Factorized Scattering in the Presence of Reflecting Boundaries

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Abstract

We formulate a general set of consistency requirements, which are expected to be satisfied by the scattering matrices in the presence of reflecting boundaries. In particular we derive an equivalent to the boostrap equation involving the W-matrix, which encodes the reflection of a particle off a wall. This set of equations is sufficient to derive explicit formulas for W, which we illustrate in the case of some particular affine Toda field theories.

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1 Introduction

The common procedure to treat the scattering of particles is to work in infinit ely extended space-time. Yet restricting the space volume to finite size may reveal interesting information, which is not observable in the infinite volum e limit. In completely integrable models the modifications arising from the p resence of boundaries can be computed exactly. We therefore direct our interest to integrable models on a finite line delimite d by perfectly reflecting mirrors. The central object is the S-matrix, whic h is factorized into 2-body S-matrices in this case. They have to satisfy Y ang-Baxter[1, 2] equations, which provide nontrivial constraints for no n-diagonal matrices. In the presence of reflecting boundaries one obtains similar factorization equations including the *wall matrix* W, which descr ibes the scattering of a particle off the wall [3, 4, 5].

The main object of the present paper is to study scattering described by diagona l S-matrices. In this case the nontrivial constraints result from the boot strap hypothesis, which we formulate for the situation with finite space volu me. Whilst in the situation without boundaries the ensueing consistency equations allow us to determine explicitly the S-matrix [6, 7], in the present situation t hey will enable us to compute the W-matrix.

The layout of this article is as follows. Firstly we extend the Zamolodchikov al gebra including the wall matrix W to take boundaries into account. In section 3 we employ it to derive the factorization equations in the presence of reflecting boundaries and in section 4 we formulate our central equations (4.15), the *wall bootstrap* equations. In section 5 we apply this framework to the 1-particle Bullough-Dodd model ($A_2^{(2)}$-affine Toda theory) and to several 2-particle affine Toda systems ($A_2^{(1)}$ and $A_4^{(2)}$). Finally we state our conclusions.

2 Zamolodchikov algebra

Factorized S-matrices, describing one-dimensional scattering, have to satisfy certain consistency conditions, which in general provide powerful tools for their explicit construction. These axioms can be extracted most easily as associ ativity conditions of the well known *Zamolodchikov algebra* [2]

$$Z_a(\theta_a)Z_b(\theta_b) = S_{ab}^{\bar{a}\bar{b}}(\theta_{ab})\, Z_{\bar{b}}(\theta_b)Z_{\bar{a}}(\theta_b) \tag{2.1}$$

$$Z_a^\dagger(\theta_a)Z_b^\dagger(\theta_b) = S_{ab}^{\bar{a}\bar{b}}(\theta_{ab})\, Z_{\bar{b}}^\dagger(\theta_b)Z_{\bar{a}}^\dagger(\theta_a) \tag{2.2}$$

$$Z_a(\theta_a)Z_b^\dagger(\theta_b) = S_{ab}^{\bar{a}\bar{b}}(\theta_{ba})\, Z_{\bar{b}}^\dagger(\theta_b)Z_{\bar{a}}(\theta_a) + 2\pi\,\delta_{ab}\,\delta(\theta_{ab}) \ , \tag{2.3}$$

where each of the operators Z_a is associated with a particle "a" and S denotes the unitary and crossing invariant two particle scattering matrix which satisfy the Yang-Baxter-(3.9) and bootstrap equation (4.13). Their dependence on the momenta is parameterized usually by the rapidities θ_a , i.e. $p_a = m(\cosh\theta_a, \sinh\theta_a)$, having the advantage that the branch cuts on the real axis in the complex plane of the Mandelstam variable unfold. Relativistic invariance demands that the scattering matrix depends only on the rapidity difference, which we denote $\theta_{ab} := \theta_a - \theta_b$.

In the present case this operator algebra has to be extended in order to include the presence of a wall. When a particle scatters off the wall, it reverses its momentum and possibly cha nges its quantum numbers. If $Z_a(\theta)$ represents particle a and $Z_w(0)$ represents the wall in Zamolodchikov's algebra, this process is encoded in th e following relation

$$Z_a(\theta)Z_w(0) = \sum_{\bar{a}} W_a^{\bar{a}}(\theta)Z_{\bar{a}}(\bar{\theta})Z_w(0), \tag{2.4}$$

where $\bar\theta = -\theta$ and the matrix $W_a^{\bar{a}}(\theta)$ describes the scattering by the wall. Notice that we do not interchange the order of Z_a and Z_w as i n (2.1)-(2.3), such that the W-matrix is not the result of a braiding like the scattering matrix. From its definition, $W_a^{\bar{a}}(\theta)$ has to satisfy the usual unitarity condition

$$\sum_{\bar{a}} W_a^{\bar{a}}(\theta)W_{\bar{a}}^{a'}(-\theta) = \delta_a^{a'}. \tag{2.5}$$

2

The algebra, involving Z_a's and $Z_a^\dagger$ only, is now very similar to the usual case (2.1) - (2.3), except that in the process $a+b \to c+d$, we have to distinguish three different situations, in which the braiding of two operators might produce:

1. $^-S_{ab}^{cd}(\theta_a, \theta_b)$: describing scattering before any particle has hit the wall;

2. $^0S_{ab}^{cd}(\theta_a, \theta_b)$: describing scattering after one particle has h it the wall;

3. $^+S_{ab}^{cd}(\theta_a, \theta_b)$: describing scattering after both particles have hit the wall.

In ^0S, it is the particle with negative rapidity, which has scattered off the wall. Notice that the wall breaks translational invariance, so that the S-matrices will not depend only on the difference of rapidities. In particular ^0S is in general a function of the sum of the rapidities $\tilde{\theta}_{ab} := \theta_a + \theta_b$.

3 Factorization equations

We now use the associativity of the previous algebra to derive consistency condi tions. Let us therefore consider the scattering of particles labelled by quantum numb ers $a, b \ldots$, with rapidities $\theta_a, \theta_b \ldots$ in the presence of a ref lecting wall, which we locate for convenience at rapidity $\theta = 0$. We start f rom a state with $\theta_a > \theta_b$, then

$$
\begin{aligned}
Z_a(\theta_a)Z_b(\theta_b)Z_w(0) =& {}^-S_{ab}^{a_1b_1}(\theta_{ab})\, Z_{b_1}(\theta_b)Z_{a_1}(\theta_a)Z_w(0) \\
=& {}^-S_{ab}^{a_1b_1}(\theta_{ab})W_{a_1}^{\bar{a}_1}(\theta_{a_1})\, Z_{b_1}(\theta_b)Z_{\bar{a}_1}(\bar{\theta}_a)Z_w(0) \\
=& {}^-S_{ab}^{a_1b_1}(\tilde{\theta}_{ab})W_{a_1}^{\bar{a}_1}(\theta_a)^0S_{b_1\bar{a}_1}^{b_2\bar{a}_2}(\tilde{\theta}_{b_1a_1})\, Z_{\bar{a}_2}(\bar{\theta}_a)Z_{b_2}(\theta_b)Z_w(0) \\
=& {}^-S_{ab}^{a_1b_1}(\theta_{ab})W_{a_1}^{\bar{a}_1}(\theta_a)^0S_{b_1\bar{a}_1}^{b_2\bar{a}_2}(\tilde{\theta}_{b_1a_1})W_{b_2}^{\bar{b}_2}(\theta_b)\, Z_{\bar{a}_2}(\bar{\theta}_a)Z_{\bar{b}_2}(\bar{\theta}_b)Z_w(0).
\end{aligned} \tag{3.6}
$$

As in the derivation of the Yang-Baxter equation, factorization now implies that the order in which the particles scatter is irrelevant too. If we go through the same

steps, but scatter particle b first from the wall, w e derive the following identity:

$$^{-}S_{ab}^{a_1 b_1}(\theta_{ab})W_{a_1}^{\bar{a}_1}(\theta_a)^0 S_{b_1 \bar{a}_1}^{b_2 \bar{a}_2}(\tilde{\theta}_{b_1 a_1})W_{b_2}^{\bar{b}_2}(\theta_b) = W_a^{\bar{b}}(\theta_b)^0 S_{a\bar{b}}^{a_1 \bar{b}_1}(\tilde{\theta}_{ab})W_{a_1}^{\bar{a}_1}(\theta_a)^+ S_{\bar{b}_1 \bar{a}_1}^{\bar{b}_2 \bar{a}_2}(\theta_{ab}).$$

(3.7)

Diagramatically this corresponds to the equation in figure 1.

The presence of the wall breaks parity invariance, which - if true - would deman d $S_{ab}^{cd}(\theta) = S_{ba}^{dc}(\theta)$. But restrictions of this kind can be genera ted, following the argumentation originally due to Cherednik [3, 4]. In the limit, when the rapidity of one of the particles vanishes, it is impossi ble to decide, whether it has or has not hit the wall before scattering off a nother particle. This imposes the additional conditions:

$$W_a^{\bar{a}}(0)^+ S_{b\bar{a}}^{b_1 \bar{a}_1}(\theta) =^0 S_{ab}^{a_1 b_1}(\theta)W_{a_1}^{\bar{a}_1}(0),$$

$$W_a^{\bar{a}}(0)^0 S_{b\bar{a}}^{b_1 \bar{a}_1}(\theta) =^- S_{ab}^{a_1 b_1}(\theta)W_{a_1}^{\bar{a}_1}(0).$$

(3.8)

To complete the scheme, we still have to consider 3-particle scattering. Howev er if equ.(3.7) is satisfied, we can always arrange rapidities, such th at all particles scatter against each other, before (or after) they hit the wall. Therefore factorization requires $^{\pm}S(\theta)$ to satisfy in additi on the usual Yang-Baxter equations

$$^{\pm}S_{\bar{a}\bar{b}}^{\bar{a}'\bar{b}'}(\theta_{ab})\,^{\pm}S_{a\bar{c}}^{\bar{a}\bar{c}'}(\theta_{ac})\,^{\pm}S_{bc}^{\bar{b}\bar{c}}(\theta_{bc}) =^{\pm} S_{\bar{b}\bar{c}}^{\bar{b}'\bar{c}'}(\theta_{bc})\,^{\pm}S_{\bar{a}c}^{\bar{a}'\bar{c}}(\theta_{ac})\,^{\pm}S_{ab}^{\bar{a}\bar{b}}(\theta_{ab}) \quad .$$

(3.9)

These equations are sufficient to determine the S- and W-matrices, unless th ey are diagonal. In this case (3.7) are trivially satisfied and we requ ire more information to determine them. Once an S-matrix posses a pole due to the propagation of a bound state particle, one can formulate the so-called

4 Bootstrap equations

For simplicity we will in the sequel consider only diagonal S, W-matrices:

$$W_a^b(\theta) = \delta_a^b W_a(\theta)$$

(4.10)

4

and similarly for the S-matrices. In this case equs.(3.7) and (3.8) are satisfied, if

$$^{0}S_{ba}(\theta) =^{0}S_{ab}(\theta) =^{-}S_{ab}(\theta) =^{+}S_{ba}(\theta) =^{-}S_{ba}(\theta) =^{+}S_{ab}(\theta). \tag{4.11}$$

Here we used unitarity equ.(2.5), which implies $W(0)^2 = 1$. Equ.(4.11) includes constraints usually coming from parity invariance. As a result of this equation we shall not distinguish anymore in the following between $^{-}S, ^{0}S, ^{+}S$ and solely refer to them as S.

When particle c is a bound state of particles a and b one assumes in addition to the Zamolodchikov algebra the validity of an operator product expansion involving the operators representing those particles

$$Z_a\left(\theta + i\eta^b_{ac} + \frac{i\varepsilon}{2}\right) \; Z_b\left(\theta - i\eta^a_{bc} - \frac{i\varepsilon}{2}\right) = \frac{i\Gamma^c_{ab}Z_c(\theta)}{\varepsilon} \;\;, \tag{4.12}$$

where Γ^k_{ij} denotes the three particle vertex on mass-shell and η^c_{ab} are the so-called fusing angles. Then multiplying this equation by $Z_d(\theta_d)$, using equation (2.1) and taking the limit $\varepsilon \to 0$ leads to a nontrivial consistency condition, which is known as the *bootstrap* equation [?]

$$S_{dc}(\theta) = S_{da}(\theta - i\eta^b_{ac}) \; S_{db}(\theta + i\eta^a_{bc}) \;. \tag{4.13}$$

It states that scattering particle d against c is equivalent to scatter d against the bound state $a+b$. Evidently there has to be an equation of this kind in the presence of reflecting boundaries. Thus let us scatter particles a, b and d with rapidities $\theta_0 + i\eta^b_{ac} + \frac{i\varepsilon}{2}, \theta_0 - i\eta^a_{bc} - \frac{i\varepsilon}{2}, \theta_d > 0$. We obtain by the same procedure as in the previous subsection

$$Z_a\left(\theta_0 + i\eta^b_{ac} + \frac{i\varepsilon}{2}\right) Z_b\left(\theta_0 - i\eta^a_{bc} - \frac{i\varepsilon}{2}\right) Z_d(\theta_d)Z_w(0) = S_{ab}\left(2\theta_0 + i\eta^b_{ac} - i\eta^a_{bc}\right)$$

$$S_{ad}\left(\theta_{0d} + i\eta^b_{ac} + \frac{i\varepsilon}{2}\right) S_{bd}\left(\theta_{0d} - i\eta^a_{bc} - \frac{i\varepsilon}{2}\right) S_{ad}\left(\tilde{\theta}_{0d} + i\eta^b_{ac} + \frac{i\varepsilon}{2}\right)$$

$$S_{bd}\left(\tilde{\theta}_{0d} - i\eta^a_{bc} - \frac{i\varepsilon}{2}\right) W_a\left(\theta_a + i\eta^b_{ac} + \frac{i\varepsilon}{2}\right) W_b\left(\theta_b - i\eta^a_{bc} - \frac{i\varepsilon}{2}\right) W_d(\theta_d)$$

$$Z_b\left(-\theta_0 + i\eta^a_{bc} + \frac{i\varepsilon}{2}\right) Z_a\left(-\theta_0 - i\eta^b_{ac} - \frac{i\varepsilon}{2}\right) Z_d(-\theta_d)Z_w(0) \;.$$

On the other hand multiplying equation (4.12) by $Z_d(\theta_d)Z_w(0)$ and performing similar operations we derive after taking the limit $\varepsilon \to 0$, the bootstrap equation in the presence of the wall

$$S_{dc}(\theta_{0d})\,S_{dc}(\tilde{\theta}_{0d})W_c(\theta_c) = W_a(\theta_a + i\eta^b_{ac})W_b(\theta_b - i\eta^a_{bc}) \tag{4.14}$$

$$S_{ab}(2\theta_0 + i\eta^b_{ac} - i\eta^a_{bc})S_{ad}(\theta_{0d} + i\eta^b_{ac})S_{bd}(\theta_{0d} - i\eta^a_{bc})S_{ad}(\tilde{\theta}_{0d} + i\eta^b_{ac})S_{bd}(\tilde{\theta}_{0d} - i\eta^a_{bc}).$$

Diagramatically we depict this situation in figure 2. Since the scattering matrix satisfies the conventional bootstrap equation (4.13), our equation for W reduces to

$$W_c(\theta) = W_a(\theta + i\eta^b_{ac})\,W_b(\theta - i\eta^a_{bc})\,S_{ab}(2\theta + i\eta^b_{ac} - i\eta^a_{bc}) \tag{4.15}$$

which we call, due to the presence of the factor $S_{ab}(2\theta_0 + i\eta^b_{ac} - i\eta^a_{bc})$, an *inhomogeneous* bootstrap equation.

The equations (3.7) and (4.14) also solve the analogous problem of factorized scattering in the presence of two walls, since the two walls do not interfere with each other.

5   The W-matrix

In this section, we shall discuss the solutions of the coupled equs. (4.15). The two particle scattering matrices to be used in this paper always factorize into the form $S(\theta) = \prod_x \{x\}_\theta$. Adopting our notation from [8], each of this block reads

$$\{x\}_\theta := \frac{[x]_\theta}{[-x]_\theta}, \tag{5.16}$$

with

$$[x]_\theta := \frac{\langle x + 1 \rangle_\theta \langle x - 1 \rangle_\theta}{\langle x + 1 - B \rangle_\theta \langle x - 1 + B \rangle_\theta} \tag{5.17}$$

and

$$\langle x \rangle_\theta := \sinh \frac{1}{2}\left(\theta + \frac{i\pi x}{h}\right). \tag{5.18}$$

B is a function, which takes its values between 0 and 2 containing the dependence on the coupling constant β of the Lagrangian field theory. h denotes the Coxeter number of the underlying Lie algebra of the theory. The S-matrices possess furthermore the property to be invariant under $B \to 2 - B$, that is under an interchange of the strong and weak coupling regime.

Alternatively each block is equivalent to the following integral representation

$$\{x\}_\theta = \exp\left(\int_0^\infty \frac{dt}{t\sinh t}\ f_{x,B}(t)\ \sinh\frac{\theta t}{i\pi} \right) \tag{5.19}$$

where

$$f_{x,B}(t) = 8\sinh\frac{tB}{2h}\sinh\frac{t}{h}\left(1 - \frac{B}{2}\right)\sinh t\left(1 - \frac{x}{h}\right). \tag{5.20}$$

Whilst (5.16) nicely exhibits the polestructure of the S-matrix, equation (5.19) is sometimes more useful for explicit evaluations and we shall require this form below.

We might now expect that the W-matrix factorizes in an analogous fashion into blocks as the S-matrix. Indeed we find a one-to-one correspondence between the blocks of the W- and S-matrix:

$$W(\theta)\ =\ \prod_x \mathcal{W}_x(\theta)\ . \tag{5.21}$$

Similar as the S-matrix, the W-matrix factorizes further into subblocks

$$\mathcal{W}_x(\theta)\ =\ \frac{w_{1-x}(\theta)w_{-1-x}(\theta)}{w_{1-x-B}(\theta)w_{-1-x+B}(\theta)}\ . \tag{5.22}$$

As demanded by the unitarity (2.5) of the W-matrix we have

$$w_x(\theta)\ w_x(-\theta)\ =\ 1. \tag{5.23}$$

Furthermore we shall verify the relations

$$w_{x-2h}(\theta)\ w_{-x}(\theta)\ =\ 1 \tag{5.24}$$

$$w_x(0) = w_{-h}(\theta)\ =\ 1 \tag{5.25}$$

$$w_x\left(\theta + \frac{iy\pi}{2h}\right)w_x\left(\theta - \frac{iy\pi}{2h}\right)\ =\ w_{x+y}(\theta)\ w_{x-y}(\theta) \tag{5.26}$$

$$w_x(\theta + i\pi)\ =\ \eta_x(\theta)\ w_x(\theta) \tag{5.27}$$

7

where the function $\eta_x(\theta)$ satisfies individually the homogeneous bootstrap equation

$$\eta_x(\theta + i\eta_{ac}^b)\,\eta_x(\theta - i\eta_{bc}^a) \;=\; \eta_x(\theta)\;. \qquad (5.28)$$

Notice that $\eta_x(\theta)$ does not contain any poles in the physical sheet $0 < \theta < i\pi$. All blocks converge to one in the asymptotic limit $\theta \to 0$, resulting from

$$\lim_{\theta \to \infty} w_x(\theta) = 1 \;\;. \qquad (5.29)$$

It turns out that the function $w_x(\theta)$ posses neither poles nor zeros in the physical strip, such that no particle creation and absorption takes place in the wall. The absence of the poles and zeroes was expected from the assumpti on that the scattering off the wall takes place in an elastic fashion.

We shall now compute some explicit examples of the W-matrix, starting with th e

5.1   The Bullough-Dodd model

The BD-model [9] ($A_2^{(2)}$-affine Toda theory) represents an integrab le quantum field theory involving one type of scalar field only, which satisfies a relativistically invariant equation in two dimensions. The model is ideal to illustrate the general principles presented in the previous sections, since the particle, say A, emerges as a bound state of itself, i.e. $A + A \to A$ is possible. Its classical Lagrangian is obtainable from a folding [10, 11] of the $D_4^{(1)}$-affine Toda theory, where the three ro ots corresponding to the degenerate particles and the negative of the highest root are identified. The resulting Dynkin diagram is the simplest example of a non-simply laced one, containing the root α, which is related to the scalar field, whose square length equals two and the root $\alpha_0 = -2\alpha$, whose square length is consequently eight. Then its classical Lagrangian density reads

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{\beta^2}\left(2e^{\beta\phi} + e^{-2\beta\phi}\right)\;, \qquad (5.30)$$

8

where m denotes the bare mass and β the coupling constant, which we assume to be real to avoid the presence of solitons. For more details on the model we refer to [12] and the references therein , but here we mainly want state the main properties we are going to employ. The scattering matrix can be obtained by the above folding procedure and it turns out to be [6]

$$S^{BD}(\theta) = \{1\}_\theta \, \{2\}_\theta \; . \tag{5.31}$$

The Coxeter number of the BD-model equals three.

The scattering matrix satisfies the homogeneous bootstrap equation in the form

$$S^{BD}(\theta) = S^{BD}(\theta + \omega) \; S^{BD}(\theta - \omega) \; , \tag{5.32}$$

with $\omega = \frac{i\pi}{3}$ and consequently the inhomogeneous bootstrap eq uation acquires the form

$$W(\theta) = W(\theta + \omega) \, W(\theta - \omega) \, S^{BD}(2\theta) \; . \tag{5.33}$$

Employing now Fourier transforms after taking the logarithm to solve such equations, we obtain

$$W(\theta) = \exp\left(\int d\theta' \, G(\theta - \theta') \ln S(2\theta') \right), \tag{5.34}$$

where the Green function G is given by

$$G(\theta) = \lim_{\eta \uparrow 1} \frac{1}{\omega\sqrt{3}} \frac{\sinh\left(\frac{2\pi}{3\omega}(\theta)\eta\right)}{\sinh\left(\frac{\pi}{3}(\theta)\eta\right)} \; . \tag{5.35}$$

The introduction of the parameter η is necessary to guarantee the convergen ce of the Fourier transform. Employing now the integral representation for the blocks of the S-matrix (5.19), we are lead to a factorization of the form (5.21) and (5.22), where each of the sublocks $w_x(\theta)$ is given by the integral representation

$$w_x(\theta) = \exp\left(\int_0^\infty \frac{dt}{t \sinh t} \frac{2 \sinh\left(1 + \frac{x}{h}\right) t \sinh \frac{2\theta t}{i\pi}}{1 - 2 \cosh \frac{2t\omega}{\pi}} \right) \; . \tag{5.36}$$

9

Solving the integral we obtain

$$
w_x(\theta) \;=\; \prod_{l=0}^{\infty} \left(\frac{\Gamma\!\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right) \Gamma\!\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} - \frac{i\theta}{\pi}\right)}{\Gamma\!\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} + \frac{i\theta}{\pi}\right) \Gamma\!\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} - \frac{i\theta}{\pi}\right)} \right)^{\frac{\sin((l+1)\omega)}{\sin\omega}} .
$$

$$(5.37)$$

This equation exhibits nicely the pole structure of $w_x(\theta)$, and therefore $W(\theta)$, and can be used to prove the functional identities (5.23) - (5.29). The function $\eta(\theta)$, which results as a shift of $i\pi$ in equation (5.27) takes on the form

$$
\eta_x(\theta) \;=\; \prod_{l=0}^{\infty} \left(\frac{\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} - \frac{i\theta}{\pi}\right)\left(-1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right)}{\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} - \frac{i\theta}{\pi}\right)\left((l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right)} \right)^{\frac{\sin((l+1)\omega)}{\sin\omega}}
$$

$$(5.38)$$

and satisfies individually the homogeneous bootstrap equation

$$
\eta_x(\theta + i\omega)\, \eta_x(\theta - i\omega) \;=\; \eta_x(\theta) .
$$

$$(5.39)$$

$\eta_x(\theta)$ does not posses any poles in the physical sheet. Furthermore we derive from (5.37) the functional equation

$$
w_x(\theta + i\omega)\; w_x(\theta - i\omega) = w_x(\theta)\frac{\langle x \rangle_{-2\theta}}{\langle x \rangle_{2\theta}}
$$

$$(5.40)$$

from which we infer the crucial identity for the blocks of the W-matrix

$$
\mathcal{W}_x(\theta) = \mathcal{W}_x(\theta + i\omega)\, \mathcal{W}_x(\theta - i\omega)\{x\}_{2\theta} .
$$

$$(5.41)$$

This equation demonstrates explicitly that the factorization of W occurs in a one-to-one fashion with respect to the factorization of the S-matrix and we finally obtain the solution for the W-matrix of the Bullough-Dodd model

$$
W(\theta) = \mathcal{W}_1(\theta)\mathcal{W}_2(\theta).
$$

$$(5.42)$$

Notice that this function posses neither poles nor zeros in the physical sheet .

10

5.2 The $A_2^{(1)}$-affine Toda theory

The $A_2^{(1)}$-affine Toda theory is the most simple example of an affine Toda theory [13, 14] involving more than one particle. It contains two particles of equal masses which are conjugate to each other, that is choosing complex scalar fields we have $\phi_1^* = \phi_2$. Its classical Lagrangian de nsity

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{\beta^2}\left(e^{\beta\sqrt{2}\phi_2} + e^{\frac{\beta}{\sqrt{2}}\left(\sqrt{3}\,\phi_1 - \phi_2\right)} + e^{-\frac{\beta}{\sqrt{2}}\left(\sqrt{3}\,\phi_1 + \phi_2\right)}\right) \tag{5.43}$$

possesses a $\mathbf{Z}_3$-symmetry, in the sense that it is left invariant under the transformation

$$\phi \to \begin{pmatrix} e^{\frac{2\pi i}{3}n} \\ & e^{\frac{4\pi i}{3}n} \end{pmatrix} \phi \qquad\qquad n = 1, 2, 3 \ . \tag{5.44}$$

From the three point couplings, which turn out to be $C_{111} = -C_{222} = -i3\beta m^2$ and $C_{112} = C_{221} = 0$ or the application of the fusing rule of affine Toda theory [15, 16, 8, 17] we obtain that the following processes are permitted

$$V_1 + V_1 \quad \to \quad V_2 = V_{\bar{1}} \tag{5.45}$$

$$V_2 + V_2 \quad \to \quad V_1 = V_{\bar{2}} \tag{5.46}$$

where we denote the particles by V_i with $i = 1, 2$. The scattering matrices are given by

$$S_{11}(\theta) = S_{22}(\theta) \quad = \quad \{1\}_\theta \tag{5.47}$$

$$S_{12}(\theta) \quad = \quad \{2\}_\theta. \tag{5.48}$$

Here the blocks are again of the form (5.16) with $h = 3$. Where $S_{11}(\theta) = S_{22}(\theta)$ have poles at $\frac{2\pi i}{3}$ describing the processes (5.45) and (5.46), whereas $S_{12}(\theta)$ has no poles in the physical sheet. Furthermore, the scattering matrix satisfies the bootstrap equations

$$S_{l2}(\theta) \quad = \quad S_{l1}(\theta + i\omega)S_{l1}(\theta - i\omega) \tag{5.49}$$

$$S_{l1}(\theta) \quad = \quad S_{l2}(\theta + i\omega)S_{l2}(\theta - i\omega) \tag{5.50}$$

11

for $l = 1, 2$, $\omega = i\pi/3$, together with the crossed versions of this. S ince the scattering matrices involved satisfy the ordinary bootstrap equation s, the wall bootstrap equations reduce to

$$W_2(\theta) \;=\; W_1(\theta + i\omega)\, W_1(\theta - i\omega)\, S_{11}(2\theta) \tag{5.51}$$

$$W_1(\theta) \;=\; W_2(\theta + i\omega)\, W_2(\theta - i\omega)\, S_{22}(2\theta) \; . \tag{5.52}$$

Together with equation (5.41) we notice that these equations are solved by

$$W_1(\theta) = W_2(\theta) = \mathcal{W}_1(\theta) \; . \tag{5.53}$$

Again $W(\theta)$ introduces no poles nor zeros in the physical sheet. The fact tha t $W_1(\theta)$ equals $W_2(\theta)$ is a consequence of the mass degeneracy of the theory, which is reflected by the automorphism of the Dynkin diagram [10, 11]. The folding

towards the Bullough-Dodd model introcuces an additional block in the W-matrix, due to the identification of particle 1 and 2, in a similar fashion as for the S-matrix.

5.3   The $A_4^{(2)}$-affine Toda theory

The $A_4^{(2)}$ affine Toda theory is the most simple example of an affine Toda theory, where the roots associated to the particles are connected by more than one lace on the Dynkin diagram. It descibes two self-conjugate real scalar fields whose classical mass ratio is $m_1^2 = (5 - \sqrt{5})/(5 + \sqrt{5})m_2^2$. The roots involved in this theory might be constructed from a $D_6^{(1)}$-affine Dynkin diagram, where the four roots forming the two handles and the two roots which are connnected to the handles are identified. The resulting roots are

$$\alpha_1 = -\frac{2\sqrt{2}}{\sqrt{5}}\left(\sin\frac{2\pi}{5}, \sin\frac{\pi}{5}\right) \qquad \text{and} \qquad \alpha_2 = \frac{4\sqrt{2}}{\sqrt{5}}\left(\sin\frac{\pi}{5}\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}\cos\frac{\pi}{5}\right) \tag{5.54}$$

where the root corresponding to the affinisation α_0 is the negative of twice the sum of this two roots. In terms of this vectors the Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{\beta^2}\left(e^{\beta\alpha_0\cdot\phi} + 2e^{\beta\alpha_1\cdot\phi} + 2e^{\beta\alpha_2\cdot\phi}\right) \quad , \tag{5.55}$$

from which we may compute the three point couplings $C_{111} = C_{222} = 0$ and $C_{221} \neq 0$, $C_{112} \neq 0$ such that the following fusing processes are possible

$$V_1 + V_1 \quad \to \quad V_2 \tag{5.56}$$

$$V_2 + V_2 \quad \to \quad V_1 \tag{5.57}$$

$$V_1 + V_2 \quad \to \quad V_1 + V_2 \; . \tag{5.58}$$

The corresponding scattering matrices turn out to be

$$S_{11}(\theta) \quad = \quad \{1\}_\theta\{4\}_\theta \tag{5.59}$$

$$S_{12}(\theta) \quad = \quad \{2\}_\theta\{3\}_\theta \tag{5.60}$$

$$S_{22}(\theta) \quad = \quad \{1\}_\theta\{2\}_\theta\{3\}_\theta\{4\}_\theta. \tag{5.61}$$

The Coxeter number h is five in this case. Here $S_{11}(\theta)$ has a single pole with negative residue at $\frac{3\pi i}{5}$ and one with positive residue at $\frac{2\pi i}{5}$ describing the process (5.56). $S_{12}(\theta)$ has single poles with negative residues at $\frac{\pi i}{5}$, $\frac{2\pi i}{5}$ and single poles with positive residue at $\frac{3\pi i}{5}$, $\frac{4\pi i}{5}$ corresponding to (5.58). $S_{22}(\theta)$ has a single pole with negative residue at $\frac{\pi i}{5}$, a single pole with positive residue at $\frac{4\pi i}{5}$ related the the fusing (5.57) and double poles at $\frac{2\pi i}{5}$ $\frac{3\pi i}{5}$. The bootstrap equation are in this case

$$S_{l2}(\theta) \quad = \quad S_{l1}(\theta + i\omega)S_{l1}(\theta - i\omega) \tag{5.62}$$

$$S_{l1}(\theta) \quad = \quad S_{l2}(\theta + i\omega)S_{l2}(\theta - i\omega) \tag{5.63}$$

$$S_{l1}(\theta) \quad = \quad S_{l1}(\theta + 3i\omega)S_{l2}(\theta + i\omega) \tag{5.64}$$

$$S_{l2}(\theta) \quad = \quad S_{l2}(\theta - i\omega)S_{l1}(\theta + 2i\omega) \tag{5.65}$$

with $l = 1, 2$. Because of the previous equations, the wall bootstrap equations reduce to

$$W_2(\theta) \quad = \quad W_1(\theta + i\omega) \, W_1(\theta - i\omega) \, S_{11}(2\theta) \tag{5.66}$$

13

$$W_1(\theta) = W_2(\theta + i\omega) W_2(\theta - i\omega) S_{22}(2\theta) \tag{5.67}$$

$$W_1(\theta) = W_2(\theta + i\omega) W_1(\theta - 3i\omega) S_{12}(2\theta) \tag{5.68}$$

$$W_2(\theta) = W_1(\theta + 2i\omega) W_2(\theta - i\omega) S_{12}(2\theta) . \tag{5.69}$$

Introducing the notation $W_{11}(\theta) := W_1(\theta)$, $W_{22}(\theta) := W_2(\theta)$ and $W_{12}(\theta) :=$ $W_2(\theta)/W_1(\theta)$, these equations decouple and we are left with the problem to solve

$$W_{ij}(\theta) = W_{ij}(\theta + i\omega) W_{ij}(\theta - i\omega) W_{ij}(\theta + 3i\omega) W_{ij}(\theta - 3i\omega) S_{ij}(2\theta) . \tag{5.70}$$

Again we utilise Fourier transforms after taking the logarithm and obtain

$$W_{ij}(\theta) = \exp\left(\int d\theta' \, G(\theta - \theta') \ln S_{ij}(2\theta') \right), \tag{5.71}$$

where the Green function $G(\theta)$ in this case is given by

$$G(\theta) = \lim_{\eta \uparrow 1} \frac{1}{\omega \sinh \frac{\pi\theta\eta}{\omega}} \left(\frac{\sinh\left(\frac{\pi}{3\omega}(\theta)\eta\right)}{\sin \frac{2\pi}{3}} + \frac{\sinh\left(\frac{4\pi}{5\omega}(\theta)\eta\right)}{\sin \omega + 3\sin 3\omega} + \frac{\sinh\left(\frac{2\pi}{5\omega}(\theta)\eta\right)}{\sin 3\omega + 3\sin 9\omega} \right) . \tag{5.72}$$

Employing now again the integral representation for the S-matrix we obtain the following integral representaion for the building blocks of the W-matrix

$$w_x(\theta) = \exp\left(I_{\frac{2\pi}{3}}(\theta) + I_\omega(\theta) + I_{3\omega}(\theta) \right) , \tag{5.73}$$

whith

$$I_a(\theta) = \frac{1}{5 - 6\sin^2 a} \int_0^\infty \frac{dt}{t \sinh t} \frac{2 \sinh\left(1 + \frac{x}{h}\right) t \sinh \frac{2\theta t}{i\pi}}{\cos a - 2 \cosh \frac{2t\omega}{\pi}} . \tag{5.74}$$

Solving the integral gives

$$w_x(\theta) = \prod_{l=0}^\infty \left(\frac{\Gamma\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right) \Gamma\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} - \frac{i\theta}{\pi}\right)}{\Gamma\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} + \frac{i\theta}{\pi}\right) \Gamma\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} - \frac{i\theta}{\pi}\right)} \right)^{P_l} . \tag{5.75}$$

with

$$P_l = \frac{\sin\left((l+1)\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} + \frac{\sin\left((l+1)\frac{\pi}{5}\right)}{2\sin\left(\frac{\pi}{5}\right)\left(5 - 6\sin^2\left(\frac{\pi}{5}\right)\right)} + \frac{\sin\left((l+1)\frac{3\pi}{5}\right)}{2\sin\left(\frac{3\pi}{5}\right)\left(5 - 6\sin^2\left(\frac{3\pi}{5}\right)\right)} \tag{5.76}$$

14

Again this equation is useful to extract the polestructure and to prove the identities (5.23) - (5.29). The function $\eta_x(\theta)$ is n ow given by

$$\eta_x(\theta) \;=\; \prod_{l=0}^{\infty} \left(\frac{\left((l+1)\frac{\omega}{\pi} - \frac{x}{2h} - \frac{i\theta}{\pi}\right)\left(-1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right)}{\left(1 + (l+1)\frac{\omega}{\pi} + \frac{x}{2h} - \frac{i\theta}{\pi}\right)\left((l+1)\frac{\omega}{\pi} + \frac{x}{2h} + \frac{i\theta}{\pi}\right)} \right)^{P_l}, \qquad (5.77)$$

satisfying the homogeneous bootstrap equation and having no poles in the physica l sheet. Further we derive the relation

$$w_x(\theta + i\omega) \; w_x(\theta - i\omega)w_x(\theta + 3i\omega)w_x(\theta - 3i\omega) = \frac{\langle x \rangle_{-2\theta}}{\langle x \rangle_{2\theta}} \qquad (5.78)$$

from which we deduce

$$\mathcal{W}_{ij_x}(\theta) = \mathcal{W}_{ij_x}(\theta + i\omega)\mathcal{W}_{ij_x}(\theta - i\omega)\mathcal{W}_{ij_x}(\theta + 3i\omega)\mathcal{W}_{ij_x}(\theta - 3i\omega)\{x\}_{2\theta}. \qquad (5.79)$$

Comparision with equation (5.70) now demonstrates that the W-matrix again factorizes in a one-to-one fashion with respect to the scattering matrix and we finally obtain the W-matrix for the $A_2^{(4)}$-affine Toda theory

$$W_1(\theta) \;=\; \mathcal{W}_1\mathcal{W}_4 \qquad (5.80)$$

$$W_2(\theta) \;=\; \mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4 \qquad (5.81)$$

From the property of the function $w_x(\theta)$ we note again that the physical sheet is free of singularities.

6   Conclusions

We have demonstrated how to formulate factorization equations and in particular the inhomogeneous bootstrap equations by employing an extended version of Zamolodchikov's algebra. Whereas in the absence of reflecting boundaries such equations could be utilised to construct the two particle scattering matrix, now they are sufficient to determine the W-matrix, which encodes the scattering of a particle off the wall. For all cases investigated $W(\theta)$ does posses neither poles nor

zeros in the physical sheet, such that the wall does not create or absorb any particles. This feature is made transparent by expressing $W(\theta)$ as infinite products of Γ functions. A ctually the structure is very similar to the one found for the minimal two pa rticle form factors $F(\theta)$ [18, 12].

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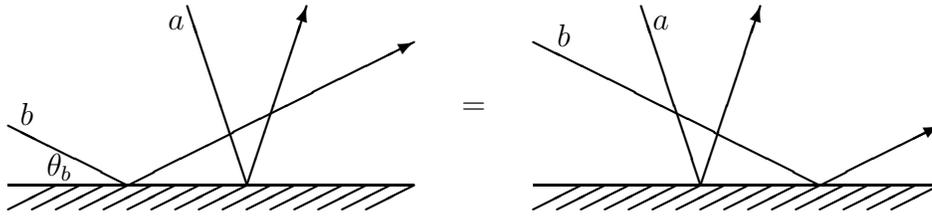
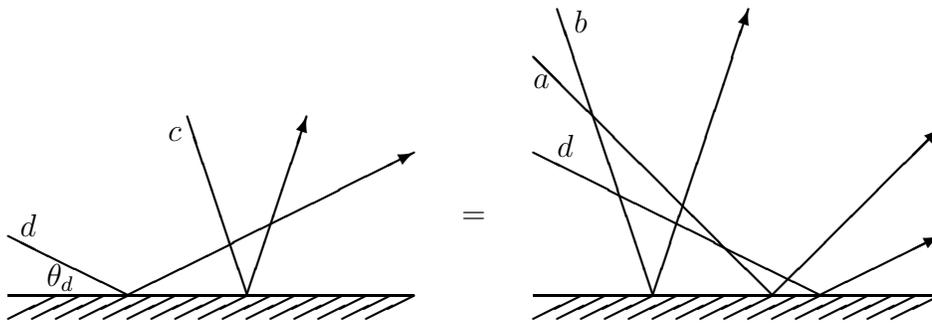Figure 1: The factorization equation in the presence of a wall

Figure 2: The inhomogeneous bootstrap equation