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**Citation:** Fring, A. & Köberle, R. (1995). Boundary Bound States in Affine Toda Field Theory. *International Journal of Modern Physics A (ijmpa)*, 10(05), pp. 739-751. doi: 10.1142/s0217751x95000346

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**Original citation:** Fring, A & Koberle, R (0030). Boundary Bound States in Affine Toda Field Theory. Int.J.Mod.Phys. A10 (1995) 739-752,

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hep-th/9404188

## Boundary Bound States in Affine Toda Field Theory

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### Abstract

We demonstrate that the generalization of the Coleman-Thun mechanism may be applied to the situation, when considering scattering processes in 1+1-dimensions in the presence of reflecting boundaries. For affine Toda field theories we find that the binding energies of the bound states are always half the sum over a set of masses having the same colour with respect to the bicolouration of the Dynkin diagram. For the case of  $E_6$ -affine Toda field theory we compute explicitly the spectrum of all higher boundary bound states. The complete set of states constitutes a closed bootstrap.

April 1994

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# 1 Introduction

In any quantum field theory its general principles will impose restrictions on their possible scattering matrices. It has turned out that in an infinite two dimensional space-time volume in particular, the equations resulting from integrability, like the Yang-Baxter- [1] and bootstrap equations [2], serve as a very powerful tool and lead to their exact determination. Various questions of physical interest, like the study of dissipative quantum mechanical systems [3], the study of the space of boundary states in open string theory [4, 5] and that of several statistical systems require the restriction to a finite space dimension. Then in addition to the S-matrix a matrix, say W, will enter the formalisms encoding the scattering off the boundary. For many theories the approach, which has been persued for infinite volume, has been successful in the finite volume case, where one solves the analogous set of equations. This yields an explicit expression for the W-matrices [6-17]. For the Sine-Gordon theory the results have been confirmed by the Bethe Ansatz [18].

In this paper we are concerned with their generalizations, affine Toda field theories [19] for which it was demonstrated in [7, 9], that it is possible to determine the W-matrices by employing the crossing, unitarity and bootstrap equations. The methods of solving the bootstrap equation in [7] was rather indirect using Fourier transforms and in [9] more explicit use has been made of the crossing unitarity relation [8]. But hitherto no sufficient explanation has been provided concerning the interpretation of the singularities occuring in the W-matrix. The natural conjecture is to assume that we may employ a bootstrap program in analogy to the bulk theory. This means that “bootstrapping” on odd order poles with positive residues should yield other matrices, such that all further matrices obtained in this manner form a closed system in the sense specified below. This prescription led to a generalization [20, 21] of the Coleman-Thun mechanism [22] in the bulk theory and provided a complete interpretation of the multipole structure in the S-matrix.

The principal aim of the following is to show that in the theory with boundaries the same program may be applied.

Our presentation is organised as follows. In section 2 we briefly review some of the main features of the S- and W-matrices. Section 3 is devoted to a general discussion of the boundary bound state bootstrap equation, from which we compute the binding energies of all boundary bound states. We illustrate the general arguments in section 4 with the concrete example of the  $E_6$ -affine Toda field theory. Here we verify that the bootstrap program may be applied to the theory in the presence of reflecting boundaries in the same fashion as in the bulk theory, which is our main result. In section 5 we state our conclusions.

## 2 Preliminaries

In order to establish our notation and to achieve a somewhat self-consistent presentation we shall briefly review some of the established properties of the S- and W-matrices, which may be derived from the Zamolodchikov algebra [2]. The W-matrix will always carry two indices, i.e.  $W_{i\alpha}(\theta)$ , the first referring to particle of type  $i$  and the second indicates that the boundary is in state  $\alpha$ . For additional distinction we shall always be referring to particles and boundary bound states by latin and greek letters, respectively. Simply by applying twice the relations for the Zamolodchikov algebra one obtains the unitarity conditions

$$S_{ij}(\theta)S_{ij}(-\theta) = 1 \quad \text{and} \quad W_{i\alpha}(\theta)W_{i\alpha}(-\theta) = 1. \quad (2.1)$$

Far less obvious are the crossing relations for the S- [2] and W-matrices [8]

$$S_{ij}(\theta) = S_{ij}(i\pi - \theta) \quad \text{and} \quad W_{i\alpha}(\theta)W_{\bar{i}\alpha}(\theta + i\pi) = S_{ii}(2\theta). \quad (2.2)$$

The most powerful restrictions for diagonal scattering matrices result from the so-called bootstrap equations. For the S-matrix they read [2]

$$S_{lk}(\theta) = S_{li}\left(\theta + i\eta_{ik}^j\right) S_{lj}\left(\theta - i\eta_{jk}^i\right) \quad (2.3)$$

and for the W-matrix [7]

$$W_{k\alpha}(\theta) = W_{i\alpha}(\theta + i\eta_{ik}^j) W_{j\alpha}(\theta - i\eta_{jk}^i) S_{ij}(2\theta + i\eta_{ik}^j - i\eta_{jk}^i) \quad . \quad (2.4)$$

Here  $\eta_{ij}^k$  denotes the fusing angle which emerges whenever the particle  $k$  can exist as a bound state in the scattering between  $i$  and  $j$ .

The solutions to this equations contain several ambiguities, like the CDD-ambiguity known from the bulk theory [23], which in the context of affine Toda field theory corresponds to a shift of the rapidity by  $i\pi$  [9]. Furthermore, it was pointed out by Sasaki [16] that whenever we have a solution to (2.4), say  $W_{n\alpha}(\theta)$  for  $n = 1, \dots, r$ , ( $r$  denoting the number of different types of particles) then, because of (2.3),  $W_{n\alpha}(\theta) \prod S_{nl}(\theta)$  will be a solution as well. However, the new solution might introduce additional poles in the physical sheet, which have to be given an interpretation. It is the

### 3 Boundary bound state bootstrap equation

which will account for these [8],

$$W_{j\beta}(\theta) = S_{ij}(\theta + i\eta_{i\alpha}^\beta) W_{j\alpha}(\theta) S_{ij}(\theta - i\eta_{i\alpha}^\beta) \quad . \quad (3.1)$$

Here  $\eta_{i\alpha}^\beta$  denotes the pole which emerges when particle  $i$  hits the boundary in the state  $\alpha$  and causes it to transit into state  $\beta$ . The interpretation of this equation is the usual one and assumes an analytic continuation in the rapidity plane. It says that the following two situations are equivalent: when  $i$  possesses precisely this rapidity, an additional particle bouncing off the wall may scatter off particle  $i$  either before or after the excitation  $i + \alpha \rightarrow \beta$  takes place. In the latter case it will pick up two S-matrices. It is this equation we shall be mainly concerned with in the following. We depict it in figure 1.

Notice that the ambiguities in  $W_{n\alpha}(\theta)$  are restricted by this equations. At first we note that the factor  $S_{nl}(\theta)$  has to be the same for all possible values of  $\alpha$ , but

more severe restrictions result from the possible additional poles introduced by it. We shall comment on this point more below.

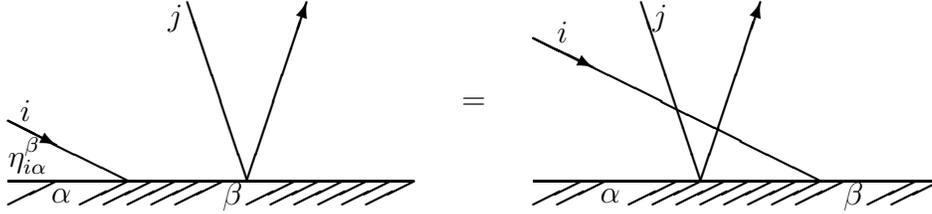


Figure 1: The boundary bound state bootstrap equation

We now like to obtain some information about the possible angles  $\eta$ . We apply equation (3.1) to the two processes  $i + \alpha \rightarrow \beta$  and  $\beta + \bar{i} \rightarrow \alpha$  and use unitarity and crossing of the S-matrix to obtain

$$\eta_{i\alpha}^{\beta} + \eta_{\bar{i}\beta}^{\alpha} = \pi . \quad (3.2)$$

A further useful equation results when considering a situation in which the state  $\beta$  results as an excitation from  $\alpha$  by particle  $i$  and  $\gamma$  can be reached from  $\alpha$  by  $j$  and from  $\beta$  by  $k$ , refer figure 2. In this case we may eliminate the W-matrices in (3.1)

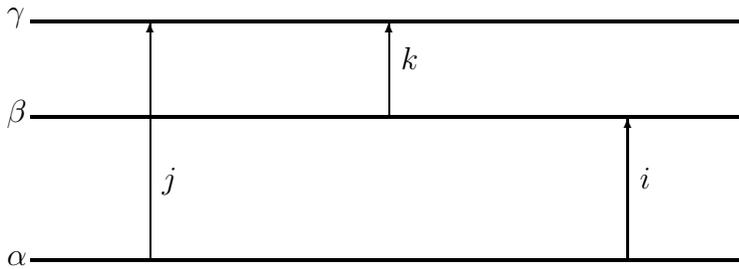


Figure 2: Boundary transitions

and obtain an equation involving solely S-matrices

$$S_{lj}(\theta + i\eta_{j\alpha}^{\gamma}) S_{lj}(\theta - i\eta_{j\alpha}^{\gamma}) = S_{li}(\theta + i\eta_{i\alpha}^{\beta}) S_{li}(\theta - i\eta_{i\alpha}^{\beta}) S_{lk}(\theta + i\eta_{k\beta}^{\gamma}) S_{lk}(\theta - i\eta_{k\beta}^{\gamma}) .$$

This equation is useful since in general we consider the situation involving the boundaries, when the bulk theory described by the S-matrices is already known. We shall employ this equation below as a consistency equation for the fusing angles. Clearly, similar equations may be derived involving other processes.

We shall now compute the binding energies for the boundary bound states. Remembering [9] that the W-matrix is constituted out of blocks  $\mathcal{W}_x$

$$\mathcal{W}_x(\theta) = \frac{w_{1-x}(\theta)w_{-1-x}(\theta)}{w_{1-x-B}(\theta)w_{-1-x+B}(\theta)} \quad , \quad (3.3)$$

with

$$w_x(\theta) = \frac{\langle \frac{x-h}{2} \rangle_\theta}{\langle \frac{x-h}{2} \rangle_{-\theta}} \quad (3.4)$$

and  $\langle x \rangle_\theta = \sinh \frac{1}{2} \left( \theta + \frac{i\pi}{h} x \right)$ ,  $0 \leq B(\beta) \leq 2$ .  $\mathcal{W}_x(\theta)$  posses poles at  $\theta_\pm = (\pm 1 - x - h)/\frac{i\pi}{2h}$ , and we therefore obtain that a pole in the physical sheet can only result when  $x$  lies between  $2h$  and  $4h$ . ( $h$  denotes here as usual the Coxeter number of the underlying Lie algebra.) Recalling further that the W-matrix was constructed by replacing the blocks  $\{x\}_\theta = [x]_\theta/[x]_{-\theta}$  ( $[x]_\theta = \langle x+1 \rangle_\theta \langle x-1 \rangle_\theta / \langle x+1-B \rangle_\theta \langle x-1+B \rangle_\theta$ ) in  $S_{i\bar{i}}$  by  $\mathcal{W}_x(\theta)$  and shifting some of the  $x$ 's by  $2h$ , we obtain that the pole  $\theta_\pm$  has a corresponding pole in  $S_{ii}(\theta)$  at  $\eta_\pm = (\pm 1 + x)\frac{i\pi}{2h}$ , which relates to one in  $W_{i\alpha}(\theta)$  at  $2\eta_\pm$ . We may see this more directly from (2.2).

In other words, if the block  $\mathcal{W}_{(x-h)+2h}(\theta)$  occurs in  $W(\theta)$ , then the block  $\{x\}_\theta$  occurs in  $S_{ii}(\theta)$  and  $\{h-x\}_\theta$  appears in  $S_{\bar{i}\bar{i}}(\theta)$ . Having found a relation between the poles in the W- and in the scattering matrix, we have shifted the problem of finding the angles  $\eta_{i\alpha}^\beta$  towards a problem of analysing the singularities in  $S_{ii}$  at  $2\eta_\pm$ . This means that we may view the two scattering matrices in (3.1) as part of a bootstrap equation. In fact we shall verify

$$S_{i\bar{i}}(\theta + i\eta_{i\alpha}^\beta) S_{\bar{i}i}(\theta - i\eta_{i\alpha}^\beta) = \prod_k S_{lk}(\theta) \quad , \quad (3.5)$$

where all the  $k$ 's will be of the same colour with respect to the bicolouration of the Dynkin diagram [24, 25, 26]. We can understand this equation by drawing the

possible graphs, which are in agreement with the generalization of the Coleman-Thun mechanism [22] in the bulk theory. Considering a situation in which two identical particles with relative rapidities  $2\eta$  scatter against each other, we observe that the only consistent graphs we may draw are box-diagrams. Then the left hand side of this equation simply corresponds to the scattering of an additional particle  $l$  with the two particles and the right hand side encodes the scattering with the intermediate particles. We illustrate this in some more detail for a double pole. The only graph we may draw is depicted in figure 3.

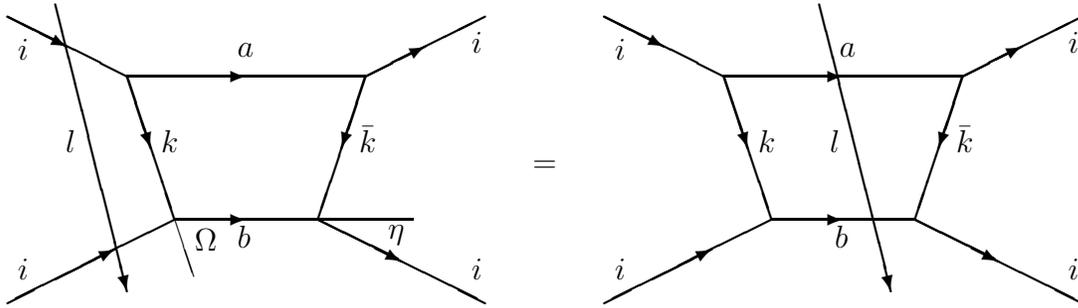


Figure 3: “Double pole bootstrap equation ”

It corresponds to the equation

$$S_{li}(\theta + i\eta) S_{li}(\theta - i\eta) = S_{la}(\theta) S_{lb}(\theta) . \quad (3.6)$$

On the other hand we could have scattered  $l$  with different rapidities such that we obtain the usual bootstrap equation involving three S-matrices only

$$S_{li}(\theta + i\eta) = S_{la}(\theta) S_{lk}(\theta + i\Omega) \quad (3.7)$$

$$S_{li}(\theta - i\eta) = S_{lb}(\theta) S_{l\bar{k}}(i\pi + \theta + i\Omega) , \quad (3.8)$$

which when multiplied with each other give equation (3.6). In the usual fashion [26] there are two fusing rules associated with them  $\sigma^{\xi(i)}\gamma_i = \sigma^{\xi(a)}\gamma_a + \sigma^{\xi(k)}\gamma_k$  and  $\sigma^{\tilde{\xi}(i)}\gamma_i = \sigma^{\xi(b)}\gamma_b + \sigma^{\tilde{\xi}(\bar{k})}\gamma_{\bar{k}}$ . After relating the  $\gamma$ 's for particle and anti-particle and

carrying out the usual identification between the angles in the bootstrap equation and the integers  $\xi(t)$ , we derive

$$\xi(i) - \xi(a) = \xi(b) - \tilde{\xi}(i) + \frac{c(i)}{2} - \frac{c(a) + c(b)}{4} .$$

Since both sides of this equation have to be integers we can deduce that the colours of  $a$  and  $b$  have to be the same, i.e.  $c(a) = c(b)$ . By eliminating particle  $k$  and  $\bar{k}$  the fusing rule related to this process reads

$$\left( \sigma^{\xi(i)-\xi(a)} + \sigma^{\xi(a)-\xi(i)+\frac{c(i)-c(a)}{2}} \right) \gamma_j = \gamma_a + \gamma_b . \quad (3.9)$$

Similar arguments hold for higher order poles and in general we will have (3.5).

Employing then the explicit expression for  $S$  [26], parameterising the fusing angle by  $\eta_{j0}^\alpha = (2l + \frac{c(i)-c(j)}{2})\frac{\pi}{h}$  and shifting the dummy variable of the product appropriately we obtain

$$\frac{\left[ 2q - \frac{c(l)+1}{2} \right]^{-\frac{1}{2}\lambda_l \cdot \left( \sigma^{q-l+\frac{c(j)-c(i)}{2}} + \sigma^{q+l} \right) \gamma_j}{\left[ -2q + \frac{c(l)+1}{2} \right]^{-\frac{1}{2}\lambda_l \cdot \left( \sigma^{q+l+\frac{c(i)-1}{2}} + \sigma^{q-l+\frac{c(j)-1}{2}} \right) \gamma_j}} = \frac{\left[ 2q - \frac{c(l)+1}{2} \right]^{-\frac{1}{2}\lambda_l \cdot \sigma^q (\gamma_{i_1} + \dots + \gamma_{i_n})}}{\left[ -2q + \frac{c(l)+1}{2} \right]^{-\frac{1}{2}\lambda_l \cdot \sigma^{q+\frac{c(i)-1}{2}} (\gamma_{i_1} + \dots + \gamma_{i_n})}}$$

from which we infer the general fusing rule

$$\left( \sigma^l + \sigma^{-l+\frac{c(j)-c(i)}{2}} \right) \gamma_i = \gamma_{i_1} + \dots + \gamma_{i_n} . \quad (3.10)$$

Implicitly we required here that all the colours on the right hand side coincide, in other words there is no fusing rule if the colours differ. Notice further that acting on this equation with  $\sigma_-$  or  $\sigma_+$ , which for the standard fusing rule yields a second solution [26], will simply turn this equation into itself and hence this equation possesses only one solution. The reason for this will be apparent below. We may now employ a relation found in [26] \*

$$q(1) \cdot \sigma^l \gamma_i = -iy_i \sin \left( \frac{\pi}{h} \right) e^{-i\frac{\pi}{h} \left( 2l + \frac{1-c(i)}{2} \right)} \quad (3.11)$$

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\*The minus sign on the right hand side was missing therein.

and project (3.10) into the velocity plane, then

$$2y_j \cos(\eta_{j0}^\alpha) = y_{i_1} + \dots + y_{i_n} . \quad (3.12)$$

Using now the proportionality between the Perron-Frobenius vector and the masses [25] yields for the binding energy of level  $\alpha$ , after setting  $E_0 = 0$ ,

$$E_\alpha = \frac{1}{2} (m_{i_1} + \dots + m_{i_n}) = m_j \cos(\eta_{j0}^\alpha) . \quad (3.13)$$

We illustrate this in figure 4. Since this mass-triangle is equilateral it becomes clear

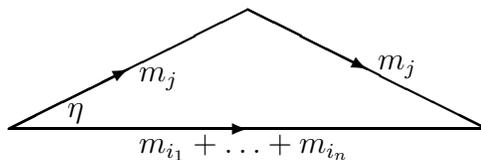


Figure 4: Mass triangle in the velocity plane

as well that there is only one possible triangle, unlike in the general case, in which one can draw an equivalent one, corresponding to the second solution. This is the geometrical reason for the fact that the fusing rule possess only one solution.

## 4 $E_6$ -Affine Toda Field Theory

In order to verify the general claims made in the previous section, we shall investigate now explicitly  $E_6$ -affine Toda field theory. It furnishes a very good example since it possesses a very rich structure and several nontrivial features in comparison with other cases. Its fusing properties are nontrivial, it possesses self-conjugate and non-self-conjugate scalar fields unlike the other members of the E-series and in addition, when folded according to (4.3), it constitutes the theory with the automorphism of the highest order, namely 3 . Its classical Lagrangian density for the

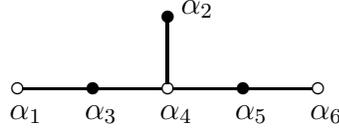
bulk theory reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{i=0}^6 n_i e^{\beta \alpha_i \cdot \phi} \quad (4.1)$$

with  $n_i$  denoting the Kac labels, being the integer coefficients in the expansion of the highest root in terms of the simple roots  $\alpha_i$  of  $E_6$ .  $\phi$  constitutes a vector whose components are the six scalar fields of the theory. A suitable basis for the simple roots in which the mass-matrix diagonalises is given by

$$\mathcal{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}. \quad (4.2)$$

Here the entry  $\mathcal{A}_{ij}$  denotes the  $j$ 's component of the simple root  $\alpha_i$ . Labelling our particles according to the following Dynkin diagram



The negative of the highest root acquires the form  $\alpha_0 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6$ . The theory remains invariant under the following transformations of the extended Dynkin diagram

$$\alpha_1 \rightarrow \alpha_6 \rightarrow \alpha_0 \quad , \quad \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_5 \quad , \quad \alpha_4 \rightarrow \alpha_4 \quad (4.3)$$

which may be used to derive the twisted theory  $D_4^{(3)}$ -theory via the folding procedure [27-30]. The masses which in all Toda theories may be organised into the Perron-Frobenius eigenvector of the Cartan matrix take on particularly simple values

$$m_1^2 = m_6^2 = (3 - \sqrt{3}) m^2 \quad m_2^2 = 2(3 - \sqrt{3}) m^2 \quad (4.4)$$

$$m_3^2 = m_5^2 = (3 + \sqrt{3}) m^2 \quad m_4^2 = 2(3 + \sqrt{3}) m^2 \quad . \quad (4.5)$$

For reasons of completeness we shall furthermore report all the two-particle scattering matrices up to the ones which can be trivially obtained from the identities

$$S_{ij}(\theta) = S_{ji}(\theta) = S_{\bar{i}\bar{j}}(\theta) \quad [21]$$

$$\begin{aligned} S_{11} &= \{1\}\{7\} & S_{12} &= \{4\}\{8\} & S_{13} &= \{2\}\{6\}\{8\} \\ S_{14} &= \{3\}\{5\}\{7\}\{9\} & S_{15} &= \{4\}\{6\}\{10\} & S_{16} &= \{5\}\{11\} \\ S_{22} &= \{1\}\{5\}\{7\}\{11\} & S_{23} &= \{3\}\{5\}\{7\}\{9\} & S_{24} &= \{2\}\{4\}\{6\}^2\{8\}\{10\} \\ S_{33} &= \{1\}\{3\}\{5\}\{7\}^2\{9\} & S_{34} &= \{2\}\{4\}^2\{6\}^2\{8\}^2\{10\} & S_{35} &= \{3\}\{5\}^2\{7\}\{9\}\{11\} \\ & & S_{44} &= \{1\}\{3\}^2\{5\}^3\{7\}^3\{9\}^2\{11\} & & . \end{aligned}$$

We have omitted here the explicit dependence on the rapidity  $\theta$ . Then by starting with one of the solutions obtained for the boundary in the ground state [9]

$$W_{10}(\theta) = W_{60}(\theta) = \mathcal{W}_5(\theta)\mathcal{W}_{35}(\theta) \quad (4.6)$$

$$W_{20}(\theta) = \mathcal{W}_1(\theta)\mathcal{W}_7(\theta)\mathcal{W}_{11}(\theta)\mathcal{W}_{29}(\theta) \quad (4.7)$$

$$W_{30}(\theta) = W_{50}(\theta) = \mathcal{W}_3(\theta)\mathcal{W}_5(\theta)\mathcal{W}_7(\theta)\mathcal{W}_{11}(\theta)\mathcal{W}_{29}(\theta)\mathcal{W}_{33}(\theta) \quad (4.8)$$

$$W_{40}(\theta) = \mathcal{W}_1(\theta)\mathcal{W}_3(\theta)\mathcal{W}_5^2(\theta)\mathcal{W}_7(\theta)\mathcal{W}_9^2(\theta)\mathcal{W}_{27}(\theta)\mathcal{W}_{29}(\theta)\mathcal{W}_{31}^2(\theta)\mathcal{W}_{35}(\theta) \quad (4.9)$$

we now construct all the higher boundary states by means of a bootstrap principle which is in close analogy to the one in the bulk theory. Employing the fact that the poles of  $\mathcal{W}_x(\theta)$  are situated at  $\theta_{\pm} = \frac{\pm 1 - x - h}{2h}i\pi$ , with  $h = 12$  and  $\pm$  referring to the sign of the residues, we observe that the only possible poles in the physical sheet are the ones resulting from the blocks which have been shifted. Letting then the odd order poles with positive residue participate in the bootstrap equation (3.1) we compute

$$W_{j\alpha}(\theta) = S_{j1} \left( \theta - \frac{i\pi}{12} \right) W_{j0}(\theta) S_{j1} \left( \theta + \frac{i\pi}{12} \right) = S_{j3}(\theta) W_{j0}(\theta) \quad (4.10)$$

$$W_{j\beta}(\theta) = S_{j2} \left( \theta - \frac{4i\pi}{12} \right) W_{j0}(\theta) S_{j2} \left( \theta + \frac{4i\pi}{12} \right) = S_{j2}(\theta) W_{j0}(\theta) \quad (4.11)$$

$$W_{j\gamma}(\theta) = S_{j3} \left( \theta - \frac{2i\pi}{12} \right) W_{j0}(\theta) S_{j3} \left( \theta + \frac{2i\pi}{12} \right) = S_{j2}(\theta) S_{j5}(\theta) W_{j0}(\theta) \quad (4.12)$$

$$W_{j\delta}(\theta) = S_{j_3} \left( \theta - \frac{4i\pi}{12} \right) W_{j_0}(\theta) S_{j_3} \left( \theta + \frac{4i\pi}{12} \right) = S_{j_5}(\theta) W_{j_0}(\theta) \quad (4.13)$$

$$W_{j\epsilon}(\theta) = S_{j_4} \left( \theta - \frac{i\pi}{12} \right) W_{j_0}(\theta) S_{j_4} \left( \theta + \frac{i\pi}{12} \right) = S_{j_2}(\theta) S_{j_3}(\theta) S_{j_5}(\theta) W_{j_0}(\theta) \quad (4.14)$$

$$W_{j\phi}(\theta) = S_{j_4} \left( \theta - \frac{3i\pi}{12} \right) W_{j_0}(\theta) S_{j_4} \left( \theta + \frac{3i\pi}{12} \right) = S_{j_3}(\theta) S_{j_5}(\theta) W_{j_0}(\theta) \quad (4.15)$$

$$W_{j\rho}(\theta) = S_{j_5} \left( \theta - \frac{2i\pi}{12} \right) W_{j_0}(\theta) S_{j_5} \left( \theta + \frac{2i\pi}{12} \right) = S_{j_2}(\theta) S_{j_3}(\theta) W_{j_0}(\theta) \quad (4.16)$$

Clearly in order to have a proper bootstrap we have to ensure that our system closes under equation (3.1) and we still have to investigate the singularity structure of (4.10) - (4.16). Keeping the same philosophy, that is bootstrapping only on odd order poles with positive residues we can account for all poles in this manner. It is worth noting that in proceeding in this fashion care has to be taken about possible zeros emerging from the unshifted blocks which might cancel some of the poles or alter their order. In the following table we present all the “fusing angles” which participate in the bootstrap:

$i \setminus \mu$	0	$\alpha$	$\beta$	$\gamma$	$\delta$	$\epsilon$	$\phi$	$\rho$
1	$1^\alpha$	$3^\gamma$	$1^\rho 5^\alpha$	$1^\epsilon 5^\phi 11^\beta$	$1^\phi 7^\beta 11^0$	$11^\rho$	$7^\rho 11^\alpha$	$9^\delta$
2	$4^\beta$	$4^\rho 6^\alpha$	$2^\phi 6^\beta 8^0$	$6^\gamma 8^\delta$	$4^\gamma 6^\delta$	$6^\epsilon 8^\phi$	$4^\epsilon 6^\phi 10^\beta$	$6^\rho 8^\alpha$
3	$2^\gamma 4^\delta$	$2^\epsilon 4^\phi 8^0$	$4^\gamma$	$6^\rho$	$6^\alpha$	$8^\gamma 10^\delta$	$8^\delta$	$4^\epsilon 8^\beta 10^0$
4	$1^\epsilon 3^\phi 5^\beta$	$5^\rho$	$3^\epsilon 7^0$	$7^\delta$	$5^\gamma$	$7^\phi 9^\beta 11^0$	$5^\epsilon 9^0$	$7^\alpha$
5	$2^\rho 4^\alpha$	$6^\delta$	$4^\rho$	$4^\epsilon 8^\beta 10^0$	$2^\epsilon 4^\phi 8^0$	$8^\rho 10^\alpha$	$8^\alpha$	$6^\gamma$
6	$1^\delta$	$1^\phi 7^\beta 11^0$	$1^\gamma 5^\delta$	$9^\alpha$	$3^\rho$	$11^\gamma$	$7^\gamma 11^\delta$	$1^\epsilon 5^\phi 11^\beta$

Table 1: The boundary fusing angles  $\eta'_{i\mu}$

Here each entry in the table indicates a fusing angle as multiple of  $\frac{i\pi}{12}$ , where the left column refers to the particle type which scatters off the boundary in the state indicated in the first row. The superscript refers to the state the boundary is changing into. We observe that all the angles match up in the way announced earlier in equation (3.2). As a further consistency check we may then employ the equation

subsequent to (3.2) together with the explicit form for the scattering matrices. In figure 5 we illustrate for some cases which particles may cause an excitation or lowering of the states in the boundary when possessing a particular value of the rapidity. So we obtain an interesting picture familiar from atomic physics suggesting that the boundary interacts with the particles in a kind of matter-radiation way. Due to the CDD-ambiguity, there is a second solution for the ground state and a similar result may be found, when starting with this other ground state. The ambiguity mentioned at the end of section 2 has now disappeared. We may start the bootstrap with an expression  $\prod SW_{i0}$ , where the product runs over a different range than the one found in our solution. For the cases we checked we do not obtain a closed bootstrap. However, there may be possibilities for which this can be achieved. Appealing to a similar principle as in the bulk theory, that is expecting the solution to be “minimal”, the presented solution seems the most natural one.

## 5 Conclusions

As our principal result we have found that the generalization of the Coleman-Thun mechanism in the bulk theory possesses an analogue in the theory with boundaries. For the  $E_6$ -affine Toda field theory we have explicitly demonstrated how the bootstrap closes. The binding energies are half the sum of the masses belonging to the same colour with respect to the bicolouration of the Dynkin diagram. We found six different energy levels in this case, of which two are degenerate.

We have carried out a similar analysis for the  $E_7$ -theory and several members of the A-series, finding that the bootstrap always closes in this fashion. We obtain in each case that the number of energy levels equals the rank of the underlying Lie algebra.

There are several immediate questions to be answered. The generalisation of this result to other theories is highly desirable. Having a closed formula for the

W-matrix in analogy to the S-matrix, valid for all algebras will probably be a necessary step into this direction. A complete investigation of the possible integrable boundary conditions is still outstanding, although steps into this direction have been undertaken in [31].

A very interesting challenge is posed by the question of how to extend this results to the situation off-shell.

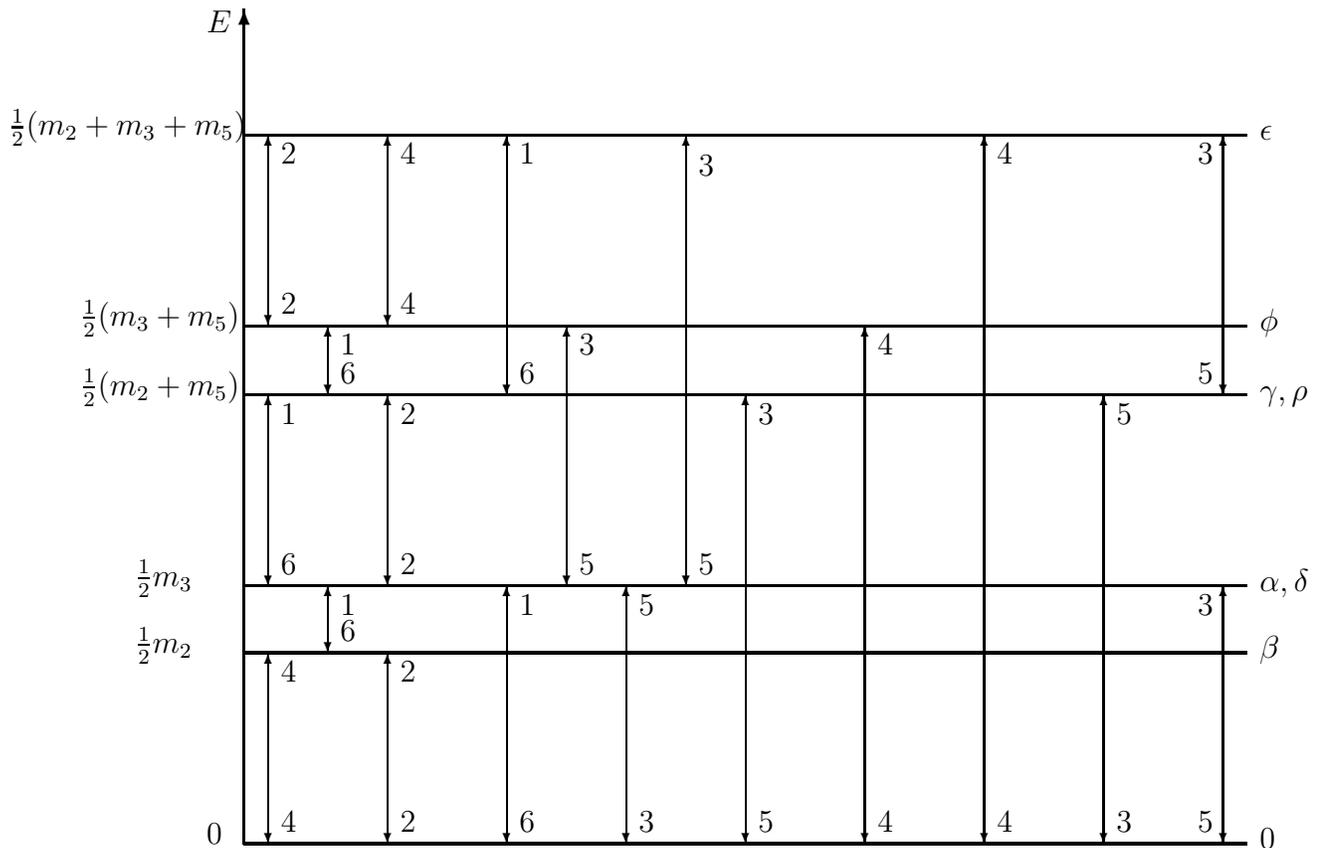


Figure 5: Boundary transitions for  $E_6$  affine Toda field theory

### Acknowledgements

A.F. would like to thank HEFCW and FAPESP(Brasil) for financial support and furthermore the Instituto de Fisica de São Carlos for its hospitality, where part

of this work has been carried out.

After the completion of our work we received a recent preprint [31], in which similar conclusions have been reached for the case of  $A_2$ -affine Toda field theory.

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