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The Mass Spectrum and Coupling in Affine Toda Theories

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Abstract

We provide a unified derivation of the mass spectrum and the three point coupling of the classical affine Toda field theories, using general Lie algebraic techniques. The masses are proportional to the components of the right Perron-Frobenius vector and the three point coupling is proportional to the area of the triangle formed by the masses of the fusing particles.

The affine Toda field theory is a relativistic field theory describing the coupling of r independent scalar fields. It has been known for about ten years that in two dimensions its classical equations exhibit remarkable integrability properties [1,2,3].

There is such a theory for each affine root system but we shall consider only those associated with the root systems of an untwisted affine Kac-Moody algebra $\hat{\mathbf{g}}$. Then r, the number of fields, equals the rank of the finite dimensional simple Lie algebra $\mathbf{g}$.

Interest in this theory was intensified by the observation of Hollowood and Mansfield [4] that affine Toda field theories were integrable deformations (both classically and quantum mechanically) of a conformally invariant theory, namely the ordinary Toda field theory associated with $\mathbf{g}$. In accordance with general ideas of Zamolodchikov [5], the infinite number of conservation laws associated with the integrability are relics of the broken conformal symmetry.

It was then noticed on the basis of case-by-case computation [1,6,7,8,9] that the masses $m_1, m_2, \ldots, m_r$ and the three point coupling C_{ijk} of the fields occurring in the classical equations of motion display remarkable patterns that are independent of the magnitude of the symmetry breaking and seem to be preserved in the quantum theory, at least if $\mathbf{g}$ is simply laced. The masses are proportional to the components of the Perron-Frobenius vector and the coupling has modulus proportional to the area of the triangle formed by the masses of the three coupling particles.

The object of this letter is to provide a unified derivation of these remarkable results, extended to be valid for any simple Lie algebra, whether simply laced or not, using results in Lie algebra theory and developing further ideas of Freeman [10] concerning the explanation of the mass pattern.

The Lagrangian of the affine Toda theory reads

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - e^{\beta\Phi} E e^{-\beta\Phi} E^\dagger\right) \qquad , \tag{1}$$

where Φ is an element of the Cartan subalgebra $\mathbf{h}$. Let T_3 be the generator in $\mathbf{h}$ of a maximal SU(2) subalgebra embedded in $\mathbf{g}$. Then if h denotes the Coxeter number of $\mathbf{g}$, that is one less than the quotient of the dimension of $\mathbf{g}$ by its rank, or equivalently one more than the height of its highest root ψ, the quantity

$$S := e^{2\pi i T_3/h} \tag{2}$$

furnishes a $\mathbb{Z}_h$ grading of $\mathbf{g}$:

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1 \oplus \cdots \oplus \mathbf{g}_{h-1} \qquad \text{where} \qquad S\mathbf{g}_n S^{-1} = e^{2\pi i n/h}\mathbf{g}_n = \omega^n \mathbf{g}_n \qquad . \tag{3}$$

We have

$$\mathbf{g}_0 = \mathbf{h} \qquad , \qquad \mathbf{g}_1 = \{E_{\alpha_1}, \ldots, E_{\alpha_r}, E_{-\psi}\} \tag{4}$$

where $\alpha_1, \ldots, \alpha_r$ are the simple roots of $\mathbf{g}$ with respect to $\mathbf{h}$. Then E occurring in the Lagrangian (1) is an element of $\mathbf{g}_1$ which has to commute with its hermitian conjugate $E^\dagger$ in order that the term linear in Φ in (1) disappears, and $\Phi = 0$ correctly describes the ground state.

It is known that E is a 'regular element' of $\mathbf{g}$ so that the set of elements commuting with it forms a unique second Cartan subalgebra, $\mathbf{h}'$ (containing $E^\dagger$). The interplay between these two Cartan subalgebras $\mathbf{h}$ and $\mathbf{h}'$, said by Kostant [11] to be in apposition, is crucial to what follows and is known to be relevant in the construction of the conserved currents, [2], which is achieved by gauging the zero curvature potentials into $\mathbf{h}'$ so that their duals become conserved.

We denote by h_i and e_α the generators of $\mathbf{h}'$ and the corresponding step operators in a Cartan-Weyl basis, respectively. It then follows that $Sx\cdot hS^{-1} = \sigma(x)\cdot h \in \mathbf{h}'$, $Se_\alpha S^{-1} \sim e_{\sigma(\alpha)}$ where σ must be an element of the Weyl group of $\mathbf{g}$ (and so orthogonal) of order h, $\sigma^h = 1$, as S^h commutes with $\mathbf{g}$. It is possible to choose phases so that, for all α,

$$Se_\alpha S^{-1} = e_{\sigma(\alpha)} \quad . \tag{5}$$

The action of σ on the system of hr roots splits them into s disjoint orbits, say, each composing at most h elements. Hence $hs \geq hr$. But the quantities $\sum_{n=1}^{h} e_{\sigma^n(\alpha)}$ associated with each orbit are linearly independent and commute with S and so, by (4) lie in $\mathbf{h} = \mathbf{g}_0$. Hence $s \leq r$. We conclude that the number of orbits is precisely r, the rank of $\mathbf{g}$, and that each orbit composes h roots. We define

$$A_i := \frac{1}{\sqrt{h}} \sum_{n=1}^{h} e_{\sigma^n(\gamma_i)} \qquad i = 1, \ldots, r \tag{6}$$

where γ_i is a representative of the i-th orbit Ω_i, to be specified later (see (23)). $-\Omega_i$ is also an orbit and so must coincide with $\Omega_{\bar{i}}$, say, where $\bar{i}$ is a permutation of order 2 of the numbers $1,\ldots r$. We then have

$$[A_i, A_j] = 0 \qquad \text{and} \qquad \mathrm{tr}(A_i A_j) = \delta_{i\bar{j}} \tag{7}$$

in view of our chosen normalisations. Since $A_i \in \mathbf{h}$ we can expand the Lie algebra valued fields

$$\Phi = \sum_{i=1}^{r} \phi_i A_i \qquad \text{, so} \qquad \phi_i^* = \phi_{\bar{i}} \quad . \tag{8}$$

The significance of this expansion is that the ϕ_i are the mass eigenstates as we now see, by considering the term in the Lagrangian (1) quadratic in Φ. After rearrangment it is

$$-\frac{\beta^2}{2}\mathrm{tr}\Big(\Phi\left[E^\dagger, [E,\Phi]\right]\Big) = -\frac{\beta^2}{2}\sum_{i,j}\phi_i\phi_j \mathrm{tr}\Big(A_i\left[E^\dagger,[E,A_j]\right]\Big) \quad . \tag{9}$$

Given $E \in \mathbf{h}'$ we can write $E = \eta \cdot h$ and calculate

$$\left[E^\dagger, [E, A_j]\right] = |\eta\cdot\gamma_j|^2 A_j \tag{10}$$

so that the above quadratic term reads

$$-\frac{\beta^2}{2}\sum_i |\eta\cdot\gamma_i|^2|\phi_i|^2 \tag{11}$$

and we recognise the particle masses

$$m_i = \beta|\eta\cdot\gamma_i| \qquad i = 1,\ldots,r \quad . \tag{12}$$

It is intriguing that this is the Higgs formula for the masses of gauge particles for a specific Higgs field in the complex adjoint representation η.

To proceed further we need to know more about η and γ_i. As $E \in \mathbf{g}_1 \cap \mathbf{h}'$, η is an eigenvector of σ with eigenvalue $\omega = \exp(2\pi i/h)$. Kostant [11] has confirmed, what is strongly suggested by the properties of σ already demonstrated, that σ is a 'Coxeter element' of the Weyl group of $\mathbf{g}$, that is, it is conjugate to a product of reflections σ_i in the simple roots α_i (henceforth with respect to $\mathbf{h}'$ rather than $\mathbf{h}$), each taken just once:

$$\sigma_i(\alpha_j) = \alpha_j - 2\frac{\alpha_j\cdot\alpha_i}{\alpha_i^2} = \alpha_j - K_{ji}\alpha_i \tag{13}$$

where K_{ji} is the Cartan matrix of $\mathbf{g}$.

The simple roots correspond to the points of the Dynkin diagram of $\mathbf{g}$ which can always be 'bicoloured' so that each point is either 'black' or 'white' and the lines link no two points of the same colour. It is then possible to choose a system of simple roots with respect to $\mathbf{h}'$ so that

$$\sigma = \sigma_B\sigma_W \tag{14}$$

where σ_B is a product of the reflections in the simple black roots, whose ordering is irrelevant as they commute, and similarly σ_W. Then it is possible to verify the identity [12]

$$(\sigma_B + \sigma_W)\alpha_i = \sum_{j=1}^{r}(2\delta_{ij} - K_{ij})\alpha_j \quad . \tag{15}$$

As $(\sigma_B+\sigma_W)^2 = 2 + \sigma + \sigma^{-1}$ this relates eigenvalues of σ and the Cartan matrix K. Let x denote the left eigenvector of K, to eigenvalue $2 - 2\cos\theta$, so

$$\sum_{i=1}^{r} x_i K_{ij} = 2(1-\cos\theta)x_j \quad . \tag{16}$$

Because of the bicolouration we also have

$$\sum_{i=1}^{r} c(i)x_i K_{ij} = 2(1+\cos\theta)c(j)x_j \quad , \tag{17}$$

where $c(i) = \pm 1$ as i is black or white. Then, defining

$$q_W := \sum_{i\in W} x_i\alpha_i \qquad q_B := \sum_{i\in B} x_i\alpha_i \tag{18}$$

we have , trivially

$$\sigma_W(q_W) = -q_W \qquad \sigma_B(q_B) = -q_B \tag{19}$$

and by (15) and (17)

$$\sigma_W(q_B) = q_B + 2\cos\theta q_W \qquad \text{and} \qquad \sigma_B(q_W) = q_W + 2\cos\theta q_B \quad . \tag{20}$$

Thus both σ_B and σ_W act on the plane in rootspace spanned by q_B and q_W and we can readily find the eigenvector of σ in this plane. We have

$$\sigma(q) = e^{2i\theta} q \qquad \text{if} \qquad q = e^{-\frac{1}{2}i\theta} q_W + e^{\frac{1}{2}i\theta} q_B \tag{21}$$

and can then calculate

$$q \cdot \alpha_j = ic(j) \sin\theta \; x_j \alpha_j^2 e^{\frac{-ic(j)\theta}{2}} \quad . \tag{22}$$

The possible values of θ are $n_i\pi/h$ where $n_1, \ldots, n_r$ are the exponents of $\mathbf{g}$ taking values between 1 and $h-1$. The eigenvalue $n_1 = 1$, i.e. $\theta = \pi/h$ yields the smallest eigenvalue of K. By the Perron-Frobenius theorem the corresponding components of x_i never vanish and can all taken to be positive. Taking into account these facts, one can deduce that the roots $c(i)\alpha_i$ lie on distinct orbits of σ. Hence we can choose as our representative of the orbit Ω_i

$$\gamma_i = c(i)\alpha_i \quad . \tag{23}$$

By (22) and (23) the masses (12) are now given by

$$m_i = \beta x_i \alpha_i^2 \sin\frac{\pi}{h} \tag{24}$$

and are thus proportional to the components of the right Perron-Frobenius vector. (As K is not symmetric if $\mathbf{g}$ is not simply laced the left and right eigenvectors to the same eigenvalue may differ.)

We can now examine the three point coupling by expanding the Lagrangian (1) to get the cubic term

$$\frac{1}{3!} C_{ijk}\phi_i\phi_j\phi_k = \frac{\beta^3}{6} \mathrm{tr}\Big(\Phi\, \big[[\Phi, E], [\Phi, E^\dagger]\big]\Big) \quad . \tag{25}$$

Expanding Φ by (8) we see

$$C_{ijk} = \beta^3 \mathrm{tr}\Big(\big[[A_i, E], [A_j, E^\dagger]\big]\, A_k\Big) \quad . \tag{26}$$

It is necessary to check that this is totally symmetric in the indices. This follows remembering that the A_i, A_j and $E, E^\dagger$ mutually commute. In fact, as the commutator in (26) commutes with S, defined in equation (2), we can write

$$\big[[E, A_i], [E^\dagger, A_j]\big] = \beta^{-3} C_{ijk} A_{\bar{k}} \tag{27}$$

But

$$\big[[E, A_i], [E^\dagger, A_j]\big] = \frac{(\eta\cdot\gamma_i)(\eta^*\cdot\gamma_j)}{h} \sum_{p,q} \omega^{q-p} [e_{\sigma^p(\gamma_i)}, e_{\sigma^q(\gamma_j)}] \tag{28}$$

which we rearrange as

$$\frac{(\eta\cdot\gamma_i)(\eta^*\cdot\gamma_j)}{h} \sum_p S^p \left(\sum_q \omega^q [e_{\gamma_i}, e_{\sigma^q(\gamma_j)}]\right) S^{-p} \quad . \tag{29}$$

The commutator of two step operators is non zero either when the roots concerned add to zero or add to a third root. In the first case we have

$$[e_{\gamma_i}, e_{\sigma^q(\gamma_j)}] = \gamma_i \cdot h \quad . \tag{30}$$

As the result of the calculation must commute with S, and this does not, we expect these terms to sum up to zero. Indeed

$$\sum_p S^p \gamma_i \cdot h S^{-p} = (1-\sigma^h)(1-\sigma)^{-1}\gamma_i \cdot h = 0 \tag{31}$$

as $\sigma^h = 1$ and $\sigma \neq 1$.

In the second case the commutator may give a step operator for a root in the orbit $\Omega_{\bar{k}}$ and hence contribute to C_{ijk}. Defining

$$Q(i,j,\bar{k}) := \{q \in \mathbb{Z} \mid \gamma_i + \sigma^q(\gamma_j) \in \Omega_{\bar{k}}\} \quad , \tag{32}$$

we have

$$\begin{aligned} \big[[E, A_i], [E^\dagger, A_j]\big] &= \frac{(\eta\cdot\gamma_i)(\eta^*\cdot\gamma_j)}{h} \sum_p S^p \Big(\sum_{\bar{k}} \sum_{q\in Q} \omega^q \varepsilon(i,j,q) e_{\gamma_i+\sigma^q(\gamma_j)}\Big) S^{-p} \\ &= \frac{(\eta\cdot\gamma_i)(\eta^*\cdot\gamma_j)}{\sqrt{h}} \sum_{\bar{k}} \sum_{q\in Q} \omega^q \varepsilon(i,j,q) A_{\bar{k}} \quad , \end{aligned} \tag{33}$$

where $\varepsilon(i,j,q)$ is the structure constant defined by

$$[e_{\gamma_i}, e_{\sigma^q(\gamma_j)}] = \varepsilon(i,j,q) e_{\gamma_i+\sigma^q(\gamma_j)} \quad . \tag{34}$$

This is the confirmation of Dorey's rule [13] for the coupling of particle $\bar{k}$ to particle i and j.

We shall now show that when the set Q (32) is not empty, it contains precisely two elements. The particle $\bar{k}$ has mass

$$\beta|\eta\cdot\gamma_{\bar{k}}| = \beta\,|\eta\cdot(\gamma_i + \sigma^q(\gamma_j))| \tag{35}$$

which can be reexpressed as

$$m_{\bar{k}} = \left|m_i + m_j e^{i\theta_{ij}(q)}\right| \tag{36}$$

using (22) and the fact that $\eta\cdot\sigma^q(\alpha_j) = \omega^{-q}\eta\cdot\alpha_j$ where

$$\theta_{ij}(q) := \left(2q + \frac{c(i)-c(j)}{2}\right)\theta, \qquad \theta = \frac{\pi}{h} \quad . \tag{37}$$

So the mass squared is

$$m_{\bar{k}}^2 = m_i^2 + m_j^2 + 2m_i m_j \cos\theta_{ij}(q) \tag{38}$$

that is m_i, m_j, $m_{\bar{k}}$ form a triangle with the angle between sides m_i and m_j being $\pi - \theta_{ij}(q)$. Any other $q' \in Q(i,j,\bar{k})$ must also couple γ_i, γ_j to the orbit $\Omega_{\bar{k}}$, and produce the same mass $m_{\bar{k}}$. Thus $\cos\theta_{ij}(q) = \cos\theta_{ij}(q')$ and q' must satisfy one of the two equations

$$\theta_{ij}(q) \pm \theta_{ij}(q') = 2\pi n \qquad n \in \mathbb{Z} \quad . \tag{39}$$

Since $0 \leq q, q' \leq h-1$, the form of eq.(37) tells us that the second equation can not be satisfied. Thus

$$q + q' + \frac{c(i)-c(j)}{2} = nh = h \quad . \tag{40}$$

The integer n cannot be zero otherwise i and j would be of different colour and either q or q' is zero. It means that $\gamma_i + \gamma_j = c(i)(\alpha_i - \alpha_j)$ is a root and α_i, α_j are simple, which is of course impossible. Therefore we can have at most two elements in Q related by eq.(40). On the other hand, the fact that A_i, A_j commute implies that

$$\sum_{q \in Q} \varepsilon(i,j,q) = 0 \tag{41}$$

and Q has to contain more than one element. As a result, when Q is not empty,

$$\varepsilon(i,j,q) + \varepsilon(i,j,q') = 0 \tag{42}$$

and

$$\begin{aligned} \beta^2 \left[[E, A_i], [E^\dagger, A_j] \right] &= \beta^2 \frac{(\eta \cdot \gamma)(\eta^* \cdot \gamma_j)}{\sqrt{h}} \sum_{\bar{k}} \varepsilon(i,j,q)(\omega^q - \omega^{q'}) A_{\bar{k}} \\ &= \sum_{\bar{k}} \frac{2i}{\sqrt{h}} m_i m_j \sin\theta_{ij}(q) \varepsilon(i,j,q) A_{\bar{k}} \end{aligned} \tag{43}$$

Comparing with equation (27) we finally obtain

$$C_{ijk} = \frac{4i\beta\varepsilon(i,j,q)}{\sqrt{h}} \Delta_{ijk} \tag{44}$$

where Δ_{ijk} denotes the area of the triangle bounded by the sides of length m_i, m_j and $m_k = m_{\bar{k}}$. The structure constants $\varepsilon(i,j,q)$ are most easily specified by saying that, if ψ is the highest root, $\sqrt{2/\psi^2}\varepsilon(i,j,q)$ takes values ± 1 unless i, j and k all correspond to short roots. Then the value is $\pm 1/\sqrt{2}$ for B, C and F_4 and $\pm 2/\sqrt{3}$ for G_2. (These facts are important for the vertex operator construction of non-simply laced Kac-Moody algebras [14].)

An easier way to see the coupling must be proportional to Δ_{ijk} is to observe that in eq.(28), interchanging i and j has the effect $\omega^{q-p} \to -\omega^{p-q}$. Since the coupling is symmetric in i and j, we are allowed to add such a term and produce a sine function. The 'colour term' in $\theta_{ij}(q)$ also comes out correctly if we take care of the factor $(\eta \cdot \gamma_i)(\eta^* \cdot \gamma_j)$. However, the above argument is still necessary to give a correct factor in eq.(44).

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