Exact Form Factors in Integrable Quantum Field Theories: the Sine-Gordon Model

H. Babujian[1,2], A. Fring[3], M. Karowski[4] and A. Zapletal[5]

Institut für Theoretische Physik

Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

In memory of Harry Lehmann

Abstract

We provide detailed arguments on how to derive properties of generalized form factors, originally proposed by one of the authors (M.K.) and Weisz twenty years ago, solely based on the assumption of "maximal analyticity" and the validity of the LSZ reduction formalism. These properties constitute consistency equations which allow the explicit evaluation of the n-particle form factors once the scattering matrix is known. The equations give rise to a matrix Riemann-Hilbert problem. Exploiting the "off-shell" Bethe ansatz we propose a general formula for form factors for an odd number of particles. For the Sine-Gordon model alias the massive Thirring model we exemplify the general solution for several operators. In particular we calculate the three particle form factor of the soliton field, carry out a consistency check against the Thirring model perturbation theory and thus confirm the general formalism.

[1]Permanent address: Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036 Armenia.
[2]e-mail: babujian@lx2.yerphi.am, babujian@physik.fu-berlin.de
[3]e-mail: fring@physik.fu-berlin.de
[4]e-mail: karowski@physik.fu-berlin.de
[5]e-mail: zapletal@physik.fu-berlin.de

1 Introduction

More than fifty years ago, Heisenberg [1] pointed out the importance of studying analytic continuations of scattering amplitudes into the complex momentum plane. The first concrete investigations in this direction were carried out by Jost [2] and Bargmann [3], initially for non-relativistic scattering processes. The original ideas turned out to be very fruitful and lead to interesting results on-shell, i.e. for the S-matrix [4], as well as off-shell, that is for the two-particle form factors, see for instance [5].

Once one restricts ones attention to 1+1 dimensional integrable theories, the n-particle scattering matrix factories into two particle S-matrices and the approach, now usually referred to as the bootstrap program, reveals its full strength. On-shell, it leads to the exact determination of the scattering matrix [6, 7], (for reviews see also [9-12]). The results obtained in this way agree with the S-matrix obtained from the extrapolation of semi-classical expressions for the Sine-Gordon model [8]. The first off-shell considerations were carried out about two decades ago by one of the authors (M.K.) et al. [13, 14], who introduced the concept of a generalized form factor and formulated several consistency equations which are expected to be satisfied by these objects. Thereafter this approach was mainly developed further and studied in the context of several explicit models by Smirnov et al. [15-23]. Recently this program has seen some revival in relation to models which arise as perturbations of certain conformal field theories [24], particularly in the context of affine Toda theories [25] and closely related models [26-48].

An entirely different method, the Bethe ansatz [49], was initially formulated in order to solve the eigenvalue problem for certain integrable Hamiltonians. The approach has found applications in the context of numerous models and has led to a detailed study of various mass spectra and S-matrices (for reviews and an extensive list of references see for instance [50]). The original techniques have been refined into several directions, of which in particular the so-called "off-shell" Bethe ansatz, which was originally formulated by one of the authors (H.B.) [51, 52], will be exploited for our purposes. This version of the Bethe ansatz paves the way to extend the approach to the off-shell physics and opens up the intriguing possibility to merge the two methods, that is the form factor approach and the Bethe ansatz. The basis for this opportunity lies in the observation [53, 54, 55], that the "off-shell" Bethe ansatz captures the vectorial structure of Watson's equations (see section 2.2 properties (i) and (ii)). These are matrix difference equations giving rise to a matrix Riemann-Hilbert problem which is solved by an "off-shell" Bethe ansatz. Furthermore, there exist interesting speculations in order to make contact with general concepts of algebraic quantum field theory [60, 61].

Conceptionally, the on and off-shell approaches are very similar. For the on-shell situation one has certain constraints resulting from general physical and in particular analytic properties (referred to as "maximal analyticity assumption"), which lead to a set of conditions which turned out to be so restrictive that they allow to construct the exact scattering matrix almost uniquely. This approach is adopted in order to determine the key off-shell quantities, i.e. the form factors. In the present manuscript we provide a detailed derivation of the consistency equations solely based on the maximal analyticity assumption and the validity of the LSZ-formalism [57] (see also [58]). Form factors are vector valued functions, representing matrix elements of some local operator $\mathcal{O}(x)$ at the origin between an in-state and the vacuum, which we denote by (refer equation (3.2) for more details)

$$F_{\underline{\alpha}}^{\mathcal{O}}\left(((p_i+p_j)^2+i\varepsilon)_{(1\leq i<j\leq n)}\right) := \langle 0\,|\mathcal{O}(0)|\,p_1,\ldots,p_n\rangle^{in}_{\alpha_1\ldots\alpha_n} \,. \tag{1.1}$$

Once all the n-particle form factors are known, one is in principle in a position to compute all correlation functions. In particular the two point function for an hermitian operator $\mathcal{O}$ in real Euclidian space reads

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle \;=\; \sum_{n=0}^{\infty}\int \frac{d\theta_1\ldots d\theta_n}{n!\,(4\pi)^n}\;|\;F_{\underline{\alpha}}^{\mathcal{O}}(\theta_1,\ldots,\theta_n)\;|^2\,\exp\left(-r\sum_{i=1}^{n} m_i\cosh\theta_i\right). \tag{1.2}$$

Here r denotes the radial distance $r = \sqrt{x_1^2+x_2^2}$ and θ is the rapidity related to the momentum via $p_i = m_i\sinh\theta_i$ (see section 3.2 for more details). The explicit evaluation of all integrals and sums remains an open challenge for almost all theories, except the Ising model[1]. Important progress towards a solution of this problem has recently been achieved in [59].

A commonly used procedure which will yield expressions which satisfy all of the consistency requirements is constituted out of the following steps: First of all one has to have solved the on-shell system, that is one requires expressions for the S-matrix. In the next step one usually makes an ansatz for the form factors of a type already introduced in [13], in which one extracts explicitly the expected singularity structure. The nature of the ansatz guarantees by construction that the generalized Watson's equations (properties (i) and (ii)) are satisfied once the scattering matrix is diagonal. For generically non-diagonal scattering matrices one may invoke also the techniques of the "off-shell" Bethe ansatz [53, 54, 55] in order to capture the vectorial structure of the form factors.

[1]Of course one may also adopt a very practical point of view and resort to the well-known fact that the series expansion of correlation functions in terms of form factors (1.2) converges very rapidly. Consequently correlations functions may be approximated very often quite well by simply including the two-particle form factor into the expansion. ¿From that point of view the form factor program is completed, since the calculation of the two-particle form factors is well understood.

The ansatz only involves the rapidity differences, apart from a possible pre-factor, which takes the spin of the local field $\mathcal{O}$ into account, and has therefore the desired behavior under Lorentz transformations (refer property (v) in section 3.2). General solutions for the so-called minimal form factors (the function which satisfies the functional equation (4.10)) are always fairly easy to find. Once the scattering matrix is non-diagonal one has also to encode the vectorial structure at this stage. What is then left, is to determine a general function which takes the complete singularity structure into account. For this purpose one may now invoke properties (iii) and (iv) (equations (3.12) and (3.13) for the bosonic case), which lead to a set of recursive equations. In principle these equations may now be solved step by step, once the first non-vanishing form factor for a particular operator is properly fixed. However, only after a few steps the expressions become usually algebraically very complex and reveal very little insight. Therefore, it is highly desirable to search for structures of a more general nature, that is in particular to seek for closed expressions for all n-particle form factors. Only such expressions may ultimately shed more light on the analytic expressions for the correlation functions (1.2). Alternatively, one may try to construct directly a representation for the creation operators of the particles in the in-state in (1.1) [62, 63, 64, 65]. Representing the local operator $\mathcal{O}$ in the same space, one may in principle also compute the form factors.

In the present manuscript we provide a general expression (see theorem 4.1) of a different kind, which solves all the consistency requirements. It is very generic by construction and, roughly speaking, captures the vectorial nature of the form factors by means of "off-shell" Bethe ansatz states and the pole structure by particular contour integrals. We exemplify this general expression for the form factors of the Sine-Gordon model involving an odd number of states, which was hitherto unknown. For the even case similar expressions may be found in [17, 18]. We present a detailed analysis of the three particle form factor.

Once solutions for the set of consistency equations are found, it is highly desirable to verify the solutions with some alternative method. Several different methods have been developed in recent years. Assuming that the theory under consideration results from the perturbation of some conformal field theory, one may carry out the following consistency checks. For instance one may take the operator in the form factor to be the trace of the energy momentum tensor and exploit the so-called c-theorem [66] in order to obtain a first indication about the result. This check is not extremely restrictive what the higher n-particle form factors concerns, since the expected value for c is usually already saturated after the two-particle contribution. Alternatively one may also compare with the perturbation theory around the conformal field theory, which is possible for all operators of the model. The latter approach has turned out to be very fruitful [27]. A further consistency check consist out of the comparison between the exact result obtained from the form factors with the predictions of the renormalisation group (that is asymptotic freedom etc. [67, 35]). In the present manuscript we present a check of our solutions against conventional perturbation theory in standard quantum field theory.

The manuscript is organized as follows: In section 2 we review the properties of the general scattering matrix and in particular the Sine-Gordon S-matrix. In section 3 we motivate the general properties of the generalized form factors, for simplicity initially only for the bosonic case, which we thereafter extend to the general situation involving also fermions. In section 4 we briefly explain the "off-shell" Bethe ansatz and state theorem 4.1, the main result of the manuscript. We present a general formula[2] (based on the "off-shell" Bethe ansatz) for form factors with an odd number of solitons or anti-solitons. Furthermore, we provide an explicit analysis of several two- and three particle form factors and carry out various consistency checks relating different form factors to each other. In section 5 we compare our solution for a three particle form factor against perturbative perturbation theory. Our conclusions are stated in section 6. In appendix A we provide the proofs of the properties of the generalized form factors. In appendix B we proof theorem 4.1 and appendix C serves as a depot for several useful formulae employed in the working.

2 The S-matrix

2.1 General Properties

In this section we briefly review some of the well known facts on the general properties of the scattering matrices. The Fock space is spanned by the in- or out-states of the particles

$$|\, p_1, \ldots, p_n \,\rangle^{in/out}_{\alpha_1 \ldots \alpha_n} = a^{in/out\,\dagger}_{\alpha_1}(p_1) \cdots a^{in/out\,\dagger}_{\alpha_n}(p_n)\,|\,0\,\rangle \tag{2.1}$$

where the $a^\dagger$'s are creation operators. The p's denote the momenta and the α's the internal quantum numbers of the particles, such as the particle type etc. We choose the normalization

$${}_{\alpha'}\langle\, p' \,|\, p \,\rangle_\alpha = \delta_{\alpha'\alpha}\, 2\omega\, 2\pi\, \delta(p'-p) = \delta_{\alpha'\alpha}\, 4\pi\, \delta(\theta'-\theta) \tag{2.2}$$

where the rapidity θ is related to the momentum by $p = m \sinh\theta$ and $\omega = \sqrt{m^2+p^2}$.

In an integrable quantum field theory in 1+1-dimensions there exists an infinite set of conservation laws. Therefore in a scattering process the sets of incoming and outgoing

[2]Our formula is similar to an analogous one of Smirnov [18] for even number of particles. This should be a starting point for a comparison of both formulae.

momenta are equal

$$\{p_1,\ldots,p_n\} = \{p'_1,\ldots,p'_{n'}\}\,.$$

The n-particle S-matrix is defined by

$$\begin{aligned} |\,p_1,\ldots,p_n\,\rangle^{in}_{\alpha_1\ldots\alpha_n} &= |\,p_1,\ldots,p_n\,\rangle^{out}_{\alpha'_1\ldots\alpha'_n} S^{(n)}{}^{\alpha'_n\ldots\alpha'_1}_{\alpha_1\ldots\alpha_n}(p_1,\ldots,p_n) \\ &= |\,p_n,\ldots,p_1\,\rangle^{out}_{\alpha'_n\ldots\alpha'_1} (\sigma^{(n)} S^{(n)})^{\alpha'_n\ldots\alpha'_1}_{\alpha_1\ldots\alpha_n}(p_1,\ldots,p_n). \end{aligned}$$

The statistics of the particles has been taken into account by the diagonal matrix $\sigma^{(n)}$. It is a product of all two particle matrices σ with entries -1 if both particles are fermions and $+1$ otherwise (see [68]). As a consequence of integrability , i.e. the existence of an infinite number of conserved quantities, the n-particle S-matrix factorizes into $n(n-1)/2$ two-particle ones

$$\sigma^{(n)} S^{(n)}(p_1,\ldots,p_n) = \prod_{i<j} \sigma S^{(2)}(p_i,p_j)\,,$$

where the product on the right hand side has to be taken in a specific order (see e.g. [7]). For this reason it is sufficient to investigate the properties of the two-particle scattering matrix. As is usual in integrable quantum field theories in 1+1-dimensions it is most convenient to regard the two-particle S-matrix as a function of the rapidity differences $\theta = |\theta_i - \theta_j|$ rather than as a function of the Mandelstam variables $s_{ij} = (p_i + p_j)^2$. In order to establish the analytic properties of the two-particle S-matrix one may employ the relations $s_{ij} = m_i^2 + m_j^2 + 2m_i m_j \cosh\theta_{ij}$, $t_{ij} = (p_i - p_j)^2 = 2m_i^2 + 2m_j^2 - s_{ij}$. Considering the scattering matrix as a function in the complex s_{ij}-plane, there will be two branch cuts present, the s-channel one for $s_{ij} > (m_i + m_j)^2$ and the t-channel one for $s_{ij} < (m_i - m_j)^2$. In figure 1 the physical s-channel and t-channel regions in the complex (a) $s-$ and (b) θ–plains are labeled by I and II, respectively. The crossing transition is depicted by an arrow. This and the transitions corresponding to the exchange of in– and out–going waves are given by:

$$\begin{array}{rlcl} I \leftrightarrow II: & s_{ij} + i\epsilon \leftrightarrow t_{ij} - i\epsilon & \Leftrightarrow & \theta \leftrightarrow i\pi - \theta \\ I \leftrightarrow III: & s_{ij} + i\epsilon \leftrightarrow s_{ij} - i\epsilon & \Leftrightarrow & \theta \leftrightarrow -\theta \\ II \leftrightarrow VI: & t_{ij} - i\epsilon \leftrightarrow t_{ij} + i\epsilon & \Leftrightarrow & i\pi - \theta \leftrightarrow i\pi + \theta \end{array}$$

(It will be important in the following to notice that the t-channel cut (II-IV) is not present for form factors as a function defined in the complex s_{ij}-plane.)

Let V be a finite dimensional vector space, whose basis vectors label all types of particles of the model. Then one considers the S-Matrix as an intertwining operator acting on the tensor product of two of these spaces

$$S_{12}(\theta) \,:\, V_1 \otimes V_2 \to V_2 \otimes V_1\,.$$

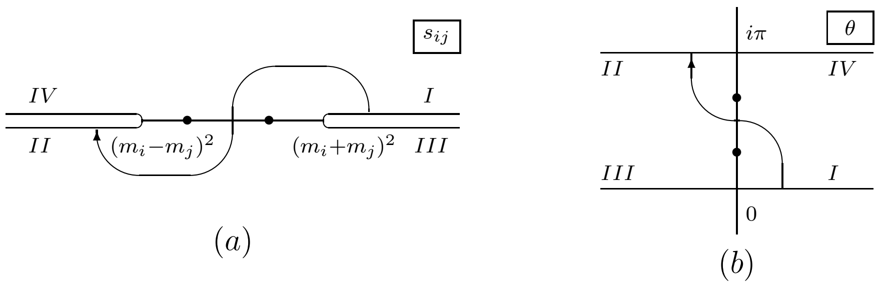


Figure 1: *The analyticity domains in the complex planes of* (a) *the Mandelstam variable* $s_{ij} = (p_i + p_j)^2$ *and* (b) *the rapidity difference variable* $\theta = |\theta_1 - \theta_2|$. *The physical regimes in the s- and t-channels are denoted by I and II, respectively. The crossing transition from the s- to the t-channel is indicated by the arrow. As explained in the main text, the interchange of* in *and* out *means transition from I to III for the s-channel, and II to IV for the t-channel. The dots denote the possible positions of poles corresponding to one particle intermediate states.*

The unitarity of the S-matrix reads

$$\sum_{\alpha'\beta'} \left(S^{\alpha''\beta''}_{\beta'\alpha'}(\theta) \right)^* S^{\beta'\alpha'}_{\alpha\beta}(\theta) = \delta_{\alpha''\alpha}\delta_{\beta''\beta} \quad \text{or} \quad S_{21}(-\theta)\, S_{12}(\theta) = 1 \tag{2.3}$$

since by analytic continuation from positive to negative variable one has $S^\dagger_{12}(\theta) = S_{21}(-\theta)$. The crossing relations are

$$S^{\delta\gamma}_{\alpha\beta}(\theta) = S^{\gamma\bar{\beta}}_{\bar{\delta}\alpha}(i\pi - \theta) = S^{\bar{\alpha}\delta}_{\beta\bar{\gamma}}(i\pi - \theta) \tag{2.4}$$

where the bar refers to the anti-particles. The Yang-Baxter equation which follows from the higher conservation laws is

$$(\sigma S)_{12}(|\theta_{12}|)(\sigma S)_{13}(|\theta_{13}|)(\sigma S)_{23}(|\theta_{23}|) = (\sigma S)_{23}(|\theta_{23}|)(\sigma S)_{13}(|\theta_{13}|)(\sigma S)_{12}(|\theta_{12}|) \tag{2.5}$$

where $\theta_{ij} = \theta_i - \theta_j$. When there are no transitions of the sort that two bosons change into two fermions, the signs given by the statistics cancel.

As usual we use here and in the following the notation for a vector with components $v^{\alpha_1 \ldots \alpha_n}$ and a matrix with elements $A^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_n}$ acting on these vector

$$v^{1\ldots n} \in V_{1\ldots n} = V_1 \otimes \cdots \otimes V_n \ , \quad A_{1\ldots n} : \ V_{1\ldots n} \to V_{1\ldots n} \tag{2.6}$$

where all vector spaces V_i are isomorphic to V and whose basis vectors label all kinds of particles. An S-matrix as S_{ij} acts nontrivial only on the factors $V_i \otimes V_j$ and in addition exchanges these factors. If we want to express the fact that a particle belongs to a multiplet of a specific type of particles, we also write $v^a \in V_a, v^b \in V_b$ etc. and consider $V = \bigoplus_a V_a$

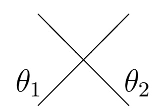
as the direct sum of all these spaces. Usually these spaces V_a are the representation spaces of a symmetry group or quantum group of the model.

The physical S-matrix in the formulas above is given for positive values of the rapidity parameter θ. For later convenience we will also consider an auxiliary matrix $\dot{S}$ regarded as a function depending on the individual rapidities of both particles $\theta_1,\ \theta_2$ or $\theta_{12} = \theta_1 - \theta_2$

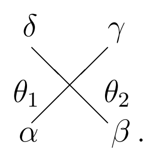
$$\dot{S}_{12}(\theta_1, \theta_2) = \dot{S}_{12}(\theta_1 - \theta_2) = \begin{cases} (\sigma S)_{12}(|\theta_1 - \theta_2|) & \text{for } \theta_1 > \theta_2 \\ (S\sigma)^{-1}_{21}(|\theta_1 - \theta_2|) & \text{for } \theta_1 < \theta_2 \end{cases} \tag{2.7}$$

with σ taking into account the statistics of the particles. Up to these statistics factors $\dot{S}$ is obviously the analytic extension of the physical S-matrix S from positive to negative values of θ, due to the unitarity (2.3).

It appears convenient to introduce a graphical representation for several of the amplitudes, which will allow us to develop a more direct graphical intuition for the derivation of several relations. The auxiliary matrix $\dot{S}$ may be depicted as

$\dot{S}_{12}(\theta_1, \theta_2) \quad =$ 

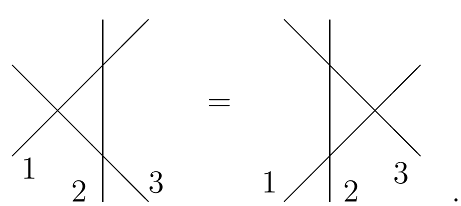
Here and in the following we associate a rapidity variable $\theta_i \in \mathbf{C}$ to each space V_i which is graphically represented by a line labeled by θ_i or simply by i. In terms of the components of the S-matrix we have

$\dot{S}^{\delta\gamma}_{\alpha\beta}(\theta_1, \theta_2) \quad =$ 

In terms of the auxiliary S-matrix the Yang-Baxter equation has the general form

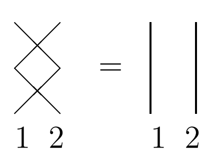
$$\dot{S}_{12}(\theta_{12})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{23}(\theta_{23}) = \dot{S}_{23}(\theta_{23})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{12}(\theta_{12})$$

which grphically simply reads

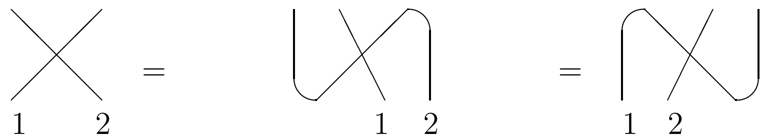


(2.8)

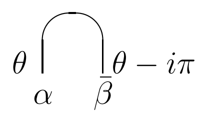
Unitarity and crossing may be written and depicted as

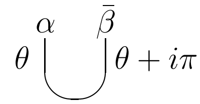
$\dot{S}_{21}(\theta_{21})\dot{S}_{12}(\theta_{12}) = 1 \ :$ 

$$\dot{S}_{12}(\theta_1-\theta_2) \;=\; \mathbf{C}^{2\bar{2}}\,\dot{S}_{\bar{2}1}(\theta_2+i\pi-\theta_1)\,\mathbf{C}_{\bar{2}2} \;=\; \mathbf{C}^{1\bar{1}}\,\dot{S}_{2\bar{1}}(\theta_2-(\theta_1-i\pi)\,\mathbf{C}^{\bar{1}1}:$$



where $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\bar{1}}$ are charge conjugation operators with components $\mathbf{C}^{\alpha\bar{\beta}} = \mathbf{C}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$. We have introduced the graphical rule, that a line changing the "time direction" also interchanges particles and antiparticles and changes the rapidity as $\theta \to \theta \pm i\pi$, as follows

$$\mathbf{C}_{\alpha\bar{\beta}} = \delta_{\alpha\beta} = $$ 

$$,\quad \mathbf{C}^{\alpha\bar{\beta}} = \delta_{\alpha\beta} = $$ 

(2.9)

Similar crossing relations will be used below to investigate the properties of form factors.

2.2 Bound states

Let the two particles labeled by 1 and 2 of mass m_1 and m_2, respectively form a bound state labeled by (12) of mass $m_{(12)}$. If the mass of the bound state is

$$m_{(12)} = \sqrt{m_1^2+m_2^2+2m_1m_2\cosh\theta_{12}^{(12)}} \quad , \quad (\operatorname{Re}\theta_{12}^{(12)} = 0,\; 0 < \operatorname{Im}\theta_{12}^{(12)} < \pi)$$

the corresponding eigenvalue of the S-matrix $S_e(\theta)$ will have a pole at $\theta = \theta_{12}^{(12)}$ such that

$$S_e(\theta) \approx \frac{R_e}{\theta-\theta_{12}^{(12)}} \quad , \quad \text{for} \;\; \theta \to \theta_{12}^{(12)}, \tag{2.10}$$

giving rise to a residue R_e. The eigenvalues are given by the diagonalization of the S-matrix

$$S_{12}(\theta) = \sum_e \varphi_e^{21}\, S_e(\theta)\, \varphi_{12}^e \tag{2.11}$$

where the projections onto the eigenspaces are given by the intertwiners (Clebsch-Gordan cofficients) φ_{12}^e with

$$\sum_e \varphi_e^{12}\varphi_{12}^e = \mathbf{1}_{12} \quad , \quad \varphi_{12}^{e'}\varphi_e^{12} = \delta_{e'e}.$$

Formula (2.10) may also be written as

$$\operatorname*{Res}_{\theta=\theta_{12}^{(12)}} S_{12}(\theta) = \varphi_{(12)}^{21} R_{(12)} \varphi_{12}^{(12)} \tag{2.12}$$

where a matrix product with respect to the space of bound states $V_{(12)}$ is assumed.

Remark: In general an eigenvalue of S may have several poles corresponding to bound states of different masses. On the other hand several eigenvalues may have poles at the same point, which means that there are several types of bound states (12) of the particles 1 and 2 with the same mass. The space of the bound states $V_{(12)}$ is then a direct sum of spaces belonging to these types of particles.

The corresponding fields are related by a normal-product relation like

$$\Psi_{(12)}(x) = \mathcal{N}[\Psi_1\Psi_2](x)\varphi^{12}_{(12)}\,.$$

The bound state S-matrix which describes the scattering of a bound state with another particle is given by [68]

$$\dot{S}_{(12)3}(\theta_{(12)3}) = \sqrt{R_{(12)}}\varphi^{(12)}_{12}\dot{S}_{13}(\theta_{13})\dot{S}_{23}(\theta_{23})\varphi^{12}_{(12)}/\sqrt{R_{(12)}}\,\Big|_{\theta_1-\theta_2=\theta^{(12)}_{12}} \tag{2.13}$$

where the rapidity $\theta_{(12)}$ is fixed by $p_1 + p_2 = p_{(12)}$. Here and below we use the phase convention that $\sqrt{R_e} = i\sqrt{-R_e}$ if $R_e < 0$.

In integrable quantum field theories there exist different types of bound state spectra which may be characterized by the absence or presence of solitons or kinks. Of course, in quantum field theory the bootstrap picture means that all particles are to be considered on the same footing. The names 'solitons', 'kinks' and 'breathers' are motivated by the classical non-linear equations associated with the quantum model. These equations may possess soliton or kink solutions, i.e localized non-singular solutions with a localized energy density. Special solutions consisting of a soliton and an antisoliton are called 'breathers' because of their oscillatory behaviour. In the quantum case we call a particle a soliton if it is a bound state of itself and another particle. Similarly (and more general), we call a particle a kink if it is a bound state of a particle with the same mass and another particle. The mass spectra of integrable quantum field theories characterized by the absence or presence of solitons are given as follows:

i) There are particles labeled by a with mass [7, 69]

$$m_a = m_1\frac{\sin\frac{\pi}{2}\nu a}{\sin\frac{\pi}{2}\nu}\ ,\quad a = 1, 2, \ldots < 2/\nu.$$

This means that two particles of mass m_a and m_b form a bound state of mass $m_{c=a+b}$. The corresponding poles of the two particle S-matrix element and the rapidities in the bound state formula (2.13) are given by

$$\theta^c_{ab} = i\tfrac{\pi}{2}\nu(a+b)\ ,\quad \theta_a = \theta_c + i\tfrac{\pi}{2}\nu b\ ,\quad \theta_b = \theta_c - i\tfrac{\pi}{2}\nu a.$$

The chiral $SU(N)$-Gross-Neveu model [72], the $Z(N)$ invariant Ising models [73] or the $SU(N)$-affine Toda field theories are examples for the above spectrum with

$\nu = 2/N$. In general the mass spectrum is more involved, for instance for affine Toda field theories (with real coupling constant) related to simply laced algebras the masses constitute the entries of the Perron-Frobenius eigenvector of the Cartan matrix [70, 71] and for theories related to non-simply laced algebras they do not even renormalise uniformly [74].

ii) If there exist kinks (solitons) of mass M labeled by A then there are three types of bound states:

   a) Particles (breathers) labeled by a are kink-antikink bound states with

$$m_a = 2M \sin \tfrac{\pi}{2}\nu a \ , \quad a = 1, 2, \ldots < 1/\nu. \tag{2.14}$$

   Here the corresponding poles of the kink-antikink S-matrix and the rapidities in the bound state formula (2.13) are given by

$$\theta^a_{AB} = i\pi(1 - a\nu) \ \ , \ \ \theta_A = \theta_c + \frac{1}{2}\theta^a_{AB} \ \ , \quad \theta_B = \theta_c - \frac{1}{2}\theta^a_{AB}.$$

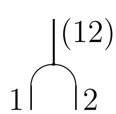
   b) The kink B may be considered as a bound state of a particle a and a kink A such that the pole of the (a-A)-S-matrix and the rapidities in the bound state formula (2.13) are

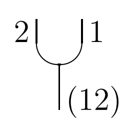
$$\theta^B_{aA} = \frac{i\pi}{2}(1 + a\nu) \ \ , \ \ \theta_a = \theta_B + \frac{i\pi}{2}(1 - a\nu) \ \ , \quad \theta_A = \theta_B - i\pi a\nu.$$

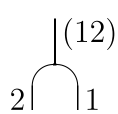
   c) In addition as in i) two particles of mass m_a and m_b form a bound state of mass $m_{c=a+b}$, however, here $a < 1/\nu$.

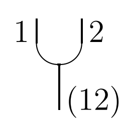
Examples for the latter case are the sine-Gordon (SU(2)-affine Toda theory) alias the massive Thirring model with $\nu = \beta^2/(8\pi - \beta^2) = \pi/(\pi + 2g)$ and the $O(2N)$-Gross-Neveu model with $\nu = 1/(N-1)$. Also in this case the mass spectrum is in general more complicated, for example all affine Toda field theories with purely imaginary coupling fall into this category [75].

The bound state formulae above may be depicted as follows: For $\theta_{12} = \theta_1 - \theta_2 = \theta^{(12)}_{12}$ with Im $\theta_{12} > 0$ we introduce

$\sqrt{R_{(12)}}\,\varphi^{(12)}_{12} \ =$ 

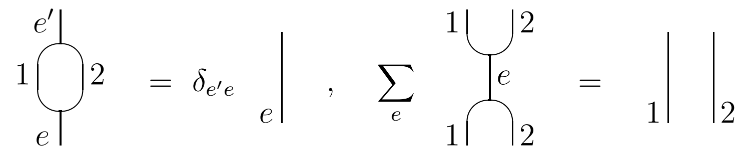
, $\quad (\sigma\varphi)^{21}_{(12)}\sqrt{R_{(12)}} \ =$ 

$1/\sqrt{R_{(12)}}\,(\varphi\sigma)^{(12)}_{21} \ =$ 

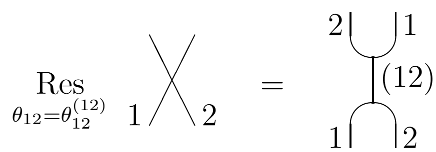
, $\quad \varphi^{12}_{(12)}\,/\sqrt{R_{(12)}} \ =$ 

.

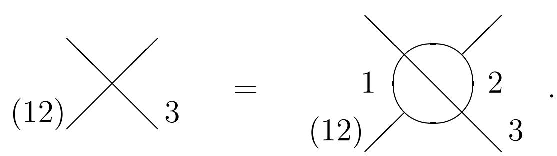
Then we have the relations



where formally we have put $R_e/R_e = 1$ even if $R_e = 0$ in case that e does not correspond to a bound state. The sum in the last formula is over all eigenspaces $V_e \subset V_1 \otimes V_2$ of the S-matrix. If we would sum only over those $e = (12)$ which correspond to bound states, we would get the projector onto the subspace of bound states in $V_1 \otimes V_2$. Moreover formula (2.12) is depicted as

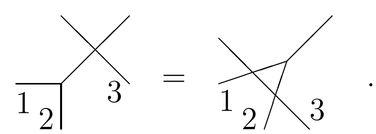


The bound state formula (2.13) may be depicted as



It implies relations of two-particle S-matrices [69] called 'pentagon equations' (also referred to as bootstrap equations) like

$$S_{(12)3}\sqrt{R_{(12)}}\,\varphi_{12}^{(12)} = \sqrt{R_{(12)}}\,\varphi_{12}^{(12)}\;S_{13}S_{23} \quad :$$



2.3 The Sine-Gordon model S-matrix

The Sine-Gordon model alias the massive Thirring model is defined by the Lagrangians

$$\mathcal{L}^{SG} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\alpha}{\beta^2}(\cos\beta\phi - 1),$$

$$\mathcal{L}^{MTM} = \bar\psi(i\gamma\partial - M)\psi - \frac{1}{2}g(\bar\psi\gamma^\mu\psi)^2,$$

respectively.

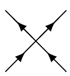
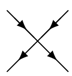
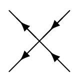
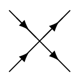
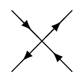
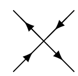
The Fermi field ψ correspond to the soliton and antisoliton and the bose field ϕ to the lowest 'breather' which is the lowest soliton antisoliton bound state. The precise relation between the related coupling constants was found by Coleman [76] within the framework of perturbation theory

$$\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{\pi}{\pi + 2g}$$

where the parameter ν is introduced for later convenience. The two-particle S-matrix is

$$S(\theta,\nu) = \begin{pmatrix} a & & & & & & \\ & b & c & & & & \\ & c & b & & & & \\ & & & a & & & \\ & & & & S_{sb} & & \\ & & & & & S_{bb} & \\ & & & & & & \ddots \end{pmatrix} \tag{2.15}$$

where the soliton-soliton amplitude $a(\theta)$ and the soliton-antisoliton forward and backward amplitudes $b(\theta)$ and $c(\theta)$

$a =$  $=$  , $b =$  $=$  , $c =$  $=$ 

are given by [8]

$$\begin{aligned} b(\theta) &= \frac{\sinh \theta/\nu}{\sinh(i\pi - \theta)/\nu}\, a(\theta)\,, \qquad c(\theta) = \frac{\sinh i\pi/\nu}{\sinh(i\pi - \theta)/\nu}\, a(\theta)\,, \\ a(\theta) &= \exp \int_0^\infty \frac{dt}{t}\, \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \, \cosh \frac{1}{2} t}\, \sinh t \frac{\theta}{i\pi}\,. \end{aligned} \tag{2.16}$$

These amplitudes fulfill 'crossing'

$$a(i\pi - \theta) = b(\theta)\ , \quad c(i\pi - \theta) = c(\theta) \tag{2.17}$$

and unitarity

$$a(-\theta)a(\theta) = 1\ , \quad b(-\theta)b(\theta) + c(-\theta)c(\theta) = 1. \tag{2.18}$$

The intertwiners φ_{ab}^{e} of section 2.1 are given by the non-vanishing components

$$\varphi_{ss}^{0} = \varphi_{\bar s \bar s}^{\bar 0} = 1\ , \quad \varphi_{s\bar s}^{\pm} = 1/\sqrt{2}\ , \quad \varphi_{\bar s s}^{\pm} = \pm 1/\sqrt{2} \tag{2.19}$$

and the corresponding S-matrix eigenvalues are

$$S_0 = S_{\bar 0} = a\ , \quad S_\pm = b \pm c\,. \tag{2.20}$$

The amplitudes $S_0 = S_{\bar 0}$ have no poles corresponding to bound states. The amplitudes $S_\pm(\theta)$ have poles at $\theta = i\pi(1 - k\nu)$ for even/odd $k < 1/\nu$ corresponding to the k-th breather as soliton-antisoliton bound states.

As examples of soliton-breather and breather-breather amplitudes those for the lowest breather are [7]

$$
\begin{aligned}
S_{sb}(\theta) &= \frac{\sinh\theta + i\sin\frac{1}{2}\pi(1+\nu)}{\sinh\theta - i\sin\frac{1}{2}\pi(1+\nu)} = -\exp\int_0^\infty \frac{dt}{t}\, 2\frac{\cosh\frac{1}{2}\nu t}{\cosh\frac{1}{2}t}\,\sinh t\frac{\theta}{i\pi}\,, \\
S_{bb}(\theta) &= \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu} = -\exp\int_0^\infty \frac{dt}{t}\, 2\frac{\cosh(\frac{1}{2}-\nu)t}{\cosh\frac{1}{2}t}\,\sinh t\frac{\theta}{i\pi}\,.
\end{aligned}
\tag{2.21}
$$

The S-matrix element S_{bb} has been dicussed before in [77]. The pole of $S_{sb}(\theta)$ at $\theta = i\pi(1+\nu)/2$ belongs to the soliton as a soliton-breather bound state and the pole of $S_{bb}(\theta)$ at $\theta = i\pi\nu$ to the second breather b_2 as a breather-breather bound state. The intertwiners φ_{ab}^e of section 2.1 are given by the non-vanishing components

$$
\varphi_{sb}^s = \varphi_{bs}^s = \varphi_{bb}^{b_2} = 1. \tag{2.22}
$$

The formulae involving higher breather may be found in [7], e.g.

$$
S_{sb_k}(\theta) = (-1)^k \exp\int_0^\infty \frac{dt}{t}\, 2\frac{\cosh\frac{1}{2}\nu t\sinh\frac{1}{2}\nu k t}{\cosh\frac{1}{2}t\sinh\frac{1}{2}\nu t}\,\sinh t\frac{\theta}{i\pi} \tag{2.23}
$$

for $k < l$

$$
S_{b_k b_l}(\theta) = \exp\int_0^\infty \frac{dt}{t}\, 4\frac{\cosh\frac{1}{2}\nu t\sinh\frac{1}{2}\nu k t\cosh\frac{1}{2}(1-\nu l)t}{\cosh\frac{1}{2}t\sinh\frac{1}{2}\nu t}\,\sinh t\frac{\theta}{i\pi} \tag{2.24}
$$

and

$$
S_{b_k b_k}(\theta) = -\exp\int_0^\infty \frac{dt}{t}\, 2\frac{\cosh\frac{1}{2}\nu t\sinh\frac{1}{2}(2k\nu-1)kt + \sinh\frac{1}{2}(1-\nu)t}{\cosh\frac{1}{2}t\sinh\frac{1}{2}\nu t}\,\sinh t\frac{\theta}{i\pi}\,. \tag{2.25}
$$

3 Properties of generalized form factors

We investigate the properties of generalized form factors, in particular for integrable quantum field theories in 1+1 dimensions. Some formulae, originally proposed in [13], are recalled and the physical arguments on how to derive them are provided in appendix A. All arguments are solely based on the validity LSZ reduction formalism [57] (see also [58]) and the additional assumption of “maximal analyticity” which means, roughly speaking, that the S-matrix and the form factors are analytic functions everywhere except at those points where they posses singularities due to physical intermediate states. In other words the entire pole structure is of physical origin and in the following we investigate it employing the arguments of [13, 14, 69] (see also [17, 18, 28]).

3.1 Form factors in momentum space and rapidity space

For simplicity we first consider the case of bosonic charged particles. The extension to the general situation will be provided below. The corresponding Fock space is spanned by the in- or out-states of particles and anti-particles given by (2.1) and (2.2). In addition to the notation of vectors and matrices of (2.6) we denote co-vectors by

$$v_{1\ldots n} \in V^{\dagger}_{1\ldots n} \tag{3.1}$$

with components $v_{\underline{\alpha}} = v_{\alpha_1\ldots\alpha_n}$.

Let now $\mathcal{O}(x)$ be a local scalar operator, the generalized form factors are defined as the co-vector valued functions given by

$$\langle\, 0\,|\,\mathcal{O}(x)\,|\,p_1,\ldots,p_n\,\rangle^{in}_{\alpha_1\ldots\alpha_n} = e^{-ix(p_1+\ldots+p_n)}\,F^{\mathcal{O}}_{\underline{\alpha}}\Big((s_{ij}+i\epsilon)_{(1\le i<j\le n)}\Big) \tag{3.2}$$

where $\underline{\alpha} = \{\alpha_1,\ldots,\alpha_n\}$ and where $s_{ij} = (p_i+p_j)^2$ is one of the Mandelstam variables, as in the previous section. There may also be anti-particles in the state. As is well known these functions are boundary values of analytic functions as indicated by the ϵ-prescription. We assume that the domain of analyticity is much larger than could be proven by means of general principles. Similar as for the scattering matrix we assume in addition at this point "maximal analyticity" meaning that there should be no redundant poles, but all singularities should be of physical origin as particle states etc. Since the x-dependence of the form factors is trivial, in the sense that we may always carry out a translation as in eq. (3.2), we consider in the following the operator always at the origin, i.e. $\mathcal{O} = \mathcal{O}(0)$.

Under the assumption that F is an analytic function, an interchange in eq. (3.2) of the *in* and *out* states leads to the replacement of $s+i\epsilon$ by $s-i\epsilon$. This means in particular that

$$\langle\, 0\,|\,\mathcal{O}\,|\,p_1,\ldots,p_n\,\rangle^{out}_{\alpha_1\ldots\alpha_n} = F^{\mathcal{O}}_{\underline{\alpha}}\Big((s_{ij}-i\epsilon)_{(1\le i<j\le n)}\Big). \tag{3.3}$$

The crossing property for the connected part of the matrix element yields

$$\,{}^{\;\;\;out}_{\alpha_1\ldots\alpha_m}\langle\, p_1,\ldots,p_m\,|\,\mathcal{O}\,|\,p_{m+1},\ldots,p_n\,\rangle^{in\ \text{conn.}}_{\alpha_{m+1}\ldots\alpha_n} = F^{\mathcal{O}}_{\underline{\alpha}}(s_{ij}+i\epsilon, t_{rs}-i\epsilon, s_{kl}+i\epsilon) \tag{3.4}$$

where $1\le i<j\le m,\ 1\le r\le m<s\le n,\ m\le k<l\le n$ and $t_{rs} = (p_r-p_s)^2$ is another Mandelstam variable. See appendix A for a proper derivation of this claim.

The most basic properties of the form factors are usually refered to as Watson's equations [78], which have been already known in the fifties. It is instructive at this point to discuss them first for the case $n=2$. Using the completeness of the *out*-states we have

$$F^{\mathcal{O}}_{\alpha_1\alpha_2}(s_{12}+i\epsilon) = \langle\, 0\,|\,\mathcal{O}\,|\,p_1,p_2\,\rangle^{in}_{\alpha_1\alpha_2} = \sum_{out}\langle\, 0\,|\,\mathcal{O}\,|\,out\,\rangle\langle\, out\,|\,p_1,p_2\,\rangle^{in}_{\alpha_1\alpha_2}. \tag{3.5}$$

For $4m^2 \le s_{12} <$ 'lowest inelastic threshold' only the two particle S-matrix contributes

$$F^{\mathcal{O}}_{\alpha_1\alpha_2}(s_{12}+i\epsilon) = F^{\mathcal{O}}_{\alpha'_1\alpha'_2}(s_{12}-i\epsilon)\, S^{\alpha'_2\alpha'_1}_{\alpha_1\alpha_2}(s_{12}) \tag{3.6}$$

and analogously starting with ${}_{\alpha_1}\langle\, p_1 \,|\, \mathcal{O} \,|\, p_2 \,\rangle_{\alpha_2}$ in (3.5) we obtain

$$F^{\mathcal{O}}_{\alpha_1\alpha_2}(t_{12}-i\epsilon) = F^{\mathcal{O}}_{\alpha_1\alpha_2}(t_{12}+i\epsilon)\,, \tag{3.7}$$

where the fact has been used that the one-particle S-matrix is always trivial. In integrable theories there are no inelastic transitions, therefore eq. (3.6) holds for all $s \ge 4m^2$. The generalized Watson's equations for $1 \le m \le n$ read as (see [13])

$$F^{\mathcal{O}}_{\underline{\alpha}}(s_{ij}+i\epsilon, t_{rs}-i\epsilon, s_{kl}+i\epsilon) = S^{\alpha_m\ldots\alpha_1}_{\alpha'_1\ldots\alpha'_m}(s_{ij})\, F^{\mathcal{O}}_{\underline{\alpha}'}(s_{ij}-i\epsilon, t_{rs}+i\epsilon, s_{kl}-i\epsilon)\, S^{\alpha'_n\ldots\alpha'_{m+1}}_{\alpha_{m+1}\ldots\alpha_n}(s_{lk}). \tag{3.8}$$

For a diagonal S-matrix these equations have been discussed before in [79].

The generalized form factors also contain singularities [13, 14, 9] which are determined by the one-particle states in all sub-channels $(\alpha_i, \ldots, \alpha_j) \subset (\alpha_1, \ldots, \alpha_n)$ (see figure 2). Poles occur if the square of the total momentum in the sub-channel equals the one-particle

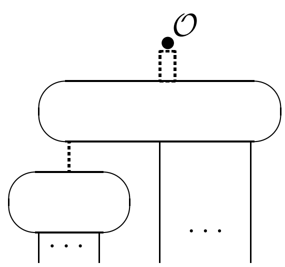


Figure 2: *A singular contribution to the n-particle form factor diagram corresponding to a sub-channel. The dashed lines belong to off-shell lines.*

mass squared. In particular there are poles, if for instance particle 1 is the anti-particle of particle 2 and particle 1 is crossed to the out-state together with $p_2 \to p_1$, which means $(p_2 + p_3 - p_1)^2 \to m_3^2$. Alternatively, if particle 3 is a bound state of particle 1 and 2, in which case $(p_2 + p_1)^2 \to m_3^2$. The residues of the form factors at these poles are related to form factors with fewer legs, as indicated in figure 2. We will discuss these facts later in detail.

Similarly as for the S-matrix we may also write the form factors (3.4) as co-vector valued analytic functions of the rapidity differences $\theta_{ij} = \theta_i - \theta_j$

$$\begin{aligned} F^{\mathcal{O}}_{\underline{\alpha}}(s_{ij}+i\epsilon, t_{rs}-i\epsilon, s_{kl}+i\epsilon) &= F^{\mathcal{O}}_{\underline{\alpha}}(|\theta_{ij}|, i\pi - |\theta_{rs}|, |\theta_{kl}|) \\ F^{\mathcal{O}}_{\underline{\alpha}}(s_{ij}-i\epsilon, t_{rs}+i\epsilon, s_{kl}-i\epsilon) &= F^{\mathcal{O}}_{\underline{\alpha}}(-|\theta_{ij}|, i\pi + |\theta_{rs}|, -|\theta_{kl}|) \end{aligned}$$

The domains of analyticity and the physical regimes in the complex planes of the Mandelstam variables and the rapidity difference variables are depicted in figure 1. However, now the branch cut between between region II and IV is absent (c.f. eqs. (3.7 and (3.8)).

3.2 The auxiliary form factor function

Furthermore, it is convenient to introduce a new co-vector valued auxiliary function $f_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ which is considered as an analytic function of the individual rapidities of the particles, instead of analytic functions of all rapidity differences (see also [17]). It coincides with the generalized form factor for a particular order of the rapidities

$$f_{\underline{\alpha}}^{\mathcal{O}}(\theta_1,\dots,\theta_n) = F_{\underline{\alpha}}^{\mathcal{O}}(|\theta_{ij}|) = \langle\, 0 \,|\, \mathcal{O} \,|\, p_1,\dots,p_n \,\rangle_{\underline{\alpha}}^{in}\,, \quad \text{for} \;\; \theta_1 > \ldots > \theta_n. \tag{3.9}$$

For all other arrangements of the rapidities the functions $f_{\underline{\alpha}}^{\mathcal{O}}(\underline{\theta})$ are given by analytic continuation. The domains of analyticity, the physical regimes and the transitions to the crossed regions in the complex planes of θ_i and θ_j for $\theta_i > \theta_j$ are depicted in figure 3.

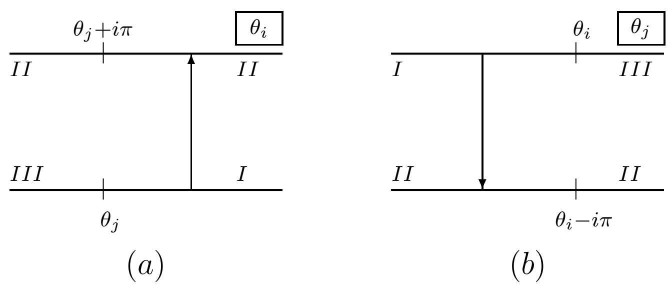


Figure 3: *The physical regimes in the complex planes of the rapidity variables* (a) θ_i *and* (b) θ_j *for* $\theta_i > \theta_k > \theta_j$ $(k = 1,\dots,n)$. *Again the crossing transitions (see appendix B) are indicated by the arrows.*

Now we formulate the main properties of generalized form factors in terms of the auxiliary functions $f_{1\dots n}^{\mathcal{O}}$ under the assumptions of "maximal analyticity".

Properties: The co-vector valued auxiliary function $f_{1\dots n}^{\mathcal{O}}(\underline{\theta})$ is meromorphic in all variables $\theta_1,\dots,\theta_n$ and

(i) fulfills the symmetry property under the permutation of both, the variables θ_i, θ_j and the spaces i, j at the same time

$$f_{\dots ij\dots}^{\mathcal{O}}(\dots,\theta_i,\theta_j,\dots) = f_{\dots ji\dots}^{\mathcal{O}}(\dots,\theta_j,\theta_i,\dots)\, S_{ij}(\theta_i - \theta_j) \tag{3.10}$$

for all possible arrangements of the θ's,

(ii) fulfills the periodicity property under the cyclic permutation of the rapidity variables and spaces

$$f_{1\dots n}^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n,) = f_{2\dots n1}^{\mathcal{O}}(\theta_2,\dots,\theta_n,\theta_1 - 2\pi i), \tag{3.11}$$

(iii) and has poles determined by one-particle states in each sub-channel (see figure 2). In particular the function $f^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} f^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n) = 2i\,\mathbf{C}_{12}\, f^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots,\theta_n)\Big(\mathbf{1} - S_{2n}\ldots S_{23}\Big) \tag{3.12}$$

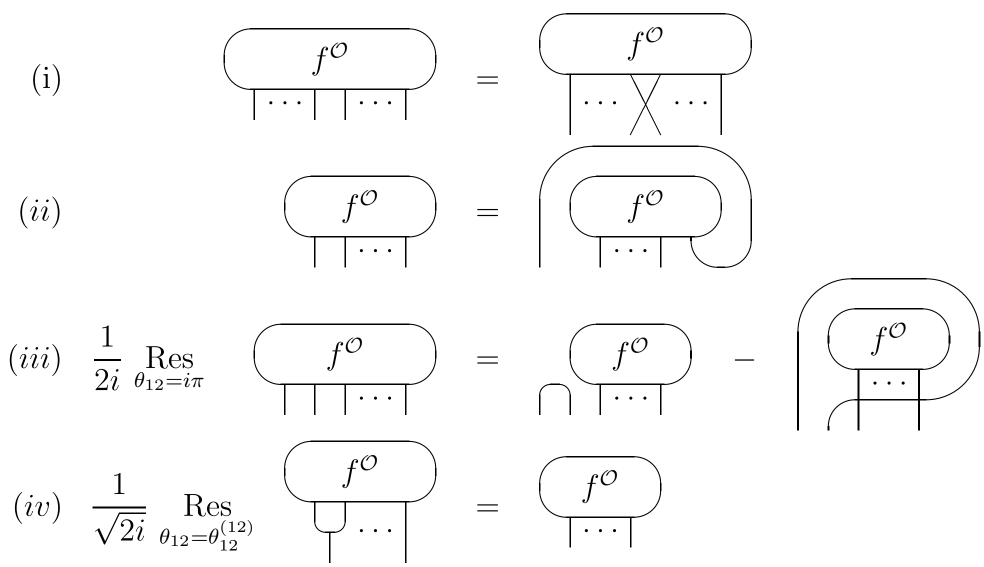
where $\mathbf{C}_{12}$ is the charge conjugation matrix with matrix elements $\mathbf{C}_{\alpha\alpha'} = \delta_{\bar\alpha\alpha'}$.

(iv) If there are also bound states in the model the function $f^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = \theta^{(12)}_{12}$ such that

$$\operatorname*{Res}_{\theta_{12}=\theta^{(12)}_{12}} f^{\mathcal{O}}_{12\ldots n}(\theta_1,\theta_2,\ldots,\theta_n)\,\varphi^{12}_{(12)} = f^{\mathcal{O}}_{(12)\ldots n}(\theta_{(12)},\ldots,\theta_n)\,\sqrt{2iR_{(12)}} \tag{3.13}$$

where the quantities $\varphi^{12}_{(12)}$, $R_{(12)}$ and the values of θ_1, θ_2, $\theta_{(12)}$ and $\theta^{(12)}_{12}$ were discussed in section 2.2.

The property (i) - (iv) may be depicted as

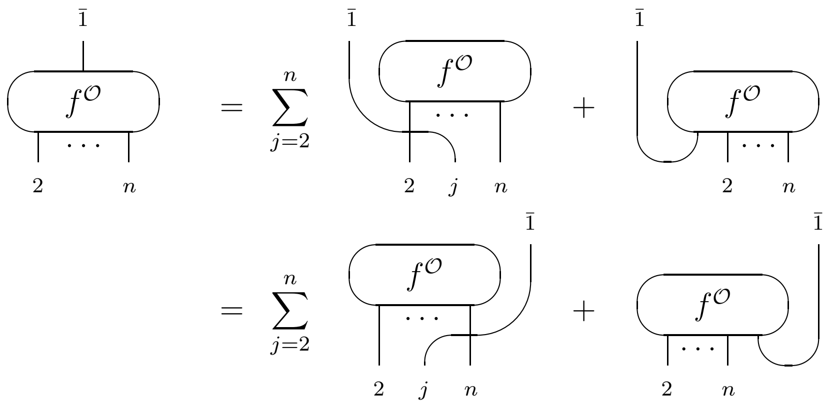


Both properties (ii) and (iii) are consequences of the general **crossing formulae**

$$\begin{aligned}
&{}_{\bar 1}\langle\, p_1 \,|\, \mathcal{O} \,|\, p_2,\ldots,p_n\,\rangle^{in}_{2\ldots n} \\
&= \sum_{j=2}^{n} {}_{\bar 1}\langle\, p_1 \,|\, p_j\,\rangle_j\, f^{\mathcal{O}}_{2\ldots\hat{j}\ldots n}\, S_{2j}\cdots S_{j-1j} + \mathbf{C}^{\bar 1 1}\, f^{\mathcal{O}}_{12\ldots n}(\theta_1 + i\pi_-,\ldots,\theta_n) \\
&= \sum_{j=2}^{n} {}_{\bar 1}\langle\, p_1 \,|\, p_j\,\rangle_j\, f^{\mathcal{O}}_{2\ldots\hat{j}\ldots n}\, S_{jn}\cdots S_{jj+1} + f^{\mathcal{O}}_{2\ldots n1}(\ldots,\theta_n,\theta_1 - i\pi_-)\,\mathbf{C}^{1\bar 1}.
\end{aligned} \tag{3.14}$$

where we introduced the notation $f^{\mathcal{O}}_{2\ldots\hat{j}\ldots n}$, meaning that the space j and the corresponding variable θ_j are missing. In terms of the components, ${}_{\bar 1}\langle\, p_1 \,|\, p_j\,\rangle_j$ means $\delta_{\bar\alpha_1\alpha_j}\, 4\pi\,\delta(\theta_1 - \theta_j)$

and $\delta^{1\bar{1}}$ means $\delta_{\alpha_1\bar{\alpha}_1}$. These are equations for distributions where on the right hand side the second terms are understood as boundary values of analytic functions with $\pi_- = \pi - \epsilon$. The crossing formulae may be depicted as



where we have again used the graphical rule (2.9), which states that a line changing the "time direction" also interchanges particles and anti-particles and changes $\theta \to \theta \pm i\pi$. Taking the analytic part of the crossing relation one obtains property (ii) and considering in addition the part with point like support one gets property (iii). The proofs of the properties (i)-(iv) and equation (3.14) are provided in appendix A.

(v) Naturally, since we are dealing with relativistic quantum field theories we finally have

$$f^{\mathcal{O}}_{1\ldots n}(\theta_1 + u, \ldots, \theta_n + u) = e^{\pm su}\, f^{\mathcal{O}}_{1\ldots n}(\theta_1, \ldots, \theta_n) \tag{3.15}$$

if the local operator transforms under Lorentz transformations as $\mathcal{O} \to e^{\pm su}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

3.3 The general bosonic and fermionic case

For the general case where the states involve also fermions and where $\mathcal{O}(x)$ is a local bosonic or fermionic operator with arbitrary spin we write the matrix elements of $\mathcal{O}(0)$ as

$$\langle\, 0 \,|\, \mathcal{O} \,|\, p_1, \ldots, p_n \,\rangle^{in}_{\alpha_1\ldots\bar{\alpha}_n} = \sum_l \bar{v}(p_i) \cdots \Gamma^{(l)}_{\mu_1\ldots\mu_k} \cdots u(p_j)\, p^{\mu_1}_{i_1} \ldots p^{\mu_k}_{i_k}\, G^{(l),\mathcal{O}}_{\underline{\alpha}}(s_{ij} + i\epsilon) \tag{3.16}$$

where the Γ are matrices in spinor space. For the invariant form factor functions $G^{\mathcal{O}}_{\underline{\alpha}}$, the Watson's equations look quite analogously to those in the bosonic case. However, sometimes it is more convenient to consider the full matrix elements and then we must take into account sign factors due to the fermions. Analogously to eq. (3.9) we introduce the co-vector valued auxiliary function $f^{\mathcal{O}}$ which determines the form factors for a specific order of the rapidities. For the general case the three main properties of the co-vector valued function $f^{\mathcal{O}}$ may be written as:

$$
\begin{array}{llcl}
\text{(i)} & \mathrm{f}^{\mathcal{O}}_{1\ldots\mathrm{ij}\ldots\mathrm{n}}(\theta_1,\ldots,\theta_\mathrm{i},\theta_\mathrm{j},\ldots,\theta_\mathrm{n}) & = & f^{\mathcal{O}}_{1\ldots ji\ldots n}(\theta_1,\ldots,\theta_j,\theta_i,\ldots,\theta_n)\,\dot{S}_{ij} \\
\text{(ii)} & & = & f^{\mathcal{O}}_{2\ldots n1}(\theta_2,\ldots,\theta_n,\theta_1-2i\pi)\,\sigma_{\mathcal{O}1} \\
\text{(iii)} & & \approx & \frac{2i}{\theta_{12}-i\pi}\,\mathbf{C}_{12}\,f^{\mathcal{O}}_{3\ldots n}(\theta_3,\ldots,\theta_n)\Big(\mathbf{1}-S_{2n}\ldots S_{23}\Big)
\end{array}
\tag{3.17}
$$

The bound state formula (iv) is in general true for the invariant part of the form factors. For the case of fermions, spinors have to be taken into account (see the examples below). In the formulae (3.17) the statistics of the operator $\mathcal{O}$ is taken into account by $\sigma_{\mathcal{O}1}=-1$ if both $\mathcal{O}$ and particle 1 are fermionic and $\sigma_{\mathcal{O}1}=1$ otherwise. The statistics of the particles is taken into account by $\dot{S}$ which means that $\dot{S}_{12}=-S_{12}$ if both particles are fermions and $\dot{S}_{12}=S_{12}$ otherwise. Again, both properties (ii) and (iii) are consequences of the **crossing formulae**, which, for the general case of bosons or fermions, reads

$$
\begin{aligned}
&{}_{\bar{1}}\langle\, p_1\,|\,\mathcal{O}\,|\,p_2,\ldots,p_n\,\rangle^{in}_{2\ldots n} \\
&= \sigma_{\mathcal{O}1}\left\{\sum_{j=2}^{n}{}_{\bar{1}}\langle\, p_1\,|\,p_j\,\rangle_j\, f^{\mathcal{O}}_{2\ldots\hat{j}\ldots n}\,\dot{S}_{2j}\cdots\dot{S}_{j-1j}+\mathbf{C}^{\bar{1}1}f^{\mathcal{O}}_{12\ldots n}(\theta_1+i\pi_-,\ldots,\theta_n)\right\} \\
&= \sum_{j=2}^{n}{}_{\bar{1}}\langle\, p_1\,|\,p_j\,\rangle_j\, f^{\mathcal{O}}_{2\ldots\hat{j}\ldots n}\,\dot{S}_{jn}\cdots\dot{S}_{jj+1}+f^{\mathcal{O}}_{2\ldots n1}(\ldots,\theta_n,\theta_1-i\pi_-)\,\mathbf{C}^{1\bar{1}}.
\end{aligned}
\tag{3.18}
$$

replacing eq. (3.14). The proof of these relations are also given in appendix A.

The appearance of $\dot{S}$ is natural in the context of factorizing S-matrices. See for example the general Yang-Baxter relation (2.5) which is essential if transitions as fermion + anti-fermion → boson + anti-boson are possible.

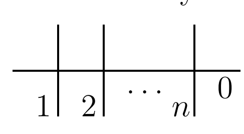
4 Solution for the sine-Gordon alias massive Thirring model Model

We will now provide a constructive and systematic way of how to solve the properties i)-v) for the co-vector valued function f once the scattering matrix is given. To capture the vectorial structure of the form factors we will employ the techniques of the “off-shell” Bethe ansatz [51, 52] which we now explain briefly.

4.1 The general formula

As usual in the context of algebraic Bethe ansatz we define the monodromy matrix

$$T_{1\ldots n,0}(\underline{\theta},\theta_0)=\dot{S}_{10}(\theta_1-\theta_0)\,\dot{S}_{20}(\theta_2-\theta_0)\cdots\dot{S}_{n0}(\theta_n-\theta_0)= \tag{4.1}$$

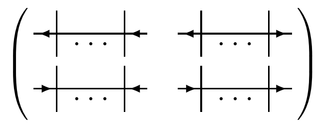


as a matrix acting in the tensor product of the "quantum space" $V_{1\dots n} = V_1 \otimes \cdots \otimes V_n$ and the "auxiliary space" V_0 (all $V_i \cong \mathbf{C}^2$ = soliton-antisoliton space). The Yang-Baxter algebra relations yield

$$T_{1\dots n,a}(\underline{\theta}, \theta_a)\, T_{1\dots n,b}(\underline{\theta}, \theta_b)\, \dot{S}_{ab}(\theta_a - \theta_b) = \dot{S}_{ab}(\theta_a - \theta_b)\, T_{1\dots n,b}(\underline{\theta}, \theta_b)\, T_{1\dots n,a}(\underline{\theta}, \theta_a) \tag{4.2}$$

which in turn implies the basic algebraic properties of the sub-matrices A, B, C, D with respect to the auxiliary space defined by

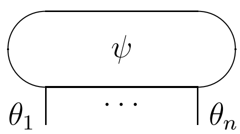
$$T_{1\dots n,0}(\underline{\theta}, \theta) \equiv \begin{pmatrix} A_{1\dots n}(\underline{\theta}, \theta) & B_{1\dots n}(\underline{\theta}, \theta) \\ C_{1\dots n}(\underline{\theta}, \theta) & D_{1\dots n}(\underline{\theta}, \theta) \end{pmatrix} \equiv$$



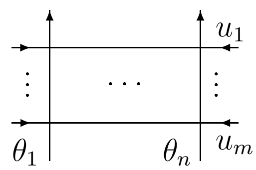
$$. \tag{4.3}$$

A Bethe ansatz co-vector in $V^{\dagger}_{1\dots n}$ is defined by

$$\psi_{1\dots n}(\underline{\theta}, u_1, \dots, u_m) \;=\; \Omega_{1\dots n} C_{1\dots n}(\underline{\theta}, u_1) \cdots C_{1\dots n}(\underline{\theta}, u_m)$$



$$= \tag{4.4}$$



where $\Omega_{1\dots n}$ is the "pseudo-vacuum" co-vector consisting only of particles of highest weight. When the monodromy matrix involves only the scattering matrix of soliton anti-solitons it is given as

$$\Omega_{1\dots n} = \uparrow \otimes \cdots \otimes \uparrow \tag{4.5}$$

consisting only of solitons and fulfilling

$$\begin{aligned} \Omega_{1\dots n}\, B_{1\dots n}(\underline{\theta}, u) &= 0 \\ \Omega_{1\dots n}\, A_{1\dots n}(\underline{\theta}, u) &= \prod_{i=1}^{n} \dot{a}(\theta_i - u)\Omega_{1\dots n} \\ \Omega_{1\dots n}\, D_{1\dots n}(\underline{\theta}, u) &= \prod_{i=1}^{n} \dot{b}(\theta_i - u)\Omega_{1\dots n}\,. \end{aligned} \tag{4.6}$$

Here the eigenvalues of the matrices A and D, i.e. $\dot{a}$ and $\dot{b}$ are related to the amplitudes of the scattering matrix (refer (2.16) via $\dot{a} = -a$ and $\dot{b} = -b$.

In the conventional Bethe ansatz [50], one is usually concerned with the computation of the eigenvalues of the transfer matrix

$$\tau_{1\dots n}(\underline{\theta}, u) = A_{1\dots n}(\underline{\theta}, u) + D_{1\dots n}(\underline{\theta}, u) \tag{4.7}$$

on a Bethe wave vector. Applying the transfer matrix to the co-vector (4.4) one obtains in general an equation of the form

$$\begin{aligned} \psi_{1\dots n}(\underline{\theta}, u_1, \dots, u_m)\, \tau_{1\dots n}(\underline{\theta}, u) \;&=\; \Lambda(u|u_1, \dots |\underline{\theta})\, \psi_{1\dots n}(\underline{\theta}, u_1, \dots, u_m) \\ &- \sum_{j=1}^{m} \Lambda_j(u_1, \dots, u_m|\underline{\theta})\, \psi^j_{1\dots n}(\underline{\theta}|u_1, \dots, u_{j-1}, u, u_{j+1}, \dots, u_m) \end{aligned} \tag{4.8}$$

where $\Lambda(u|u_1,\ldots|\underline{\theta})$ and $\Lambda_j(u_1,\ldots,u_m|\underline{\theta})$ are some complex valued functions. The co-vectors $\psi^j_{1\ldots n}(\underline{\theta}|u_1,\ldots,u_{j-1},u,u_{j+1},\ldots,u_m)$ are not proportional to the Bethe ansatz vectors $\psi_{1\ldots n}(\underline{\theta},u_1,\ldots,u_m)$. Hence in general, that is for an arbitrary set of the spectral parameter, the Bethe ansatz vector is not an eigenvector of the transfer matrix. To achieve this one usually imposes the validity of the Bethe ansatz equations, i.e. $\Lambda_j(u_1,\ldots,u_m|\underline{\theta}) = 0$ $(j = 1,\ldots,m)$ such that the so-called "unwanted terms" vanish and one obtains a genuine eigenvalue equation for the transfer matrix with eigenvalue $\Lambda(u|u_1,\ldots|\underline{\theta})$. In analogy to the one particle situation one may refer to such Bethe vectors as being "on-shell" in contrast to the generic situation (4.8) which is referred to as "off-shell" [51, 52]. In order to construct solutions to the properties (i)-(v) we shall employ the Bethe vector (4.4) in its "off-shell" version.

Let us now consider the auxiliary form factor function given by

$$f^{\mathcal{O}}_{\underline{\alpha}}(\theta_1,\ldots,\theta_n) = \langle\, 0\,|\,\mathcal{O}\,|\,p_1,\ldots,p_n\,\rangle^{in}_{\underline{\alpha}}\ ,\quad \text{for}\ \ \theta_1 > \ldots > \theta_n. \tag{4.9}$$

where the indices $\underline{\alpha}$ refer to solitons and antisolitons.

Theorem 4.1 *The co-vector valued function $f^{\mathcal{O}}_{1\ldots n}(\underline{\theta})$ fulfills the conditions* (i), (ii) *and* (iii) *of section 3 (see eqs. (3.10-3.12)) if it is represented by the following generalized Bethe ansatz [55]*

$$f^{\mathcal{O}}_{1\ldots n}(\underline{\theta}) = N^{\mathcal{O}}_n \int_{\mathcal{C}_{\underline{\theta}}} du_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} du_m\, g(\underline{\theta},\underline{u})\,\Omega_{1\ldots n} C_{1\ldots n}(\underline{\theta},u_1)\cdots C_{1\ldots n}(\underline{\theta},u_m) \tag{4.10}$$

with a normalization constant $N^{\mathcal{O}}_n$ and the scalar function

$$g(\underline{\theta},\underline{u}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \phi(\theta_i - u_j) \prod_{1\le i<j\le m} \tau(u_i-u_j)\, e^{\pm\tilde{s}\left(2\sum u_j - \sum \theta_i\right)} \tag{4.11}$$

where $\tilde{s} = s/q$ and s is the "spin" (c.f. eq. (3.15)) and $q = n-2m$ is the charge of the operator $\mathcal{O}$. The number $\tilde{s}$ is assumed to fulfil $\exp(2\pi i\tilde{s}) = (-1)^n$*. The function $F(\theta)$ (see (4.14)) is a soliton-soliton form factor fulfilling Watson's equations*

$$F(\theta) = -F(-\theta)\,a(\theta) = F(2\pi i - \theta) \tag{4.12}$$

with the soliton-soliton scattering amplitude $a(\theta)$ (see 2.16). The scalar functions $\phi(u)$ and $\tau(u)$ are defined as

$$\phi(u) = \frac{1}{F(u)\,F(u+i\pi)}\quad ,\quad \tau(u) = \frac{1}{\phi(u)\,\phi(-u)}\ . \tag{4.13}$$

The integration contour $\mathcal{C}_{\underline{\theta}}$ consists of several pieces (see figure 4):

a) A line from $-\infty$ to ∞ avoiding all poles such that $\Im\theta_i - \pi - \epsilon < \Im u_j < \Im\theta_i - \pi$.

b) Clock wise oriented circles around all poles (of the $\phi(\theta_i - u_j)$) at $u_j = \theta_i$.

In addition we assume that the number of particles involved, i.e. n, to be odd.

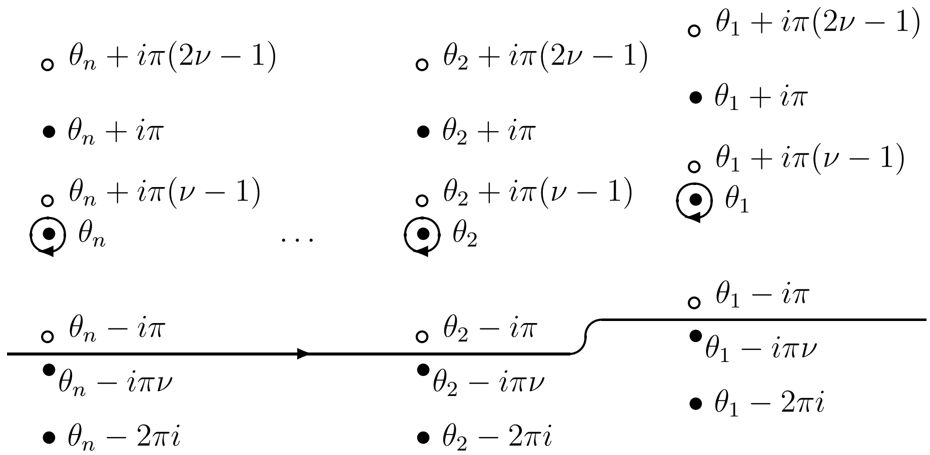


Figure 4: *The integration contour $\mathcal{C}_{\underline{\theta}}$ (for the repulsive case $\nu > 1$). The bullets belong to poles of the integrand resulting from $a(\theta_i - u_j)\,\phi(\theta_i - u_j)$ and the small open circles belong to poles originating from $b(\theta_i - u_j)$ and $c(\theta_i - u_j)$.*

This theorem is proven in appendix B.

Remarks:

- The minus sign in eq. (4.12) is due to the fermionic statistics of the solitons.
- A solution of the Watson's equations (4.12) is

$$F(\theta) = -i \sinh \tfrac{1}{2}\theta \, f_{ss}^{min}(\theta) \tag{4.14}$$

where the 'minimal' soliton-soliton form factor function is given as

$$f_{ss}^{min}(\theta) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \, \cosh \frac{1}{2}t} \frac{1 - \cosh t(1 - \theta/(i\pi))}{2 \sinh t}.$$

The corresponding functions $\phi(u)$ and $\tau(u)$ are (see appendix C)

$$\phi(u) = const. \, \frac{1}{\sinh u} \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{1}{2}(1-\nu)t \left(\cosh t(\frac{1}{2} - u/(i\pi)) - 1 \right)}{\sinh \frac{1}{2}\,\nu t \sinh t}$$

$$\tau(u) = const. \; \sinh u \sinh u/\nu$$

- Using Watson's equations (4.12) for $F(u)$, crossing (2.17) and unitarity (2.18) for the sine-Gordon amplitudes one derives the following identities for the scalar functions $\phi(u)$ and $\tau(u)$

$$\phi(u) = \phi(i\pi - u) = -\frac{1}{b(u)}\,\phi(u - i\pi) = \frac{a(u-2\pi i)}{b(u)}\,\phi(u-2\pi i)\ , \tag{4.15}$$

$$\tau(u) = \tau(-u) = \frac{b(u)}{a(u)}\,\frac{a(2\pi i - u)}{b(2\pi i - u)}\,\tau(u - 2\pi i) \tag{4.16}$$

  where $b(u)$ is the soliton-antisoliton scattering amplitude related to $a(u)$ by crossing $b(u) = a(i\pi - u)$.

- The number of C-operators m depends on the charge $q = n - 2m$ of the operator $\mathcal{O}$, e.g. $m = (n-1)/2$ for the soliton field $\psi(x)$ with charge $q = 1$.

- The integrals in eq. (4.11) converge if $\frac{1}{2}(1 + 1/\nu)q \mp 2\tilde{s} + 2/\nu + 1 > 0$.

- Note that other sine-Gordon form factors can be calculated from the general formula (4.10) using the bound state formula (3.13).

We shall now apply the general formula (4.10) to an explicit example and exploit the fact that the properties (i)-(iv) relate several different form factors to each other. This will permit us to carry out various consistency checks.

4.2 The two particle form factors

We repeat some well known results (see for example [13, 69]). According to equation (3.17), the auxiliary function for the two particle form factor $f^{\mathcal{O}}_{\alpha\beta}(\theta_1, \theta_2)$ has to satisfy

$$f^{\mathcal{O}}_{\alpha\beta}(\theta_1, \theta_2) = \sum_{\alpha'\beta'} f^{\mathcal{O}}_{\beta'\alpha'}(\theta_2, \theta_1)\,\dot{S}^{\beta'\alpha'}_{\alpha\beta}(\theta_{12}) = f^{\mathcal{O}}_{\beta\alpha}(\theta_2, \theta_1 - 2\pi i)\,\sigma_{\mathcal{O}1}.$$

These matrix equations may be solved easily by diagonalization of the S-matrix. If there are only bosons involved we have to solve the scalar "Watson's equations"

$$f^{\mathcal{O}}_e(\theta) = f^{\mathcal{O}}_e(-\theta)\,S_e(\theta) = f^{\mathcal{O}}_e(2\pi i - \theta) \tag{4.17}$$

where $S_e(\theta)$ are the eigenvalues of the S-matrix given by eq. (2.11). In [13] it was shown that the general solution of these equations is of the form

$$f^{\mathcal{O}}_e(\theta) = N^{\mathcal{O}}_e K_e(\theta)\,f^{min}_e(\theta) \tag{4.18}$$

where $N_e^{\mathcal{O}}$ is a normalization factor, $f_e^{min}(\theta)$ is the minimal solution of Watson's equations without any poles or zeroes in the physical strip $0 \leq \mathrm{Im}\,\theta \leq \pi$ and $K_e(\theta)$ is an even periodic function with period $2\pi i$. If the S-matrix eigenvalue is given by

$$S_e(\theta) = \exp \int_0^\infty dt\, f(t) \sinh t\theta/i\pi \tag{4.19}$$

the minimal solution of Watson's equations is given as

$$f_e^{min}(\theta) = \exp \int_0^\infty dt\, f(t)\, \frac{1 - \cosh t(1 - \theta/i\pi)}{2 \sinh t}\,. \tag{4.20}$$

If there are also fermions involved Watson's equations (4.17) hold for the invariant form factors (c.f. (3.16)). For the full matrix elements the representation (4.18) holds with additional factors $\exp(\pm\theta_i/2)$ on the right hand side for all fermions.

The poles of $f_e^{\mathcal{O}}(\theta)$ in the physical strip are determined by the one-particle states in the channel corresponding to the S-matrix eigenvalue. In [13] the minimality assumption was made, meaning that there are only these poles and no zeroes in $0 < \Im\,\theta < \pi$. This implies that

$$K_e(\theta) = \prod_{k=1}^{L} \frac{1}{\sinh \frac{1}{2}(\theta - \theta_k) \sinh \frac{1}{2}(\theta + \theta_k)}\,, \quad (\Re\,\theta_k = 0,\ 0 < \Im\,\theta_k < \pi)\,.$$

For several examples this assumption was checked against perturbation theory.

Examples

We present two-particle form factors for several local operators and several particle states of the sine-Gordon quantum field theory. Some of them were already calculated in [13] (see also [17]). Up to normalizations the problem is solved by eqs. (4.18-4.20) since the sine-Gordon S-matrix (2.15) is diagonal except of the soliton-anti-sector where the eigenvalues are given by eqs. (2.16) and (2.20).

4.2.1 The two-breather form factor

The simplest sine-Gordon form factor is that for a scalar operator $\phi^2(x) = \mathcal{N}\phi^2(x)$ connecting the two-particle lowest breather state to the vacuum

$$f_{bb}^{\phi^2}(\theta_{12}) = \langle\, 0 \,|\, \phi^2 \,|\, p_1, p_2 \,\rangle_{bb}^{in} = N_{bb}^{\phi^2}\, K_{bb}(\theta_{12})\, f_{bb}^{min}(\theta_{12})\,.$$

According to (4.19) and (4.20) the minimal form factor function combined with (2.21) reads

$$f_{bb}^{min}(i\pi x) = -i \sinh \tfrac{1}{2}\theta\, \exp \int_0^\infty \frac{dt}{t}\, 2 \frac{\cosh(\frac{1}{2} - \nu)t}{\cosh \frac{1}{2} t}\, \frac{1 - \cosh t(1 - x)}{2 \sinh t}\,.$$

The "minimality assumption" implies that the 'pole function'

$$K_{bb}(\theta) = \frac{1}{\sinh\frac{1}{2}(\theta - i\pi\nu)\sinh\frac{1}{2}(\theta + i\pi\nu)}$$

only possesses the pole corresponding to the second breather b_2 as a bound state of two lowest breathers b. The normalization constant can be calculated by means of the asymptotic behavior [13]. Weinberg's power counting implies that in the limit of infinite momentum transfer the form factor tends to its free value

$$\langle\, 0 \,|\, \phi^2 \,|\, p_1, p_2 \,\rangle_{bb}^{in} \to \langle\, 0 \,|\, :\phi^2: \,|\, p_1, p_2 \,\rangle_{bb}^{free} = 2\, Z^\phi \quad \text{as} \quad (p_1 + p_2)^2 \to -\infty \;.$$

Here Z^ϕ is the wave-function renormalization constant of the fundamental sine-Gordon field which has been calculated in [13]

$$Z^\phi = (1+\nu)\frac{\frac{\pi}{2}\nu}{\sin\frac{\pi}{2}\nu}\, E(\nu)\;. \tag{4.21}$$

We introduce the function

$$\begin{aligned} E(x) &= \exp\left(-\frac{1}{\pi}\int_0^{\pi x}\frac{t}{\sin t}dt\right) \qquad (4.22)\\ &= \exp\left\{-\frac{1}{\pi}\Big(i\,\mathrm{L}\left(e^{i\pi x}\right) + i\,\mathrm{L}\big(e^{i\pi x}+1\big) - x\ln\left(e^{i\pi x}+1\right) + \frac{i\pi}{12}\Big)\right\} \end{aligned}$$

where $L(x) = \sum\limits_{n=1}^{\infty}\frac{x^n}{n^2} + \frac{1}{2}\ln x\ln(1-x)$ denotes the Rogers dilogarithm [80]. Notice, that this wave function renormalization constant satisfies (compare figure 5) $0 \le Z^\phi \le 1$, which is a general consequence of positivity [56] (see also e.g. [58] p. 204). For the free boson case, that is $\beta = 0,\ \nu = 0$, we have $Z^\phi = 1$. For the free soliton case, i.e. $g = 0,\ \nu = 1$, where the breather decays into soliton-antisoliton pairs, we have $Z^\phi = 0$. Using the asymptotic formula the normalization has been calculated in [13]

$$N_{bb}^{\phi^2} = -2(1+\nu)\frac{\pi}{2}\nu\cot\frac{\pi}{2}\nu.$$

In addition we may now employ this result and compute a further renormalization constant by means of the bound state formula (3.13). With $\varphi_{b_2}^{bb} = 1$ (see eq. (2.22)) we calculate the wave function renormalization constant Z^{ϕ^2} via

$$\operatorname*{Res}_{\theta=\theta_0} f_{bb}^{\phi^2}(\theta)\Big(2i\operatorname*{Res}_{\theta=\theta_0} S_{bb}(\theta)\Big)^{-1/2} = f_{b_2}^{\phi^2} = \langle\, 0 \,|\, \phi^2 \,|\, p_1 + p_2 \,\rangle_{b_2}^{in} = \sqrt{Z^{\phi^2}}$$

where the fusing angle is $\theta_0 = i\pi\nu$. The wave function renormalization constant turns out to be

$$Z^{\phi^2} = \left(Z^\phi\right)^2\frac{\cos\pi\nu}{\cos^2\frac{\pi}{2}\nu}\, E(1-2\nu). \tag{4.23}$$

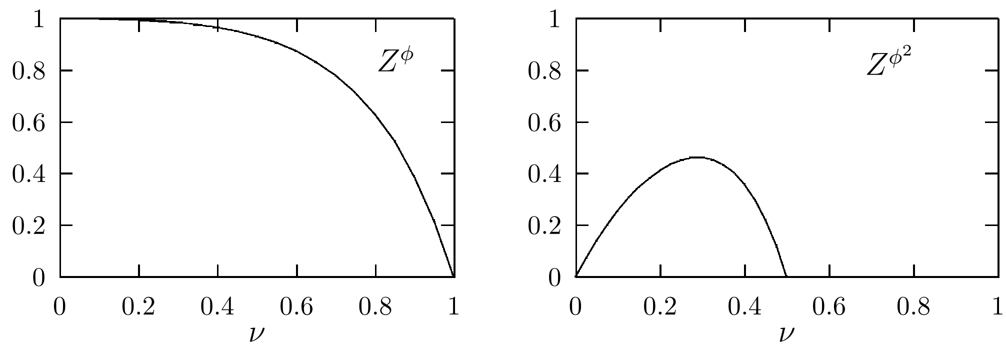


Figure 5: *The wave function renormalization constants Z^{ϕ} and Z^{ϕ^2} as a function of the coupling* $\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{1}{1+2g/\pi}$.

Again we have $0 \leq Z^{\phi^2} \leq 1$ and now Z^{ϕ^2} vanishes at $\nu = 0$ and $\nu = 1/2$, which is to be expected since at these values (compare for instance with the mass formula (2.14)) the second breather decays into the two lowest breathers or a soliton-antisoliton pair, respectively.

4.2.2 The breather-soliton form factor

We now choose $\mathcal{O}(x)$ to be the fermi-field $\psi(x)$ of the massive Thirring model which annihilates the soliton. We assume the breather-soliton form factor related to this field

$$f^{\psi}_{bs}(\theta_1, \theta_2) = \langle\, 0 \,|\, \psi \,|\, p_1, p_2 \,\rangle^{in}_{bs}$$

to acquire the following form

$$f^{\psi}_{bs}(\theta_1, \theta_2) = N^{\psi}_{bs} \Big(1 + N^{\psi}_5 \, \gamma^5 \, \coth \tfrac{1}{2}\theta_{12}\Big) u(p_2) \, K_{bs}(\theta_{12}) \, f^{min}_{bs}(\theta_{12}) \,, \tag{4.24}$$

that is consisting out of a scalar and a pseudoscalar coupling part. For our conventions concerning spinors and the γ-matrices see section 5. Upon employing (4.19) and (4.20), the minimal form factor function reads together with equation (2.21)

$$f^{min}_{bs}(i\pi x) = \sin \tfrac{\pi}{2} x \, \exp \int_0^\infty \frac{dt}{t} \, 2 \frac{\cosh \frac{1}{2}\nu t}{\cosh \frac{1}{2} t} \, \frac{1 - \cosh t(1-x)}{2 \sinh t} \,.$$

Extracting explicitly the expected pole structure the “pole function” reads

$$K_{bs}(\theta) = \frac{1}{\sinh \frac{1}{2}(\theta - i\pi \frac{1+\nu}{2}) \, \sinh \frac{1}{2}(\theta + i\pi \frac{1+\nu}{2})} \,.$$

Once more we may use the bound state formula (3.13) in order to compute the normalization constants, with $\varphi_s^{bs} = 1$ and $\theta_0 = i\frac{\pi}{2}(\nu+1)$ (see eq. (2.22)) to obtain

$$\operatorname*{Res}_{\theta_{12}=\theta_0} f_{bs}^{\psi}(\theta_1,\theta_2)\left(2i\operatorname*{Res}_{\theta_{12}=\theta_0} S_{bs}(\theta_{12})\right)^{-1/2} = f_s^{\psi} = \langle\, 0\,|\,\psi\,|\,p_1+p_2\,\rangle_s^{in} = u(p_1+p_2)\,.$$

This determines the normalization constant to be

$$N_{bs}^{\psi} = \frac{\cos^2\frac{\pi}{2}\nu}{E\left(\frac{1}{2}(1-\nu)\right)}\sqrt{\frac{E(\nu)}{\sin\frac{\pi}{2}\nu}}$$

and the ratio of pseudo-scalar and scalar coupling to be

$$N_5^{\psi} = -\tan\tfrac{\pi}{2}\nu\,\tan\tfrac{\pi}{4}(1+\nu).$$

Note that formula (4.24) may alternatively also be written as

$$\begin{aligned} f_{bs}^{\psi}(\theta_1,\theta_2) &= N_{bs}^{\psi}\cos^2\tfrac{\pi}{4}(1-\nu)\,\frac{i}{\gamma(p_1+p_2)-m}\left(1+\gamma^5\,\coth\tfrac{1}{2}\theta_{12}\right)u(p_2)\,f_{bs}^{min}(\theta_{12}) \\ &= N_{bs}^{\psi}\,\frac{\sin\frac{\pi}{4}(1+\nu)}{\cos^2\frac{\pi}{2}\nu}\left(\frac{e^{-\frac{1}{2}i\pi\nu\gamma^5}}{\sinh\frac{1}{2}(\theta_{12}+\theta_0)}+\frac{e^{\frac{1}{2}i\pi\nu\gamma^5}}{\sinh\frac{1}{2}(\theta_{12}-\theta_0)}\right)\frac{u(p_2)\,f_{bs}^{min}(\theta_{12})}{\sinh\frac{1}{2}\theta_{12}}\,. \end{aligned}$$

4.2.3 Soliton-antisoliton form factors

Having an in-state, which involves a soliton and an antisoliton, we have several options for the operator $\mathcal{O}(x)$ such that the two-partice form factor is non vanishing.

a) Let $\mathcal{O}(x) = j^{\mu}(x) = \mathcal{N}\bar{\psi}\gamma^{\mu}\psi(x)$ the electromagnetic current [13]

$$f_{s\bar{s}}^{j^{\mu}}(\theta_1,\theta_2) = \langle\, 0\,|\,j^{\mu}\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} = \bar{v}(p_2)\gamma^{\mu}u(p_1)\,f_-(\theta_{12}).$$

The function $f_-(\theta)$ fulfills Watsons equations with the negative C-and P-parity S-matrix eigenvalue $S_-(\theta) = -\frac{\cosh\frac{1}{2}(\theta+i\pi)/\nu}{\cosh\frac{1}{2}(\theta-i\pi)/\nu}\,a(\theta)$ (see eqs. (2.16-2.20)). Taking the singularity structure into account we obtain [13], with the help of (4.19), (4.20) and (2.16)

$$f_-(\theta) = \frac{\cosh\frac{1}{2}(i\pi-\theta)}{\cosh\frac{1}{2}(i\pi-\theta)/\nu}\,f_{ss}^{min}(\theta)$$

and

$$f_{ss}^{min}(i\pi x) = \exp\int_0^{\infty}\frac{dt}{t}\,\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t}\,\frac{1-\cosh t(1-x)}{2\sinh t}\,.$$

b) Let $\mathcal{O}(x)=\phi(x)$ the fundamental sine-Gordon field which correspond to the lowest breather [13]

$$f^{\phi}_{s\bar{s}}(\theta_1,\theta_2)=\langle\, 0\,|\,\phi\,|\,p_1,p_2\,\rangle^{in}_{s\bar{s}}=N^{\phi}_{s\bar{s}}\,\bar{v}(\theta_2)u(\theta_1)\,\frac{1}{\sinh\theta_{12}}f_-(\theta_{12})\,.$$

The function $f_-(\theta)$ is the same as in a). Since Coleman's correspondence [76] relates the field ϕ and the current j^μ by

$$\epsilon^{\mu\nu}\partial_\nu\phi=-\frac{2\pi}{\beta}j^\mu \tag{4.25}$$

the normalization constant turns out to be [13]

$$N^{\phi}_{s\bar{s}}=\frac{2\pi i}{\beta M}\,.$$

We may now carry out a consistency check and compute once more the wave function renormalization constant now starting, however, from a different form factor. For this purpose we use once again the bound state formula (3.13) with $\varphi^{s\bar{s}}_b=-\varphi^{\bar{s}s}_b=1/\sqrt{2}$ (see eq. (2.19)) and calculate

$$\operatorname*{Res}_{\theta=\theta_0}f^{\phi}_{s\bar{s}}(\theta)\,\sqrt{2}\Big(2i\operatorname*{Res}_{\theta=\theta_0}S_-(\theta)\Big)^{-1/2}=f^{\phi}_b=\langle\,0\,|\,\phi\,|\,p\,\rangle^{in}_b=\sqrt{Z^\phi}$$

where the fusing angle is $\theta_0=i\pi(1-\nu)$. This computation leads to the value for wave function renormalization constant of the previous subsection (4.21) which has been obtained in [13] by slightly different arguments.

c) Let $\mathcal{O}(x)=\mathcal{N}\phi^2(x)$

$$f^{\phi^2}_{s\bar{s}}(\theta_1,\theta_2)=\langle\,0\,|\,\phi^2\,|\,p_1,p_2\,\rangle^{in}_{s\bar{s}}=N^{\phi^2}_{s\bar{s}}\,\bar{v}(\theta_2)u(\theta_1)\,f_+(\theta_{12})\,.$$

The function $f_+(\theta)$ fulfills Watsons equations with the positive C-and P-parity S-matrix eigenvalue $S_+(\theta)=-\frac{\sinh\frac{1}{2}(\theta+i\pi)/\nu}{\sinh\frac{1}{2}(\theta-i\pi)/\nu}\,a(\theta)$ (see eqs. (2.16-2.20)). With (4.19) and (4.20) we obatin [13] together with the explicit expression for the integral representation of this amplitude of the scattering matrix (2.16)

$$f_+(\theta)=\frac{\sinh\frac{1}{2}(i\pi-\theta)}{\sinh\frac{1}{2}(i\pi-\theta)/\nu}\,f^{min}_{ss}(\theta)\,.$$

We still have to fix the normalization constant $N^{\phi^2}_{s\bar{s}}$, which may be achieved by employing the bound state formula (3.13). Taking $\varphi^{s\bar{s}}_{b_2}=\varphi^{\bar{s}s}_{b_2}=1/\sqrt{2}$ (see eq. (2.19)) we calculate the wave function renormalization constant to be

$$\operatorname*{Res}_{\theta=\theta_0}f^{\phi^2}_{s\bar{s}}(\theta)\,\sqrt{2}\,\Big(2i\operatorname*{Res}_{\theta=\theta_0}S_+(\theta)\Big)^{-1/2}=f^{\phi^2}_{b_2}=\langle\,0\,|\,\phi^2\,|\,p\,\rangle^{in}_{b_2}=\sqrt{Z^{\phi^2}}$$

where $\theta_0 = i\pi(1-2\nu)$. The wave function renormalization constant Z^{ϕ^2} was calculated in the previous subsection (4.23), such that we obtain the normalization constant

$$N_{s\bar{s}}^{\phi^2} = \frac{(1+\nu)\,\pi}{8M\,\sin^3\frac{\pi}{2}\nu}\,.$$

4.3 Three particle form factors

We shall now analyse the general expression proposed in theorem 4.1 for an explicit example. First we recall the three breather form factor which was already calculated in [13] and apply a consistency checks using (iii). Furthermore we calculate the three soliton form factor using the general formula (4.10) and apply some consistency checks using (iii) and (iv).

4.3.1 The three breather form factor

We choose the operator $\mathcal{O}(x)$ to be the fundamental sine-Gordon field $\phi(x)$ which corresponds to the lowest breather and consider the form factor

$$f_{bbb}^{\phi}(\theta_1,\theta_2,\theta_3) = \langle\, 0\,|\,\phi\,|\,p_1,p_2,p_3\,\rangle_{bbb}^{in}.$$

The minimality assumption suggests the proposal [13]

$$f_{bbb}^{\phi}(\theta_1,\theta_2,\theta_3) = N_{bbb}^{\phi}\,K_{bbb}(\theta_1,\theta_2,\theta_3)\,f_{bb}^{min}(\theta_{12})\,f_{bb}^{min}(\theta_{13})\,f_{bb}^{min}(\theta_{23})$$

with the "pole function"

$$K_{bbb}(\theta_1,\theta_2,\theta_3) = \frac{1}{\cosh\frac{1}{2}\theta_{12}\,\cosh\frac{1}{2}\theta_{13}\,\cosh\frac{1}{2}\theta_{23}}\,K_{bb}(\theta_{12})\,K_{bb}(\theta_{13})\,K_{bb}(\theta_{23})\;.$$

The "two-breather pole function" $K_{bb}(\theta)$ and the minimal form factor $f_{bb}^{min}(\theta)$ were already provided above. We use property (iii), i.e. the recursion relation (3.12), and calculate

$$\operatorname*{Res}_{\theta_{12}=i\pi} f_{bbb}^{\phi}(\theta_{12},\theta_{13},\theta_{23}) = 2i\sqrt{Z^{\phi}}\,(1-S_{bb}(\theta_{23}))$$

which determines the normalization constant [13]

$$N_{bbb}^{\phi} = \tfrac{1}{2}\pi^2\nu^2(1+\nu)^2\cot\tfrac{\pi}{2}\nu\,\cos^4\tfrac{\pi}{2}\nu\left(Z^{\phi}\right)^{-3/2}.$$

4.3.2 The three (anti)-soliton form factor

We now choose $\mathcal{O}(x)$ to be the fermi-field $\psi^{\pm}(x)$ of the massive Thirring model which annihilates the soliton. We consider the form factor

$$f_{123}^{\psi^{\pm}}(\theta_1,\theta_2,\theta_3) = \langle\, 0\,|\,\psi^{\pm}\,|\,p_1,p_2,p_3\,\rangle_{123}^{in}\,.$$

Here $\pm$ refers to the first or second component of the spinor, respectively. Nonvanishing matrix elements contain two solitons and one antisoliton. Taking in the general formula (4.10) for $n=3$ and $m=1$ we obtain

$$f_{123}^{\psi^{\pm}}(\underline{\theta}) = N_3^{\psi^{\pm}} \prod_{1\le i<j\le 3} F(\theta_{ij}) \int_{\mathcal{C}} du \prod_{i=1}^{3} \phi(\theta_i - u)\, e^{\pm\left(\sum u - \sum \theta_i/2\right)}\, \Omega_{123}\, C_{123}(\underline{\theta},u)\,. \tag{4.26}$$

Here the function $F(\theta)$, which fulfills Watsons equations

$$F(\theta) = -F(-\theta)\, a(\theta) = F(2\pi i - \theta)\,,$$

is closely related to the minimal form factor which was computed above

$$F(\theta) = -i \sinh\tfrac{1}{2}\theta\, f_{ss}^{min}(\theta)\,.$$

The scalar function $\phi(u)$ reads

$$\phi(u) = \frac{1}{F(u)F(u+i\pi)}\,.$$

We now use property (iii), i.e. the recursion relation (3.12) and calculate

$$\operatorname*{Res}_{\theta_{12}=i\pi} f_{123}^{\psi^{\pm}}(\underline{\theta}) = 2i\, C_{12}\, f_3^{\psi^{\pm}}\, (1 - S_{23}(\theta_{23}))$$

which determines the normalization constant

$$N_3^{\psi^{\pm}} = \pm i \frac{\sqrt{M}}{4\pi} \left(f_{ss}^{min}(0)\right)^2. \tag{4.27}$$

Note that this follows also from the general recursion relation (B.7). The form factor is now fixed with all its constants. However, we also expect the bound state formula (3.13) to hold and we may employ it now as a consistency check. We calculate with φ_-^{12} given by eq. (2.19) and the fusing angle given by $\theta_0 = i\pi(1-\nu)$

$$\operatorname*{Res}_{\theta_{12}=\theta_0} f_{123}^{\psi^{\pm}}(\theta_1,\theta_2,\theta_3)\, \varphi_-^{12} \left(2i \operatorname*{Res}_{\theta=\theta_0} S_-(\theta)\right)^{-1/2} = f_{b3}^{\psi^{\pm}}(\theta_{(12)},\theta_3)\,.$$

The result of this computation coincides with the form factor proposed in (4.24). Having convinced ourselves of the mutual consistency of several solutions we shall now carry out an additional check and compare the results with conventional perturbation theory.

5 Perturbation theory: Massive Thirring model

In order to check the three particle form factor of the fundamental fermi field corresponding to the soliton in perturbation theory we calculate the four point vertex function. We start with the Lagrangian

$$\mathcal{L}^{MTM} = \bar{\psi}(i\gamma\partial - M)\psi - \frac{1}{2}g(\bar{\psi}\gamma^\mu\psi)^2 \ .$$

The fermi field $\psi(x)$ annihilates a soliton and creates an antisoliton with the following normalisation

$$\langle\, 0 \,|\, \psi(x) \,|\, p \,\rangle_\alpha = \delta_{\alpha s}\, e^{-ipx}\, u(p) \ , \quad \langle\, 0 \,|\, \bar{\psi}(x) \,|\, p \,\rangle_\alpha = \delta_{\alpha \bar{s}}\, e^{ipx}\, \bar{v}(p). \tag{5.28}$$

We use the following conventions for the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.29}$$

and for the spinors

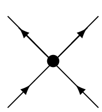
$$u(p) = \sqrt{M}\begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix} \ , \quad v(p) = \sqrt{M}\, i \begin{pmatrix} e^{-\theta/2} \\ -e^{\theta/2} \end{pmatrix} \quad \text{with} \quad p^\mu = M\begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix} \ . \tag{5.30}$$

We also employ the formulae

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \ , \quad [\gamma^\mu, \gamma^\nu] = 2\epsilon^{\mu\nu}\gamma^5 \ , \quad (\epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \ , \ \epsilon^{01} = 1)$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = 0 \ , \quad \epsilon^{\mu\rho}\epsilon^{\nu\sigma} = g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma} \ , \quad \gamma^5\gamma^\mu = \epsilon^{\mu\rho}\gamma_\rho \ .$$

The Lagrangian implies the Feynman rules of figure 6



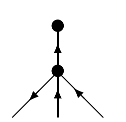
$= -ig\gamma^\mu \otimes \gamma_\mu$,

$= \dfrac{i}{\gamma k - M}$.

Figure 6: *The Feynman rules for the massive Thiring model.*

The three particle matrix element of the fermi field up to order g turns out to be

$\langle\, 0 \,|\, \psi(0) \,|\, p_1, p_2, p_3 \,\rangle^{in}_{\bar{s}ss} =$



$$\begin{aligned} &= -ig\,\frac{i}{\gamma(p_1+p_2+p_3) - M}\Big(\gamma^\mu u(p_2)\,\bar{v}(p_1)\gamma_\mu u(p_3) - \gamma^\mu u(p_3)\,\bar{v}(p_1)\gamma_\mu u(p_2)\Big) + O(g^2) \\ &= -ig\,\sinh\tfrac{1}{2}\theta_{23}\,\frac{u(p_2)\cosh\frac{1}{2}\theta_{12} + u(p_3)\cosh\frac{1}{2}\theta_{13}}{\cosh\frac{1}{2}\theta_{12}\,\cosh\frac{1}{2}\theta_{13}\,\cosh\frac{1}{2}\theta_{23}} + O(g^2)\ . \end{aligned}$$

Note that for the soliton-soliton scattering amplitude this implies

$$a(\theta) = 1 - ig \tanh \tfrac{1}{2}\theta + O(g^2)$$

in agreement with eq. (2.16).

To calculate the exact form factor up to this order we start from the general formula (4.26)

$$f^{\psi^\pm}_{\bar{s}ss}(\theta_1, \theta_2, \theta_3) = \int_{\mathcal{C}_{\underline{\theta}}} du\, I(\underline{\theta}, u)$$

with the integrand

$$I(\underline{\theta}, u) = N_3^{\psi^\pm} \prod_{1 \le i < j \le 3} F(\theta_{ij}) \prod_{i=1}^{3} \phi(\theta_i - u)\, e^{\pm\left(\sum u - \sum \theta_i/2\right)} \left(\Omega C(\underline{\theta}, u)\right)_{\bar{s}ss}.$$

Using the residue theorem the integral may be written as

$$\begin{aligned}\int_{\mathcal{C}_{\underline{\theta}}} du\, I(\underline{\theta}, u) \quad &= \quad 2\pi i \left(\operatorname*{Res}_{\theta_1 - i\pi} - \tfrac{1}{2}\Big(\operatorname*{Res}_{\theta_1} + \operatorname*{Res}_{\theta_2} + \operatorname*{Res}_{\theta_3} - \operatorname*{Res}_{\theta_1 + i\pi(\nu-1)} \Big)\right) I(\underline{\theta}, u) \\ &\quad + \tfrac{1}{2} \int_{\mathcal{C}_0} du \left(I(\underline{\theta}, u) + I(\underline{\theta}, u + i\pi) \right)\end{aligned}$$

where the integration contour $\mathcal{C}_0$ is a line from $-\infty$ to ∞ avoiding all poles such that $\Im\theta_i + \pi(\nu - 2) < \Im u < \Im\theta_i$ (for $\nu > 1,\ \nu = 1/(1 + 2g/\pi) \approx 1$). The integral on the right hand side is of higher order in g and the residues give

$$f^{\psi^\pm}_{\bar{s}ss}(\theta_1, \theta_2, \theta_3) = N_3^{\psi^\pm} \frac{\mp 4\pi g}{\sqrt{M}} \sinh \tfrac{1}{2}\theta_{23} \frac{u^\pm(p_2) \cosh \frac{1}{2}\theta_{12} + u^\pm(p_3) \cosh \frac{1}{2}\theta_{13}}{\cosh \frac{1}{2}\theta_{12}\, \cosh \frac{1}{2}\theta_{13}\, \cosh \frac{1}{2}\theta_{23}} + O(g^2)$$

which is consistent with the result of the Feynman graph calculation because of equation (4.27) and $f^{min}_{ss}(0) = 1 + O(g)$. Hence we obtain mutual consistency between the solutions of the form factor equations and conventional perturbation theory.

6 Conclusions

We have outlined in detail the so-called "form factor program". Using only the "maximal analyticity assumption" and the validity of the LSZ formalism we have derived general properties of form factors. The properties are expressed in terms of the equations (i)–(v). We provide a solution for these equations in a closed form, which captures the vectorial structure by means of the "off-shell" Bethe ansatz and the singularity structure in term of certain contour integrals. The validity of this solution has been checked by constructing various explicit two and three particle form factors. We have compared our solution for the three particle form factor of the fundamental fermi field with the expressions obtained from perturbation theory in the massive Thirring model. We find complete aggrement between these two approaches and we take this as a further indication for the validity of the “form factor program” formalism.

The vectorial nature of the form factors we present, is encapsulated in the “off-shell” Bethe ansatz and the singularities are encoded in certain contour integrals. We assume that this structure is of a universal nature and will allow to construct further solutions of other integrable theories. It will be highly interesting to work out such solutions explicitly. This task and the detailed study of the correlation functions obtained from these solutions is left to future investigations.

We have applied the general formula (4.10) to an explicit example and exploit the fact that the properties (i)–(iv) relate several different form factors to each other. This permits us to carry out various consistency checks. We have for instance the following relations

$$
\begin{tikzcd}
f^{\psi}_{b_2 s} \arrow[d, "(iii)"] & f^{\psi}_{\bar{s}ss} \arrow[l, "(iv)"'] \arrow[r, "(iv)"] \arrow[d, "(iii)"] & f^{\psi}_{bs} \arrow[d, "(iv)"] \\
f^{\psi}_{s} \arrow[r, phantom, "="] & f^{\psi}_{s} \arrow[r, phantom, "="] & f^{\psi}_{s}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^{\phi}_{bbb} \arrow[r, "(iii)"] \arrow[d, "(iv)"] & f^{\phi}_{b} \arrow[r, phantom, "="] & f^{\phi}_{b} \\
f^{\phi}_{b_2 b} & f^{\phi}_{s\bar{s}b} \arrow[l, "(iv)"'] \arrow[r, "(iv)"] \arrow[u, "(iii)"] & f^{\phi}_{s\bar{s}} \arrow[u, "(iv)"] \arrow[d, leftrightarrow, "(4.25)"] \\
 & & f^{j^{\mu}}_{s\bar{s}}
\end{tikzcd}
\qquad
\begin{tikzcd}
f^{\phi^2}_{s\bar{s}b} \arrow[r, "(iv)"] \arrow[d, "(iv)"] & f^{\phi^2}_{bb} \arrow[d, "(iv)"] \\
f^{\phi^2}_{s\bar{s}} \arrow[r, "(iv)"] & f^{\phi^2}_{b_2}
\end{tikzcd}
$$

Several of these form factor relations and consistency checks have been presented in this paper. The proof of further relations will be published elsewhere.

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Appendix

A Derivation of properties of generalized form factors

In this appendix we derive the formulae for form factors of section 3. We use LSZ techniques [57] (see e.g. [58]) and assume in addition “maximal analyticity” which means that all singularities originate from physical intermediate states. For simplicity we consider only particles with the same mass m. Generalizations to the case of particles with different masses are obvious.

A.1 Properties of generalized form factors for the pure bosonic case

As usual we write the in-field as

$$\phi^{in}_{\alpha}(x) = \int \frac{dp}{2\pi 2\omega} \left(a^{in}_{\alpha}(p)\, e^{-ipx} + a^{in\,\dagger}_{\bar\alpha}(p)\, e^{ipx} \right). \tag{A.1}$$

It fulfills the Klein-Gordon equation $(\partial^2 + m^2)\,\phi(x) = 0$ and when acting on states of the form (2.1) it creates anti-particles and annihilates particles. The commutation rules of the creation and annihilation operators are

$$\begin{aligned} [\, a^{in}_{\alpha'}(p')\,,\, a^{in}_{\alpha}(p)] &= 0 &\text{(A.2)}\\ [\, a^{in}_{\alpha'}(p')\,,\, a^{in\,\dagger}_{\alpha}(p)] &= \delta_{\alpha'\alpha}\, 2\omega\, 2\pi\, \delta(p'-p) = \delta_{\alpha'\alpha}\, 4\pi\, \delta(\theta'-\theta). &\text{(A.3)} \end{aligned}$$

Corresponding formulae hold also for the out-field.

For the matrix elements of a local scalar operator $\mathcal{O} = \mathcal{O}(0)$ we have the LSZ-reduction formulae [57]

$$\begin{aligned} &{}^{\;out}_{\dots\bar\alpha'_1}\langle \dots, p'_1 \,|\, \mathcal{O} \,|\, p_1, \dots \rangle^{in}_{\alpha_1\dots} &\text{(A.4)}\\ &= {}^{\;out}_{\dots\bar\alpha'_1}\langle \dots, p'_1 \,|\, a^{out\,\dagger}_{\alpha_1}(p_1)\, \mathcal{O} \,|\, \dots \rangle^{in}_{\dots} + i \int d^2x \; {}^{\;out}_{\dots\bar\alpha'_1}\langle \dots, p'_1 \,|\, T\left[\mathcal{O}\, j^{\dagger}_{\alpha_1}(x) \right] |\, \dots \rangle^{in}_{\dots}\, e^{-ip_1x} \\ &= {}^{\;out}_{\dots}\langle \dots \,|\, \mathcal{O}\, a^{in}_{\alpha'_1}(p'_1) \,|\, p_1, \dots \rangle^{in}_{\alpha_1\dots} + i \int d^2x \; {}^{\;out}_{\dots}\langle \dots \,|\, T\left[\mathcal{O}\, j^{\dagger}_{\alpha'_1}(x) \right] |\, p_1, \dots \rangle^{in}_{\alpha_1\dots}\, e^{ip'_1x} \end{aligned}$$

where T is the time ordering operator and the source term $j(x) = (\partial^2 + m^2)\,\phi(x)$ is given by the interpolating field $\phi(x)$. If p_1 (p'_1) corresponds to an anti-particle (a particle) $j^\dagger$ has to be replaced by j. On further reductions and combined with the assumption of maximal analyticity the LSZ-formulae imply the crossing formula (3.4) for the connected part of the matrix element. We call a contribution to a matrix element ${}^{out}\langle \dots \,|\, \mathcal{O} \,|\, p_1, \dots \rangle^{in}$

- *disconnected* with respect to p_1, if its support as a distribution with respect to p_1 is point like and
- *connected* with respect to p_1 if it is a boundary value of an analytic function of the Mandelstam variables s_{1j}.

With this notation the first terms in eqs. (A.4) are disconnected and the second ones are connected with respect to p_1 or p'_1, respectively.

If we interchange in eqs. (A.4) *in* and *out* the time ordering is replaced by anti-time ordering. Comparing eq. (3.3) with (3.2) this means that $s + i\epsilon$ is replaced by $s - i\epsilon$.

Combined with the completeness of the in- and out-states we obtain the general Watson's equations (3.8). However, from integrability follows a stronger formula. To show this, we consider the branch point $s_{12} = (m_1 \pm m_2)^2$ separately. For simplicity we assume $m_1 = m_2$.

Lemma A.1 *Let the S-matrix factorize as denoted in (3.7), and let s_{12} be in a neighbourhood of $4m^2$ or 0 and all other s_{ij} away from $4m^2$, then*

$$
\begin{aligned}
F^{\mathcal{O}}_{\underline{\alpha}}(s_{12}+i\epsilon, s_{ij}+i\epsilon) &= F^{\mathcal{O}}_{\underline{\alpha}'}(s_{12}-i\epsilon, s_{ij}+i\epsilon)\, S^{\alpha_2'\alpha_1'}_{\alpha_1\alpha_2}(s_{12}) \quad \text{for} \quad s_{12}\approx 4m^2 && \text{(A.5)}\\
F^{\mathcal{O}}_{\underline{\alpha}}(s_{12}+i\epsilon, s_{ij}+i\epsilon) &= F^{\mathcal{O}}_{\underline{\alpha}'}(s_{12}-i\epsilon, s_{ij}+i\epsilon) \quad \text{for} \quad s_{12}\approx 0 && \text{(A.6)}
\end{aligned}
$$

with $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\underline{\alpha}' = (\alpha_1', \alpha_2', \dots, \alpha_n)$ and $2 < i < j \le n$. Corresponding formulae hold for all other branch points $s_{ij} = 4m^2$.

Proof: By means of formula (3.4) we may cross all particles except 1 and 2 to the left hand side. Using again LSZ we have for the full matrix element

$$
\begin{aligned}
&{}^{out}\langle\, p_3,\dots,p_n \,|\, \mathcal{O} \,|\, p_1,p_2 \,\rangle^{in} && \text{(A.7)}\\
&= {}^{out}\langle\, p_3,\dots,p_n \,|\, a^{out\,\dagger}(p_1)\, \mathcal{O} \,|\, p_2 \,\rangle + i\int d^2x\; {}^{out}\langle\, p_3,\dots,p_n \,|\, T\left[\mathcal{O}\, j^\dagger(x)\right] \,|\, p_2 \,\rangle\, e^{-ip_1x}\\
&{}^{out}\langle\, p_3,\dots,p_n \,|\, \mathcal{O} \,|\, p_1,p_2 \,\rangle^{out} && \text{(A.8)}\\
&= {}^{out}\langle\, p_3,\dots,p_n \,|\, a^{in\,\dagger}(p_1)\, \mathcal{O} \,|\, p_2 \,\rangle - i\int d^2x\; {}^{out}\langle\, p_3,\dots,p_n \,|\, T^*\left[\mathcal{O}\, j^\dagger(x)\right] \,|\, p_2 \,\rangle\, e^{-ip_1x}
\end{aligned}
$$

where we have omitted the indices α and T^* means anti-time ordering. The term ${}^{out}\langle\, p_3,\dots,p_n \,|\, a^{out\,\dagger}(p_1)\, \mathcal{O} \,|\, p_2 \,\rangle$ is disconnected as in eq. (A.4), whereas ${}^{out}\langle\, p_3,\dots,p_n \,|\, a^{in\,\dagger}(p_1)\, \mathcal{O} \,|\, p_2 \,\rangle$ in general contains also connected contributions. However, for factorizing S-matrices this term is given by

$$
{}^{out}\langle\, p_3,\dots,p_n \,|\, a^{in\,\dagger}(p_1) \,|\, q_1,\dots,q_m \,\rangle^{in\;in}\langle\, q_m,\dots,q_1 \,|\, \mathcal{O} \,|\, p_2 \,\rangle
$$

which is disconnected with respect to p_1. Therefore, if we take the connected parts of eqs. (A.7) and (A.8) we obtain as in equations (3.2) and (3.3)

$$
\begin{aligned}
{}^{out}\langle\, p_3,\dots,p_n \,|\, \mathcal{O} \,|\, p_1,p_2 \,\rangle^{in}_{conn.} &= F^{\mathcal{O}}_{\underline{\alpha}}(s_{12}+i\epsilon, (t_{rs})_{(1\le r\le 2<s\le n)}, (s_{kl}+i\epsilon)_{(2<k<l\le n)}) && \text{(A.9)}\\
{}^{out}\langle\, p_3,\dots,p_n \,|\, \mathcal{O} \,|\, p_1,p_2 \,\rangle^{out}_{conn.} &= F^{\mathcal{O}}_{\underline{\alpha}}(s_{12}-i\epsilon, (t_{rs})_{(1\le r\le 2<s\le n)}, (s_{kl}+i\epsilon)_{(2<k<l\le n)}) && \text{(A.10)}
\end{aligned}
$$

which implies the first claim. Moreover, crossing in equations (A.7) and (A.8) in addition also particle 2 to the left hand side, by the same arguments we confirm the second claim, since $|\, p_1 \,\rangle^{in} = |\, p_1 \,\rangle^{out}$.

As a consequence of this lemma and the bose statistics of the particles we have the property (i) (c.f. eq. (3.10))

$$f^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = f^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\, S_{ij}(\theta_i-\theta_j). \tag{A.11}$$

Iterating this formula we find that for $\theta_1 < \dots < \theta_n$ the auxiliary function $f^{\mathcal{O}}_{\underline{\alpha}}(\theta)$ yields the matrix element for an *out*-state

$$f^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n) = f^{\mathcal{O}}_{\alpha'_n\dots\alpha'_1}(\theta_n,\dots,\theta_1)\, S^{\alpha'_n\dots\alpha'_1}_{\alpha_1\dots\alpha_n}(\underline{\theta}) = \langle\, 0\,|\,\mathcal{O}\,|\,p_1,\dots,p_n\,\rangle^{out}_{\underline{\alpha}}. \tag{A.12}$$

We obtain property (ii) (see eq. (3.11)) by comparing the analytic parts of the crossing relations of the following lemma.

Lemma A.2 *In terms of the auxiliary functions crossing for the full matrix elements reads*

$$\begin{aligned} &{}_{\bar 1}\langle\, p_1\,|\,\mathcal{O}\,|\,p_2,\dots,p_n\,\rangle^{in}_{2\dots n} \\ &= \begin{cases} {}_{\bar 1}\langle\, p_1\,|\,p_2\,\rangle_2\, f^{\mathcal{O}}_{3\dots n}(\theta_3,\dots,\theta_n) \\ \qquad\qquad + \mathbf{C}^{\bar 1 1}\, f^{\mathcal{O}}_{1\dots n}(\theta_1+i\pi_-,\dots,\theta_n) & \text{for } \theta_1\geq\theta_2>\dots>\theta_n \\ {}_{\bar 1}\langle\, p_1\,|\,p_n\,\rangle_n\, f^{\mathcal{O}}_{2\dots n-1}(\theta_2,\dots,\theta_{n-1}) \\ \qquad\qquad + f^{\mathcal{O}}_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1-i\pi_-)\,\mathbf{C}^{1\bar 1} & \text{for } \theta_2>\dots>\theta_n\geq\theta_1 \end{cases} \end{aligned} \tag{A.13}$$

where $\pi_- = \pi-\epsilon$.

Together with property (i) this lemma implies the general crossing formulae (3.14) for arbitrary ordering of the rapidities.

Proof: The disconnected contributions in eq. (A.13) follow directly from the LSZ - formulae (A.4). Moreover the LSZ - formulae imply that crossing of particle 1 means $p_1 \to -p_1$. In terms of the Mandelstam variables s or the rapidity differences (see also figure 3 for the analytic properties) this means $s_{1j}+i\epsilon \to t_{1j}-i\epsilon$ or $\theta_{1j}\to i\pi-\theta_{1j}$. However, since there is no branch cut separating region II and IV (see (A.6)), this is equivalent to considering $s_{1j}+i\epsilon \to t_{1j}+i\epsilon$ or $\theta_{1j}\to i\pi+\theta_{1j}$. Hence, because of $\theta_{1j} = |\theta_1-\theta_j|$ we have the equivalences

$$\begin{aligned} \theta_{1j}\to i\pi+\theta_{1j} \quad &\Leftrightarrow \quad \theta_1\to\theta_1+i\pi \quad \text{for}\ \ \theta_1>\theta_j \\ \theta_{1j}\to i\pi+\theta_{1j} \quad &\Leftrightarrow \quad \theta_1\to\theta_1-i\pi \quad \text{for}\ \ \theta_1<\theta_j \end{aligned}$$

which imply the claim.

The form factors have poles determined by one-particle states in any subchannel $(\alpha_i,\dots,\alpha_j)\subset(\alpha_1,\dots,\alpha_n)$, if the square of the total momentum of all particles in the subchannel equals the one-particle mass squared. We follow the arguments of [13] and in particular of [69] (see also [17, 28]). A particular type of poles is always present, even if there are no boundstates. These poles are often refered to as kinematic poles. If for instance in (A.13) particle 1 is the anti-particle of 2, then one-particle intermediate states with the quantum numbers of all other particles j ($2 < j \leq n$) yield contributions to a pole at $p_1 \approx p_2$, since then $(p_2 + p_j - p_1)^2 \approx m_j^2$ in eq. (A.13), (see figure 7). The residue

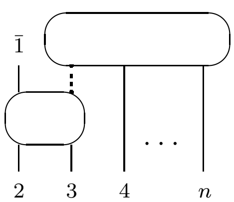


Figure 7: *A graph contributing to a pole of a form factor. The dashed lines denotes an off-shell line with the propagator* $\frac{i}{(p_2+p_3-p_1)^2-m^2}$.

of this pole is given by property (iii) of eq. (3.12). This can be seen as follows. By (ii) we have for $\theta_1 \approx \theta_2$ and $\theta_1 \neq \theta_j$, $(j = 3, \dots, n)$

$$\begin{aligned} f^{\mathcal{O}}_{12\dots n}(\theta_1 + i\pi_-, \dots) &\approx \frac{1}{\theta_1 - i\epsilon - \theta_2} \mathbf{C}_{12}\, g(\theta_2, \dots, \theta_n) \\ f^{\mathcal{O}}_{2\dots n1}(\dots, \theta_1 - i\pi_-) &\approx \frac{1}{\theta_1 + i\epsilon - \theta_2} \mathbf{C}_{12}\, g(\theta_2, \dots, \theta_n) \end{aligned}$$

for some function $g(\theta_2, \dots, \theta_n)$. Employing now the well known identity $\frac{1}{a \pm i\epsilon} = \frac{P}{a} \mp i\pi\delta(a)$, with P denoting the principal value, the general crossing relations (3.14) imply for the full matrix element for $\theta_1 \approx \theta_2$ and $\theta_1 \neq \theta_j$, $(j = 3, \dots, n)$)

$$\begin{aligned} {}_{\bar{1}}\langle\, p_1 \,|\, \mathcal{O} \,|\, p_2, \dots, p_n \,\rangle^{in}_{2\dots n} &\approx {}_{\bar{1}}\langle\, p_1 \,|\, p_2 \,\rangle_2\, f^{\mathcal{O}}_{3\dots n} + \left(\frac{P}{\theta_1 - \theta_2} + i\pi\delta(\theta_1 - \theta_2) \right) \mathbf{C}_{12}\, g \\ &\approx {}_{\bar{1}}\langle\, p_1 \,|\, p_2 \,\rangle_2\, f^{\mathcal{O}}_{3\dots n}\, S_{2n} \cdots S_{23} + \left(\frac{P}{\theta_1 - \theta_2} - i\pi\delta(\theta_1 - \theta_2) \right) \mathbf{C}_{12}\, g. \end{aligned}$$

Comparing the delta-function parts we obtain

$$g(\theta_2, \dots, \theta_n) = 2i\, f^{\mathcal{O}}_{3\dots n}(\theta_3, \dots \theta_n) \left(\mathbf{1} - S_{2n} \cdots S_{23} \right)$$

which yields (iii).

If there are also bound states there are additional poles [13] and we have also property (iv). The latter is obvious from figure 2 and eq. (2.10) up to a normalization. The normalization follows from the following argument developed in [13]. Let us consider a model with a bound state of type c of two particles of type a and b such that the attractive region is connected analytically (by a coupling constant) to a repulsive region, where the bound state decays. For simplicity we consider first the two-particle form factor

$$f^{\mathcal{O}}_{ab}(\theta_{ab}) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_a, p_b \,\rangle^{in}_{ab} \,, \quad (\theta_a > \theta_b) \tag{A.14}$$

such that the (scalar hermitian) operator $\mathcal{O}$ connects the bound state c with the vacuum

$$f^{\mathcal{O}}_{c}(\theta_c) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_c \,\rangle_c = \sqrt{Z^{\mathcal{O}}} \neq 0. \tag{A.15}$$

Then in the atractive region of the coupling the two-point Wightman function reads

$$\langle\, 0 \,|\, \mathcal{O}(x)\mathcal{O}(y) \,|\, 0 \,\rangle = Z^{\mathcal{O}} \Delta_+(x-y, m_c^2) + \text{contributions from other masses} \tag{A.16}$$

or the time ordered two-point function in momentum space fulfills

$$\langle\, 0 \,|\, \tilde{\mathcal{O}}(k)\mathcal{O}(0) \,|\, 0 \,\rangle \approx Z^{\mathcal{O}} \frac{i}{k^2 - m_c^2 + i\epsilon} \quad \text{at} \quad k^2 \approx m_c^2 \tag{A.17}$$

where $Z^{\mathcal{O}}$ is a wave function renormalization function. In the repulsive region the contribution from the two-particle intermediate states ab is given by

$$\begin{aligned}
\langle\, 0 \,|\, \mathcal{O}(x)\mathcal{O}(y) \,|\, 0 \,\rangle &= \int \frac{dp_a}{4\pi\omega_a} \frac{dp_b}{4\pi\omega_b} \frac{1}{2} \langle\, 0 \,|\, \mathcal{O}(x) \,|\, p_a, p_b \,\rangle^{in}_{ab} \; {}^{in}\langle\, p_a, p_b \,|\, \mathcal{O}(y) \,|\, 0 \,\rangle^{ab} + \dots \\
&= \frac{1}{8\pi} \int_{-\infty}^{\infty} d\theta\, f^{\mathcal{O}}_{ab}(\theta) f^{\mathcal{O}ab}(\theta) \Delta_+(x-y, s_{ab}) + \dots
\end{aligned} \tag{A.18}$$

where summation over the multiplets a and b is assumed with $s_{ab} = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta$ and $\Delta_+(x, m^2) = (2\pi)^{-2} \int d^2k e^{ikx} \Theta(k_0) 2\pi\delta(k^2 - m^2)$. In the repulsive region the functions $f^{\mathcal{O}}_{ab}(\theta)$ and $f^{\mathcal{O}ab}(\theta)$ have poles in the unphysical sheet at $\pm\theta^c_{ab}$ ($\mathrm{Im}\;\theta^c_{ab} < 0$), respectively. If we move to the attractive region these poles will cross the integration path and by analytic continuation we get

$$\begin{aligned}
\langle\, 0 \,|\, \mathcal{O}(x)\mathcal{O}(y) \,|\, 0 \,\rangle &= \frac{1}{8\pi} \left\{ -\oint_{\theta^c_{ab}} + \oint_{-\theta^c_{ab}} + \int_{-\infty}^{\infty} \right\} d\theta\, f^{\mathcal{O}}_{ab}(\theta) f^{\mathcal{O}ab}(\theta) \Delta_+(x-y, s_{ab}) + \dots \\
&= \frac{1}{4i} \left(\operatorname*{Res}_{\theta=\theta^c_{ab}} - \operatorname*{Res}_{\theta=-\theta^c_{ab}} \right) f^{\mathcal{O}}_{ab}(\theta) f^{\mathcal{O}ab}(\theta) \Delta_+(x-y, m_c^2) + \dots
\end{aligned} \tag{A.19}$$

Both residues give the same contribution, because

$$\begin{aligned}
\operatorname*{Res}_{\theta=\theta^c_{ab}} f^{\mathcal{O}}_{ab}(\theta) f^{\mathcal{O}ab}(\theta) &= \operatorname*{Res}_{\theta=\theta^c_{ab}} f^{\mathcal{O}}_{ba}(-\theta)\, S_{ab}(\theta) f^{\mathcal{O}ab}(\theta) = f^{\mathcal{O}}_{ba}(-\theta^c_{ab}) \varphi^{ba}_c R_c\, \varphi^c_{ab} f^{\mathcal{O}ab}(\theta^c_{ab}) \\
\operatorname*{Res}_{\theta=-\theta^c_{ab}} f^{\mathcal{O}}_{ab}(\theta) f^{\mathcal{O}ab}(\theta) &= \operatorname*{Res}_{\theta=-\theta^c_{ab}} f^{\mathcal{O}}_{ab}(-\theta)\, S_{ba}(-\theta) f^{\mathcal{O}ba}(-\theta) = -f^{\mathcal{O}}_{ab}(-\theta^c_{ab}) \varphi^{ab}_c R_c\, \varphi^c_{ba} f^{\mathcal{O}ba}(\theta^c_{ab})
\end{aligned}$$

where property (i) eq. (3.10) and the residue formulae for the S-Matrix (2.10) and (2.11) have been used. Using this and comparing eqs. (A.16) and (A.19) we obtain

$$
\begin{aligned}
Z^{\mathcal{O}} = f_c^{\mathcal{O}}(\theta_c)\, f^{\mathcal{O}c}(\theta_c) &= \frac{1}{4i}\left(\operatorname*{Res}_{\theta=\theta_{ab}^c} - \operatorname*{Res}_{\theta=-\theta_{ab}^c}\right) f_{ab}^{\mathcal{O}}(\theta) f^{\mathcal{O}ab}(\theta) \\
&= \left(\operatorname*{Res}_{\theta=\theta_{ab}^c} f_{ab}^{\mathcal{O}}(\theta)\varphi_c^{ab}\frac{1}{\sqrt{2iR_c}}\right)\left(\frac{1}{\sqrt{2iR_c}}\operatorname*{Res}_{\theta=-\theta_{ab}^c}\varphi_{ab}^c f^{\mathcal{O}ab}(\theta)\right)
\end{aligned}
$$

which agrees with (iv). The general case may be proven similarly.

A.2 Properties of form factors for the general case

We now consider the case in which the particles are taken to be fermions and the operators may be of fermionic or bosonic nature. Again we use LSZ techniques [57] and "maximal analyticity". The two component fermionic in-field is

$$
\psi_\alpha^{in}(x) = \int \frac{dp}{2\pi 2\omega}\left(a_\alpha^{in}(p)\, u(p)\, e^{-ipx} + a_{\bar\alpha}^{in\,\dagger}(p)\, v(p)\, e^{ipx}\right) \tag{A.20}
$$

and fulfills the Dirac equation $(i\gamma\partial - m)\,\psi(x) = 0$. The anti-commutation relations are

$$
\begin{aligned}
\{\, a_{\alpha'}^{in}(p')\,,\, a^{in}{}_\alpha(p)\} &= 0 \\
\{\, a_{\alpha'}^{in}(p')\,,\, a_\alpha^{in\,\dagger}(p)\} &= \delta_{\alpha'\alpha}\, 2\omega\, 2\pi\, \delta(p'-p) = \delta_{\alpha'\alpha}\, 4\pi\, \delta(\theta'-\theta).
\end{aligned} \tag{A.21}
$$

Corresponding formulae hold for the out-field. We use the conventions for the γ-matrices and the spinors of (5.29) and (5.30).

The LSZ-reduction formulas for fermions read

$$
\begin{aligned}
&{}^{\;out}_{\dots\bar\alpha_1'}\langle \dots, p_1' \,|\, \mathcal{O} \,|\, p_1, \dots\rangle^{in}_{\alpha_1\dots} \\
&\quad= \sigma_{\mathcal{O}\alpha_1}\; {}^{\;out}_{\dots\bar\alpha_1'}\langle \dots, p_1' \,|\, a_{\alpha_1}^{out\,\dagger}(p_1)\, \mathcal{O} \,|\, \dots\rangle^{in}_{\dots} \\
&\qquad + i\int d^2x\; {}^{\;out}_{\dots\bar\alpha_1'}\langle \dots, p_1' \,|\, T\,[\mathcal{O}\,\bar\jmath_{\alpha_1}(x)] \,|\, \dots\rangle^{in}_{\dots}\, u(p_1)\, e^{-ip_1x} && \text{(A.22)} \\
&\quad= \sigma_{\mathcal{O}\alpha_1'}\; {}^{\;out}_{\dots}\langle \dots \,|\, \mathcal{O}\, a_{\alpha_1'}^{in}(p_1') \,|\, p_1, \dots\rangle^{in}_{\alpha_1\dots} \\
&\qquad - i\,\sigma_{\mathcal{O}\alpha_1'}\int d^2x\; {}^{\;out}_{\dots}\langle \dots \,|\, T\left[\mathcal{O}\,\bar\jmath_{\alpha_1'}(x)\right] \,|\, p_1, \dots\rangle^{in}_{\alpha_1\dots}\, v(p_1')\, e^{ip_1'x} && \text{(A.23)}
\end{aligned}
$$

where $\bar\jmath(x) = \bar\psi(x)\,(i\gamma\overleftarrow{\partial} + m)$ and $\sigma_{\mathcal{O}\alpha} = -1$ if $\mathcal{O}$ is fermionic and $\sigma_{O\alpha} = 1$ otherwise. Obviously $\sigma_{\mathcal{O}\alpha} = (-1)^n$ if n is the total number of fermions in the states. A similar formula holds if we interchange particles and anti-particles. The invariant form factors $G^{(l)\,\mathcal{O}}(s_{ij}+i\epsilon)$ defined by eq. (3.13) are again boundary values of analytic functions. Again, if we interchange *in* and *out* time ordering is replaced by anti-time ordering, which means again that $s_{ij}+i\epsilon$ is replaced by $s_{ij}-i\epsilon$. The crossing relation for the connected part of the matrix element reads

$$\begin{aligned}
& {}^{out}_{\bar\alpha_1\dots\alpha_m}\langle\, p_1,\dots,p_m\,|\,\mathcal{O}\,|\,p_{m+1},\dots,p_n\,\rangle^{in\ \mathrm{conn.}}_{\alpha_{m+1}\dots\bar\alpha_n} \\
= \;& (-1)^m \prod_{i=1}^{m}\sigma_{\mathcal{O}\alpha_i}\sum \bar v(p_n)\cdots\bar u(p_m)\,\Gamma_{\mu_1\dots\mu_k}\,u(p_{m+1})\cdots v(p_1)\,p_{i_1}^{\mu_1}\dots p_{i_k}^{\mu_k} \\
& G^{\mathcal{O}}_{\alpha_1\dots\bar\alpha_n}(s_{ij}+i\epsilon,t_{rs}-i\epsilon,s_{kl}+i\epsilon)
\end{aligned} \tag{A.24}$$

Watson's equations for the invariant form factor functions G acquire the same form as those for F for the bosonic case (3.8). Also Lemma A.1 holds for the invariant form factors G.

Analogously to the bosonic case it is convenient to introduce the vector valued auxiliary function $f^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ which is considered as an analytic function of the rapidities of the particles. Its components coincide again with the physical matrix elements for a particular order of the rapidities.

$$f^{\mathcal{O}}_{\underline{\alpha}}(\theta_1,\dots,\theta_n) := \langle\, 0\,|\,\mathcal{O}\,|\,p_1,\dots,p_n\,\rangle^{in}_{\underline{\alpha}}\ ,\quad \text{for}\ \ \theta_1>\dots>\theta_n. \tag{A.25}$$

In the other sectors the function $f^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$ is again given by analytic continuation. Again, as a consequence of Lemma A.1 and the fermi statistics of the particles we have now the property (i) in the form (c.f. eq. (3.17))

$$f^{\mathcal{O}}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = f^{\mathcal{O}}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\,(-S)_{ij}(\theta_i-\theta_j). \tag{A.26}$$

The LSZ-formulae (A.22) and (A.23) imply the general crossing formulae (3.18). Note that some signs in these formulae depend on the choice of the relative phases of the u- and the v-spinors taken in eq. (5.30). The crossing formulae again, as for the bosonic case, implies the properties (ii) and (iii) as given by (3.17), where we have used the fact that $\sigma_{\mathcal{O}1}=\sigma_{2n}\cdots\sigma_{23}$, if particle 1 has the same statistics as particle 2.

B Proof of the theorem 4.1

In the proof of theorem 4.1 we follow [55] (see also [53]).

Proof that eq. (4.10) fulfills (i):

Property (i) (c.f. eq. (3.17)) follows directly from the Yang-Baxter equations, the definitions of the soliton-soliton S-matrix (2.15) and the pseudo-ground state Ω and Watson's equations for $F(\theta)$

$$\begin{aligned}
F(\theta_{ji})\,\Omega_{\dots ji\dots}\,C_{\dots ji\dots}(\dots\theta_j,\theta_i\dots)\,\dot S_{ij}(\theta_{ij}) &= F(\theta_{ji})\,\Omega_{\dots ji\dots}\,\dot S_{ij}(\theta_{ij})\,C_{\dots ij\dots}(\dots\theta_i,\theta_j\dots)\\
&= -F(\theta_{ji})\,a(\theta_{ij})\,\Omega_{\dots ij\dots}\,C_{\dots ij\dots}(\dots\theta_i,\theta_j\dots)\\
&= F(\theta_{ij})\,\Omega_{\dots ij\dots}\,C_{\dots ij\dots}(\dots\theta_i,\theta_j\dots).
\end{aligned}$$

The minus sign is due to the fermi statistics of the solitons (c.f. (3.1))

Proof that eq. (4.10) fulfills (ii):

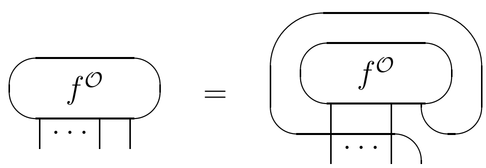
Using (i) the property (ii) (c.f. eq. (3.17)) may be rewritten as a difference equation

$$f^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = f^{\mathcal{O}}_{1\dots n}(\underline{\theta}')\,Q_{1\dots n}(\underline{\theta}) \tag{B.1}$$

where $\underline{\theta}' = (\theta_1,\dots,\theta'_n = \theta_n - 2\pi i)$ and $Q(\underline{\theta})$ is the trace of the monodromy matrix (4.1) over the auxiliary space for the specific value of the spectral parameter $\theta_0 = \theta_n$

$$Q_{1\dots n}(\underline{\theta}) = \mathrm{tr}_0 T^Q_{1\dots n}(\underline{\theta})\ ,\quad \text{with}\ \ T^Q_{1\dots n}(\underline{\theta}) = T_{1\dots n}(\underline{\theta},\theta_n) \tag{B.2}$$

since $S_{n0}(0) = P_{n0}$ is the permutation operator. This may be depicted as



In the following we will suppress the indices $1\dots n$. The Yang-Baxter relations (4.2) imply the well known commutation rules for the matrices A, C, D defined in eq. (4.3)

$$\begin{aligned}
C(\underline{\theta},u)C(\underline{\theta},v) &= C(\underline{\theta},v)C(\underline{\theta},u)\\
C(\underline{\theta},u)A(\underline{\theta},\theta) &= \frac{a(\theta-u)}{b(\theta-u)}A(\underline{\theta},\theta)C(\underline{\theta},u) - \frac{c(\theta-u)}{b(\theta-u)}A(\underline{\theta},u)C(\underline{\theta},\theta)\\
C(\underline{\theta},u)D(\underline{\theta},\theta) &= \frac{a(u-\theta)}{b(u-\theta)}D(\underline{\theta},\theta)C(\underline{\theta},u) - \frac{c(u-\theta)}{b(u-\theta)}D(\underline{\theta},u)C(\underline{\theta},\theta)
\end{aligned} \tag{B.3}$$

In addition there are commutation rules where also the matrices A^Q, C^Q, D^Q defined by

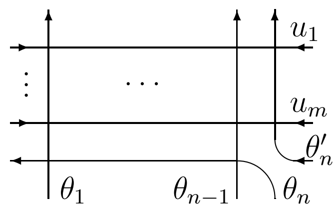
$$T^Q(\underline{\theta}) = \begin{pmatrix} A^Q(\underline{\theta}) & B^Q(\underline{\theta})\\ C^Q(\underline{\theta}) & D^Q(\underline{\theta})\end{pmatrix}$$

are involved [14]

$$\begin{aligned}
C(\underline{\theta}',u)A^Q(\underline{\theta}) &= \frac{a(\theta'_n-u)}{b(\theta_n-u)}A^Q(\underline{\theta})C(\underline{\theta},u) - \frac{c(\theta_n-u)}{b(\theta_n-u)}A(\underline{\theta}',u)C^Q(\underline{\theta})\\
C(\underline{\theta}',u)D^Q(\underline{\theta}) &= \frac{a(u-\theta_n)}{b(u-\theta'_n)}D^Q(\underline{\theta})C(\underline{\theta},u) - \frac{c(u-\theta'_n)}{b(u-\theta'_n)}D(\underline{\theta}',u)C^Q(\underline{\theta})\,.
\end{aligned} \tag{B.4}$$

To analyze the right hand side of eq. (B.1) we proceed as follows: We apply the trace of $T^Q(\underline{\theta})$ to the co-vector $f^{\mathcal{O}}(\underline{\theta}')$ as given by eq. (4.10) and the Bethe ansatz (4.4). In the contribution from $A^Q(\underline{\theta})$

$$\Omega\, C(\underline{\theta}', u_1) \cdots C(\underline{\theta}', u_m)\, A^Q(\underline{\theta}) =$$



because of charge conservation only the amplitudes $a(\theta_n' - u_j)$ appear in the S-matrices $\dot{S}(\theta_n' - u_j)$ which are constituents of the C-operators. Therefore we may shift all u_j-integration contours $\mathcal{C}_{\underline{\theta}'}$ to $\mathcal{C}_{\underline{\theta}}$ without changing the values of the integrals, because the functions $a(\theta_n' - u_j)\phi(\theta_n' - u_j)$ are holomorphic inside $\mathcal{C}_{\underline{\theta}'} - \mathcal{C}_{\underline{\theta}}$.

We now proceed as usual in the algebraic Bethe ansatz and push the $A^Q(\underline{\theta})$ and $D^Q(\underline{\theta})$ through all the C-operators using the commutation rules (B.4) and obtain

$$C(\underline{\theta}', u_1) \cdots C(\underline{\theta}', u_m)\, A^Q(\underline{\theta}) = \prod_{j=1}^{m} \frac{a(\theta_n' - u_j)}{b(\theta_n - u_j)}\, A^Q(\underline{\theta})\, C(\underline{\theta}, u_1) \cdots C(\underline{\theta}, u_m) + uw_A\,, \quad \text{(B.5)}$$

$$C(\underline{\theta}', u_1) \cdots C(\underline{\theta}', u_m)\, D^Q(\underline{\theta}) = \prod_{j=1}^{m} \frac{a(u_j - \theta_n)}{b(u_j - \theta_n')}\, D^Q(\underline{\theta})\, C(\underline{\theta}, u_1) \cdots C(\underline{\theta}, u_m) + uw_D\,. \quad \text{(B.6)}$$

The "wanted terms" written explicitly originate from the first term in the commutations rules (B.4); all other contributions yield the so-called "unwanted terms". If we insert these equations into the representation (4.10) of $f(\underline{\theta}')$ we find that the desired contribution from A^Q already gives the result we are looking for. The wanted contribution from D^Q applied to Ω gives zero. The unwanted contributions cancel after integration over the u_j. All these three facts can be seen as follows. We have

$$\Omega\, A^Q(\underline{\theta}) = \prod_{i=1}^{n} \dot{a}(\theta_i - \theta_n)\, \Omega\,, \quad \Omega\, D^Q(\underline{\theta}) = 0$$

which follow from eq. (4.6).

The relations (4.15) for $\phi(u)$ and (4.12) for $F(\theta)$ imply that the wanted term from A^Q yields $f(\underline{\theta})$. The commutation relations (B.3) and (B.4) imply that the unwanted terms are proportional to a product of C-operators, where exactly one $C(\underline{\theta}, u_j)$ is replaced by $C^Q(\underline{\theta})$. Because of the commutativity of the Cs it is sufficient to consider only the unwanted terms for $j = m$ which are denoted by $uw_A^m(\underline{u})$ and $uw_D^m(\underline{u})$. They come from the second term in (B.4) when $A^Q(\underline{\theta})$ is commuted with $C(\underline{\theta}, u_m)$. Then the resulting $A(\underline{\theta}, u_m)$

pushed through the other Cs and taking only the first terms in (B.3) into account and correspondingly for $D^Q(\underline{\theta}; u_m)$ we arrive at

$$uw_A^m(\underline{u}) = -\frac{c(\theta_n - u_m)}{b(\theta_n - u_m)} \prod_{j<m} \frac{a(u_m - u_j)}{b(u_m - u_j)} A(\underline{\theta}', u_m)\, C(\underline{\theta}', u_1) \dots C^Q(\underline{\theta})$$

$$uw_D^m(\underline{u}) = -\frac{c(u_m - \theta_n')}{b(u_m - \theta_n')} \prod_{j<m} \frac{a(u_j - u_m)}{b(u_j - u_m)} D(\underline{\theta}', u_m)\, C(\underline{\theta}', u_1) \dots C^Q(\underline{\theta}).$$

Using again (4.6) and the relations (4.15) and (4.16) for $\phi(u)$ and $\tau(u)$ we obtain

$$g(\underline{\theta}', \underline{u})\, \Omega\, uw_D^m(\underline{u}) = -g(\underline{\theta}', \underline{u}')\, \Omega\, uw_A^m(\underline{u}')$$

where also $c(u)/b(u) = -c(-u)/b(-u)$ has been used and $\underline{u}' = (u_1, \dots, u_m' = u_m + 2\pi i)$. Therefore after integration of the A-unwanted term along $\mathcal{C}_{\underline{\theta}}$ and the D-unwanted term along $\mathcal{C}_{\underline{\theta}'}$ both cancel.

Proof that eq. (4.10) fulfills (iii):

We will prove that eq. (4.10) fulfills (iii) (see eq. (3.17)) in the form of

$$\operatorname*{Res}_{\theta_{1n}=i\pi} f^{\mathcal{O}}_{1\dots n}(\theta_1, \dots, \theta_n) = -\sigma_{\mathcal{O}n}\, 2i\, \mathbf{C}_{1n}\, f^{\mathcal{O}}_{2\dots n-1}(\theta_2, \dots, \theta_{n-1}) \Big(\mathbf{1}_{2\dots n-1} - S_{2n} \cdots S_{n-1n}\Big)$$

which is equivalent to eq. (3.17) due to (i). We consider the n-particle form factor function given by eq. (4.10)

$$f^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = \prod_{i=1}^{m} \left(\int_{\mathcal{C}_{\underline{\theta}}} du_i \right) g_n(\underline{\theta}, \underline{u})\, \Omega_{1\dots n} C_{1\dots n}(\underline{\theta}, u_1) \cdots C_{1\dots n}(\underline{\theta}, u_m)$$

with the scalar function

$$\begin{aligned} g_n(\underline{\theta}, \underline{u}) \;\; = \;\; & \frac{N_n^{\mathcal{O}}}{N_{n-2}^{\mathcal{O}}}\, g_{n-2}(\underline{\tilde{\theta}}, \underline{\tilde{u}}) \\ & \times F(\theta_1 - \theta_n) \prod_{i=2}^{n-1} \Big(F(\theta_1 - \theta_i)\, F(\theta_i - \theta_n) \Big) \prod_{i=1}^{n} \phi(\theta_i - u_m) \\ & \times \prod_{j=1}^{m-1} \Big(\phi(\theta_1 - u_j)\, \phi(\theta_n - u_j)\, \tau(u_j - u_m) \Big)\, e^{\pm \tilde{s}(2u_m - \theta_1 - \theta_n)} \end{aligned}$$

where $\underline{\tilde{\theta}} = \theta_2, \dots, \theta_{n-1}$ and $\underline{\tilde{u}} = u_1, \dots, u_{m-1}$. We calculate the residue of this function at $\theta_1 = \theta_n + i\pi$. It consists of three terms

$$\operatorname*{Res}_{\theta_1=\theta_n+i\pi} f^{\mathcal{O}}_{1\dots n}(\underline{\theta}) = R_1 + R_2 + R_3$$

This is because each of the m integration contours will be "pinched" at three points:

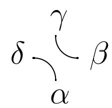
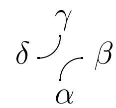
(1) $u_j = \theta_n = \theta_1 - i\pi$,

(2) $u_j = \theta_n + i\pi = \theta_1$

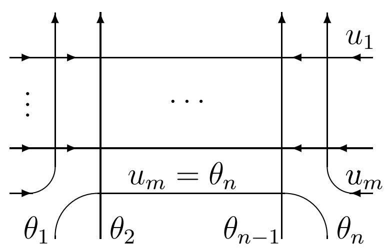
(3) $u_j = \theta_n - i\pi = \theta_1 - 2i\pi$.

Due to symmetry it is sufficient to determine the contribution from the u_m-integration and multiply the result by m.

The contribution of (1) is given by u_m-integration along the small circle around $u_m = \theta_n$ (see figure 4). The S-matrix $S(\theta_n - u_m)$ yields the permutation operator $S(0) = P$ and $S(\theta_1 - u_m)$ the annihilation-creation operator $S(i\pi) = K$

$$S^{\delta\gamma}_{\alpha\beta}(0) = \delta_{\alpha\delta}\,\delta_{\beta\gamma} =$$  $$, \qquad S^{\delta\gamma}_{\alpha\beta}(i\pi) = \mathbf{C}_{\alpha\beta}\,\mathbf{C}^{\delta\gamma} = \delta_{\alpha\bar\beta}\,\delta_{\delta\bar\gamma} =$$  .

Therefore we have for $u_m = \theta_n = \theta_1 - i\pi$

$$\Omega_{1\dots n}\,C_{1\dots n}(\underline{\theta}, u_1)\cdots C_{1\dots n}(\underline{\theta}, \theta_n) =$$ 

$$= \prod_{j=1}^{m-1}\Big(\dot b(\theta_1 - u_j)\dot a(\theta_n - u_j)\Big)\,\mathbf{C}_{1n}\,\Omega_{2\dots n-1}C_{2\dots n-1}(\underline{\tilde\theta}, u_1)\cdots$$
$$\cdots C_{2\dots n-1}(\underline{\tilde\theta}, u_{m-1})\,\dot S_{2n}\cdots\dot S_{n-1n}$$

where $\mathbf{C}_{1n}$ is the charge conjugation matrix with $\mathbf{C}_{\alpha\beta} = \delta_{\bar\alpha\beta}$. We have used the fact that because of charge conservation the amplitude $b(\cdot)$ only contributes to the S-matrices $S(\theta_1 - u_j)$ and $a(\cdot)$ to the S-matrices $S(\theta_n - u_j)$.

We combine this with the scalar function g_n and after having performed the remaining u_j-integrations we obtain

$$\begin{aligned}
R^{(1)}_{1\dots n} &= \mathbf{C}_{1n}\,f^{\mathcal{O}}_{2\dots n-1}(\underline{\tilde\theta})\,\dot S_{2n}\cdots\dot S_{n-1n}\\
&\times m\,\frac{N^{\mathcal{O}}_n}{N^{\mathcal{O}}_{n-2}}\operatorname*{Res}_{\theta_1=\theta_n+i\pi}(-2\pi i)\operatorname*{Res}_{u_m=\theta_n}\dot a(\theta_n - u_m)\phi(\theta_n - u_m)\,\dot b(\theta_1 - u_m)\phi(\theta_1 - u_m)\\
&\times F(i\pi)\prod_{i=2}^{n-1}\Big(F(\theta_1 - \theta_i)F(\theta_i - \theta_n)\phi(\theta_i - u_m)\Big)\\
&\times\prod_{j=1}^{m-1}\Big(\dot b(\theta_1 - u_j)\phi(\theta_1 - u_j)\dot a(\theta_n - u_j)\phi(\theta_n - u_j)\tau(u_j - u_m)\Big)\,e^{\pm\tilde s(2u_m - \theta_1 - \theta_n)}\\
&= 2i\,\sigma_{\mathcal{O}n}\,\mathbf{C}_{1n}\,f^{\mathcal{O}}_{2\dots n-1}(\underline{\tilde\theta})\,S_{2n}\cdots S_{n-1n}
\end{aligned}$$

if we relate the normalization constants by the recursion relation

$$N_n^{\mathcal{O}} = N_{n-2}^{\mathcal{O}} e^{\pm i\pi\tilde{s}} \frac{\left(f_{ss}^{min}(0)\right)^2}{4\pi m}\,. \tag{B.7}$$

We have also used that $\sigma_{\mathcal{O}n} = (-1)^n$ and

$$F(u)\,F(u+i\pi)\,\phi(u) = 1 \quad \text{and} \quad b(u+i\pi)\,\phi(u+i\pi)\,a(u)\,\phi(u)\,\tau(-u) = 1$$

which follows from the definitions (4.12) and (4.13). Finally we have used the normalization $F(i\pi) = 1$ and $\operatorname{Res}_{u=0} \dot{a}(u)\phi(u) = \operatorname{Res}_{u=i\pi} \dot{b}(u)\phi(u) = -2i/f_{ss}^{min}(0)$ because of eq. (4.14). Note also that the signs from the $\dot{a}$s and $\dot{b}$s cancel and $\sigma_{\mathcal{O}n} = (-1)^{n-2}$, since all particles are fermions.

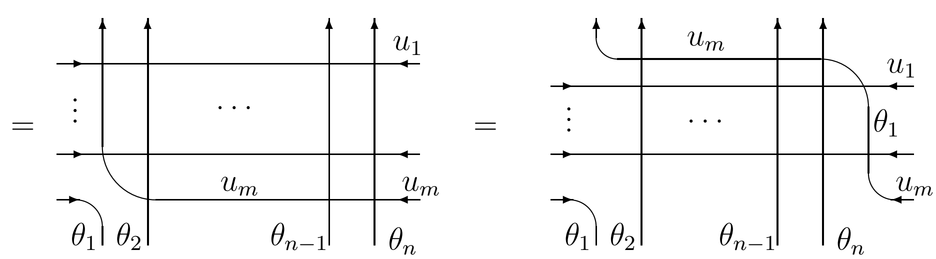
The remaining contribution to (iii) is due to R_2 and R_3

$$R_{1\ldots n}^{(2)} + R_{1\ldots n}^{(3)} = 2i\,\mathbf{C}_{1n}\, f_{2\ldots n-1}^{\mathcal{O}}(\underline{\tilde{\theta}}).$$

If both particles at 1 and n are solitons both vanish. If one particle at 1 or n is an soliton and the other an anti-soliton one term gives the desired expression and the other vanish. If both particles at 1 and n are anti-solitons both terms cancel. These fact can be proven as follows.

The contribution of (2) is given by the u_m-integration along the small circle around $u_M = \theta_1$ (see again figure 4). Now $S(\theta_1 - u_m)$ yields the permutation operator $S(0) = P$ and the co-vector part of this contribution for $u_m = \theta_1 = \theta_n + i\pi$ is

$$\Omega_{1\ldots n}\, C_{1\ldots n}(\underline{\theta}, u_1) \cdots C_{1\ldots n}(\underline{\theta}, u_m = \theta_1)\, P_n(s)$$



$$= \prod_{i=1}^{n} \dot{a}(\theta_i - u_m) \prod_{j=1}^{m-1} \Big(\dot{a}(\theta_1 - u_j)\dot{a}(\theta_n - u_j)\Big)$$
$$\times\, \mathbf{C}_{1n}\, \Omega_{2\ldots n-1} C_{2\ldots n-1}(\underline{\tilde{\theta}}, u_1) \cdots C_{2\ldots n-1}(\underline{\tilde{\theta}}, u_{m-1})\, P_1(\bar{s})\, P_n(s) \tag{B.8}$$

where the Yang-Baxter relation (2.8) has been used iteratively. $P_1(\bar{s})$ and $P_n(s)$ project onto the components where the particle at 1 is a soliton and at n is an anti-soliton, respectively. The remaining components if both particles at 1 and n are anti-solitons are calculated below.

We combine this with the scalar function g_n and after having performed the remaining u_j-integrations we obtain

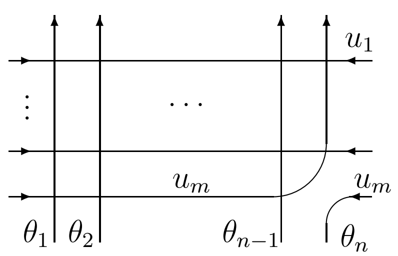
$$
\begin{aligned}
R^{(2)}_{1\ldots n}\, P_n(s) \;&=\; \mathbf{C}_{1n}\, f^{\mathcal{O}}_{2\ldots n-1}(\underline{\tilde\theta})\, P_1(\bar s)\, P_n(s) \\
&\quad \times m\, \frac{N^{\mathcal{O}}_n}{N^{\mathcal{O}}_{n-2}} \operatorname*{Res}_{\theta_1=\theta_n+i\pi} (-2\pi i) \operatorname*{Res}_{u_m=\theta_1} \dot a(\theta_1-u_m)\phi(\theta_1-u_m)\, \dot a(\theta_n-u_m)\phi(\theta_n-u_m) \\
&\quad \times F(i\pi) \prod_{i=2}^{n-1} \Big(F(\theta_1-\theta_i)F(\theta_i-\theta_n)\dot a(\theta_i-u_m)\phi(\theta_i-u_m) \Big) \\
&\quad \times \prod_{j=1}^{m-1} \Big(\dot a(\theta_1-u_j)\phi(\theta_1-u_j)\dot a(\theta_n-u_j)\phi(\theta_n-u_j)\tau(u_j-u_m) \Big)\, e^{\pm\tilde s(2u_m-\theta_1-\theta_n)} \\
&= \;\; -2i\, \sigma_{\mathcal{O}n}\, \mathbf{C}_{1n}\, f^{\mathcal{O}}_{2\ldots n-1}(\underline{\tilde\theta})\, P_1(\bar s)\, P_n(s)\ .
\end{aligned}
$$

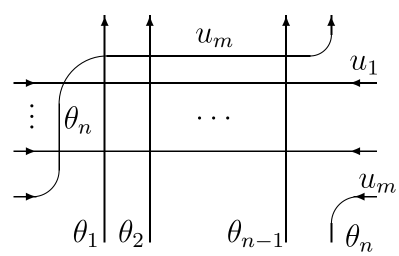
if we apply the condition $\exp(2\pi\tilde s) = (-1)^n = \sigma_{\mathcal{O}n}$ and if we relate the normalization constants as above. We have used the identities

$$
F(-u)\,F(u+i\pi)\,\dot a(u)\,\phi(u) = 1 \quad\text{and}\quad a(u)\,\phi(u)\,a(u-i\pi)\,\phi(u-i\pi)\,\tau(-u) = 1. \tag{B.9}
$$

The contribution of (3) is given by u_m-integration along the small circle around $u_m = \theta_n - i\pi$ (see again figure 4). Now $S(\theta_n - u_m)$ yields the annihilation-creation operator $S(i\pi) = K$ and the co-vector part of this contribution for $u_m = \theta_1 - 2\pi i = \theta_n - i\pi$ is

$$
\Omega_{1\ldots n}\, C_{1\ldots n}(\underline\theta, u_1)\cdots C_{1\ldots n}(\underline\theta, u_m=\theta_n-i\pi)\, P_1(s)
$$

$=$ 

$=$ 

$$
\begin{aligned}
&= \prod_{i=1}^{n} \dot b(\theta_i-u_m) \prod_{j=1}^{m-1} \Big(\dot b(\theta_1-u_j)\dot b(\theta_n-u_j) \Big) \\
&\qquad \times \mathbf{C}_{1n}\, \Omega_{2\ldots n-1} C_{2\ldots n-1}(\underline{\tilde\theta}, u_1)\cdots C_{2\ldots n-1}(\underline{\tilde\theta}, u_{m-1})\, P_1(s)\, P_n(\bar s)
\end{aligned} \tag{B.10}
$$

if the particle at 1 is a soliton and that at n an anti-soliton. This contribution obviously vanishes if n is a soliton. Again the remaining components if both particles at 1 and n are anti-solitons are calculated below.

Again we combine this with the scalar function g_n and after having performed the remaining u_j-integrations we obtain

$$
R^{(3)}_{1\ldots n}\, P_1(s) \;=\; \mathbf{C}_{1n}\, f^{\mathcal{O}}_{2\ldots n-1}(\underline{\tilde\theta})\, P_1(s)\, P_n(\bar s)
$$

$$
\begin{aligned}
&\times m\,\frac{N_n^{\mathcal{O}}}{N_{n-2}^{\mathcal{O}}}\,\operatorname*{Res}_{\theta_1=\theta_n+i\pi}(-2\pi i)\operatorname*{Res}_{u_m=\theta_n-i\pi}\dot{b}(\theta_n-u_m)\phi(\theta_n-u_m)\,\dot{b}(\theta_1-u_m)\phi(\theta_1-u_m)\\
&\times F(i\pi)\prod_{i=2}^{n-1}\Big(F(\theta_1-\theta_i)F(\theta_i-\theta_n)\dot{b}(\theta_i-u_m)\phi(\theta_i-u_m)\Big)\\
&\times\prod_{j=1}^{m-1}\Big(\dot{b}(\theta_1-u_j)\phi(\theta_1-u_j)\dot{b}(\theta_n-u_j)\phi(\theta_n-u_j)\tau(u_j-u_m)\Big)\,e^{\pm\bar{s}(2u_m-\theta_1-\theta_n)}\\
=\quad&-2i\,\sigma_{\mathcal{O}n}\,\mathbf{C}_{1n}\,f^{\mathcal{O}}_{2\ldots n-1}(\underline{\tilde{\theta}})\,P_1(s)\,P_n(\bar{s})
\end{aligned}
$$

provided that we fix the normalization constants as above. We have used the identities

$$
\dot{b}(u)\,F(-u)\,F(u+i\pi)\,\phi(u)=1\quad\text{and}\quad a(u)\,\phi(u)\,a(u-i\pi)\,\phi(u-i\pi)\,\tau(-u)=1. \tag{B.11}
$$

Finally we calculate $R^{(2)}_{1\ldots n}+R^{(3)}_{1\ldots n}$ for the case that both particles at 1 and at n are anti-solitons. Instead of eq. (B.8) we have now for $u_m=\theta_1=\theta_n+i\pi$

$$
\begin{aligned}
&\prod_{i=1}^{n}\dot{a}(\theta_i-u_m)\prod_{j=1}^{m-1}\Big(\dot{a}(\theta_1-u_j)\dot{a}(\theta_n-u_j)\Big)\frac{c(\theta_n-u_{m-1})}{a(\theta_n-u_{m-1})}\\
&\qquad\times\,\Omega_{2\ldots n-1}C_{2\ldots n-1}(\underline{\tilde{\theta}},u_1)\cdots C_{2\ldots n-1}(\underline{\tilde{\theta}},u_{m-2})\,D_{2\ldots n-1}(\underline{\tilde{\theta}},u_{m-1})\,P_1(\bar{s})\,P_n(\bar{s})+\ldots
\end{aligned}
$$

and because of the Yang-Baxter relations (B.6)

$$
\begin{aligned}
&C(\underline{\tilde{\theta}},u_1)\cdots C(\underline{\tilde{\theta}},u_{m-2})\,D(\underline{\tilde{\theta}},u_{m-1})\\
&\qquad=\prod_{j=1}^{m-2}\frac{a(u_j-u_{m-1})}{b(u_j-u_{m-1})}\,D(\underline{\tilde{\theta}},u_{m-1})\,C(\underline{\tilde{\theta}},u_1)\cdots C(\underline{\tilde{\theta}},u_{m-2})+\ldots
\end{aligned}
$$

where the dots refer to similar terms with $D_{2\ldots n-1}(\underline{\tilde{\theta}},u_j)$, $(j<m-1)$. Because of symmetry with respect to the C-operators it is sufficient to consider only this term.

Similarly we get instead of eq. (B.10) $u_m=\theta_1-2\pi i=\theta_n-i\pi$

$$
\begin{aligned}
&\prod_{i=1}^{n}\dot{b}(\theta_i-u_m)\prod_{j=1}^{m-1}\Big(\dot{a}(\theta_1-u_j)\dot{a}(\theta_n-u_j)\Big)\frac{c(\theta_1-u_{m-1})}{b(\theta_1-u_{m-1})}\\
&\qquad\times\,\Omega_{2\ldots n-1}C_{2\ldots n-1}(\underline{\tilde{\theta}},u_1)\cdots C_{2\ldots n-1}(\underline{\tilde{\theta}},u_{m-2})\,A_{2\ldots n-1}(\underline{\tilde{\theta}},u_{m-1})\,P_1(\bar{s})\,P_n(\bar{s})+\ldots
\end{aligned}
$$

and because of the Yang-Baxter relations (B.6)

$$
\begin{aligned}
&C(\underline{\tilde{\theta}},u_1)\cdots C(\underline{\tilde{\theta}},u_{m-2})\,A(\underline{\tilde{\theta}},u_{m-1})\\
&\qquad=\prod_{j=1}^{m-2}\frac{a(u_{m-1}-u_j)}{b(u_{m-1}-u_j)}\,A(\underline{\tilde{\theta}},u_{m-1})\,C(\underline{\tilde{\theta}},u_1)\cdots C(\underline{\tilde{\theta}},u_{m-2})+\ldots
\end{aligned}
$$

where the dots again refer to similar terms with $A_{2\ldots n-1}(\underline{\tilde{\theta}}, u_j)$, $(j < m-1)$. We apply $D_{2\ldots n-1}(\underline{\tilde{\theta}}, u_{m-1})$ and $A_{2\ldots n-1}(\underline{\tilde{\theta}}, u_{m-1})$ to the pseudo-vacuum, use as above the identities (B.9) and (B.11), and find that the sum $R^{(2)} + R^{(3)}$ is proportional to (with $u = u_{m-1}$)

$$\int_{\mathcal{C}_{\underline{\tilde{\theta}}}} du \left\{ \frac{c(\theta_n - u)}{a(\theta_n - u)} \prod_{i=2}^{n-1} \dot{b}(\theta_i - u)\phi(\theta_i - u) \prod_{j=1}^{m-2} \frac{a(u_j - u)}{b(u_j - u)} \tau(u_j - u) \right.$$
$$\left. + \frac{c(\theta_1 - u)}{b(\theta_1 - u)} \prod_{i=2}^{n-1} \dot{a}(\theta_i - u)\phi(\theta_i - u) \prod_{j=1}^{m-2} \frac{a(u - u_j)}{b(u - u_j)} \tau(u_j - u) \right\} = I$$

Due to crossing we have

$$\frac{c(\theta_n - u_{m-1})}{a(\theta_n - u_{m-1})} = -\frac{c(\theta_1 - u_{m-1} - 2\pi i)}{b(\theta_1 - u_{m-1} - 2\pi i)}.$$

In addition we use the quasi-periodicity properties (4.15) and (4.16) of $\phi(\cdot)$ and $\tau(\cdot)$ and get

$$\begin{aligned} I &= \left\{ -\int_{\mathcal{C}_{\underline{\tilde{\theta}}} + 2\pi i} + \int_{\mathcal{C}_{\underline{\tilde{\theta}}}} \right\} du_{m-1} \frac{c(\theta_1 - u_{m-1})}{b(\theta_1 - u_{m-1})} \\ & \quad \prod_{i=2}^{n-1} \dot{a}(\theta_i - u_{m-1})\phi(\theta_i - u_{m-1}) \prod_{j=1}^{m-2} \frac{a(u_{m-1} - u_j)}{b(u_{m-1} - u_j)} \tau(u_j - u_{m-1}) \end{aligned}$$

which vanishes since the integrand is holomorphic inside the contour $\mathcal{C}_{\underline{\tilde{\theta}}} - (\mathcal{C}_{\underline{\tilde{\theta}}} + 2\pi i)$.

C Some useful formulae

In this appendix we provide some explicit formulae (which partly may also be found elsewhere in the literature) for typical scattering matrices, minimal form factors and some auxiliary functions which we frequently employed in the explicit computations. We state some typical integral representation, which are very useful since via (4.19) and (4.20) they relate the scattering matrix and the minimal form factors effortlessly. The infinite product representations in terms of Gamma functions, obtained from the evaluation of the integrals or the direct solution of the functional relations, make the singularity structure more transparent. For numerical purposes it is often more useful to express the Gamma functions with the help of Euler's product representation in terms of rational functions at the cost of an additional infinite product.

A typical S-matrix eigenvalue is (for $a > 0$)

$$S(i\pi x, a) = \frac{a + x}{a - x} = \exp \int_0^\infty \frac{dt}{t}\, 2e^{-ta} \sinh tx\ .$$

According to (4.19) and (4.20) the corresponding minimal form factor function is therefore

$$
\begin{aligned}
f^{min}(i\pi x,a) &= \exp\int_0^\infty \frac{dt}{t}\,2e^{-ta}\frac{1-\cosh t(1-x)}{2\sinh t}\\
&= \prod_{l=0}^{\infty}\frac{(2l+2+a-x)(2l+a+x)}{(2l+1+a)^2} = \frac{\Gamma^2(\frac{1}{2}+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-\frac{x}{2})\Gamma(\frac{a}{2}+\frac{x}{2})}.
\end{aligned}
$$

In particular, for $a=0$ we recover the scattering matrix of the Ising model

$$
S=-1 \quad\to\quad f^{min}(i\pi x)=\sin\frac{\pi}{2}x
$$

For negative values of a we use $S(\theta,a)=1/S(\theta,-a)$ and $f^{min}(\theta,a)=1/f^{min}(\theta,-a)$. A further typical S-matrix eigenvalue[3] is (for $0<a<1$)

$$
S(i\pi x,a)=\frac{\sin\frac{\pi}{2}(a+x)}{\sin\frac{\pi}{2}(a-x)}=\exp\int_0^\infty\frac{dt}{t}\,2\frac{\sinh t(1-a)}{\sinh t}\sinh tx\ .
$$

with the corresponding minimal form factor function

$$
\begin{aligned}
f^{min}(i\pi x,a) &= \exp\int_0^\infty\frac{dt}{t}\,2\frac{\sinh t(1-a)}{\sinh t}\,\frac{1-\cosh t(1-x)}{2\sinh t}\\
&= \prod_{k=0}^{\infty}\prod_{l=0}^{\infty}\frac{2l+2k+a+x}{2l+2k+2-a+x}\,\frac{2l+2k+2+a-x}{2l+2k+4-a-x}\left(\frac{2l+2k+3-a}{2l+2k+1+a}\right)^2\\
&= \prod_{k=0}^{\infty}\frac{\Gamma(k+1-\frac{a}{2}+\frac{x}{2})}{\Gamma(k+\frac{a}{2}+\frac{x}{2})}\,\frac{\Gamma(k+2-\frac{a}{2}-\frac{x}{2})}{\Gamma(k+1+\frac{a}{2}-\frac{x}{2})}\left(\frac{\Gamma(k+\frac{1}{2}+\frac{a}{2})}{\Gamma(k+\frac{3}{2}-\frac{a}{2})}\right)^2\ .
\end{aligned}
$$

The sine-Gordon soliton-soliton S-matrix reads

$$
\begin{aligned}
&S_{ss}(i\pi x)=a(i\pi x)=\exp\int_0^\infty\frac{dt}{t}\,\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t}\,\sinh tx\\
&= \prod_{k=0}^{\infty}\prod_{l=0}^{\infty}\frac{2l+\nu+k\nu+x}{2l+\nu+k\nu-x}\,\frac{2l+2+k\nu+x}{2l+2+k\nu-x}\,\frac{2l+1+\nu+k\nu-x}{2l+1+\nu+k\nu+x}\,\frac{2l+1+k\nu-x}{2l+1+k\nu+x}\\
&= \prod_{k=0}^{\infty}\frac{\Gamma((\nu+k\nu-x)/2)}{\Gamma((\nu+k\nu+x)/2)}\,\frac{\Gamma((2+k\nu-x)/2)}{\Gamma((2+k\nu+x)/2)}\,\frac{\Gamma((1+\nu+k\nu+x)/2)}{\Gamma((1+\nu+k\nu-x)/2)}\,\frac{\Gamma((1+k\nu+x)/2)}{\Gamma((1+k\nu-x)/2)}
\end{aligned}
$$

with

$$
a(i\pi(x+\nu))=-\cot\frac{\pi}{2}x\,\cot\frac{\pi}{2}(x+\nu)\,a(i\pi x).
$$

Consequently, the minimal form factor function is

[3] For instance almost all diagonal scattering matrices related to perturbation of certain conformal field theories may be built out of these elementary blocks.

$$
\begin{aligned}
f_{ss}^{min}(i\pi x) &= \exp\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t}\frac{1-\cosh t(1-x)}{2\sinh t}\\
&= \prod_{k=0}^\infty\prod_{l=0}^\infty\prod_{m=0}^\infty \frac{2m+2l+2+\nu+k\nu-x}{2m+2l+3+\nu+k\nu-x}\,\frac{2m+2l+\nu+k\nu+x}{2m+2l+1+\nu+k\nu+x}\\
&\qquad\times\frac{2m+2l+4+k\nu-x}{2m+2l+3+k\nu-x}\,\frac{2m+2l+2+k\nu+x}{2m+2l+1+k\nu+x}\\
&\qquad\times\left(\frac{2m+2l+2+\nu+k\nu}{2m+2l+1+\nu+k\nu}\,\frac{2m+2l+2+k\nu}{2m+2l+3+k\nu}\right)^2\\
&= \prod_{k=0}^\infty\prod_{l=0}^\infty \frac{\Gamma\Big(l+\frac{1}{2}(3+\nu+k\nu-x)\Big)}{\Gamma\Big(l+1+\frac{1}{2}(\nu+k\nu-x)\Big)}\,\frac{\Gamma\Big(l+\frac{1}{2}(1+\nu+k\nu+x)\Big)}{\Gamma\Big(l+\frac{1}{2}(\nu+k\nu+x)\Big)}\\
&\qquad\times\frac{\Gamma\Big(l+\frac{1}{2}(3+k\nu-x)\Big)}{\Gamma\Big(l+2+\frac{1}{2}(k\nu-x)\Big)}\,\frac{\Gamma\Big(l+\frac{1}{2}(1+k\nu+x)\Big)}{\Gamma\Big(l+1+\frac{1}{2}(k\nu+x)\Big)}\\
&\qquad\times\frac{\Gamma^2\Big(l+\frac{1}{2}(1+\nu+k\nu)\Big)}{\Gamma^2\Big(l+1+\frac{1}{2}(\nu+k\nu)\Big)}\,\frac{\Gamma^2\Big(l+\frac{1}{2}(3+k\nu)\Big)}{\Gamma^2\Big(l+1+\frac{1}{2}k\nu\Big)}
\end{aligned}
$$

with asymptotic behavior for $|\Re\theta|\to\infty$, $(|\Im\theta-\pi|<\frac{\pi}{2}(3+\nu-|1-\nu|))$

$$
f_{ss}^{min}(i\pi-\theta) = c_{ss}\left(e^{\frac{1-\nu}{4\nu}|\theta|}+o(1)\right)
$$

with the constant

$$
c_{ss} = \exp\frac{1}{2}\int_0^\infty\frac{dt}{t}\left(\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t\,\sinh t}-\frac{1-\nu}{\nu t}\right).
$$

The corresponding functions $\phi(u)=\Big(F(u)F(i\pi+u)\Big)^{-1}$ and $\tau(u)=\Big(\phi(u)\phi(-u)\Big)^{-1}$ with $F(i\pi x)=\sin(\frac{\pi}{2}x)f_{ss}^{min}(i\pi x)$ are

$$
\begin{aligned}
\phi(i\pi x) &= \frac{1}{F^2\left(\frac{i\pi}{2}\right)}\frac{1}{\sin(\pi x)}\exp\int_0^\infty\frac{dt}{t}\frac{\sinh\frac{1}{2}(1-\nu)t\left(\cosh t(\frac{1}{2}-x)-1\right)}{\sinh\frac{1}{2}\nu t\sinh t}\\
&= \frac{1}{F^2\left(\frac{i\pi}{2}\right)}\prod_{k=0}^\infty\prod_{l=0}^\infty\frac{2l+1+\nu+k\nu+x}{2l+k\nu+x}\,\frac{2l+2+\nu+k\nu-x}{2l+1+k\nu-x}\left(\frac{2l+k\nu+\frac{1}{2}}{2l+\nu+k\nu+\frac{3}{2}}\right)^2\\
&= \frac{1}{F^2\left(\frac{i\pi}{2}\right)}\prod_{k=0}^\infty\frac{\Gamma\Big(\frac{1}{2}(k\nu+x)\Big)}{\Gamma\Big(\frac{1}{2}(1+\nu+k\nu+x)\Big)}\,\frac{\Gamma\Big(\frac{1}{2}(1+k\nu-x)\Big)}{\Gamma\Big(\frac{1}{2}(2+\nu+k\nu-x)\Big)}\,\frac{\Gamma^2\left(\frac{\nu+k\nu}{2}+\frac{3}{4}\right)}{\Gamma^2\left(\frac{k\nu}{2}+\frac{1}{4}\right)}
\end{aligned}
$$

with

$$
\phi(i\pi(x+\nu)) = \frac{\sin\frac{\pi}{2}x}{\cos\frac{\pi}{2}(x+\nu)}\,\phi(i\pi x)
$$

and

$$\tau(i\pi x) = \frac{F^2(i\pi/2)F^2(-i\pi/2)}{\sin\frac{\pi}{2\nu}}\ \sin\pi x\ \sin(\pi x/\nu)\,.$$

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