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Braid Relations in Affine Toda Field Theory

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Abstract

We provide explicit realizations for the operators which when exchanged give rise to the scattering matrix. For affine Toda field theory we present two alternative constructions, one related to a free bosonic theory and the other formally to the quantum affine Heisenberg algebra of $U_q(\hat{Sl}_2)$.

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1 Introduction

Integrable theories in 1+1 space-time dimensions represent the hitherto best understood examples of relativistic quantum field theories. Whereas for the classical theories integrable refers to the existence of an infinite number of conserved quantities, such that the equations of motion are solvable, in the quantum theory one expects that the eigenvalue spectrum of the Hamiltonian can be determined.

Conformal field theories are particularly well understood examples of this concept. They may be viewed in two alternative ways, either one may view the massive theories as perturbations of the conformally invariant theories [1] and thus explain the origin of mass, or one may regard the massless theories as limiting cases of the former. Toda field theories [2] are relativistically invariant scalar field theories which provide elegant unifying examples allowing for both viewpoints. Many of its characteristics reflect properties of the underlying Lie or affine Lie algebra for the massless or massive case already at the classical level, respectively. Due to the remarkable renormalisation properties many quantities survive the quantization procedure, in particular the classical mass ratios for theories related to simply laced Lie algebras.

Since the formulation of the Wightman axioms it has been attempted to set up axiomatic systems which are so restrictive, that they allow for a unique construction of an exact quantum field theory, opposed to a perturbative approach. The bootstrap program [3] provides such a system and permits to determine the scattering matrix. This approach may in principle be extended to the situation off-shell [4, 5] and allows to compute form-factors and hence correlation functions. Recently the extension of this approach to the situation in which boundaries are present has attracted much attention.

All axioms may be derived when one postulates the existence of a certain set of operators, which implies that they already contain all the information of the theory. It seems therefore desirable to obtain explicit representations for these operators and illuminate their structure, which in particular with regard to the form-factors may lead to their explicit construction.

The paper is organized as follows: In section 2 we state the general properties of the operators and in section 3 we demonstrate how they may be employed to derive the axiomatic system for the scattering matrix. We illustrate this approach with the explicit example of affine Toda field theory for which we present in section 4 a derivation of the expression for the S-matrix as a phase and in section 5 and 6 explicit realizations of these operators. In section 7 we state our conclusions.

2 Zamolodchikov algebra

About sixteen years ago it was suggested by Zamolodchikov and Zamolodchikov [3] that one may place a postulate concerning the existence of a certain set of creation and annihilation type operators $Z_i(\theta_i)$ and $Z_i^\dagger(\theta_i)$ at the center of the formulation of a massive quantum field theory in 1+1 space-time dimensions. These operators are associated to the particles of the theory in a one-to-one fashion and are characterized by their quantum number i and their dependence on the rapidity θ_i. It is common to parameterize the two-momenta of the particle as $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$ since then the branch cuts in the complex Mandelstam s-plane unfold. Any product of these operators is thought to constitute an element of an associative algebra. Assuming the existence of a vacuum $|0\rangle$ one may construct a Hilbert space by a successive action of $Z_i^\dagger(\theta_i)$ on it

$$Z_1^\dagger(\theta_1)\ldots Z_n^\dagger(\theta_n)|0\rangle \quad . \tag{2.1}$$

Analogously one defines the dual space to this by acting with $Z_i(\theta_i)$ successively from the right on $\langle 0|$. For the in- and out-states one can then select a set of linearly independent vectors in the way, that one orders the in-states with decreasing and the out-states with increasing real parts of the rapidities. Every state which is not ordered in this way is regarded as being neither an in- nor an out-state.

Then the n-particle scattering matrix is defined to be the unitary matrix which relates the in- and the out-states

$$Z_n^\dagger(\theta_n)\ldots Z_1^\dagger(\theta_1)|0\rangle_{\rm out} = \prod_{1\le i<j\le n} S_{ij}(\theta_i-\theta_j) Z_1^\dagger(\theta_1)\ldots Z_n^\dagger(\theta_n)|0\rangle_{\rm in} \quad , \tag{2.2}$$

with $\mathrm{Re}(\theta_1) > \ldots > \mathrm{Re}(\theta_n)$. Here we have assumed that the quantum numbers of the in- and out-state do not differ, which implies that the S-matrix is diagonal. This is the case for the theories we are concerned with in the following, i.e. affine Toda field theory with real coupling constant. Furthermore we have employed the fact that in an integrable theory the n-particle S-matrices factorize into $n(n-1)/2$ two particle-S-matrices and it will therefore be sufficient to characterize their properties. Thus one may view the n-particle scattering as the outcome of the re-ordering of the operators $Z^\dagger$ or more specifically for an integrable theory one may regard the two-particle scattering matrix as the result of the braiding of two of these operators

$$Z_i(\theta_i)Z_j(\theta_j) \;=\; S_{ij}(\theta_i-\theta_j)\, Z_j(\theta_j)Z_i(\theta_i) \tag{2.3}$$

$$Z_i^\dagger(\theta_i)Z_j^\dagger(\theta_j) \;=\; S_{ij}(\theta_i-\theta_j)\, Z_j^\dagger(\theta_j)Z_i^\dagger(\theta_i) \;\;. \tag{2.4}$$

It remains to be specified what the braiding of Z with $Z^\dagger$ will lead to. Viewing these operators as a generalization of a Bose-Fermi algebra, one may postulate that

$$Z_i(\theta_i)Z_j^\dagger(\theta_j) = S_{ij}(\theta_j-\theta_i)\, Z_j^\dagger(\theta_j)Z_i(\theta_i) + \;\delta_{ij}\;\delta(\theta_i-\theta_j)\,. \tag{2.5}$$

In this case one recovers for $S \to \pm 1$ the usual bosonic commutation and fermionic anticommutation relations, respectively. This limit may be obtained either by letting the rapidity go to zero or the effective coupling to the value corresponding to the free theory. Viewing the asymptotic states as representations of the Z's, we may interpret $Z_i^\dagger(\theta)$ and $Z_i(\theta)$ as particle creation and annihilation operators, respectively. Besides the fact that one needs relations connecting the Hilbert space with its dual, the latter equation may be employed as well for the derivation of form-factor axioms [4, 5, 6, 7], that is in particular for the derivation of the kinematic residue equation (refer appendix of [5]).

In addition to the braiding properties we require a characterization of the singularity structure of the Z's in order to obtain a complete description. Annihilation processes are characterized by the operator product expansion

$$\lim_{\bar\theta\to\theta} Z_i(\bar\theta + i\pi)Z_{\bar\imath}(\theta) = c_i \;\;, \tag{2.6}$$

with c_i being some constant. Clearly the same relations hold for $Z_i^\dagger(\theta)$.

Possible fusing processes, $Z_i + Z_j \to Z_k$, are incorporated via the following operator product expansion

$$\lim_{\bar\theta\to\theta}(\bar\theta-\theta)Z_i\left(\bar\theta + i\bar\eta_{ik}^j\right)\; Z_j\left(\theta - i\bar\eta_{kj}^i\right) = \Gamma_{ij}^k Z_k(\theta) \;\;. \tag{2.7}$$

Here the $\bar{\eta}_{ij}^k = \pi - \eta_{ij}^k$ denote the fusing angles for bound states and Γ_{ij}^k the three-particle vertex on mass-shell defined via

$$\text{Res } \mathrm{S_{ij}}(\theta)|_{\mathrm{i}\eta_{\mathrm{ij}}^{\mathrm{k}}} = \left(\Gamma_{\mathrm{ij}}^{\mathrm{k}}\right)^2 \quad . \tag{2.8}$$

One may still find solutions to the preceding set of equations which will however lack a definite interpretation as a quantum field theory. One further restriction emerges from the existence of higher order poles which have to be given an interpretation in form of a generalization of the Coleman-Thun mechanism [8, 9]. On the level of the scattering matrix it is in general a very cumbersome procedure to verify whether all the required graphs exist and a more concise argumentation is highly desirable. For the operators of the Zamolodchikov algebra we expect for a second order poles

$$\text{Res}|_{\theta=\theta'} \; Z_i(\theta + i\eta) Z_j(\theta' - i\eta') \;=\; \Gamma_{i\bar{k}}^l \Gamma_{jk}^m \; Z_l(\theta) Z_m(\theta) \tag{2.9}$$

whereas a third order pole may be incorporated into

$$\text{Res}|_{\theta=\theta'} \; Z_i(\theta + i\tilde{\eta}) Z_j(\theta' - i\tilde{\eta}') \;=\; \Gamma_{i\bar{k}}^l \Gamma_{jk}^m \Gamma_{lm}^n \; Z_n(\theta) \quad . \tag{2.10}$$

The angles $\eta + \eta'$ and $\tilde{\eta} + \tilde{\eta}'$ correspond to the second and third order pole, respectively, in the scattering matrix S_{ij} and have definite relations to the more fundamental fusing angles which occur in (2.8). In general one obtains for an N^{th} order pole

$$\text{Res}|_{\theta=\theta'} \; Z_i(\theta + i\hat{\eta}) Z_j(\theta' - i\hat{\eta}') \;=\; \Gamma_1 \dots \Gamma_N \prod_l Z_l(\theta) \quad . \tag{2.11}$$

Further restrictions on possible fusing processes and therefore on the operators result when a symmetry is present in the theory. For instance invariance of the operator product expansion under $Z(N)$-symmetry, i.e. $Z_l(\theta) \to \exp \frac{l\, 2\pi i}{N} Z_l(\theta)$ [10] will put severe restrictions on the possible labels for the Z's. In affine Toda theories these symmetries may all be identified with the automorphisms of the Coxeter-Dynkin diagrams.

The associativity of the algebra (2.3) - (2.5) together with the operator product expansion imply certain consistency conditions for the two-particle S-matrix, like the Yang-Baxter equations [11] and the bootstrap equations [3], as we shall demonstrate in the next section. It is worth noting that this ideas extend to the case in which we have a reflecting boundary in space simply by adding one more generator to the algebra which then represents a wall or a defect [12]. These operators are however not quite on the same footing, since for instance for the case of pure reflection, they will always remain on the right in every product.

Having broken the conformal symmetry one nonetheless expects the presence of an operator, say d', which generates a Lorentz boost

$$e^{\Delta d'} Z_i(\theta) e^{-\Delta d'} \;=\; Z_i(\theta+\Delta) \;\; . \tag{2.12}$$

This is realized if it satisfies the relation

$$[d', Z_i(\theta)] \;=\; \frac{d}{d\theta} Z_i(\theta) \quad . \tag{2.13}$$

In addition, due to the integrability of the theory under consideration, we presume that by integrating a certain set of local densities we obtain an infinite set of conserved, real and commuting charges Q_s, which are graded by their spin s. The possible values of this spins depend naturally on the theory under consideration, for instance in affine Toda field theories, the values of this spins are known to equal the exponents modulo the Coxeter number h of the underlying Lie algebra [13] at the classical level. The charges are expected to act in the space of states (2.2) as well and are assumed to diagonalise them [3]

$$Q_s \; Z_1^\dagger(\theta_1) \ldots Z_n^\dagger(\theta_n) |0\rangle_{\substack{\text{in} \\ \text{out}}} \;=\; \sum_{i=1}^{n} \omega_s^i(\theta_i) \; Z_1^\dagger(\theta_1) \ldots Z_n^\dagger(\theta_n) |0\rangle_{\substack{\text{in} \\ \text{out}}} \quad . \tag{2.14}$$

Carrying out a spatial reflection relates positive and negative spins and we may derive the important relation for the eigenvalues of Q_s on the one-particle states

$$\omega_s^i(\theta_i) \;=\; \omega_{-s}^i(-\theta_i) \quad . \tag{2.15}$$

Acting with Q_s on the representation of (2.7) we immediately obtain a relation found originally in [14]

$$\omega_s^i(i\bar{\eta}_{ik}^j) + \omega_s^j(-i\bar{\eta}_{kj}^i) = \omega_s^k(0) \;\; . \tag{2.16}$$

On the other hand, acting on the representation of (2.6) one obtains

$$\omega_s^i(0) \;=\; (-1)^{1+s} \omega_s^{\bar{\imath}}(0) \;\; , \tag{2.17}$$

which yields the well known relation for the electrical charge with $s=0$ and the masses with $s=1$. This equation may be employed as a very powerful tool, i.e. knowing the values of the spins s one is able to solve these equations for the possible values of $\bar{\eta}$ and hence construct the S-matrix via the bootstrap principle [1].

Although the concept described above is relatively old, it has remained on a rather formal level and explicit realizations of the Zamolodchikov algebra, in particular for Lagrangian field theories, have hitherto been found only for few theories [15, 16, 17, 18, 19, 20].

Evidently such an algebra will crucially depend on the explicit nature of the theory and in order to proceed with an explicit construction one has to consider a concrete theory. Nonetheless, without possessing any explicit realization, one is able to make a few model independent statements. It is instructive to see how the previous relations lead to axiomatic systems for the on-shell quantities of any integrable theory. This may also be extended to the situation off-shell and a realization of the algebra may be employed to construct explicit solutions for instance for the form-factors in form of trace functions [18].

3 S-matrix axioms

For the case at hand, an integrable quantum field theory, the on-shell properties are characterized by the two-particle scattering matrix for which the axioms arise in a straightforward manner. Parity invariance and "unitarity" may be obtained without having any explicit realization for the Z's in a fairly simple way. Employing (2.3) twice for $Z_i(\theta_i)\ Z_j(\theta_j)$ yields

$$S_{ij}(\theta_{ij})S_{ji}(\theta_{ji}) \;=\; 1 \quad . \tag{3.1}$$

This equations may be satisfied if both parity, i.e. symmetry in i and j and "unitarity" hold

$$S_{ij}(\theta_{ij}) \;=\; S_{ji}(\theta_{ij}) \;=\; S_{ij}(\theta_{ji})^{-1} \quad . \tag{3.2}$$

By Lorentz invariance, the scattering matrix depends only on the rapidity difference which we denote from now on by $\theta_{ij} := \theta_i - \theta_j$. Similarly straightforward is to derive the crossing relation, which relates the scattering of particle i and j to the scattering of i with $\bar{\jmath}$. Considering

$$Z_j(\theta_j + i\pi)Z_{\bar{\jmath}}(\theta_j)Z_i(\theta_i) \tag{3.3}$$

we may by associativity either annihilate the Z_j and $Z_{\bar{\jmath}}$ first by relation (2.6) or commute the $Z_i(\theta_i)$ to the left and then contract. Equating the results implies the crossing relation for the S-matrix

$$S_{i\bar{j}}(\theta_{ij} + i\pi) S_{ij}(\theta_{ij}) \ = \ 1 \ \ . \tag{3.4}$$

Finally one may derive the bootstrap equation for the scattering matrix

$$S_{li}\left(\theta + i\bar{\eta}_{ik}^{j}\right) \ S_{lj}\left(\theta - i\bar{\eta}_{kj}^{i}\right) \ = \ S_{lk}\left(\theta\right) \quad . \tag{3.5}$$

By considering the product

$$Z_i\left(\bar{\theta} + i\bar{\eta}_{ik}^{j}\right) \ Z_j\left(\tilde{\theta} - i\bar{\eta}_{kj}^{i}\right) \ Z_l(\theta') \ \ , \tag{3.6}$$

we may either commute Z_l with Z_i, Z_j and then compute the residue for $\bar{\theta} \to \tilde{\theta}$ or alternatively contract first Z_i and Z_j and subsequently commute Z_k with Z_l. Then upon denoting $\theta = \tilde{\theta} - \theta'$ and equating the results leads directly to the bootstrap equation for the S-matrix.

When the residue (2.8) is positive, having assumed a bosonic theory, the poles $i\eta_{ij}^{k}$ are related to the bound state Z_k in the direct channel of the process $Z_i + Z_j$, satisfying the usual relation for the masses of the fusing particles [3]

$$m_k^2 = m_i^2 + m_j^2 + 2 m_i m_j \cos \eta_{ij}^{k} \ \ . \tag{3.7}$$

For this equation to hold it is crucial that all masses renormalise uniformly, i.e. the classical mass ratios pertain. This is a somewhat unappealing feature which still plagues the bootstrap approach, that one still has to resort to perturbative arguments in order to judge whether a solution to the set of axioms is suitable or not. One further restriction emerges for theories with higher order poles, like for instance affine Toda field theories in which we may have poles up to order 12. Then the Coleman-Thun mechanism [8] may be generalized to the extend that all odd order poles with positive residues are permitted to participate in the bootstrap [9, 21]. The bootstrap equations corresponding to this poles may be derived from (3.6), after suitably re-interpreting the fusing angles, together with (2.9) - (2.11) in the same fashion. Hence one has excluded all redundant poles and zeros. Further ambiguities, the so-called CCD-factors [22], may be excluded by the additional assumption that $-i \ln S(\theta)$ is damped for $\theta \to \infty$.

In a similar fashion one may employ the associativity of the algebra and derive Yang-Baxter equations, which however, due to the diagonality of the two particle S-matrices we are concerned with, only contain trivial information and may therefore be ignored in the present context.

4 The S-matrix as a phase

For our purpose and in various other contexts, like for instance when computing finite size effects via the Thermodynamic Bethe Ansatz, one requires the S-matrix to be represented in form of a phase. In this section we therefore present an explicit derivation of it. Adopting the notation of [23, 24] the S-matrix of affine Toda field theories related to simply laced Lie algebras [25, 10, 1, 26, 27, 28, 9, 21, 29, 24, 30] may be cast into the very compact form

$$S_{ij}(\theta) \;=\; \prod_{q=1}^{h} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_{\theta}^{-\frac{1}{2}\lambda_i\cdot\sigma^q\gamma_j} \;=\; \frac{\chi_{ij}(\theta)}{\chi_{ij}(-\theta)} \quad , \tag{4.1}$$

with

$$\begin{aligned} \chi_{ij}(\theta) \;&:=\; \prod_{q=1}^{h} \left[2q+1+\frac{c(i)-c(j)}{2} \right]_{\theta}^{\sigma^q\lambda_i\cdot\lambda_j} \qquad (4.2) \\ &= \prod_{q=1}^{h} \left(\frac{\left(1-\omega^{-q}e^{-\theta-\frac{i\pi}{h}\left(2+\frac{c(i)-c(j)}{2}\right)}\right)\left(1-\omega^{-q}e^{-\theta-\frac{i\pi}{h}\left(\frac{c(i)-c(j)}{2}\right)}\right)}{\left(1-\omega^{-q}e^{-\theta-\frac{i\pi}{h}\left(2-B+\frac{c(i)-c(j)}{2}\right)}\right)\left(1-\omega^{-q}e^{-\theta-\frac{i\pi}{h}\left(B+\frac{c(i)-c(j)}{2}\right)}\right)} \right)^{\sigma^q\lambda_i\cdot\lambda_j} \quad . \end{aligned}$$

The $\{\}_\theta$ are building blocks consisting out of sinh-functions, i.e. $\{x\}_\theta = [x]_\theta/[x]_{-\theta}$, $[x]_\theta = < x+1 >_\theta < x-1 >_\theta \,/\, < x+1-B >_\theta < x-1+B >_\theta$ and $< x >_\theta = \sinh \frac{1}{2}\left(\theta + \frac{i\pi x}{h}\right)$. $B(\beta) = \frac{2\beta^2}{4\pi+\beta^2}$ is the effective coupling constant which takes values between 0 and 2 when β is taken to be purely real. h denotes the Coxeter number and ω is the h^{th} root of unity. σ denotes the particular Coxeter element in the conventions of [24], $c(i)$ the colour values related to the bicolouration of the Dynkin diagram, λ_i a fundamental weight and γ_i is $c(i)$ times a simple root.

In [31] it was noted that the function $X_{ij}(z,\xi) = \prod_{q=1}^{h}\left(1-e^{-\frac{2\pi i}{h}q}\left(\frac{\xi}{z}\right)\right)^{\sigma^q\gamma_i\cdot\gamma_j}$, which arose from the normal ordering of two vertex operators associated with classical solitons, exhibits close similarities to the two-particle scattering amplitude. The precise relation will become more apparent in this section. In analogy to the expressions of section 5.2 in [32] we obtain for $|\xi| < |z|$ the identity

$$\exp\left(-\sum_{n\in\mathbf{e}} \frac{(\lambda_i \cdot q^*(n))\,(\lambda_j \cdot q(n))}{n} \left(\frac{\xi}{z}\right)^n\right) \;=\; \prod_{q=1}^{h} \left(1 - e^{-\frac{2\pi i}{h} q} \left(\frac{\xi}{z}\right)\right)^{\sigma^q \lambda_i \cdot \lambda_j} \quad . \tag{4.3}$$

Here the $q(n)$ form an orthogonal and complete set of eigenvalues of the Coxeter element with eigenvalue ω^n and $\mathbf{e}$ denotes a set of positive integers, the exponents of the Lie algebra modulo h. Employing this relations, a few manipulations lead to

$$\ln \chi_{ij}(\theta_{ij}) = 4 \sum_{n\in\mathbf{e}} \frac{(\lambda_i \cdot q^*(n))\,(\lambda_j \cdot q(n))}{n} e^{-n\left(\theta_{ij} - \frac{i\pi}{h}\left(\frac{c(i)-c(j)}{2}+1\right)\right)} \sin\frac{n\pi}{2h} B \sin(2-B)\frac{n\pi}{2h}. \tag{4.4}$$

Keeping in mind that we used $\mathrm{Re}(\theta_i) > \mathrm{Re}(\theta_j)$ we cannot simply interchange the two rapidities in order to derive the expression involving $\chi_{ij}(\theta_{ji})$. Careful analysis then yields for the phase of the S-matrix defined as $\delta_{ij}(\theta) = -i \ln S_{ij}(\theta)$

$$\begin{aligned} \delta_{ij}(\theta) \;&=\; \varepsilon_\theta \sum_{n\in\mathbf{e}} f(n,B) \frac{(\lambda_i \cdot q^*(n))\,(\lambda_j \cdot q(n))}{n} e^{-n\left(\varepsilon_\theta \theta - \frac{i\pi}{h}\left(\frac{c(i)-c(j)}{2}\right)\right)} && (4.5)\\ &=\; \varepsilon_\theta \sum_{n\in\mathbf{e}} \frac{\hat{x}_i(n)\hat{x}_j(n)}{n} f(n,B) e^{-n\varepsilon_\theta \theta} && (4.6) \end{aligned}$$

with

$$f(n,B) := \sin\frac{n\pi}{2h} B \;\sin(2-B)\frac{n\pi}{2h} \tag{4.7}$$

and $\varepsilon_\theta = \pm 1$ for $\mathrm{Re}\theta > 0$ and $\mathrm{Re}\theta < 0$, respectively. We have evaluated the inner product of the fundamental weights and the eigenvectors of the Coxeter element

$$\sigma^l \lambda_i \cdot q(n) = x_i(n) \exp\left(-\frac{i\pi}{h} n \left(\frac{c(i)+1}{2} + 2l\right)\right) \quad . \tag{4.8}$$

The $x_i(n)$ denote the left eigenvectors of the Cartan matrix and $\hat{x}_i(n) = \sqrt{2/\sin(n\pi/h)}\,x_i(n)$. This expression may be derived from the relation between the fundamental weights and roots $\gamma_i = (\sigma_- - \sigma_+)\lambda_i$ and the inner product of $\sigma^l \gamma_i \cdot q(n)$ [24]. Similar expressions for S have been quoted in [33, 34].

In order to convince ourselves that no mistakes have been made in the derivation we can check for consistency whether (4.5) and (4.6) still satisfy the S-matrix axioms. First one may verify the crucial relation *

$$\lim_{\theta\to 0} \delta_{ij}(\theta) \;=\; -\delta_{ij}\,\pi \; . \tag{4.9}$$

* In general we have that $\lim_{\theta\to 0} \{x\}_\theta = 1$, except for $x = 1$ in which case we obtain -1. Due to the presence of the block $\{1\}_\theta$ only in S_{ii} one obtains $S_{ij}(0) = (-1)^{\delta_{ij}}$.

We do not have an analytic proof of this relation, but it may be verified easily numerically. This relation confirms in particular that we have chosen the correct normalization. The parity invariance, i.e. the symmetry in i and j, and the unitarity relations (3.2) follow trivially from (4.6). Not quite so obvious is the crossing relation. Relating the fundamental weights of the anti-particles to the ones of the particles in a similar way as the representatives of the orbits Ω_i and $\Omega_{\bar{\imath}}$ [23], we obtain

$$\lambda_{\bar{\imath}} = -\sigma^{-\frac{h}{2}+\frac{c(\bar{\imath})-c(i)}{4}}\lambda_i \tag{4.10}$$

and hence the crossing relation

$$\begin{aligned} S_{i\bar{\jmath}}(\theta_{ij}) &= \exp\left(-\sum_{n\in\mathbf{e}} f(n,B)\frac{(\lambda_i\cdot q^*(n))\left(\lambda_j\cdot\sigma^{\frac{h}{2}+\frac{c(j)-c(\bar{\jmath})}{4}}q(n)\right)}{n}e^{-n\left(\theta_{ij}-\frac{i\pi}{h}\left(\frac{c(i)-c(j)}{2}\right)\right)}\right) \\ &= S_{ij}(i\pi-\theta_{ij}) \ . \end{aligned} \tag{4.11}$$

Alternatively, we could have used $x_{\bar{\imath}}(n) = -x_i(n)e^{ni\pi}$ in (4.6). Finally we would like to verify the bootstrap equation, which after the usual identification of the fusing angles is most suitable in the form

$$\prod_{t=i,j,k} S_{lt}\left(\theta + i\eta(t)\right) \;=\; 1 \ . \tag{4.12}$$

Here the fusing angles are given by $\eta(i) = -\frac{\pi}{h}\left(2\xi(i)+\frac{1-c(i)}{2}\right)$ and the integers $\xi(t)$ comprise two independent conjugacy classes of integers which provide solutions to the fusing rule [29, 23, 24] $\sum_{t=i,j,k}\sigma^{-\xi(t)}\lambda_t = 0$. Then considering

$$\lambda_t\cdot q(n)e^{-n\theta-in\eta(t)}e^{n\frac{i\pi}{h}\left(\frac{c(t)-c(l)}{2}\right)} = \sigma^{-\xi(t)}\lambda_t\cdot q(n)e^{-n\frac{i\pi}{h}\left(\frac{1+c(l)}{2}\right)}$$

yields, by summing over i,j,k and by employing the fusing rule, precisely the bootstrap equation (4.12). So indeed all the S-matrix axioms are satisfied. We shall be content here with this check and do not report on the pole structure of (4.5) or (4.6).

Using the fact that $f(n,B) = -\frac{n\beta^2}{4h}\sin\frac{n\pi}{h}+\mathcal{O}(\beta^4)$ one obtains the precise relation between the function $X_{ij}(\theta)$ [31] which played an important role in the study of the classical soliton solutions and the phase of the scattering matrix

$$\delta_{ij}(\theta) \;=\; -\frac{\beta^2}{2h}\frac{d}{d\theta}\ln X_{ij}(\theta) \ . \tag{4.13}$$

This is the reversed situation as in the soliton case, in which the phase shift may be obtained by integration of the classical time delay [35].

5 Bosonic Representation

As already mentioned the algebra (2.3) - (2.5) may be regarded as a generalization of a Bose-Fermi algebra. It is known since some time that it is possible to find operators inside the bosonic Fock space which obey fermionic commutation relations and vice versa there exist operators of bosonic nature which may be expressed in terms of fermionic operators [36]. This observation, among others, led to the notion that in two dimensional space-time bosons and fermions are regarded as equivalent. In [37] it was shown that this construction may be generalized to arbitrary spins. It is interesting to note that one may further extend this in such a way that one obtains an explicit representation of the Zamolodchikov algebra in terms of free bosonic fields $a_i(\theta)$. This may be achieved by replacing the spin by the rapidity dependent phase of the S-matrix and therefore turning the conventional expression into a convolution. So defining now the operators

$$Z_i(\theta) \;:=\; a_i(\theta)\;\exp\left(i\int_\theta^\infty d\theta'\delta_{il}(\theta-\theta')\;n_l(\theta')\right) \tag{5.1}$$

$$Z_i^\dagger(\theta) \;:=\; a_i^\dagger(\theta)\;\exp\left(-i\int_\theta^\infty d\theta'\delta_{il}(\theta-\theta')\;n_l(\theta')\right) \tag{5.2}$$

we have obtained a solution of the algebra (2.3) - (2.5) in terms of free bosonic operators, which are assumed to satisfy as usual

$$[a_i(\theta), a_j^\dagger(\theta')] \;=\; \delta_{ij}\;\delta(\theta-\theta') \tag{5.3}$$

$$[a_i(\theta), a_j(\theta')] \;=\; \left[a_i^\dagger(\theta), a_j^\dagger(\theta')\right] \;=\; 0 \quad . \tag{5.4}$$

One may verify this as usual by employing the Baker-Campbell-Hausdorff formula. Here we denote with $n_i(\theta) = a_i^\dagger(\theta)a_i(\theta)$ the number density operator. The scattering matrix should then be of the form

$$\ln S_{ij}(\theta) \;=\; i\varepsilon_\theta\delta_{ij}(\varepsilon_\theta\theta) \quad . \tag{5.5}$$

For vanishing effective coupling one expects that, $\delta \to \pi n$, such that the algebra reproduces the usual bosonic or fermionic algebra. This is indeed the case when $\delta_{ij}(\theta)$ is taken to be the phase of the S-matrix of affine Toda field theory, as may be verifies easily with the expressions of the previous section. This solution is completely model independent and may be employed for all diagonal theories. On the possible extension to the non-diagonal case we report elsewhere.

6 Vertex operator construction

When relaxing relation (2.5) and solely demanding that the braiding of two Z's will pick up the scattering matrix one may construct conceptually entirely different representations. In fact this relation was only employed in order to derive the kinematic residue equation for the form factors.

6.1 The exchange relations for the Z's

We shall now try to find explicit representations for the operators Z which obey the algebra (2.3) for affine Toda field theories. Such a construction has originally been suggested by Corrigan and Dorey [17]. Here we shall propose an alternative construction. Due to the close resemblance of the factor $X_{ij}(\theta)$ and the phase of the scattering matrix, which was demonstrated in section 4 it is suggestive to consider the vertex operator construction formulated in [32] for the classical solitons in affine Toda field theory. With this motivation in mind we may take the function F introduced in [32] as a prototype representative of the Zamolodchikov algebra

$$F_i(\xi) = \exp\left(\sum_{n\in\mathbf{e}} \frac{\gamma_i \cdot q(n)}{n} \xi^n E_{-n}\right) \exp\left(-\sum_{n\in\mathbf{e}} \frac{\gamma_i \cdot q^*(n)}{n} \xi^{-n} E_n\right) \quad . \tag{6.1}$$

Here γ_i denotes again a representative of the orbit Ω_i generated by the Coxeter element σ by acting on the root space, $q(n)$ an eigenvector of this element as introduced in section 4 and the E_n denote the generators of a principal Heisenberg subalgebra of level one

$$[E_n, E_m] \;=\; n \;\; \delta_{n+m,0} \quad . \tag{6.2}$$

In [32] it was demonstrated that the operator product expansion of two F's yields

$$F_i(z)F_j(\xi) = X_{ij}(z,\xi) \;\; : F_i(z)F_j(\xi) : \tag{6.3}$$

with : . . . : denoting the normal ordering with respect to the principal Heisenberg subalgebra and

$$X_{ij}(z,\xi) = \prod_{q=1}^{h} \left(1 - e^{-\frac{2\pi i q}{h}} \frac{\xi}{z}\right)^{\sigma^q \gamma_i \cdot \gamma_j} \qquad . \tag{6.4}$$

In order to compare this expression with the one known for the S-matrix, it is useful to convert this expression into one involving sinh-functions. Choosing $z = \exp(\theta_i)$, $\xi = \exp(\theta_j)$, employing the relation

$$\sigma^q \gamma_i \cdot \gamma_j \;=\; 2\sigma^{q+\frac{c(j)-c(i)}{2}} \lambda_i \cdot \lambda_j - \sigma^{q+\frac{c(j)-c(i)}{2}-1} \lambda_i \cdot \lambda_j - \sigma^{q+\frac{c(j)-c(i)}{2}+1} \lambda_i \cdot \lambda_j \tag{6.5}$$

and shifting the dummy variable q in (6.4) appropriately one obtains

$$X_{ij}(\theta) = \prod_{q=1}^{h} \left(\frac{< 2q + c(i) - c(j) >_\theta^2}{< 2q + c(i) - c(j) + 2 >_\theta < 2q + c(i) - c(j) - 2 >_\theta} \right)^{\sigma^q \lambda_i \cdot \lambda_j} \qquad . \tag{6.6}$$

This formula already resembles very closely the expressions for part of the S-matrix (4.1) and by making the assumption, which appears now to be quite natural, that the Z's are a kind of quantum deformation of the F's we are forced to split up the blocks (6.6) and manipulate them individually, i.e. essentially we have to replace a root by a weight. Substituting now the inner product $\gamma_i \cdot q(n)$ by the normalised left eigenvector of the Cartan matrix $\hat{x}_i(n)$, we obtain the $2\pi i$-periodic operator

$$Z_i(\theta) = \exp\left(\sum_{n \in \mathbf{e}} \frac{\hat{x}_i(n)}{n} e^{n\theta} \hat{E}_{-n} \right) \, \exp\left(- \sum_{n \in \mathbf{e}} \frac{\hat{x}_i(n)}{n} e^{-n\theta} \hat{E}_n \right) \quad . \tag{6.7}$$

Assuming now that the $\hat{E}$'s obey a deformed version of the Heisenberg subalgebra

$$[\hat{E}_n, \hat{E}_m] \; = in \; f(n,B) \; \delta_{n+m,0} \quad , \tag{6.8}$$

we may carry out the operator product expansion and obtain together with (4.6) and (4.8)

$$Z_i(\theta_i) Z_j(\theta_j) = \chi_{ij}(\theta_{ij}) \; : Z_i(\theta_i) Z_j(\theta_j) : \quad . \tag{6.9}$$

Then employing the OPE in opposite order and canceling the normal ordered terms gives rise to the equation (2.3) and we have found indeed a representative for the Z's.

The commutation relations may be related directly to the quantum Heisenberg algebra

$$[\alpha_{in}, \alpha_{jm}] \;=\; \delta_{n+m,0} \, \frac{1}{n} \, \left(q^{nK_{ij}} - q^{-nK_{ij}}\right) \, \left(q^{nc} - q^{-nc}\right) \tag{6.10}$$

which was introduced by Drinfel'd [38] in the context of a mode expansion for the generators of the quantum affine algebras. Here q denotes the deformation parameter, K the Cartan matrix of the affine Lie algebra and c the level. Considering now $U_q(\hat{Sl_2})$ and identifying $q = e^{\frac{i\pi B}{4h}}$, $\hat{E}_n = n\alpha_n\sqrt{i}$ the two algebras become formally isomorphic at the level

$$c \ = \ \frac{4}{B} - 2 \ \ . \tag{6.11}$$

For the conformally invariant theory the phenomenon of a coupling constant dependent central charge is known for the Virasoro algebra. The level one module is obtained at the self-dual point of the theory, whilst for the free theory the level tends to zero or infinity. The algebra exhibits explicitly the strong-weak duality, i.e. invariance under $B \to 2 - B$, which is present in conformal Toda theories [39] and is known to survive the breaking of the conformal symmetry [25, 10]. In the limit to the free theory $B \to 0$ or $B \to 2$, the algebra (6.8) becomes Abelian, and the Z's become representatives of the usual bosonic algebra.

In comparison with [17] we have overcome the feature that the minimal theory and the coupling constant dependent part do not interfere with each other. This is achieved by eliminating four of the Heisenberg subalgebras required in the construction of [17] and replacing it by one, namely (6.8). We also do not require any "delocalisation" in the rapidities. In [34] a construction which employs one Heisenberg subalgebra has been proposed. Here the effective coupling has been moved into the operators themselves, which has a consequence that one does not obtain the bosonic algebra for the free theory.

6.2 Operator product expansions

Considering now the product of $Z_i(\theta_i)Z_{\bar{\imath}}(\theta_i + i\pi)$ will not lead to any nontrivial information since the S-matrix which is picked up by the braiding gives $S_{i\bar{\imath}}(i\pi) = S_{ii}(0) = -1$. So we have to consider explicitly the normal ordering which yields

$$: \ Z_i(\theta)Z_{\bar{\imath}}(\theta + i\pi) \ := \ 1 \ \ . \tag{6.12}$$

We then obtain

$$Z_i(\theta)Z_{\bar{\imath}}(\theta + i\pi) \ = \ \exp\left(i \sum_{n\in \mathbf{e}} \frac{\hat{x}_i^2(n)}{n} f(n,B) \right) = c_i = i \ \ , \tag{6.13}$$

and hence confirming relation (2.6). Once again we do not have an analytic proof for the convergence of this series, but is may be verified numerically.

We now like to verify the operator product expansion which leads to a three particle fusing process. Considering the product

$$Z_i(\theta + i\bar{\eta}_{ik}^j) Z_j(\theta' - i\bar{\eta}_{kj}^i) \tag{6.14}$$

we obtain for $\theta' \to \theta$

$$\exp\left(\sum_{n\in\mathbf{e}} \left(\hat{x}_i(n) e^{in\bar{\eta}_{ik}^j} + \hat{x}_j(n) e^{-in\bar{\eta}_{kj}^i} \right) \frac{e^{n\theta}}{n} \hat{E}_{-n} \right)$$
$$\exp\left(-\sum_{n\in\mathbf{e}} \left(\hat{x}_i(n) e^{in\bar{\eta}_{ik}^j} + \hat{x}_j(n) e^{-in\bar{\eta}_{kj}^i} \right) \frac{e^{-n\theta}}{n} \hat{E}_{-n} \right) \exp\left(i \sum_{n\in\mathbf{e}} \frac{\hat{x}_i(n)\hat{x}_j(n)}{n} f(n,B) e^{-in\bar{\eta}_{ij}^k} \right).$$

Upon identifying $\bar{\eta}_{ik}^j = \eta(i) - \eta(k) = -\frac{\pi\xi(n)}{h} \left(2 \left(\xi(i) - \xi(j) + \frac{c(j)-c(i)}{2} \right) \right)$, noting that the second solution of the fusing rule corresponds to $-\eta(i) + \eta(k)$ together with the identity

$$x_k(n) = x_i(n) e^{in(\eta(i)-\eta(k))} + x_j(n) e^{in(\eta(j)-\eta(k))} \tag{6.15}$$

yields $Z_k(\theta)$ for the first two factors and Γ_{ij}^k for the latter. Hence we confirm (2.7) and (2.8). One may also proceed further and check the operator product expansions corresponding to the higher order poles.

6.3 Lorentz boost

This far we have not specified the operator which generates the Lorentz boost. For this purpose we define now an operator in analogy to the Sugawara-Sommerfield construction [40, 41], but now involving only generators of the quantum affine Heisenberg algebra

$$d' := -\sum_{n\in\mathbf{e}} i\, f(n,B)^{-1}\, \hat{E}_{-n} \hat{E}_n \quad . \tag{6.16}$$

Here $f(n,B)$ denotes the function introduced in (4.7) . Then together with

$$[\hat{E}_n, Z_i(\theta)] \;=\; i x_i(n) f(n,B) e^{n\theta} Z_i(\theta) \tag{6.17}$$

it is easy to verify that $-d'$ counts the principal grade of $\hat{E}_l$

$$[d', \hat{E}_l] \;=\; -l \hat{E}_l \quad . \tag{6.18}$$

A little bit of algebra then shows that the commutator of d' with $Z_i(\theta_i)$ is the same as the derivative of it with respect to the rapidity and hence d' indeed satisfies (2.13) and may be viewed as the generator of the Lorentz boost.

6.4 Conserved Charges

An explicit construction for the conserved charges in affine Toda field theory has unfortunately not yet been carried out for the quantum case. However, guided by the fact that classically the conserved quantities are known to be graded by the exponents of the Lie algebra [13] modulo h, it seems very suggestive to conjecture that they may be realized as

$$Q_n \sim \hat{E}_n \quad . \tag{6.19}$$

Notice that this is real when choosing the convention $(\hat{E}_n)^\dagger = \hat{E}_n$ and a real constant of proportionality. This choice is possible without upsetting (6.8). Furthermore, all this charges will commute with each other for positive n. We then compute

$$[Q_n, Z_i(\theta)] \sim e^{n\theta} Z_i(\theta) \quad . \tag{6.20}$$

When the constant of proportionality is even in n we observe that (2.15) holds. Notice further that in the free theory, i.e. $B = 0$ or $B = 2$, the vertex operators commute with the charges. Carrying out a further consistency check and acting with Q_n on the state

$$\prod_{t=i,j,k} Z_t(\theta + i\eta(t))|0\rangle \tag{6.21}$$

implies

$$\sum_{t=i,j,k} [Q_n, Z_t(\theta + i\eta(t))] = 0 \tag{6.22}$$

and therefore

$$\sum_{t=i,j,k} x_i(n) e^{in\eta(t)} = 0 \quad . \tag{6.23}$$

This equation corresponds to equation (6.15), which resulted as a projection of the fusing rule into the velocity plane [24]. It is interesting to note that this equation results as well as an ambiguity of the solution for the form factor axioms and corresponds to the consistency equation derived by Zamolodchikov [14] to decide whether operators of certain spins may be present in a particular theory.

Certainly the expressions for the charges need to be put on a more solid ground and a proper derivation of them is still required. Nonetheless, it is quite intriguing that the expression has aready the expected properties.

7 Conclusions

We have demonstrated that it is possible to find explicit realisations for the operators which when exchanged will give rise to the scattering matrix. We provided two alternative constructions. One realisation may be obtained by extending the construction which relates particles of different spins and statistic in two dimensions. Starting from the classical soliton operators we construct a second realisation in terms of the generators of the quantum Heisenberg algebra related to $U_q(\hat{Sl_2})$. It was shown that this operators contain all the information of the scattering matrix and therefore permit to regard them as the central objects, rather than S.

The operators may be employed to extract information for the off-shell properties of the theory, like form-factors and ultimately correlation functions. It remains as very interesting issue to clarify how the operators may be transformed into real space and possibly utilise them in order to obtain the braid relations for the Toda fields themselves. A further interesting problem, on which we report elsewhere, is provided by the opportunity to extend this construction to the non-diagonal situation. This may solve the still outstanding problem for the form of the scattering matrix for affine Toda field theories related to purely complex β by constructing first the operators instead of the S-matrix.

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