability can be obtained from the following equation:

$$p(t) = \begin{cases} x(t) \geq ct : & \frac{e^{-r(T-t)} - e^{-(r+c)(T-t)}}{e^{-r(T-t)}} = 1 - e^{-c(T-t)}; \\ x(t) < ct : & \frac{e^{-r(T-t)} - e^{-r(T-t)-cT+x(t)}}{e^{-r(T-t)}} = 1 - e^{x(t)-cT}; \end{cases} \tag{5.3}$$

Where $x(t)$ is the default risk adjusted rate defined for the default model in Chapter 4.

Note that we assume the actual loss is only determined by the recovery rate R upon default, i.e. in case of zero recovery zero coupon risky bond the investor losses the par value. For further extensions covering a stochastic loss process, one may change the assumption of fixed recovery with a stochastic version in the model setup.

5.2 Simulated Dynamic Portfolio Loss for CDOs

In order to estimate the time dependent movements of the spreads over the life time of the CDO contract, multi time-step spread simulation is probably the only way that this could be achieved with the further aim to develop hedging strategies. In general, we need to calculate both the loss and remaining notional to obtain the final spread.

One may argue that for the calculation of the remaining notional, possible questions come from two main fronts: the first one is whether we should use the whole notional on each tranche or the notional left after defaults over time; the second one is more complex and worth a lot more work in further research, that is how one should weight the remaining notional against the aggregate loss under the random recovery/interest-rate assumption.

Here we focus on developing a dynamic hedging technique within a vanilla framework. Meanwhile, considering the trading needs for hedging and risk reducing purposes, we calculate the spreads and the basis of a ratio of aggregate future loss on the remaining notional at each time step. Both the recovery and risk-free rates are fixed in this study. In other words, pricing a CDO future is not that different than pricing a normal CDO; the difference is that when it comes to each time step we consider only forward loss and notional, excluding the deductions due to prior defaults. For example, a 125-name standard 5-year CDO type index may become a 100-name 3-year CDO by the end of the second year.

Mathematically, for a contract ending at time T with n payment steps, one can define the premium leg[1] as following:

$$\sum_{i=1}^{n} \Delta t_i \cdot \omega \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}}, \tag{5.4}$$

t_i refers to the n payment steps, $i = 1, \ldots, n$, ω is the spread and $EL_{t_{i-1}}$ is the expected future loss of the tranche at current payment date while $B_{t_{i-1}}$ can be seen as the price of a risk-free government bond or simply the discount rate at time t_{i-1}.

Similarly, the protection leg value is given by:

$$\sum_{i=1}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i} \tag{5.5}$$

Hence, the spread of a normal CDO product is given by:

$$\omega = \frac{\sum_{i=1}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i}}{\sum_{i=1}^{n} \Delta t_i \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}}} \tag{5.6}$$

The tranche loss and notional of future time steps can be calculated in the same way. For time step t_j where $j \leq i$ and $0 < t_1 \leq \ldots \leq t_j \leq t_i \leq \ldots \leq t_{n-1} \leq t_n = T$, we have:

$$\text{premium}_{t_j} = \sum_{i=j}^{n} \Delta t_i \cdot \omega_{t_i} \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}} \tag{5.7}$$

[1]*Premium Leg* is the cash-flow of periodic payments from the contract holder to the issuer. *Protection Leg* is the one off payment at time of default from the contract issuer to the holder.

and future value for the protection leg is:

$$\text{protect}_{t_j} = \sum_{i=j}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i} \tag{5.8}$$

Finally, we are able to obtain the future prices of the CDO contract for $j \in (0, T]$:

$$\omega_{t_j} = \frac{\sum\limits_{i=j}^{n} (EL_{t_i} - EL_{t_{i-1}}) \cdot B_{t_i}}{\sum\limits_{i=j}^{n} \Delta t_i \cdot (1 - EL_{t_{i-1}}) \cdot B_{t_{i-1}}} \tag{5.9}$$

Having obtained the time dependent prices of the CDO portfolio, we are now able to consider its changes overtime, and thus the Greeks[2].

5.3 Simulated Dynamic Portfolio Loss for CDS Baskets

As for basket type CDS indices, the spread is calculated as the sum of each of the included individual CDSs. The technique of pricing a single name CDS with fixed recovery and flat interest rate is simple and easy for us to start with. We consider the discrete case at this early stage.

Say we have a single name CDS with fixed recovery rate R, the notional is given by N, the contract is knocked out on default and the claim is adjusted according to recovery, the life of contract starts at time 0 and the time interval contain n payment dates, i.e., $0 < t_1 \leq \ldots \leq t_i \leq \ldots \leq t_{n-1} \leq t_n = T$. Given no default until maturity, the premium leg is[3]:

$$s \cdot (1 - p) \cdot N \cdot (e^{-rt_1} + \ldots + e^{-rt_i} + \ldots + e^{-rt_n}) \tag{5.10}$$

[2]Only *Delta* and *Gamma* factors are derived and discussed.

[3]Here we suppose the credit spread is fixed at the initial purchase price.

As for the protection leg of the same contract, we have:

$$p \cdot (1 - R)N \cdot (e^{-rt_1} + \ldots + e^{-rt_i} + \ldots + e^{-rt_n}) \tag{5.11}$$

The credit spread for this CDS is given by:

$$s = \frac{p \cdot (1 - R)N \cdot (e^{-rt_1} + \ldots + e^{-rt_i} + \ldots + e^{-rt_n})}{(1 - p) \cdot N \cdot (e^{-rt_1} + \ldots + e^{-rt_i} + \ldots + e^{-rt_n})} = (1 - R) \cdot \frac{p}{1 - p} \tag{5.12}$$

The variable p in the above equations is the probability of default, here for simplicity, we treat it as fixed. In our dynamic environment, the spot probability of default is available from the simulations and if the default time t_τ is given, denote the time of default by τ, the premium leg is found as:

$$s \cdot N \cdot \sum_{i=1}^{\tau} (1 - p_i) \cdot e^{-rt_i} \tag{5.13}$$

Meanwhile, the protection leg is:

$$(1 - R)N \cdot \sum_{i=1}^{\tau} p_i \cdot e^{-rt_i} \tag{5.14}$$

The credit spread is straightforward to calculated:

$$s = \frac{(1 - R)N \cdot \sum_{i=1}^{\tau} p_i \cdot e^{-rt_i}}{N \cdot \sum_{i=1}^{\tau} (1 - p_i) \cdot e^{-rt_i}} = (1 - R) \cdot \frac{\sum_{i=1}^{\tau} p_i \cdot e^{-rt_i}}{\sum_{i=1}^{\tau} (1 - p_i) \cdot e^{-rt_i}} \tag{5.15}$$

Further, for the CDS spreads in future payment date along the time line, for any payment date t_j, $0 < j \leq i \leq n$, the CDS spread can be obtained from the following:

$$s_j = (1 - R) \cdot \frac{\sum_{i=j}^{\tau} p_i \cdot e^{-rt_i}}{\sum_{i=j}^{\tau} (1 - p_i) \cdot e^{-rt_i}} \tag{5.16}$$

We are able now to calculate the spread of the whole CDS basket. Note that every single name CDS is knocked out once the referenced underlying defaults, and the name is subtracted from the basket name list as well as the notional. One may record the basket spread Greeks and evaluate the hedging effect using the CDS basket against the CDO.

Furthermore, corporate bonds which match the names included in the CDO contract, in principle can also be used as a hedging instrument for the CDO held. However, if one take into account the cheap initial payment for a CDS basket, hedging using a basket of CDS might be more preferable for market players. In this study we will simply consider a portfolio consisting of a CDO and a CDS basket both sharing the same underlying name list. The idea and the development of the hedging technique according to the simulated results can be easily adopted to other complex combinations.

5.4 Implementation with Monte Carlo

Here we illustrate the Monte Carlo simulation environment used to implement the model, and more importantly, we provide details as tohow the hedge is calculated.

First we describe the simulation process in steps:

1. Generate value $x(t)$ using equation (5.1) and (5.2) for each underlying company over the whole contract time.
2. Calculate the default adjusted asset value growth rate factor $e^{rt+x(t)}$ for each time $0 \leq t \leq T$.

3. Calculate the time dependent default probability from equation (5.3) for all companies over the whole time period.

4. Calculate the expected time t asset value for all companies using the default probability from above step.

5. Calculate the default barrier using equation:

$$K = \Phi^{-1}(1 - p_k).$$

Here K is the default condition discussed in Li (2000), and p_k is the probability of default over the whole time interval in the market model.

6. Compare the expected asset value and the default barrier and determine a credit event under condition:

$$\Phi^{-1}(1 - p_t) < \Phi^{-1}(1 - p_k).$$

Defaulted companies are knocked out from the contract.

7. Repeat and indicate all the defaults in the portfolio and the loss given default is mapped for this trial.

We illustrate the idea behind the detailed calculations of Delta and Gamma cure with a simple example.

As shown in Table (5.1), we start with a homogeneous portfolio of three companies A_1, A_2 and A_3. The number of time steps we have in this example is four, the cumulative number of defaults given in the entry of L_t, and the adjusted default is indicated by L_t'. Here the adjusted default stands only for the manually adjusted 'imaginary' defaults happened to the portfolio, so that the portfolio delta can be calculated after the spread is calculated using both expected losses.

Table 5.1: **Trial Example of Portfolio with Three Companies over Four Periods**

	T_1	T_2	T_3	T_4
A_1			1	
A_2				
A_3	1			
L_t	1	1	2	2
L'_t	2	2	3	3

It is obvious that for any given default probability and observed payment time, the calculation of spread is straightforward. To observe the spread change using a new plug in of default probability driver will cause inconsistencies in the model itself. Meanwhile, suppose we arrive to just one step before the five year maturity of a CDO, the spread one may agree to pay is only the expected loss during this one quarter against the remaining portfolio value. No one would rationally care about the default time of previous defaults as long as the defaulted names are kicked out of the CDO.

Coming back to the simulation procedure described previously, we add this 'extra' default to the existing simulation result and finish this trial, then repeat until the collected simulation sample size is large enough, in this study, the simulation standard error is customized to be 1%, and the relative number of simulation trials can be calculated using: standard error= $\frac{\sigma_s}{\sqrt{n}}$, where n is the number of trials and σ_s is the sample standard deviation of the CDO tranche spreads.

8. Track the defaults at each time step, then given the default and value write down of the whole portfolio, re-calculate aggregated loss over remaining time from this extra default.

9. Calculate the expected time t asset value for all companies after loss from extra default is deducted.

10. Find out the spread for each tranche and store data for further use (gamma, hedge ratio, etc.).

11. Repeat the above process for a large number of times and average the final tranche spread to find out the spread change over time.

As for hedging instruments, we use the CDS basket containing the same names covered by the CDO portfolio. The CDS contracts are priced by equalizing the two cash-flow legs in line with the default probability over the whole time interval. Similarly, 'imaginary' defaults are adjusted according to each name at each time step.

As we now have the value and default-adjusted value of both our CDO and CDS products, we can derive the time dependent deltas by finding the difference between the two values at each time step, thus obtaining the hedge ratio.

In this framework, the probability of default and loss on tranches are all available to us, both products can then be valued on the predetermined payment dates, and the Greeks are finally calculated. Meanwhile, the framework allows for multiple defaults on each payment day, and as an open specification of a dynamic structural model, extra assumptions can be easily incorporated to finally obtain the spread distribution of either each individual tranche or the CDO product on bulk.

5.5 Numerical Results

In this secion, we demonstrate the calibration of a five-year iTraxx Europe CDO contract. The total number of underlying names is 125 and the five considered tranches are: 0-3% *Equity*, 3%$\sim$6% *Junior Mezzanine*, 6%$\sim$9% *Senior Mezzanine*, 9%$\sim$12% *Senior* and 12%$\sim$22% *Super Senior*. The observed payment dates are set quarterly and the risk-free interest rate is flat on 5% while the revoery rate is fixed at 40%.

5.5.1 Credit Spread

For the simulation needs, the coupon rate c is set to be homogeneous for all underlying companies at 1.6% based on market data for 31^{st} Aug 2008, the drift factor a and volatility σ are set 1.25% and 0.85% respectively, using the same dataset as for the coupon rate c. The total number of simulation trials is $100,000$, and the final spread is calculated as the average of all occuring tranche spreads.

The time dependent result for the CDO spread is given in basis point in the table below:

Quarters	**0%$\sim$3%**	**3%$\sim$6%**	**6%$\sim$9%**	**9%$\sim$12%**	**12%$\sim$22%**
Q1	2440.31	369.13	225.49	160.19	118.84
Q2	2971.89	552.41	285.29	188.53	130.71
Q3	1526.71	787.19	390.16	229.29	145.09
Q4	351.90	794.17	567.02	294.07	162.91
Q5	51.38	454.51	733.76	405.50	185.67

Continued on next page

Table 5.2 – continued from previous page

Quarters	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Q6	6.13	161.60	682.89	563.98	215.60
Q7	0.94	41.88	423.01	670.17	254.48
Q8	0.30	8.66	183.65	606.31	301.10
Q9	0.07	1.98	63.29	399.89	352.63
Q10	0	0.38	18.71	202.81	409.62
Q11	0	0.08	5.91	88.54	475.39
Q12	0	0.03	1.75	35.87	554.06
Q13	0	0.01	0.48	14.36	642.72
Q14	0	0	0.19	5.52	717.76
Q15	0	0	0.06	1.93	729.88
Q16	0	0	0.03	0.84	645.25
Q17	0	0	0.0148	0.27	501.17
Q18	0	0	0.0113	0.11	345.24
Q19	0	0	0.0043	0.0502	213.91
Q20	0	0	0.0019	0.0241	120.81

Table 5.2: **Discrete CDO Spreads on Payment Dates.**

We can see from above Table (5.2) that the Equity tranche vanishes around the middle of the third year of the contract. The Junior Mezzanine tranche vanishes around mid-fourth year, while the remaining tranches have a bigger chance to survive through out the contract life time.

To better understand the simulation results, we take some snap shots of the spreads in scatter output as shown in Figure (5.1) below:

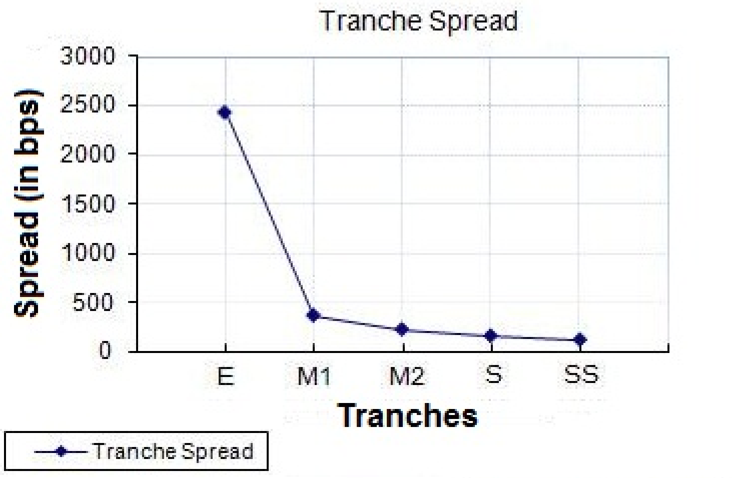

Figure 5.1: *Trading spread at the beginning of contract.*

Next, we put the spreads from Table (5.2) of the five observed tranches during the contract time together, to obtain Figure (5.2):

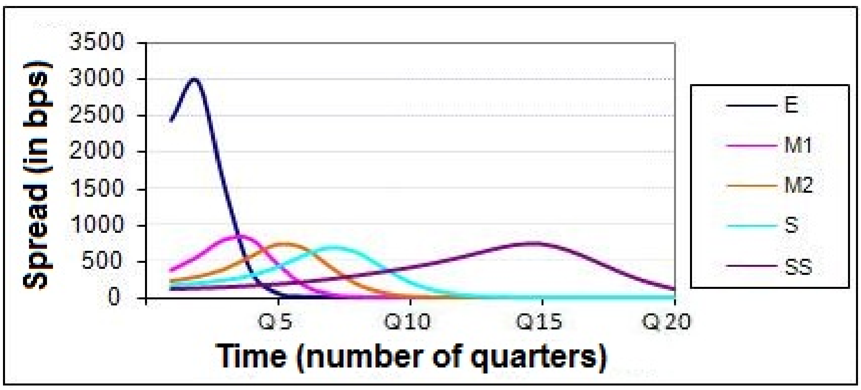

Figure 5.2: *Simulated Tranche Spreads Till Maturity.*

It is easy to see that the tranche prices over time are skewed, and the 'peaks' of the first four tranches take place about every half a year. As our results are obtained via simulation, the spreads are calculated as the ratio of the expected cumulative loss from the observed payment period and the tranche notional outstanding for the same period.

In the meantime, as we are also trying to observe the change of spreads at each time step on a spread-per-default basis, the time t_i CDO spread of each tranche with an extra default added on the payment date is given in Table (5.3). This would enable us to obtain the Delta factor of the CDO spread.

Quarters	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Q1	2766.64	409.01	240.83	168.19	123.36
Q2	2866.63	628.89	310.33	199.62	136.09
Q3	990.23	844.96	436.09	245.91	151.64
Q4	167.16	721.13	631.64	321.97	171.07
Q5	19.29	334.56	760.79	452.41	196.18
Q6	1.99	100.33	618.87	616.06	229.46
Q7	0.34	21.98	332.71	679.97	272.16
Q8	0.06	4.29	126.68	554.14	322.05
Q9	0	0.87	39.64	327.01	376.51
Q10	0	0.16	11.14	149.92	437.82
Q11	0	0.0443	3.53	61.85	509.23
Q12	0	0.0369	0.8872	23.91	592.88
Q13	0	0.0063	0.2143	9.08	682.32
Q14	0	0	0.0884	3.18	738.68
Q15	0	0	0.0435	1.33	715.98

Continued on next page

Table 5.3 – continued from previous page

Quarters	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Q16	0	0	0.0362	0.4275	597.52
Q17	0	0	0.0161	0.1685	434.42
Q18	0	0	0.0039	0.0665	283.98
Q19	0	0	0.0019	0.0269	167.47
Q20	0	0	0.0008	0.0035	92.13

Table 5.3: **Spreads with Added Default.**

Comparing the numerical results given in Tables (5.2) and (5.3), we observe a significant increase in spreads of the same tranche at the same time. In other words, one could expect in Table (5.3) that the loss will be larger and the extra defaults 'consume' more of the notional value in the tranche. The numerical results given in above Table (5.3) are plotted in the figure below:

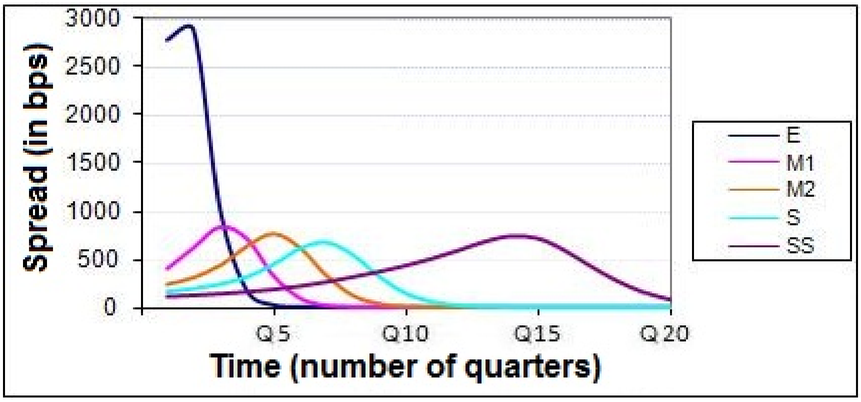

Figure 5.3: *Default Adjusted Tranche Spreads for CDO Delta.*

Comparing the 'peak' positions of the spreads over time, we may see that there is a sharper decrease in the Equity tranche. Here we see that the value of the highest spread of each tranche over contract time is higher than the results in the previous Table (5.2), yet the time when these highest spreads occur is earlier or at least remain the same. This results a sharper decrease in the spreads which greatly affect the value of the hedging factors.

For simulations on 'extra defaults', we move on along the tree branch and observe the default-able nodes. The spreads in Table (5.4) below are calculated using same model and structure as before:

Quarters	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Q1	3072.51	457.84	258.29	176.94	128.05
Q2	2484.21	712.91	340.04	211.97	141.72
Q3	536.09	874.91	490.51	264.98	158.54
Q4	67.11	611.57	696.37	354.98	179.78
Q5	5.86	228.17	759.38	504.82	207.56
Q6	0.53	58.01	534.38	662.24	244.52
Q7	0.1619	10.89	247.28	670.71	291.04
Q8	0.036	1.87	84.99	484.72	344.04
Q9	0	0.4257	23.48	254.21	401.72
Q10	0	0.0394	6.07	106.71	468.23
Q11	0	0.0143	1.92	41.46	545.27
Q12	0	0.0024	0.4581	15.27	633.48
Q13	0	0	0.1277	5.59	715.67
Q14	0	0	0.0467	2.01	746.27
Q15	0	0	0.0382	0.6816	686.26
Q16	0	0	0.0168	0.2162	539.24
Q17	0	0	0.0052	0.0963	366.81

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Table 5.4 – continued from previous page

Quarters	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Q18	0	0	0.0039	0.0518	227.72
Q19	0	0	0.0012	0.0046	129.51
Q20	0	0	0.0009	0.0034	69.49

Table 5.4: **Spreads with Added Default for Gamma.**

One may observe that when extra defaults are occurring at each time step, the left-moving (right-skewing) effect continues. The spread of the Equity tranche column in Table (5.4) is strictly decreasing this time, and is wiped out during the same payment period, i.e., by the end of 2^{nd} year, but with a lower ending spread.

Similar results are found for the higher tranches as well. Therefore, it is safe to conclude that the left-moving (right-skewing) effect continues, here for Junior Mezzanine, Senior Mezzanine, Senior and Super Senior tranches, with the 'peaks' occuring at the 3^{rd}, 5^{th}, 7^{th} and 14^{th} quarters respectively.

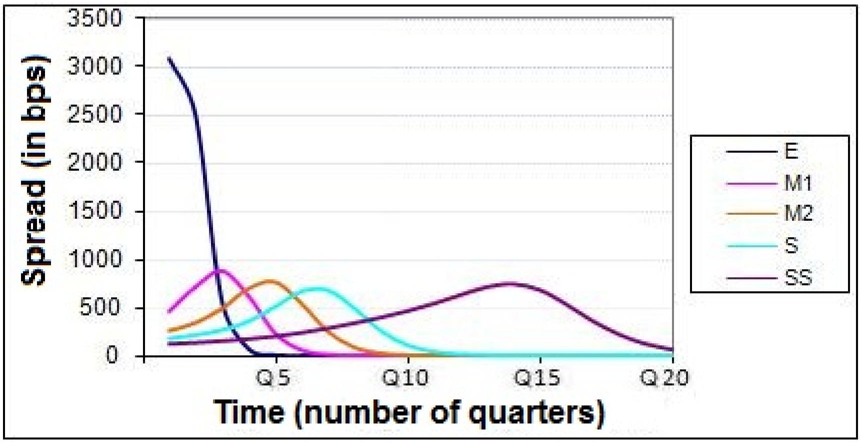

Figure 5.4: *Default Adjusted Tranche Spreads for CDO Gamma.*

One may notice that in Tables[4] (5.3) and (5.4), the highest spreads for tranches from Junior Mezzanine to Super Senior are found at the same payment period and further we observe that on the left tail[5] the extra default results to a larger spread due to larger loss, but on the right tail until the end of contract, the spread in Table (5.4) is lower than that in Table (5.3).

5.5.2 Hedging Factors

Before we move on to the hedging factors, we first observe the time and default dependent spread change by tranche-wisely graphing the default based spread curves together, figures (5.5) - (5.9) plot the evolution of the 5 tranche spreads.

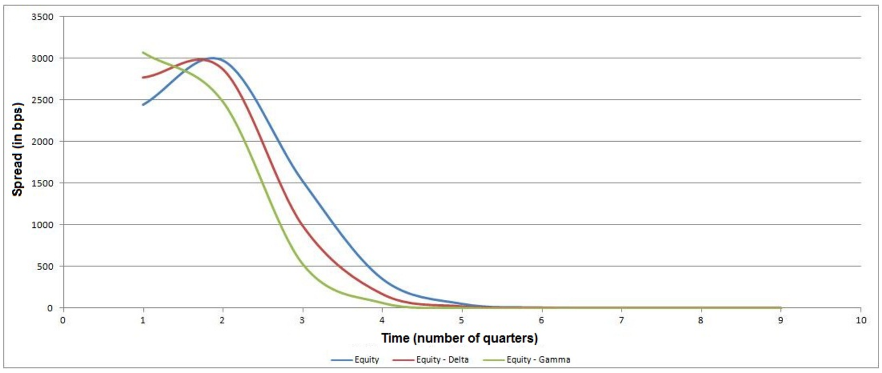

Figure 5.5: *Equity Tranche Spread.*

[4]Data plots shown in Figures (5.3) and (5.4)

[5]on the earlier payment dates.

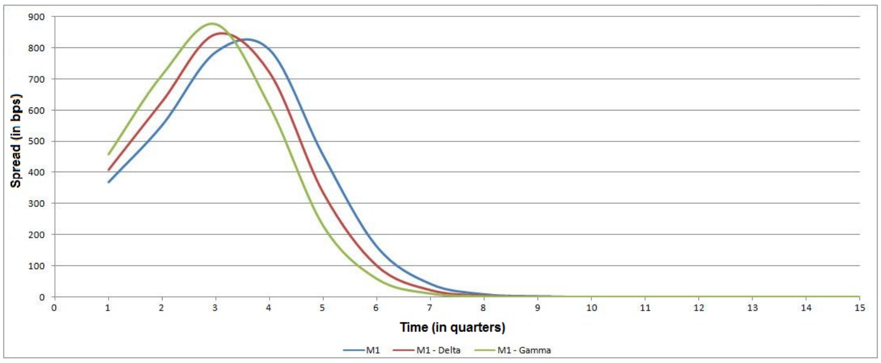

Figure 5.6: *Junior Mezzanine Tranche Spread.*

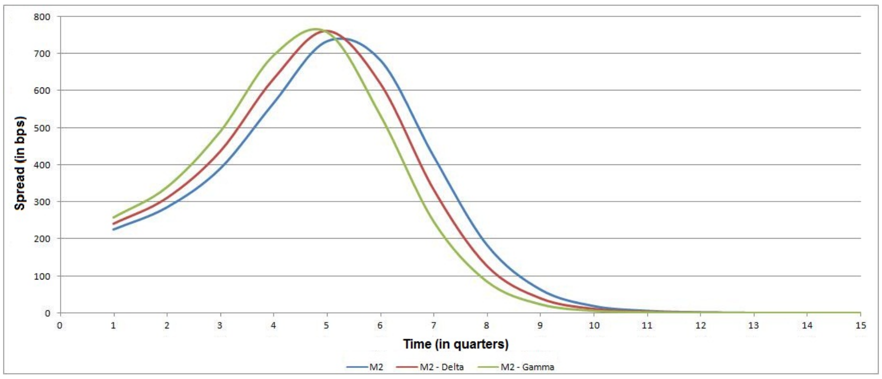

Figure 5.7: *Senior Mezzanine Tranche Spread.*

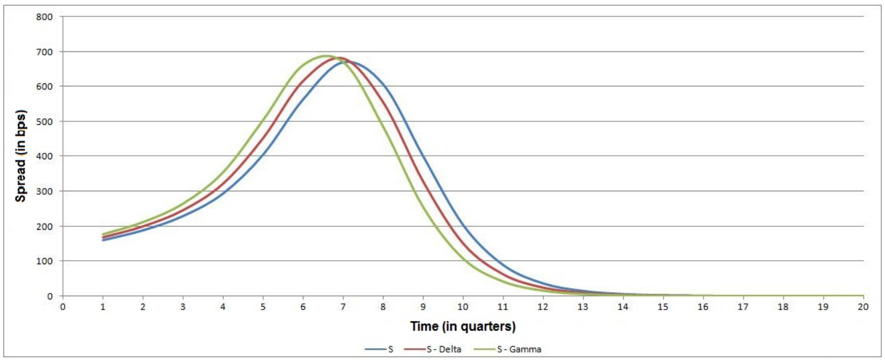

Figure 5.8: *Senior Tranche Spread.*

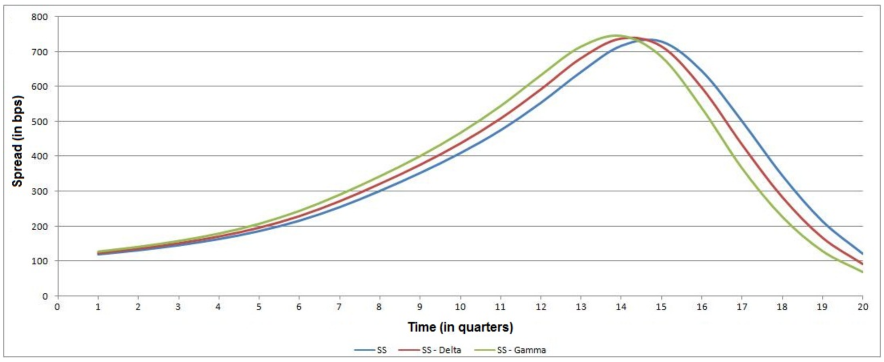

Figure 5.9: *Super Senior Tranche Spread.*

CDO Delta

For the Equity tranche the difference between the 'Equity' curve and 'Equity - Delta' curve is first increasing and then converge to zero. A similar pattern can be found also in the right tail of all the other 4 tranches as shown in Figure (5.6) to Figure (5.9). The largest gap between the original and 'Delta' curves is in the shoulder part, where one can conclude an up-then-down movement in 'Delta' on the right hand side of the 'peaks'.

For the Equity tranche, only a right tail is evidenced in Figure (5.5). Furthermore, there is a sharp drop within one year from the time that the highest spread occurs. In this case, the larger default probability will result to a sharper decrease of the tranche spread over time. Thus, one may expect a big delta change during the extreme market conditions that we examine.

As for the left tails of the tranches except the Equity tranche, as shown in Figure (5.6) to Figure (5.9), the largest gap takes place in the shoulder part as well, the difference this time is obtained in negative values due to the lower position of the original curve. Thus, we see a negative to positive sign change in CDO Delta which is coincide with the position of highest spreads - the 'peaks'. In other words, the first positive change in spread indicates the 'head' while the largest difference on the shoulders is positioned within half a year's time for each of the four tranches respectively.

For the evaluation of the CDO Delta, we calculate the difference between the tranche spreads and adjusted tranche-delta spreads over the entire observed time period of *20 Quarters*, denote the CDO Delta for the ith quarter as Δ_i for $i = 1, \ldots, 20$, as given in Table (5.5).

Δ	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Δ_1	105.27	-76.48	-25.04	-11.09	-5.38
Δ_2	536.47	-57.77	-45.93	-16.62	-6.54
Δ_3	184.75	73.03	-64.62	-27.90	-8.16
Δ_4	32.09	119.95	-27.03	-46.91	-10.51
Δ_5	4.13	61.28	64.03	-52.07	-13.86
Δ_6	0.60	19.91	90.29	-9.81	-17.68
Δ_7	0.27	4.37	56.96	52.17	-20.95

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Table 5.5 – continued from previous page

Δ	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Δ_8	0.072	1.11	23.66	72.87	-23.87
Δ_9	0	0.2214	7.61	52.89	-28.19
Δ_{10}	0	0.0295	2.38	26.69	-33.84
Δ_{11}	0	-0.0048	0.8578	11.96	-38.82
Δ_{12}	0	-0.0060	0.2703	5.27	-39.61
Δ_{13}	0	0.0024	0.1042	2.33	-20.91
Δ_{14}	0	0	0.0192	0.5961	13.89
Δ_{15}	0	0	-0.0031	0.4092	47.73
Δ_{16}	0	0	-0.0047	0.1025	66.75
Δ_{17}	0	0	0.0011	0.0473	61.25
Δ_{18}	0	0	-0.0016	0.0233	46.44
Δ_{19}	0	0	-0.0033	-0.0011	28.68

Table 5.5: **Delta for CDO Product.**

As shown in Table (5.5), the $0 \sim 3\%$ Equity tranche vanished between quarter 8 and quarter 9 with a positive Delta. For the $3 \sim 6\%$ Junior Mezzanine tranche, we observe two negative values at quarters 11 and 12, then the tranche is wiped out between quarters 13 and 14 with a positive Delta of 0.0024 at Q13. For the $6 \sim 9\%$ Senior Mezzanine tranche, the delta value is negative from Q15 to Q20 except Q17. The negative Deltas of the two Mezzanine tranches before maturity are obtained due to the fat-tail effect. The two positive 'noises': Δ_{13} of Junior Mezzanine tranche and Δ_{17} of Senior Mezzanine tranche mentioned above could possibly obtained due to simulation error. This error is no longer found after application of variance reduction technique[6], considering that the error reduction in the simulated scenarios has a square root convergence, we also increased the number of sample paths to 500,000 to meet the requirement of large sample paths in order to obtain an accurate estimation.

For the Junior Mezzanine tranche spread in Figure (5.6), the spreads given for the first quarter of the contract time are on the left 'Torso' of the curve, thus the large negative delta in the beginning of Junior Mezzanine column. In the meantime, we observe from Figure (5.9) that there is hardly a right tail for the Super Senior tranche, hence we obtained a series of negative value before the fourteenth quarter.

The CDO Delta values are summarized as:

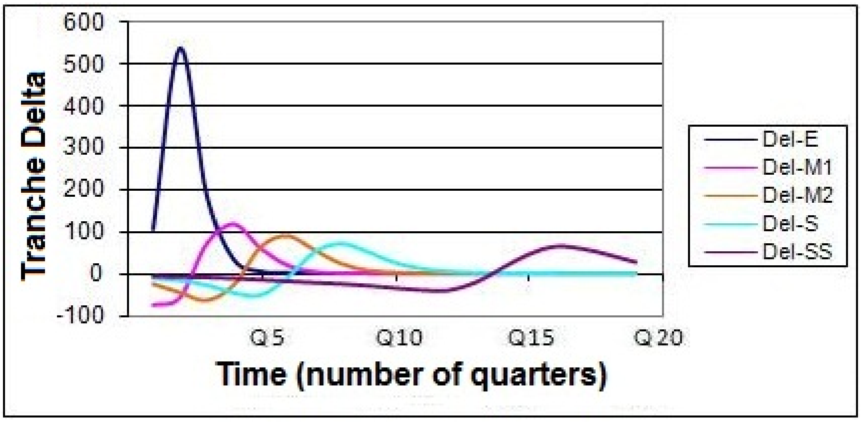

Figure 5.10: *Delta for CDO Product.*

We see that the peak values of the different tranches vary, yet there is a positive relationship between the number of defaults at the observed time period and the high spreads over contract time.

[6]The *Antithetic Variates* method in particular. The random number Z_t generated for $x(t)$ is considered as $\|Z_t\|$, thus the absolute value is used twice as Z_t and $-Z_t$ for generating two different $x(t)$s.

CDO-Gamma

In order to depict how delta is evolving before maturity, we look into the Gamma value of the spreads, Gamma is defined as the difference in Deltas:

$$\Gamma_t = \Delta_{t+1}^{T-\Delta} - \Delta_{t+1}^{\Delta-\Gamma}$$

where:

$$\Delta_t^{T-\Delta} = S_{t+1}^{T} - S_{t+1}^{\Delta}$$

and

$$\Delta_t^{\Delta-\Gamma} = S_{t+1}^{\Delta} - S_{t+1}^{\Gamma}$$

The results for the same observation period used for the Delta calculations are presented in the Table (5.6) below:

Γ	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Γ_1	82.33	-27.82	8.50	2.45	0.3613
Γ_2	84.70	-36.53	0.1035	5.11	0.5540
Γ_3	18.65	13.55	-28.44	5.50	0.8721
Γ_4	2.66	18.96	-20.46	-5.88	1.21
Γ_5	0.4219	8.83	4.87	-19.07	1.20
Γ_6	0.2673	1.98	15.28	-17.25	1.04
Γ_7	0.072	0.6739	7.51	0.0713	1.34
Γ_8	0	0.0999	2.57	9.68	2.21
Γ_9	0	0.0264	0.7646	6.29	2.20
Γ_{10}	0	-0.0178	0.4286	3.32	1.78
Γ_{11}	0	-0.0420	0.1837	1.77	-6.25
Γ_{12}	0	0.0024	0.0625	1.15	-13.32
Γ_{13}	0	0	0.0138	-0.0549	-15.84

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Table 5.6 – continued from previous page

Γ	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Γ_{14}	0	0	-0.0224	0.1978	-10.55
Γ_{15}	0	0	-0.0156	0.0304	-0.8635
Γ_{16}	0	0	0.0009	0.0326	4.99
Γ_{17}	0	0	-0.0016	0.0010	8.48
Γ_{18}	0	0	-0.0033	0.0018	6.03

Table 5.6: **Gamma for CDO Product.**

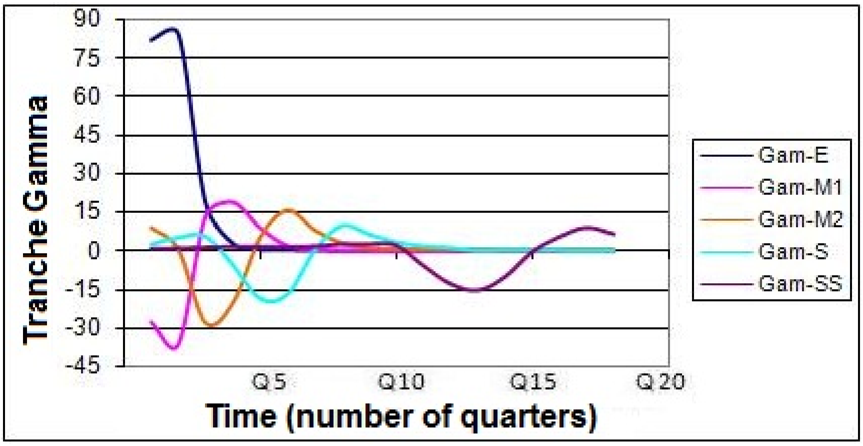

Figure 5.11: *Gamma for CDO Product.*

For the CDO Gamma, one may conclude that as the default probability increases, the sharper is the rise and fall in the spreads and thus a faster movement of the whole curve to the left hand side is observed. Further, as shown in Figure (5.11), the direction of movement and the position of 'peak' values again coincide with those of the CDO Delta and the spread curves.

CDS Basket Delta

In order to obtain the hedging ratio, the last step is to calculate the Delta factor of the hedging instrument, the CDS basket in our analysis. The basket CDS delta is calculated by taking the difference of the averaged values of the single name CDS spreads with and without the adjusted extra default on payment dates.

The average CDS spread and CDS delta are given in Table (5.7) below:

Quarters	**Spreads**	**CDSΔ**
Q1	332.28	26.86
Q2	305.42	25.53
Q3	279.89	24.16
Q4	255.73	22.55
Q5	233.18	20.43
Q6	212.75	17.89
Q7	194.87	15.11
Q8	179.75	13.32
Q9	166.43	12.41
Q10	154.02	11.09
Q11	142.92	9.65
Q12	133.27	7.66

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Table 5.7 – continued from previous page

Quarters	**Spreads**	**CDS - Δ**
Q13	125.61	6.46
Q14	119.15	5.53
Q15	113.62	4.14
Q16	109.48	3.04
Q17	106.44	1.91
Q18	104.54	0.91
Q19	103.63	0.33
Q20	103.30	–

Table 5.7: **Discrete CDS Spread and Delta on Payment Dates.**

Note that the spreads in this section are calculated for hedging purposes only. Thus the spreads in the figures are the simulated trading spreads of a 5-year contract traded at a generic time t with maturity still fixed at five years as seen from time 0. For instance, in Figure (5.12) below, the spread value of 154.02 for **Q10** represents the value of a 5-year CDS contract with only two and half years left to maturity.

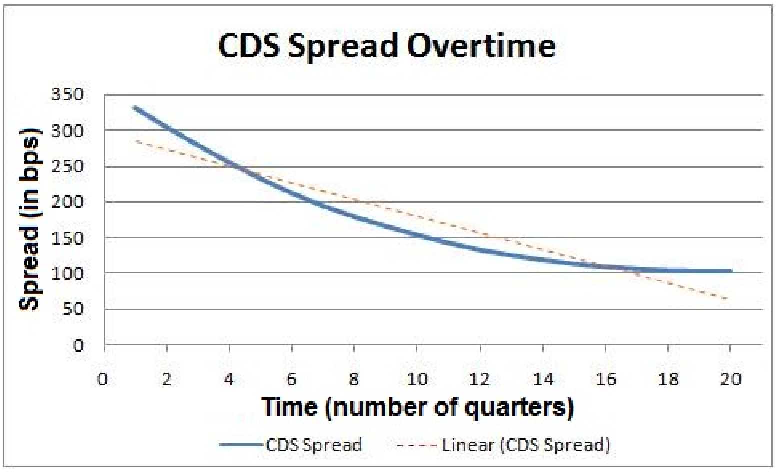

Figure 5.12: *Average single CDS spread.*

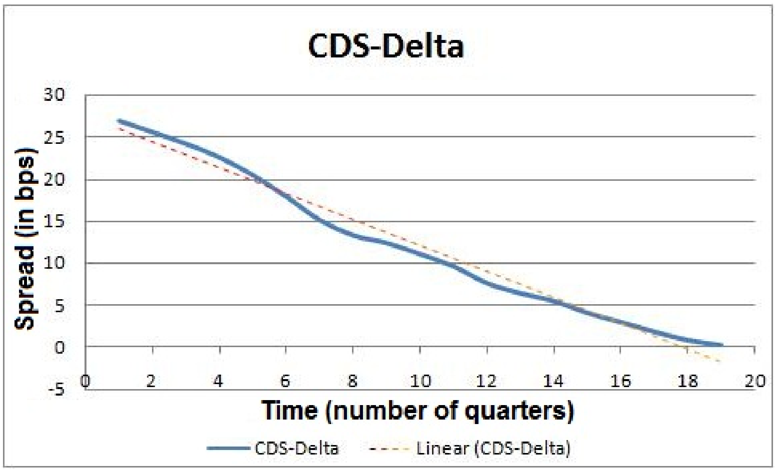

Figure 5.13: *Average single name spread delta of the basket.*

With the CDS Delta shown in Figure (5.13), we are now able to calculate the hedging ratio of the entire portfolio, in similar ways as the widely recognized portfolio *Delta* and *Gamma* for equity derivatives. The hedging ratio is given by:

$$\Delta_P = \frac{\Delta_{CDO}}{\Delta_{CDS}} \quad \text{while} \quad \Gamma_P = \frac{\Gamma_{CDO}}{\Delta_{CDS}^2} \tag{5.17}$$

Numerical results are shown below:

$\Delta(\times 10^{-3})$	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Δ_P^1	1045.1	-759.3	-248.6	-110.1	-16
Δ_P^2	5326.1	-573.5	-455.9	-165	-19.4
Δ_P^3	1834.2	725	-641.6	-276.9	-24.3
Δ_P^4	318.6	1190.8	-268.3	-465.7	-31.3
Δ_P^5	41	608.4	635.7	-517.9	-41.28
Δ_P^6	5.9	197.6	896.4	-97.4	-52.6
Δ_P^7	2.7	43.4	565.5	517.9	-62.4
Δ_P^8	0.7	11	234.9	723.4	-71.1
Δ_P^9	0	2.1	75.6	525.1	-83.9
Δ_P^{10}	0	0.3	23.7	264.9	-100.7
Δ_P^{11}	0	-0.05	8.5	118.7	-115
Δ_P^{12}	0	-0.06	2.68	052.3	-117.9
Δ_P^{13}	0	0.023	1.03	23.13	-62.28
Δ_P^{14}	0	0	0.19	5.9	41.37
Δ_P^{15}	0	0	-0.03	4.06	142.16
Δ_P^{16}	0	0	-0.047	1.018	198.8
Δ_P^{17}	0	0	0.011	0.473	182.4
Δ_P^{18}	0	0	-0.016	0.23	138.3
Δ_P^{19}	0	0	-0.033	-0.011	85.4

Table 5.8: **Delta-hedge Ratio for CDO Product. (Ratio in** 10^{-3})

$\Delta(\times 10^{-3})$	**0%∼3%**	**3%∼6%**	**6%∼9%**	**9%∼12%**	**12%∼22%**
Γ_P^1	304	-102	31.4	9.1	0.4
Γ_P^2	313.1	-135	0.38	18.9	0.6
Γ_P^3	68.9	50.1	-105.1	20.33	0.96
Γ_P^4	9.83	70.1	-75.6	-21.7	1.34
Γ_P^5	1.55	32.6	18	-70.5	1.33
Γ_P^6	0.98	7.3	56.5	-63.8	1.15
Γ_P^7	0.266	2.5	27.8	0.264	1.48
Γ_P^8	0	0.369	9.5	35.8	2.45
Γ_P^9	0	0.0976	2.8	23.3	2.4
Γ_P^{10}	0	-0.066	1.6	12.3	1.97
Γ_P^{11}	0	-0.16	0.68	6.54	-6.9
Γ_P^{12}	0	0.0088	0.23	4.3	-14.8
Γ_P^{13}	0	0	0.051	-0.2	-17.6
Γ_P^{14}	0	0	-0.083	0.73	-11.69
Γ_P^{15}	0	0	-0.058	0.11	-0.958
Γ_P^{16}	0	0	0.003	0.12	5.53
Γ_P^{17}	0	0	-0.0059	0.0037	9.4
Γ_P^{18}	0	0	-0.012	0.0066	6.69

Table 5.9: **Gamma-hedge Ratio for CDO Product.(Ratio in 10^{-4})**

The plots of the hedging ratios (CDO Delta and CDO Gamma) are depicted in Figure (5.14) and (5.15):

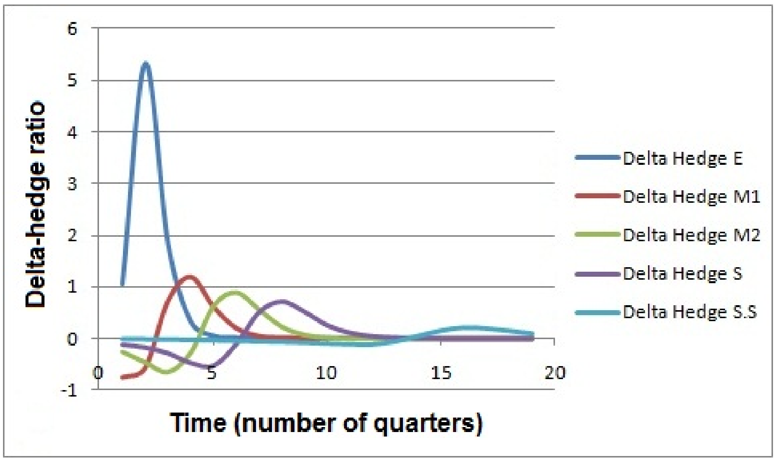

Figure 5.14: *Hedge Ratio Plot for CDO Delta.*

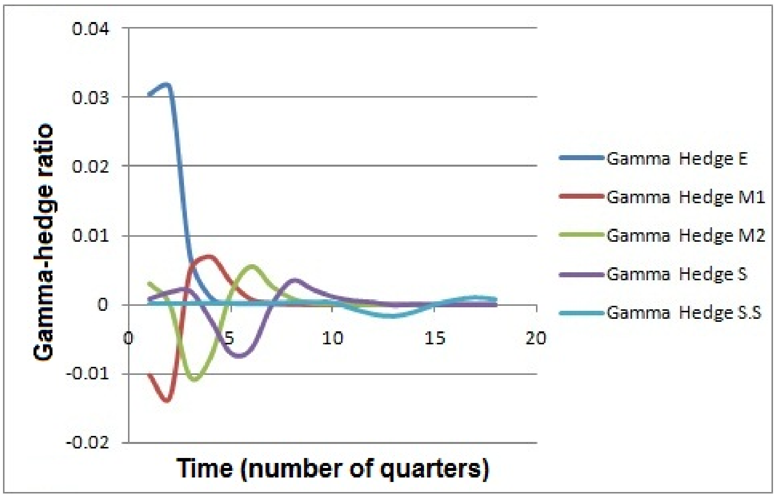

Figure 5.15: *Hedge Ratio Plot for CDO Gamma.*

The hedging ratios in Tables (5.8) and (5.9) are expressed in the same way as Delta/Gamma hedging ratio of traditional derivatives such as options. In other words, a portfolio Delta hedge ratio of 5326.1×10^{-3} on $0\%\sim3\%$ tranche at the 2^{nd} quarter means the price of the hedging derivative (CDS basket in this case) will rise about 5.3% if the price of the underlying CDO increases. Therefore, with the time dependent Delta and Gamma hedge ratios obtained, portfolio managers are now able to actively adjust the proportion of CDS basket in the whole portfolio to reach a ‘Delta Neutral position. Considering the fact that current market standard model is not able to provide the time evolution of spread, here we finally have an approach to attempt dynamic portfolio management for portfolios of complex credit derivatives.

To conclude, a novel contribution of our analytic and computational methodology is that with the hedging parameters in hand, one may construct efficient hedging strategies based on different risk exposure and tolerance of possible loss. This significantly advances the relevant literature and provides flexibility in the management of structured portfolios of credit products.

Chapter 6

Summary, Conclusions and Suggestions for Further Research

6.1 Synopsis of the Thesis

In chapter 2 we reviewed major credit default products and structured portfolio credit products in detail and proposed a new Monte Carlo simulation framework for the pricing of *Collateralized Debt Obligations* (CDOs).

We first provided an overview of the structure of the main building blocks which come in the form of *Credit Default Swaps* (CDSs) and how they work as a risk transferring tool in the finance industry. Two other types of credit portfolio products were subsequently reviewed and the introduction of main stream credit risk models was provided. We carried on by summarizing the common simulation methods for pricing CDOs, and then we proposed an alternative methodology that is based on an economical sense of the models and market observables. Such simulation method provides a testing environment which houses the asset value based models with reliable assumptions. In later sections of Chapter 2, detailed implementation procedures were listed together with numerical results compared to market data.

We mainly focus on the utilization of default probabilities and recovery rates through our simulation framework. One important conclusion is that dynamic models can be discretely simulated using marginal default probabilities and structural models thus can be calibrated as a barrier based model.

It is well known that static models of credit risk are inadequate to meet the increasing demand for hedging credit derivatives since they fail to track the credit risk profile of a structured portfolio over multiple time periods. Chapter 3 provided an overview of the credit market together with the *Principal Component Analysis* (PCA). Our findings suggest that although the market standard model is powerful in many aspects, it is not performing well during extreme market conditions such as the current credit crunch. The reason is that although one can always increase the implied correlation to obtain a more accurate spread according to a particular distribution, the concomitant result is in increase in the “noise”.

In chapter 4, we proposed a dynamic credit risk model based on asset growth rate. The model is easy to calibrate as inputs are either obtained directly or easily derived from market data. It turns out that our framework can accommodate both bearish and bullish credit markets and fits market quotes reasonably well. We provide two alternative candidates for default conditions and we evaluated them. We illustrated our model with a CDO-type contract. As a dynamic structural model, our approach does not rely on certain types of distributions. Notably, further extensions can be made to assess the effects of exogenous factors such that pairwise correlation and interest rate.

Following the recent market slide and the ongoing credit crunch, our belief is that an “old school” - type intuitive approach will be valuable for market practitioners who are increasingly focusing on new routes to mitigate and hedge risk exposure. Thus, we test drive our model in chapter 5 with the multi-step simulation framework to demonstrate the time evolution of spread tranches for a CDO type product. The numerical results suggested the presence of a unified spread change for all tranches and a ‘right tailing’ effect on per default basis. We believe that for the first time in the literature the time evolution of the hedge ratio between a basket of simple CDS contracts and a CDO from a dynamic model is provided. This type of information, in analogy to the expected future cash-flow/interest rate for fixed income derivatives is crucial since players in the credit field will be able to combine simple instruments with portfolio credit products for hedging purposes.

6.2 Summary of our Contributions

In order to improve the dynamic aspect of portfolio credit models, our very first idea of development has been inspired by interest rate models: In the early 1990s, the Heath-Jarrow-Merton (HJM) framework was developed under the principle of taking the market forward rate term structure as given, and eventually model the no-arbitrage evolution of entire forward yield curve. As it is well known today, the HJM framework is widely used to price exotic interest rate derivatives.

During the last decade, new credit portfolio products have been developed where the payoff depends not only on the times of default but also on the credit spread levels. One such product is an European option on a single tranche CDO. In order to address complex credit derivative products, a totally different to the standard market model approach should be developed that takes into account the time evolution of tranche spreads.

Our primary efforts in this thesis focus on the development of a valuation and hedging framework that meets the following requirements:

- Fit all quoted tranche spreads across both the term and capital structure.
- Offer an unified time evolution of tranche spreads.
- Explain tranche spreads using historical movements.
- Take into account how defaults in the future are likely to affect tranche spreads and the volatility of spreads.
- Most importantly, develop a relatively simple and intuitive model which is realistic in its parameterization and efficient from a computational perspective.

In this thesis, our first goal was to create a discrete multi-step Monte Carlo simulation framework which houses different assumptions from the main incumbent models. The framework was designed to work with cumulative asset loss models such as the Merton (1974) asset value model. However, one of our main contribution is that through the discrete steps of the simulation one may construct the loss distribution in a manner similar in nature but more pragmatic than the Gaussian copula model. Detailed analysis of the nature of this framework shows that the simulation process is capable of handling computation tasks under multi-state assumptions, hence the time evolution of both spreads and spread based ratings are captured.

Second, we proposed in Chapter 4 a new approach for modelling portfolio credit risk under the *Dynamic Growth Rate Model*. The novelty of our attempt is that, this model is constructed using the expected asset growth rate. From the principal component analysis in Chapter 3, we observed that the market is generally underestimating the risk embedded in complex portfolio credit derivatives. While the popular copula models focus on correlation and types of copula functions, we chose instead to model default probabilities. For example, for a CDO type product, the market standard one factor Gaussian copula model implies one correlation factor to price five tranches. Wagner, Bluhm & Overbeck (2003) has shown that changing the implied correlation parameter results in different default distribution for each of the individual tranches. However, the expected loss of the entire underlying pool of the CDO is not affected. A further yet more important problem of taking dynamic correlation as input in factor models is that the cumulative loss distribution under changing correlation might decrease in certain cases, thus the model is logically violated and encounters arbitrage issues.

We derived a close-form formula for the probability of default. We further carried out a numerical test based on input parameters we retrieved from market data. This time we compared the simulation results with market spreads only, and the analysis proved that our model is capable of capturing market spreads in both bullish and bearish markets, although results suggest a slightly higher tolerable spread compared to the recorded market data at the time of this study.

The Strength of our framework is shown in Chapter 5 where we combine the multi-step Monte Carlo simulation process with the dynamic growth rate model to generate tranche spreads for any tenor across the life of the contract.

6.3 Conclusions

In the thesis we demonstrated that our multi-step Monte Carlo simulation framework together with the dynamic growth rate model is capable to address time dimension of the modeled product, in other words, one may now observe the time evolution of tranche spreads in the case of pricing a CDO type portfolio credit derivative.

The numerical delta and gamma analysis of the hedging strategy using a basket of CDS contracts shows that the tranche spread change is well captured in the case of unexpected defaults.

Stochastic growth rate and the resulted stochastic default threshold allow the portfolio loss to increase in times of market depression while allowing idiosyncratic risk to determine the health of a firm in times of market prosperity. The model produced CDO tranche spreads that were very close to those observed in the CDO market. An explicit expression for the stochastic default probability was found and a closed form solution was derived for the portfolio loss distribution for a large homogeneous portfolio. Closed form expressions were also found for the expected loss on a tranche that allows rapid pricing of CDO tranches.

The growth rate model presented in this thesis is, in theory, capable of pricing any portfolio credit derivative where the payoff is a function of the default times and default probability, with recovery rate and risk free rates as model inputs. It is always possible to enrich our model with more complex assumptions such as time depending stochastic recovery rates or interest rates. We opted not to do this in our work because we believe that simpler and tractable models better suit to address the credit derivatives market.

6.4 Future Research

Our research points to a number of interesting issues that can be addressed in the future. One possible extension is related to the scope of optimization. As discussed in Chapter 1 section (1.4), hedging a credit risky CDS with a bond would result a change in wealth in one's trading book as:

$$V = -V_b + (RN_b + (1 - R)N_{cds})e^{-r\tau} + \sum_{t=0}^{\tau}(cN_b - sN_{cds})e^{-rt}$$

where V is the value of portfolio, V_b is the value of bond, N_b is the bond notional, N_{cds} the CDS notional, R is the rate of recovery at the default time τ, c here is the bond coupon rate and s is the CDS spread.

Meanwhile, choosing CDS or CDS index contracts as hedging instruments will end up having the change of wealth as a periodic difference in the 'premium'. Furthermore, hedging strategies using CDS options one need to deduct the cost of option from the premium as well.

For example, if the portfolio consists of a long position in a risky bond with notional N_b, the hedge will be composed by shorting a CDS with notional N_{cds} and longing a CDS option with notional N_{op}, where $N_{cds} = N_b + N_{op}$. The amount of N_b is fully covered, and the notional of N_{op} is exposed to credit risk subject to the CDS spread. The investor will benefit from a tightening spread but has his loss limited in a bearish market. However, the cost of hedging with CDS options is conditioned on the exercise time and option maturity, more importantly, longing and shorting CDS options may eventually turn the portfolio into an actively managed credit market position, and thus the total change in wealth for a credit portfolio involving CDS options can only be examined according to assumptions on options.

Suppose that defaults take place on payment days[1]:

$$0 < t_1 < t_2 < \ldots < \tau < \ldots < T$$

Here τ is the time of default. For simplicity assume that at-the-money call options are purchased on the payment days with maturity of one fee period, i.e., $T_{op} = t_{i+1} - t_i$. The change in wealth for the above portfolio is:

$$\begin{aligned} V &= -V_b + (RN_b - (1-R)N_{cds} + RN_{op})e^{-r\tau} \\ &\quad + \sum_{0}^{\tau-1}(cN_b + sN_{cds} - s_{t_i}N_{op})e^{-rt_i} \\ &\quad + (cN_b + sN_{cds} - s_{\tau-1}N_{op})e^{-r\tau} \end{aligned}$$

Here V_b is the initial payment for risky bond, R is the recovery rate, V the total change in wealth of the portfolio, r the risk-free interest rate and s is the spread of the CDS contract we sold.

For a multi-name credit default risky product such as CDO, one may hedge using a combination of the underlying bond and/or the underlying CDS and/or options on underlying CDS. If we write down the final expected change in wealth of the whole portfolio, it is possible to perform an optimization search by aiming:

$$E[V] = 0 \quad \text{or} \quad N_{cdo} = N_{cds} + N_b + N_{op}$$

The conditions can be set on the change in wealth equation together with:

$$\begin{cases} N_{cds} \geq 0; \\ N_b \geq 0; \\ N_{op} \geq 0; \end{cases}$$

The line of research, possible combined with optimal value-at-risk measures, as the credit exposure might be covered only partially by aiming $E[V] = 0$, directly extends our findings and can be carried out in the future.

[1] Equations including accrual payments can be derived accordingly.

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