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Affine Toda Field Theory in the Presence of Reflecting Boundaries

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Abstract

We show that the "boundary crossing-unitarity equation" recently proposed by Ghoshal and Zamolodchikov is a consequence of the boundary bootstrap equation for the S-matrix and the wall-bootstrap equation. We solve this set of equations for all affine Toda theories related to simply laced Lie algebras, obtaining explicit formulas for the W-matrix which encodes the scattering of a particle with the boundary in the ground state. For each theory there are two solutions to these equations, related by CDD-ambiguities, each giving rise to different kind of physics.

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1 Introduction

The central element in a quantum field theory, on-shell, is its scattering matrix, which relates the asymptotic in- and out states. As shown in the seminal paper by Zamolodchikov and Zamolodchikov[1] integrable field theories in 1+1 dimensions posses an n-particle S-matrix, which factorises into 2-particle S-matrices, which can then be determined exactly.

Several problems in quantum mechanics, like dissipative systems[2] can be understood as a quantum field theory in the presence of boundaries [3]. In particular, one might get some more insight into the space of boundary states in open string theory [4]. The general scheme of this approach was initiated by Cherednik [5], who studied the S-matrix describing the scattering of part icles off a wall. In particular the question of how to find and solve the factorisation (Yang-Baxter) equation in the presence of a reflecting wall has been answered. Recent research has added bootstrap[6] and crossing/unitarity equations[7], whic h complete the set of equations necessary to compute the 2-particle S-matrix in a theory with boundaries.

For a theory to remain integrable the boundary conditions should maintain a suf ficient number of conservation laws. For example in conformal field theories [8], integrability demands the boundary condi tions not to introduce any length scale. This allows conformal invariance to be imposed, requiring now momentum conservation parallel to the boundary to b e hold. As discussed in [7] a breaking of conformal invariance has to respect the conservation laws, selecting in this way *integrable* boundary condition s. Formally the integrals of motion are found to be generalizations of the Hamiltonian to higher spins

$$H_s \;=\; \int_{-\infty}^{0} dx\left(T^{\bar z\bar z}_{s+1}+T^{\bar z z}_{s-1}-T^{zz}_{s+1}-\bar T^{\bar z z}_{s-1}\right)+\phi_B(y) \tag{1.1}$$

where the *boundary perturbation* $\phi_B(y)$ denotes some local field. The quantities H_s

do not depend on y, if

$$\frac{d\phi_B(y)}{dy} \ = \ T_{s+1}^{\bar{z}\bar{z}} + T_{s-1}^{\bar{z}z} - T_{s+1}^{zz} - \bar{T}_{s-1}^{\bar{z}z}|_{x=0} \quad . \tag{1.2}$$

In the present paper we shall be concerned with affine Toda field theories (ATFTs) [9] and we will assume the existe nce of such boundary conditions*. ATFTs constitute an important example of completely integrable models, since they are explicit Lagrangian versions of integrable deformations of conformal field theories. The breaking of the conformal symmetry simply corresponds to an affinisation of the Lie algebra **g**, underlying the conformal invariant theory. Many other features can be expressed as well very neatly in terms of Lie algebraic quantities, which makes them interesting objects to study even from a purely mathematical point of view.

Our presentations falls into two main sections. In the following we review some of the features of the scattering matrix of ATFTs in order to establish our notation and to formulate the equations needed, in particular (2.10). Section three is the central part of our manuscript. We derive a *pair-bootstrap* equation for the S-matrix and show that, together with the wall-bootstrap equation, t hey imply the boundary crossing-unitarity eqution. We then present the full set of constraining equations for the W-matrix, which encodes scattering due to the wall, and solve it for all simply laced ATFTs. Finally we state our conclusio ns.

2 The S-matrix of Affine Toda Field Theory

In the original formulation by Cherednik [5] of a scattering theory which includes boundaries, a distinction is made between the scattering matrices encoding the process before or after some particles have hit the boundary. In general these matrices turn out to be different, but are related by some relatively simple relations [5, 10, 11, 12, 13, 14]. Assuming diagonality of all scattering matrices involved,

*We hope to report elsewhere on a proper investigation of this question.

this distinction becomes obsolete [6] and one solely has to deal with one type of S-matrix. Presuming further the existence of some suitable boundary conditions, in the sense of (1.2), the theory will again be purely elastic. The S-matrix will then be the product of two factors: one encodes scattering off the boundary and the other coincides with the S-matrix in the absence of boundaries. It is this second factor, which we review in this section.

As the result of a sequence of investigations [15, 16, 17, 18, 19, 20, 21, 22] the S-matrix of affine Toda field theories related to simply laced Lie algebras has been found to take on a very compact form [21, 22] † . Adopting the notation of [24, 22] it reads

$$S_{ij}(\theta) = \prod_{q=a}^{b} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_{\theta}^{-\lambda_i \cdot \sigma^q \gamma_j} \quad . \tag{2.3}$$

Here θ denotes the relative rapidities, where θ_i parameterises as usual the momenta $p_i = m_i(\cosh\theta_i, \sinh\theta_i)$. The limits are taken to be $a = \frac{c(i)+1}{2}$ and $b = \frac{h-1}{2} + \frac{c(i)+c(\bar{\imath})}{4}$. $\{\}_\theta$ is a building block comprised out of sinh-functions, i.e. $\{x\}_\theta = [x]_\theta/[x]_{-\theta}$, $[x]_\theta =< x+1 >_\theta < x-1 >_\theta \; / < x+1-B >_\theta < x-1+B >_\theta$ and $< x >_\theta = \sinh\frac{1}{2}\left(\theta + \frac{i\pi x}{h}\right)$. $B(\beta)$ is a real function between 0 and 2 incorporating the coupling constant dependence of the Lagrangian field theory, which is assumed to be real in the following. In finding the solution of functional equations it is useful to employ the following equivalent integral representation for each block

$$\{x\}_\theta = \exp\left(\int_0^\infty \frac{dt}{t \sinh t} \, f_{x,B}(t) \quad \sinh\frac{\theta t}{i\pi} \right) \tag{2.4}$$

where

$$f_{x,B}(t) = 8 \sinh\frac{tB}{2h} \sinh\frac{t}{h}\left(1 - \frac{B}{2}\right) \sinh t \left(1 - \frac{x}{h}\right) . \tag{2.5}$$

In each affine Toda theory related to a simple Lie algebra **g** there are $r \equiv$ rank of **g** scalar fields, describing r particles, which can either be associated with a fundamental weight λ_i or a simple root α_i. The map σ, emerging in the exponent of

† For the non-simply laced case several candidates exist for an S-matrix. Refer to [23] and references therein.

(2.3), denotes the Coxeter element of the Weyl group, being a product of reflections in a complete set of simple roots. In order to define a "universal" element σ, that is an element which can be written in a general form, independent of what Lie algebra $\mathbf{g}$ one is concerned with, one associates signs + and − to the vertices of the Dynkin diagrams and defines a function $c(i) = \pm 1$ which identifies them. Then the two elements $\sigma_\pm$ of the Weyl group consisting of reflections related to simple roots associated with ±-signs can be used to define uniquely and unambiguously a particular Coxeter element $\sigma := \sigma_- \sigma_+$ for any Lie algebra $\mathbf{g}$. The Coxeter element splits all the roots into r disjoint orbits containing each h, being the Coxeter number, elements. The roots $\gamma_i = c(i)\alpha_i$ lie all in distinct orbits Ω_i [24] and one can therefore alternatively associate a whole orbit to a particle.

It can be shown [22] that $S_{ij}(\theta)$ (2.3) is a meromorphic function which satisfies the usual crossing and unitarity relations demanded from the two particle S-matrix

$$S_{ij}(\theta) S_{ij}(-\theta) = 1 \qquad \text{and} \qquad S_{i\bar{j}}(\theta) = S_{ij}(i\pi - \theta) \ . \tag{2.6}$$

In the case the three point coupling C_{ijk} is non-zero, $S_{ij}(\theta)$ will posses an odd order pole with positive residue due to the propagation of a bound state particle k. Then (2.3) satisfies the so-called bootstrap equation formulated originally in [1] in order to find constraining equations, which pose an equivalent to the Yang-Baxter equation [25] for diagonal S-matrices

$$S_{li}\left(\theta + i\eta_i\right) \ S_{lj}\left(\theta + i\eta_j\right) \ S_{lk}\left(\theta + i\eta_k\right) \ = \ 1 \ . \tag{2.7}$$

Here the "fusing angles" η_t, for $t = i, j, k$, are given by $\eta_t = -\frac{\pi}{h}\left(2\xi(t) + \frac{1-c(t)}{2}\right)$ and are related to the fusing rule, conjectured in [21], which decides whether the coupling constant C_{ijk} is vanishing or not. It states that only if there exist equivalence classes of integers $(\xi(i), \xi(j), \xi(k))$ and $(\xi'(i), \xi'(j), \xi'(k))$ satisfying

$$\sum_{t=i,j,k} \sigma^{\xi(t)} \gamma_t = 0 \qquad \text{and} \qquad \sum_{t=i,j,k} \sigma^{\xi'(t)} \gamma_t = 0 \ , \tag{2.8}$$

then the three-point coupling C_{ijk} will be non-zero. The two classes correspond to the only two inequivalent solutions which are related via $\xi'(t) = -\xi(t) + \frac{c(t)-1}{2}$ [22]. Classically this rule simply corresponds to the non-vanishing of the commutator involving two stepoperators of the Lie algebra $\mathbf{g}$ $\left[E_{\sigma^{\xi(i)}\gamma_i}, E_{\sigma^{\xi(j)}\gamma_j}\right]$ [24], whereas quantum mechanically it becomes equivalent to the bootstrap equation (2.7).

Making use of the identity $\gamma_i = (\sigma_- - \sigma_+)\lambda_i$ we can re-express (2.8) in terms of fundamental weights

$$\sum_{t=i,j,k} \sigma^{-\xi'(t)}\lambda_t = 0 \qquad \text{and} \qquad \sum_{t=i,j,k} \sigma^{-\xi(t)}\lambda_t = 0 \tag{2.9}$$

which will be of importance below.

Further we require a particular combination of the boostrap equations in the next section. Selecting the three equations in (2.7) for $l = i, j, k$ and substituting them into each other we derive the *pair-bootst rap* equation

$$S_{kk}\left(\theta + i2\eta_k\right) = S_{ii}\left(\theta + i2\eta_i\right)\; S_{jj}\left(\theta + i2\eta_j\right)\; S_{ij}^2\left(\theta + i\eta_i + i\eta_j\right) \quad . \tag{2.10}$$

This means having a pair of two particles i and j, possessing rapidities which will give rise to a bound state, it will be equivalent to scattering either the two pairs of particles or the two bound states against each other. We depict this situation in figure 1.

This equation posses as well a geometrical formulation, in the same manner as (2.7) can be re-expressed geometrically in terms of (2.8) and (2.9).

To see this we consider

$$S_{ii}(\theta + 2i\eta_i) = \prod_{q=1}^{h} \left\{2q - c(i)\right\}_{\theta + i2\eta_i}^{-\frac{1}{2}\lambda_i\cdot\sigma^q\gamma_i} = \prod_{q=1}^{h}\left(\frac{[2q - 4\xi(i) - 1]_{2\theta}}{[-2q - 4\xi(i) + 2c(i) - 1]_{2\theta}}\right)^{-\frac{1}{2}\lambda_i\cdot\sigma^q\gamma_i}$$

where we have employed the relation $\{x\}_{\theta + \frac{\pi i y}{h}} = \frac{[x+y]_\theta}{[y-x]_\theta}$. Shifting now in the numerator and denominator by

$$q \to q + 2\xi(i) \qquad \text{and} \qquad q \to q - 2\xi(i) + c(i) - 1$$

respectively, gives

$$S_{ii}(\theta+2i\eta_i) \;=\; \prod_{q=1}^{h} \frac{[2q-1]_\theta^{-\frac{1}{2}\lambda_i\cdot\sigma^{q+2\xi(i)}\gamma_i}}{[-2q+1]_\theta^{-\frac{1}{2}\lambda_i\cdot\sigma^{q+2\xi'(i)}\gamma_i}} \quad . \tag{2.11}$$

Performing similar manipulations on the expression for the square of the S-matrix yields

$$S_{ij}^2(\theta+i\eta_i+i\eta_j) \;=\; \prod_{q=1}^{h} \frac{[2q-1]_\theta^{-\lambda_i\cdot\sigma^{q+\xi(i)+\xi(j)}\gamma_j}}{[-2q+1]_\theta^{-\lambda_i\cdot\sigma^{q+\xi'(i)+\xi'(j)}\gamma_j}} \quad . \tag{2.12}$$

Employing now the first equation in (2.8) and the second in (2.9) we derive the relation

$$\lambda_i\cdot\sigma^{q+2\xi(i)}\gamma_i+\lambda_j\cdot\sigma^{q+2\xi(j)}\gamma_j+2\lambda_i\cdot\sigma^{q+\xi(i)+\xi(j)}\gamma_j=\lambda_k\cdot\sigma^{q+2\xi(k)}\gamma_k \;. \tag{2.13}$$

A similar equation can be obtained for the second solution for the fusing rule involving $\xi'(t)$ instead of $\xi(t)$. Assembling now this results, that is multiplying the expression (2.11) for i and j with (2.12), the power of the building blocks will take on half the left hand side of (2.13), such that (2.10) is satisfied.

The scattering matrix exhibits the usual CDD-ambiguity [26] $S_{ij}(\theta)\to\psi_{ij}(\theta)$ $S_{ij}(\theta)$, being determined only up to the factor $\psi_{ij}(\theta)$. Equations (2.6) will pose the following constraint on this function

$$\psi_{ij}\left(\theta+\frac{i\pi}{2}\right)\psi_{i\bar{j}}\left(\theta-\frac{i\pi}{2}\right) \;=\; 1 \quad , \tag{2.14}$$

which is solved by any function of the form

$$\psi_{ij}(\theta)=\prod_x \frac{<x>_\theta<h-x>_\theta}{<x>_{-\theta}<h-x>_{-\theta}} \quad . \tag{2.15}$$

Further restrictions come from the bootstrap equations

$$\psi_{li}\left(\theta+i\eta_i\right)\;\psi_{lj}\left(\theta+i\eta_j\right)\;\psi_{lk}\left(\theta+i\eta_k\right) \;=\; 1 \tag{2.16}$$

such that the scattering matrix is fixed up to the function ψ_{ij}, which we require not to introduce new poles in the physical sheet, since these w ould have to participate in the bootstrap.

It should be noted that a scattering matrix can be thought of as resulting from the braiding of two operators $Z_i(\theta)$ and $Z_j(\theta)$, associated to particle i and j, respectively, in the Zamolodchikov algebra [1]

$$Z_i(\theta)Z_j(\theta) \ = \ S_{ij}(\theta)Z_j(\theta)Z_i(\theta) \tag{2.17}$$

of which an explicit representation for the matrix (2.3) has been found in [27]. Assuming the existence of some vacuum $|0\rangle$ these operators can be used to construct the Hilbert space by successive action of $Z_i^\dagger(\theta)$ on this state, that is the Hilbert space is spanned by the operators $\prod_{i=1}^{h} Z_i^\dagger(\theta)|0\rangle$. Hence by interpreting the left hand side of (2.17) as an in-state and the two operators on the right hand side as an out-state, the S-matrix acquires its original sense as defining the superposition of out-states as in-states.

3 The W-matrix of Affine Toda Field Theory

In the presence of reflecting boundaries the introduction of an additional matrix $W_{1\ldots N}^{1'\ldots N'}(\theta)$, which encodes the scattering of particles off the boundary, is required. We assume the existence of some conserved quantities such that this matrix factorises, into one-particle amplitudes $W_i^{i'}(\theta)$, in a similar fashion as the S-matrix. Furthermore we presume its diagonality, such that particle $Z_i(\theta)$ does not change its quantum numbers while scattering from the wall, but solely reverses its momentum. Then extending the algebra (2.17) by an operator $Z_w(0)$, representing the wall, the one-particle reflection amplitude $W_i(\theta)$ results from

$$Z_i(\theta)Z_w(0) \ = \ W_i(\theta)Z_i(-\theta)Z_w(0) \ \ . \tag{3.18}$$

The operator $Z_w(0)$ is thought to define the ground state of the Hilbert space in the presence of the boundary, i.e. $|W\rangle := Z_w(0)|0\rangle$, such that now the superposition of in-states in terms of out-states is governed by a product of S- and W-matrices.

In analogy to a derivation of the first equation in (2.6), that is applying twice (2.17), from its very definition (3.18), the equivalent unitarity equation for W_i results to be

$$W_i(\theta)W_i(-\theta) \;=\; 1 \quad . \tag{3.19}$$

As in the theory without boundaries, the associativity of the algebra (2.17) gives rise to some factorization equations [5, 6], which however for diagonal S-and W-matrices contain no information. Instead we require the analogue to the bootstrap equation (2.7) in the presence of a reflecting boundary, which was derived in [6]. Relating again the fusing angles to the integer powers which occur in the fusing rule, this equation reads now

$$W_k(\theta + i\eta_{\bar{k}}) \;=\; W_i(\theta + i\eta_i)W_j(\theta + i\eta_j)S_{ij}(2\theta + i\eta_i + i\eta_j) \quad . \tag{3.20}$$

As was pointed out by Ghoshal and Zamolodchikov [7] the equivalent to the second equation in (2.6), namely the crossing relation, is far less obvious. We shall provide an alternative derivation of their crossing unitarity equation and demonstrating that in fact this equation is implied by the wall bootstrap equation. Shifting θ by $i\pi$ in (3.20) and subsequently multiplying the resulting equation by (3.20) yields

$$\begin{aligned} W_k(\theta + i\eta_{\bar{k}})W_k(\theta + i\eta_{\bar{k}} + i\pi) \;&=\; W_i(\theta + i\eta_i)W_i(\theta + i\eta_i + i\pi)W_j(\theta + i\eta_j) \\ &\quad \times W_j(\theta + i\eta_j + i\pi)S^2_{ij}(2\theta + i\eta_i + i\eta_j) \quad . \end{aligned}$$

Comparision of this equation with (2.10) shows, that it is solved if

$$W_i(\theta)W_{\bar{\imath}}(\theta + i\pi) \;=\; S_{ii}(2\theta) \tag{3.21}$$

is true. This is precisely the "cross-unitarity equation" originally derived in [7], employing the fact that the same correlation f unction can be viewed in two ways related by interchanging space and time co ordinates.

From the symmetry of the S-matrix in its indices we obtain with (3.21) that the W-matrix for particles and anti-particles coincide

$$W_{\bar{\imath}}(\theta) \;=\; W_i(\theta) \;\; . \tag{3.22}$$

Furthermore from the $2\pi i$-periodicity of the S-matrix we obtain by means of (3.21) that W is $2\pi i$-periodic in θ too. Using this facts we may rewrite (3.21) as

$$W_{\bar{\imath}}\left(\theta + \frac{i\pi}{2}\right) W_i\left(\theta - \frac{i\pi}{2}\right) S_{i\bar{\imath}}(2\theta) \;=\; 1 \;\; . \tag{3.23}$$

Equation (3.21) can be given the following meaning. Consider the p article i living in the half-space $-\infty < x \leq 0$ delimited by a wall at $x = 0$. Let it scatter with rapidity $\theta_1 = \theta$ against the wall, this process being described by $W_i(\theta)$. Now the other factor of the left hand side of equ.(3.21) $W_{\bar{\imath}}(\theta + i\pi)$ represents the scattering of the anti-particle $\bar{\imath}$, but with rapidity increased by $i\pi$, which is the same as particle i at rapidity $\theta_2 = -\theta$ penetrating the boundary. This corresponds to a process in a different physical region, namely scattering from the wall at $x = 0$ by a particle living in the half-space $0 \geq x < \infty$. Both processes are shown in fig. 2. Now $W_i(\theta)$ is not relativistically invariant, but describes the scattering off the wall in the prefered frame in which the wall is at rest. We may now look at the scattering of two particles i with rapidities θ_1, θ_2 in this particular frame with the same asymptotic configuration as the one described by the left hand side of equ.(3.21) and also illustrated in fig. 2. We have $\theta_1 = +\theta$, $\theta_2 = -\theta$ and the corresponding S-matrix is $S_{ii}(2\theta)$, which is exactly the right hand side of equ.(**??**). This equation tells us therefore, that an observer far away from the wall cannot tell whether two incoming particles with opposite rapidities scat ter against each other at $x = 0$ or whether they have been reflected by a double-sided mirror, which has been placed at the origin.

One might now worry whether equations (3.21) and (3.20) are compatible for all possible angles. We observe that taking $j = \bar{\imath}$ in the wall bootstrap equation and shifting θ by $-i\frac{\eta_i}{2} - i\frac{\eta_{\bar{\imath}}}{2}$, the right hand side of (3.20) takes on exactly the same form as (3.23) since mod$2\pi i$ $\eta_i - \eta_{\bar{\imath}} = \pm\pi$. However in that case we have $\xi(i) - \xi(j) = \pm\frac{h}{2} + \frac{c(i)-c(j)}{4}$, which is precisely the power of the Coxeter number required to relate the representatives of the orbit Ω_i and $\Omega_{\bar{\imath}}$, related to particles and antiparticles, respectively [22]. Hence this case does not pose a problem, since the fusing rule will never give rise to these angles.

In [6] we solely had equations (3.19) and (3.20) at our disposal, which we solved for some specific Toda models. It was demonstrated there that it is in principle possible to find solutions of this system of equations, but due to the lack of (3.21) it is a rather involved procedure. As demonstrated above, together with the homogeneous bootstrap equation, one equation can be derived from the other. Hence instead of solving (3.20) we now solve first the "crossing-unitarity" equation (3.21). Employing the standard technique of Fourier transforms, we obtain an integral representation for the W-matrix:

$$W_i(\theta) = \exp\left(\frac{-1}{2\pi} \quad \int d\theta' \, \frac{1}{\cosh(\theta-\theta')} \ln S_{i\bar{\imath}}(2\theta')\right) \ . \tag{3.24}$$

For the theories in mind we know that the S-matrix will always be of the form $\prod_x \{x\}_\theta$ and we therefore obtain that the W-matrix will acquire the same form

$$W(\theta) \ = \ \prod_x \mathcal{W}_x(\theta) \ \ , \tag{3.25}$$

where the blocks $\mathcal{W}_x(\theta)$ are in one-to-one correspondence to the ones in S. Using the integral representation (2.4) we can carry out the θ'-integration in (**??**) and obtain

$$\mathcal{W}_x(\theta) \ = \ \frac{w_{1-x}(\theta)w_{-1-x}(\theta)}{w_{1-x-B}(\theta)w_{-1-x+B}(\theta)} \ \ , \tag{3.26}$$

where the subblocks $w_x(\theta)$ are given by

$$w_x(\theta) = \frac{<\frac{x-h}{2}>_\theta}{<\frac{x-h}{2}>_{-\theta}} \ . \tag{3.27}$$

One easily verifies the relations

$$
\begin{aligned}
w_x(\theta)\ \ w_x(-\theta) &= 1 && (3.28)\\
w_{x-2h}(\theta)\ w_{-x}(\theta) &= 1 && (3.29)\\
w_x(0) = w_{-h}(\theta) &= 1 && (3.30)\\
w_x\left(\theta + \frac{iy\pi}{2h}\right)\ w_x\left(\theta - \frac{iy\pi}{2h}\right) &= w_{x+y}(\theta)\ w_{x-y}(\theta) && (3.31)\\
w_x\left(\theta + \frac{i\pi}{2}\right)\ w_x\left(\theta - \frac{i\pi}{2}\right) &= \frac{< x >_{2\theta}}{< x >_{-2\theta}} && (3.32)
\end{aligned}
$$

from which we deduce the unitarity

$$\mathcal{W}_x(\theta)\mathcal{W}_x(-\theta) = 1 \tag{3.33}$$

and the crossing-unitarity relation

$$\mathcal{W}_x\left(\theta + \frac{i\pi}{2}\right)\mathcal{W}_x\left(\theta - \frac{i\pi}{2}\right)\{x\}_{2\theta} = 1 \tag{3.34}$$

for each block in (3.25). Furthermore we derive

$$
\begin{aligned}
\mathcal{W}_{x+h}(\theta)\mathcal{W}_{x-h}(\theta)\{x\}_{2\theta} &= 1 && (3.35)\\
\mathcal{W}_{2h-x}(\theta)\mathcal{W}_x(\theta) &= 1 && (3.36)\\
\mathcal{W}_x(\theta + 2\pi i) &= \mathcal{W}_x(\theta)\ . && (3.37)
\end{aligned}
$$

The zeros and poles of each block $\mathcal{W}_x(\theta)$ are simple and lie on the imaginary θ-axis. The poles lie mod$2\pi i$ at

$$\theta_\pm = \frac{\pm 1 - x - h}{2h} i\pi \qquad \text{and} \qquad \theta_\pm^B = \frac{\pm B \mp 1 + x + h}{2h} i\pi \tag{3.38}$$

whereas the zeros are situated at

$${}^0\theta_\pm = \frac{\pm 1 + x + h}{2h} i\pi \qquad \text{and} \qquad {}^0\theta_\pm^B = \frac{\pm 1 \mp B - x - h}{2h} i\pi\ . \tag{3.39}$$

Notice that for $0 < x < h$, $\theta_\pm$ will never lie in the physical sheet, but the coupling constant dependent pole can now, on the contrary to the case of the S-matrix, move inside $0 <$ Im $\theta < i\pi$.

Analogous to the S-matrix, we also have a CDD-ambiguity related to the W-matrix: $W_i(\theta) \rightarrow W_i(\theta)\psi_i(\theta)$, where $\psi_i(\theta)$ satisfies the homogeneous equations (2.14), with $j = \bar{\jmath} = 0$ indicating the ground state, and (2.16), with $l = 0$. Thus the CDD-ambiguity for the S- and W-matrices turn out to be restricted by the same equations. However in the latter case it can be obtained in simple way. Whereas for the S-matrix a shift of θ by $i\pi$ gives simply rise to essentially the inverse, the same shift for the W-matrix gives rise to a new function. We observe that if $\mathcal{W}_x(\theta)$ solves equation (3.34), then $\mathcal{W}_{x+2h}(\theta) = \mathcal{W}_x(\theta + i\pi)$ will be a solution as well, which from (3.35) is evidently not simply the inverse. It is easy to verify that the function $\psi_x(\theta)$ which relates these two solutions

$$\mathcal{W}_x(\theta + i\pi) \;=\; \mathcal{W}_{x+2h}(\theta) \;=\; \mathcal{W}_x(\theta)\psi_x(\theta) \;, \tag{3.40}$$

indeed blockwise solves (2.14) and (2.16). Introduc ing a product over these blocks we observe that if $\mathcal{W}_t(\theta)$ for $t = i, j, k$ are solutions of (3.21) and (3.20) then $\mathcal{W}_t(\theta + i\pi)$ will solve them likewise and hence the function

$$\psi_i(\theta) \;=\; \frac{\mathcal{W}_i(\theta + i\pi)}{\mathcal{W}_i(\theta)} \tag{3.41}$$

obeys the CDD-constraints (2.14) and (2.16).

Drawing a close analogy to the S-matrix (2.3) of affine Toda theory we might now conjecture its W-matrix to be of the form

$$W_i(\theta) = \prod_{q=a}^{h-1+a} \left(\mathcal{W}_{2q-\frac{c(i)+c(\bar{\imath})}{2}}(\theta) \right)^{-\lambda_i \cdot \sigma^q \gamma_{\bar{\imath}}} \quad . \tag{3.42}$$

In the following we shall verify that this function, apart from the bootstrap equation, indeed satisfies the general consistency requirement expected from it.

Since each block $W_x(\theta)$ individually satisfies the unitarity relation (3.34), $W_i(\theta)$ will do so likewise. In order to satisfy the requirement that the W-matrix for particle and anti-particle coincide (3.22), we may simply use the fact that $\lambda_i \cdot \sigma^q \gamma_j = \lambda_j \cdot \sigma^q \gamma_i$ [22]. Meromorphicity follows from the same argument employed in [22] for the S-matrix. Because of relation (3.36), each block occurs twice in the product, where the values of q are related by $q + q' = h + \frac{c(i)+c(\bar{\imath})}{2}$. Then the total power of each block turns out to be $-\lambda_i \cdot \sigma^q \gamma_{\bar{\imath}}$, which for simply laced algebras will always be an integer.

Next we verify (3.21). We have

$$W_{\bar{\imath}}(\theta + i\pi) W_i(\theta) = \prod_{q=a}^{h-1+a} \left(\mathcal{W}_{2q - \frac{c(i)+c(\bar{\imath})}{2}}(\theta) \mathcal{W}_{2q+2h-\frac{c(i)+c(\bar{\imath})}{2}}(\theta) \right)^{-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_{\bar{\imath}}} \quad . \tag{3.43}$$

Although $W_x(\theta)$ is not $2h$-periodic in x, the expression in the bracket of this equation is. Together with the fact that the Coxeter element has period h, we are permitted to shift the dummy variable q by

$$q \to q - \frac{h}{2} + \frac{c(\bar{\imath}) - c(i)}{4} \quad . \tag{3.44}$$

Then employing (3.35) and the relation between simple roots associated to particle and antiparticle $\gamma_{\bar{\imath}} = -\sigma^{-\frac{h}{2}+\frac{c(i)+c(\bar{\imath})}{4}} \gamma_i$ [22], the expression becomes

$$\prod_{q=1}^{h} \{2q - c(i)\}_{2\theta}^{-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_i} \;=\; S_{ii}(2\theta) \tag{3.45}$$

and hence (3.42) solves the "crossing unitarity" relation.

In order to establish that this function satisfies the wall-boootstrap equation (3.20), we would like to use a similar trick as in the case of the S-matrix: we would like to be able to shift the dummy variable q. However, since $\mathcal{W}_{2q+\text{const}}(\theta)$ has period $2h$ in q and σ^q has period h, a shift in q by some integer value will alter the expression for $W_i(\theta)$. In fact it turns out that this expression does no t solve the wall-bootstrap equation. However, the above arguments exhibit the the reason for the following assumption:

We presume that the blocks $\mathcal{W}_x$, which constitute the W-matrix are, *up to a shift of* $2h$ *in* x, in one-to-one correspondence to the blocks $\{x\}_\theta$ which build up the S-matrix. Additional factors can only be CDD-ambiguities, which we will as usual ignore. The problem which remains is to determine which of the blocks are shifted by $2h$ and which are not. In ot her words, the bootstrap equation (3.20) will determine whether the block $\{x\}$ in the S-matrix is to be replaced by $\mathcal{W}_x(\theta)$ or $\mathcal{W}_{x+2h}(\theta)$.

At present we do not have a general unified argument valid for all Toda theories at hand and we shall therefore turn to a case-by-case analysis. The W-matrix for all Toda theories related to simply laced Lie algebras will now be obtained in following way: Writing down first the whole set of W-bootstrap equations, with t he fusing angles for instance obtained from [20] or an explicit computation of the fusing rules, we seek equations involving only one particular $W_i(\theta)$. We then find the most general solution of this equation. Having found one $W_i(\theta)$ we seek an equation involving one additional $W_j(\theta)$, which then is computed. Proceeding in this fashion we are able to construct $W_i(\theta)$ for $i = 1, \ldots, r$. The remaining equations then have to be satisfied identically. In this manner we obtain the following solutions:

3.1 $\mathbf{a_n^{(1)}}$

Using as convention that $c(1)$ is always -1, we derive for $i = 1, \ldots, r$

$$W_i(\theta) \;=\; \prod_{l=1}^{\mu(i)} \mathcal{W}_{r+2\nu(l)-2\mu(i)}(\theta) \quad . \tag{3.46}$$

Here $\mu(i)$ is a function which takes the $\mathbb{Z}_2$-symmetry of the Dynkin diagram into account, i.e.

$$\mu(i) := \begin{cases} i & \text{for } i \leq [\frac{h}{2}] \\ h-i & \text{for } i > [\frac{h}{2}] \end{cases} \;, \tag{3.47}$$

and ν is a function defined as

$$\nu(n) := \begin{cases} n & \text{for } n \text{ odd} \\ n+h & \text{for } n \text{ even} \end{cases} \;. \tag{3.48}$$

At present we do not know a solid mathematical proof for this formula to any order, but we have checked its validity to high order in n.

3.2 $\mathbf{d_n^{(1)}}$

In this case we adopt the same convention $c(1) = -1$, such that we obtain for the two "spinors" $c(r) = c(r-1) = 1$ or $c(r) = c(r-1) = -1$, depending on whether r is odd or even, respectively. Then we obtain for $i = 1, \ldots, r-2$

$$
\begin{aligned}
W_i(\theta) &= \prod_{l=1}^{i} \mathcal{W}_{2i-2l+1}(\theta)\, \mathcal{W}_{h-2i+2\nu(l)-1}(\theta) && (3.49)\\
W_r(\theta) &= W_{r-1}(\theta) = \prod_{l=1}^{[\frac{r}{2}]} \mathcal{W}_{4l-3}(\theta)\, \mathcal{W}_{h-4l+3}(\theta) && (3.50)
\end{aligned}
$$

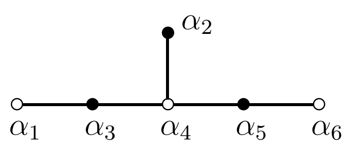
Again some inductive proof is still required.

3.3 $\mathbf{e_6^{(1)}}$

In order to avoid cumbersome expressions we introduce the following symbol

$$
{}_l[\mathcal{W}_x(\theta)]^y_m \equiv \mathcal{W}_x^{y-l}(\theta)\, \mathcal{W}_{h-x}^{y-m}(\theta)\, \mathcal{W}_{x+2h}^{l}(\theta)\, \mathcal{W}_{3h-x}^{m}(\theta) \tag{3.51}
$$

Our conventions are illustrated in the following Dynkin diagram

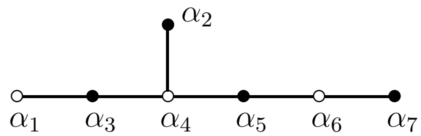


We then derive

$$
\begin{aligned}
W_1(\theta) &= W_6(\theta) = \mathcal{W}_5(\theta)\, \mathcal{W}_{35}(\theta) && (3.52)\\
W_2(\theta) &= [\mathcal{W}_1(\theta)]^1\ {}_1[\mathcal{W}_5(\theta)]^1 && (3.53)\\
W_3(\theta) &= W_5(\theta) = \mathcal{W}_3(\theta)\, \mathcal{W}_5(\theta)\, \mathcal{W}_7(\theta)\, \mathcal{W}_{11}(\theta)\, \mathcal{W}_{29}(\theta)\, \mathcal{W}_{33}(\theta) && (3.54)\\
W_4(\theta) &= [\mathcal{W}_1(\theta)]^1_1\ {}_1[\mathcal{W}_3(\theta)]^2\ {}_1[\mathcal{W}_5(\theta)]^3_2 \quad . && (3.55)
\end{aligned}
$$

3.4 $\mathbf{e_7^{(1)}}$

Again we depict our conventions in a Dynkin diagram



and obtain

$$W_1(\theta) = [\mathcal{W}_1(\theta)]_1^1 \ [\ calW_7(\theta)]^1 \tag{3.56}$$

$$W_2(\theta) = [\mathcal{W}_1(\theta)]^1 \ [\mathcal{W}_5(\theta)]^1 \ [\mathcal{W}_7(\theta)]_1^1 \ \mathcal{W}_9(\theta) \tag{3.57}$$

$$W_3(\theta) = [\mathcal{W}_1(\theta)]^1 \ [\mathcal{W}_3(\theta)]_1^1 \ [\mathcal{W}_5(\theta)]^1 \ {}_1[\mathcal{W}_7(\theta)]_1^2 \ [\mathcal{W}_9(\theta)]^1 \tag{3.58}$$

$$W_4(\theta) = [\mathcal{W}_1(\theta)]_1^1 \ [\mathcal{W}_3(\theta)]^2 \ {}_1[\mathcal{W}_5(\theta)]_2^3 \ {}_1[\mathcal{W}_7(\theta)]_1^4 \ {}_1[\mathcal{W}_9(\theta)]_1^2 \tag{3.59}$$

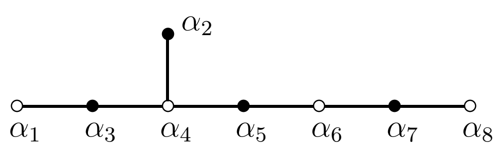
$$W_5(\theta) = [\mathcal{W}_1(\theta)]^1 \ [\mathcal{W}_3(\theta)]_1^1 \ [\mathcal{W}_5(\theta)]^2 \ {}_1[\mathcal{W}_7(\theta)]_1^2 \ [\mathcal{W}_9(\theta)]_1^1 \ \mathcal{W}_9(\theta) \tag{3.60}$$

$$W_6(\theta) = [\mathcal{W}_1(\theta)]_1^1 \ [\mathcal{W}_3(\theta)]^1 \ [\mathcal{W}_7(\theta)]^1 \ [\mathcal{W}_9(\theta)]_1^1 \tag{3.61}$$

$$W_7(\theta) = [\mathcal{W}_1(\theta)]^1 \ \mathcal{W}_9(\theta) \ . \tag{3.62}$$

3.5 $\mathbf{e_8^{(1)}}$

Together with the notations



we obtain

$$W_1(\theta) = [\mathcal{W}_1(\theta)]_1^1 \ [\mathcal{W}_7(\theta)]^1 \ [\mathcal{W}_{11}(\theta)]^1 \ [\mathcal{W}_{13}(\theta)]_1^1 \tag{3.63}$$

$$W_2(\theta) = [\mathcal{W}_1(\theta)]^1 \ [\mathcal{W}_5(\theta)]^1 \ [\mathcal{W}_7(\theta)]_1^1 \ [\mathcal{W}_9(\theta)]^1 \ {}_1[\mathcal{W}_{11}(\theta)]_1^2 \ [\mathcal{W}_{13}(\theta)]^1 \ [\mathcal{W}_{15}(\theta)]_1^1 \tag{3.64}$$

$$W_3(\theta) = [\mathcal{W}_1(\theta)]^1 \ [\mathcal{W}_3(\theta)]_1^1 \ [\mathcal{W}_5(\theta)]^1 \ {}_1[\mathcal{W}_7(\theta)]_1^2 \ [\mathcal{W}_9(\theta)]^2 \ {}_1[\mathcal{W}_{11}(\theta)]_2^3$$

$$
\begin{aligned}
& {}_1[\mathcal{W}_{13}(\theta)]_1^3\ [\mathcal{W}_{15}(\theta)]_1^1 && (3.65)\\
W_4(\theta) &= [\mathcal{W}_1(\theta)]_1^1\ [\mathcal{W}_3(\theta)]^2\ {}_1[\mathcal{W}_5(\theta)]_2^3\ {}_1[\mathcal{W}_7(\theta)]_1^4 \\
& {}_2[\mathcal{W}_9(\theta)]_3^5\ {}_2[\mathcal{W}_{11}(\theta)]_2^6\ {}_3[\mathcal{W}_{13}(\theta)]_3^6\ [\mathcal{W}_{15}(\theta)]_2^3 && (3.66)\\
W_5(\theta) &= [\mathcal{W}_1(\theta)]^1\ [\mathcal{W}_3(\theta)]_1^1\ [\mathcal{W}_5(\theta)]^2\ {}_1[\mathcal{W}_7(\theta)]_2^3\ {}_1[\mathcal{W}_9(\theta)]_1^3\ {}_2[\mathcal{W}_{11}(\theta)]_2^4 \\
& {}_1[\mathcal{W}_{13}(\theta)]_1^4\ [\mathcal{W}_{15}(\theta)]_2^2 && (3.67)\\
W_6(\theta) &= [\mathcal{W}_1(\theta)]_1^1\ [\mathcal{W}_3(\theta)]^1\ [\mathcal{W}_5(\theta)]_1^1\ [\mathcal{W}_7(\theta)]^1\ {}_1[\mathcal{W}_9(\theta)]_1^2\ {}_1[\mathcal{W}_{11}(\theta)]_1^3 \\
& {}_1[\mathcal{W}_{13}(\theta)]_1^2\ [\mathcal{W}_{15}(\theta)]^1 && (3.68)\\
W_7(\theta) &= [\mathcal{W}_1(\theta)]^1\ [\mathcal{W}_3(\theta)]_1^1\ [\mathcal{W}_9(\theta)]^1\ {}_1[\mathcal{W}_{11}(\theta)]_1^2\ {}_1[\mathcal{W}_{13}(\theta)]^1 && (3.69)\\
W_8(\theta) &= [\mathcal{W}_1(\theta)]_1^1\ [\mathcal{W}_{11}(\theta)]^1\ \ . && (3.70)
\end{aligned}
$$

We have here reported only one solution. As explained above a second solution can always be obtained by multiplication of the CDD-factor or equivalently by shifting all the xs by $2h$ in $\mathcal{W}_x$. Both relations will give rise to entirely different physics due to the different positions of the poles in the physical sheet. Depending on the order and the sign of the residue several of these states may find an interpretation as stable states in the bound ary.

4 Conclusion

We have demonstrated how the set of consistency equations for the scattering matrices in the presence of reflecting boundaries can be employed in order to compute the W-matrices for affine Toda field theories.

Evidently the completion of the picture requires further investigations and several interesting questions have still been left unanswered. Concerning ATFTs, a detailed study of possible integrable boundary conditions is desirable, which might illuminate further their relation to integrable deformed conformal field theories. It is also expected that the W-matrix can be cast into a more general unified formula, analogous to the S-matrix, which might lead to a deeper Lie algebraic understanding. Since most of the solutions for the W-matrix exhibit the possibility of stable bound states, we may relax our assumption that the wall is always in ground state and determine the matrices which incorporate these states. Finally the question, of whether it is possible to parallel the argumentation in the case of the absence of boundaries and use the knowledge obtained on-shell in order to determine off-shell properties of the theory, poses an interesting problem.

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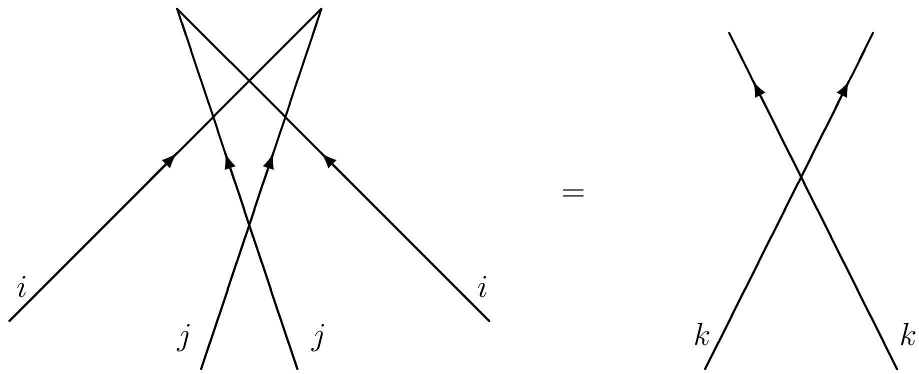

Figure 1: The bootstrap equation (2.10)

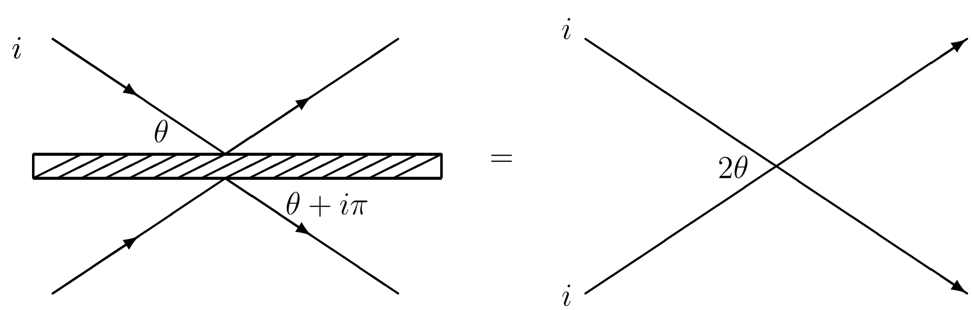

Figure 2: The "crossing unitarity" relation