# Heterotic Compactification, An Algorithmic Approach

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**Abstract**

We approach string phenomenology from the perspective of computational algebraic geometry, by providing new and efficient techniques for proving stability and calculating particle spectra in heterotic compactifications. This is done in the context of complete intersection Calabi-Yau manifolds in a single projective space where we classify positive monad bundles. Using a combination of analytic methods and computer algebra we prove stability for all such bundles and compute the complete particle spectrum, including gauge singlets. In particular, we find that the number of anti-generations vanishes for all our bundles and that the spectrum is manifestly moduli-dependent.

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# 1 Introduction

Compactification of the $E_8 \times E_8$ heterotic string on Calabi-Yau three-folds [1, 2] is one of the oldest approaches to particle phenomenology from string theory. Heterotic models have a number of phenomenologically attractive features typically not shared by alternative string constructions. Most notably, gauge unification is "automatic" and standard model families originate from an underlying spinor-representation of SO(10). However, despite its long history and substantial recent progress [3]–[8], heterotic model

building is still a long way away from one of its major goals: finding an example which does not merely have standard model spectrum but reproduces the standard model exactly, including detailed properties such as, for example, Yukawa couplings.

One of the main obstacles in achieving this goal is the inherent mathematical difficulty of heterotic models. In addition to a Calabi-Yau three-fold $X$, heterotic models require two holomorphic (semi)-stable vector bundles $V$ and $\tilde{V}$ on $X$. Except for the simple case of standard embedding, where $V$ is taken to be the tangent bundle $TX$ of the Calabi-Yau space and $\tilde{V}$ is trivial, construction of these vector bundles is often not straightforward and the computation of their properties is usually involved. For example, stability of these bundles, an essential property if the model is to preserve supersymmetry, is notoriously difficult to prove. In addition, when searching for realistic particle physics from heterotic string theory, these mathematical obstacles have to be resolved for a large number of Calabi-Yau spaces and associated bundles, as every single model (or even a small number of models) is highly likely to fail when confronted with the detailed structure of the standard model. The main purpose of this paper is to present an algorithmic approach to this problem by combining analytic methods and computer algebra. By an algorithmic approach we mean a set of techniques which allow us to construct classes of vector bundles on (certain) Calabi-Yau spaces systematically, prove their stability and compute the resulting low-energy particle spectra completely. In this paper we will focus on developing the necessary computational methods by concentrating on the five Calabi-Yau manifolds which can be obtained by intersections in an ordinary projective space. A generalization of these methods to more general complete intersection Calabi-Yau manifolds and a detailed analysis of the particle physics properties of these models will be the subject of forthcoming publications [9].

Starting with the pioneering work in [10, 11], there has been continuing activity on Calabi-Yau based non-standard embedding models over the years. Recently, there has been significant progress both from the mathematical and the model-building viewpoint, leading to models edging closer and closer towards the standard model [4, 5]. Two types of constructions, one based on elliptically fibered Calabi-Yau spaces with bundles of the Friedman-Morgan-Witten type [12] and generalizations [3]–[8], [13]–[15], the other based on complete intersection Calabi-Yau spaces with monad bundles [11], [16]–[19], have been pursued in the literature. In this paper, we will work within the context of the second approach using complete intersection Calabi-Yau manifolds and monad bundles. To explain our motivation for this choice we remind the reader of the usual "two-step" symmetry breaking in heterotic models. In the first step, the $E_8$ gauge group is broken to one of the standard grand unified groups $E_6$, SO(10) or SU(5) by a bundle $V$ with structure group SU(3), SU(4) or SU(5), respectively. Then a Wilson line is introduced to break further to the standard model group (times possible $U(1)$ factors). This second breaking requires a non-trivial first fundamental group of the Calabi-Yau space $X$ which is normally achieved by dividing $X$ by a discrete symmetry. For complete intersection Calabi-Yau manifolds, this last procedure of dividing by a

discrete symmetry group is greatly facilitated by the presence of an ambient projective space. This is one of our main motivations for working with this class of manifolds, although analyzing discrete symmetries and Wilson line breaking explicitly will be the subject of a future publication [9]. Another major reason for our choice of models is that all relevant objects can be readily described in the language of commutative algebra and, therefore, lent themselves to an analysis based on computer algebra.

In this paper, we will construct all positive monad bundles of rank 3, 4 and 5 on the five complete intersection Calabi-Yau spaces in a single projective space subject to two additional constraints. First, the bundles should be such that heterotic anomaly cancellation can be accomplished and second, their chiral asymmetry should be a (non-zero) multiple of three. We find 37 examples in total. We then prove stability for all these bundles using a variant of a simple criterion due to Hoppe [21]. Recently, this criterion has been used [22], although in a slightly different way from the present paper, to prove stability for a class of positive bundles on the quintic [19]. Further, we compute the complete spectrum for all bundles, including gauge singlet fields. It turns out that a common feature of our models is that they only lead to generations but no anti-generations. While the present paper deals with a relatively small number of examples, we have shown that the relevant methods can be applied in a systematic and algorithmic way. We expect that a significantly larger class of complete intersection Calabi-Yau spaces and bundles on them can be treated in a similar way (see [20] for a recent constraint on classifying bundles in general). This generalization and the analysis of the particle physics of the resulting models will be the subject of future work [9].

The plan of the paper is as follows. In the next section, we will briefly review the main general features of $E_8 \times E_8$ heterotic compactifications. In Section 3, we discuss the monad construction, its main properties and prove a number of general results for such bundles. In Section 4, we classify the positive monad bundles on our five Calabi-Yau spaces, prove their stability and compute the spectra. After our conclusions in Section 5, Appendix A follows with a short summary of the relevant tools in commutative algebra and how they are applied in the context of the Macaulay computer algebra package [23]. The final Appendices contain several useful technical results.

# 2 Heterotic Compactification and Physical Constraints

To set the scene, we would now like to briefly review the basic structure of $E_8 \times E_8$ heterotic vacua on Calabi-Yau three-folds (see Ref. [28, 6, 8]).

In addition to a Calabi-Yau three-fold $X$ with tangent bundle, $TX$, we need two holomorphic vector bundles $V$ and $\tilde{V}$ with associated structure groups which are subgroups of $E_8$. In the present context, we will be interested in bundles with rank $n = 3, 4, 5$ and corresponding structure group $G = \mathrm{SU(n)}$. In general, heterotic vacua can also

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contain five-branes which appear as M five-branes in the 11-dimensional strong-coupling limit and as NS 5-branes in the 10-dimensional weakly coupled theory. In either case, for a supersymmetric compactification, the five-branes have to wrap a holomorphic curve in the Calabi-Yau space $X$, whose second homology class we denote by $W \in H_2(X, \mathbb{Z})$.

Two additional conditions need to be imposed on this data if the associated compactification is to preserve $N = 1$ supersymmetry in four dimensions. First, the two bundles $V$ and $\tilde{V}$ need to be (semi-) stable bundles [29]. To introduce the notion of stability, we define the slope

$$\mu(F) = \frac{1}{\mathrm{rk}(F)} \int_X c_1(F) \wedge J \wedge J \qquad (1)$$

of a (coherent) sheaf $F$ on $X$, where $J$ is the Kähler form on $X$ and $\mathrm{rk}(F)$ and $c_1(F)$ are the rank and the first Chern class of the sheaf, respectively. A bundle $V$ is now called stable (resp. semi-stable) if for all sub-sheafs $F \subset V$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(V)$ the slope satisfies $\mu(F) < \mu(V)$ (resp. $\mu(F) \leq \mu(V)$). It is worth mentioning that a bundle $V$ is semi-stable exactly if its dual $V^*$ is and that $h^0(X, V) = h^3(X, V) = 0$ for a stable bundle $V$. To preserve supersymmetry, semi-stability of the bundle is sufficient, although in practice one often requires stability. For specific examples, either condition is typically very hard to check and the stability proof for the bundles considered in this paper, is one of our main results. In addition, for supersymmetry to be preserved, the five-brane class $W$ needs to be an effective class. This means that there indeed exists a holomorphic curve with class $W$ in $X$.

Finally, heterotic models need to satisfy a well-known anomaly condition. For the case of bundles $V$ and $\tilde{V}$ with vanishing first Chern classes, $c_1(V) = c_1(\tilde{V}) = 0$, which we consider in this paper this condition reads

$$c_2(TX) - c_2(V) - c_2(\tilde{V}) = W \ . \qquad (2)$$

Next, we turn to the general structure of the low-energy particle spectrum. In addition to the dilaton, $h^{1,1}(X)$ Kähler moduli and $h^{2,1}(X)$ complex structure moduli of the Calabi-Yau space, each of the $E_8$ gauge theories as well as the five-branes give rise to a sector of particles in the low-energy theory. Here, we will focus on the "observable" sector, associated to the first $E_8$ gauge theory with vector bundle $V$ and structure group $G$. We will not explicitly consider the particle content in the other "hidden" sectors.

The low-energy gauge group $H$ in the observable sector is given by the commutant of the structure group $G$ within $E_8$. For $G = \mathrm{SU}(3)$, $\mathrm{SU}(4)$, $\mathrm{SU}(5)$ this implies the standard grand unified groups $H = E_6$, $\mathrm{SO}(10)$, $\mathrm{SU}(5)$, respectively. In order to find the matter field representations, we have to decompose the adjoint $\mathbf{248}$ of $E_8$ under $G \times H$. In general, this decomposition can be written as

$$248 \rightarrow (1, \mathrm{Ad}(H)) \oplus \bigoplus_i (R_i, r_i) \qquad (3)$$

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| $E_8 \to G \times H$ | Residual Group Structure |
|---|---|
| $\mathrm{SU}(3) \times \mathrm{E}_6$ | $\mathbf{248} \to (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1})$ |
| $\mathrm{SU}(4) \times \mathrm{SO}(10)$ | $\mathbf{248} \to (\mathbf{1}, \mathbf{45}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{16}}) \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{15}, \mathbf{1})$ |
| $\mathrm{SU}(5) \times \mathrm{SU}(5)$ | $\mathbf{248} \to (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \overline{\mathbf{10}}) \oplus (\overline{\mathbf{5}}, \mathbf{10}) \oplus (\mathbf{10}, \mathbf{5}) \oplus (\overline{\mathbf{10}}, \overline{\mathbf{5}}) \oplus (\mathbf{24}, \mathbf{1})$ |

Table 1: *Breaking patterns of $E_8$ and decompositions of the $\mathbf{248}$ adjoint representation.*

| Decomposition | Cohomologies |
|---|---|
| $\mathrm{SU}(3) \times \mathrm{E}_6$ | $n_{27} = h^1(V), n_{\overline{27}} = h^1(V^*) = h^2(V), n_1 = h^1(V \otimes V^*)$ |
| $\mathrm{SU}(4) \times \mathrm{SO}(10)$ | $n_{16} = h^1(V), n_{\overline{16}} = h^2(V), n_{10} = h^1(\wedge^2 V), n_1 = h^1(V \otimes V^*)$ |
| $\mathrm{SU}(5) \times \mathrm{SU}(5)$ | $n_{10} = h^1(V^*), n_{\overline{10}} = h^1(V), n_5 = h^1(\wedge^2 V), n_{\overline{5}} = h^1(\wedge^2 V^*)$ |
| | $n_1 = h^1(V \otimes V^*)$ |

Table 2: *Computation of low-energy particle spectra.*

where $\mathrm{Ad}(H)$ denotes the adjoint representation of $H$ and $\{(R_i, r_i)\}$ is a set of representations of $G \times H$. The adjoint representation of $H$ corresponds to the low-energy gauge fields while the low-energy matter fields transform in the representations $r_i$ of $H$. For the three relevant structure groups these matter field representations are explicitly listed in Table 1. We may ask how many supermultiplets will occur in the low energy theory for each representation $r_i$? It turns out that this number is given by $n_{r_i} = h^1(X, V_{R_i})$, the dimension of the cohomology group $H^1(X, V_{R_i})$ of the vector bundle $V$ in the specific $G$ representation $R_i$ which is paired up with the $H$ representation $r_i$ in the decomposition (3). For $G = \mathrm{SU}(n)$, the relevant representations $R_i$ can be obtained by appropriate tensor products of the fundamental representation and one ends up having to compute $h^1(X, V \otimes V^*)$, $h^1(X, V)$, $h^1(X, V^*)$, $h^1(X, \wedge^2 V)$, and $h^1(X, \wedge^2 V^*)$. Using Serre duality, $h^1(X, V^*) = h^2(X, V)$, the number the low-energy representations can then be computed as summarized in Table 2. Further, the Atiyah-Singer index theorem [40], applied to the case $c_1(TX) = c_1(V) = 0$, tells us that the index of $V$ can be expressed as

$$\mathrm{ind}(V) = \sum_{p=0}^{3} (-1)^p \, h^p(X, V) = \frac{1}{2} \int_X c_3(V) \,, \tag{4}$$

where $c_3(V)$ is the third Chern class of $V$. For a stable bundle, we have $h^0(X, V) = h^3(X, V) = 0$ and comparison with Table 2 shows that, in this case, the index counts the chiral asymmetry, that is, the difference of the number of generations and anti-generations. The index is usually easier to compute than individual cohomologies and is useful to impose a physical constraint on the chiral asymmetry.

The heterotic models considered in this paper will be constructed as follows. After choosing a Calabi-Yau space $X$ (which we will take to be one of the five Calabi-Yau

| Intersection | $\mathcal{A}$ | Configuration | $\chi(X)$ | $h^{1,1}(X)$ | $h^{2,1}(X)$ | $d(X)$ | $\tilde{c}_2(TX)$ |
|---|---|---|---|---|---|---|---|
| Quintic | $\mathbb{P}^4$ | [4\|5] | $-200$ | 1 | 101 | 5 | 10 |
| Quadric and quartic | $\mathbb{P}^5$ | [5\|2 4] | $-176$ | 1 | 89 | 8 | 7 |
| Two cubics | $\mathbb{P}^5$ | [5\|3 3] | $-144$ | 1 | 73 | 9 | 6 |
| Cubic and 2 quadrics | $\mathbb{P}^6$ | [6\|3 2 2] | $-144$ | 1 | 73 | 12 | 5 |
| Four quadrics | $\mathbb{P}^7$ | [7\|2 2 2 2] | $-128$ | 1 | 65 | 16 | 4 |

Table 3: *The five complete intersection Calabi-Yau manifolds in a single projective space. Here, $\chi(X)$ is the Euler number, $h^{1,1}(X)$ and $h^{2,1}(X)$ are the Hodge numbers, $d(X)$ is the intersection number and $c_2(TX) = \tilde{c}_2(TX)J^2$ is the second Chern class. The normalization of the Kähler form $J$ is defined in the main text.*

spaces realized as intersections in a single ordinary projective space), we will scan over a certain, well-defined class of (monad) bundles, $V$, on $X$. We will think of these bundles as bundles in the observable sector and take the hidden bundle $\tilde{V}$ to be trivial. The anomaly condition (2) can then be satisfied by including five-branes as long as $c_2(TX) - c_2(V)$ is an effective class on $X$. This is precisely what we will require. In addition, we will only consider bundles $V$ whose index is a (non-zero) multiple of three. Only such bundles have a chance, after dividing out by a discrete symmetry, of producing a model with chiral asymmetry three. We will then prove stability for all such bundles and compute their complete low-energy spectrum.

# 3 Monad Construction of Vector Bundles

To begin our systematic construction of vector bundles for heterotic compactifications, we will make use of a standard and powerful technique for defining bundles, known as the *monad construction*. On complex projective varieties, this method of constructing vector bundles dates back to the early works on $\mathbb{P}^4$ by [33] and systematic approaches by [34] and [35]. This construction defines a vast class of vector bundles; in fact, every bundle on $\mathbb{P}^n$ can be expressed as a monad [30, 33]. Bundles defined as monads have been widely used in the mathematics and physics literature. The reader is referred to [36] for the most general construction of monads and their properties. In this work we will use a restricted form prevalent in the physics literature.

## 3.1 The Calabi-Yau Spaces

Our monad bundles will be constructed on complete intersection Calabi-Yau manifolds, $X$, which are defined in a single projective ambient space $\mathcal{A} = \mathbb{P}^m$. There are five such Calabi-Yau manifolds [31] and their properties are summarized in Table 3. They are most conveniently described by the configurations $[m|q_1, \ldots, q_K]$ listed in the Table,

where $m$ refers to the dimension of the ambient space $\mathbb{P}^m$ and the numbers $q_a$ indicate the degree of the defining polynomials. In this notation the Calabi-Yau condition $c_1(TX) = 0$ translates to $\sum_{a=1}^{K} q_a = m + 1$. Furthermore, note that $h^{1,1}(X) = 1$ for all five cases. Hence, these manifolds have their Picard group, $\text{Pic}(X)$, being isomorphic to $\mathbb{Z}$. Such manifolds are called *cyclic* [32]. The Kähler form $J$ descends from the the ambient space $\mathbb{P}^n$ and is normalized as

$$\int_{\mathbb{P}^n} J^m = 1 \ . \tag{5}$$

Integrals over $X$ of any three-form $w$, defined on $\mathcal{A} = \mathbb{P}^m$, can be reduced to integrals over the ambient space using the formula

$$\int_X w = d(X) \int_{\mathbb{P}^m} w \wedge J^{m-3} \ , \tag{6}$$

where $d(X)$ are the intersection numbers listed in Table 3. The second homology $H_2(X, \mathbb{Z})$ is dual to the integer multiples of $J \wedge J$ and the Mori cone of $X$ corresponds to all positive multiples of $J \wedge J$ [25].

For our subsequent analysis it is useful to discuss some properties of line bundles on the above Calabi-Yau manifolds. We denote by $\mathcal{O}(k)$ the $k^{\text{th}}$ power of the hyperplane bundle, $\mathcal{O}(1)$, on the ambient space $\mathbb{P}^m$ and by $\mathcal{O}_X(k)$ its restriction to the Calabi-Yau space $X$. The normal bundle $\mathcal{N}$ of $X$ in the ambient space is then given by

$$\mathcal{N} = \bigoplus_{a=1}^{K} \mathcal{O}(q_a) \ . \tag{7}$$

In general, one finds, for the Chern characters of line bundles on $X$,

$$\text{ch}_1(\mathcal{O}_X(k)) = c_1(\mathcal{O}_X(k)) = kJ \ , \tag{8}$$

$$\text{ch}_2(\mathcal{O}_X(k)) = \frac{1}{2}k^2 J^2 \ , \tag{9}$$

$$\text{ch}_3(\mathcal{O}_X(k)) = \frac{1}{6}k^3 J^3 \ . \tag{10}$$

From the Atiyah-Singer index theorem the index of $\mathcal{O}_X(k)$ is given by

$$\begin{aligned}
\text{ind}(\mathcal{O}_X(k)) &\equiv \sum_{q=0}^{3} (-1)^q h^q(X, \mathcal{O}_X(k)) \\
&= \int_X \left[ \text{ch}_3(\mathcal{O}_X(k)) + \frac{1}{12} c_2(TX) \wedge c_1(\mathcal{O}_X(k)) \right] \\
&= \frac{d(X)k}{6} \left( k^2 + \frac{1}{2}\tilde{c}_2(TX) \right) \ , \tag{11}
\end{aligned}$$

where the numbers $\tilde{c}_2(TX)$ characterize the second Chern class of $X$ and $d(X)$ are the intersection numbers. The values for these quantities can be read off from Table 3.

We recall that the Kodaira vanishing theorem [40] states that on a Kähler manifold $X$, $H^q(X, L \otimes K_X)$ vanishes for $q > 0$ and $L$ a positive line bundle. Here, $K_X$ is the canonical bundle on $X$. For Calabi-Yau manifolds $K_X$ is of course trivial and, hence,

the only non-vanishing cohomology for positive line bundles on Calabi-Yau manifolds is $H^0$. The dimension of this cohomology group can then be computed from the index theorem. In fact, inserting the values for the intersection numbers and the second Chern class from Table 3 into Eq. (11) we explicitly find, for the five Calabi-Yau spaces and for line bundles $\mathcal{O}_X(k)$ with $k > 0$, that

$$h^0([4|5], \mathcal{O}_X(k)) = \frac{5}{6}(k^3 + 5k) , \tag{12}$$

$$h^0([5|2\,4], \mathcal{O}_X(k)) = \frac{2}{3}(2k^3 + 7k) , \tag{13}$$

$$h^0([5|3\,3], \mathcal{O}_X(k)) = \frac{3}{2}(k^3 + 3k) , \tag{14}$$

$$h^0([6|3\,2\,2], \mathcal{O}_X(k)) = 2k^3 + 5k , \tag{15}$$

$$h^0([7|2\,2\,2\,2], \mathcal{O}_X(k)) = \frac{8}{3}(k^3 + 2k) . \tag{16}$$

For negative line bundles $L = \mathcal{O}_X(-k)$, where $k > 0$, it follows from Serre duality on the Calabi-Yau three-fold $X$, $h^q(X, L) = h^{3-q}(X, L^*)$, that only $H^3(L, X)$ can be non-zero and that its dimension $h^3(X, \mathcal{O}_X(-k)) = h^0(X, \mathcal{O}_X(k))$ is given by one of the explicit expressions (12)–(16). Finally, we have

$$h^0(X, \mathcal{O}_X) = h^3(X, \mathcal{O}_X) = 1 , \qquad h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0 . \tag{17}$$

Now we explicitly know the cohomology for all line bundles on the five Calabi-Yau manifolds under consideration. In particular, we conclude that $h^0(X, \mathcal{O}_X(k)) > 0$ precisely for $k \geq 0$ and, hence, that only the line bundles $\mathcal{O}_X(k)$ with $k \geq 0$ have a non-trivial section. This is one of the underlying conditions for the validity of Hoppe's criterion which will play a central role in the stability proof for our bundles.

## 3.2   Constructing the Monad

Having discussed the manifold $X$ and line bundles thereon, we now construct the requisite vector bundles $V$. Our construction proceeds as follows. On a Calabi-Yau manifold $X$, a monad bundle $V$ is defined by the short exact sequence

$$0 \to V \xrightarrow{f} B \xrightarrow{g} C \to 0 , \tag{18}$$

where $B$ and $C$ are bundles on $X$. It is standard to take $B$ and $C$ to be direct sums of line bundles over $X$, that is

$$B = \bigoplus_{i=1}^{r_B} \mathcal{O}_X(b_i) , \qquad C = \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) . \tag{19}$$

Here, $r_B$ and $r_C$ are the ranks of the bundles $B$ and $C$, respectively. The exactness of (18) implies that $\ker(g) = \text{im}(f)$ and $\ker(f) = 0$, so that the bundle $V$ can be expressed as

$$V = \ker(g) .$$

The map $g$ is a morphism between bundles and can be defined as a $r_C \times r_B$ matrix whose entries, $(i, j)$, are sections of $\mathcal{O}_X(c_i - b_j)$. As we have seen in the previous subsection, such sections exist iff $c_i \geq b_j$ and so this is what we should require. In fact, if $c_i = b_j$ for an index pair $(i, j)$ the two corresponding line bundles can simply be dropped from $B$ and $C$ without changing the resulting bundle $V$. In the following, we will, therefore, assume the stronger condition $c_i > b_j$ for all $i$ and $j$.

The Calabi-Yau manifolds discussed in this paper are complete intersections in a single projective space $\mathbb{P}^m$. We can, therefore, write down an analogous short exact sequence

$$0 \to \mathcal{V} \xrightarrow{\tilde{f}} \mathcal{B} \xrightarrow{\tilde{g}} \mathcal{C} \to 0 , \tag{20}$$

on the ambient space where

$$\mathcal{B} = \bigoplus_{i=1}^{r_B} \mathcal{O}(b_i) , \qquad \mathcal{C} = \bigoplus_{i=1}^{r_C} \mathcal{O}(c_i) . \tag{21}$$

The map $\tilde{g}$ can be viewed as a $r_C \times r_B$ matrix whose entries, $(i, j)$, are homogeneous polynomials of degree $c_i - b_j$. This sequence defines a vector bundle $\mathcal{V}$ on the ambient space whose restriction to $X$ is $V$. Further, the map $g$ can be seen as the restriction of its ambient space counterpart $\tilde{g}$ to $X$. Unless explicitly stated otherwise, we will assume throughout that this map is generic.

It is natural to enquire whether $V$ thus defined is always a bona fide bundle rather than a sheaf. We are assured on this point by the following theorem [42].

THEOREM 3.1 *Over any smooth variety $X$, if $g : B \to C$ is a morphism between locally free sheaves $B$ and $C$, then $\ker(g)$ is locally free.*

Now, by definition, a locally free sheaf of constant rank is a vector bundle. Therefore, by the above theorem, it only remains to check whether $\ker(g)$ has constant rank on $X$. Indeed, $g$ could be less than maximal rank on a singular (sometimes called 'degeneracy') locus. We note that exactness of the sequence, that is $\text{coker}(g) = 0$, is equivalent to this degeneracy locus being empty.

To show that the degeneracy locus is empty for our bundles, it turns out to be convenient to consider the dual bundle $V^*$ defined by the dual sequence

$$0 \to C^* \xrightarrow{g^T} B^* \longrightarrow V^* \to 0 , \tag{22}$$

where

$$V^* = \text{coker}(g^T) . \tag{23}$$

We can now apply the following theorem [22, 45].

THEOREM 3.2 *Let $\phi : E \to F$ be a morphism of vector bundles on a variety of dimension $N$ and let $e = \text{rk}(E)$, $f = \text{rk}(F)$ and $e \leq f$. If $E^* \otimes F$ is globally generated and $f - e + 1 > N$, then generic maps $\phi$ have a vanishing degeneracy locus.*

Therefore, take $\phi = g^T$, $E = C^*$ and $F = B^*$. For all our bundles of interest, $N = 3$ and $e < f$. In fact, $f - e$ is the rank of $V$, which is 3, 4, or 5 for the bundles of interest in heterotic compactifications. Finally, $E^* \otimes F$ is globally generated because $B$ and $C$ are direct sums of line bundles with $c_i > b_j$ for all $i, j$. Hence, all the conditions in the theorem are obeyed and we see that the degeneracy locus of $g^T$, and hence the one for $g$, is vanishing for the bundles of interest on the Calabi-Yau. However, one should note that this criterion will not always be satisfied when writing monad sequences on the higher dimensional ambient spaces, as in Eq. (20). (Such issues will be discussed further in section 4.4). For more on the degeneracy locus of bundle maps, and why Theorem 3.2 guarantees its vanishing in the dual monad, see e.g. [43, 44].)

For later reference we present the formulae for the Chern classes of $V$ (see Ref. [31]). Simplifying the expressions for $c_2(V)$ and $c_3(V)$ by imposing the vanishing of the first Chern class, we have

$$\mathrm{rk}(V) = r_B - r_C \,, \tag{24}$$

$$c_1(V) = \left( \sum_{i=1}^{r_B} b_i - \sum_{i=1}^{r_C} c_i \right) J \equiv 0 \,, \tag{25}$$

$$c_2(V) = -\frac{1}{2} \left( \sum_{i=1}^{r_B} b_i^2 - \sum_{i=1}^{r_C} c_i^2 \right) J^2 \,, \tag{26}$$

$$c_3(V) = \frac{1}{3} \left( \sum_{i=1}^{r_B} b_i^3 - \sum_{i=1}^{r_C} c_i^3 \right) J^3 \,. \tag{27}$$

Hence, from Eq. (4) and the above expression for the third Chern class, the index of $V$ is explicitly given by

$$\mathrm{ind}(V) = \sum_{p=0}^{3} (-1)^p \, h^p(X, V) = \frac{d(X)}{6} \left( \sum_{i=1}^{r_B} b_i^3 - \sum_{i=1}^{r_C} c_i^3 \right) \,. \tag{28}$$

Within this paper, we will make extensive use of the computer algebra system [23] in analyzing the monads in (18). Utilizing this powerful tool we are able to catalog efficiently bundle cohomologies previously too difficult to be calculated. Indeed, computing particle spectra, that is, sheaf cohomology, is ordinarily a tremendous task even for a single bundle, and it would be unthinkable to attempt to calculate by hand the hundreds of such cohomologies necessary in a systematic study of monad bundles. However, the recent advances in algorithmic algebraic geometry allow us to explicitly and efficiently compute the requisite cohomology groups for a certain class of bundles. For the first time, we describe in detail how to use this technology in the context of string compactification.

With this approach in mind, we recall that in computational algebraic geometry [38], sheafs are expressed in the language of graded modules over polynomial rings. If $X$ is embedded in $\mathbb{P}^m$ with homogeneous coordinates $[x_0 : x_1 : \ldots : x_m]$, we can let $R$ be the coordinate ring $\mathbb{C}[x_0, x_1, \ldots, x_m]/(X)$ where $(X)$ is the ideal associated with $X$. The bundles $B$ and $C$ are then described by free-modules of $R$ with appropriate

degrees (grading). We leave to the Appendix a detailed tutorial of the sheaf-module correspondence and the construction and relevant computation of monad bundles using computer algebra.

## 3.3 Stability of Monad Bundles

As mentioned in the previous section, (semi-)stability of the vector bundle is of central importance to heterotic compactifications. In general, proving stability is an overwhelming technical obstacle and a systematic analysis has so far been elusive. However, for a class of manifolds, a sufficient but by no means necessary condition is of great utility; this is the so-called Hoppe's criterion [21, 37]:

THEOREM 3.3 *[Hoppe's Criterion] Over a projective manifold $X$ with Picard group $Pic(X) \simeq \mathbb{Z}$ (i.e., $X$ is cyclic), let $V$ be a vector bundle with $c_1(V) = 0$. If $H^0(X, \bigwedge^p V) = 0$ for all $p = 1, 2, \ldots, \mathrm{rk}(V) - 1$, then $V$ is stable.*

We also recall that for the Calabi-Yau manifolds used in this paper all positive line bundles have a section, an underlying assumption for the validity of Hoppe's theorem which is, hence, satisfied.

The strategy is therefore clear. To prove stability for the monad bundles (18) over cyclic manifolds $X$ using Hoppe's criterion, we need to show the vanishing of $H^0(X, \wedge^p V)$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$. In the following paragraphs, we will outline the basis for this stability proof and make note of certain results and properties that are of particular use.

One additional assumption which we will make is that all line bundles involved in the definition of the bundles $V$ are positive, that is, for all $i$,

$$b_i > 0 \quad \text{and} \quad c_i > 0 . \tag{29}$$

We will refer to this property as "positivity" of the bundle $V$. While this is not required for a consistent definition of the bundle or the associated heterotic model, it turns out to be a crucial technical assumption which facilitates the stability proof. The essential point is that positivity of $V$ allows one to use Kodaira vanishing when applying Hoppe's criterion to the dual bundle $V^*$. To see how this works, recall that the dual bundle is defined by the sequence $0 \to C^* \longrightarrow B^* \longrightarrow V^* \to 0$ and that its stability is equivalent to that of $V$. The associated long exact sequence in cohomology is

$$
\begin{aligned}
0 \quad &\to \quad H^0(X, C^*) \to H^0(X, B^*) \to \boxed{H^0(X, V^*)} \\
&\to \quad H^1(X, C^*) \to H^1(X, B^*) \to H^1(X, V^*) \\
&\to \quad H^2(X, C^*) \to H^2(X, B^*) \to H^2(X, V^*) \\
&\to \quad H^3(X, C^*) \to H^3(X, B^*) \to H^3(X, V^*) \to 0 .
\end{aligned} \tag{30}
$$

Given that we are dealing with positive bundles $V$, it follows that $B^*$ and $C^*$ are sums of negative line bundles and, hence, $H^0(X, B^*)$ and $H^1(X, C^*)$ in the above sequence are zero due to Kodaira vanishing. This means the "boxed" cohomology $H^0(X, V^*)$

also vanishes. (For later considerations we note that Kodaira vanishing also implies $H^1(X, B^*) = H^2(X, C^*) = 0$ and, hence, $H^1(X, V^*) \simeq H^2(X, V) = 0$.) In order to prove stability of $V^*$ by applying Hoppe's criterion we have to show that $H^0(X, \wedge^p V^*) = 0$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$ and we have just completed the first step for $p = 1$.

Next, we need to compute the cohomologies $H^0 X, \wedge^p V^*)$ for $p > 1$. However, a further simplification occurs because we are dealing with unitary bundles. In fact, for an $SU(n)$ bundle $V$, we have

$$\wedge^{n-1} V^* \simeq V \tag{31}$$

(see, for example Ref. [41]) . Therefore, to cover the case $p = n - 1$, the highest exterior power relevant to Hoppe's criterion, we only need to show that $H^0(X, V) = 0$. This is indeed the case for all bundles considered in this paper and the explicit proof, which is somewhat lengthy, is presented in Appendix B.2. This completes the stability proof for the rank 3 bundles.

For rank 4 and 5 bundles we have to look at further exterior powers of $V^*$, namely $\Lambda^p V^*$ for $p = 2, \ldots, \mathrm{rk}(V) - 2$. To deal with those we consider the standard long exact ("exterior power") sequence [22, 40] for $\Lambda^p V^*$

$$
\begin{aligned}
0 \quad &\to \quad S^p C^* \to S^{p-1} C^* \otimes B^* \to S^{p-2} C^* \otimes \wedge^2 B^* \to \ldots \\
&\to \quad A \otimes \wedge^{p-1} B^* \to \wedge^p B^* \to \wedge^p V^* \to 0 ,
\end{aligned}
\tag{32}
$$

which is induced by the short exact sequence (22). Here $S^i$ is the $i$-th symmetrised tensor power of a bundle. Such a sequence does not itself induce a long exact sequence in cohomology; we need to slice it up into groups of three. In other words, we introduce co-kernels $K_i$ such that (32) becomes the following set of short exact sequences

$$
\begin{aligned}
0 \quad &\to \quad S^p C^* \to S^{p-1} C^* \otimes B^* \to K_1 \to 0 , \\
0 \quad &\to \quad K_1 \to S^{p-2} C^* \otimes \wedge^2 B^* \to K_2 \to 0 , \\
&\quad \vdots \\
0 \quad &\to \quad K_{p-1} \to \wedge^p B^* \to \wedge^p V^* \to 0 .
\end{aligned}
\tag{33}
$$

Each of the above now induces a long exact sequence in cohomology in analogy to (30):

$$0 \to H^0(X, S^p C^*) \to H^0(X, S^{p-1} C^* \otimes B^*) \to H^0(X, K_1) \to H^1(X, S^p C^*) \to \ldots \to 0 ,$$
$$0 \to H^0(X, K_1) \to H^0(X, S^{p-2} C^* \otimes \wedge^2 B^*) \to H^0(X, K_2) \to H^1(X, K_1) \to \ldots \to 0 ,$$
$$\vdots$$
$$0 \to H^0(X, K_{p-1}) \to H^0(X, \wedge^p B^*) \to \boxed{H^0(X, \wedge^p V^*)} \to H^1(X, K_{p-1}) \to \ldots \to 0 .$$
$$\tag{34}$$

The term we need is boxed and we need to trace through the various sequences, using the readily computed cohomologies of the symmetric and antisymmetric powers of $B^*$ and $C^*$, to arrive at the answer. Let us now do this explicitly for the case $p = 2$, that is, $H^0(X, \Lambda^2 V^*)$. The long exact sequence (32) then specializes to

$$0 \to S^2 C^* \to C^* \otimes B^* \to \Lambda^2 B^* \to \Lambda^2 V^* \to 0 , \tag{35}$$

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which needs to be broken up into the two short exact sequences

$$0 \quad \to S^2 C^* \to C^* \otimes B^* \to K \to 0 \qquad (36)$$

$$0 \quad \to K \to \Lambda^2 B^* \to \Lambda^2 V^* \to 0 \,. \qquad (37)$$

From the first of these we have the long exact sequence

$$
\begin{aligned}
0 \quad &\to \quad H^0(X, S^2 C^*) \to H^0(X, C^* \otimes B^*) \to H^0(X, K) \\
&\to \quad H^1(X, S^2 C^*) \to H^1(X, C^* \otimes B^*) \to H^1(X, K) \\
&\to \quad H^1(X, S^2 C^*) \to \dots \,.
\end{aligned}
\qquad (38)
$$

Since $B^*$ and $C^*$ are sums negative line bundles, so are their various tensor products which appear in the above sequences. From Kodaira vanishing all cohomologies of such bundles vanish except for the third. Applying this to (38) we immediately deduce that $H^0(X, K) = H^1(X, K) = 0$. Using this information in the long exact sequence

$$0 \to H^0(X, K) \to H^0(X, \Lambda^2 B^*) \to H^0(X, \Lambda^2 V^*) \to H^1(X, K) \to \dots \qquad (39)$$

which follows from (37) we find $H^0(X, \Lambda^2 V^*) = 0$, as desired. This completes the stability proof for rank 4 bundles [1].

Finally, for rank 5 bundles, we still need to compute $H^0(X, \Lambda^3 V^*)$. Repeating the above steps for this case one finds that Kodaira vanishing on $X$ alone does not quite provide sufficient information to conclude that $H^0(X, \Lambda^3 V^*) = 0$. In this case, we need to employ the additional technique of *Koszul sequences* [31, 40] which rely on the embedding of the Calabi-Yau manifold in an ambient space $\mathcal{A}$. Specifically, for a vector bundle $\mathcal{W}$ on $\mathcal{A}$ the Koszul sequence reads

$$0 \to \wedge^K \mathcal{N}^* \otimes \mathcal{W} \to \dots \to \wedge^2 \mathcal{N}^* \otimes \mathcal{W} \to \mathcal{N}^* \otimes \mathcal{W} \to \mathcal{W} \xrightarrow{\rho} \mathcal{W}|_X \to 0 \,, \qquad (40)$$

where $\mathcal{W}|_X$ denotes the restriction of $\mathcal{W}$ to $X$ and $\rho$ is the associated restriction map. Here $\mathcal{N}^*$ is the dual of the Calabi-Yau normal bundle, defined in Eq. (7). As will be shown in the next section, the Koszul sequence can be used to compute the relevant cohomologies directly from the ambient space. This will allow us to complete the stability proof for rank 5 bundles.

# 4 Classification and Examples

Armed with the general information about the five Calabi-Yau manifolds and monad bundles we can now proceed to classify such bundles, prove their stability and compute their spectrum.

---

[1]Together with $H^0(X, V^*) = 0$, which we have shown earlier, it also provides an independent argument for the stability of rank 3 bundles.

## 4.1 Classification of Configurations

For the monad bundles defined by the short exact sequence (18), we can immediately formulate a classification scheme. Recall that, taking the bundles $B$ and $C$ to be direct sums of line-bundles over the manifold $X$, we have

$$0 \to V \to \bigoplus_{i=1}^{r_B} \mathcal{O}_X(b_i) \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \to 0 \ , \qquad V \simeq \ker(g) \ . \tag{41}$$

From our discussion so far these bundles are subject to a number of physical and mathematical constraints which can be summarised as follows:

1. As discussed earlier we require all $b_i$ and $c_i$ to be positive; this is a technical assumption which will significantly simplify our computations.

2. We furthermore require that $b_i < c_j$ for all $i$ and $j$; this is to ensure that the map $g$, which consists of sections of $\mathcal{O}_X(c_j - b_i)$, has no zero entries. Further, we require the map $g$ to be generic. Then, all conditions of Theorem (3.2) are met and we are guaranteed that $V$, as defined by the sequence (41), is indeed a bundle.

3. Since we are dealing with special unitary bundles we impose $c_1(V) = 0$.

4. For a given Calabi-Yau space $X$ and a bundle $V$ we need to ensure that the anomaly condition (2) can be satisfied. To do this we impose the condition that $c_2(TX) - c_2(V)$ must be effective. Then, we can choose a trivial hidden bundle $\tilde{V}$ and a five-brane wrapping a holomorphic curve with homology class $c_2(TX) - c_2(V)$. In practice, this condition simply means that the coefficient of $J^2$ in $c_2(TX) - c_2(V)$ must be non-negative [2] .

5. We require that the index of $V$ is a non-zero multiple of three. Only such models may lead to three generations after dividing by a discrete symmetry.

6. Since we are interested in low-energy grand unified groups we consider bundles $V$ with structure group $\mathrm{SU}(n)$, where $n = \mathrm{rk}(V) = 3, 4, 5$.

Therefore, an integer partitioning problem immediately presents itself to us: find partitions $\{b_i\}_{i=1,\dots,r_C+n}$ and $\{c_i\}_{i=1,\dots,r_C}$ of positive integers $b_i > 0$, $c_i > 0$ satisfying $b_i < c_j$ for all $i$, $j$ and subject to the condition $\sum_{i=1}^{r_B} b_i - \sum_{i=1}^{r_C} c_i = 0$ for vanishing first Chern class of $V$ (see Eq. (25)). Further, we demand that the index of $V$, Eq. (28), is non-zero and divisible by three and that the coefficient of $J^2$ in $c_2(TX) - c_2(V)$ be non-negative, in order to ensure the existence of a holomorphic five-brane curve. From Eq. (26) the last constraint can be explicitly written as

$$0 \le -\frac{1}{2}\Big( \sum_{i=1}^{r_C+n} b_i - \sum_{i=1}^{r_C} c_i \Big) \le \tilde{c}_2(TX) \ , \tag{42}$$

---

[2]However, for a given example there may well be other ways to satisfy the anomaly condition which involve a non-trivial hidden bundle $\tilde{V}$.

where the numbers $\tilde{c}_2(TX)$ for the second Chern class of $X$ are given in Table 3. Since $b_i < c_j$ for all $i$, $j$ it is clear that this constraint implies an upper bound on $b_i$ and $c_j$ and, hence, that the number of vector bundles in our class is finite [3]. To derive this bound explicitly we slightly modify an argument from Appendix B of Ref. [19]. Define the quantity

$$S = \sum_{i=1}^{r_C+n} b_i = \sum_{i=1}^{r_C} c_i \ , \tag{43}$$

and consider the following chain of inequalities

$$2\,\tilde{c}_2(TX) \geq \sum_{i=1}^{r_C} c_i^2 - \sum_{i=1}^{r_C+n} b_i^2 \ \geq \ (b_{max}+1)\sum_{i=1}^{r_C} c_i - \sum_{i=1}^{r_C+n} b_i^2$$

$$= \ S + \sum_{i=1}^{r_C+n} b_{max} b_i - \sum_{i=1}^{r_C+n} b_i^2 \geq S \ .$$

From Table 3, $\tilde{c}_2(TX)$ is at most 10 and, hence, the sum $S$ cannot exceed 20, thereby placing an upper bound on our partitioning problem.

Given the finiteness of the problem, the classification of all positive monad bundles subject to the above constraints is now easily computerisable. Given these conditions, we found 37 bundles on the five Calabi-Yau manifolds in question, 20 for rank 3, 10 for rank 4 and 7 for rank 5. Had we relaxed the condition that $c_3$ should be divisible by 3, we would have found 43, 15, 10, 6, and 3 bundles, respectively on the 5 cyclic manifolds, for a total of 77. A complete list of all such bundles for the five Calabi-Yau manifolds of concern is given in the Tables 4–8.

## 4.2   $E_6$-GUT Theories

The first case we shall analyse is $E_6$-GUT theories which arise from $SU(3)$ bundles. We have already seen in Section 3.3 that all such bundles are indeed stable. This result has been explicitly confirmed by a computer algebra computation of $H^0(X,V^*)$ and $H^0(X,\Lambda^2 V^*)$ along the lines described in Appendix A. We can, therefore, directly turn to a computation of their particle spectrum.

### 4.2.1   Particle Content

The number of **27** and $\overline{\textbf{27}}$ representation of $E_6$ is easy to obtain. Since $V$ is stable we already know that $H^0(X,V) = H^3(X,V) = 0$. From the long exact sequence (30) we have deduced earlier that $H^2(X,V) \simeq H^1(X,V^*) = 0$ so that $H^1(X,V)$ is the only non-vanishing cohomology. Its dimension can be directly computed from the index (28), so that

$$n_{27} = h^1(X,V) = -\,\mathrm{ind}(V) \ , \qquad n_{\overline{27}} = h^2(X,V) = 0 \ . \tag{44}$$

---

[3]The constraint (42) arises because we require $N = 1$ supersymmetry in four dimensions. If we relaxed this condition and allowed for anti-five branes there would be no immediate bound on the number of vector bundles. However, in this case, the stability of such non-supersymmetric models has to be analyzed carefully [27].

| Rank | $\{b_i\}$ | $\{c_i\}$ | $c_2(V)/J^2$ | ind$(V)$ |
|------|-----------|-----------|--------------|----------|
| 3 | (2, 2, 1, 1, 1) | (4, 3) | 7 | -60 |
| 3 | (2, 2, 2, 1, 1) | (5, 3) | 10 | -105 |
| 3 | (3, 2, 1, 1, 1) | (4, 4) | 8 | -75 |
| 3 | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 3 | -15 |
| 3 | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 6 | -45 |
| 3 | (3, 3, 3, 1, 1, 1) | (4, 4, 4) | 9 | -90 |
| 3 | (2, 2, 2, 2, 2, 2, 2, 2) | (4, 3, 3, 3, 3) | 10 | -90 |
| 3 | (2, 2, 2, 2, 2, 2, 2, 2, 2) | (3, 3, 3, 3, 3, 3) | 9 | -75 |
| 4 | (2, 2, 1, 1, 1, 1) | (4, 4) | 10 | -90 |
| 4 | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 5 | -30 |
| 4 | (2, 2, 2, 1, 1, 1, 1) | (4, 3, 3) | 9 | -75 |
| 4 | (2, 2, 2, 2, 1, 1, 1, 1) | (3, 3, 3, 3) | 8 | -60 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1) | (3, 3, 2) | 7 | -45 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1) | (4, 2, 2) | 8 | -60 |
| 5 | (2, 2, 2, 2, 2, 1, 1, 1, 1, 1) | (3, 3, 3, 3, 3) | 10 | -75 |

Table 4: *Positive monad bundles on the quintic,* [4|5].

| Rank | $\{b_i\}$ | $\{c_i\}$ | $c_2(V)/J^2$ | ind$(V)$ |
|------|-----------|-----------|--------------|----------|
| 3 | (2, 2, 1, 1, 1) | (4, 3) | 7 | -96 |
| 3 | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 3 | -24 |
| 3 | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 6 | -72 |
| 4 | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 5 | -48 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1) | (3, 3, 2) | 7 | -72 |

Table 5: *Positive monad bundles on* [5|2 4].

| Rank | $\{b_i\}$ | $\{c_i\}$ | $c_2(V)/J^2$ | ind$(V)$ |
|------|-----------|-----------|--------------|----------|
| 3 | (1, 1, 1, 1) | (4) | 6 | -90 |
| 3 | (1, 1, 1, 1, 1) | (3, 2) | 4 | -45 |
| 3 | (2, 1, 1, 1, 1) | (3, 3) | 5 | -63 |
| 3 | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 3 | -27 |
| 3 | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 6 | -81 |
| 4 | (1, 1, 1, 1, 1, 1) | (3, 3) | 6 | -72 |
| 4 | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 5 | -54 |
| 4 | (1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2) | 4 | -36 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1, 1) | (3, 2, 2, 2) | 6 | -63 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2, 2) | 5 | -45 |

Table 6: *Positive monad bundles on* [5|3 3].

| Rank | $\{b_i\}$ | $\{c_i\}$ | $c_2(V)/J^2$ | ind$(V)$ |
|------|-----------|-----------|--------------|----------|
| 3 | (1, 1, 1, 1, 1) | (3, 2) | 4 | -60 |
| 3 | (2, 1, 1, 1, 1) | (3, 3) | 5 | -84 |
| 3 | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 3 | -36 |
| 4 | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 5 | -72 |
| 4 | (1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2) | 4 | -48 |
| 5 | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2, 2) | 5 | -60 |

Table 7: *Positive monad bundles on* [6|2 2 3].

| Rank | $\{b_i\}$ | $\{c_i\}$ | $c_2(V)/J^2$ | ind$(V)$ |
|------|-----------|-----------|--------------|----------|
| 3 | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 3 | -48 |

Table 8: *Positive monad bundles on* [7|2 2 2 2].

Therefore, for the rank 3 bundles in Tables 4–8, the (negative of the) right-most column gives the number of **27** representations. This result also provides the first example of what is a general feature of positive monad bundles, namely the absence of anti-generations. The numbers $n_{27}$ have been independently verified by computer algebra.

What about the $E_6$ singlets? These correspond to the cohomology $H^1(X, \mathrm{ad}(V)) = H^1(X, V \otimes V^*)$. We begin by tensoring the defining sequence (22) for $V^*$ by $V$. This leads to a new short exact sequence

$$0 \to C^* \otimes V \to B^* \otimes V \to V^* \otimes V \to 0 \; . \tag{45}$$

One can produce two more short exact sequences by multiplying (22) with $B$ and $C$. Likewise, three short exact sequences can be obtained by multiplying the original sequence (18) for $V$ with $V^*$, $B^*$ and $C^*$. The resulting six sequences can then be arranged into the following web of three horizontal sequences $h_I$, $h_{II}$, $h_{III}$ and three vertical ones $v_I$, $v_{II}$, $v_{III}$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^* \otimes V & \to & B^* \otimes V & \to & V^* \otimes V & \to & 0 \qquad h_I \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^* \otimes B & \to & B^* \otimes B & \to & V^* \otimes B & \to & 0 \qquad h_{II} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^* \otimes C & \to & B^* \otimes C & \to & V^* \otimes C & \to & 0 \qquad h_{III} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & v_I & & v_{II} & & v_{III} & &
\end{array}
\tag{46}
$$

The long exact sequence in cohomology induced by $h_I$ reads

$$
\begin{aligned}
0 \quad &\to \quad H^0(X, C^* \otimes V) \to H^0(X, B^* \otimes V) \to H^0(X, V^* \otimes V) \\
&\to \quad H^1(X, C^* \otimes V) \to H^1(X, B^* \otimes V) \to \boxed{H^1(X, V^* \otimes V)} \\
&\to \quad H^2(X, C^* \otimes V) \to \dots
\end{aligned}
\tag{47}
$$

and we have boxed the term which we would like to compute. We will also need the long exact sequences which follow from $v_I$ and $v_{II}$. They are given by

$$
\begin{aligned}
0 \quad &\to \quad H^0(X, C^* \otimes V) \to H^0(X, C^* \otimes B) \to H^0(X, C^* \otimes C) \\
&\to \quad H^1(X, C^* \otimes V) \to H^1(X, C^* \otimes B) \to H^1(X, C^* \otimes C) \\
&\to \quad H^2(X, C^* \otimes V) \to H^2(X, C^* \otimes B) \to H^2(X, C^* \otimes C) \to \dots
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
0 \quad &\to \quad H^0(X, B^* \otimes V) \to H^0(X, B^* \otimes B) \to H^0(X, B^* \otimes C) \\
&\to \quad H^1(X, B^* \otimes V) \to H^1(X, B^* \otimes B) \to H^1(X, B^* \otimes C) \\
&\to \quad H^2(X, B^* \otimes V) \to H^2(X, B^* \otimes B) \to H^2(X, B^* \otimes C) \to \dots
\end{aligned}
\tag{49}
$$

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Now, because of the integers defining $B$ and $C$ satisfy $b_i < c_j$, the tensor product $C^* \otimes B$ is a direct sum of negative line bundles and, hence, all its cohomology groups vanish except the third. Further, the middle cohomologies $H^1$ and $H^2$ of $B^* \otimes B$ and $C^* \otimes C$ vanish. From the sequence (48) this implies

$$H^0(X, C^* \otimes V) = H^2(X, C^* \otimes V) = 0 , \qquad H^1(X, C^* \otimes V) = H^0(X, C^* \otimes C) . \quad (50)$$

Vanishing of $H^2(X, C^* \otimes V)$ means that the long exact sequence (47) breaks after the second line and we get

$$h^1(X, V^* \otimes V) = h^1(X, B^* \otimes V) - h^1(X, C^* \otimes V) + h^0(X, V^* \otimes V) - h^0(X, B^* \otimes V) . \quad (51)$$

Using the additional information

$$h^1(X, B^* \otimes V) - h^0(X, B^* \otimes V) = h^0(X, B^* \otimes C) - h^0(X, B^* \otimes B) . \quad (52)$$

which follows from the sequence (49) and the fact that $h^0(X, V^* \otimes V) = 1$ (see Theorem B.1 of Ref. [31]) Eq. (51) can be re-written as

$$h^1(X, V^* \otimes V) = h^0(X, B^* \otimes C) - h^0(B^* \otimes B) - h^0(C^* \otimes C) + 1 . \quad (53)$$

This equation, together with Eqs. (12)–(16) and (17), allows us to directly compute the number $n_1$ of $E_6$-singlets and the results are given in Table 9. For reference, we have also included the number of **27**-representations (the number of $\overline{\bf 27}$ particles, we recall, is zero). In addition, the results for $h^1(X, V^* \otimes V)$ have been independently confirmed using Macaulay [23], following the procedure outlined in Appendix A. We note that the above derivation of Eq. (53) is independent of the rank of the vector bundle $V$ and, hence, it remains valid for rank 4 and 5 bundles.

## 4.3  $SO(10)$-GUT Theories

Grand Unified theories with gauge group SO(10) are obtained from rank 4 bundles with structure group SU(4). We have already shown the stability of positive rank 4 monad bundles $V$ in Section 3.3. As before, we have explicitly confirmed this general result for the rank 4 bundles in our classification with Macaulay [23], by showing that $H^0(X, \Lambda^p V^*)$ for $p = 1, 2, 3$ vanishes. We proceed to analyze the particle content of SO(10) GUT theories.

### 4.3.1  Particle Content

Recall from Table 2, that for $SO(10)$-GUT theories we need to compute $n_{16} = h^1(X, V)$, $n_{\overline{16}} = h^1(X, V^*) = h^2(X, V)$, $n_{10} = h^1(X, \wedge^2 V)$ and $n_1 = h^1(X, V \otimes V^*)$.

Let us begin with the generations and anti-generations in **16** and $\overline{\bf 16}$. As in the case of rank 3 bundles, stability implies that $H^0(X, V) = H^3(X, V) = 0$ and, further, from the sequence (30), also $H^2(X, V) = H^1(X, V^*)$ is zero. Hence, as before, the number

| $X$ | $\{b_i\}$ | $\{c_i\}$ | $n_{27}$ | $n_1$ |
|---|---|---|---|---|
| [4\|5] | (2, 2, 1, 1, 1) | (4, 3) | 60 | 141 |
| | (2,2,2,1,1) | (5, 3) | 105 | 231 |
| | (3, 2, 1, 1, 1) | (4, 4) | 75 | 171 |
| | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 15 | 46 |
| | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 45 | 109 |
| | (3, 3, 3, 1, 1, 1) | (4, 4, 4) | 90 | 199 |
| | (2, 2, 2, 2, 2, 2, 2, 2) | (4, 3, 3, 3, 3) | 90 | 180 |
| | (2, 2, 2, 2, 2, 2, 2, 2, 2) | (3, 3, 3, 3, 3, 3) | 75 | 154 |
| [5\|2 4] | (2, 2, 1, 1, 1) | (4, 3) | 96 | 206 |
| | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 24 | 64 |
| | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 72 | 154 |
| [5\|3 3] | (1, 1, 1, 1) | (4) | 90 | 200 |
| | (1, 1, 1, 1, 1) | (3, 2) | 45 | 103 |
| | (2, 1, 1, 1, 1) | (3, 3) | 63 | 136 |
| | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 27 | 64 |
| | (2, 2, 2, 1, 1, 1) | (3, 3, 3) | 81 | 163 |
| [6\|2 2 3] | (1, 1, 1, 1, 1) | (3, 2) | 60 | 132 |
| | (2, 1, 1, 1, 1) | (3, 3) | 84 | 174 |
| | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 36 | 82 |
| [7\|2 2 2 2] | (1, 1, 1, 1, 1, 1) | (2, 2, 2) | 48 | 100 |

*Table 9: The particle content for the $E_6$-GUT theories arising from our classification of stable, positive $SU(3)$ monad bundles $V$ on the Calabi-Yau threefold $X$. The number $n_{\overline{27}}$ of anti-generations vanishes.*

of anti-generations vanishes and the number of generations can be computed from the index, so that

$$n_{16} = h^1(X, V) = -\operatorname{ind}(V) , \qquad n_{\overline{16}} = 0 . \tag{54}$$

Thus, for the rank 4 bundles in Tables 4–8, the (negative) of the right-most column gives the number of **16** representations.

Next, we need to compute the Higgs content which is given by $n_{10} = h^1(X, \wedge^2 V)$. It can be shown in general that for generic maps $g : B \to C$ the number of **10** representations always vanishes, that is

$$n_{10} = 0 . \tag{55}$$

The proof is somewhat technical and can be found in Appendix B.2. Again, this result can be readily verified using computer algebra.

Finally, we need to compute the number $n_1$ of SO(10) singlets which is easily obtained from Eq. (53). The results for the spectrum from rank 4 bundles are summarized in Table 10.

A vanishing number, $n_{10}$, of Higgs particles is not desirable from a particle physics viewpoint. One might, therefore, wonder whether more specific choices of the map $g$ in (18) could produce a non-zero value for $n_{10}$. This problem has been encountered in Ref. [5, 24, 6] where the spectrum of compactification was shown to depend on the region of moduli space. Specifically, it was shown that the spectrum takes a generic form with possible enhancements in special regions of the moduli space; this was dubbed the "jumping phenomenon" in [24, 6].

To see that a similar phenomenon can arise for monad bundles, let is consider the following $SU(4)$ bundle on the quintic, [4|5].

$$0 \to V \to \mathcal{O}_X^{\oplus 2}(2) \oplus \mathcal{O}_X^{\oplus 4}(1) \xrightarrow{g} \mathcal{O}_X^{\oplus 2}(4) \to 0 \ . \tag{56}$$

This bundle and its particle content for a generic map $g$ is given in the first line of Table 10. Now we explicitly define the map $g$ by

$$g = \begin{pmatrix} 4x_3^2 & 9x_0^2 + x_2^2 & 8x_2^3 & 2x_3^3 & 4x_1^3 & 9x_1^3 \\ x_0^2 + 10x_2^2 & x_1^2 & 9x_2^3 & 7x_3^3 & 9x_1^3 + x_2^3 & x_1^3 + 7x_4^3 \end{pmatrix} \ . \tag{57}$$

where $x_0, \ldots, x_4$ are the homogeneous coordinates of $\mathbb{P}^4$. This choice for $g$ is no longer completely generic, although the sequence (56) is still exact. Following the steps in Appendix A.4, we can use Macaulay to calculate the spectrum for this case. We find

$$n_{16} = 90 \ , \qquad n_{\overline{16}} = 0 \ , \qquad n_{10} = 13 \ , \qquad n_1 = 277 \ . \tag{58}$$

This is identical to the generic result in Table 10, except for the number of **10** representations which has changed from 0 to 13.

## 4.4 $SU(5)$-GUT Theories

Finally, we should consider SU(5) GUT theories which originate from rank 5 bundles with structure group SU(5). To demonstrate their stability from Hoppe's criterion we have to show that $H^0(X, \Lambda^p V^*)$ for $p = 1, 2, 3, 4$ vanish. For $p = 1, 2, 4$ this has already been accomplished in Section 3.3, so it remains to deal with the case $p = 3$.

Unfortunately, for $p = 3$ the long exterior power sequences (34) together with Kodaira vanishing are not quite sufficient to prove that $H^0(X, \Lambda^3 V^*) = 0$. Indeed, writing down (33) for $p = 3$ we find

$$0 \to S^3 C^* \to S^2 C^* \otimes B^* \to K_1 \to 0 \ ,$$
$$0 \to K_1 \to C^* \otimes \wedge^2 B^* \to K_2 \to 0 \ , \tag{59}$$
$$0 \to K_2 \to \wedge^3 B^* \to \wedge^3 V^* \to 0 \ .$$

| $X$ | $\{b_i\}$ | $\{c_i\}$ | $n_{16}$ | $n_1$ |
|---|---|---|---|---|
| $[4\vert5]$ | (2, 2, 1, 1, 1, 1) | (4, 4) | 90 | 277 |
| | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 30 | 112 |
| | (2, 2, 2, 1, 1, 1, 1) | (4, 3, 3) | 75 | 236 |
| | (2, 2, 2, 2, 1, 1, 1, 1) | (3, 3, 3, 3) | 60 | 193 |
| $[5\vert2\ 4]$ | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 48 | 159 |
| $[5\vert3\ 3]$ | (1, 1, 1, 1, 1, 1) | (3, 3) | 72 | 213 |
| | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 54 | 166 |
| | (1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2) | 36 | 113 |
| $[6\vert2\ 2\ 3]$ | (1, 1, 1, 1, 1, 1, 1) | (3, 2, 2) | 72 | 213 |
| | (1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2) | 48 | 145 |

*Table 10: The particle content for the $SO(10)$-GUT theories arising from our classification of stable, positive, $SU(4)$ monad bundles $V$ on the Calabi-Yau threefold $X$. The number $n_{\overline{16}}$ of anti-generations vanishes. The number $n_{10}$ vanishes for generic choices of the map $g$ in the monad sequence (18), but can be made non-vanishing with particular choices of $g$.*

Now, using the 3 intertwined long exact sequences in cohomology induced by the above 3 sequences, together with Kodaira vanishing for the negative bundles formed from the symmetric and anti-symmetric powers of $B^*$ and $C^*$, we can only conclude that

$$H^0(X, \wedge^3 V^*) \simeq H^2(X, K_1) . \tag{60}$$

We will now show that the stability proof can be completed by applying Koszul resolutions to our rank 5 bundles. This technique makes explicit use of the embedding in the ambient space $\mathcal{A} = \mathbb{P}^m$ and its complexity grows with the number of co-dimensions of the Calabi-Yau manifold $X$ in $\mathcal{A}$. We, therefore, start with the quintic, $X = [4\vert5]$, the only co-dimension one example among the five Calabi-Yau manifolds under consideration, before we proceed to the more complicated examples.

### 4.4.1 Stability for Rank 5 Bundles on the Quintic

For the quintic, the normal bundle is simply given by $\mathcal{N} = \mathcal{O}(5)$ and the Koszul sequence (40), applied to $\mathcal{W} = \Lambda^3 \mathcal{V}^*$, explicitly reads

$$0 \to \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \to \wedge^3 \mathcal{V}^* \to \wedge^3 V^* \to 0 . \tag{61}$$

From this, we have the long exact sequence in cohomology,

$$0 \to H^0(\mathcal{A}, \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) \to H^0(\mathcal{A}, \wedge^3 \mathcal{V}^*) \to H^0(X, \wedge^3 V^*) \to H^1(\mathcal{A}, \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) \to \dots \tag{62}$$

Thus, if we knew $H^0(\mathcal{A}, \wedge^3 \mathcal{V}^*)$ and $H^1(\mathcal{A}, N^* \otimes \wedge^3 \mathcal{V}^*)$, we could hope to determine $H^0(X, \wedge^3 V^*)$ itself. In fact, we can show that $H^0(\mathcal{A}, \wedge^3 \mathcal{V}^*) = H^1(\mathcal{A}, N^* \otimes \wedge^3 \mathcal{V}^*) = 0$ by

writing down the ambient space version of the exterior power sequences (59) tensored by $\mathcal{N}^*$.

$$
\begin{aligned}
0 &\rightarrow \mathcal{N}^* \otimes S^3 \mathcal{C}^* \xrightarrow{h} \mathcal{N}^* \otimes S^2 \mathcal{C}^* \otimes \mathcal{B}^* \rightarrow \mathcal{K}_1 \rightarrow 0 \;, \\
0 &\rightarrow \mathcal{N}^* \otimes \mathcal{K}_1 \rightarrow \mathcal{N}^* \otimes \mathcal{C}^* \otimes \wedge^2 \mathcal{B}^* \rightarrow \mathcal{K}_2 \rightarrow 0 \;, \\
0 &\rightarrow \mathcal{K}_2 \rightarrow \mathcal{N}^* \otimes \wedge^3 \mathcal{B}^* \rightarrow \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow 0 \;.
\end{aligned}
\tag{63}
$$

Since $\mathcal{B}^*$, $\mathcal{C}^*$ and $\mathcal{N}^*$ are all negative bundles, it follows that $H^0(\mathcal{A}, \wedge^3 \mathcal{V}^*) = 0$ and $h^1(\mathcal{A}, \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) = h^3(\mathcal{A}, \mathcal{K}_1) = \ker(h')$, where $h' : H^4(\mathcal{A}, \mathcal{N}^* \otimes S^3 \mathcal{C}^*) \rightarrow H^4(\mathcal{A}, \mathcal{N}^* \otimes S^2 \mathcal{C}^* \otimes \mathcal{B}^*)$ is the map induced from $h$ above. Now, we note that since the ranks of the maps in the defining monads were chosen, by construction, to be maximal rank, it follows that the induced map $h$ in the exterior power sequence is also maximal rank. To proceed further, we finally observe that for any generic, maximal rank map $h : \mathcal{U} \rightarrow \mathcal{W}$ between two ambient space bundles $\mathcal{U}$ and $\mathcal{W}$ the induced map $\tilde{h} : H^0(\mathcal{A}, \mathcal{U}) \rightarrow H^0(\mathcal{A}, \mathcal{W})$ is also maximal rank (see Appendix B.2). Since the sequences above are all defined over the ambient space and $h$ is maximal rank, it follows from the above argument that $h'$ is maximal rank and $\ker(h') = 0$. Therefore,

$$
h^1(\mathcal{A}, \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) = 0.
\tag{64}
$$

Thus, returning to (62), we find that $H^0(X, \wedge^3 V^*) = 0$ and by Hoppe's criterion, *all generic, positive $SU(5)$ bundles are stable on the quintic.*

### 4.4.2 The Co-dimension 2 and 3 Manifolds

The stability proof for our remaining rank 5 bundles is similar in approach, but slightly more lengthy than that given in the previous subsection. In the interests of space, we will only give an overview of it here. We recall from Subsection 4.1 that the remaining Calabi-Yau manifolds with rank 5 bundles are defined by two and three constraints in $\mathbb{P}^5$ and $\mathbb{P}^6$ respectively. We first look at the co-dimension two case.

For co-dimension two, the normal bundle takes the form $N = \mathcal{O}(q_1) \oplus \mathcal{O}(q_2)$ with $q_1, q_2 > 0$. This time the Koszul sequence (40) is no longer short-exact, but reads

$$
0 \rightarrow \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow \wedge^3 \mathcal{V}^* \xrightarrow{\rho} \wedge^3 V^* \rightarrow 0 \;.
\tag{65}
$$

It can be split into two short exact sequences,

$$
\begin{aligned}
0 &\rightarrow \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow \mathcal{K} \rightarrow 0 \;, \\
0 &\rightarrow \mathcal{K} \rightarrow \wedge^3 \mathcal{V}^* \xrightarrow{\rho} \wedge^3 V^* \rightarrow 0 \;.
\end{aligned}
\tag{66}
$$

From the long cohomology sequences of these two resolutions, we find that $H^0(X, \wedge^3 V^*) \simeq H^2(\mathcal{A}, \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*)$ (since $H^0(\mathcal{A}, \wedge^3 \mathcal{V}^*) = H^0(\mathcal{A}, \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) = 0$ by the same arguments as before). Next, the exterior power sequence (32), multiplied by $\Lambda^2 \mathcal{N}^*$ and

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written over $\mathbb{P}^5$ yields,

$$
\begin{aligned}
0 & \rightarrow \wedge^2 \mathcal{N}^* \otimes S^3 \mathcal{C}^* \xrightarrow{h} \wedge^2 \mathcal{N}^* \otimes S^2 \mathcal{C}^* \otimes \mathcal{B}^* \rightarrow \mathcal{K}_1 \rightarrow 0 \ , \\
0 & \rightarrow \mathcal{K}_1 \rightarrow \wedge^2 \mathcal{N}^* \otimes \mathcal{C}^* \otimes \wedge^2 \mathcal{B}^* \rightarrow \mathcal{K}_2 \rightarrow 0 \ , \\
0 & \rightarrow \mathcal{K}_2 \rightarrow \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{B}^* \rightarrow \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^* \rightarrow 0 \ .
\end{aligned}
\tag{67}
$$

Once again, we find that $H^2(\mathcal{A}, \wedge^2 \mathcal{N}^* \otimes \wedge^3 \mathcal{V}^*) \simeq H^4(\mathcal{A}, \mathcal{K}_1)$ and $h^4(\mathcal{A}, \mathcal{K}_1) = \ker(h')$ where $h' : H^5(\mathcal{A}, \wedge^2 \mathcal{N}^* \otimes S^3 \mathcal{C}^*) \rightarrow H^5(\mathcal{A}, \wedge^2 \mathcal{N}^* \otimes S^2 \mathcal{C}^* \otimes \mathcal{B}^*)$. As before, it follows from our definition of the monad that $h'$ is maximal rank and $\ker(h') = 0$. Therefore, all positive rank 5 bundles on the manifolds [5|2 4] and [5|3 3] are stable.

With this analysis complete, we are left with only one rank 5 bundle on the co-dimension 3 manifold, [6|2 2 3], to consider. In this case, we could directly apply the Koszul resolution techniques as above, with a normal bundle, $\mathcal{N} = \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(3)$, and higher antisymmetric powers in the Koszul resolution (40). Note, however, that in this case we are not assured that the dual sequence (22) is well defined on the ambient space, since the numeric criteria in Theorem 3.2 are not satisfied on $\mathbb{P}^6$. However, we can still compute the cohomology of the relevant sheaves on $\mathbb{P}^6$. The calculation is lengthy, but straightforward.

It is worth noting that there is an alternative approach to this case. Instead of viewing the Koszul resolution as describing the restriction of objects on $\mathbb{P}^m$ to the Calabi-Yau, we may view $X = [6|2 2 3]$ as a sub-variety in the 4-fold $Y = [6|2 2]$. Then we may apply the Koszul techniques exactly as before, viewing the normal bundle to the Calabi-Yau as a line bundle, $\mathcal{O}_Y(3)$ in [6|2 2]. The analysis then reduces to that described for the co-dimension 1 case (61) (that is, that of the rank 5 bundles on the quintic). A straightforward calculation shows that $H^0(X, \wedge^3 V^*) = 0$ and the final rank 5 bundle is stable.

### 4.4.3 Particle Content

We have shown, using the Koszul sequence, that all positive rank 5 bundles in our classification are stable. Let us now analyze their particle spectrum. From Table 2, we need to compute $n_{10} = h^1(X, V)$, $n_{\overline{10}} = h^1(X, V^*) = h^2(X, V)$, $n_5 = h^1(X, \wedge^2 V)$, $n_{\overline{5}} = h^1(X, \wedge^2 V^*) = h^2(X, \wedge^2 V)$, and $n_1 = h^1(X, V \otimes V^*)$. As for rank 4 and 5 bundles, we have $h^0(X, V) = h^3(X, V) = 0$ from stability and $h^2(X, V) = 2$ from the sequence (30). Consequently, we find

$$
n_{10} = h^1(X, V) = -\operatorname{ind}(V) \ , \qquad n_{\overline{10}} = 0 \ .
\tag{68}
$$

As before, we have no anti-generations and the (negative) of the index, listed in right-most column of Tables 4–7, gives the number $n_{10}$ for all rank 5 bundles. We include these in Table 11 for reference.

Next, we need to compute the $H^1(X, \wedge^2 V)$ and $H^2(X, \wedge^2 V)$. From the above ar-guments we know that $V$ is stable; hence $\wedge^2 V$ is also stable and thus $H^0(X, \wedge^2 V)$ and

25

| $X$ | $\{b_i\}$ | $\{c_i\}$ | $n_{10}$ | $n_1$ |
|---|---|---|---|---|
| [4|5] | (1, 1, 1, 1, 1, 1, 1, 1) | (3, 3, 2) | 45 | 202 |
| | (1, 1, 1, 1, 1, 1, 1, 1) | (4, 2, 2) | 60 | 262 |
| | (2, 2, 2, 2, 2, 1, 1, 1, 1, 1) | (3, 3, 3, 3, 3) | 75 | 301 |
| [5|2 4] | (1, 1, 1, 1, 1, 1, 1, 1) | (3, 3, 2) | 72 | 288 |
| [5|3 3] | (1, 1, 1, 1, 1, 1, 1, 1, 1) | (3, 2, 2, 2) | 63 | 243 |
| | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2, 2) | 45 | 176 |
| [6|2 2 3] | (1, 1, 1, 1, 1, 1, 1, 1, 1) | (2, 2, 2, 2, 2) | 60 | 226 |

*Table 11: The particle content for the $SU(5)$-GUT theories arising from our classification of stable, positive, $SU(5)$ monad bundles $V$ on the Calabi-Yau threefold $X$. The number of anti-generations, $n_{\overline{10}}$, vanishes. Further, $n_5 = n_{10}$. Moreover, $n_{\overline{5}} = 0$ for generic choices of the map $g$ in Eq. (18), and can be made non-vanishing in special regions of moduli space.*

$H^3(X, \wedge^2 V)$ both vanish (recall that we have already shown explicitly that $H^0(X, \wedge^2 V^*) = H^3(X, \wedge^2 V)$ vanishes). Therefore, applying the index theorem (4) to $\Lambda^2 V$ we have

$$- h^1(X, \wedge^2 V) + h^2(X, X, \wedge^2 V) = \text{ind}(\wedge^2 V) = \frac{1}{2} \int_X c_3(\wedge^2 V) . \tag{69}$$

For $SU(n)$ bundles one has (see Eq. (339) of Ref. [6]),

$$c_3(\wedge^2 V) = (n - 4)c_3(V) . \tag{70}$$

Hence, combining (69) and (70), we find the relation

$$- n_5 + n_{\overline{5}} = \text{ind}(V) = -n_{10} . \tag{71}$$

We still need to compute one of the numbers $n_5$ and $n_{\overline{5}}$. Macaulay [23] can very easily calculate $n_{\overline{5}} = h^1(X, \wedge^2 V^*) = h^2(X, \wedge^2 V)$. It turns out that

$$n_{\overline{5}} = 0 \tag{72}$$

for all rank 5 bundles and generic choices [4] of the map $g$. From Eq. (71) this implies

$$n_5 = n_{10} , \tag{73}$$

and, hence, the complete spectrum is determined by $n_{10}$ and $n_1$. We have listed these numbers in Table 11.

## 5 Conclusion

In this paper, we have presented a classification of positive $SU(n)$ monad bundles on the five Calabi-Yau manifolds defined by complete intersections in a single projective space.

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[4]Presumably, $n_{\overline{5}}$ can be different from zero for non-generic choices of the map $g$, similar to the case of $n_{\overline{10}}$ for rank 4 bundles.

We have required that these bundles can be incorporated into a consistent heterotic compactification where the heterotic anomaly cancellation condition can be satisfied by including an appropriately wrapped five-brane. In addition, we have imposed two "physical" conditions, namely that the rank of bundle be $n = 3, 4, 5$ (in order to obtain a suitable grand unification group) and that the index of the bundle (that is, the chiral asymmetry) is a non-zero multiple of three. Given these conditions, we found 37 bundles on the five Calabi-Yau manifolds in question, 20 for rank 3, 10 for rank 4 and 7 for rank 5. Using a simple criterion due to Hoppe, we have shown that all these bundles are stable and, hence, lead to supersymmetric compactifications. We have also computed the full particle spectrum for all 37 cases, including the number of gauge singlets. A generic feature of all our bundles is that the number of anti-generations vanishes.

These results show that a combination of analytic computations and computer algebra can be used to analyze a class of models algorithmically. In particular, we have seen that the notoriously difficult problem of proving stability can be addressed systematically and that the full particle spectra can be obtained for all cases. Although the final number of models is still relatively small we expect that these methods can be extended to much larger classes of Calabi-Yau manifolds, such as complete intersections in products of projective spaces and in weighted projective spaces. Such a large-scale analysis which is currently underway [9] will lead to a substantial number of examples with broadly the right physical properties. This class of models can then be used to implement more detailed particle physics requirements and to systematically search for examples close to the standard model.

# Acknowledgments

# A  Monads, Sheaf Cohomology and Computational Algebraic Geometry

In this Appendix, we briefly outline some basics of commutative algebra as relevant for computing sheaf cohomology (see Refs. [38, 40]). In most computer algebra packages such as Macaulay2 [23], of which we make extensive use in this paper, these techniques are essential. Computational algebraic geometry has also been recently used in string

phenomenology in [26] and the reader is referred to tutorials in these papers as well for a quick introduction.

## A.1   The Sheaf-Module Correspondence

Since we are concerned with compact manifolds, we will focus on projective varieties in $\mathbb{P}^m$. A projective algebraic variety is the zero locus of a set of homogeneous polynomials in $\mathbb{P}^m$ with coordinates $[x_0 : x_1 : \ldots : x_m]$. In the language of commutative algebra, projective varieties correspond to homogeneous ideals, $I$, in the polynomial ring $R_{\mathbb{P}^n} = \mathbb{C}[x_0, \ldots, x_m]$. An ideal $I \subset R_{\mathbb{P}^n}$, associated to a variety, is generated by the defining polynomials of the variety and consists of all polynomials which vanish on this variety. The quotient ring $A = R_{\mathbb{P}^n}/I$ is called the *coordinate ring* of the variety.

In general, a ring $R$ is called *graded* if

$$R = \bigoplus_{i \in \mathbb{Z}} R_i, \qquad \text{such that } r_i \in R_i, r_j \in R_j \Rightarrow r_i r_j \in R_{i+j} \ .$$

For the polynomial ring $R_{\mathbb{P}^n}$ the $R_i$ consists of the homogeneous polynomials of degree $i$. In analogy to vector spaces over a field, one can introduce $R$-modules $M$ over the ring $R$. In practice, one can think of $M$ as consisting of vectors with polynomial entries with $R$ acting by polynomial multiplication. A module is called *graded* if

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \qquad \text{such that } r_i \in R_i, m_j \in M_j \Rightarrow r_i m_j \in M_{i+j} \ .$$

The graded ring $R$ is itself a graded $R$-module, $M(R)$. Similarly, an ideal $I$ in a graded ring $R$ is a graded $R$-module and a submodule of $M(R)$. Another important example of a graded $R$ module is $R(k)$ which denotes the ring $R$ with degrees shifted by $-k$. For example, $x^2 y \in R_{\mathbb{P}^n}$ is of degree 3, but seen as an element of the module $R_{\mathbb{P}^n}(-2)$, its degree is $3 + 2 = 5$.

Sheafs over a (projective) variety can also be described as a module by virtue of the *sheaf-module correspondence*. Given the graded ring $R$ and a finitely generated graded $R$-module $M$, one defines an associated sheaf $\widetilde{M}$ as follows. On an open set $U_g$, given by the complement of the zero locus of $g \in R$, the sections over $U_g$ are $\widetilde{M}(U_g) = \{m/g^n | m \in M, \text{degree}(m) = \text{degree}(g^n)\}$. On $\mathbb{P}^m$, this looks concretely as follows. A sufficiently fine open cover of $\mathbb{P}^m$ is provided by $U_{x_i}$, the open sets where $x_i \neq 0$. Let us first consider the module $M(R_{\mathbb{P}^n})$, that is, the ring $R_{\mathbb{P}^n}$ seen as a module. Then $\widetilde{M(R_{\mathbb{P}^n})}(U_{x_i}) = \{f/x_i^m, f \text{ homogeneous of degree } n\}$ and, hence, $\widetilde{M(R_{\mathbb{P}^n})} = \mathcal{O}_{\mathbb{P}^m}$, where $\mathcal{O}_{\mathbb{P}^m}$ is the trivial sheaf on $\mathbb{P}^n$. Similarly, for the modules $R_{\mathbb{P}^n}(k)$ one has

$$\mathcal{O}_{\mathbb{P}^m}(k) \simeq \widetilde{R_{\mathbb{P}^n}(k)} \ .$$

For projective varieties $X \subset \mathbb{P}^m$ and associated ideal $I$, the story is similar. Now, one needs to consider the graded modules over the coordinate ring $A = R/I$. In particular, for line bundles $\mathcal{O}_X(k)$ on $X$ one has

$$\mathcal{O}_X(k) = \widetilde{A(k)} \ .$$

## A.2 Constructing Monads using Computer Algebra

Recall from (41), that we wish to construct bundles $V$ defined by

$$0 \to V \xrightarrow{f} \bigoplus_{i=1}^{r_B} \mathcal{O}_X(b_i) \xrightarrow{g} \bigoplus_{i=1}^{r_C} \mathcal{O}_X(c_i) \longrightarrow 0 \;, \tag{74}$$

over the manifold $X$. In this subsection, we outline how one may proceed with this construction using commutative algebra packages such as [23] and applying the Sheaf-Module correspondence discussed above. Let $A$ be the coordinate ring of $X$. For example, for the quintic, [4|5] we can write

$$A = \mathbb{C}[x_0, \ldots, x_4] / \left( \sum_{i=0}^{5} x_i^5 + \psi x_0 x_1 x_2 x_3 x_4 \right) \;. \tag{75}$$

where the round brackets denote the ideal generated by the enclosed polynomial. In practice, we will randomize $\psi$, the complex structure and in fact work over the ground field $\mathbb{Z}/p\mathbb{Z}$ for some large prime $p$ instead of $\mathbb{C}$ in order to speed up computation. The free modules corresponding to the bundles $B$, $C$ are given by $\oplus_{i=1}^{r_B} A(b_i)$, $\oplus_{i=1}^{r_B} A(c_i)$ with grading $\{b_1, b_2, \ldots, b_{r_B}\}$, $\{c_1, c_2, \ldots, c_{r_C}\}$ and ranks $r_B$, $r_C$ [5]. At the level of modules, the map $g$ can then be specified by an $r_C \times r_B$ matrix whose entries, $g_{ij}$ are homogeneous polynomials of degree $c_i - b_j$, that is $g_{ij} \in \mathcal{O}_X(c_i - b_j)$. Indeed, the degrees of the entries of $g$ are so as preserve the gradings of $B$ and $C$ and our choice $c_i \geq b_j$ ensures that such polynomials indeed exist. Moreover, we choose these polynomials to be random; this corresponds to the genericity assumption for $g$ used repeatedly in the main text.

## A.3 Algorithms for Sheaf Cohomology

We shall not delve into the technicalities of this vast subject and will only mention that for commutative algebra packages such as [23], there are built-in routines for computing cohomology groups of sheafs (modules). The standard algorithm is based on the so-called Bernstein-Gel'fand-Gel'fand correspondence and on Tate resolutions of exterior algebras. The interested reader is referred to the books [38] and [39] for details.

## A.4 A Tutorial

Let us explicitly present a Macaulay2 code [23] for one of the examples from our classification. This will serve to illustrate the power and relative ease with which computer algebra assists in the proof of stability and the calculation of the particle spectrum.

Let us take the first rank 4 example for $X = $ [4|5] in Table 10, which was further discussed around Eq. (56). It is defined by

$$B = \mathcal{O}_X^{\oplus 2}(2) \oplus \mathcal{O}_X^{\oplus 4}(1) \;, \qquad C = \mathcal{O}_X^{\oplus 2}(4) \;. \tag{76}$$

---

[5] In most computer packages, the convention is to actually take the grading to be negative, viz., $\{-b_1, -b_2, \ldots, -b_{r_B}\}$.

We work over the polynomial Ring $R_{\mathbb{P}^4}$ with variables $x_0, \ldots, x_4$ and the ground field $\mathbb{Z}/27449$. The (projective) coordinate ring $A$ of a smooth quintic $X$ is then defined following Eq. (75). In Macaulay this reads

```
R   =   ZZ/27449[x_{0}..x_{4}];
A   =   Proj( R/ideal(x_{0}^5 + x_{1}^5 + x_{2}^5 + x_{3}^5 + x_{4}^5 +
            2*x_{0}*x_{1}*x_{2}*x_{3}*x_{4}));
```

Next, we define o, the trivial sheaf (line-bundle) over $A$, and the $A$-modules associated to the bundles B and C.

```
o = OO_(A);
B = module (o^2 (2) ++ o^4 (1));
C = module (o^2 (4));
```

Subsequently, a random, generic map, gmap, can be constructed between B and C (note that in Macaulay, maps are defined backwards):

```
gmap = map(C, B, random(C, B));
```

Finally, we can define $V^*$ as the co-kernel of the transpose of fmap:

```
Vdual = sheaf coker transpose fmap;
```

We can check that $V^*$ has the expected rank 4 using the command

```
print rank Vdual;
```

The cohomologies of Vdual are easily obtained, for example,

```
print rank HH^2 Vdual;
```

produces 90, precisely as expected. Likewise, one can verify that HH^0 Vdual gives 0, as is required by stability. To compute $n_{10} = h^1(X, \wedge^2 V^*)$, one only needs the following command

```
print rank HH^1 exteriorPower(2, Vdual);
```

which gives 0, as indicated in Table 10. For the non-generic map (57), one can define

```
gmap   =   map (C, B, matrix{{4*x_{3}^2, 9*x_{0}^2 + x_{2}^2, 8*x_{2}^3,
            2*x_{3}^3,4*x_{1}^3,9*x_{1}^3}, {x_{0}^2 + 10*x_{2}^2, x_{1}^2,
            9*x_{2}^3, 7*x_{3}^3,9*x_{1}^3 + x_{2}^3, x_{1}^3 +7*x_{4}^3 } } );
```

One can then check that the cohomologies of $V^*$ remain unchanged with respect to the generic case, that is, $h^0(X, V^*) = h^1(X, V^*) = h^3(X, V^*) = 0$ and $h^2(X, V^*) = 90$ while rank HH^1 exteriorPower(2, Vdual) now results in $n_{10} = h^1(X, \Lambda^2 V^*) = 13$.

The singlets are also easy to compute. The group $H^1(X, V \otimes V^*)$ can be thought of as the global Ext-group $Ext^1(V, V) \simeq Ext^1(V^*, V^*)$; this is, again, implemented in [23]. The command "print rank Ext^1(Vdual, Vdual);" will give us 277.

# B  Some useful technical results

## B.1  Genericity of Maps

We first state a helpful fact regarding the genericity of maps in the ambient space. Consider a morphism $h : \mathcal{B} \to \mathcal{C}$ between two sums of line bundles $\mathcal{B} = \oplus_{i=1}^{r_B} \mathcal{O}(b_i)$ and $\mathcal{C} = \oplus_{i=1}^{r_C} \mathcal{O}(c_i)$ on $\mathcal{A} = \mathbb{P}^m$. The map $h$ can explicitly be specified by a $r_C \times r_B$ matrix $h_{ij} \in \mathcal{O}(c_i - b_j)$ and it induces a map $\tilde{h} : H^0(\mathcal{A}, \mathcal{B}) \to H^0(\mathcal{A}, \mathcal{C})$. The induced map $\tilde{h}$ is also described by $h_{ij}$ acting on the sections of $\mathcal{B}$ and, hence, if the matrix $(h_{ij})$ has maximal rank (almost everywhere) then $\tilde{h}$ has maximal rank.

## B.2  Proof of $H^0(X, V) = 0$

In this section we will provide a proof that $H^0(X, V) = 0$ for all the bundles defined by positive monads on cyclic complete intersection Calabi-Yau manifolds. The proof is similar in spirit to the stability proof of Section 4.5 in that we approach the problem from the point of view of an embedding space and use Koszul sequences (40) to determine the necessary cohomology.

Because the Koszul resolutions depend on the normal bundle to the Calabi-Yau, the length of the calculation increases with the co-dimension of the embedded Calabi-Yau. For conciseness, we will only provide the proof for the co-dimension 1 case (the quintic) here, however the higher co-dimension cases follow by an entirely analogous construction. Consider a positive bundle defined by (18) on the quintic ($X = [4|5]$). Clearly, the normal bundle is simply $\mathcal{N} = \mathcal{O}(5)$, and from (40) we just obtain the short exact sequence,

$$0 \to \mathcal{N}^* \otimes \mathcal{V} \to \mathcal{V} \to V \to 0 . \tag{77}$$

As before, we have the long exact sequence in cohomology,

$$0 \to H^0(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{V}) \to H^0(\mathcal{A}, \mathcal{V}) \to H^0(X, V) \to H^1(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{V}) \to \ldots \tag{78}$$

So, in order to compute $H^0(X, V)$, we must first find $H^0(\mathcal{A}, \mathcal{V})$ and $H^1(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{V})$. To do this, we will define the following short exact sequences on the ambient space:

$$0 \to \mathcal{V} \to \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0 , \tag{79}$$

and the same sequence tensored with the dual of the normal bundle,

$$0 \to \mathcal{N}^* \otimes \mathcal{V} \to \mathcal{N}^* \otimes B \xrightarrow{h} \mathcal{N}^* \otimes \mathcal{C} \to 0 . \tag{80}$$

From the Bott Vanishing formula [40] we have the following formula for the cohomology of line bundles on the ambient space.

$$h^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n}{n} & q = 0 \quad k > -1 \\ 1 & q = n \quad k = -n-1 \\ \binom{-k-1}{-k-n-1} & q = n \quad k < -n-1 \\ 0 & \text{otherwise} \end{cases} . \tag{81}$$

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Using this to compute elements of the various long exact cohomology sequences corresponding to (79) and (80) we find the following

$$0 \to H^0(\mathcal{A}, \mathcal{V}) \to H^0(\mathcal{A}, \mathcal{B}) \xrightarrow{g'} H^0(\mathcal{A}, \mathcal{C}) \to H^1(\mathcal{A}, \mathcal{V}) \to 0 \ , \tag{82}$$
$$0 \to H^0(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{V}) \to H^0(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{B}) \xrightarrow{h'} H^0(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{C}) \to H^1(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{V}) \to 0 \ .$$

Now, we note that since the maps in (79) and (80) were chosen to be maximal rank on $\mathcal{A}$ (that is, the sequences were constructed to be exact) it follows from the arguments in Appendix B.1 that the induced cohomology maps $g'$ and $h'$ on the ambient space are also maximal rank. With these results in hand, we then find that

$$h^0(\mathbb{P}^n, \mathcal{V}) = h^0(\mathbb{P}^n, \mathcal{B}) - \mathrm{rk}(g') \ ,$$
$$h^1(\mathbb{P}^n, \mathcal{N}^* \otimes \mathcal{V}) = h^0(\mathbb{P}^n, \mathcal{N}^* \otimes \mathcal{C}) - \mathrm{rk}(h') \ . \tag{83}$$

Clearly, we will have $h^0(\mathbb{P}^n, \mathcal{V}) = 0$ and $h^1(\mathbb{P}^n, \mathcal{N}^* \otimes \mathcal{V}) = 0$ if $g'$ and $h'$ are maximal rank and if

$$h^0(\mathbb{P}^n, \mathcal{B}) \leq h^0(\mathbb{P}^n, \mathcal{C}) \ ,$$
$$h^1(\mathbb{P}^n, \mathcal{N}^* \otimes \mathcal{C}) \leq h^1(\mathbb{P}^n, \mathcal{N}^* \otimes \mathcal{B}) \ . \tag{84}$$

However, using (81) and the defining sums of line bundles on the ambient space (18) we find by direct calculation that (84) is satisfied for all the rank 4 bundles on the quintic. Therefore, we find that $h^0(X, V) = 0$ for all rank 4 bundles on the quintic.

The analysis for all the other ranks is similar in construction (with additional wedge powers of the normal bundle in (77)) as long as Theorem 3.2 is satisfied and we can consistently define our monads on the ambient space. That is, we must be able to write the short exact sequences (18) on the ambient space with maps whose degeneracy loci vanish. Thus, the technique above can only be applied directly to rank 4 and 5 bundles on $\mathbb{P}^4$ and rank 5 bundles on $\mathbb{P}^5$. For all the other rank 4 and 5 bundles in the list, we again apply the techniques of Koszul sequences, but to a 4-fold in the embedding space rather than $\mathbb{P}^m$ itself. For the rank 3 bundles in our list we cannot apply these methods, nor do we make use of $h^0(X, V) = 0$ in those stability proofs, however we can verify that the identity holds for all the rank 3 bundles as well by computer calculation. Thus, we find that $h^0(X, V) = 0$ for all the bundles in our list and verify this by computer algebra using [23].

## B.3   Proof that $n_1 = h^1(X, \wedge^2 V^*) = 0$ for the $SO(10)$ Models

For simplicity, we provide here the argument for rank 4 bundles on the quintic. As in the previous discussion, the proof is easily extended to the other cases. We begin once again with the Koszul sequence in the co-dimension 1 case, this time for $\wedge^2 V^*$:

$$0 \to \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^* \to \wedge^2 \mathcal{V}^* \to \wedge^2 V^* \to 0 \ . \tag{85}$$

From this, we have the long exact sequence in cohomology,

$$... \to H^1(\mathcal{A}, \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^*) \to H^1(\mathcal{A}, \wedge^2 \mathcal{V}^*) \to H^1(X, \wedge^2 V^*) \to H^2(\mathcal{A}, \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^*) \to ...$$

(86)

We will show that $h^1(X, \wedge^2 V^*) = 0$ by proving that $h^1(\mathcal{A}, \wedge^2 \mathcal{V}^*)$ and $h^2(\mathcal{A}, \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^*)$ both vanish.

We begin with $h^1(\mathcal{A}, \wedge^2 \mathcal{V}^*)$. To proceed, we have the exterior power sequences

$$0 \to S^2 \mathcal{C}^* \to \mathcal{C}^* \otimes \mathcal{B}^* \to \Lambda^2 \mathcal{B}^* \to \Lambda^2 \mathcal{V}^* \to 0 \ , \tag{87}$$

$$0 \to \mathcal{N}^* \otimes S^2 \mathcal{C}^* \to \mathcal{N}^* \otimes \mathcal{C}^* \otimes \mathcal{B}^* \to \mathcal{N}^* \otimes \Lambda^2 \mathcal{B}^* \to \mathcal{N}^* \otimes \Lambda^2 \mathcal{V}^* \to 0 \ , \tag{88}$$

which we can split into the short exact sequences:

$$0 \to S^2 \mathcal{C}^* \to \mathcal{C}^* \otimes \mathcal{B}^* \to K_1 \to 0 \ ,$$
$$0 \to K_1 \to \Lambda^2 \mathcal{B}^* \to \Lambda^2 \mathcal{V}^* \to 0 \ , \tag{89}$$

and similarly,

$$0 \to \mathcal{N}^* \otimes S^2 \mathcal{C}^* \to \mathcal{N}^* \otimes \mathcal{C}^* \otimes \mathcal{B}^* \to K_2 \to 0 \ ,$$
$$0 \to K_2 \to \mathcal{N}^* \otimes \Lambda^2 \mathcal{B}^* \to N^* \otimes \Lambda^2 \mathcal{V}^* \to 0 \ . \tag{90}$$

Each of these generates a long exact sequence in cohomology. Using the familiar results for the cohomologies of positive and negative line bundles on the ambient space, from (89) we immediately obtain $h^1(\mathcal{A}, \Lambda^2 \mathcal{V}^*) = h^2(K_1) = 0$. Likewise, the cohomology sequence of (90) leads us to $h^2(\mathcal{A}, \mathcal{N}^* \otimes \Lambda^2 \mathcal{V}^*) = h^3(K_2) = 0$ and

$$0 \to H^3(\mathcal{A}, K_2) \to H^4(\mathcal{A}, \mathcal{N}^* \otimes S^2 \mathcal{C}^*) \xrightarrow{f} H^4(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{C}^* \otimes \mathcal{B}^*) \to H^4(\mathcal{A}, K_2) \to 0 \ . \tag{91}$$

Combining these results we find

$$h^2(\mathcal{A}, \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^*) = h^4(\mathcal{A}, \mathcal{N}^* \otimes S^2 \mathcal{C}^*) - \mathrm{rk}(f) \ . \tag{92}$$

Now, as before we note that by maximal rank arguments of B.1 and Serre duality,

$$\mathrm{rk}(f) = \min(h^4(\mathcal{A}, \mathcal{N}^* \otimes S^2 \mathcal{C}^*), h^4(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{C}^* \otimes \mathcal{B}^*)) \tag{93}$$

By direct computation using [23] we find that $h^4(\mathcal{A}, \mathcal{N}^* \otimes S^2 \mathcal{C}^*) < h^4(\mathcal{A}, \mathcal{N}^* \otimes \mathcal{C}^* \otimes \mathcal{B}^*)$ for all the bundles in our list. Thus, $h^2(\mathcal{A}, \mathcal{N}^* \otimes \wedge^2 \mathcal{V}^*) = 0$ and we may conclude that

$$h^1(\mathcal{A}, \wedge^2 \mathcal{V}^*) = 0 \ . \tag{94}$$

The argument is the same in spirit for the other manifolds in our list. The only key difference being the length of the starting Koszul sequence (which will containing higher wedge powers of $\mathcal{N}^*$). The resulting cohomology analysis follows straightforwardly.

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