Monad Bundles in Heterotic String Compactifications

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Abstract

In this paper, we study positive monad vector bundles on complete intersection Calabi-Yau manifolds in the context of $E_8 \times E_8$ heterotic string compactifications. We show that the class of such bundles, subject to the heterotic anomaly condition, is finite and consists of about 7000 models. We explain how to compute the complete particle spectrum for these models. In particular, we prove the absence of vector-like family anti-family pairs in all cases. We also verify a set of highly non-trivial necessary conditions for the stability of the bundles. A full stability proof will appear in a companion paper. A scan over all models shows that even a few rudimentary physical constraints reduces the number of viable models drastically.

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1 Introduction

As string and M-theory continue to develop, it remains a problem of central importance to produce models that are relevant to 4-dimensional particle phenomenology. While many approaches to this goal have been explored over the years, a string model with exactly the particle content and detailed properties of the standard model remains elusive. One of the first and currently most successful approaches to this challenge has been provided by heterotic string theory. Because they naturally incorporate gauge unification, heterotic models are particularly well suited for use in string phenomenology. The vector bundles with $\mathrm{SU}(n)$ structure group used in heterotic models lead to the gauge groups of grand unified theories (GUTs) in 4-dimensions and under suitable symmetry breaking (that is, Wilson lines, etc) can contain the symmetry of the standard model.

More specifically, compactification of 10-dimensional heterotic string theory on Calabi-Yau three-folds equipped with (poly-)stable holomorphic $SU(n)$ vector bundles leads to $N = 1$ supersymmetric versions of GUTs.

Despite substantial recent progress [12–22], heterotic model-building continues to present a number of formidable mathematical obstacles. In addition to a Calabi-Yau three-fold X, heterotic models require two holomorphic (poly-)stable vector bundles V and $\tilde{V}$. Except for the simplest case of the so-called "standard embedding" (in which V is taken to be the tangent bundle to the Calabi-Yau and $\tilde{V}$ is trivial) explicit constructions of both the Calabi-Yau three-fold and the vector bundle, V are generally hard to obtain and difficult to analyze mathematically. It is our goal in this work to present techniques studying a large class of heterotic models in detail. We utilize the well-known monad construction of vector bundles to build bundles over the set of compete intersection Calabi-Yau manifolds.

It is our hope that by formulating a systematic construction of a large class of vector bundles over an explicit and relatively simple set of Calabi-Yau manifolds, we can build a substantial number of heterotic models which can be thoroughly scanned for physically relevant properties. This program was begun in [1] in which we laid out an algorithmic approach to bundle constructions over cyclic Calabi-Yau three-folds defined as complete intersections in a single projective space. In this work, we greatly extend our class of bundles and manifolds by generalizing the techniques to Calabi-Yau manifolds obtained as complete intersections in products of (un-weighted) projective spaces. From the 7890 such complete intersection Calabi-Yau manifolds (CICYs) classified in [2–7], we consider the 4500 or so "favourable" ones, by which we mean CICYs whose second cohomology entirely descends from the ambient space. In this paper, we focus on the "traditional" class of positive monads, that is, monads defined using strictly positive line bundles only. In a forthcoming publication [45] we will show that this condition of positivity can, in fact, be somewhat relaxed.

This paper has two main objectives. First, we will show that there is a finite number of positive monads bundles on favourable CICYs and provide a complete classification. Second, we will develop algorithms to calculate the complete particle spectrum of all such monads and apply these methods to carry out a statistical analysis and identify promising particle physics models. Finally, we perform a number of checks for the stability of positive monad bundles. A systematic proof of stability will be presented in the companion paper [46].

The plan of the paper is as follows. In Sections 2, 3 and 4 we briefly review some general facts about heterotic model building, complete intersection Calabi-Yau manifolds and the monad construction, respectively. Section 5 summarises the various physical and mathematical constraints on positive monads and why the number of such monads is finite; we then present a complete classification. Some non-trivial checks for the stability of these bundles are carried out in Section 6. Computation of the particle spectrum is discussed in Section 7, before we conclude in Section 8. To simplify the discussion in the main body of the paper, many of the underlying mathematical methods and technical results have been collected in appendices. Appendix A summarises our notation and conventions throughout the paper. In Appendix B we review some mathematical methods and formulae which are essential for our calculations. Appendix C collects technical results on CICYs, most of them well-known, some, such as the identification of redundancies in the CICY list, new.

2 Heterotic Calabi-Yau Model Building

To set the scene, we start by briefly reviewing the basics facts on $E_8 \times E_8$ heterotic Calabi-Yau model building. For a more complete discussion see for example [8–10, 12].

A heterotic Calabi-Yau model is specified by four pieces of data, a Calabi-Yau manifolds X, the observable and hidden holomorphic vector bundles V and $\tilde{V}$ on X, each with a structure group contained in E_8 and a holomorphic curve $C \subset X$ with associated homology class $W = [C] \in H_2(X, \mathbb{Z})$. Physically, the curve C is wrapped by five-branes stretching across the four-dimensional uncompactified space-time. While models without five-branes are of course possible we would like to maintain a general viewpoint and include this possibility. On this data three physical constraints have to be imposed.

- Anomaly cancellation: Anomaly cancellation in the heterotic string imposes a topological condition which relates the Calabi-Yau manifold X, the two vector bundles and the five-brane class W. For the case of bundles V and $\tilde{V}$ with $c_1(V) = c_1(\tilde{V}) = 0$ it can be written as

$$c_2(TX) - c_2(V) - c_2(\tilde{V}) = W \; . \tag{2.1}$$

- Effectiveness: To ensure four-dimensional $N = 1$ supersymmetry the five-brane has to wrap a holomorphic curve. Hence, the five-brane class W must be chosen such that it indeed has a holomorphic curve representative C, with $W = [C]$. Classes $W \in H_2(X, \mathbb{Z})$ with this property are called *effective*.

- Stability: The Donaldson-Uhlenbeck-Yau theorem [11] guarantees the existence of a connection satisfying the hermitian Yang-Mills equations (and, hence, preserving $N = 1$ supersymmetry) on a holomorphic vector bundle, provided this bundle is *(poly)-stable*. Hence, both V and $\tilde{V}$ must be (poly)-stable holomorphic vector bundles on X.

Since the notion of stability of a vector bundle is perhaps not overly familiar it is worth providing a definition. In this paper, we will not be concerned with poly-stability but only with the slightly stronger condition of *stability*. To define this condition, one needs to introduce the *slope* of a bundle (or coherent sheaf) V by

$$\mu(V) \equiv \frac{1}{\text{rk}(V)} \int_X c_1(V) \wedge J \wedge J \; , \tag{2.2}$$

where J is the Kähler form on X. Then, a bundle V is called stable if $\mu(\mathcal{F}) < \mu(V)$ for any coherent sub-sheaf $\mathcal{F} \subset V$ with $0 < \text{rk}(\mathcal{F}) < \text{rk}(V)$.

Within the above set-up we will make a number of standard model-building choices. We are mostly interested in the observable sector and for the associated vector bundle V we require a structure group $G = \text{SU}(n)$, where $n = 3, 4, 5$. This means that the rank of V should be $\text{rk}(V) = n = 3, 4, 5$ and, in order to have a special unitary rather than just a unitary structure group, we need

$$c_1(V) = 0 \; . \tag{2.3}$$

This class of bundles also enjoy nice properties with regard to stability. Examining (2.2), we see that an $SU(n)$ bundle is stable if and only if all its proper sub-sheafs have *strictly* negative slope. An immediate consequence of stability of V is that $H^0(X, V)$ vanishes [1]: Another useful property is that a bundle V is stable if and only if its dual $V^\star$ is stable [23, 27]. Thus, $H^0(X, V^\star)$ also vanishes for our bundles. In summary, for stable bundles V we necessarily have

$$H^0(X, V) = H^0(X, V^\star) = 0 \; . \tag{2.4}$$

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[1] This ensues from the following simple argument. Since if $H^0(X, V) = H^0(X, V \otimes \mathcal{O}_X^\star) = \hom_X(\mathcal{O}, V) \neq 0$, then $\mathcal{O}_X$ is a proper sub-line-bundle of V. However, $c_1(\mathcal{O}_X) = 0$ and $\text{rk}(\mathcal{O}_X) = 1$ so $\mu(\mathcal{O}_X) = 0$ and not strictly negative, whereby making $\mathcal{O}_X$ a proper de-stablising subsheaf and V would be unstable. Similarly, $H^0(X, V^\star) = 0$.

In fact (2.4) is only the first of a set of vanishing conditions (see Ref. [12]): if an $SU(n)$ bundle V is stable, it is further true that

$$H^0(X, \wedge^p V) = 0 \qquad \forall\, p = 1, \ldots, \mathrm{rk}(V) - 1 \ . \tag{2.5}$$

Note that since for $SU(n)$ bundles, $\wedge^p V \simeq \wedge^q V^\star$ for all $p + q = n = \mathrm{rk}(V)$ (cf. Eq.(B.4)), (2.5) is equivalent to saying that $H^0(X, \wedge^p V^\star) = 0$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$.

The definition of stability involves all coherent sub-sheafs of a given bundle and is, therefore, typically not easy to prove. In this paper, we will be content with performing a "check" for stability by verifying the necessary and highly non-trivial (but generally not sufficient) conditions (2.5). A full stability proof of the monad bundles considered in this paper will appear in Ref. [46].

Of course, we have to make sure that there exist a solution to the anomaly condition (2.1). An effective way to guarantee this which does not require searching for suitable hidden bundles $\tilde{V}$ is to demand that

$$c_2(TX) - c_2(V) \text{ is an effective class on } X \ . \tag{2.6}$$

In this case, both the anomaly and effectiveness conditions are satisfied [2] for a trivial hidden bundle $\tilde{V}$ and a five-brane class $W = c_2(TX) - c_2(V)$.

The observable low-energy particle content from such a model is summarized in Table 1. For the three choices of structure group $G = SU(3), SU(4)$ or $SU(5)$ one obtains low-energy GUTs with gauge group $H = E_6, SO(10)$ or $SU(5)$, respectively. The representations of H which arise in the effective four-dimensional theory are obtained by decomposing the **248** adjoint representation of E_8 under $G \times H$. The number of matter fields in the various representations is given by the dimension $h^1(X, U)$ of the bundle cohomology groups, where $U = V, V^\star, \wedge^2 V, \wedge^2 V^\star, V \otimes V^\star$, as indicated in Table 1. A particularly useful quantity is the index $\mathrm{ind}(V) \equiv h^0(X, V) - h^1(X, V) + h^2(X, V) - h^3(X, V)$ of a bundle V. A stable bundle V satisfies Eq. (2.4) and, hence, the index equals $\mathrm{ind}(V) = -h^1(X, V) + h^2(X, V)$. Comparing with Table 1, this is precisely the chiral asymmetry, that is the difference of the number of anti-families and families. From the Atiyah-Singer index theorem it can be computed in terms of the third Chern class $c_3(V)$ of the bundle V as

$$\mathrm{ind}(V) = -h^1(X, V) + h^2(X, V) = \frac{1}{2} \int_X c_3(V) \ . \tag{2.7}$$

Provided that $H^0(X, \Lambda^2 V) = H^0(X, \Lambda^2 V^\star) = 0$, as will indeed be the case for our bundles and will be explicitly checked later on, the index theorem applied to $\Lambda^2 V$ together with the relation $c_3(\Lambda^2 V) = (n - 4)c_3(V)$ (see the appendix of Ref. [12]) leads to

$$(n - 4)\, \mathrm{ind}(V) = -h^1(X, \wedge^2 V) + h^1(X, \wedge^2 V^\star) \ . \tag{2.8}$$

This result will be useful for the SU(5) case and it implies that the chiral asymmetry between $\bar{\mathbf{5}}$ and $\mathbf{5}$ representations is the same as the one between $\mathbf{10}$ and $\bar{\mathbf{10}}$. Hence, the chiral part of the spectrum always comes in pairs of $\mathbf{10}$ and $\bar{\mathbf{5}}$ (or $\bar{\mathbf{10}}$ and $\mathbf{5}$), that is in complete SU(5) families (or anti-families).

For a realistic model the GUT group H will have to be eventually broken to the standard model group. This is usually accomplished by dividing the Calabi-Yau manifold X by a freely-acting discrete symmetry Γ and then introducing Wilson lines on the quotient space X/Γ. In this paper, we will not carry this step out explicitly. However, when we analyse the properties of our models later

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[2]Of course, there may be other choices which involve a non-trivial hidden bundle $\tilde{V}$. Since we are mostly interested in the observable sector at this stage the important point for now is the existence of a viable hidden sector.

$G \times H$	Breaking Pattern: $\mathbf{248} \rightarrow$	Particle Spectrum
$SU(3) \times E_6$	$(\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1})$	$\begin{aligned} n_{27} &= h^1(X, V) \\ n_{\overline{27}} &= h^1(X, V^\star) = h^2(X, V) \\ n_1 &= h^1(X, V \otimes V^\star) \end{aligned}$
$SU(4) \times SO(10)$	$(\mathbf{1}, \mathbf{45}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{16}}) \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{15}, \mathbf{1})$	$\begin{aligned} n_{16} &= h^1(X, V) \\ n_{\overline{16}} &= h^1(X, V^\star) = h^2(X, V) \\ n_{10} &= h^1(X, \wedge^2 V) \\ n_1 &= h^1(X, V \otimes V^\star) \end{aligned}$
$SU(5) \times SU(5)$	$(\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{24}, \mathbf{1})$	$\begin{aligned} n_{10} &= h^1(X, V) \\ n_{\overline{10}} &= h^1(X, V^\star) = h^2(V) \\ n_5 &= h^1(X, \wedge^2 V^\star) \\ n_{\overline{5}} &= h^1(X, \wedge^2 V) \\ n_1 &= h^1(X, V \otimes V^\star) \end{aligned}$

Table 1: A vector bundle V with structure group G can break the E_8 gauge group of the heterotic string into a GUT group H. The low-energy representation are found from the branching of the $\mathbf{248}$ adjoint of E_8 under $G \times H$ and the low-energy spectrum is obtained by computing the indicated bundle cohomology groups.

on we will impose an important physical constraint which follows from this construction. Assuming that the bundle V descends to the quotient space X/Γ the "downstairs" chiral asymmetry is given by $\mathrm{ind}(V)/|\Gamma|$, where $|\Gamma|$ is the order of the discrete symmetry group. Clearly, the downstairs chiral asymmetry should be three, so we should require that

$$\mathrm{ind}(V) \text{ is divisible by } 3 . \tag{2.9}$$

Assuming this is the case, the order of the discrete group Γ one needs is given by $|\Gamma| = \mathrm{ind}(V)/3$. The Calabi-Yau manifold X can only be quotiented by such a group if the Euler number $\chi(X)$ is divisible by its order $|\Gamma|$. Hence, in addition we demand that

$$\chi(X) \text{ is divisible by } \mathrm{ind}(V)/3 \tag{2.10}$$

These two conditions are clearly necessary for successful Wilson line breaking to a model with three families but by no means sufficient. Nevertheless, we will see that they already impose strong constraints on monad bundles.

3 Complete Intersection Calabi-Yau Threefolds

To begin our construction of vector bundles for heterotic models, we first discuss the relevant class of compact Calabi-Yau manifolds. Ever since the realization that Calabi-Yau three-folds played a central role in superstring compactification [8], constructions of so-called "complete intersection Calabi-Yaus" (CICYs) [2–7] have been a topic of interest. Indeed, this method of Calabi-Yau construction was used in one of the first attempts to systematically study families of Calabi-Yau manifolds. Subsequent work, especially in light of mirror symmetry, was carried out in explicit mathematical detail [4,6,7,29] for half a decade, culminating in the pedagogical text [30] on the subject. The manifolds in [1], used to illustrate a new algorithmic approach in heterotic compactification, are special cases of these CICYs.

Unfortunately, much of the original data was stored on computer media, such as magnetic tapes at CERN, which have been rendered obsolete by progress. Partial results, including, luckily, the list of the CICY threefolds itself, can be found on the Calabi-Yau Homepage [31]. In this section, we shall resurrect some of the useful facts concerning the CICY threefolds, which will be of importance to our bundle constructions later. We will present only the essentials, leaving most of the details to Appendix C.

3.1 Configuration Matrices and Classification

We are interested in manifolds X which can be described as algebraic varieties, that is, as intersections of the zero loci of K polynomials $\{p_j\}_{j=1,\ldots,K}$ in an ambient space $\mathcal{A}$. For our purpose, we will consider ambient spaces $\mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$ given by a product of m ordinary projective spaces with dimensions n_r. We denote the projective coordinates of each factor $\mathbb{P}^{n_r}$ by $\mathbf{x}^{(r)} = [x_0^{(r)} : x_1^{(r)} : \ldots : x_{n_r}^{(r)}]$, its Kähler form by J_r and the kth power of the hyperplane bundle by $\mathcal{O}_{\mathbb{P}^{n_r}}(k)$. The Kähler forms are normalised such that

$$\int_{P^{n_r}} J_r^{n_r} = 1 \ . \tag{3.1}$$

The manifold X is called a *complete intersection* if the dimension of X is equal the dimension of $\mathcal{A}$ minus the number of polynomials. This is, in a sense, the optimal way in which polynomials can intersect. To obtain threefolds X from complete intersections we then need

$$\sum_{r=1}^{m} n_r - K = 3 \ . \tag{3.2}$$

Each of the defining homogeneous polynomials p_j can be characterised by its multi-degree $\mathbf{q}_j = (q_j^1, \ldots, q_j^m)$, where q_j^r specifies the degree of p_j in the coordinates $\mathbf{x}^{(r)}$ of the factor $\mathbb{P}^{n_r}$ in $\mathcal{A}$. A convenient way to encode this information is by a *configuration matrix*

$$\begin{bmatrix} \mathbb{P}^{n_1} & q_1^1 & q_2^1 & \cdots & q_K^1 \\ \mathbb{P}^{n_2} & q_1^2 & q_2^2 & \cdots & q_K^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_1^m & q_2^m & \cdots & q_K^m \end{bmatrix}_{m \times K} \ . \tag{3.3}$$

Note that the jth column of this matrix contains the multi-degree of the polynomial p_j. In order that the resulting manifold be Calabi-Yau, the condition

$$\sum_{j=1}^{K} q_j^r = n_r + 1 \qquad \forall r = 1, \ldots, m \tag{3.4}$$

needs to imposed (essentially to guarantee that $c_1(TX)$ vanishes). Henceforth, a CICY shall mean a Calabi-Yau threefold, specified by the configuration matrix (3.3), satisfying the conditions (3.2) and (3.4). In fact, the condition (3.4) even obviates the need for the first column $\mathbb{P}^{n_1} \ldots \mathbb{P}^{n_m}$ in the configuration matrix. Subsequently, we will frequently need the normal bundle $\mathcal{N}$ of X in $\mathcal{A}$ which is given by

$$\mathcal{N} = \bigoplus_{j=1}^{K} \mathcal{O}_{\mathcal{A}}(\mathbf{q}_j) \ . \tag{3.5}$$

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Here and in the following we employ the short-hand notation $\mathcal{O}_\mathcal{A}(\mathbf{k}) = \mathcal{O}_{\mathbb{P}^{n_1}}(k^1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^{n_r}}(k^r)$ for line bundles on the ambient space $\mathcal{A}$.

As an archetypal example, the famous quintic in $\mathbb{P}^4$ is simply denoted as "[4|5]", or, even more succinctly, as "[5]". One might immediately ask how many possible non-isomorphic (one obvious isomorphism being row and column permutations) configurations could there be. This question was nicely settled in [2, 6] and the number is, remarkably, finite. A total of 7890 is found and can be accessed at [31]. This was the first large data-set of Calabi-Yau manifolds (cf. [21]).

We have compiled an electronic list of these CICYs which contains all the essential information including configuration matrices, Euler numbers $\chi(X)$, second Chern classes $c_2(TX)$, Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$ and allows for easy calculation of triple intersection numbers. It also contains previously unknown information, in particular about redundancies within the CICY list. This data underlies many of the subsequent calculations for monad bundles on CICYs. For more details on this "legacy" subject see Appendix C.

3.2 Favourable Configurations

Our choice of complete intersection Calabi-Yau manifolds is motivated largely by the explicit and relatively simple nature of the constructions. Perhaps the most valuable advantage of the presence of the ambient space $\mathcal{A}$ is the existence of relatively straightforward methods to identify discrete symmetries, a crucial step for the implementation of Wilson line breaking. To take maximal advantage of the presence of the ambient space we will focus on CICYs for which this explicit embedding is particularly useful. For some CICYs, the second cohomology $H^2(X)$ is not entirely spanned by the restrictions of the ambient space Kähler forms J_r. For example, in the case of the well-known Tian-Yau manifold, $X = \begin{bmatrix} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix}$, there are two Kähler forms descending from the two $\mathbb{P}^3$'s, but $h^{1,1}(X) = 14$. Here, we would like to focus on CICYs X for which the second cohomology is entirely spanned by the ambient space Kähler forms and which are, hence, characterised by

$$h^{1,1}(X) = m = \# \text{ of } \mathbb{P}^n\text{'s}.$$

We shall call manifolds with this property *favourable*. Such favourable CICYs offer a number of considerable practical advantages. There are 5 manifolds with $h^{1,1}(X) = m = 1$. These are also referred to as *cyclic* CICYs and they constitute the subject of Ref. [1].

The Kähler cone, that is the set of allowed Kähler forms J on X, is simply given by $\{J = t^r J_r \,|\, t^r \geq 0\}$, where t^r are the Kähler moduli. Further, the set of all line bundles on X, the Picard group $\mathrm{Pic}(X)$, is isomorphic to $\mathbb{Z}^m$, so line bundles on X can be characterised by an integer vector $\mathbf{k} = (k^1, \ldots, k^m)$. We denote these line bundles by $\mathcal{O}_X(\mathbf{k})$ and they can be obtained by restricting their ambient space counterparts $\mathcal{O}_\mathcal{A}(\mathbf{k})$ to X.

We can also introduce a dual basis $\{\nu^r\}$ of $H^4(X, \mathbb{Z})$, satisfying

$$\int_X \nu_r \wedge J_s = \delta^r_s \,, \tag{3.6}$$

and, via Poincaré duality $H^4(X, \mathbb{Z}) \simeq H_2(X, \mathbb{Z})$, we can use this basis to describe the second integer homology of X. The effective classes $W \in H_2(X, \mathbb{Z})$ then correspond precisely to the positive integer linear combinations of ν^r, that is $w_r \nu^r$ with $w_r \geq 0$. This property makes checking our version of the anomaly cancellation condition (2.6) very simple. If we expand second Chern classes in the basis $\{\nu^r\}$, writing $c_2(U) = c_{2r}(U)\nu^r$ for any bundle U, then the condition (2.6) simply turns into the inequalities

$$c_{2r}(V) \leq c_{2r}(TX) \quad \forall \, r = 1, \ldots, m. \tag{3.7}$$

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Details on the computation of Chern classes on CICYs are given in Appendix C.

Scanning through the CICY data, we find that there is a total of 4515 CICYs which are favourable. This is still a large dataset and we shall henceforth restrict our attention to these.

3.3 Line Bundles on CICYs

As we will see line bundles on CICYs are the main building blocks of the monad bundles considered in this paper, so we need to know their detailed properties. In particular we need to be able to fully determine the cohomology of line bundles on CICYs. We will return to this problem shortly after briefly reviewing a few more elementary properties. For an ambient space $\mathcal{A}$ with m projective factors, we consider a generic line bundle $L = \mathcal{O}_X(\mathbf{k})$ on a CICY X, where $\mathbf{k} = (k^1, \ldots, k^m)$ is an m-dimensional integer vector. The Chern characters of such a line bundle are given by

$$
\begin{aligned}
\mathrm{ch}_1(L) &= c_1(L) = k^r J_r \\
\mathrm{ch}_2(L) &= \tfrac{1}{2} k^r k^s J_r \wedge J_s \\
\mathrm{ch}_3(L) &= \tfrac{1}{6} k^r k^s k^t J_r \wedge J_s \wedge J_t \ ,
\end{aligned}
\tag{3.8}
$$

with implicit summation in $r, s, t = 1, \ldots, m$. Note that every line bundle on a CY 3-fold is uniquely classified by its first Chern class, as can be seen explicitly from the above expression for ch_1. The dual of the line bundle L is simply given by $L^\star = \mathcal{O}_X(-\mathbf{k})$. Using the Atiyah-Singer index theorem, the index of L can be written as

$$
\mathrm{ind}(L) \equiv \sum_{q=0}^{3} (-1)^q h^q(X, L) = \int_X \mathrm{ch}(L) \wedge \mathrm{Td}(X) = \int_X \left[\mathrm{ch}_3(L) + \frac{1}{12} \mathrm{ch}_2(TX) \wedge c_1(L) \right]
$$
$$
= \frac{1}{6} \left(d_{rst} k^r k^s k^t + \frac{1}{2} k^r c_{2r}(TX) \right) \ .
\tag{3.9}
$$

A special class of line bundles are the so-called *positive line bundles* which, in the present case, are the line bundles $L = \mathcal{O}_X(\mathbf{k})$ with all $k^r > 0$. The Kodaira vanishing theorem (B.8) applies to such positive bundles and (given the canonical bundle K_X of a Calabi-Yau manifold is trivial) it implies that $h^q(X, L) = 0$ for all $q \neq 0$. This means that $h^0(X, L)$ is the only non-vanishing cohomology and it can, hence, be easily calculated from the index (3.9) since $h^0(X, L) = \mathrm{ind}(L)$. The situation is just as simple for *negative line bundles* L, that is line bundles for which $L^\star$ is positive. In our case, the negative line bundles $L = \mathcal{O}_X(\mathbf{k})$ are of course the ones with all $k^r < 0$. Applying the Kodaira vanishing theorem to $L^\star = \mathcal{O}(-\mathbf{k})$ and then using Serre duality it follows that $h^3(X, L)$ is the only non-vanishing cohomology of a negative line bundle. Again, it can be computed from the index using $h^3(X, L) = -\mathrm{ind}(V)$. These result for positive and negative line bundles can also be checked using the techniques of spectral sequences. In this case, the dimension of the single non-zero cohomology can be computed without explicitly knowledge of the Leray maps d_i between cohomologies.

One more general statement can be made. It turns out that semi-positive line bundles, that is line bundles $L = \mathcal{O}_X(\mathbf{k})$, where $k^r \geq 0$ for all r, always have at least one section, so $h^0(X, L) > 0$. One might be tempted to conclude that the line bundles with sections are precisely the semi-positive ones. While this is indeed the case for some CICYs it is by no means always true and for some CICYs the class of line bundles with a section is genuinely larger than the class of semi-positive line bundles.

Further quantitative statements about the cohomology of line bundles $L = \mathcal{O}(\mathbf{k})$ containing "mixed" or zero entries k^r are not so easily obtained. For a general line bundle with mixed sign or

zero entries, computing the dimensions $h^q(X, \mathcal{O}_X(\mathbf{k}))$ does require explicit information about the dimensions of kernels and ranks of the Leray maps d_i. Fortunately, this information can be obtained based on a computational variation of the Bott-Borel-Weil theorem. In this way, we are able to calculate all line bundle cohomologies on favourable CICYs explicitly. We do not particularly require this computation in the present paper and will defer a full discussion on the matter to Ref. [46]. The general result involves a large number of case distinctions, analogous to but significantly more complex than the Bott formula (B.6) for line bundle cohomology over $\mathbb{P}^n$.

As an illustration, we provide a "generalised Bott formula" for mixed line bundles of the form $\mathcal{O}_X(-k, m)$ with $k \geq 1$, and $m \geq 0$ on the manifold $X = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array} \right]$. We find that

$$
h^q(X, \mathcal{O}_X(-k, m)) = \begin{cases} (k+1)\binom{m}{3} - (k-1)\binom{m+3}{3} & q = 0 \quad k < \frac{(1+2m)(6+m+m^2)}{3(2+3m(1-m))} \\ (k-1)\binom{m+3}{3} - (k+1)\binom{m}{3} & q = 1 \quad k > \frac{(1+2m)(6+m+m^2)}{3(2+3m(1-m))} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{3.10}
$$

where $\binom{n}{m}$ is the usual binomial coefficient with the convention that $\binom{0}{m} = 1$.

It should be clear from the above example that the explicit formulae for mixed line bundle cohomology are complicated and, in practice, have to be implemented as a computer program. The outline of our algorithm for computer implementation will be presented in [46].

4   The Monad Construction on CICYs

As was discussed in Ref. [1], large classes of vector bundles can be constructed over projective varieties using a variant of Horrock's monad construction [24–26]. The monad-bundles have been used extensively in string compactification throughout the years [22, 32–36]. Vector bundles defined through the monad short exact sequences can be thought of as kernels of maps between direct sums of line bundles. For reviews of this construction and some of its applications, see Ref. [27, 28, 34]. The *monad bundles* V considered in this paper are defined through the short exact sequence

$$
0 \to V \to B \xrightarrow{f} C \to 0 \ , \text{ where}
$$

$$
B = \bigoplus_{i=1}^{r_B} \mathcal{O}_X(\mathbf{b}_i) \ , \quad C = \bigoplus_{j=1}^{r_C} \mathcal{O}_X(\mathbf{c}_j) \tag{4.11}
$$

are sums of line bundles with ranks r_B and r_C, respectively. From the exactness of (4.11), it follows that the bundle V is defined as

$$
V = \ker(f) \ . \tag{4.12}
$$

The rank n of V is easily seen, by exactness of (4.11), to be

$$
n = \mathrm{rk}(V) = r_B - r_C \ . \tag{4.13}
$$

Because the Calabi-Yau manifolds discussed in this work are defined as complete intersection hypersurfaces in a product of projective spaces, we can write a short exact sequence analogous to (4.11) but over the ambient space, $\mathcal{A}$.

$$
0 \to \mathcal{V} \to \mathcal{B} \xrightarrow{\tilde{f}} \mathcal{C} \to 0 \ , \text{ where}
$$

$$
\mathcal{B} = \bigoplus_{i=1}^{r_B} \mathcal{O}_\mathcal{A}(\mathbf{b}_i) \ , \quad \mathcal{C} = \bigoplus_{j=1}^{r_C} \mathcal{O}_\mathcal{A}(\mathbf{c}_j) \ . \tag{4.14}
$$

Here, the map $\tilde{f}$ is a matrix whose entries are homogeneous polynomials of (multi-)degree $\mathbf{c}_j - \mathbf{b}_i$. The sequence (4.14) defines a coherent sheaf $\mathcal{V}$ on $\mathcal{A}$ whose restriction to X is V (and additionally the map f can be viewed as the restriction of $\tilde{f}$).

A number of mathematical constraints should be imposed on the above monad construction.

Bundleness: It is not a priori obvious that the exact sequence (4.11) indeed defines a bundle rather than a coherent sheaf. However, thanks to a theorem of Fulton and Lazarsfeld [37] this is the case provided two conditions are satisfied (see also [1]). First, all line bundles in C should be greater or equal than all line bundles in B. By this we mean that $c_j^r \geq b_i^r$ for all r, i and j. Second, the map $f : B \to C$ should be sufficiently generic [3]. Phrased in terms of ambient space language this means that the map $\tilde{f} : \mathcal{B} \to \mathcal{C}$ should be made up from sufficiently generic homogeneous polynomials of degree $\mathbf{c}_j - \mathbf{b}_i$. We will henceforth require these two conditions. An immediate consequence of V being a bundle is that (4.11) can be dualized to the short exact sequence

$$0 \to C^\star \xrightarrow{f^T} B^\star \to V^\star \to 0 \; , \tag{4.15}$$

so that the dual bundle $V^\star$ is given by

$$V^\star = \mathrm{coker}(f^T) \, . \tag{4.16}$$

Non-triviliaty: The above constraint on the integers c_j^r and b_i^r can be slightly strengthened. Suppose that a monad bundle V is defined by the short exact sequence

$$0 \to V \to B \oplus R \xrightarrow{f'} C \oplus R \to 0 \; , \tag{4.17}$$

where the repeated summand R is a line bundle or direct sum of line bundles. The so-defined bundle V is indeed equivalent to the one defined by the sequence (4.11), so the common summand R is, in fact, irrelevant[4]. To exclude common line bundles in B and C we should demand that all line bundles in C are strictly greater than all line bundles in B. By this we mean that $c_j^r \geq b_i^r$ for all r, i and j and, in addition, that for all i and j strict inequality, $c_j^r > b_i^r$, holds for at least one r (which can depend on i and j).

Positivity: We require that all line bundles in B and C are positive, that is $b_i^r > 0$ and $c_j^r > 0$ for all i, j and r. Monads discussed in the physics literature [22, 32–35] have typically been of this type and we will refer to them as *positive monads*. The reasons for this constraint are mainly of a practical nature. We have seen in our discussion of line bundles on CICYs that the cohomology of positive line bundles is particularly simple and easy to calculate from the index theorem. This fact significantly simplifies the analysis of positive monads.

Furthermore, experience seems to indicate that non-positive bundles are "more likely" to be unstable. As an extreme case, one can easily show that monads constructed only from negative line

---

[3]The actual condition of Fulton and Lazarfeld's theorem, apart from genericity of f, is that $C^\star \otimes B$ is globally generated so has at least $r_B r_C$ sections. This is indeed the case if $c_j^r \geq b_i^r$ for all r, i and j since, in this case, the line bundles $\mathcal{O}_X(\mathbf{c}_j - \mathbf{b}_i)$ which make up $C^\star \otimes B$ are semi-positive so have at least one section each. On some CICYs the line bundles with sections extend beyond the semi-positive ones, as discussed earlier, and for those CICYs one can likely allow monads where some of the entries in C are smaller than the ones in B and still preserve "bundleness" of V. In the present paper, we will not pursue this very case-dependent possibility further.

[4]This follows directly from the Snake Lemma [23], using the obvious injections of B, C into $B \oplus R$ and $C \oplus R$.

bundles are unstable because they explicitly have non-vanishing $H^0(X, V^\star)$. Of course we are not implying that all non-positive monads are unstable. In fact, in a forthcoming paper [45] we will show that allowing zero entries can still be consistent with stability. However, from the point of view of stability, starting with positive monads seems the safest bet, and we will focus on this class in the present paper.

In addition to the constraints of a more mathematical nature above we should consider physical constraints. To formulate them we need explicit expressions for the Chern classes of monad bundles. One finds

$$
\begin{aligned}
\mathrm{rk}(V) &= r_B - r_C = n , \\
c_1^r(V) &= \sum_{i=1}^{r_B} b_i^r - \sum_{j=1}^{r_C} c_j^r , \\
c_{2r}(V &= \frac{1}{2} d_{rst} \left(\sum_{j=1}^{r_C} c_j^s c_j^t - \sum_{i=1}^{r_B} b_i^s b_i^t \right) , \\
c_3(V) &= \frac{1}{3} d_{rst} \left(\sum_{i=1}^{r_B} b_i^r b_i^s b_i^t - \sum_{j=1}^{r_C} c_j^r c_j^s c_j^t \right) ,
\end{aligned}
\tag{4.18}
$$

where d_{rst} are the triple intersection numbers (C.13) on X and the relations for $c_{2r}(V)$ and $c_3(V)$ have been simplified assuming that $c_1^r(V) = 0$. Then we need to impose two physics constraints.

Correct structure group: To have bundles with structure group $\mathrm{SU}(n)$ where $n = 3, 4, 5$ we first of all need that $n = r_B - r_C = 3, 4, 5$. In addition, the first Chern class of V needs to vanish which, from the second Eq. (4.18), can be expressed as

$$
S^r := \sum_{i=1}^{r_C+n} b_i^r = \sum_{j=1}^{r_C} c_j^r \qquad \forall r = 1, \ldots, k .
\tag{4.19}
$$

We have defined the quantities S^r which represent the first Chern classes of B and C and will be useful for the classification of positive monads below.

Anomaly cancellation/effectiveness: As we have seen this condition can be stated in the simple form (3.7). Inserting the above expression for the second Chern class gives

$$
d_{rst} \left(\sum_{j=1}^{r_C} c_j^s c_j^t - \sum_{i=1}^{r_B} b_i^s b_i^t \right) \le 2 c_{2r}(TX) \qquad \forall r .
\tag{4.20}
$$

In addition, we should of course prove stability of positive monads, a task which will be systematically dealt with in Ref. [46]. This completes the set-up of monads bundles. To summarise, we will consider monad bundles V of rank 3, 4 or 5, defined by the short exact sequence (4.11) with positive line bundles only. In addition, all line bundles in C must be strictly greater than all line bundles in B and the two constraints (4.19) and (4.20) must be satisfied.

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5   Classification of Positive Monads on CICYs

An obvious question is whether the class of monads defined in the previous section is finite. In this section, we show that this is indeed the case and subsequently classify all such monads.

We start by stating the classification problem in a more formal way. For any favourable CICY manifold X with second Chern class $c_{2r}(TX)$ and triple intersection numbers d_{rst}, defined in a product of m projective spaces, and for any $n = 3, 4, 5$, we wish to find all sets of integers b_i^r and c_j^r, where $r = 1, \ldots, m$, $i = 1, \ldots, r_B = r_C + n$ and $j = 1, \ldots, r_C$ satisfying the conditions

1. $b_i^r \geq 1$, $\quad c_j^r \geq 1$, $\quad \forall i, j, r$;

2. $c_j^r \geq b_j^r$, $\quad \forall i, j, r$;

3. $\forall i, j$ there exists at least one r such that $c_j^r > b_i^r$;

4. $\displaystyle\sum_{i=1}^{r_B} b_i^r = \sum_{j=1}^{r_C} c_j^r = S^r$, $\quad \forall r$; $\hfill (5.1)$

5. $d_{rst} \left(\displaystyle\sum_{j=1}^{r_C} c_j^s c_j^t - \sum_{i=1}^{r_B} b_i^s b_i^t \right) \leq 2c_{2r}(TX) \qquad \forall r$.

Our first task is to show that this defines a finite class. Although all that is involved are simple manipulations of inequalities it is not complete obvious at first which approach to take. We start by defining the maxima $b_{\max}^r = \max_i \{b_i^r\}$, minima $c_{\min}^r = \min_j \{c_j^r\}$ and their differences $\theta^r = c_{\min}^r - b_{\max}^r \geq 0$ which are of course positive for all r. Then we can write

$$b_i^r = b_{\max}^r - T_i^r, \qquad c_j^r = c_{\min}^r + D_j^r, \qquad (5.2)$$

where T_i^r and D_j^r are the deviations from the maximum and minimum values. It is also useful to introduce the sums

$$T^r = \sum_{i=1}^{r_B} T_i^r, \quad D^r = \sum_{j=1}^{r_C} D_j^r \qquad (5.3)$$

of these deviations. Given theses definitions, it is easy to see that

$$S^r = b_{\max}^r r_B - T^r, \quad S^r = c_{\min}^r r_C + D^r. \qquad (5.4)$$

Subtracting these two equations and using $r_B = r_C + n$ it follows that

$$\theta^r r_C + (D^r + T^r) = n b_{\max}^r. \qquad (5.5)$$

We will use this identity shortly. Next, from the definition (4.19), and since all $c_s^j \geq 1$, we obtain the two inequalities

$$S^r \geq \sum_{j=1}^{r_C} \mathbb{I}^r = r_C \mathbb{I}^r, \quad S^r \leq \sum_{i=1}^{r_B} b_{\max}^r = b_{\max}^r r_B, \quad \forall \, r, \qquad (5.6)$$

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where $\mathbb{I}_s$ is a vector with all entries being 1. After this preparation, we come to the key part of the argument which involves working out the consequences of condition 5 in (5.1).

$$
\begin{aligned}
2c_{2r}(TX) &\geq d_{rst}\left(\sum_{j=1}^{r_C}c_j^s c_j^t - \sum_{i=1}^{r_B}b_i^s b_i^t\right) \\
&= d_{rst}\left(\sum_{j=1}^{r_C}(c_{\min}^s + D_j^s)c_j^t - \sum_{i=1}^{r_B}(b_{\max}^s - T_i^s)b_i^t\right) && \text{inserting (5.2)} \\
&= d_{rst}\left((c_{\min}^s - b_{\max}^s)S^t + \sum_{j=1}^{r_C}D_j^s c_j^t + \sum_{i=1}^{r_B}T_i^s b_i^t\right) && \text{using (4.19)} \\
&\geq d_{rst}\left(\theta^s S^t + (D^s + T^s)\mathbb{I}_t\right) && \text{since } c_j^t, b_i^t \geq 1 \text{ , using (5.3)} \\
&\geq d_{rst}\left(\theta^s(r_C\mathbb{I}^t) + (D^s + T^s)\mathbb{I}^t\right) && \text{by first inequality (5.6)} \\
&= d_{rst}\left(nb_{\max}^s\mathbb{I}^t\right) && \text{from (5.5)} \\
&\geq \frac{n}{r_B}d_{rst}\left(S^s\mathbb{I}^t\right) && \text{by second inequality}
\end{aligned}
\tag{5.7}
$$

From the second last line in the above chain of inequalities, we can also express this result as a bound in the variables $b_{\max}^r$ (the maximum entries the bundle B can have in each projective space), resulting in

$$
2c_{2r}(TX) \geq n\sum_{s,t}d_{rst}b_{\max}^s \ . \tag{5.8}
$$

It turns out that the matrices $\sum_t d_{rst}$ are always non-singular, so this equation provides an upper bound for $b_{\max}^r$. Moreover, since each $b_{\max}^r \in \mathbb{Z}_{\geq 1}$, and since the matrix $n\sum_t d_{rst}$ has entries in $\mathbb{Z}_{\geq 0}$, Eq. (5.8) may not have solutions for all manifolds. In fact, of the 4515 favourable CICYs, Eq. (5.8) immediately eliminates all but 63 which include the 5 cyclic ones studied in Ref. [1]. One finds that the values for $b_{\max}^r$ are very small indeed and never exceed 6.

So far, we have bounded the maximal entries of the bundle B. What about r_B, the rank of B? It turns out there are various ways to derive an upper bound on r_B. First note that, from the third condition in (5.1), for all $j \in \{1, \ldots, r_C\}$, there exists a $\sigma \in \{1, \ldots, m\}$, call it $\sigma(j)$, such that

$$
c_j^r - b_{\max}^r \geq \delta^{r\sigma(j)} \ . \tag{5.9}
$$

Introduce

$$
\nu^r = \sum_{j=1}^{r_C}\delta^{r\sigma(j)} \ , \tag{5.10}
$$

the number of line bundles in C which are bigger than the ones in B due to the r-th entry. Since all line bundles in C are bigger than the ones in B it follows that

$$
\sum_{r=1}^{m}\nu^r = r_C = r_B + n \ . \tag{5.11}
$$

We conclude that

$$
r_B b_{\max}^r \geq \sum_{i=1}^{r_B}b_i^r = \sum_{j=1}^{r_C}c_j^r \geq \sum_{j=1}^{r_C}(b_{\max}^r + \delta^{r\sigma(j)}) = r_c b_{\max}^r + \nu^r \tag{5.12}
$$

and, hence, that $nb_{\max}^r \geq \nu^r$. Summing this result over r one easily finds that

$$
r_B \leq n\left(1 + \sum_{r=1}^{m}b_{\max}^r\right) \ . \tag{5.13}
$$

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Since we have already bounded $b^r_{\max}$ (independently of r_B) this provides an upper bound for r_B. This shows that our class of bundles is indeed finite. While the above bound is simple, for the practical purpose of classifying all bundles it often turns out to be too weak, and requires computationally expensive scanning of monads with large r_B and, hence, a large number of integer entries. Based on Eq. (5.13) alone, a classification on a desktop machine is likely impossible. Fortunately, one can derive other constraints on r_B which in many cases turn out to be stronger. Using $nb^r_{\max} \geq \nu^r$ in Eq. (5.8) leads to

$$\sum_{r,s} d_{rst}\nu^t \leq 2c_{2r}(TX) \ . \tag{5.14}$$

For each CICY, one can find all integer solutions (ν^r) (subject to the constraint $\nu^r \geq 0$, of course) to this equation and then calculate the maximal possible value for r_B from Eq. (5.11). Finally, starting again from condition 5 of (5.1) we find

$$
\begin{aligned}
2c_{2r}(TX) \ &\geq\ d_{rst}\left[\sum_{j=1}^{r_C} c_j^s c_j^t - \sum_{i=1}^{r_B} b_i^s b_i^t\right] \\
&\geq\ d_{rst}\left[\sum_{j=1}^{r_C}(b^s_{\max} + \delta^{s\sigma(j)})(b^t_{\max} + \delta^{t\sigma(j)}) - \sum_{i=1}^{r_B} b_i^s b_i^t\right] \\
&=\ d_{rst}\left[\sum_{j=1}^{r_C} b^s_{\max} b^t_{\max} - \sum_{i=1}^{r_B} b_i^s b_i^t + 2\nu^s b^t_{\max} + \delta_s^t \nu^t\right] \\
&\geq\ d_{rst}\left[-nb^s_{\max} b^t_{\max} + 2\nu^s b^t_{\max} + \delta_s^t \nu^t\right] \ .
\end{aligned}
\tag{5.15}
$$

Rewriting this as an system of linear inequalities for ν^s, we have that

$$\sum_s \left(2\sum_t d_{rst} b^t_{\max} + d_{rss}\right)\nu^s \leq 2c_{2r}(TX) + nd_{rst}b^s_{\max}b^t_{\max} \ . \tag{5.16}$$

Again, this equation can be solved for all non-negative integers ν^r since b_s^{max} is bounded from (5.8) and, subsequently, we can compute the maximal r_B from Eq. (5.11). In practice, we evaluate all three bounds (5.13), (5.14), (5.16) for every CICY and use the minimum value obtained. In this way we find maximal values for r_B ranging from 8 to 22 depending on the CICY.

The explicit classification is now simply a matter of computer search. For each of the 63 CICYs with solutions to the inequality (5.8) we scan over all allowed values of n, r_B and over all values for S^r subject to the last inequality in (5.7). For each fixed set of these quantities we then generate all multi-partitions of entries b_i^r and c_j^r eliminating, of course, trivial redundancies due to permutations. Upon performing this scan, we find that positive monad bundles only exist over 36 favourable CICYs (out of the 63 which passed the initial test). These 36 manifolds, together with the number of monad bundles over them, are listed in Table 2.

In total, we find 7118 positive monad bundles. These include the 77 positive monad bundles on the 5 cyclic CICYs (these are the CICYs with $h^{1,1}(X) = 1$) found in Ref. [1]. Some explicit examples are listed in Table 3. Focusing on the different ranks of V considered, we find 5680 bundles of rank 3, 1334 of rank 4, and 104 of rank 5 on these 36 manifolds. To get an idea of the distribution, in part (a) of Fig. 1 we have plotted the number of monads as a function of the index $\mathrm{ind}(V)$. It seems, at first glance, that the distribution is roughly Gaussian. For comparison, in part (b) of Fig. 1, we have plotted the number of monads which satisfy the two 3-generation constraints (2.9) and (2.10). The same data, but split up into the three cases $n = 3, 4, 5$ for the rank of V, is shown in Fig. 2. The total numbers of bundles in all cases has been collected in Table 4.

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Config	No.Bundles	Config	No.Bundles	Config	No.Bundles	Config	No.Bundles
$\begin{bmatrix}5\end{bmatrix}$	$(20,14,9)$	$\begin{bmatrix}3 & 3\end{bmatrix}$	$(5,3,2)$	$\begin{bmatrix}4 & 2\end{bmatrix}$	$(7,5,3)$	$\begin{bmatrix}3 & 2 & 2\end{bmatrix}$	$(3,2,1)$
$\begin{bmatrix}2 & 2 & 2 & 2\end{bmatrix}$	$(2,1,0)$	$\begin{bmatrix}2\\4\end{bmatrix}$	$(611,308,56)$	$\begin{bmatrix}3\\3\end{bmatrix}$	$(62,43,14)$	$\begin{bmatrix}0 & 2\\2 & 3\end{bmatrix}$	$(80,12,0)$
$\begin{bmatrix}0 & 2\\3 & 2\end{bmatrix}$	$(12,5,0)$	$^{(4)}\begin{bmatrix}0 & 2\\4 & 1\end{bmatrix}$	$(126,17,0)$	$\begin{bmatrix}1 & 1\\3 & 2\end{bmatrix}$	$(15,8,0)$	$\begin{bmatrix}1 & 1\\4 & 1\end{bmatrix}$	$(153,35,19)$
$\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$	$(3,0,0)$	$\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$	$(5,0,0)$	$\begin{bmatrix}2 & 1\\3 & 1\end{bmatrix}$	$(13,2,0)$	$^{(2)}\begin{bmatrix}0 & 0 & 2\\2 & 2 & 2\end{bmatrix}$	$(5,0,0)$
$^{(3)}\begin{bmatrix}0 & 0 & 2\\3 & 2 & 1\end{bmatrix}$	$(5,0,0)$	$^{(2)}\begin{bmatrix}0 & 1 & 1\\2 & 2 & 2\end{bmatrix}$	$(5,0,0)$	$\begin{bmatrix}0 & 1 & 1\\2 & 3 & 1\end{bmatrix}$	$(12,5,0)$	$\begin{bmatrix}0 & 1 & 1\\3 & 2 & 1\end{bmatrix}$	$(8,0,0)$
$^{(4)}\begin{bmatrix}0 & 1 & 1\\4 & 1 & 1\end{bmatrix}$	$(126,17,0)$	$\begin{bmatrix}0 & 2 & 1\\2 & 2 & 1\end{bmatrix}$	$(2,0,0)$	$\begin{bmatrix}1 & 1 & 1\\3 & 1 & 1\end{bmatrix}$	$(2,0,0)$	$\begin{bmatrix}2 & 1 & 1\\2 & 1 & 1\end{bmatrix}$	$(1,0,0)$
$\begin{bmatrix}0 & 0 & 1 & 1\\2 & 2 & 2 & 1\end{bmatrix}$	$(3,0,0)$	$^{(3)}\begin{bmatrix}0 & 0 & 1 & 1\\3 & 2 & 1 & 1\end{bmatrix}$	$(5,0,0)$	$\begin{bmatrix}2\\2\\3\end{bmatrix}$	$(553,232,0)$	$\begin{bmatrix}0 & 2\\1 & 2\\1 & 2\end{bmatrix}$	$(8,0,0)$
$\begin{bmatrix}1 & 1\\0 & 2\\1 & 3\end{bmatrix}$	$(74,0,0)$	$^{(1)}\begin{bmatrix}1 & 1\\0 & 2\\2 & 2\end{bmatrix}$	$(9,0,0)$	$\begin{bmatrix}1 & 1\\1 & 1\\1 & 3\end{bmatrix}$	$(25,0,0)$	$^{(1)}\begin{bmatrix}1 & 1\\1 & 1\\2 & 2\end{bmatrix}$	$(9,0,0)$
$\begin{bmatrix}1 & 1\\1 & 2\\0 & 3\end{bmatrix}$	$(34,0,0)$	$\begin{bmatrix}1 & 1\\2 & 1\\2 & 1\end{bmatrix}$	$(3,0,0)$	$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\3 & 1 & 1\end{bmatrix}$	$(9,0,0)$	$\begin{bmatrix}2\\2\\2\\2\end{bmatrix}$	$(3665,625,0)$

Table 2: The 36 manifolds which admit positive monads. The No.Bundles column next to each manifold is a triple, corresponding to the respective numbers of ranks 3,4, and 5 monads. Identical numbers in brackets to the left of a configuration matrices indicate equivalent configurations as identified in Appendix C.

CICY X	B	C	$\mathrm{rk}(V)$	$\begin{bmatrix}c_2(TX)\\c_2(V)\end{bmatrix}$	$\mathrm{ind}(V)=\frac{1}{2}c_3(V)$
$\begin{bmatrix}1 & 2\\1 & 2\\1 & 2\\1 & 2\end{bmatrix}$	$\mathcal{O}_X(1,1,1,1)^{\oplus 8}$	$\mathcal{O}_X(5,1,1,1)$ $\oplus\mathcal{O}_X(1,5,1,1)$ $\oplus\mathcal{O}_X(1,1,5,1)$ $\oplus\mathcal{O}_X(1,1,1,5)$	4	$\begin{bmatrix}(24,24,24,24)\\(24,24,24,24)\end{bmatrix}$	-64
$\begin{bmatrix}1 & 1 & 1\\2 & 2 & 1\\2 & 2 & 1\end{bmatrix}$	$\mathcal{O}_X(1,1,1)^{\oplus 10}$	$\mathcal{O}_X(1,1,2)^{\oplus 3}$ $\oplus\mathcal{O}_X(1,2,1)^{\oplus 3}$ $\oplus\mathcal{O}_X(4,1,1)$	3	$\begin{bmatrix}(24,36,36)\\(24,36,36)\end{bmatrix}$	-69
$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$	$\mathcal{O}_X(1,1)^{\oplus 11}$	$\mathcal{O}_X(6,1)\oplus\mathcal{O}_X(1,2)^{\oplus 5}$	5	$\begin{bmatrix}(24,44)\\(20,30)\end{bmatrix}$	-40
$\begin{bmatrix}4 & 5\end{bmatrix}$	$\mathcal{O}_X(1)^{\oplus 6}$	$\mathcal{O}_X(2)^{\oplus 3}$	3	$\begin{bmatrix}(50)\\(15)\end{bmatrix}$	-15

Table 3: Some examples from the 7118 positive monads on favourable CICYs.

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	Bundles	$\text{ind}(V) = 3k$	$\text{ind}(V) = 3k$ and k divides $\chi(X)$	$\text{ind}(V) = 3k$ $|\text{ind}(V)| < 40$ and k divides $\chi(X)$
rank 3	5680	3091	458	19
rank 4	1334	207	96	2
rank 5	104	52	5	0
Total	7118	3350	559	21

Table 4: The number of positive monad bundles on favourable CICYs. Imposing that the index, $\text{ind}(V)$, is divisible by 3 reduces the number and requiring, in addition, that $\text{ind}(V)/3$ divides the Euler number of the corresponding CICY leads to a further reduction.
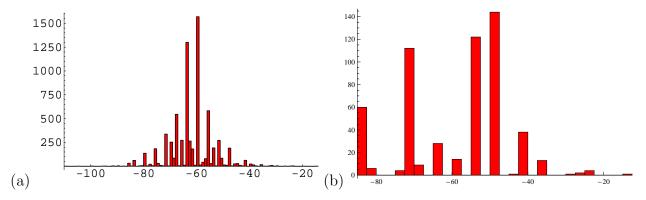

Figure 1: (a) Histogram for the index, $\text{ind}(V)$, of the 7118 positive monads found over 36 favourable CICYs: the horizontal axis is $\text{ind}(V)$ and the vertical, the number of bundles; (b) the same data set, but only taking into account monads with $\text{ind}(V) = 3k$ for some positive integer k, such that k divides the Euler number of the corresponding CICY.

It is clear from this table that even the two very rudimentary physical constraints (2.9) and (2.10) lead to a very substantial reduction of the number of viable bundles. If these two constraints are combined with a "sensible" limit on the index, for example $\text{ind}(V) < 40$ (assuming that the discrete symmetries one is likely to find are of order ≤ 13), then part (b) of the figures show that the number of remaining bundles is very small indeed: there are only 21 of these. Remarkably these, perhaps of the most physical interest, only exist on the cyclic manifolds discussed in [1] as well as the transposes of these configuration matrices (cf. [21]).

6   Stability

In this paper we will prove the set of highly non-trivial vanishing conditions (2.5) to test the stability of our bundles. These conditions are a generalization of Hoppe's criterion [47], used in [1] to prove stability in the case of cyclic Calabi-Yau manifolds. In the cyclic case, the conditions in (2.5) are sufficient for stability, while for general CICYs, the vanishing of these cohomologies is necessary, but no longer sufficient. None-the-less, the generalized Hoppe condition still provides an important check of stability. As mentioned earlier, Eq. (2.5) is equivalent to the same condition but written in terms of the dual bundle, which turns out to be technically simpler. Hence, in this section we prove
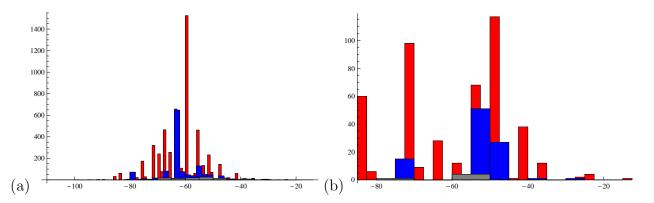
Figure 2: (a) Histogram for the index, $\mathrm{ind}(V)$, of the positive monads, 5680 of rank 3 (in red), 1334 of rank 4 (in blue), and 104 of rank 5 (in gray), found over 36 favourable CICYs: the horizontal axis is $\mathrm{ind}(V)$ and the vertical, the number of bundles; (b) the same data set, but only taking into account monads with $\mathrm{ind}(V) = 3k$ for some positive integer k, such that k divides the Euler number of the corresponding CICY.

that

$$H^0(X, \wedge^p V^\star) = 0, \qquad p = 1, \ldots, \mathrm{rk}(V) - 1 . \tag{6.1}$$

As mentioned previously, the full proof of stability of our bundles will appear in Ref. [46].

We shall prove that condition (6.1) is satisfied in two steps. First, we demonstrate that the vanishing of certain ambient space cohomologies (given in (6.5)) associated to a Koszul resolution (B.10) guarantee that $H^0(X, \wedge^p V^\star) = 0$. After specifying these necessary cohomology groups, as a second step, we will show that they are in fact zero for all positive monads. This is accomplished by studying an exterior power sequence (B.11) on $\mathcal{A}$. We derive conditions (6.10), (6.12), and (6.15) which hold for all the bundles in our classification.

6.1   Step 1: Using the Koszul Sequence

Let us begin with the Koszul resolution for $\wedge^p V$, which, from Eq. (B.10), reads

$$0 \to \wedge^K \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star \to \wedge^{K-1} \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star \to \ldots \to \wedge^p \mathcal{V}^\star \to \wedge^p \mathcal{V}^\star|_X \to 0 . \tag{6.2}$$

In the above, K is the co-dimension of the CICY X and $\wedge^p \mathcal{V}^\star$ is defined on the ambient product projective space $\mathcal{A}$ while $\wedge^p \mathcal{V}^\star|_X = \wedge^p V^\star$ lives on X. The normal bundle, $\mathcal{N}$ of X in $\mathcal{A}$ has been defined in Eq. (3.5).

We can break this long exact sequence into K inter-related short exact ones by introducing $K-1$ (co-)kernels Q_i such that

$$
\begin{array}{ccccccccc}
0 & \to & \wedge^K \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star & \to & \wedge^{K-1} \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star & \to & Q_1 & \to & 0 \\
0 & \to & Q_1 & & \to & \wedge^{K-2} \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star & \to & Q_2 & \to & 0 \\
& & & \vdots & & & & & \\
0 & \to & Q_{K-2} & & \to & \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star & \to & Q_{K-1} & \to & 0 \\
0 & \to & Q_{K-1} & & \to & \wedge^p \mathcal{V}^\star & \to & \wedge^p V^\star & \to & 0
\end{array}
\tag{6.3}
$$

Now, each of the above short exact sequences induces a long exact sequence in cohomology. Adopting the convention that $Q_0 := \wedge^K \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star$ and $Q_K := \wedge^p V^\star$ (keeping in mind that $\wedge^j \mathcal{N}^\star \simeq \mathcal{O}_{\mathcal{A}}$ for

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$j = 0$), the j-th long exact sequence takes the form

$$
\begin{array}{ccccccc}
0 & \to & H^0(\mathcal{A}, Q_{j-1}) & \to & H^0(\mathcal{A}, \wedge^{K-j}\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star) & \to & H^0(\mathcal{A}, Q_j) & \to \\
& \to & H^1(\mathcal{A}, Q_{j-1}) & \to & H^1(\mathcal{A}, \wedge^{K-j}\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star) & \to & H^1(\mathcal{A}, Q_j) & \to \\
& & & \vdots & & & \\
& \to & H^{K+2}(\mathcal{A}, Q_{j-1}) & \to & H^{K+2}(\mathcal{A}, \wedge^{K-j}\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star) & \to & H^{K+2}(\mathcal{A}, Q_j) & \to \\
& \to & H^{K+3}(\mathcal{A}, Q_{j-1}) & \to & H^{K+3}(\mathcal{A}, \wedge^{K-j}\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star) & \to & H^{K+3}(\mathcal{A}, Q_j) & \to \quad 0 \ .
\end{array}
\tag{6.4}
$$

We have used the fact that $\mathcal{A}$ is of dimension $K+3$ since X has co-dimension K and hence the highest cohomology group is $K + 3$. To ensure vanishing of $H^0(X, \wedge^p V^\star)$ it suffices to have $H^0(\mathcal{A}, \wedge^p\mathcal{V}^\star)$ and $H^1(\mathcal{A}, Q_{K-1})$ be zero. The latter vanishes, in turn, if $H^1(\mathcal{A}, \mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star)$ and $H^2(\mathcal{A}, Q_{K-2})$ are both zero. Thus arguing inductively, it is sufficient (though not necessary) for the vanishing of $H^0(X, \wedge^p V^\star)$ that

$$
H^j(\mathcal{A}, \wedge^j\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star) = 0 \qquad \text{for } j = 0, \dots, K \ .
\tag{6.5}
$$

Indeed, these constitute $K + 1$ vanishing conditions. For $j > K$, $\wedge^j\mathcal{N}^\star = 0$ since $\mathcal{N}$ by definition is rank K and the cohomologies are zero automatically.

6.2 Step 2: Using the Exterior Power Sequence

How can we demonstrate that (6.5) is satisfied? We recall the definition (4.14) of the monad on the ambient space, whose dual is given by

$$
0 \to \mathcal{C}^\star \to \mathcal{B}^\star \to \mathcal{V}^\star \to 0 \ .
$$

The p-th exterior power of $\mathcal{V}^\star$ can be extracted from the exterior-power sequence (B.11), which here reads

$$
0 \to S^p\mathcal{C}^\star \to S^{p-1}\mathcal{C}^\star \otimes \mathcal{B}^\star \to \dots \to \mathcal{C}^\star \otimes \wedge^{p-1}\mathcal{B}^\star \to \wedge^p\mathcal{B}^\star \to \wedge^p\mathcal{V}^\star \to 0 \ .
\tag{6.6}
$$

By S^j we denote the j-th symmetric tensor power. We can tensor this sequence by $\wedge^j\mathcal{N}^\star$ for $j = 0, \dots, K$. Each of the resulting $K + 1$ sequences can be broken up into p short exact ones, by introducing (co-)kernels q_i^j, where $i = 0, \dots, p-1$ and $j = 0, \dots, K$. This leads to

$$
\begin{array}{ccccccc}
0 & \to & \wedge^j\mathcal{N}^\star \otimes S^p\mathcal{C}^\star & \to & \wedge^j\mathcal{N}^\star \otimes S^{p-1}\mathcal{C}^\star \otimes \mathcal{B}^\star & \to & q_1^j & \to & 0 \\
0 & \to & q_1^j & \to & \wedge^j\mathcal{N}^\star \otimes S^{p-2}\mathcal{C}^\star \otimes \wedge^2\mathcal{B}^\star & \to & q_2^j & \to & 0 \\
& & & \vdots & & & & \\
0 & \to & q_{p-2}^j & \to & \wedge^j\mathcal{N}^\star \otimes \mathcal{C}^\star \otimes \wedge^{p-1}\mathcal{B}^\star & \to & q_{p-1}^j & \to & 0 \\
0 & \to & q_{p-1}^j & \to & \wedge^j\mathcal{N}^\star \otimes \wedge^p\mathcal{B}^\star & \to & \boxed{\wedge^j\mathcal{N}^\star \otimes \wedge^p\mathcal{V}^\star} & \to & 0 \ ,
\end{array}
\tag{6.7}
$$

where we have boxed the term whose j-th cohomology group on $\mathcal{A}$ needs to vanish.

Next, we consider the cohomology associated to (6.7). From Kodaira vanishing on $\mathcal{A}$ (Eq. (B.9)), a negative bundle $\mathcal{L}^\star$ satisfies the vanishing conditions

$$
H^m(\mathcal{A}, \mathcal{L}^\star) = 0 \text{ unless } m = \dim(\mathcal{A}) = K + 3 \ .
\tag{6.8}
$$

Our bundles $\mathcal{B}^\star$, $\mathcal{C}^\star$ as well as their tensors and powers are of course direct sums of strictly negative bundles, and hence obey (6.8). Each of the short exact sequences in Eq. (6.7) induces a long exact sequence in cohomology which are intertwined by the (co-)kernels. It will be helpful to consider (6.7) and its cohomology for each value of p individually. The results are immediate for the first two cases under consideration.

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For $p = 1$, we quickly see that $H^m(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star) = 0$ for $m = 0, \ldots, K + 1$ as these are all sandwiched between two vanishing terms, namely $H^m(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^{p-1} \mathcal{C}^\star \otimes \mathcal{B}^\star)$ and $H^{m+1}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^p \mathcal{C}^\star)$. Thus (6.5) is automatically satisfied for $p = 1$. Similarly, for $p = 2$, $H^m(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star) = 0$ for $m = 0, \ldots, K$, again satisfying Eq. (6.5).

For longer exterior power sequences the result requires a little more analysis. For $p = 3$, $H^m(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star)$ vanishes automatically only for $m = 0, \ldots, K - 1$, one short of the upper bound of j required in Eq. (6.5). Nevertheless, we find the equivalence $H^K(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star) \simeq H^{K+2}(\mathcal{A}, q_1^j)$, and the latter cohomology group resides in the four-term exact sequence

$$0 \to H^{K+2}(\mathcal{A}, q_1^j) \to H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star) \xrightarrow{g} H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^2 \mathcal{C}^\star \otimes \mathcal{B}^\star) \to H^{K+3}(\mathcal{A}, q_1^j) \to 0 \,.$$
(6.9)

The single case which remains to be checked is $j = K$. It was argued in Appendix B of Ref. [1] that on the ambient space, the map g above, induced from the defining map of the monad (which we recall, by construction, is generic), is also generic. Therefore, if g is injective, then the requisite term $H^{K+2}(\mathcal{A}, q_1^j)$ vanishes. Injectivity simply requires that

$$h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star) \leq h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^2 \mathcal{C}^\star \otimes \mathcal{B}^\star) \,. \tag{6.10}$$

At last, for the final case of $p = 4$, $H^m(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star)$ vanishes automatically only for $m = 0, \ldots, K - 2$, two short of the upper bound for j in Eq. (6.5). However, we have that $H^{K-1}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star) \simeq H^{K+2}(\mathcal{A}, q_1^j)$ the latter of which resides in a four-term exact sequence

$$0 \to H^{K+2}(\mathcal{A}, q_1^j) \to H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^4 \mathcal{C}^\star) \xrightarrow{g_1} H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star \otimes \mathcal{B}^\star) \to H^{K+3}(\mathcal{A}, q_1^j) \to 0 \,;$$
(6.11)

The relevant case $j = K - 1$. As before, the map g_1 is generic and the requisite term $H^{K+2}(\mathcal{A}, q_1^j)$ vanishes if g_1 is injective, or if

$$h^{K+3}(\mathcal{A}, \wedge^{K-1} \mathcal{N}^\star \otimes S^4 \mathcal{C}^\star) \leq h^{K+3}(\mathcal{A}, \wedge^{K-1} \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star \otimes \mathcal{B}^\star) \,. \tag{6.12}$$

Similarly, we have that $H^K(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star) \simeq H^{K+2}(\mathcal{A}, q_2^j)$ and the latter resides in

$$0 \to H^{K+2}(\mathcal{A}, q_2^j) \to H^{K+3}(\mathcal{A}, q_1^j) \xrightarrow{g_2} H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^2 \mathcal{C}^\star \otimes \wedge^2 \mathcal{B}^\star) \to H^{K+3}(\mathcal{A}, q_2^j) \to 0 \,, \tag{6.13}$$

where we need to focus on the case $j = K$. The cohomology $H^{K+3}(\mathcal{A}, q_1^j)$ again resides in a four-term exact sequence

$$0 \to H^{K+2}(\mathcal{A}, q_1^j) \to H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^4 \mathcal{C}^\star) \xrightarrow{h} H^{K+3}(\mathcal{A}, \wedge^j \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star \otimes \mathcal{B}^\star) \to H^{K+3}(\mathcal{A}, q_1^j) \to 0 \,.$$
(6.14)

The maps g_2 and h are generic, as before. Therefore, the cohomology at the end of Eq. (6.14), $H^{K+3}(\mathcal{A}, q_1^K) \simeq \mathrm{coker}(h)$ (which also appears as the second term of (6.13)) has dimension $h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star \otimes \mathcal{B}^\star) - h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^4 \mathcal{C}^\star)$. For injectivity of g_2, this dimension should not exceed $h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^2 \mathcal{C}^\star \otimes \wedge^2 \mathcal{B}^\star)$, so we have the condition

$$h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^3 \mathcal{C}^\star \otimes \mathcal{B}^\star) - h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^4 \mathcal{C}^\star) \leq h^{K+3}(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes S^2 \mathcal{C}^\star \otimes \wedge^2 \mathcal{B}^\star) \,. \tag{6.15}$$

This condition then guarantees the vanishing of $H^{K+3}(\mathcal{A}, q_1^K)$ and subsequently that of $H^K(\mathcal{A}, \wedge^K \mathcal{N}^\star \otimes \wedge^p \mathcal{V}^\star)$.

We need not consider cases with $p > 4$ since our bundles are maximally of rank 5. In summary then, the conditions (6.10), (6.12) and (6.15) suffice to guarantee Eqs. (6.5) and hence our main claim, Eq. (6.1). These conditions on the ambient space cohomology can be readily checked algorithmically using the Bott formula (B.6) and the Künneth formula (B.7). We have done so for all our positive monads using computer scans and find these conditions are always satisfied.

In conclusion, for all positive monad bundles V, $H^0(X, \wedge^p V^\star) = 0$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$. This concludes our non-trivial check of stability.

7 Computing the Particle Spectrum

7.1 Bundle Cohomology

While computing the full cohomology of monad bundles is generally a difficult task, it will become clear in the following that significant simplifications arise for positive monads. This computational advantage is of course one of the motivations to consider positive monads and it will lead to a number of general statements about their cohomology.

7.1.1 Number of Families and Anti-families in $H^1(X, V)$ and $H^1(X, V^\star)$

The defining short exact sequence (4.11) of the monad bundle V induces the long exact sequence

$$
\begin{array}{rcccccl}
0 & \to & H^0(X, V) & \to & H^0(X, B) & \to & H^0(X, C) \\
  & \to & H^1(X, V) & \to & H^1(X, B) & \to & H^1(X, C) \\
  & \to & H^2(X, V) & \to & H^2(X, B) & \to & H^2(X, C) \\
  & \to & H^3(X, V) & \to & H^3(X, B) & \to & H^3(X, C) \to 0
\end{array}
\tag{7.16}
$$

Since both B and C are sums of positive line bundles we know from Kodaira vanishing that the cohomologies $H^q(X, C) = H^q(X, B) = 0$ for all $q > 0$. The above long exact sequence then immediately implies that $H^2(X, V) = 0$. In the previous Section we have already shown that $H^0(X, V) = H^3(X, V) = 0$ always, so that the only non-vanishing cohomology of positive monads is $H^1(X, V)$. The dimension $h^1(X, V)$ of this first cohomology can then be calculated from the index theorem (2.7) or indeed the above long exact sequence. In summary, one finds

$$
h^1(X, V) = h^0(X, C) - h^0(X, B) = -\mathrm{ind}(V) , \quad h^q(X, V) = 0 \text{ for } q \neq 1 .
\tag{7.17}
$$

This means that the number of anti-families always vanishes and that the number of families can easily be read off from the index in Figs. 1 and 2. The absence of vector-like pairs of families might be considered an attractive feature and is certainly a pre-requisite for compactifications with the exact standard model spectrum. We stress that this property is directly linked to the property of positivity and will not generally hold if we allowed zero or negative integer entries in the line bundles defining the monad.

7.1.2 Computing $H^1(X, \wedge^2 V^\star)$ and Number of Higgs Multiplets

For SU(3) bundles we have $V \simeq \Lambda^2 V^\star$ and, hence, the cohomology groups $H^1(X, \wedge^2 V)$ and $H^1(X, \wedge^2 V^\star)$ contain no new information. However, for SU(4) and SU(5) this is not the case and we have to perform another calculation. In the case of rank four, $\wedge^2 V \simeq \wedge^2 V^\star$, so that $H^1(X, \wedge^2 V) \simeq H^1(X, \wedge^2 V^\star)$. For rank five the situation is less trivial, but from Eq. (2.8) we know that $h^1(X, \wedge^2 V)$ and $h^1(X, \wedge^2 V^\star)$ are related by the index, $\mathrm{ind}(V)$, of V. Hence, in both the rank four and five cases it is enough to compute one of $H^1(X, \wedge^2 V)$ and $H^1(X, \wedge^2 V^\star)$ and, in the following, we will opt for $H^1(X, \wedge^2 V^\star)$.

To calculate this cohomology, we proceed as in Section 6. Since the arguments therein were stated for general anti-symmetric power p, it is instructive to be more explicit here. We start by writing down the Koszul resolution (B.10) for $\wedge^2 V^\star$ which is given by

$$
0 \to \wedge^2 \mathcal{V}^\star \otimes \wedge^K \mathcal{N}^\star \to \wedge^2 \mathcal{V}^\star \otimes \wedge^{K-1} \mathcal{N}^\star \to \ldots \to \wedge^2 \mathcal{V}^\star \otimes \mathcal{N}^\star \to \wedge^2 \mathcal{V}^\star \to \wedge^2 V^\star \to 0 .
\tag{7.18}
$$

Recall that K is the co-dimension of the CICY X embedded in the ambient space $\mathcal{A}$ and $\mathcal{N}$ is the normal bundle (3.5) of X in $\mathcal{A}$. As a first step we will now derive vanishing theorems for the

cohomologies of the bundles $\wedge^2 V^\star \otimes \wedge^j \mathcal{N}^\star$ which appear in the above Koszul sequence. To do this, we start the exact sequence for antisymmetric products of bundles from (B.11):

$$0 \to S^2\mathcal{C}^\star \to \mathcal{C}^\star \otimes \mathcal{B}^\star \to \wedge^2\mathcal{B}^\star \to \wedge^2\mathcal{V}^\star \to 0 \ , \tag{7.19}$$

which is induced from the dual sequence

$$0 \to \mathcal{C}^\star \to \mathcal{B}^\star \to \mathcal{V}^\star \to 0 \ . \tag{7.20}$$

We can then tensor (7.19) by $\wedge^j \mathcal{N}^\star$ for $j = 0, \ldots, K$ and break the resulting 4-term exact sequence into two short exact sequences

$$
\begin{aligned}
0 &\to S^2\mathcal{C}^\star \otimes \wedge^j\mathcal{N}^\star \to \mathcal{C}^\star \otimes \mathcal{B}^\star \otimes \wedge^j\mathcal{N}^\star \to Q_j \to 0 \ ; \\
0 &\to Q_j \to \wedge^2\mathcal{B}^\star \otimes \wedge^j\mathcal{N}^\star \to \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star \to 0 \ ;
\end{aligned}
\qquad j = 0, \ldots, K \ , \tag{7.21}
$$

where Q_j are approriate (co)kernels. This induces two inter-related long exact sequences in cohomology on $\mathcal{A}$ which are given by

$$\tag{7.22}$$

Note that since X is of codimension K, the ambient space has dimension $K + 3$ and hence there are no non-vanishing cohomology groups above H^{K+3}. Moreover, the bundles $\mathcal{N}^\star$, $\mathcal{B}^\star$ and $\mathcal{C}^\star$ as well as their various tensor and wedge products are all negative and, hence, all their cohomologies except the highest one, namely $K + 3$, vanish by (6.8); we have marked this explicitly in Eq. (7.22).

Therefore, the sequences (7.22) immediately imply that for all j,

$$
\begin{aligned}
&H^i(\mathcal{A}, Q_j) = 0 \ , && i = 0, \ldots, K + 1 \ ; \\
&H^i(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) \simeq H^{i+1}(\mathcal{A}, Q_j) = 0 \ , && i = 0, \ldots, K \ ; \\
&H^{K+1}(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) \simeq H^{K+2}(\mathcal{A}, Q_j)
\end{aligned}
\tag{7.23}
$$

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as well as two 4-term exact sequences:

$$0 \to H^{K+2}(\mathcal{A}, Q_j) \to H^{K+3}(\mathcal{A}, S^2\mathcal{C}^\star \otimes \wedge^j\mathcal{N}^\star) \xrightarrow{g} H^{K+3}(\mathcal{A}, \mathcal{C}^\star \otimes \mathcal{B}^\star \otimes \wedge^j\mathcal{N}^\star) \to H^{K+3}(\mathcal{A}, Q_j) \to 0 \ ;$$

$$0 \to H^{K+2}(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) \to H^{K+3}(\mathcal{A}, Q_j) \to H^{K+3}(\mathcal{A}, S^2\mathcal{B}^\star \otimes \wedge^j\mathcal{N}^\star) \to H^{K+3}(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) \to 0 \ . \tag{7.24}$$

In Eq. (7.24) we have introduced a map g which is induced from the defining map f of the monad in Eq. (4.11). As in the previous subsection, g is generic and thus has maximal rank. The top sequence then implies that $H^{K+2}(\mathcal{A}, Q_j) = 0$ and, hence, by Eq. (7.23), $H^{K+1}(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star)$ vanishes as well. To summarise then, we find the vanishing cohomology groups

$$H^i(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) = 0 \ , \qquad \forall \ i = 0, \dots K+1, \ j = 0, \dots, K \ . \tag{7.25}$$

Equipped with these results, we can re-examine the Koszul sequence (7.18). It has $K+2$ terms and we can break it up into K short exact sequences, introducing (co)kernels much like we did above. Then, the vanishing of the cohomology groups

$$H^{j+1}(\mathcal{A}, \wedge^2\mathcal{V}^\star \otimes \wedge^j\mathcal{N}^\star) = 0 \ , \qquad \forall \ j = 0, \dots, K \ , \tag{7.26}$$

which represent a subset of the vanishing theorems (7.25), implies that

$$H^1(X, \wedge^2 V^\star) = 0 \ . \tag{7.27}$$

We emphasize that the assumption of a generic map f, which defines the monad in (4.11), is crucial to arrive at this result. For rank four bundles with low-energy gauge group SO(10) it implies (see Table 1) that

$$n_{10} = h^1(X, \wedge^2 V) = 0 \ , \tag{7.28}$$

and, hence, a vanishing number of Higgs multiplets. For rank five bundles with low-energy gauge group SU(5) we have

$$n_5 = h^1(X, \wedge^2 V^\star) = 0 \ , \quad n_{\bar 5} = -\text{ind}(V) \ , \tag{7.29}$$

where Eq. (2.8) has been used. This means the number of **10** and **$\bar 5$** representations is the same, forming complete SU(5) families and there are no vector-like pairs of **5** and **$\bar 5$** representations. The absence of Higgs multiplets in the SO(10) and SU(5) models is a phenomenologically problematic feature which was already observed in Ref. [1]. There, it has also been shown that the number of Higgs multiplets can be non-zero once the assumption of a generic map f is dropped. A similar situation was encountered in [16].

We expect a similar bundle-moduli dependence of the spectrum, as first discussed in [13], for the more general class of models considered in this paper. It remains a matter of a more detailed analysis, focusing on physically promising models within our classification, to decide if a realistic particle spectrum can be obtained from such a mechanism.

7.1.3 Singlets and $H^1(X, V \otimes V^\star)$

Finally, we need to calculate the number of gauge group singlets which correspond to the cohomology $H^1(X, \text{ad}(V)) = H^1(X, V \otimes V^\star)$. We begin by tensoring the defining sequence (4.15) for $V^\star$ by V. This leads to a new short exact sequence

$$0 \to C^\star \otimes V \to B^\star \otimes V \to V^\star \otimes V \to 0 \ . \tag{7.30}$$

One can produce two more short exact sequences by multiplying (4.15) with B and C. Likewise, three short exact sequences can be obtained by multiplying the original sequence (4.11) for V with $V^\star$, $B^\star$ and $C^\star$. The resulting six sequences can then be arranged into the following web of three horizontal sequences h_I, h_{II}, h_{III} and three vertical ones v_I, v_{II}, v_{III}.

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & C^\star \otimes V & \to & B^\star \otimes V & \to & V^\star \otimes V & \to & 0 & & h_I \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & C^\star \otimes B & \to & B^\star \otimes B & \to & V^\star \otimes B & \to & 0 & & h_{II} \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & C^\star \otimes C & \to & B^\star \otimes C & \to & V^\star \otimes C & \to & 0 & & h_{III} \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & & \\
 & & v_I & & v_{II} & & v_{III} & & & &
\end{array}
\tag{7.31}
$$

The long exact sequence in cohomology induced by h_I reads

$$
\begin{aligned}
0 & \to & H^0(X, C^\star \otimes V) \to H^0(X, B^\star \otimes V) \to H^0(X, V^\star \otimes V) \\
 & \to & H^1(X, C^\star \otimes V) \to H^1(X, B^\star \otimes V) \to \boxed{H^1(X, V^\star \otimes V)} \\
 & \to & H^2(X, C^\star \otimes V) \to \ldots
\end{aligned}
\tag{7.32}
$$

and we have boxed the term which we would like to compute. We will also need the long exact sequences which follow from v_I and v_{II}. They are given by

$$
\begin{aligned}
0 & \to & H^0(X, C^\star \otimes V) \to H^0(X, C^\star \otimes B) \to H^0(X, C^\star \otimes C) \\
 & \to & H^1(X, C^\star \otimes V) \to H^1(X, C^\star \otimes B) \to H^1(X, C^\star \otimes C) \\
 & \to & H^2(X, C^\star \otimes V) \to H^2(X, C^\star \otimes B) \to H^2(X, C^\star \otimes C) \\
 & \to & H^3(X, C^\star \otimes V) \to H^3(X, C^\star \otimes B) \to H^3(X, C^\star \otimes C) \to 0 \ ;
\end{aligned}
\tag{7.33}
$$

$$
\begin{aligned}
0 & \to & H^0(X, B^\star \otimes V) \to H^0(X, B^\star \otimes B) \to H^0(X, B^\star \otimes C) \\
 & \to & H^1(X, B^\star \otimes V) \to H^1(X, B^\star \otimes B) \to H^1(X, B^\star \otimes C) \\
 & \to & H^2(X, B^\star \otimes V) \to H^2(X, B^\star \otimes B) \to H^2(X, B^\star \otimes C) \\
 & \to & H^3(X, B^\star \otimes V) \to H^3(X, B^\star \otimes B) \to H^3(X, B^\star \otimes C) \to 0 \ .
\end{aligned}
\tag{7.34}
$$

To make progress we need information about the cohomologies of $B^\star \otimes B$, $C^\star \otimes C$ and $C^\star \otimes B$. For the cyclic cases (the CICYs with $h^{1,1}(X) = 1$) discussed in Ref. [1] all line bundles L on X have vanishing middle cohomologies, that is $H^1(X, L) = H^2(X, L) = 0$ and, hence, the same is true for $B^\star \otimes B$, $C^\star \otimes C$ and $C^\star \otimes B$. For the general case discussed here this is no longer necessarily true since $B^\star \otimes B$, $C^\star \otimes C$ and $C^\star \otimes B$ may contain "mixed" line bundles with different sign or zero entries which may have non-vanishing middle cohomologies. This means in some cases there will not be sufficiently many zero entries in the above long exact sequences to compute $h^1(X, V \otimes V^\star)$ without additional input, for example about the rank of maps.

However, a general formula can be derived for all monads satisfying

$$
H^1(X, C^\star \otimes C) = H^2(X, C^\star \otimes B) = 0 \ .
\tag{7.35}
$$

Since we can compute all line bundle cohomologies we can explicitly check for each given example whether these conditions are actually satisfied. Let us focus on models where this is the case. Then

the sequence (7.33) implies that $H^2(X, C^\star \otimes V) = 0$ which means that (7.32) breaks after the second line and this 6-term exact sequence implies:

$$h^1(X, V^\star \otimes V) = h^1(X, B^\star \otimes V) - h^1(X, C^\star \otimes V) + h^0(X, V^\star \otimes V) - h^0(X, B^\star \otimes V) + h^0(X, C^\star \otimes V) \,. \tag{7.36}$$

In the above, we have used the fact that for any long exact sequence, whatever the number of terms, the total alternating sum of the dimensions of the terms vanishes.

We can apply a similar trick to the other 2 long exact sequences. Using our assumptions $H^1(X, B^\star \otimes C) \simeq H^2(X, C^\star \otimes B) = 0$ in the sequence (7.34) and $H^1(X, C^\star \otimes C) = 0$ in the sequence (7.33) gives the two relations

$$\begin{aligned}
h^1(X, B^\star \otimes V) - h^0(X, B^\star \otimes V) &= h^0(X, B^\star \otimes C) - h^0(X, B^\star \otimes B) + h^1(X, B^\star \otimes B) \\
h^0(X, C^\star \otimes V) - h^1(C^\star \otimes V) &= h^0(X, C^\star \otimes B) - h^0(X, C^\star \otimes C) - h^1(X, C^\star \otimes B) \,.
\end{aligned}$$

Inserting these into Eq. (7.36) and using the fact that for a stable $SU(n)$ bundle V, $h^0(X, V \otimes V^\star) = 1$ (see Section 4.2 of [1]) gives the final result

$$\begin{aligned}
n_1 = h^1(X, V^\star \otimes V) &= h^0(X, B^\star \otimes C) - h^0(X, B^\star \otimes B) - h^0(X, C^\star \otimes C) \\
&\quad + h^0(X, C^\star \otimes B) - h^1(X, C^\star \otimes B) + h^1(X, B^\star \otimes B) + 1 \tag{7.37}
\end{aligned}$$

for the number of singlets. We emphasize that this is result is valid provided the monad satisfies the two conditions (7.35). In this case, Eq. (7.37) allows an explicit calculation of the number of singlets from the known line bundle cohomologies.

As an example, we consider the manifold $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and the rank 4 monad bundle defined by

$$B = \mathcal{O}_X(1,1)^{\oplus 6} \oplus \mathcal{O}_X(2,1)^{\oplus 2} \,, \quad C = \mathcal{O}_X(2,3)^{\oplus 2} \oplus \mathcal{O}_X(3,1)^{\oplus 2} \,. \tag{7.38}$$

It can be checked from the known line bundle cohomologies that this bundle indeed satisfies the conditions (7.35). The number of singlets, calculated from Eq. (7.37), is then given by $n_1 = 241$.

For bundles which do not satisfy (7.35) other methods can be employed. In favourable cases, the cohomologies of $B^\star \otimes B$, $C^\star \otimes C$ and $C^\star \otimes B$ may have a different pattern of zeros which still allows the derivation of a formula for n_1 analogous to Eq. (7.37) by combining appropriate parts of the sequences (7.32), (7.33) and (7.34). If this is not possible one has to resort to ambient space methods and Koszul resolutions in combination with our results for the ranks of maps in Leray spectral sequences. Here, we will not present such a calculation which is likely to be complicated and, if required at all, should probably be only carried out for physically promising models. However, we stress that all the necessary technology is available so that the number of singlets can, not just in principle but in practice, be obtained for all positive monads on favourable CICYs.

8 Conclusions and Prospects

In this paper, we have analysed positive monad bundles with structure group $SU(n)$ (where $n = 3, 4, 5$) on favourable CICY manifolds in the context of $N = 1$ supersymmetric compactifications of the $E_8 \times E_8$ heterotic string. We have shown that the class of these bundles, subject to the heterotic anomaly condition, is finite and consists of 7118 examples. More specifically, we find that these 7000 or so monads are concentrated on only 36 CICYs. All other of the 4500 or so CICYs do not allow positive monads which satisfy the anomaly condition.

As a highly non-trivial test for the stability of these bundles we have shown that $H^0(X, \wedge^p V^*) = 0$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$ for all our examples. A systematic stability proof will be presented in Ref. [46]. We have also shown how to calculate the complete particle spectrum for these models. In particular, we found that the number of anti-families always vanishes so that there are no vector-like family anti-family pairs present in any of the models. For low-energy groups SO(10) and SU(5) ($n = 4, 5$) the number of Higgs fields vanishes at generic points in the bundle moduli space. However, as was shown in Ref. [1], for non-generic values of the bundle moduli, Higgs multiplets can arise. The details of this moduli-dependence of the spectrum (see Ref. [13]) have to be analysed for specific models, preferably focusing on physically promising examples. Furthermore, we have shown that the number of gauge singlets can be calculated, in many cases in terms of a generic formula, or else by applying more elaborate methods.

Based on the results for the particle spectrum, we have scanned the 7118 bundles imposing two rudimentary physical conditions. First, the number of families should equal $3k$ for some non-zero integer k, so there is a chance to obtain three families after dividing by a discrete symmetry of order k. In addition, the Euler number of the Calabi-Yau space should be divisible by k. It turns out that only 559 out of the 7118 bundles pass this basic test. If, in addition, one demands that the order k of the symmetry does not exceed 13 one is left with only 21 models.

This drastic reduction of the number of viable models due to a few basic physical constraints is not uncharacteristic and has been observed in the context of other string constructions [43, 44]. In our case, the main reason for this reduction is the relatively large values for the Euler characteristic of our models (roughly, a Gaussian distribution with a maximum at about 60, see Fig. 1) in conjunction with the empirical fact that large discrete symmetries of Calabi-Yau manifolds are hard to find. In order to make this statement more precise a systematic analysis of discrete symmetries Γ on CICYs X (which lead to a smooth quotient X/Γ) has to be carried out and the results of this analysis have to be combined with the results of the present paper. We are planning to perform this explicitly in the near future. However, even in the absence of such a classification of discrete symmetries we find it likely that the vast majority of positive monads will fail to produce three-family models on X/Γ given the large number of families on the "upstairs" manifold X.

These large numbers are, of course, directly related to the property of positivity. An obvious course of action is, therefore, to relax this condition and also allow zero or even slightly negative integers b_i^r and c_j^r in the definition (4.11) of the monad. The number of these non-positive monads is vastly larger than the number of positive ones and it turns out the distribution of their Euler characteristics is peaked at smaller values, as expected. Crucially, as will be shown in Ref. [46], some of these non-positive monads are still stable and, hence, lead to supersymmetric models. We, therefore, believe that the generalisation to non-positive monads is a crucial step towards realistic models within this framework and work in this direction is underway [45].

Acknowledgments

A   Notation and conventions

Throughout the paper we will adhere to the following notations:

X	Calabi-Yau threefold embedded in ambient space $\mathcal{A} = \mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_m}$
$[q_j^r]_{j=1,\ldots,K}^{r=1,\ldots,m}$	configuration matrix for co-dimension K CICY in product of m projective spaces
$\mathcal{O}_\mathcal{A}(\mathbf{k})$	product of line bundles $\mathcal{O}_\mathcal{A}(k^1) \otimes \cdots \otimes \mathcal{O}_\mathcal{A}(k^m)$ on $\mathcal{A}$
$\mathcal{O}_X(\mathbf{k})$	restriction of $\mathcal{O}_\mathcal{A}(\mathbf{k})$ to X
$\mathcal{N}$	Normal bundle of X in $\mathcal{A}$
TX	Tangent bundle of X; similarly, $T\mathcal{A}$ is the tangent bundle of $\mathcal{A}$
V	Vector bundle on X, the dual bundle is denoted $V^\star$
B, C	Sum of line bundles $\bigoplus_i \mathcal{O}_X(\mathbf{b}_i)$ and $\bigoplus_a \mathcal{O}_X(\mathbf{c}_a)$ on X
$\mathcal{V}$	Vector bundle on ambient $\mathcal{A}$ which restricts to V on X
$\mathcal{B}, \mathcal{C}$	Sums of line bundles $\bigoplus_i \mathcal{O}_\mathcal{A}(\mathbf{b}_i)$ and $\bigoplus_a \mathcal{O}_\mathcal{A}(\mathbf{c}_a)$ on $\mathcal{A}$

B   Some Mathematical Preliminaries

In this appendix, we collect some useful mathematical facts which will be of importance throughout the paper. These can be found in standard references such as [30, 40–42].

Serre Duality: For a vector bundle V on a manifold M of complex dimension n, Serre duality relates the cohomology groups of V with those of its dual as:

$$H^i(M, V) \simeq H^{n-i}(X, V^\star \otimes K_M) \qquad i = 0, 1, \ldots, n , \tag{B.1}$$

where $K_M = \bigwedge^n TM^\star$ is the canonical bundle of M. For a Calabi-Yau threefold X, the canonical bundle K_X is the trivial bundle $\mathcal{O}_X$ and, hence, Serre duality takes the particularly simple form

$$H^i(X, V) \simeq H^{3-i}(X, V^\star) \qquad i = 0, 1, 2, 3 . \tag{B.2}$$

Atiyah-Singer Index Theorem: For a unitary bundle V on a Calabi-Yau threefold X, the index theorem relates the index, or the alternating sum of dimensions of the cohomology groups of V with the characteristic classes of the bundle and the manifold:

$$\text{ind}(V) = \sum_{i=0}^{3} (-1)^i h^i(X, V) = \int_X \text{ch}(V) \wedge \text{Td}(X) = \frac{1}{2} \int_X c_3(V) , \tag{B.3}$$

where $\text{Td}(X)$ is the Todd class for the tangent bundle of X. Only in the last equality have we used the fact the both $c_1(TX)$ and $c_1(V)$ vanish.

Higher Exterior Powers: For $SU(n)$ bundles we have the equivalences

$$\wedge^p V \simeq \wedge^q V^\star \qquad p + q = n \tag{B.4}$$

and the relation (see Appendix B of Ref. [12]),

$$c_3(\wedge^2 V) = (n - 4) c_3(V) . \tag{B.5}$$

The Bott Formula: The cohomology of line-bundles over a projective space $\mathbb{P}^n$ is given by a simple formula, the so-called Bott formula (see, for example, Ref. [32]), which dictates that

$$h^q(\mathbb{P}^n, (\wedge^p T\mathbb{P}^n) \otimes \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n+p+1}{p}\binom{k+n}{n-p} & q=0 & k > -p-1, \\ 1 & q=n-p & k=-n-1, \\ \binom{-k-p-1}{-k-n-1}\binom{-k-n-2}{p} & q=n & k < -n-p-1, \\ 0 & \text{otherwise} & . \end{cases} \tag{B.6}$$

Künneth formula: The Künneth formula gives the cohomology of bundles over direct product of spaces. For a product $\mathcal{A} = \mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_m}$ of projective spaces and $\mathbf{k} = (k^1, \ldots, k^m)$, it states that

$$H^n(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(\mathbf{k})) = \bigoplus_{q_1+\ldots+q_m=n} H^{q_1}(\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(k^1)) \times \ldots \times H^{q_m}(\mathbb{P}^{n_m}, \mathcal{O}_{\mathbb{P}^{n_m}}(k^m)), \tag{B.7}$$

Kodaira Vanishing Theorem: For positive line bundle L on a Kahler manifold M the Kodaira vanishing theorem states that

$$H^q(M, L \otimes K_M) = 0 \quad \forall \, q > 0, \tag{B.8}$$

where K_M is the canonical bundle on M. For a Calabi-Yau manifold, X, K_X is trivial and therefore the only non-vanishing cohomology for a positive line bundle, L, on X is $H^0(X, L)$. On the ambient space $\mathcal{A}$ it is useful to look at the the Serre dual of Eq. (B.8). For positive line bundles $\mathcal{L}$ on $\mathcal{A}$ Eq. (B.1) this leads to $H^q(\mathcal{A}, \mathcal{L} \otimes K_{\mathcal{A}}) \simeq H^{\dim(\mathcal{A})-q}(\mathcal{A}, \mathcal{L}^\star \otimes K_{\mathcal{A}}^\star \otimes K_{\mathcal{A}})$. The canonical bundle $K_{\mathcal{A}}$ and its dual tensor to $\mathcal{O}_{\mathcal{A}}$ and we have the important fact that

$$H^q(\mathcal{A}, \mathcal{L}^\star) = 0 \text{ unless } q = \dim(\mathcal{A}). \tag{B.9}$$

Koszul Resolution   The standard method of computing the cohomology of a vector bundle $V = \mathcal{V}|_X$ obtained by restricting the bundle (or sheaf) $\mathcal{V}$ on the ambient space $\mathcal{A}$ to the variety X is the so-called *Koszul Resolution* of V. In general, if X is a smooth hypersurface of co-dimension K, which is the zero locus of a holomorphic section s of the bundle $\mathcal{N}$, then the following exact sequence exists:

$$0 \to \mathcal{V} \otimes \wedge^K \mathcal{N}^\star \to \mathcal{V} \otimes \wedge^{K-1} \mathcal{N}^\star \to \ldots \to \mathcal{V} \otimes \mathcal{N}^\star \to \mathcal{V} \to V \to 0. \tag{B.10}$$

Thus, if the cohomology of the bundles $\wedge^j \mathcal{N}^\star \otimes \mathcal{V}$ are known on the ambient space, we can use the Koszul sequence to determine the cohomology of V. We recall that for a CICY, the normal bundle is given in terms of the configuration matrix, as in Eq. (3.5).

Exterior-Power Sequence: Given a short exact sequence of vector bundles A, B and C on any manifold:
$$0 \to A \to B \to C \to 0,$$

there exists a long exact sequence for the p-th exterior power of C, derivable from a so-called Eagon-Northcott complex. This sequence reads:

$$0 \to S^p A \to S^{p-1} A \otimes B \to \ldots \to A \otimes \wedge^{p-1} B \to \wedge^p B \to \wedge^p C \to 0. \tag{B.11}$$

C More on CICYs

We have introduced basic facts about CICYs in the main text. In this appendix, we present some more detailed properties relevant to our investigation. Many of these are standard results which can be found, for example, in Ref. [30] but we also discuss some new aspects, in particular the redundancy in the CICY list.

C.1 Chern Classes and Intersection Form

We focus on a class of CICYs X, defined as the common zero locus of K polynomials in an ambient space $\mathcal{A} = \mathbb{P}^{n_1} \otimes \cdots \otimes \mathbb{P}^{n_m}$ with m projective factors of dimension n_r.. This CICY is characterised by a configuration matrix $[q_j^r]_{j=1,\dots,K}^{r=\dots,m}$, as in Eq. (3.3), where q_j^r denotes the degree of the jth polynomial in the variables of the rth projective space. These degrees are subject to the complete intersection condition (3.2) and the condition (3.4) which ensures the vanishing of the first Chern class $c_1(TX)$. Integration over X can be reduced to integration over the ambient space $\mathcal{A}$ using the formula

$$\int_X \cdot = \int_{\mathcal{A}} \mu \wedge \cdot \,, \qquad \mu := \wedge_{j=1}^K \left(\sum_{r=1}^m q_r^j J_r \right) \,. \tag{C.12}$$

In this way, one can compute the triple intersection numbers

$$d_{rst} = \int_X J_r \wedge J_s \wedge J_t \tag{C.13}$$

where J_r are the Kahler forms of the ambient space projective factors $\mathbb{P}^{n_r}$. The Chern classes are given as simple functions of the entries in the configuration matrix [30]. The total Chern class can be expanded in terms of the ambient space Kahler forms as

$$c(TX) = c_1^r(TX)J_r + c_2^{rs}(TX)J_rJ_s + c_3^{rst}(TX)J_rJ_sJ_t \,, \tag{C.14}$$

where

$$c_1^r(TX) = 0 \tag{C.15}$$

$$c_2^{rs}(TX) = \frac{1}{2} \left[-\delta^{rs}(n_r + 1) + \sum_{j=1}^K q_j^r q_j^s \right] \tag{C.16}$$

$$c_3^{rst}(TX) = \frac{1}{3} \left[\delta^{rst}(n_r + 1) - \sum_{j=1}^K q_j^r q_j^s q_j^t \right] \,. \tag{C.17}$$

The second Chern class should be expressed as $c_2(TX) = c_{2r}(TX)\nu^r$ relative to a basis ν^r of $H^4(X, \mathbb{Z})$, as defined in Eq. (3.6). The conversion from the coefficients $c_2^{rs}(TX)$ above can be accomplished by contraction with the intersection numbers

$$c_{2r}(TX) = d_{rst} c_2^{st}(TX) \,. \tag{C.18}$$

Similarly, the Euler number $\chi(X)$ is obtained from

$$\chi(X) = d_{rst} c_3^{rst}(TX) \,. \tag{C.19}$$

C.2 Hodge Numbers

We wish to know the full topological data of X including the Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$, whose difference, by the Index Theorem (B.3), is the Euler number $\chi(X)$:

$$h^{1,1}(X) - h^{2,1}(X) = \frac{1}{2}\chi(X) \ . \tag{C.20}$$

Therefore, it suffices to compute either one of these two Hodge numbers. This calculation is the subject of Ref. [4] and it this turns out to be much more involved than calculating the Euler number. While this paper explains the basic method, sadly, the actual data for these Hodge numbers seems to have been lost. Both for the purpose of reconstructing this data and because related techniques can be applied to monad bundles it is useful to review the methods of Ref. [4].

Recalling that

$$H^{p,q}(X) \simeq H^q(X, \wedge^p T^\star X) \ , \tag{C.21}$$

where $T^\star X$ is the cotangent bundle of X, we can write the desired cohomologies as

$$H^{1,1}(X) = H^1(M, T^\star X), \qquad H^{2,1}(X) \simeq H^{1,2}(X) = H^2(X, T^\star X) \simeq H^1(X, TX) \ . \tag{C.22}$$

In the second part of the above expression, we have used Serre duality, (B.1), to establish the isomorphism between $H^2(X, T^\star X)$ and $H^1(X, TX)$.

We can therefore concentrate on the computing $H^1(X, TX)$. We invoke the Euler sequence which states that, for an embedding of X into an ambient space $\mathcal{A}$, there is a short exact sequence

$$0 \to TX \to T\mathcal{A}|_X \to \mathcal{N}|_X \to 0 \ , \tag{C.23}$$

where $\mathcal{N}$ is the normal bundle of X in $\mathcal{A}$ and $T\mathcal{A}$ is the the tangent bundle of $\mathcal{A}$. The bar and the subscript, X, denotes restriction of the bundle to the Calabi-Yau manifold X. This induces a long exact sequence in cohomology as

$$\begin{array}{ccccccc}
0 & \to & H^0(X, TX) & \to & H^0(X, T\mathcal{A}|_X) & \to & H^0(X, \mathcal{N}|_X) & \to \\
& \to & H^1(X, TX) & \overset{d}{\to} & H^1(X, T\mathcal{A}|_X) & \to & H^1(X, \mathcal{N}|_X) & \to \\
& \to & H^2(X, TX) & \to & \cdots &&&
\end{array} \tag{C.24}$$

Since X is a Calabi-Yau manifolds it follows that $H^0(X, TX) = H^{1,3}(X) = 0$. Using this, the relations (C.22), and the fact that $\mathrm{rk}(d) = 0$ (see Eq. (6.1) of Ref. [4]), we have the short exact sequence

$$0 \to H^0(X, T\mathcal{A}|_X) \to H^0(X, \mathcal{N}|_X) \to H^{2,1}(X) \to 0 \ , \tag{C.25}$$

and, consequently,

$$h^{2,1}(X) = h^0(X, \mathcal{N}|_X) - h^0(X, T\mathcal{A}|_X) \ . \tag{C.26}$$

C.2.1 Hodge Number Obstructions

Making use of the essential techniques of Leray tableaux and Koszul resolutions, one can, in principle compute the two terms in Eq. (C.26) and, hence, obtains the Hodge numbers of complete intersection 3-folds. However, direct calculation shows that one quickly encounters certain obstructions to the computation which will naturally divide our set of 7890 configurations.

Trivial Direct Products First of all, we recognize that there are trivial cases in the list, comprising of CICYs which are simply direct products of lower-dimensional Calabi-Yau manifolds, viz., $K3 \times T^2$ and T^6. These generically have reduced holonomy and we shall not consider them. The identifiers for these are 31-52, a total of 22 cases. Therefore, our list is immediately reduced to be of length 7868.

Normal Bundle and Obstructions The Leray $E_1^{j,k}(\mathcal{N}_X)$ tableaux is readily established for the normal bundle $\mathcal{N}_X$ according to (B.6) and (B.7). It turns out that if there exists $j \leq j'$ in $[-K, 0]$ such that

$$E_1^{j,j}(\mathcal{N}|_X) \neq 0 \text{ and } E_1^{j',-j'}(\mathcal{N}|_X) \neq 0 \ , \tag{C.27}$$

then, the spectral sequence cannot be iterated to obtain E_∞ without the knowledge of the ranks of some maps. Such a case, which we call "normal bundle obstructed" needs to be addressed separately [4]. For all remaining cases, the Leray spectral sequence actually terminates at E_1 and we can read off the required cohomology as [4]:

$$h^0(X, \mathcal{N}|_X) = \sum_{j=0}^{K} e_1^{j,j}(\mathcal{N}|_X) + \sum_{j=1}^{K} \sum_{l=0}^{j-1} (-1)^{j+l} e_1^{l,j}(\mathcal{N}|_X) \ . \tag{C.28}$$

In the above, we have used, and shall henceforth adopt, the notation that h^j is the dimension of the cohomology group H^j, $e_r^{j,k}$ is the dimension of $E_r^{j,k}$.

Now, we find a total of 12 normal bundle obstructed cases, namely the CICYs with identifiers 1443, 1877, 2569, 2980, 3747, 4228, 4448, 4757, 6174, 6229, 7236 and 7243. For these, Ref. [4] gives a rule to replace the configuration matrix by an isomorphic one which does not have a normal bundle obstruction. To this equivalent configuration, Eq. (C.28) can then be directly applied.

Tangent Bundle and Obstructions Like the normal bundle spectral sequence the tangent bundle spectral sequence can, in general, be obstructed, that is, one cannot compute E_∞ without knowledge of specific maps. However, for the case of compete intersection calabi-yau manifolds we are saved from this difficulty by several useful results.

The first such result is that for a particular class of configurations (those without a decomposing $(n-1)$-leg, see Ref. [4] for a description of the dot/leg diagrams and notation), $E_1^{q+k,k}(V)$ vanishes for $q \geq n-1$ for any bundle V on X. It turns out that if a diagram representing a Calabi-Yau 3-fold has no decomposing 1-legs, $H^1(X, T\mathcal{A})$ vanishes and no decomposing 2-legs implies that $H^2(X, N) = 0$ so that the sequences

$$0 \rightarrow H^0(X, T\mathcal{A}) \rightarrow H^0(X, N_X) \rightarrow H^1(X, TX) \rightarrow 0$$
$$0 \rightarrow H^1(X, N_X) \rightarrow H^2(X, TX) \rightarrow H^2(X, T\mathcal{A}) \rightarrow 0 \tag{C.29}$$

are exact [4].

For 3-folds with decomposing 1-legs the hodge numbers can be computed by relying on the classification of complex surfaces (see Eq. (2.4)in Ref. [4]). Simple formulas for these Hodge numbers in terms of sub-diagrams were found in [4]. For the bulk of cases, however, the diagrams have no decomposing 1-legs.

Further, it can be shown that an n-fold configuration with the property of a decomposing $(n-1)$-leg is equivalent to another one with no decomposing $(n-1)$-leg [4]. So in analysing configurations

representing Calabi-Yau 3-folds it is sufficient to look only at configurations with no decomposing 2-legs. This leads to the following structure

$$E_1^{0,0}(T\mathcal{A}) = \bigoplus_{r=1}^{m} H^0(\mathbb{P}_r^{n_r}, T(\mathbb{P}_r^{n_r}));$$

$$E_1^{K+2,K}(T\mathcal{A}) \approx C^m;$$

$$E_1^{q+k,k}(T\mathcal{A}) \ni H^0(\mathbb{P}_r^{n_r}, T(\mathbb{P}_r^{n_r}) \otimes h_r^{-1}) \approx C^{n_r+1}, \quad \forall\{A, r : \sum_{a \in S} q_a^r = 1, k = |S| + 1\}$$

where $|S|$ denotes the cardinality of S, the set of indices labeling a subset of constraints which act only in a $(q + k)$-dimensional factor of the ambient space $\mathcal{A}$. With these results in hand, we can compute the Hodge numbers of X.

C.3 Redundancy in the CICY list

It is worth observing that the 7890 CICYs which appear in the original list are presumably not all topologically distinct [39]. This is a relatively new observation and should be pointed out.

Wall's theorem (see, for example, Ref. [30]) states that for real six-folds, the intersection form and the second Potryagin class suffice to distinguish non-isormophism. Though for complex threefolds, these are not enough, the two quantities are good indicators (and will be enough to distinguish our heterotic models). Therefore, we propose a simple check for redundancy. We compare the basic topological invariants Hodge numbers $h^{1,1}(X)$, $h^{2,1}(X)$, second Chern class $c_{2r}(TX)$, and intersection numbers d_{rst}, and identify any two CICYs with identical sets, up to permutation in the indices r, s.

Upon implementing such a scan one finds, of the 7890 in the original list, that there are 378 sets of redundancies, consisting of equivalent pairs, triples, or even n-tuples for n as large as 6. These are expected to have isomorphism. In all, 813 manifolds are involved; taking one representative from each of the 378 sets, a total of 435 CICY seem redundant. Throughout the rest of the paper, however, we will adhere to the original identifier names of the manifolds to avoid confusion and shall point out explicitly, where necessary, the equivalences.

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