A note on the tensor product of restricted simple modules for algebraic groups

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Abstract

Let G be a semisimple simply connected algebraic group over an algebraically closed field of positive characteristic p. Denote by G_1 its first Frobenius kernel. In this note, we determine for which group G the restriction to G_1 of any indecomposable G-summand of the tensor product of any two restricted simple G-modules remains indecomposable.

Keywords: semisimple algebraic group, Frobenius kernel, tensor product

1 Introduction and notations

Let G be a semisimple simply connected algebraic group over an algebraically closed field k of positive characteristic p. Assume that G is defined and split over the prime subfield $\mathbf{F_p}$ of p elements. Let $F : G \to G$ be the corresponding Frobenius morphism and denote by $G_1 := \mathrm{Ker}(F)$ the first Frobenius kernel of G. We recall the basic definitions and notation needed here. More details can be found in Jantzen [8].

Let T be an F-stable split maximal torus of G and let $W = N_G(T)/T$ be the Weyl group. Let B be an F-stable Borel subgroup containing T (and denote by B^+ the opposite Borel subgroup) and let U (resp. U^+) be the unipotent radical of B (resp. of B^+). We denote by T_1 and B_1 the corresponding subgroups (schemes) of G_1.

Let $X = X(T)$ be the weight lattice and fix a non-singular, symmetric positive definite W-invariant form on $X \otimes_{\mathbf{Z}} \mathbf{R}$, denoted by $\langle .,. \rangle$. Let Φ be the root system, Φ^+ the set of positive roots which makes B the negative Borel and let Π be the set of simple roots. Define the set of dominant weights by

$$X^+ = \{\lambda \in X \mid \langle \lambda, \check{\alpha} \rangle \geq 0 \ \forall \alpha \in \Pi\}$$

where $\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$ for $\alpha \in \Phi$. Define also the set of restricted weights X_1 by

$$X_1 = \{\lambda \in X^+ \mid \langle \lambda, \check{\alpha} \rangle < p \ \forall \alpha \in \Pi\}.$$

The weight lattice has a natural partial ordering: for $\lambda, \mu \in X$ we write $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of simple roots. Let w_0 be the longest element in the Weyl group W. We denote by α_0 the highest short root of Φ

and by ρ half the sum of the positive roots. The Coxeter number associated to the root system Φ is given by $h = \langle \rho, \check{\alpha}_0 \rangle + 1$.

For $\lambda \in X$, let k_λ be the one dimensional B-module on which T acts via λ and denote by $\nabla(\lambda)$ the induced module $\mathrm{Ind}_B^G k_\lambda$. Then $\nabla(\lambda)$ is finite dimensional and it is non-zero if and only if $\lambda \in X^+$. For $\lambda \in X^+$, the socle $L(\lambda)$ of $\nabla(\lambda)$ is simple and furthermore $\{L(\lambda) \mid \lambda \in X^+\}$ is a complete set of non-isomorphic simple G-modules. For $\lambda \in X^+$, we denote by $\Delta(\lambda)$ the Weyl module given as $\Delta(\lambda) := \nabla(-w_0\lambda)^*$. A rational G-module M is said to have a good filtration if it has a filtration

$$\{0\} = M_0 \subseteq M_1 \subseteq ... \subseteq M_k = M$$

such that each quotient M_i/M_{i+1} is isomorphic to an induced module $\nabla(\mu_i)$ for some $\mu_i \in X^+$. A rational G-module T is called a tilting module if both T and T^* have a good filtration. Indecomposable tilting modules have been classified (see Ringel [9] and Donkin [3]), they are parametrized by the set of dominant weights X^+. For each $\lambda \in X^+$, we denote the corresponding indecomposable tilting module by $T(\lambda)$. For the dominant weight $(p-1)\rho$ we have $\nabla((p-1)\rho) = \Delta((p-1)\rho) = L((p-1)\rho) = T((p-1)\rho)$, this module is called the Steinberg module and is denoted by St. The restriction to G_1 of the set of restricted simple G-modules $\{L(\lambda) \mid \lambda \in X_1\}$ gives a complete set of non-isomorphic simple G_1-modules.

We shall also make use of the theory of G_1T-modules (see Janzten [8]II.9). In particular, for $\lambda \in X$ we consider the induced module $\hat{Z}_1'(\lambda) := \mathrm{Ind}_{B_1T}^{G_1T} k_\lambda$.

The Steinberg module St is simple and injective when restricted to G_1 and one suspects that for all $\lambda \in X_1$ the injective hull of $L(\lambda)$ as a G_1-

module can be obtained by restricting the indecomposable G-summand of $St \otimes L((p-1)\rho + w_0\lambda)$ containing the highest weight $2(p-1)\rho + w_0\lambda$. This is known to be true when $p \geq 2h-2$ (see Jantzen [7] section 4). It was first shown for $p \geq 3h-3$ by Ballard in [2]. Stephen Doty suggested to look at a more general problem (see [5]), namely the restriction to G_1 of arbitrary indecomposable G-summands of the tensor product of arbitrary restricted simple G-modules. More precisely, he asked the following question: For which group G does the following condition hold?

Condition (*): For all restricted weights λ and μ, the indecomposable G-summands of the tensor product $L(\lambda) \otimes L(\mu)$ remain indecomposable upon restriction to G_1.

For $G = SL_2(k)$, it is well known that Condition (*) holds. In [6], Stephen Doty and Anne Henke used this fact to express the indecomposable G-summand of the tensor product of arbitrary (not necessarily restricted) simple modules as a twisted tensor product of certain "small" tilting modules.

In this paper, we answer Doty's question completely. We assume from now on, and without loss of generality, that the root system of the group G is *irreducible*. We will show that, in fact, Condition (*) only holds in very few cases, namely:

Theorem 1 *Condition (*) holds if and only if G has Dynkin type A_1, or $p = 2$ and G has Dynkin type A_2 or $B_2 = C_2$.*

This result is given by Propositions 2 and 3 below.

2 Proof

Proposition 1 *Let $\lambda \in X_1$. Assume that all indecomposable G-summands of $L(\lambda) \otimes St$ remain indecomposable upon restriction to G_1. Then there is no non-zero weight τ of $L(\lambda)$ of the form $\tau = p\mu$ for some $\mu \in X$.*

Proof: Note that if all indecomposable G-summands of $L(\lambda) \otimes St$ remain indecomposable as G_1-modules then they also remain indecomposable as G_1T-modules. Considered as a G_1T-module, $L(\lambda) \otimes St$ has a filtration with quotients $\hat{Z}_1'((p-1)\rho + \nu)$ with $\nu \in X$ occuring $\dim L(\lambda)_\nu$ times, where $L(\lambda)_\nu$ denotes the ν-weight space of the module $L(\lambda)$ (see Jantzen [8]II.9.19). Now if $\nu = p\mu$ is a weight of $L(\lambda)$ then $\hat{Z}_1'((p-1)\rho+\nu) \cong St \otimes p\mu$ is projective and injective so it must occur as a G_1T-summand of $L(\lambda) \otimes St$. Thus, by assumption, $L(\lambda) \otimes St$ must have a G-summand whose restriction to G_1T is $St \otimes p\mu$. But, for $\mu \neq 0$, the simple G_1T-module $St \otimes p\mu$ does not lift to G. Hence μ must be zero. QED

Remark: We now give a different proof of Proposition 1 by considering the G_1-Steinberg block component of $L(\lambda) \otimes St$. Using Jantzen [8]II.10.4, it is isomorphic, as G-modules, to $St \otimes Z^F$ for some G-module Z. As every indecomposable G-summand of $L(\lambda) \otimes St$ remains indecomposable as G_1-modules, Z must be a trivial module and we have

$$\mathrm{Hom}_G(St, L(\lambda) \otimes St) \cong \mathrm{Hom}_{G_1}(St, L(\lambda) \otimes St).$$

But we always have

$$\mathrm{Hom}_G(St, L(\lambda) \otimes St) \subseteq \mathrm{Hom}_{G_1T}(St, L(\lambda) \otimes St) \subseteq \mathrm{Hom}_{G_1}(St, L(\lambda) \otimes St).$$

Hence,

$$\mathrm{Hom}_{G_1T}(St, L(\lambda)\otimes St) = \mathrm{Hom}_{G_1}(St, L(\lambda)\otimes St).$$

Now as G_1-modules we have

$$\begin{aligned} St\otimes St &\cong St\otimes \mathrm{Ind}_{B_1}^{G_1} k_{(p-1)\rho} \\ &\cong \mathrm{Ind}_{B_1}^{G_1}(St\otimes k_{(p-1)\rho}) \\ &\cong \mathrm{Ind}_{B_1}^{G_1}(\mathrm{Ind}_{T_1}^{B_1} k) \\ &\cong \mathrm{Ind}_{T_1}^{G_1} k. \end{aligned}$$

Similarly, as G_1T-modules, we have $St\otimes St \cong \mathrm{Ind}_T^{G_1T} k$. So

$$L(\lambda)^T \cong \mathrm{Hom}_{G_1T}(St, L(\lambda)\otimes St) = \mathrm{Hom}_{G_1}(St, L(\lambda)\otimes St) \cong L(\lambda)^{T_1}.$$

Now the T_1- fixed points space of $L(\lambda)$ is exactly the sum of the weight spaces corresponding to weights of the form $p\mu$ for some $\mu \in X$. As it has to coincide with the T- fixed points, we have that every weight of $L(\lambda)$ of the form $p\mu$ for some $\mu\in X$ must in fact be zero.

Proposition 2 *Assume that the root system of G is irreducible. If Condition (*) holds then either G has Dynkin type A_1 or $p = 2$ and G has Dynkin type A_2 or $B_2 = C_2$.*

Before proving this proposition, let us first make a note on truncation of simple modules. Let Γ be a subset of the set of simple roots Π and let G_Γ be the corresponding Levi subgroup i.e. G_Γ is the subgroup generated by T and the root subgroups U_α with $\pm\alpha \in \Gamma$. The simple G_Γ-modules are parametrized by $X_\Gamma^+ = \{\lambda \in X \mid \langle \lambda, \check{\alpha}\rangle \geq 0 \ \forall \alpha \in \Gamma\}$, we denote them by $L_\Gamma(\lambda)$, $\lambda \in X_\Gamma^+$. For $\lambda \in X^+$ and $\mu \in X$, we write $L(\lambda)_\mu$ to denote the μ-weight space of the G-module $L(\lambda)$. Then the truncation functor Tr_Γ^λ gives

$$Tr_\Gamma^\lambda L(\lambda) := \bigoplus_{(m_\alpha)\in \mathbf{Z}^{|\Gamma|}} L(\lambda)_{\lambda - \sum_{\alpha\in\Gamma} m_\alpha \alpha} \cong L_\Gamma(\lambda)$$

(see Jantzen [8]II.2.11)

Proof: We shall consider the cases $p > 2$ and $p = 2$ separately. Let us start with the case $p > 2$. Note that for any irreducible root system of rank at least 2, we can choose $\alpha \in \Pi$ such that α has non-zero inner product with precisely one other simple root, say β, and $\langle \alpha, \check{\beta} \rangle = -1$. Let ω_α and ω_β be the corresponding fundamental weights. Then we have $\alpha = 2\omega_\alpha - \omega_\beta$ and so

$$p\omega_\beta = (2\omega_\alpha + (p-1)\omega_\beta) - \alpha.$$

We claim that $p\omega_\beta$ occurs as a weight of $L(2\omega_\alpha + (p-1)\omega_\beta)$. This follows from the remark on truncation of simple modules mentioned above, taking $\Gamma = \{\alpha\}$, and the fact that when $p > 2$, 0 occurs as a weight of the simple $SL_2(k)$-module $L(2)$. Hence, by Proposition 1, Condition (*) doesn't hold in this case.

We now turn to the case $p = 2$. Here we shall use the remark on truncation of simple modules with Γ generating a root system of type A_2, and noting that when $p = 2$, the simple $SL_3(k)$-module $L(1,1)$ has non-zero 0-weight space.

First consider G of the following Dynkin type: A_n, $n \geq 3$; B_n, $n \geq 4$; C_n, $n \geq 3$; D_n, $n \geq 5$; $E_{6,7,8}$; F_4. In all these cases, we can find simple roots α, β and γ such that

$$\langle \alpha, \check{\beta} \rangle = -1, \ \langle \alpha, \check{\eta} \rangle = 0 \ \forall \alpha, \beta \neq \eta \in \Pi$$

$$\langle\beta,\check{\alpha}\rangle = \langle\beta,\check{\gamma}\rangle = -1,\ \langle\beta,\check{\eta}\rangle = 0\ \forall\alpha,\beta,\gamma \neq \eta \in \Pi.$$

Let $\omega_\alpha, \omega_\beta, \omega_\gamma$ be the corresponding fundamental weights. Then we have $\alpha = 2\omega_\alpha - \omega_\beta$, $\beta = -\omega_\alpha + 2\omega_\beta - \omega_\gamma$ and so

$$2\omega_\gamma = (\omega_\alpha + \omega_\beta + \omega_\gamma) - \alpha - \beta.$$

So we have that $2\omega_\gamma$ is a weight of $L(\omega_\alpha + \omega_\beta + \omega_\gamma)$ and hence by Proposition 1, Condition (*) does not hold for such groups.

We are left with three types of groups, B_3, D_4 and G_2. For type B_3, we take $\Pi = \{\alpha, \beta, \gamma\}$ such that

$$\langle\alpha,\check{\beta}\rangle = \langle\beta,\check{\alpha}\rangle = -1,\ \langle\beta,\check{\gamma}\rangle = -2.$$

Then

$$2\omega_\gamma = (\omega_\alpha + \omega_\beta) - \alpha - \beta$$

and we can argue as before.

For type D_4, let $\Pi = \{\alpha, \beta, \gamma, \delta\}$ with

$$\langle\alpha,\check{\beta}\rangle = \langle\beta,\check{\alpha}\rangle = \langle\beta,\check{\gamma}\rangle = \langle\beta,\check{\delta}\rangle = -1.$$

Then we have that $2\omega_\gamma + 2\omega_\delta = (\omega_\alpha + \omega_\beta + \omega_\gamma + \omega_\delta) - \alpha - \beta$ and we can argue as before.

For type G_2, write $\Pi = \{\alpha, \beta\}$ such that

$$\langle\alpha,\check{\beta}\rangle = -1,\ \langle\beta,\check{\alpha}\rangle = -3.$$

then we note that

$$2\omega_\alpha = (\omega_\alpha + \omega_\beta) - \alpha - \beta.$$

As $L(\omega_\alpha + \omega_\beta) = St = \nabla(\omega_\alpha + \omega_\beta)$ and $2\omega_\alpha$ is a dominant weight, it does occur as a weight of $L(\omega_\alpha + \omega_\beta)$. This completes the proof. QED

Proposition 3 *Condition (*) holds for G of Dynkin type A_1 for all primes and for G of Dynkin type A_2 and $B_2 = C_2$ when $p = 2$.*

Proof: Type A_1: Let $0 \leq m, n \leq p-1$ and consider the tensor product of the two simple modules $L(m) \otimes L(n)$. It is a tilting module and all its weights are less or equal to $2p-2$. So any indecomposable G-summand is either simple or indecomposable projective (injective) when restricted to G_1. Thus condition (*) clearly holds here.

Type A_2, $p = 2$: Note that all restricted simple modules are tilting modules in this case. Direct calculations using characters show that we have the following decomposition as G-modules and that each summand has simple G_1-socle.

$$L(1,0) \otimes L(0,1) \cong k \oplus L(1,1)$$
$$L(1,0) \otimes L(1,0) \cong T(2,0) \quad \text{with } G_1\text{-socle } L(0,1)$$
$$L(0,1) \otimes L(0,1) \cong T(0,2) \quad \text{with } G_1\text{-socle } L(1,0)$$
$$L(1,0) \otimes L(1,1) \cong T(2,1) \quad \text{with } G_1\text{-socle } L(1,0)$$
$$L(0,1) \otimes L(1,1) \cong T(1,2) \quad \text{with } G_1\text{-socle } L(0,1)$$
$$L(1,1) \otimes L(1,1) \cong T(2,2) \oplus 2St \quad \text{where } T(2,2) \text{ has } G_1\text{-socle } k.$$

Type $B_2 = C_2$, $p = 2$: Choose the following ordering on the set of simple roots: $\langle \alpha_1, \check{\alpha_2} \rangle = -1$ and $\langle \alpha_2, \check{\alpha_1} \rangle = -2$. Note that all restricted simple modules are tilting except $L(0,1)$ which occurs as a submodule of $\nabla(0,1)$ with quotient k. Now, direct calculations using characters show that we have the following decompositions as G-modules and that each summand has simple G_1-socle.

$$L(1,0) \otimes L(0,1) \cong L(1,1)$$

$$L(1,0) \otimes L(1,0) \cong T(2,0) \quad \text{with } G_1\text{-socle } k$$

$$L(0,1) \otimes L(0,1) \cong M \quad \text{with } G_1\text{-socle } k$$

$$L(1,0) \otimes L(1,1) \cong T(2,1) \quad \text{with } G_1\text{-socle } L(0,1)$$

$$L(0,1) \otimes L(1,1) \cong T(1,2) \quad \text{with } G_1\text{-socle } L(1,0)$$

$$L(1,1) \otimes L(1,1) \cong T(2,2) \quad \text{with } G_1\text{-socle } k.$$

QED

Remark: Note that the proof of Theorem 1 given here can easily be generalized to the case where G is a reductive group (with irreducible root system) such that its derived subgroup is simply connected.

In this case, Proposition 1 tells us that there is no weight τ of $L(\lambda)$ satisfying $\tau \notin \{\nu \in X \mid \langle \nu, \check{\alpha} \rangle = 0 \ \forall \alpha \in \Pi\}$ and $\tau = p\mu$ for some $\mu \in X$. For the proofs of Propositions 2 and 3, it is clear that we can reduce the calculations to the derived subgroup.

3 Remarks on some tilting modules

In the remark following Proposition 1, we considered the G_1-Steinberg block component $St \otimes Z^F$ of the G-module $L(\lambda) \otimes St$. There, we showed that if condition (*) holds then Z is a trivial module. We now investigate the G-module Z in the general case.

Note that when $p \geq 2h - 2$, the module $L(\lambda) \otimes St$ is tilting for any restricted weight λ (see [1] 2.5 Corollary). As any summand of a tilting module is a tilting module, we see that $St \otimes Z^F$ is also a tilting module. So by definition $St \otimes Z^F$ and $St \otimes (Z^*)^F$ have a good filtration. Now using

Donkin [4], this is equivalent to saying that for all $\lambda \in X^+$ we have

$$\mathrm{Ext}^1_G(\Delta(\lambda), St \otimes Z^F) = 0$$

and similarly for Z^*. In particular, for all $\mu \in X^+$ we have $\Delta((p-1)\rho + p\mu) \cong St \otimes \Delta(\mu)^F$ and so

$$\mathrm{Ext}^1_G(\Delta((p-1)\rho + p\mu), St \otimes Z^F) \cong \mathrm{Ext}^1_G(\Delta(\mu), Z) = 0$$

and similarly for Z^*. Hence Z is a tilting module. We have seen in Proposition 2 that in many cases, Z is not a trivial module. In this section we investigate some of its properties.

Proposition 4 *For $p \geq 2h - 2$, the tilting module Z is semisimple.*

Proof: Let μ be any dominant weight of the G-module Z. Then μ satisfy $(p-1)\rho + p\mu \leq (p-1)\rho + \lambda$ and so $p\mu \leq \lambda$. We want to show that any such λ belong to the lowest alcove $C = \{\lambda \in X^+ \mid 0 < \langle \lambda + \rho, \check{\alpha}_0 \rangle < p\}$. By the linkage principle, this would imply that the module Z is semisimple. First note that as λ is restricted, for any simple root α, we have $\langle \lambda, \check{\alpha} \rangle \leq p - 1 = \langle (p-1)\rho, \check{\alpha} \rangle$. So we have that

$$\langle \lambda, \check{\alpha}_0 \rangle \leq \langle (p-1)\rho, \check{\alpha}_0 \rangle = (p-1)(h-1).$$

Now as $p\mu \leq \lambda$, we have

$$p\langle \mu, \check{\alpha}_0 \rangle \leq \langle \lambda, \check{\alpha}_0 \rangle \leq (p-1)(h-1).$$

This implies that $\langle \mu, \check{\alpha}_0 \rangle < (h-1)$ and hence

$$\langle \mu + \rho, \check{\alpha}_0 \rangle < (h-1) + (h-1) = 2h - 2 \leq p$$

by assumption. On the other hand, as μ is dominant, we have that $\langle \mu + \rho, \check{\alpha} \rangle > 0$ for all simple root α. Hence, μ belongs to the lowest alcove as required. QED

Let us now specialise to the case where $L(\lambda) = St$. So we are looking at the G_1-Steinberg block component $St \otimes Z^F$ of $St \otimes St$. Note that the module $St \otimes St$ is tilting for all primes, and hence so is Z. We are going to deduce the dimension of the G-module Z from the following proposition. Although we only need a very particular case of it, namely the dimension of the T_1-fixed points of the Steinberg module, we give a result about any T_1-weight spaces of any induced G_1T-module $\hat{Z}_1'(\lambda)$.

Proposition 5 *For $\lambda \in X$, all non-zero T_1-weight spaces of $\hat{Z}_1'(\lambda)$ have the same dimension, namely*

$$p^{|\Phi^+|}/|\mathbf{Z}\Phi/(\mathbf{Z}\Phi \cap pX)| = p^{|\Phi^+|-r(p)}$$

where $r(p)$ denotes the rank of the Cartan matrix of G over $\mathbf{F_p}$.

Proof: Using Jantzen [8]II.9.16, we see that the set of T-weights (with multiplicities) of $\hat{Z}_1'(\lambda)$ is given by

$$\Lambda = \{\lambda - \sum_{\alpha \in \Phi^+} m_\alpha \alpha,\ 0 \leq m_\alpha \leq p-1\}.$$

Let $\mu = \lambda - \sum_{\alpha \in \Phi^+} n_\alpha \alpha \in \Lambda$. Consider the set of weights $\nu \in \Lambda$ congruent to μ modulo pX. So we want to find all solutions (m_α) of the equation

$$\lambda - \sum_{\alpha \in \Phi^+} m_\alpha \alpha \equiv \lambda - \sum_{\alpha \in \Phi^+} n_\alpha \alpha \mod pX$$

so

$$\sum_{\alpha\in\Phi^+} m_\alpha\alpha \equiv \sum_{\alpha\in\Phi^+} n_\alpha\alpha \mod pX.$$

View it as a system of linear equations over $\mathbf{F_p}$. Then any solution is obtained by adding to μ a solution of the homogeneous system of linear equations

$$\sum_{\alpha\in\Phi^+} m_\alpha\alpha = 0 \quad \text{in } X/pX.$$

The dimension of the $\mathbf{F_p}$-vector space of solutions is $|\Phi^+|-r(p)$ so the number of solution is $p^{|\Phi^+|-r(p)}$. In particular, we see that each non-zero T_1-weight space has the same dimension, as the result is independant of μ. We can also write this dimension as the dimension of $\hat{Z}_1'(\lambda)$, namely $p^{|\Phi^+|}$, divided by the number of distinct T_1-weights, namely $|\mathbf{Z}\Phi/(\mathbf{Z}\Phi\cap pX)|$. QED

Corollary 1 *Let $St\otimes Z^F$ be the G_1-Steinberg block component of the G-module $St\otimes St$. Assume Z is non-zero. Then the dimension of Z is given by*

$$\dim_k Z = p^{|\Phi^+|-r(p)}$$

where $r(p)$ denotes the rank of the Cartan matrix of G over $\mathbf{F_p}$.

Proof: Note that $\dim Z = \dim \mathrm{Hom}_{G_1}(St, St\otimes St)$ and as G_1-modules $St\otimes St \cong \mathrm{Ind}_{T_1}^{G_1} k$, so we have

$$\begin{aligned}\dim Z &= \dim \mathrm{Hom}_{G_1}(St, \mathrm{Ind}_{T_1}^{G_1} k)\\ &= \dim \mathrm{Hom}_{T_1}(St, k)\\ &= \dim St^{T_1}.\end{aligned}$$

Hence the result follows from Proposition 4. QED

Remark: For $p > 2$, the Steinberg weight $(p-1)\rho$ belongs to $\mathbf{Z}\Phi$ so we can always find $(m_\alpha) \in \mathbf{Z}^{|\Phi^+|}$ such that $(p-1)\rho - \sum_{\alpha\in\Phi^+} m_\alpha \alpha \equiv 0 \bmod pX$. Thus in this case the module Z is non-zero.

For $p = 2$, explicit calculations shows that $Z = 0$ if and only if G has type A_n with $n \equiv 1 \bmod 4$, B_n with $n \equiv 1, 2 \bmod 4$, C_n all n or D_n with $n \equiv 2 \bmod 4$.

In all other cases, Z is a non-zero tilting module whose character can in principle be computed. Very few indecomposable tilting modules are known in general so it would be very interesting to determine the decomposition of Z into indecomposable tilting modules.

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