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The Fusing Rule and the Scattering Matrix of Affine Toda Theory

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Abstract

Affine Toda theory is an integrable theory with many interesting features. Classically, the presence of trilinear couplings is given by Dorey's "fusing rule", whatever the simple Lie algebra concerned. This paper discusses the structure of this rule, alternative solutions and formulations, and the relationship to the quantum conservation laws. This insight is applied to the conjectured scattering matrix of the quantum theory. The crossing and bootstrap properties are verified in a general way, valid for any Lie algebra, but the analyticity properties require the extra assumption that the algebra be simply laced. Various identities satisfied by a Coxeter element play a crucial role.

1. Introduction

To each simple Lie algebra $\mathbf{g}$ there corresponds an affine Toda theory in which $r \equiv$ rank $\mathbf{g}$ scalar fields satisfy r relativistically invariant coupled equations in two dimensions. For more than ten years, the many interesting features of these theories have been a source of fascination [1,3,2] which has been intensified by the observation [4,5] that they promise to furnish a concrete example of Zamolodchikov's concept of an integrable perturbation of a conformal field theory [6]. Recently, Dorey [7] has formulated a neat "fusing rule" as the necessary and sufficient condition for the existence of a trilinear coupling between three of the fields corresponding to mass eigenstates. At the classical level, a proof of this rule has recently been given [8] which works uniformly for all simple $\mathbf{g}$.

The nearest available approach to defining the quantum theory is the specification of the scattering matrix for the r particles created by the fields. It is sufficient to give only the elastic two body S-matrices providing they satisfy certain "bootstrap" properties [9] reflecting the integrability of the theory, in addition to the usual unitarity, hermitian analyticity and crossing properties of a relativistic theory. Although such an S-matrix has been given [10,11,12,13,14,15,16,17,18], the verification of the desired properties has been on a case-by-case basis. Here again, we present a unified treatment, particularly of the crossing and bootstrap properties, which are the least obvious, in the hope of understanding better the structure of these theories. In the course of the proof of the fusing rule, it was found that, given one solution to the equation, there always existed precisely one other inequivalent solution [8]. Here we clarify this result and show the role it plays in verifying the bootstrap property which we formulate in a new, crossing symmetric way. The crossing property itself relies on an identity satisfied by the Coxeter element of the Weyl group of $\mathbf{g}$ (in the special choice previously found convenient).

The presentation of results falls into three subsequent sections. The first discusses the fusing rule, its two solutions and alternative formulations. The second section discusses in detail the connection between the fusing rule and the r linearly independent conservation laws that apply to the three particle process implied by the coupling. Particular attention is paid to the crossing of one of the particles. Finally the elastic S-matrix elements are presented in terms of a universal formula whose properties can be checked uniformly, making use of the previous discussion. Particular attention is paid to the crossing and bootstrap properties and to the location of the singularities, distinguishing between the physical and unphysical sheets. It is shown that the poles corresponding to the propagation of bound state particles are of odd order and possess residues of definite sign. The proof of these meromorphy properties requires $\mathbf{g}$ to be simply laced, but it is interesting that this condition is not necessary for the crossing and bootstrap properties.

2. The Fusing Rule

Not surprisingly, many features of the affine Toda equations associated with the simple Lie algebra $\mathbf{g}$ can be understood in terms of the structure of the root system of that algebra. If σ is a Coxeter element, (a product of reflections in some complete set of simple roots), it splits the roots into precisely r orbits $\Omega_1, \Omega_2, \ldots \Omega_r$, each containing $h \equiv \frac{\dim g}{r} - 1$ elements. The r particles correspond to these orbits as explained in [8]. We can choose the set of simple roots so that $\sigma = \sigma_- \sigma_+$, where σ_+ denotes the product of reflections in "white" simple roots α_i endowed with the colour value $c(i) = 1$ and σ_- the product of reflections in the "black" simple roots α_i endowed with the colour value $c(i) = -1$. As it is understood that the two colours, black and white, are assigned to the vertices of the Dynkin diagram of g in such a way that no two vertices of the same colour are linked, there is no ambiguity in σ_+ and σ_-, once the set of simple roots is chosen. (Our convention $c(W) = 1 = -c(B)$ differs in sign from [8] for reasons of convenience that become clear later). The roots $\gamma_i = c(i)\alpha_i$ can be shown to lie in distinct orbits and we therefore adjust our labeling so that $\gamma_i \in \Omega_i$.

Dorey [7] proposed the "fusing rule" namely that particles i, j and k couple, so that their totally symmetric coupling constant does not vanish, $C_{ijk} \neq 0$, if and only if there are representatives of Ω_i, Ω_j and Ω_k which sum to zero. Explicitly there exists a triplet of integers $(\xi(i), \xi(j), \xi(k))$ so that

$$\sum_{t=i,j,k} \sigma^{\xi(t)} \gamma_t = 0. \tag{2.1}$$

Such triplets form what we call equivalence classes $\xi(t) \to \xi(t) + m$, as is seen multiplying (2.1) by σ^m. In the course of the classical proof that the modulus of C_{ijk} is proportional to the area of the triangle formed by the masses of the three fusing particles, [8], the existence of a second equivalence class was found to be important. We shall now construct it explicitly from the first. Understanding the notation that $\sigma_{c(i)} = \sigma_+$ or σ_- according as i is white or black, we have the trivial identity

$$\sigma_{c(i)} \sigma_{c(j)} = \sigma^{\frac{c(j)-c(i)}{2}}. \tag{2.2}$$

But, as $\sigma_{c(j)} \gamma_j = -\gamma_j$, we find, acting on γ_j with (2.2), the identity

$$\sigma_{c(i)} \gamma_j = -\sigma^{\frac{c(j)-c(i)}{2}} \gamma_j. \tag{2.3}$$

Using this and $\sigma_{c(l)} \sigma^q = \sigma^{-q} \sigma_{c(l)}$, the action of $\sigma_{c(l)}$ on the fusing equation (2.1) leads to the new equation

$$\sum_{t=i,j,k} \sigma^{-\xi(t) + \frac{c(t)-c(l)}{2}} \gamma_t = 0,$$

which defines a new triplet of integers $(\xi'(i), \xi'(j), \xi'(k))$ which we shall show to be inequivalent to the previous triplet. Whether l is black or white is irrelevant as the effect is the difference in a common power of σ. Choosing l to be white we obtain the following relation between the two triplet representatives

$$\xi'(t) = -\xi(t) + \frac{c(t)-1}{2} \quad \text{for} \quad t = i, j, k. \tag{2.4}$$

With this choice the second version of the fusing equation reads

$$\sum_{t=i,j,k} \sigma^{\xi'(t)} \gamma_t = 0. \tag{2.5}$$

If the two solutions ξ and ξ' are equivalent, then $\xi(t) - \xi'(t) = 2\xi(t) + \frac{1-c(t)}{2}$ is the same for $t = i, j$ or k. Thus, in particular,

$$2(\xi(i) - \xi(j)) = \frac{c(i) - c(j)}{2}.$$

As the left hand side is an even integer, and the right hand side can only take the values ± 1 or 0, we conclude that i, j and k all have the same colour and that $\xi(i), \xi(j)$ and $\xi(k)$ are equal. It follows that $\alpha_i + \alpha_j + \alpha_k = 0$ which is impossible for three simple roots. Thus the two solutions are indeed inequivalent. In section 3. we shall prove that no more inequivalent solutions exist for the fusing of the same three particles.

In proving the Clebsch-Gordan property of the coupling, [19], Braden pointed out that the fusing could equally be reexpressed in terms of the fundamental weights, λ_i. From the trivial identities

$$\sigma_{-c(i)} \lambda_i = \lambda_i, \qquad \sigma_{c(i)} \lambda_i = \lambda_i - \alpha_i, \tag{2.6}$$

and (2.2), we obtain the following relation between γ_i and λ_i

$$\gamma_i = (1 - \sigma^{-1}) \sigma^{\frac{1-c(i)}{2}} \lambda_i \tag{2.7a}$$
$$= (\sigma_- - \sigma_+) \lambda_i. \tag{2.7b}$$

It is also useful to record the analogue of (2.3) for fundamental weights

$$\sigma_{c(i)} \lambda_j = \sigma^{-\frac{c(i)+c(j)}{2}} \lambda_j. \tag{2.8}$$

Substituting (2.7a) in (2.1) and dividing off the factor $(1 - \sigma^{-1})$, which never vanishes, [20], gives the new, fundamental weight version of the fusing equation

$$\sum_{t=i,j,k} \sigma^{-\xi'(t)} \lambda_t = 0. \tag{2.9}$$

It is the convenience of the overall sign in relation (2.7a) which lead us to change our convention for the sign of $c(i)$, since $\lambda_i - \sigma^{-1}\lambda_i$ is a sum of positive roots as λ_i is dominant.

Similarly, substituting (2.7a) in (2.4) (with $c(l) = 1$), yields the alternative fusing rule for the fundamental weights

$$\sum_{t=i,j,k} \sigma^{-\xi(t)} \lambda_t = 0. \tag{2.10}$$

All four of these versions, (2.1), (2.5), (2.9) and (2.10), of the fusing rule will play a role in proving the properties of the hypothetical S-matrix. Before demonstrating this and before considering the relation to the conservation laws for the three particle process admitted by the coupling, we shall consider the "crossing" of one of the particles into its antiparticle.

$-\Omega_i$ is an orbit of the Coxeter element denoted $\Omega_{\bar{i}}$, and in the classical theory the field creating particle i is the complex conjugate of that creating $\bar{i}$, thereby suggesting that the particles are antiparticles of each other. As $-\gamma_i$ and $\gamma_{\bar{i}}$ belong to the same orbit under the action of σ, there must be a power of σ converting one to the other. Indeed, as is shown in the next section,

$$\gamma_{\bar{i}} = -\sigma^{-\frac{h}{2}+\frac{c(i)-c(\bar{i})}{4}}\gamma_i. \tag{2.11}$$

The power of σ is indeed integral as $c(i)c(\bar{i}) = (-1)^h$. We can therefore rewrite (2.1) as

$$\sigma^{\xi(i)}\gamma_i + \sigma^{\xi(j)}\gamma_j = \sigma^{\xi(k)}\gamma_k, \tag{2.12}$$

where

$$\xi(\bar{k}) = \xi(k) + \frac{h}{2} - \frac{c(k)-c(\bar{k})}{4}. \tag{2.13}$$

3. Conservation Laws for Three Particle Processes

Adding the two relations (2.6) gives $(\sigma_+ + \sigma_-)\lambda_i = 2\lambda_i - \alpha_i$. Remembering $\alpha_i = K_{ij}\lambda_j$, where K is the Cartan matrix, yields the fundamental equation [23]

$$(\sigma_+ + \sigma_-)\alpha_i = (2\delta_{ij} - K_{ij})\alpha_j.$$

Formally squaring this relation gives

$$2 + \sigma + \sigma^{-1} = (2-K)^2 \tag{3.1}$$

which relates eigenvalues of the Coxeter element σ and the Cartan matrix K. As σ is a real $r \times r$ matrix of order h its eigenvalues occur in complex conjugate pairs which are integer powers of $e^{\frac{2\pi i}{h}}$. These integer powers, taken to be between 1 and $h-1$, in order, are called the exponents $s(1), \ldots s(r)$ of $\mathbf{g}$ and must satisfy

$$s(j) + s(r+1-j) = h. \tag{3.2}$$

According to (3.1), the possible eigenvalue -1 of σ would correspond to the eigenvalue 2 of K, whilst the complex conjugate pairs of eigenvalues $\exp\frac{\pm 2\pi i s(j)}{h} = \exp\frac{\mp 2\pi i s(r+1-j)}{h}$ of σ correspond to the eigenvalues $2(1 \mp \cos\frac{\pi s(j)}{h})$ of K. Notice this makes K positive definite, as it should be, with smallest eigenvalue $4\sin^2\frac{\pi}{2h}$, corresponding to $s(1) = 1$.

Just as equation (3.1) connects pairs of eigenvalues of σ and K, it is also possible to relate the corresponding pairs of eigenvectors [8]. Let us denote $\theta(n) = \frac{s(n)\pi}{h}$ and $x_i(n)$ the components of the left eigenvector of K, so

$$x_i(n)K_{ij} = 4\sin^2\frac{\theta(n)}{2}\,x_j(n), \tag{3.3}$$

then the eigenvector of the Coxeter element σ satisfying

$$\sigma q(n) = e^{2i\theta(n)}q(n), \tag{3.4}$$

is given by

$$q(n) = \sum_{k\in B} x_k(n)\alpha_k + e^{-i\theta(n)}\sum_{k\in W} x_k(n)\alpha_k. \tag{3.5}$$

Complex conjugating (3.4) tells us that $q^*(n)$ is proportional to $q(r+1-n)$ when the eigenvalue is not degenerate (this complication only occurs when $\mathbf{g} = so(4N)$ and the eigenvalue is -1 and will henceforth be ignored for simplicity of exposition). Comparing with (3.5) and remembering that the x_k are real, we see that for n fixed but k variable

$$x_k(r+1-n) \sim c(k)x_k(n). \tag{3.6}$$

Conversely to (3.5), remembering the change in sign convention for $c(i)$ relative to [8], we have for the scalar product of $q(n)$ with an element of Ω_i

$$q(n)\sigma^l(\gamma_i) = iy_i(n)\,\sin\theta(n)\,e^{-i\theta(n)(2l+\frac{1-c(i)}{2})}, \tag{3.7}$$

where $y_j(n) = x_j(n)\alpha_j^2$ is the right eigenvector of K to the same eigenvalue as in (3.3).

As Dorey [7] pointed out, taking such scalar products with the fusing equation (2.1), yields equations which can be regarded as conservation laws for the three particle process admitted by the coupling C_{ijk}. We now look at this in more detail, and, in particular, see that only r linearly independent conservation laws result for this three particle process, even though in general there are expected to be an infinite number of conservation laws, reflecting the integrability of the theory. In fact, using (3.7), the fusing equation (2.1) yields

$$\sum_{t=i,j,k} y_t(n)\,e^{i\eta(n,t)} = 0, \tag{3.8}$$

where the phase

$$\eta(n,t) = -\theta(n)\left(2\xi(t) + \frac{1-c(t)}{2}\right). \tag{3.9}$$

Complex conjugating (3.8) simply reverses the sign of η which is equivalent to interchanging $\xi(t)$ and $\xi'(t)$, the two inequivalent solutions to the powers of the Coxeter element in the fusing equation seen earlier.

K is a symmetrisable matrix with negative or zero entries off the diagonal. Hence, by the Perron-Frobenius theorem, the eigenvector corresponding to the lowest eigenvalue is non degenerate and each of its components can be taken to be strictly positive. These are the quantities $y_k(1)$ which, in the classical theory, were proven to be proportional to the particle masses (using the relations between eigenvectors of σ and K, (3.5) and (3.7) above) [22,8]. Assuming this to be true also in the quantum theory, equation (3.8) must be interpreted as a component of energy momentum conservation for the process $i + j + k \to 0$ permitted by the fusing rule. The relation (3.8) can be viewed geometrically. The three complex numbers summing to zero can represent the sides of a triangle in the complex plane in two ways. When n=1, the length of the side of the triangle is proportional to the mass of the particle to which it corresponds. The particles i, j and k therefore maintain their stability with respect to the decay of any one into the antiparticles of the other two allowed by the coupling C_{ijk} since its mass is less than the sum of the masses of the other two by the triangular inequality. This is an important consequence of the fusing rule. Corresponding to the complex conjugation of (3.8) is another pair of triangles with sides of the same length. All four are depicted in the figure.

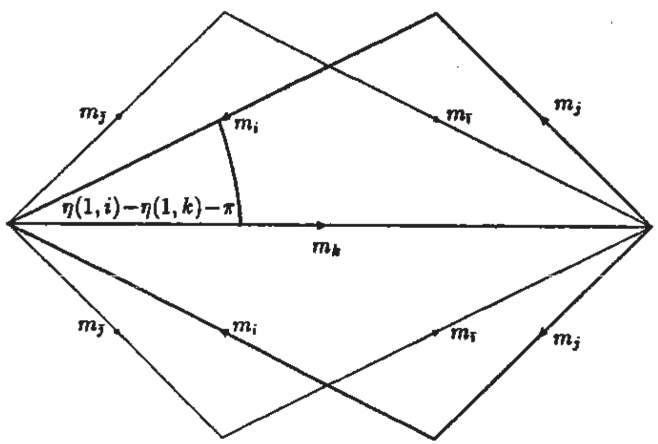

Figure 1: The mass triangles in the complex plane

Since the triangles in this figure are the only ones which can be constructed from three sides with fixed modulus, the proof for the non-existence of a third solution to equations (2.1), (2.5), (2.9) and (2.10) follows immediately. One might worry about the arbitrariness $\eta(1,t) \to \eta(1,t) + 2\pi k$ in (3.8) for k taken to be an integer, but from (3.9) we see that this is the same as saying that $\xi(t) \to \xi(t) - kh$. Since $\sigma^{-kh}\gamma_t = \gamma_t$, this will not lead to a further equivalence class.

In the kinematic situation just described, it is possible to take the particle momenta to be real and Euclidean instead of Lorentzian so that they correspond to the directed sides of the triangle. As the rapidity of the particles is imaginary in Euclidean space, the relative rapidity of two particles can be viewed as i times the exterior angle

to the mass triangle at the vertex where the corresponding lines intersect. The two velocity is a unit vector represented by the phase in (3.8), with n=1. So, for particle t

$$v^+(t) \equiv \frac{v_1(t) + iv_2(t)}{\sqrt{2}} = e^{i\eta(1,t)}, \tag{3.10}$$

and $\eta(1,t)$ is seen to be the imaginary rapidity of particle t. The equations (3.8) for the remaining values of $n > 1$ read

$$\sum_{t=i,j,k} y_t(n)((v^+(t))^{s(n)} = 0, \tag{3.11}$$

and so $y_t(n)$ can be interpreted as the generalised mass of particle t of "spin" $s(n)$ that enters the higher conservation laws. Notice that these masses, unlike the $s(1) = 1$ ones are not all positive and that there are only r of them despite the infinite number of conservation laws that apply in general. Now let us cross particle k into its antiparticle $\bar{k}$. The conservation law for the process $i + j \to \bar{k}$ should read

$$y_i(n)(v^+(i))^{s(n)} + y_j(n)(v^+(j))^{s(n)} = y_{\bar{k}}(n)(v^+(\bar{k}))^{s(n)}. \tag{3.12}$$

As $v(\bar{k}) = -v(k)$, we see comparing with (3.11) that

$$y_{\bar{k}}(n) = (-1)^{1+s(n)} y_k(n). \tag{3.13}$$

This is familiar enough for conventional masses which have $s(1) = 1$ and for electric charge which can be thought of as a mass with $s = 0$. That the eigenvectors of the Cartan matrix actually possess this property (3.13) can be verified directly from (3.7) and (2.11) and we conclude that the underlying group theory "knows" about the crossing of particles into antiparticles.

We can now derive relation (2.11), announced earlier, from (3.7) with $n = 1$. Projecting both sides of the equation $\gamma_{\bar{i}} = -\sigma^l \gamma_i$ onto $q(1)$ and remembering that the Perron-Frobenius theorem guarantees that the coefficients $y_i(1)\sin\frac{\pi}{h}$ can be taken to be positive, we can equate phases to determine the unknown power l as $-\frac{h}{2} + \frac{c(i)-c(\bar{i})}{4}$ mod $2h$. As l has to be an integer this both determines (2.11) and the fact that $c(\bar{i}) = (-1)^h c(i)$.

By (3.10), $v^-(t) = v^+(t)^*$ so that the other, minus, light cone component of the conservation equation (3.11) follows by complex conjugation. But equally, $v^-(t) = v^+(t)^{-1}$ so that we can think of these minus component conservation laws as being versions of the plus component ones (3.11) with negative exponents. Using (3.2), (3.9) and (3.6)

$$y_t(n)(v^+(t))^{-s(n)} = y_t(n)(v^+(t))^{h+s(r+1-n)} = c(t)y_t(n)(v^+(t))^{s(r+1-n)}$$

$$\sim y_t(r+1-n)(v^+(t))^{s(r+1-n)}.$$

So we see that the conservation laws for negative exponents are actually linearly related to the ones for positive exponents. In a similar way the conservation laws for any spin equal to an exponent modulo h turns out to be proportional to one of the fundamental conservation equations (3.11) whenever a three particle process is considered. This is the "collapse" of the conservation laws mentioned earlier.

4. The S-Matrix

Because of the restrictive consequences of energy and momentum conservation in two dimensions, the outcome for the scattering of two particles can only be the same two particles each with energy and momentum unchanged. Thus the scattering of particles i and j say is elastic and, by Lorentz invariance, the corresponding S-matrix element, $S_{ij}(\theta)$ depends only on the relative rapidity, θ. The order of i and j is irrelevant and combining this with hermitian analyticity, we have

$$S_{ij}(\theta) = S_{ji}(\theta) = S_{ij}(-\theta)^{-1}. \tag{4.1}$$

Furthermore the analytic continuation of the unitarity condition reads

$$S_{ij}(\theta)S_{ij}(\theta^*)^* = 1, \tag{4.2}$$

so that $S_{ij}(\theta)$ is simply a phase factor when θ is real. The rapidity variable is useful as it unfolds the square root unitarity branch cuts in both the direct and crossed channels so that $S_{ij}(\theta)$ is meromorphic in θ with period $2i\pi$. The physical sheet corresponds to the imaginary part of θ lying between 0 and π and the unphysical sheet to the range between π and 2π. Physical particles occurring as bound states should appear as poles of odd order on the physical sheet with θ imaginary. Because the theory is relativistic, we have the crossing property

$$S_{ij}(\theta) = S_{i\bar{j}}(i\pi - \theta), \tag{4.3}$$

where, as argued previously, $\bar{j}$ denotes the antiparticle of j.

If three or more particles scatter, the individual momenta are no longer necessarily conserved unless the theory is integrable and so restricted by further conservation laws beyond energy and momentum. Since the classical affine Toda theory possesses such conservation laws, it is natural to suppose they extend to the quantum theory. Then the individual momenta are conserved and the S-matrix for the scattering of a number of particles "factorises" into a product of two particle S-matrices. Usually the ambiguities in such a procedure lead to Yang-Baxter equations which have to be satisfied for consistency, but this is not so in affine Toda theory owing to the lack of degeneracy in the particle spectrum apart from that due to antiparticles. But according to the fusing rule (2.1), $S_{ij}(\theta)$ must possess a pole due to the propagation of particle $\bar{k}$. This has led Zamolodchikov [9], to formulate a "bootstrap" equation which we find convenient to express in the following crossing symmetric form

$$S_{li}(\theta + i\eta(1,i))S_{lj}(\theta + i\eta(1,j))S_{lk}(\theta + i\eta(1,k)) = 1. \tag{4.4}$$

This holds whenever $C_{ijk} \neq 0$, that is whenever the fusing rule (2.1) is satisfied, and corresponds to the diagrammatic equation of the second figure.

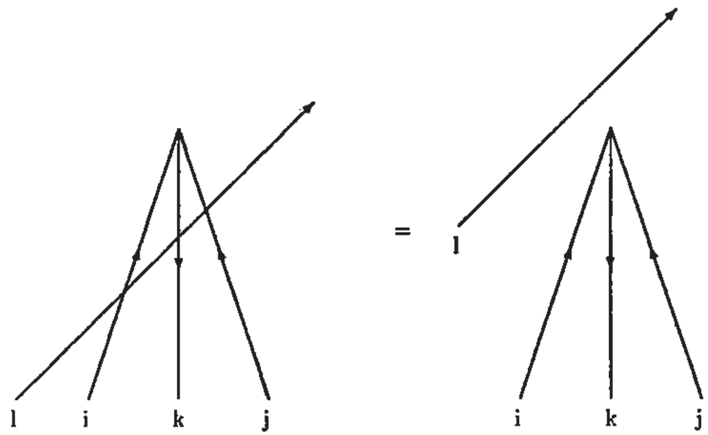

Figure 2: The bootstrap equation

$\eta(1,i)$, $\eta(1,j)$ and $\eta(1,k)$ are the imaginary rapidities of particles i, j and k defined in equations (3.9) and (3.10). Using the crossing relation (4.3), unitarity (4.2) and suitably redefining θ yields the usually quoted form of the bootstrap relation. As we shall see, (4.4) fits in particularly well with the fusing equation (2.1). Another version of (4.4), with the signs of $\eta(1,t)$ reversed in the arguments, follows on complex conjugating (4.4) and using the analytic continuation of the unitarity relation (4.2).

The culmination of a series of inspired guesses has led to a solution of the above conditions satisfied by the two-particle S-matrix in affine Toda theory [7]. In our earlier notation this reads

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q - \frac{c(i)+c(j)}{2} \right\}_{\theta}^{-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_j}. \tag{4.5}$$

Our aim is to show in a unified way, for any simple Lie algebra $\mathbf{g}$, that (4.5) satisfies the requirements (4.1) to (4.4) and has the required meromorphy if, in addition, $\mathbf{g}$ is simply laced. First we must explain the "building block" notation used in (4.5) as introduced before [13]. Let

$$< x >_\theta = \sinh\frac{1}{2}(\theta + \frac{i\pi x}{h}), \tag{4.6}$$

so

$$< x + 2h >_\theta = - < x >_\theta = < -x >_{-\theta} .$$

Further let

$$[x]_\theta = \frac{< x+1 >_\theta < x-1 >_\theta}{< x+1-B >_\theta < x-1+B >_\theta}, \tag{4.7}$$

where B depends on the coupling constant of the affine Toda theory and takes real values between 0 and 2 for real coupling [10,13]. So

$$[x+2h]_\theta = [x]_\theta = [-x]_{-\theta}.$$

Finally, define the building block appearing in (4.5)

$$\{x\}_\theta = \frac{[x]_\theta}{[x]_{-\theta}} = \frac{[x]_\theta}{[-x]_\theta}. \tag{4.8}$$

Then

$$\{x+2h\}_\theta = \{x\}_\theta = \{-x\}_\theta^{-1} = \{-x\}_{-\theta}, \tag{4.9}$$

and

$$\{0\}_\theta = \{h\}_\theta = 1. \tag{4.10}$$

Furthermore, as

$$[x]_{\theta + \frac{i\pi y}{h}} = [x+y]_\theta,$$

$$\{x\}_{\theta + \frac{i\pi y}{h}} = \frac{[y+x]_\theta}{[y-x]_\theta}. \tag{4.11}$$

So, in particular,

$$\{x\}_{\theta + i\pi} = \{x+h\}_\theta. \tag{4.12}$$

The property $< x >_\theta^* = < -x >_{\theta^*}$ which leads to

$$\{x\}_{\theta^*}^* \{x\}_\theta = 1. \tag{4.13}$$

The zeros and poles of the building blocks $\{x\}_\theta$ are all simple and lie on the imaginary θ axis where $\{x\}_\theta$ is real. Mod $2i\pi$, the poles of $\{x\}_\theta$ lie at

$$\theta = \theta(x)_\pm \equiv i(x \pm 1)\frac{\pi}{h} \qquad \text{and} \qquad i(-x \pm (B-1))\frac{\pi}{h}$$

except when $x = 0$ and h (mod $2h$) when there are no poles and when $x = \pm 1$ and $x = h \pm 1$ mod ($2h$) when the poles at $\theta(x)_\mp$ do not occur owing to a cancellation with zeros which are usually distinct.

The limit of $\{x\}_\theta$ as $B \to i\infty$ exists, and is denoted by $\{x\}_\theta^{min}$, being given by putting equal to unity the factors involving B. This quantity still satisfies the properties (4.9)-(4.13) as long as we understand $[x]_\theta^{min}$ to be defined by (4.7) with the B dependent factors put equal to unity. We now verify that expression (4.5) satisfies the above properties required of the two-particle S-matrix, using the above properties of the building blocks. The argument therefore applies equally to the minimal theory defined by replacing $\{x\}$ by $\{x\}^{min}$ in (4.5).

4.i) *The powers of the building blocks*

Each building block $\{x\}$ satisfies properties (4.1) and (4.2) individually and is meromorphic in the relative rapidity θ with period $2i\pi$. So for S_{ij} to satisfy these properties it is only necessary to check the circumstances under which each occurs to an integer power which is symmetric under interchange of i and j. Twice the power of $\{2q - \frac{1}{2}(c(i)+c(j))\}_\theta$ is $-\lambda_i \cdot \sigma^q \gamma_j$ which, by (2.7b), equals $-\lambda_i \cdot \sigma^q \sigma_- \lambda_j + \lambda_i \cdot \sigma^q \sigma_+ \lambda_k$. Since the elements of the Weyl group are represented by real orthogonal matrices, we see that $\sigma^q \sigma_-$ and $\sigma^q \sigma_+$ are separately symmetric. Hence $S_{ij} = S_{ji}$. But because of the factor $\frac{1}{2}$ this power is not obviously an even integer if $\mathbf{g}$ is simply laced and the roots all have squared length 2. However each building block occurs twice in the product in (4.5) because $\{x\} = \{2h - x\}^{-1}$, by (4.9). The corresponding two values of q, say q and q' are related by

$$q + q' = \frac{c(i) + c(j)}{2} + h. \tag{4.14}$$

Hence the total power of the building block $\{2q - \frac{1}{2}(c(i) + c(j))\}_\theta$ is

$$-\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_j + \frac{1}{2}\lambda_i \cdot \sigma^{q'} \gamma_j.$$

That these two terms are equal follows on inserting

$$\sigma^q = \sigma_{c(k)}^{-1} \sigma^q \sigma_{c(k)} \tag{4.15}$$

in the first term and eliminating $\sigma_{c(k)}$ by means of (2.3) and (2.8). So the resultant power is $-\lambda_i \cdot \sigma^q \gamma_j$ which is an integer, at least if $\mathbf{g}$ is simply laced and the squared root lengths equal 2. If q and q' are equal, then by (4.14) the building block concerned occurs only once but equals $\{h\}$ which, by (4.10), is 1.

4.ii) *Crossing Property (4.3)*

By (4.9) and (4.12), $\{x\}_{i\pi - \theta} = \{x+h\}_\theta^{-1}$ and so by (4.5)

$$S_{ij}(i\pi - \theta) = \prod_{q=1}^{h} \left\{ 2q + h - \frac{c(i) + c(j)}{2} \right\}_\theta^{\frac{1}{2}\lambda_i \cdot \sigma^q \gamma_j} .$$

Because $\{2q+\text{const}\}$ and σ^q each have period h in q, expressions like the above product are unaffected by integer shifts in the dummy variable q. So substituting

$$q = \bar{q} - \frac{h}{2} + \frac{c(j) - c(\bar{j})}{4},$$

$$S_{ij}(i\pi - \theta) = \prod_{\bar{q}=1}^{h} \left\{ 2\bar{q} - \frac{c(i) + c(\bar{j})}{2} \right\}_{\theta}^{\frac{1}{2}\lambda_i \cdot \sigma^{\bar{q}} \sigma^{-\frac{h}{2} + \frac{c(j)-c(\bar{j})}{4}} \gamma_j} = S_{i\bar{j}}(\theta)$$

by (2.13) and (4.5). This is the crossing property (4.3).

4.iii) *Bootstrap Property (4.4)*

Considering one of the building blocks (4.8) in one of the factors $S_{lt}(\theta + i\eta(1,t))$ of (4.4), where $t = i, j$ or k, we have using (3.9), (4.11) and (2.4)

$$\left\{ 2q - \frac{c(l) + c(t)}{2} \right\}_{\theta - \frac{i\pi}{h}\left(2\xi(t) + \frac{1-c(t)}{2}\right)}^{-\frac{1}{2}\lambda_l \cdot \sigma^q \gamma_t}$$

$$= \left(\frac{\left[2(q - \xi(t)) - \frac{c(l)+1}{2}\right]_{\theta}}{\left[-2(q - \xi'(t)) - \frac{c(l)+1}{2}\right]_{\theta}} \right)^{-\frac{1}{2}\lambda_l \cdot \sigma^q \gamma_t}.$$

Making different replacements for the dummy variable q in numerator and denominator

$$p = q - \xi \qquad \text{and} \qquad p' = q - \xi',$$

yields

$$= \frac{\left[2p - \frac{c(l)+1}{2}\right]_{\theta}^{-\frac{1}{2}\lambda \cdot \sigma^p \sigma^{\xi(t)} \gamma_t}}{\left[-2p' - \frac{c(l)+1}{2}\right]_{\theta}^{-\frac{1}{2}\lambda_l \cdot \sigma^{p'} \sigma^{\xi'(t)} \gamma_t}}.$$

Now we see that the square bracket factor in the numerator is independent of t and so occurs in each of the factors S_{li}, S_{lj} and S_{lk}. Furthermore the total power of these factors sums to zero by the fusing relation (2.1). Similarly the total power of the denominator factor sums to zero by the second version of the fusing equation, (2.5). Since this applies to all factors on the left hand side of (4.4), the bootstrap result follows.

4.iv) *Alternative Expressions for the S-matrix*

Changing $q \to -q$ in (4.5) and using (4.9) we can also write

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q + \frac{c(i) + c(j)}{2} \right\}_{\theta}^{\frac{1}{2}\lambda_i \cdot \sigma^{-q} \gamma_j}. \tag{4.16}$$

A second rearrangement is to eliminate γ_j in favour of λ_j using (2.7a) in (4.16) thereby expressing $S_{ij}(\theta)$ as a product of two factors

$$\prod_{q=1}^{h} \left\{ 2q + \frac{c(i) + c(j)}{2} \right\}_{\theta}^{\frac{1}{2}\lambda_i \cdot \sigma^{-q + \frac{1-c(j)}{2}} \lambda_j} \prod_{q=1}^{h} \left\{ 2q + \frac{c(i) + c(j)}{2} \right\}_{\theta}^{-\frac{1}{2}\lambda_i \cdot \sigma^{-q - \frac{1+c(j)}{2}} \lambda_j}.$$

Replacing $q \to q' = q + \frac{c(j)-1}{2}$ in the first factor yields the expression

$$\prod_{q=1}^{h} \left\{ 2q + 1 + \frac{c(i) - c(j)}{2} \right\}_{\theta}^{\frac{1}{2}\lambda_i \cdot \sigma^{-q} \lambda_j}. \tag{4.17}$$

Making the corresponding substitution in the second factor, namely, $q \to q' = q + \frac{c(j)+1}{2}$ yields an expression which becomes equal to expression (4.17) with the labels i and j interchanged upon the further substitution $q \to -q$. The last step is to show that (4.17) is actually symmetric under the interchange of i and j. The result is that the square of (4.17) yields the S-matrix element

$$S_{ij}(\theta) = \prod_{q=1}^{h} \left\{ 2q + 1 + \frac{c(i) - c(j)}{2} \right\}_{\theta}^{\lambda_i \cdot \sigma^{-q} \lambda_j}. \tag{4.18}$$

The significance of this is that Corrigan and Dorey have constructed vertex operators whose reordering gives rise to expression (4.18) [25]. Our remaining step is to demonstrate the symmetry of (4.17) or (4.18) under the interchange of i and j. This follows from the identity

$$\lambda_i \cdot \sigma^{-q} \lambda_j = \lambda_i \cdot \sigma^{q + \frac{c(i)-c(j)}{2}} \lambda_j = \lambda_j \cdot \sigma^{-q + \frac{c(j)-c(i)}{2}} \lambda_i.$$

The first equality follows from (2.8) and (4.15) and the second by the orthogonality of the matrices representing elements of the Weyl group.

4.v) *Location and Residues of Singularities of S_{ij}*

In view of the identity $\lambda_j \cdot \sigma^q \gamma_i = -\lambda_j \cdot \sigma^{q'} \gamma_i$ proven in 4.i), with q and q' related as in (4.14), we can write the expression for S_{ij} as

$$S_{ij}(\theta) = \prod{}' \left\{ 2q - \frac{c(i) + c(j)}{2} \right\}_{\theta}^{-\lambda_j \cdot \sigma^q \gamma_j},$$

where the product $\prod{}'$ is restricted to those values of q (mod h) for which the power of the building block is positive. We shall calculate this range of q and show that, for it, $x \equiv 2q - \frac{c(i)+c(j)}{2}$ satisfies, mod $2h$, the inequalities $0 \le x \le h$. It follows that $S_{ij}(\theta)$

can only have poles at the points $\theta = \frac{i\pi}{h}, \frac{2i\pi}{h}, \ldots \frac{(h-1)i\pi}{h}$, mod $2i\pi$, that is, on the physical sheet, or else at points on the imaginary axis between $i\pi$ and $2i\pi$, mod $2i\pi$, with positions depending on the coupling constant B. These are on the unphysical sheet.

Our task is to divide each orbit Ω_i into positive and negative roots. Consider $\phi_i \equiv \sigma^{\frac{c(i)-1}{2}}\gamma_i = (1-\sigma^{-1})\lambda_i$, by (2.7a). This lies in Ω_i and , as λ_i is dominant, is positive. But $\sigma\phi_i = -(1-\sigma)\lambda_i$ and so is a negative root. Now σ has length r and so, by general theory, there are only r positive roots which σ transforms into negative roots. As there are r distinct orbits, Ω_i, there is one and only one such root in each orbit, namely ϕ_i. Likewise there is only one negative root in each orbit Ω_i which is transformed into a positive root by σ and this is

$$\psi_i = -\phi_{\bar{i}} = -\sigma^{\frac{c(\bar{i})-1}{2}}\gamma_{\bar{i}} = \sigma^{\frac{h-1}{2}+\frac{c(i)+c(\bar{i})}{4}}\gamma_i,$$

by (2.11). As λ_j is dominant, it follows that the power $-\lambda_j\cdot\sigma^q\gamma_i$ has the sign given as follows

$$-\lambda_j\cdot\sigma^q\gamma_i \begin{cases} \geq 0 & \text{for} \quad \dfrac{c(i)+1}{2} \leq q \leq \dfrac{h-1}{2} + \dfrac{c(i)+c(\bar{i})}{4} \\ \leq 0 & \text{for} \quad \dfrac{h+1}{2} + \dfrac{c(i)+c(\bar{i})}{4} \leq q \leq \dfrac{c(i)-1}{2}, \end{cases}$$

with the inequalities on q holding mod h. Remembering that this power is symmetric under the interchange of i and j and obeys the equality quoted in the first line of this section forces the power to vanish for some values of q. Taking this into account we check the statement above and show that the singularities lie as stated. The fact that the B dependent singularities lie on the unphysical sheet follows likewise, remembering B lies between 0 and 2.

4.vi) *Particle Interpretation of Physical Sheet Poles*

Let us define ϕ in terms of the relative rapidity, θ, by $\theta = i\phi$. Then $S_{ij}(\phi)$ is real on the real ϕ axis, possesses period 2π, and by virtue of unitarity, satisfies

$$S_{ij}(-\phi) = S_{ij}(\phi)^{-1}. \tag{4.20}$$

We have seen that its singularities on the "physical sheet" ($0 < Re\phi < \pi$) can only be poles lying at the points $\phi = \frac{x\pi}{h}$ where $x = 1, 2, \ldots h-1$. By (4.20), there are corresponding zeros on the unphysical sheet at $\phi = -\frac{x\pi}{h}$, mod 2π. Some of these poles correspond to the propagation of particles k which are ij bound states ($C_{ijk} \neq 0$), others which are $i\bar{j}$ bound states ($C_{i\bar{j}k} \neq 0$). These poles ought to possess an order which is odd, if not one, and residues of definite sign depending on whether they are ij or $i\bar{j}$ bound states. We shall check these features in this section but will have nothing to say about other poles which may be anomalous thresholds [24].

Consider a pole or zero of S_{ij} at $\phi = \frac{x\pi}{h}$, where x is an integer. These can only occur in the building blocks $\{x+1\}$, $\{x-1\}$, $\{-x-1\} = \{x+1\}^{-1}$ and $\{-x+1\} =$

$\{x-1\}^{-1}$ of (4.5) as all the other building blocks are positive at $\frac{x\pi}{h}$. Indeed, near $\frac{x\pi}{h}$

$$\{x\pm1\}_\phi \sim \frac{\pm1}{\phi-\frac{x\pi}{h}}\left(\frac{\sin\frac{\pi(x\pm1)}{h}\sin\frac{\pi x}{h}\sin\frac{B\pi}{2h}\sin\frac{(2-B)\pi}{2h}}{\sin\frac{\pi}{h}\sin\frac{\pi(2x\pm B)}{2h}\sin\frac{\pi(2x\pm(2-B))}{2h}}\right). \tag{4.21}$$

As x is an integer, and B lies between 0 and 2, the bracketed factor is ≥ 0 and vanishes only if $x = \pm1$, 0, h or $h\pm1$, agreeing with our earlier statements.

Let P_+ and P_- denote the powers of $\{x\pm1\}$ occurring in S_{ij} when it is understood that $\{-x\pm1\}$ are expressed in terms of them. When g is simply laced, these powers are integers but their sign is unknown (unless we say more about x). By the remarks above, we have, near $\frac{x\pi}{h}$

$$S_{ij}(\phi) \sim \frac{(-1)^{P_-}}{\left(\phi-\frac{x\pi}{h}\right)^{P_++P_-}}A, \tag{4.22}$$

where A is positive (or zero). By unitarity, (4.20), we have near $-\frac{x\pi}{h}$

$$S_{ij}(\phi) \sim (-1)^{P_+}\left(\phi+\frac{x\pi}{h}\right)^{P_++P_-}A^{-1}. \tag{4.23}$$

This is quite general, but we shall show below that if the points $\pm\frac{x\pi}{h}$ correspond to an ij bound state, then the fusing rule (2.1) implies that

$$P_+ = P_- + 1. \tag{4.24}$$

when g is simply laced. In this case P_+ and P_- are certainly integers and we conclude that $P_+ + P_-$ is an odd integer of unknown sign. If it is positive, equal to $2n+1$, say, S_{ij} has a pole of that order at $\frac{x\pi}{h}$ whose residue has sign $(-1)^n$. If, on the other hand, it is negative, the pole is situated at $-\frac{x\pi}{h}$. If the order of this pole is $2n+1$ then its residue again has sign $(-1)^n$ by (4.23). Thus the sign of the residue depends only on the order of the pole and not where it is situated. By the crossing relation (4.3) poles corresponding to $i\bar{j}$ bound states would have opposite sign and thus be distinguished.

In order to deduce (4.24) we recall how, in section 3., if $C_{ijk} \neq 0$, a projection of the roots summing to zero in the fusing relation (2.1) gave a mass triangle with sides of length m_i, m_j and m_k, the relevant particle masses. The relative rapidities of particles i and j was $\sqrt{-1}$ times the exterior angle to the triangle at the vertex of intersection of lines i and j, and so, by (3.9) and (4.20), given by

$$\phi = \pm\frac{\pi}{h}\left(\eta(1,i)-\eta(1,j)\right) = \pm\frac{\pi}{h}\left(2\xi(i)-2\xi(j)-\frac{c(j)-c(i)}{2}\right). \tag{4.25}$$

As the overall sign is unclear, depending upon which of the two inequivalent solutions to the fusing relations (2.1) or (2.5) we consider, we retain both possibilities. Notice that (4.25) is indeed an integer multiple of $\frac{\pi}{h}$, say $\pm\frac{x\pi}{h}$, for definiteness, in agreement with the poles and zeros of $S_{ij}(\theta)$ given by (4.5).

There are four building blocks which could possess poles or zeros at the point (4.25), namely $\{x+1\}$, $\{x-1\}$, $\{-x-1\} = \{x+1\}^{-1}$ and $\{-x+1\} = \{x-1\}^{-1}$.

Comparing with (4.5), we find four corresponding possible values of q, mod2π, which satisfy

$$2q - \frac{c(i)+c(j)}{2} = \pm\left(2\xi(i) - 2\xi(j) - \frac{c(i)-c(j)}{2}\right) \pm 1.$$

The solutions are, making use of (2.4)

$$\begin{aligned} q_1 &= \xi(i) + \xi'(j) + 1, \\ q_2 &= \xi(i) + \xi'(j), \\ q_3 &= \xi(j) + \xi'(i), \\ q_4 &= \xi(j) + \xi'(i) + 1. \end{aligned}$$

Notice q_1 and q_3 satisfy (4.14) as do q_2 and q_4. This means we can concentrate on $\{x+1\}^{P_+}\{x-1\}^{P_-}$ without loss of generality, where

$$P_+ = -\lambda_j \cdot \sigma^{q_1}\gamma_i \qquad P_- = -\lambda_j \cdot \sigma^{q_2}\gamma_i.$$

Equation (4.24) results from squaring the fusing rule (2.1) after taking the k term to the opposite side of the equation, to get

$$\gamma_j \cdot \sigma^{-\xi(j)+\xi(i)}\gamma_i = -1.$$

The final result (4.24) follows from the substitution of γ_j in terms of λ_j by (2.7).

In conclusion we confirm that expression (4.5) reproduces the properties required of the two particle scattering matrix in affine Toda theory, at least when g is simply laced. We believe we have checked a wider range of properties than hitherto and have found general and uniform arguments to achieve this. It relies on an improved understanding of the fusing rule (as discussed in section 2) and identities satisfied by the Coxeter element, in particular (2.11). Since the properties, excluding meromorphy, hold equally when g is not simply laced, it would be intriguing to find whether only a simple modification is required there, perhaps fermions [25,26].

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